

Robotics Assignment #02

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Task 2.1. 1) The manipulator transformation is a series of multiple rotational $Rot_z(\theta_i)$ and translational $Trans_{x_i}(a_i)$ transformations. That means, ${}^0T_3 = {}^0A_1 {}^1A_2 {}^2A_3$ is given by

$${}^0A_1 = Rot_z(\theta_1) \cdot Trans_{x_1}(a_1)$$

$${}^1A_2 = Rot_z(\theta_2) \cdot Trans_{x_2}(a_2)$$

$${}^2A_3 = Rot_z(\theta_3) \cdot Trans_{x_3}(a_3)$$

$$\text{where } Rot_z(\theta_i) = \begin{bmatrix} C_i & -S_i & 0 & 0 \\ S_i & C_i & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{ and } Trans_{x_i}(a_i) = \begin{bmatrix} 1 & 0 & 0 & a_i \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

With this in mind, we can calculate the partial homogeneous transformations:

$${}^0A_1 = \begin{bmatrix} C_1 & -S_1 & 0 & C_1 a_1 \\ S_1 & C_1 & 0 & S_1 a_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad {}^1A_2 = \begin{bmatrix} C_2 & -S_2 & 0 & C_2 a_2 \\ S_2 & C_2 & 0 & S_2 a_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad {}^2A_3 = \begin{bmatrix} C_3 & -S_3 & 0 & C_3 a_3 \\ S_3 & C_3 & 0 & S_3 a_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

For intermediate results, we first calculate ${}^0T_2 = {}^0A_1 {}^1A_2$ and then ${}^0T_3 = {}^0T_2 {}^2A_3$.

$$\begin{aligned} {}^0T_2 &= \begin{bmatrix} C_1 & -S_1 & 0 & C_1 a_1 \\ S_1 & C_1 & 0 & S_1 a_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} C_2 & -S_2 & 0 & C_2 a_2 \\ S_2 & C_2 & 0 & S_2 a_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} C_1 C_2 - S_1 S_2 & -C_1 S_2 - S_1 C_2 & 0 & C_1 C_2 a_2 - S_1 S_2 a_2 + C_1 a_1 \\ S_1 C_2 + C_1 S_2 & -S_1 S_2 + C_1 C_2 & 0 & S_1 C_2 a_2 + C_1 S_2 a_2 + S_1 a_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} C_{1+2} & -S_{1+2} & 0 & C_{1+2} a_2 + C_1 a_1 \\ S_{1+2} & C_{1+2} & 0 & S_{1+2} a_2 + S_1 a_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\ {}^0T_3 &= \begin{bmatrix} C_{1+2} & -S_{1+2} & 0 & C_{1+2} a_2 + C_1 a_1 \\ S_{1+2} & C_{1+2} & 0 & S_{1+2} a_2 + S_1 a_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} C_3 & -S_3 & 0 & C_3 a_3 \\ S_3 & C_3 & 0 & S_3 a_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} C_{1+2} C_3 - S_{1+2} S_3 & -C_{1+2} S_3 - S_{1+2} C_3 & 0 & C_{1+2} C_3 a_3 - S_{1+2} S_3 a_3 + C_{1+2} a_2 + C_1 a_1 \\ S_{1+2} C_3 + C_{1+2} S_3 & -S_{1+2} S_3 + C_{1+2} C_3 & 0 & S_{1+2} C_3 a_3 + C_{1+2} S_3 a_3 + S_{1+2} a_2 + S_1 a_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

$$= \begin{bmatrix} C_{1+2+3} & -S_{1+2+3} & 0 & C_{1+2+3}a_3 + C_{1+2}a_2 + C_1a_1 \\ S_{1+2+3} & C_{1+2+3} & 0 & S_{1+2+3}a_3 + S_{1+2}a_2 + S_1a_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Because the relation $\theta_1 + \theta_2 + \theta_3 = 180^\circ$ is given, we can simplify the results. For $\cos(180^\circ) = -1$ and $\sin(180^\circ) = 0$:

$${}^0T_3 = \begin{bmatrix} -1 & 0 & 0 & -a_3 + C_{1+2}a_2 + C_1a_1 \\ 0 & -1 & 0 & S_{1+2}a_2 + S_1a_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Furthermore, $\cos(\alpha) = -\cos(180^\circ - \alpha)$ and $\sin(\alpha) = \sin(180^\circ - \alpha)$ for $\alpha \in [0^\circ, 180^\circ]$. These facts result in the given transformation matrix:

$${}^0T_3 = \begin{bmatrix} -1 & 0 & 0 & C_1a_1 - C_3a_2 - a_3 \\ 0 & -1 & 0 & S_1a_1 + S_3a_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- 2) To rotate around x_0 , we need to use Euler angles and append the rotation transformation matrix to the left side of 0T_3 , because we need to rotate the manipulator before the other transformation can be started.

$$\begin{aligned} {}^0T_3' &= Rot_{x_0}(\theta_0) \cdot {}^0T_3 \\ &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & C_0 & -S_0 & 0 \\ 0 & S_0 & C_0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 & C_1a_1 - C_3a_2 - a_3 \\ 0 & -1 & 0 & S_1a_1 + S_3a_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} -1 & 0 & 0 & C_1a_1 - C_3a_2 - a_3 \\ 0 & -C_0 & -S_0 & C_0S_1a_1 + C_0S_3a_2 \\ 0 & -S_0 & C_0 & S_0S_1a_1 + S_0S_3a_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

For the second rotation around x_3 by θ_4 , we append the rotation matrix to the right side of 0T_3 .

$$\begin{aligned} {}^0T_3'' &= {}^0T_3 \cdot Rot_{x_3}(\theta_4) \\ &= \begin{bmatrix} -1 & 0 & 0 & C_1a_1 - C_3a_2 - a_3 \\ 0 & -1 & 0 & S_1a_1 + S_3a_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & C_4 & -S_4 & 0 \\ 0 & S_4 & C_4 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} -1 & 0 & 0 & C_1a_1 - C_3a_2 - a_3 \\ 0 & -C_4 & S_4 & S_1a_1 + S_3a_2 \\ 0 & S_4 & C_4 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

Task 2.2. 1)

2)

Task 2.3.

Link i	θ_i	α_{i-1}	a_{i-1}	d_i
1	θ_1	0°	L_1	0
2	θ_2	0°	L_2	0
3	θ_3	0°	L_3	0

Task 2.4.

- 1) The manipulator shown in the figure follows the Denavit-Hartenberg-convention. Every z-axis is given by the axis of rotation of its joint. Every x-axis shows in the directions of the common normal of both z-axes, or better in the direction of the shortest connection between both z-axes, since there are no common normals for parallel z-axes. Every y-axis and every rotation direction is then given by the fact, that we want to have a right handed coordinate system.
- 2) The homogeneous transformation ${}^{Base}T_{Tool}$ is given by concatenation of the partial transformations from one frame to the next:

$${}^{Base}T_{Tool} = {}^0T_1 {}^1T_2 {}^2T_3 {}^3T_4$$

We determine the partial transformation matrices as follows and convert milimeters to meters to avoid big numbers:

$$\begin{aligned}
{}^0T_1 &= Trans_{(0,0,877mm)} \cdot Rot_{z_0}(\theta_1) \cdot Trans_{(425mm,0,0)} \cdot Rot_{x_0}(\pi) \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0.877 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos(\theta_1) & -\sin(\theta_1) & 0 & 0 \\ \sin(\theta_1) & \cos(\theta_1) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
&\quad \begin{pmatrix} 1 & 0 & 0 & 0.425 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\pi) & -\sin(\pi) & 0 \\ 0 & \sin(\pi) & \cos(\pi) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} \cos(\theta_1) & -\sin(\theta_1) & 0 & 0 \\ \sin(\theta_1) & \cos(\theta_1) & 0 & 0 \\ 0 & 0 & 1 & 0.877 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0.425 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
&\quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} \cos(\theta_1) & -\sin(\theta_1) & 0 & 0.425 \cdot \cos(\theta_1) \\ \sin(\theta_1) & \cos(\theta_1) & 0 & 0.425 \cdot \sin(\theta_1) \\ 0 & 0 & 1 & 0.877 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} \cos(\theta_1) & \sin(\theta_1) & 0 & 0.425 \cdot \cos(\theta_1) \\ \sin(\theta_1) & -\cos(\theta_1) & 0 & 0.425 \cdot \sin(\theta_1) \\ 0 & 0 & -1 & 0.877 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
{}^1T_2 &= Rot_{z_1}(\theta_2) \cdot Trans_{(375mm,0,0)} \\
&= \begin{pmatrix} \cos(\theta_2) & -\sin(\theta_2) & 0 & 0 \\ \sin(\theta_2) & \cos(\theta_2) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0.375 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
&= \begin{pmatrix} \cos(\theta_2) & -\sin(\theta_2) & 0 & 0.375 \cdot \cos(\theta_2) \\ \sin(\theta_2) & \cos(\theta_2) & 0 & 0.375 \cdot \sin(\theta_2) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
{}^2T_3 &= Trans_{(0,0,d_3)} \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
{}^3T_4 &= Trans_{(0,0,200mm)} \cdot Rot_{z_3}(\pi/2 + \theta_4) \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0.2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos(\pi/2 + \theta_4) & -\sin(\pi/2 + \theta_4) & 0 & 0 \\ \sin(\pi/2 + \theta_4) & \cos(\pi/2 + \theta_4) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} \cos(\pi/2 + \theta_4) & -\sin(\pi/2 + \theta_4) & 0 & 0 \\ \sin(\pi/2 + \theta_4) & \cos(\pi/2 + \theta_4) & 0 & 0 \\ 0 & 0 & 1 & 0.2 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\end{aligned}$$

Now we can compute the final transformation matrix:

$$\begin{aligned}
{}^{Base}T_{Tool} &= {}^0T_1 {}^1T_2 {}^2T_3 {}^3T_4 \\
&= \begin{pmatrix} C_1C_2 + S_1S_2 & -S_2C_1 + S_1C_2 & 0 & 0.375C_1C_2 + 0.375S_1S_2 + 0.425C_1 \\ S_1C_2 - C_1S_2 & -S_1S_2 - C_1C_2 & 0 & 0.375S_1C_2 - 0.375C_1S_2 + 0.425S_1 \\ 0 & 0 & -1 & 0.877 \\ 0 & 0 & 0 & 1 \end{pmatrix} {}^2T_3 {}^3T_4 \\
&= \begin{pmatrix} C_1C_2 + S_1S_2 & -S_2C_1 + S_1C_2 & 0 & 0.375C_1C_2 + 0.375S_1S_2 + 0.425C_1 \\ S_1C_2 - C_1S_2 & -S_1S_2 - C_1C_2 & 0 & 0.375S_1C_2 - 0.375C_1S_2 + 0.425S_1 \\ 0 & 0 & -1 & -d_3 + 0.877 \\ 0 & 0 & 0 & 1 \end{pmatrix} {}^3T_4 \\
&= \begin{pmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{pmatrix}
\end{aligned}$$

with:

$$\begin{aligned}
n_x &= (C_1C_2 + S_1S_2) \cos(\pi/2 + \theta_4) + (-S_2C_1 + S_1C_2) \sin(\pi/2 + \theta_4) \\
n_y &= (S_1C_2 - C_1S_2) \cos(\pi/2 + \theta_4) + (-S_1S_2 - C_1C_2) \sin(\pi/2 + \theta_4) \\
n_z &= 0 \\
o_x &= -(C_1C_2 + S_1S_2) \sin(\pi/2 + \theta_4) + (-S_2C_1 + S_1C_2) \cos(\pi/2 + \theta_4) = n_y \\
o_y &= -(S_1C_2 - C_1S_2) \sin(\pi/2 + \theta_4) + (-S_1S_2 - C_1C_2) \cos(\pi/2 + \theta_4) = -n_x \\
o_z &= 0 \\
a_x &= 0 \\
a_y &= 0 \\
a_z &= -1 \\
p_x &= 0.375C_1C_2 + 0.375S_1S_2 + 0.425C_1
\end{aligned}$$

$$p_y = 0.375S_1C_2 - 0.375C_1S_2 + 0.425S_1$$

$$p_z = -0.2 - d_3 + 0.877$$

where $C_i = \cos(\theta_i)$ and $S_i = \sin(\theta_i)$.

- 3) Before we can calculate the result, we need to insert the variables into ${}^{Base}T_{Tool}$. **Note:** Since we work with *meters* and not with *mm* we need to convert 120 to 0.12. Overall, this leads to:

$${}^{Base}T_{Tool} \approx \begin{pmatrix} -0.7071 & -0.9659 & 0 & 0.2035 \\ -0.9659 & 0.7071 & 0 & 0.6627 \\ 0 & 0 & -1 & 0.557 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Assuming the origin of the base frame is the origin of the world coordinate system (at $(0, 0, 0)$), we can calculate the coordinates of the tool center point as follows:

$$\begin{aligned} {}^{Base}T_{Tool} \cdot \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} &\approx \begin{pmatrix} -0.7071 & -0.9659 & 0 & 0.2035 \\ -0.9659 & 0.7071 & 0 & 0.6627 \\ 0 & 0 & -1 & 0.557 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \\ &\approx \begin{pmatrix} 0.2035 \\ 0.6627 \\ 0.557 \\ 1 \end{pmatrix} \end{aligned}$$

The tool center point is at the point $(203.5, 662.7, 557)$ of the base coordinate system.