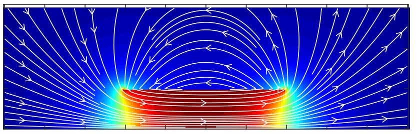
magnetic fields



field and inductance of coils, force between magnets

a) *Bz_current_loop_BiotSavart.mlx* derives the magnetic field on the axis of a current loop
b) *A_current_loop_vectorpotential.mlx* derives the overall magnetic field of a current loop

problem 1: Helmholtz current loops

Discrete superposition of solution a).

problem 2: Magnetic field on the axis of a coil/magnet

Continuous superposition of solution a).

problem 3: Magnetic field on the axis of a multilayer coil

Superposition of the solution of problem 2.

problem 4: Magnetic field on the axis of a spherical coil/magnet

Continuous superposition of solution a).

problem 5: Magnetic field of a current loop

Implement solution b) as function
2D visualization of field lines

problem 6: Magnetic field of a coil

Superposition of the solution of problem 5.
2D visualization of field lines

problem 7: Magnetic field of a multilayer coil

Superposition of the solution of problem 6.
2D visualization of field lines

problem 8: Force between two magnets

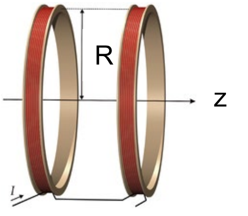
Solution of problem 5 allows to calculate the Lorentz force between two conductor loops.
Superposition of this solution gives the force between two coils or magnets.

problem 9: Inductance of a cylindrical coil

Integrating the solution of problem 6 or b) gives the magnetic flux or magnetic energy of a coil. Both results can be used to calculate the inductance of a coil.

Many thanks to Linda Riedl for the hand-drawn illustrations!

Analytical field calculations



1) Helmholtz coil

a) Define the magnetic field on the axis of a current loop with a symbolic function H(z) and plot the magnetic flux density of two loops in distance.

```
% parameter
mu0=4e-7*pi; % vacuum permeability [H/m]
I=1;         % current [A]
R=0.1;       % radius [m]
d=2*R;       % distance of current loops
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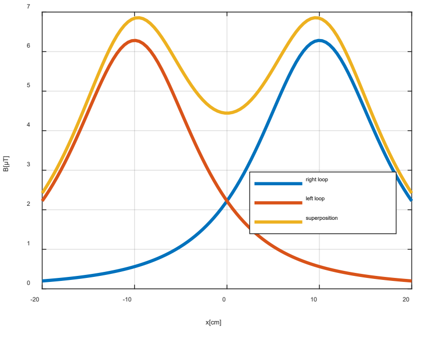
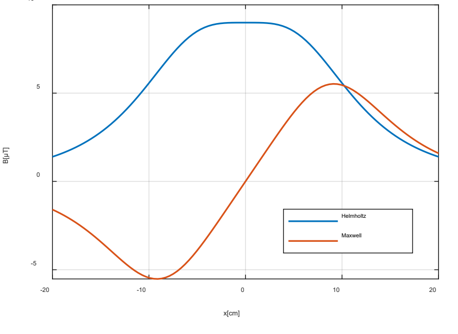
$$H_z = \frac{\mathrm{I}\,R^2}{2\,(R^2+z^2)^{3/2}}$$



b) What is special about the Helmholtz and Maxwell arrangement?
http://de.wikipedia.org/wiki/Helmholtzspule
Calculate results analytically and plot results.



Helmholtz distance
Find the distance d_H of two loops with parallel currents, so that the second derivative of the magnetic field in the center of both loops vanishes.

Maxwell distance
Find the distance d_M of two loops with antiparallel currents , so that the third derivative of the magnetic field in the center of both loops vanishes.

2) Magnetic field on the axis of a cylindrical coil or magnet

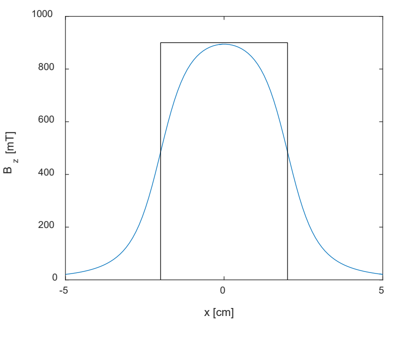
a) Calculate symbolically the magnetic field on the axis of a single-layer coil (N windings, length L, radius R, current I). According to problem 1, a short piece with length dx of the coil generates the magnetic field $dB_z = \frac{\mu_0 I R^2}{2r^3} \cdot \frac{N}{L} dx$

Superimpose the contributions of the individual turns by integrating over the coil length.

solution $$B_z(z) = \mu_0 \frac{I\,N}{2\,L} \left[\frac{\frac{L}{2} - z}{\sqrt{\left(\frac{L}{2} - z\right)^2 + R^2}} + \frac{\frac{L}{2} + z}{\sqrt{\left(\frac{L}{2} + z\right)^2 + R^2}} \right]$$

This formula is also valid for the **magnetic field of a permanent magnet** of the same dimension. In a magnetic material of remanence B_r there is a microscopic surface current density per length of $N \cdot I/L = B_r/\mu_0$.

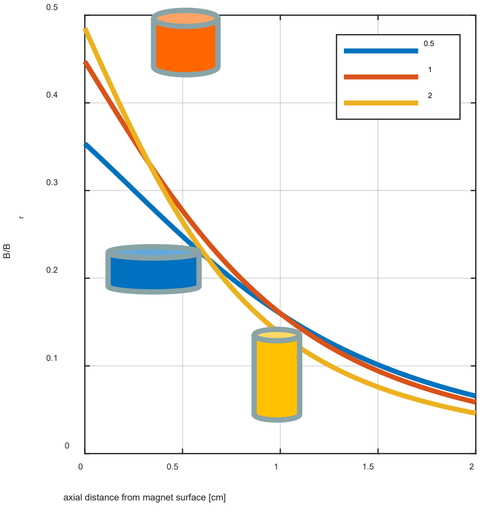
B_r=1.26T is easily achieved by strong magnets made of NdFeB. To generate this field with a coil, e.g. 100 turns with 100A on a length of 1cm are necessary ($B_r/\mu_0 = 10^6$ A/m) !



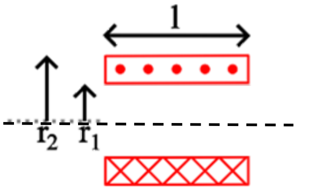
b) Plot the field curve on the axis of a permanent magnet with B_r=1T, R=1cm, L=4cm

Optional

c) The price of a permanent magnet is determined by the magnet volume. What influence does the aspect ratio of a cylindrical permanent magnet have on the axial field characteristics. Check aspect ratios length/diameter of ½, 1 and 2 for a magnet volume of 4π cm³.



3) Multilayer coil



a) Calculate analytically the magnetic field Bz(z) on the axis of a multilayer coil (N_z axial windings on length L, N_r radial windings on width r_2-r_1).

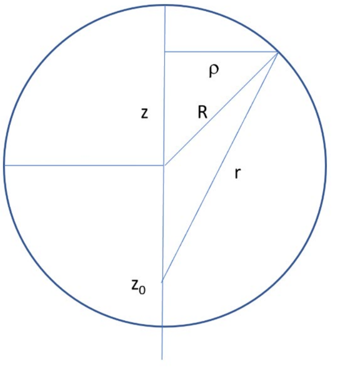
solution in coil center:
$$B_z = \frac{I\, N_r\, N_z\, \mu_0 \left(\operatorname{asinh}\left(\frac{2\, r_1}{L}\right) - \operatorname{asinh}\left(\frac{2\, r_2}{L}\right)\right)}{2\,(r_1 - r_2)}$$

Matlab's solution looks different. Therefore compare by inserting numerical values: (N_z=400, N_r=100, I=1A, L=4cm, r_1=1cm, r_2=2cm)

solution: B_z=1T

Alternative analytical solution

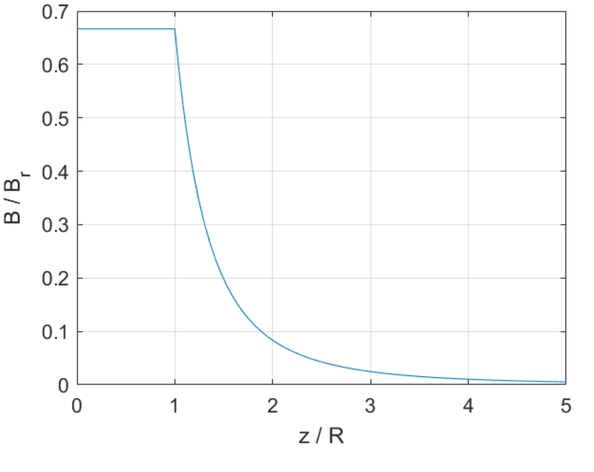
4) Spherical coil or magnet



Calculate analytically **and plot** the magnetic field Bz(z) on the axis of a magnetic sphere (Radius R, remanenz B_r). This is equivalent to the magnetic field of a coil of spherical shape and equidistant windings in axial direction. Superimpose the field contribution of each winding with radius ρ:
$$dB_z = \frac{B_r \rho^2}{2r^3} dz$$

solution on axis:
$$B_z = \frac{B_r}{3} \frac{z_0^3\,(|R - z_0| + (R - z_0)) + R^3\,(|R - z_0| - (R - z_0))}{z_0^3 |R - z_0|} \begin{cases} = \frac{2}{3} B_r & z_0 < R \\ = \frac{2}{3} B_r \frac{R^3}{z_0^3} & z_0 > R \end{cases}$$

Outside of the sphere the field is equivalent to the field of a magnetic dipole located in the center of the sphere.



5) Magnetic field of a current loop

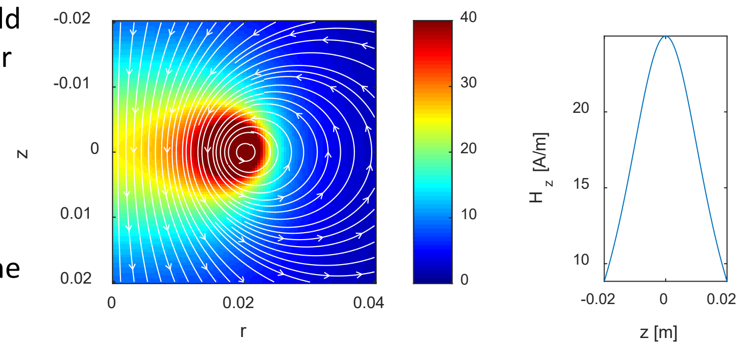
a) Calculate the magnetic field of a current loop at point P. This point is now not on the axis as in problem 1. Using the formula at the bottom, write the function
[Hz, Hr] = Hloop(R,r,z).

Hz is the axial and Hr the radial magnetic field component of a current loop (I=1A) with radius R at axial distance z and radial distance r from the center of the current loop.

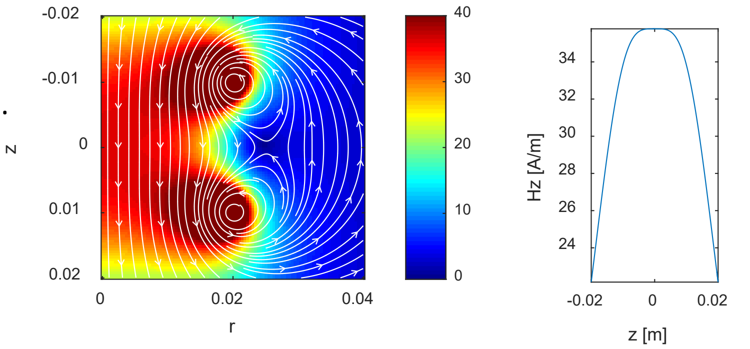
check: Hloop(1, 1, 1) → H_z = 0.0768 (A/m), H_r = 0.091 (A/m)

b) Draw the magnitude of the magnetic field of the conductor loop in the r-z plane for R=2cm with *imagesc* (*colormap jet*) and superimpose the magnetic field vector with *streamslice* .

On the right, draw the z-component of the magnetic field on the axis (r=0) of the current loop.



c) Superimpose the magnetic fields of two current loops with R=2cm at distance d=2cm and generate the same plot as b). The current direction is identical in both loops.



Flux density vector at location ρ, z in cylindrical coordinates of a current loop (center at origin, radius R and current I) according to Wikipedia

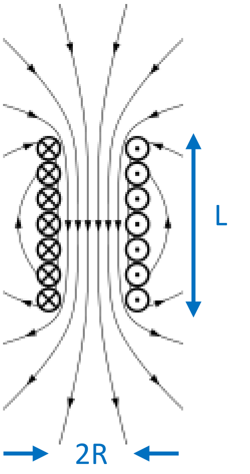
$$B_\rho = \frac{I\mu_0}{2\pi} \frac{1}{\sqrt{(R+\rho)^2+z^2}} \cdot \frac{z}{\rho} \left(\frac{R^2+\rho^2+z^2}{(R-\rho)^2+z^2} E(k^2) - K(k^2) \right)$$

$$B_z = \frac{I\mu_0}{2\pi} \frac{1}{\sqrt{(R+\rho)^2+z^2}} \cdot \left(\frac{R^2-\rho^2-z^2}{(R-\rho)^2+z^2} E(k^2) + K(k^2) \right)$$

$K(k^2)$ and $E(k^2)$ are elliptic integrals of first and second kind and

$$k^2 = \frac{4R\rho}{(R+\rho)^2+z^2}$$

6) Magnetic field of a cylindrical coil or magnet



a) Write the function [Hz Hr]=Hcoil(N,R,L,r,z) This gives the axial component Hz and radial component Hr of the magnetic field of a single-layer coil (I=1A) of radius R and length L with N turns at axial distance z and radial distance r from the coil center.

<u>Note</u>: Integrate the contributions of the individual windings as in 2a) or add them as in 5c).

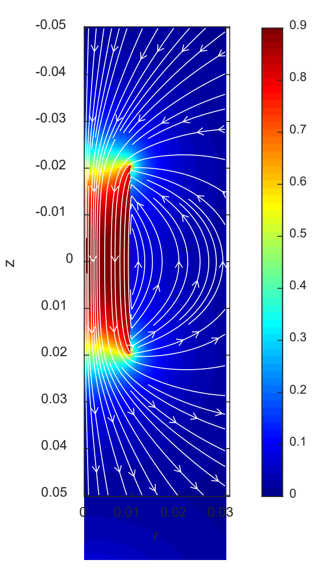
check: Hcoil(100, 0.005, 0.01, 0.003, 0.005)

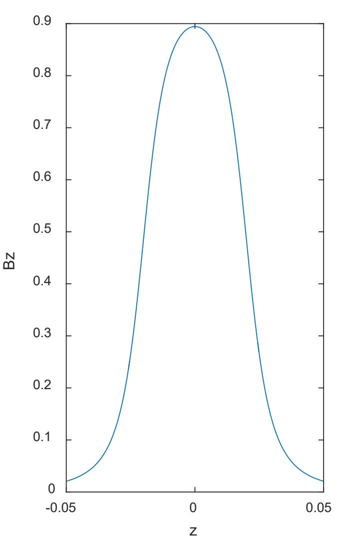
$\rightarrow$ H_r = 1641,1 A/m, H_z = 4517,7 A/m with integral

$\rightarrow$ H_r = 1625,7 A/m, H_z = 4546,8 A/m with sum

b) Calculate the magnetic field of a single layer coil with I=1A, radius R=1cm, length L=4cm and number of turns N = 31831 (corresponding to permanent magnet with $B_r = \mu_0 \, N \cdot I/L$ = 1T). Visualize the field as in problem 5.

c) Compare the field variation on the axis with the analytical solution (problem 2a).





Optional:
Compare the result with the analytical solution on Wikipedia (*Hcoil_Wiki.m*) for the parameter values from a).

$$B_\rho = \frac{\mu_0}{4\pi}\frac{NI}{l}\frac{1}{\rho}\left[\sqrt{(\rho+R)^2+\zeta^2}\Big((2-m)\,K(m) - 2\,E(m)\Big)\right]_{\zeta=z+l/2}^{\zeta=z-l/2}$$

$$B_z = \frac{\mu_0}{2\pi}\frac{NI}{l}\left[\frac{\zeta}{\sqrt{(\rho+R)^2+\zeta^2}}\left(\frac{\rho-R}{\rho+R}\Pi(n,m) - K(m)\right)\right]_{\zeta=z+l/2}^{\zeta=z-l/2}$$

$$m = 4R\rho/\left((\rho+R)^2+\zeta^2\right), \quad n = 4R\rho/(\rho+R)^2$$

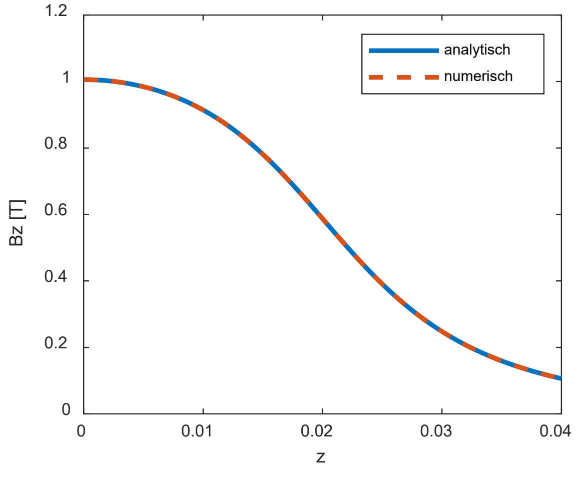
$$K(m) = \int_0^{\pi/2}\frac{1}{\sqrt{1-m\sin^2\varphi}}\,\mathrm{d}\varphi \qquad E(m) = \int_0^{\pi/2}\sqrt{1-m\sin^2\varphi}\,\mathrm{d}\varphi \qquad \Pi(n,m) = \int_0^{\pi/2}\frac{1}{(1-n\sin^2\varphi)\sqrt{1-m\sin^2\varphi}}\,\mathrm{d}\varphi$$

Note: From Dec. 2017 till Feb. 2019 there was a small error in the formula published in wikipedia.

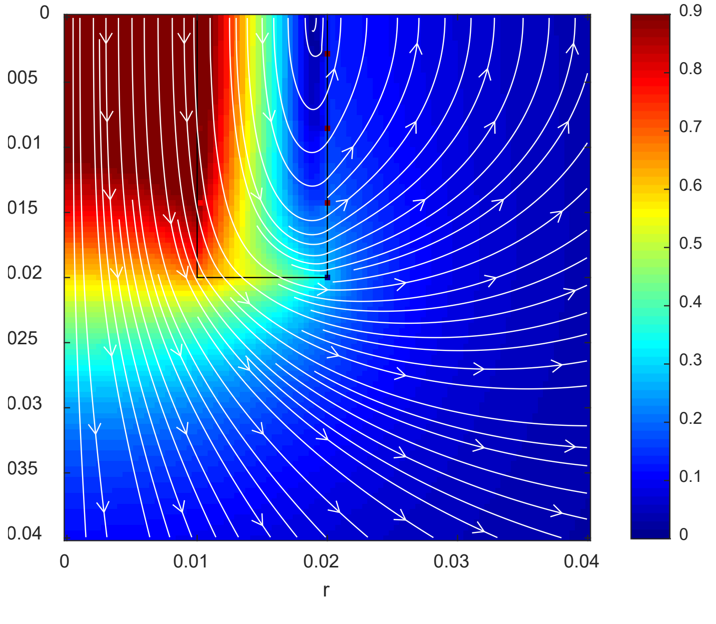
7) Magnetic field of a cylindrical multilayer coil

a) Write the function [Hz Hr] = Hmcoil(Nr,r1,r2,Nz,L,r,z).
This gives the axial component Hz and the radial component Hr of the magnetic field of a multilayer coil (I=1A, Nz axial turns on length L, Nr radial turns on width r2-r1).

Calculate the magnetic field on the coil axis for these parameters
(Nz=400, Nr=100, I=1A, L=4cm, r1=1cm, r2=2cm)
and compare with the analytical solution from problem 3.



b) Visualize the magnetic field of the coil with the parameters from above.
Hint: Sum up the contribution of each winding and do not integrate.



Force between two magnets

8) Force between two magnets

With what force do two axially magnetized cylinder magnets (Br=1.3T, diameter=length=15mm) attract each other? Consider the equivalent problem of two current-carrying coils (problem 6).

a) First calculate the force between two current loops with currents $I_1=I_2=1A$, loop radius $R_1=R_2=7{,}5mm$ and loop distance z=15mm. Applying this formula for the Lorentz force $\vec{F} = I(\vec{l} \times \vec{B})$ results in $F_z = I_2 \cdot 2\pi R_2 \cdot B_{r,1}$

solution: F = 0,1527µN

b) Calculate the force between a single current loop from a) and a coil or magnet by integrating over the force contribution of each winding of coil 2.

solution: F = 3,4mN

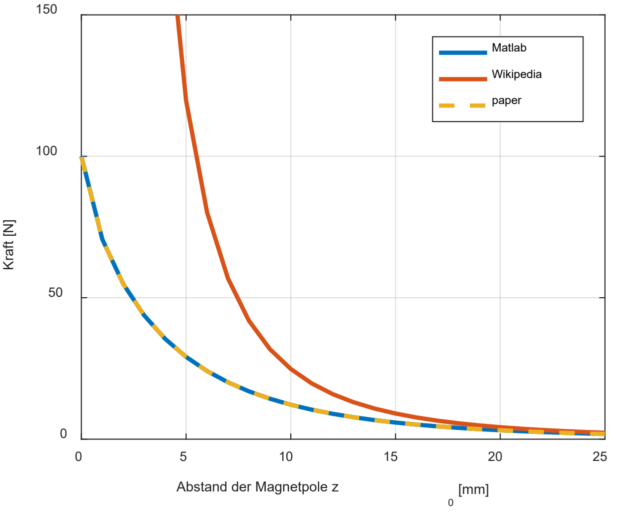
c) Calculate the force between two touching magnets, by integrating the force contributions of all conductor loops of both coils (double integral).

solution: F = 100N

d) Calculate the force for different distances between the magnets and compare the result with an approximation from Wikipedia

$$F(z_0) = \frac{\pi\mu_0}{4}\frac{N^2I^2}{L^2}R^4\left[\frac{1}{z_0^2} + \frac{1}{(z_0+2L)^2} - \frac{2}{(z_0+L)^2}\right]$$

and with equation 4 from the paper of Vokoun et al. **„Magnetostatic interactions and forces between cylindrical permanent magnets“**



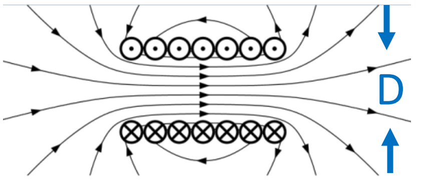
$$F_z = 4\pi\mu_0\frac{N^2I^2}{L^2}R^2\int_0^\infty \frac{1}{q}J_1^2(q)\,\sinh(\frac{qL}{2R})\,\sinh(\frac{qL}{2R})e^{-qz/R}$$

J_1 = Besselfunction first kind

coil inductance

9) Inductance of a cylindrical coil

Calculate the inductance of a coil with diameter D=2r=2cm, length l =1cm and N=1000 windings.



a) with the analytical solution from wikipedia

$$L = \frac{\mu_0 r^2 N^2}{3l}\left(-8w + 4\frac{\sqrt{1+m}}{m}\left[K\left(\sqrt{\frac{m}{1+m}}\right) - (1-m)\,E\left(\sqrt{\frac{m}{1+m}}\right)\right]\right)$$

solution: L=20,7mH

b) numerically with the concatenated flux $\Psi = L \cdot I = \sum_{i=1}^{N} \iint_{A_{loop}} \mathbf{B} \cdot d\mathbf{a} = \frac{N}{L} \iiint_{V_{coil}} B_z \; dV$

c) numerically with a volume integral of magnetic energy density $w = \frac{1}{2}\mu_0 H^2$

As the fields become small far from the coil, you can limit the integral to a small space around the coil.

$$W = \frac{1}{2}LI^2 = \iiint_0^\infty w \; dV$$

d) numerically with a surface integral over the windings of the magnetic energy density from current density j and vector potential A.

$$w = \frac{1}{2} jA$$

$$L = \frac{\mu_0 N^2 A}{l}.$$

e) Vary the length of the coil and compare the analytical solution from a) with this approximate solution from GE1 for long coils

$$L = \frac{\mu_0 N^2 A}{\alpha l + \beta r}$$

f) The analytical solution from a) is bulky, the approximation from d) is inaccurate for short coils. Determine the parameters α and β of an empirical approximation so that the error becomes minimal for l / r > 0.5.
Compare the result from a), e) and f) graphically