Robotics Assignment #02

Paul Bienkowski, Konstantin Kobs

May 5, 2015

Task 2.1. 1) The manipulator transformation is a series of multiple rotational $Rot_z(\theta_i)$ and translational $Trans_{x_i}(a_i)$ transformations. That means, ${}^0T_3 = {}^0A_1{}^1A_2{}^2A_3$ is given by

$$\begin{aligned}
{}^0A_1 &= Rot_z(\theta_1) \cdot Trans_{x_1}(a_1) \\
{}^1A_2 &= Rot_z(\theta_2) \cdot Trans_{x_2}(a_2) \\
{}^2A_3 &= Rot_z(\theta_3) \cdot Trans_{x_3}(a_3)
\end{aligned}$$

where $Rot_z(\theta_i) = \begin{bmatrix} C_i & -S_i & 0 & 0 \\ S_i & C_i & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ and $Trans_{x_i}(a_i) = \begin{bmatrix} 1 & 0 & 0 & a_i \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

With this in mind, we can calculate the partial homogeneous transformations:

$$ {}^0A_1 = \begin{bmatrix} C_1 & -S_1 & 0 & C_1a_1 \\ S_1 & C_1 & 0 & S_1a_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad {}^1A_2 = \begin{bmatrix} C_2 & -S_2 & 0 & C_2a_2 \\ S_2 & C_2 & 0 & S_2a_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad {}^2A_3 = \begin{bmatrix} C_3 & -S_3 & 0 & C_3a_3 \\ S_3 & C_3 & 0 & S_3a_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

For intermediate results, we first calculate ${}^0T_2 = {}^0A_1{}^1A_2$ and then ${}^0T_3 = {}^0T_2{}^2A_3$.

$$\begin{aligned}
{}^0T_2 &= \begin{bmatrix} C_1 & -S_1 & 0 & C_1a_1 \\ S_1 & C_1 & 0 & S_1a_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} C_2 & -S_2 & 0 & C_2a_2 \\ S_2 & C_2 & 0 & S_2a_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} C_1C_2 - S_1S_2 & -C_1S_2 - S_1C_2 & 0 & C_1C_2a_2 - S_1S_2a_2 + C_1a_1 \\ S_1C_2 + C_1S_2 & -S_1S_2 + C_1C_2 & 0 & S_1C_2a_2 + C_1S_2a_2 + S_1a_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} C_{1+2} & -S_{1+2} & 0 & C_{1+2}a_2 + C_1a_1 \\ S_{1+2} & C_{1+2} & 0 & S_{1+2}a_2 + S_1a_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
{}^0T_3 &= \begin{bmatrix} C_{1+2} & -S_{1+2} & 0 & C_{1+2}a_2 + C_1a_1 \\ S_{1+2} & C_{1+2} & 0 & S_{1+2}a_2 + S_1a_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} C_3 & -S_3 & 0 & C_3a_3 \\ S_3 & C_3 & 0 & S_3a_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} C_{1+2}C_3 - S_{1+2}S_3 & -C_{1+2}S_3 - S_{1+2}C_3 & 0 & C_{1+2}C_3a_3 - S_{1+2}S_3a_3 + C_{1+2}a_2 + C_1a_1 \\ S_{1+2}C_3 + C_{1+2}S_3 & -S_{1+2}S_3 + C_{1+2}C_3 & 0 & S_{1+2}C_3a_3 + C_{1+2}S_3a_3 + S_{1+2}a_2 + S_1a_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\end{aligned}$$

$$
= \begin{bmatrix} C_{1+2+3} & -S_{1+2+3} & 0 & C_{1+2+3}a_3 + C_{1+2}a_2 + C_1a_1 \\ S_{1+2+3} & C_{1+2+3} & 0 & S_{1+2+3}a_3 + S_{1+2}a_2 + S_1a_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

Because the relation $\theta_1+\theta_2+\theta_3 = 180°$ is given, we can simplify the results. For $\cos(180°) = -1$ and $\sin(180°) = 0$:

$$
{}^0T_3 = \begin{bmatrix} -1 & 0 & 0 & -a_3 + C_{1+2}a_2 + C_1a_1 \\ 0 & -1 & 0 & S_{1+2}a_2 + S_1a_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

Furthermore, $\cos(\alpha) = -\cos(180° - \alpha)$ and $\sin(\alpha) = \sin(180° - \alpha)$ for $\alpha \in [0°, 180°]$. These facts result in the given transformation matrix:

$$
{}^0T_3 = \begin{bmatrix} -1 & 0 & 0 & C_1a_1 - C_3a_2 - a_3 \\ 0 & -1 & 0 & S_1a_1 + S_3a_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

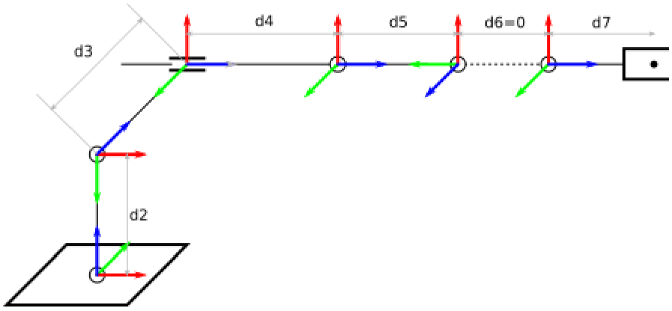
2) To rotate around x_0, we need to use Euler angles and append the rotation transformation matrix to the left side of ${}^0T_3$, because we need to rotate the manipulator before the other transformation can be started.

$$
\begin{aligned}
{}^0T_3{}' &= Rot_{x_0}(\theta_0) \cdot {}^0T_3 \\
&= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & C_0 & -S_0 & 0 \\ 0 & S_0 & C_0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 & C_1a_1 - C_3a_2 - a_3 \\ 0 & -1 & 0 & S_1a_1 + S_3a_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} -1 & 0 & 0 & C_1a_1 - C_3a_2 - a_3 \\ 0 & -C_0 & -S_0 & C_0S_1a_1 + C_0S_3a_2 \\ 0 & -S_0 & C_0 & S_0S_1a_1 + S_0S_3a_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\end{aligned}
$$

For the second rotation around x_3 by θ_4, we append the rotation matrix to the right side of ${}^0T_3$.

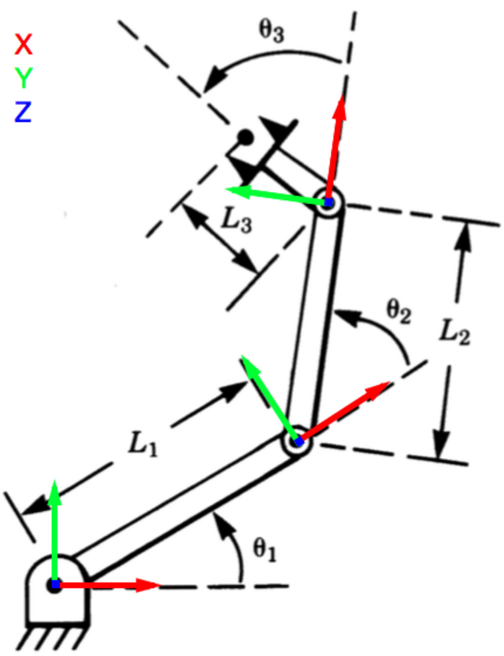
$$
\begin{aligned}
{}^0T_3{}'' &= {}^0T_3 \cdot Rot_{x_3}(\theta_4) \\
&= \begin{bmatrix} -1 & 0 & 0 & C_1a_1 - C_3a_2 - a_3 \\ 0 & -1 & 0 & S_1a_1 + S_3a_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & C_4 & -S_4 & 0 \\ 0 & S_4 & C_4 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} -1 & 0 & 0 & C_1a_1 - C_3a_2 - a_3 \\ 0 & -C_4 & S_4 & S_1a_1 + S_3a_2 \\ 0 & S_4 & C_4 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\end{aligned}
$$

Task 2.2. As by convention, we color code the axes in the figure as follows: x (red), y (green), z (blue). We show the specified coordinate frames and the DH parameters with * marking the variable of the joint.



Link i	d_i	θ_i	r_{i-1}	α_{i-1}
1	d_2	*	0	$-\frac{\pi}{2}$
2	d_3	*	0	$-\frac{\pi}{2}$
3	*	0	0	0
4	d_5	*	0	$-\frac{\pi}{2}$
5	0	*	0	$\frac{\pi}{2}$
6	d_T	*	0	0

Task 2.3. The coordinate frames are specified via the figure **??**. In this drawing, every z-axis points towards the viewer, in order to complete a right-handes coordinate system. We show the specified coordinate frames and the DH parameters:



Link i	θ_i	α_{i-1}	a_{i-1}	d_i
1	θ_1	$0°$	L_1	0
2	θ_2	$0°$	L_2	0
3	θ_3	$0°$	L_3	0

Task 2.4. 1) The manipulator shown in the figure follows the Denavit-Hartenberg-convention. Every z-axis is given by the axis of rotation of its joint. Every x-axis shows in the directions of the common normal of both z-axes, or better in the direction of the shortest connection between both z-axes, since there are no common normals for parallel z-axes. Every y-axis and every rotation direction is then given by the fact, that we want to have a right handed coordinate system.

2) The homogeneous transformation ${}^{Base}T_{Tool}$ is given by concatenation of the partial transformations from one frame to the next:

$$ {}^{Base}T_{Tool} = {}^0T_1 {}^1T_2 {}^2T_3 {}^3T_4 $$

We determine the partial transformation matrices as follows and convert milimeters to meters to avoid big numbers:

$$
\begin{aligned}
{}^0T_1 &= Trans_{(0,0,877mm)} \cdot Rot_{z_0}(\theta_1) \cdot Trans_{(425mm,0,0)} \cdot Rot_{x_0}(\pi) \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0.877 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos(\theta_1) & -\sin(\theta_1) & 0 & 0 \\ \sin(\theta_1) & \cos(\theta_1) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
&\quad \begin{pmatrix} 1 & 0 & 0 & 0.425 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\pi) & -\sin(\pi) & 0 \\ 0 & \sin(\pi) & \cos(\pi) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} \cos(\theta_1) & -\sin(\theta_1) & 0 & 0 \\ \sin(\theta_1) & \cos(\theta_1) & 0 & 0 \\ 0 & 0 & 1 & 0.877 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0.425 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
&\quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} \cos(\theta_1) & -\sin(\theta_1) & 0 & 0.425 \cdot \cos(\theta_1) \\ \sin(\theta_1) & \cos(\theta_1) & 0 & 0.425 \cdot \sin(\theta_1) \\ 0 & 0 & 1 & 0.877 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} \cos(\theta_1) & \sin(\theta_1) & 0 & 0.425 \cdot \cos(\theta_1) \\ \sin(\theta_1) & -\cos(\theta_1) & 0 & 0.425 \cdot \sin(\theta_1) \\ 0 & 0 & -1 & 0.877 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
{}^1T_2 &= Rot_{z_1}(\theta_2) \cdot Trans_{(375mm,0,0)} \\
&= \begin{pmatrix} \cos(\theta_2) & -\sin(\theta_2) & 0 & 0 \\ \sin(\theta_2) & \cos(\theta_2) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0.375 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} \cos(\theta_2) & -\sin(\theta_2) & 0 & 0.375 \cdot \cos(\theta_2) \\ \sin(\theta_2) & \cos(\theta_2) & 0 & 0.375 \cdot \sin(\theta_2) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
{}^2T_3 &= Trans_{(0,0,d_3)} \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
{}^3T_4 &= Trans_{(0,0,200mm)} \cdot Rot_{z_3}(\pi/2 + \theta_4)
\end{aligned}
$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0.2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos(\pi/2+\theta_4) & -\sin(\pi/2+\theta_4) & 0 & 0 \\ \sin(\pi/2+\theta_4) & \cos(\pi/2+\theta_4) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \cos(\pi/2+\theta_4) & -\sin(\pi/2+\theta_4) & 0 & 0 \\ \sin(\pi/2+\theta_4) & \cos(\pi/2+\theta_4) & 0 & 0 \\ 0 & 0 & 1 & 0.2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Now we can compute the final transformation matrix:

$$\begin{aligned}
{}^{Base}T_{Tool} &= {}^0T_1{}^1T_2{}^2T_3{}^3T_4 \\
&= \begin{pmatrix} C_1C_2+S_1S_2 & -S_2C_1+S_1C_2 & 0 & 0.375C_1C_2+0.375S_1S_2+0.425C_1 \\ S_1C_2-C_1S_2 & -S_1S_2-C_1C_2 & 0 & 0.375S_1C_2-0.375C_1S_2+0.425S_1 \\ 0 & 0 & -1 & 0.877 \\ 0 & 0 & 0 & 1 \end{pmatrix} {}^2T_3{}^3T_4 \\
&= \begin{pmatrix} C_1C_2+S_1S_2 & -S_2C_1+S_1C_2 & 0 & 0.375C_1C_2+0.375S_1S_2+0.425C_1 \\ S_1C_2-C_1S_2 & -S_1S_2-C_1C_2 & 0 & 0.375S_1C_2-0.375C_1S_2+0.425S_1 \\ 0 & 0 & -1 & -d_3+0.877 \\ 0 & 0 & 0 & 1 \end{pmatrix} {}^3T_4 \\
&= \begin{pmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{pmatrix}
\end{aligned}$$

with:

$$\begin{aligned}
n_x &= (C_1C_2+S_1S_2)\cos(\pi/2+\theta_4) + (-S_2C_1+S_1C_2)\sin(\pi/2+\theta_4) \\
n_y &= (S_1C_2-C_1S_2)\cos(\pi/2+\theta_4) + (-S_1S_2-C_1C_2)\sin(\pi/2+\theta_4) \\
n_z &= 0 \\
o_x &= -(C_1C_2+S_1S_2)\sin(\pi/2+\theta_4) + (-S_2C_1+S_1C_2)\cos(\pi/2+\theta_4) = n_y \\
o_y &= -(S_1C_2-C_1S_2)\sin(\pi/2+\theta_4) + (-S_1S_2-C_1C_2)\cos(\pi/2+\theta_4) = -n_x \\
o_z &= 0 \\
a_x &= 0 \\
a_y &= 0 \\
a_z &= -1 \\
p_x &= 0.375C_1C_2+0.375S_1S_2+0.425C_1 \\
p_y &= 0.375S_1C_2-0.375C_1S_2+0.425S_1 \\
p_z &= -0.2-d_3+0.877
\end{aligned}$$

where $C_i = \cos(\theta_i)$ and $S_i = \sin(\theta_i)$.

3) Before we can calculate the result, we need to insert the variables into ${}^{Base}T_{Tool}$.
   Note: Since we work with *meters* and not with *mm* we need to convert 120 to 0.12. Overall, this leads to:

$$
{}^{Base}T_{Tool} \approx \begin{pmatrix} -0.7071 & -0.9659 & 0 & 0.2035 \\ -0.9659 & 0.7071 & 0 & 0.6627 \\ 0 & 0 & -1 & 0.557 \\ 0 & 0 & 0 & 1 \end{pmatrix}
$$

   Assuming the origin of the base frame is the origin of the world coordinate system (at $(0,0,0)$), we can calculate the coordinates of the tool center point as follows:

$$
\begin{aligned}
{}^{Base}T_{Tool} \cdot \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} &\approx \begin{pmatrix} -0.7071 & -0.9659 & 0 & 0.2035 \\ -0.9659 & 0.7071 & 0 & 0.6627 \\ 0 & 0 & -1 & 0.557 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \\
&\approx \begin{pmatrix} 0.2035 \\ 0.6627 \\ 0.557 \\ 1 \end{pmatrix}
\end{aligned}
$$

   The tool center point is at the point $(203.5, 662.7, 557)$ of the base coordinate system.