Robotics Assignment #04

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2. Juni 2015

Task 4.1. To calculate the Jacobian matrix, we need the position of the end effector based on the joint angles. Therefore, we need to determine the homogeneous transformation from the base to the end effector point. We can reuse the general transformation matrix given in assignment 2 task 1, because the manipulator shown there is the same as the one in this task. Only a_i needs to be changed to l_i. Then we have

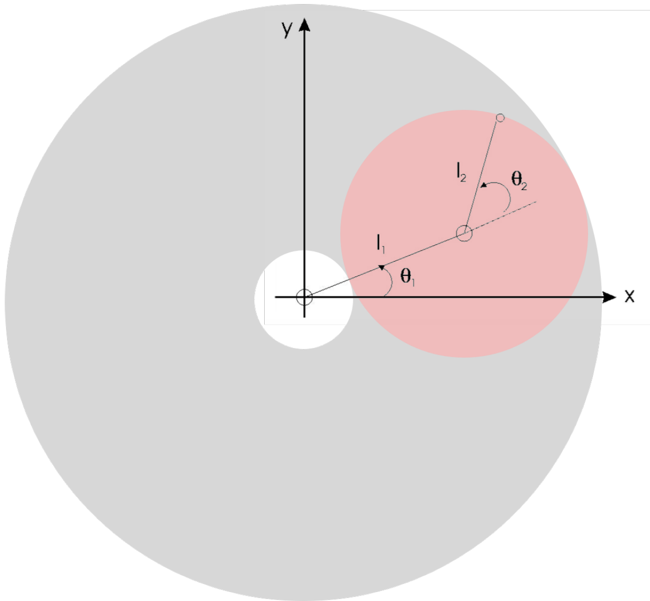
$$
{}^0T_3 = \begin{pmatrix} C_{1+2+3} & -S_{1+2+3} & 0 & C_{1+2+3}l_3 + C_{1+2}l_2 + C_1 l_1 \\ S_{1+2+3} & C_{1+2+3} & 0 & S_{1+2+3}l_3 + S_{1+2}l_2 + S_1 l_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
$$

Because this is a planar manipulator, we just need the x and y coodinates. These are $x = C_{1+2+3} \cdot l_3 + C_{1+2} \cdot l_2 + C_1 \cdot l_1$ and $y = S_{1+2+3} \cdot l_3 + S_{1+2} \cdot l_2 + S_1 \cdot l_1$. The Jacobian matrix has to be a 2×3 matrix, because we have 2 degrees of freedom in cartesian space and 3 degrees of freedom in joint space. The entries of this matrix are the partial derivatives of x and y.

$$
\begin{aligned}
J &= \begin{pmatrix} \frac{\partial x}{\partial \theta_1} & \frac{\partial x}{\partial \theta_2} & \frac{\partial x}{\partial \theta_3} \\ \frac{\partial y}{\partial \theta_1} & \frac{\partial y}{\partial \theta_2} & \frac{\partial y}{\partial \theta_3} \end{pmatrix} \\
&= \begin{pmatrix} -S_{1+2+3} \cdot l_3 - S_{1+2} \cdot l_2 - S_1 \cdot l_1 & -S_{1+2+3} \cdot l_3 - S_{1+2} \cdot l_2 & -S_{1+2+3} \cdot l_3 \\ C_{1+2+3} \cdot l_3 + C_{1+2} \cdot l_2 + C_1 \cdot l_1 & C_{1+2+3} \cdot l_3 + C_{1+2} \cdot l_2 & C_{1+2+3} \cdot l_3 \end{pmatrix}
\end{aligned}
$$

Task 4.2.

1) The visualization can be found in the following figure. The grey area that looks like a DVD is the reachable workspace of the arm. The red area depicts the space the second link can reach.



2) We need the x and y coordinates of the end effector. We can determine these by calculating the transformation matrix and extract the respective entries:

$$
\begin{aligned}
{}^0T_1 &= Rot_z(\theta_1) \cdot Trans_{x_1}(l_1) \\
&= \begin{pmatrix} C_1 & -S_1 & 0 & l_1 \cdot C_1 \\ S_1 & C_1 & 0 & l_1 \cdot S_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
{}^1T_2 &= Rot_z(\theta_2) \cdot Trans_{x_2}(l_2) \\
&= \begin{pmatrix} C_2 & -S_2 & 0 & l_2 \cdot C_2 \\ S_2 & C_2 & 0 & l_2 \cdot S_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
{}^0T_2 &= {}^0T_1 \cdot {}^1T_2 \\
&= \begin{pmatrix} C_{1+2} & -S_{1+2} & 0 & C_1 \cdot l_1 + C_{1+2} \cdot l_2 \\ S_{1+2} & C_{1+2} & 0 & S_1 \cdot l_1 + S_{1+2} \cdot l_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\end{aligned}
$$

So $x = C_1 \cdot l_1 + C_{1+2} \cdot l_2$ and $y = S_1 \cdot l_1 + S_{1+2} \cdot l_2$.
Given that

$$J = \begin{pmatrix} \frac{\partial x}{\partial \theta_1} & \frac{\partial x}{\partial \theta_2} \\ \frac{\partial y}{\partial \theta_1} & \frac{\partial y}{\partial \theta_2} \end{pmatrix}$$

this leads to

$$J = \begin{pmatrix} -S_1 \cdot l_1 - S_{1+2} \cdot l_2 & -S_{1+2} \cdot l_2 \\ C_1 \cdot l_1 + C_{1+2} \cdot l_2 & C_{1+2} \cdot l_2 \end{pmatrix}$$

3) If the determinant of the Jacobian matrix is zero, then the matrix is not invertible and therefore singular. This is why we need to calculate the determinant of the previously calculated matrix and set it equal to zero:

$$\begin{aligned}
\det(J) &= 0 \\
&= (-S_1 \cdot l_1 - S_{1+2} \cdot l_2) \cdot (C_{1+2} \cdot l_2) - (-S_{1+2} \cdot l_2) \cdot (C_1 \cdot l_1 + C_{1+2} \cdot l_2) \\
&= -(S_1 \cdot l_1 \cdot C_{1+2} \cdot l_2) - (S_{1+2} \cdot l_2 \cdot C_{1+2} \cdot l_2) + (S_{1+2} \cdot l_2 \cdot C_1 \cdot l_1) + (S_{1+2} \cdot l_2 \cdot C_{1+2} \cdot l_2) \\
&= -(S_1 \cdot l_1 \cdot C_{1+2} \cdot l_2) + (S_{1+2} \cdot l_2 \cdot C_1 \cdot l_1) \\
&= l_1 \cdot l_2 \cdot (S_{1+2} \cdot C_1 - C_{1+2} \cdot S_1) \\
&\overset{\text{sum_rule}}{=} l_1 \cdot l_2 \cdot S_{1+2-1} \\
&= l_1 \cdot l_2 \cdot S_2 = 0
\end{aligned}$$

We assume that both lengths l_1 and l_2 are bigger than zero, therefore S_2 has to be zero. That happens, when $\theta_2 = 0°$ or $\theta_2 = 180°$. So these are the singular configurations of the manipulator.

4) If θ_2 is zero or 180 degree, the manipulator will be at the edge of its workspace. So the possibility of moving any further than that is not there. The singularity is when the joint velocity becomes infinite to maintain cartesian velocity. That is the case, because we cannot go any further than this.

Task 4.3. (a) If θ_3 is zero, assuming that the figure shows the default configuration of the robot arm, then the TCP is at the edge of its workspace. Therefore, the manipulator has a singularity at this configuration.

(b) Assuming the length a_2 is much longer than the length d_4, then the arm can fold back in at $\theta_3 = 180°$ and therefore, we have a workspace boundary singularity here as well.

Task 4.4. We will combine rotations, that we can easily calculate, to get the overall matrix around an arbitrary vector k. This will be established by:

(a) rotating the vector k about the global z axis to align it with the (x,z) plane (we call this angle α).

(b) rotating the vector about the global y axis to align it with the global x axis (we call this angle β). Now rotating about k is equal to rotating about the global x axis.

(c) rotating about the global x axis/k by θ.

(d) reversing the rotation about the global y axis.

(e) reversing the rotation about the global z axis.

Because we use Roll-Pitch-Yaw angles, we need to append a new rotation matrix on the left side. Finally, we can calculate the resulting matrix with this formula:

$$Rot_{k,\theta} = Rot_z(-\alpha) \cdot Rot_y(-\beta) \cdot Rot_x(\theta) \cdot Rot_y(\beta) \cdot Rot_z(\alpha)$$

This leads to:

$$Rot_{k,\theta} = Rot_z(-\alpha) \cdot Rot_y(-\beta) \cdot Rot_x(\theta) \cdot Rot_y(\beta) \cdot Rot_z(\alpha)$$

$$= \begin{pmatrix} C_{-\alpha} & -S_{-\alpha} & 0 & 0 \\ S_{-\alpha} & C_{-\alpha} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} C_{-\beta} & 0 & S_{-\beta} & 0 \\ 0 & 1 & 0 & 0 \\ -S_{-\beta} & 0 & C_{-\beta} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & C_\theta & -S_\theta & 0 \\ 0 & S_\theta & C_\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$
$$\begin{pmatrix} C_\beta & 0 & S_\beta & 0 \\ 0 & 1 & 0 & 0 \\ -S_\beta & 0 & C_\beta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} C_\alpha & -S_\alpha & 0 & 0 \\ S_\alpha & C_\alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} C_{-\alpha}C_{-\beta} & -S_{-\alpha} & C_{-\alpha}S_{-\beta} & 0 \\ S_{-\alpha}C_{-\beta} & C_{-\alpha} & S_{-\alpha}S_{-\beta} & 0 \\ -S_{-\beta} & 0 & C_{-\beta} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & C_\theta & -S_\theta & 0 \\ 0 & S_\theta & C_\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$
$$\begin{pmatrix} C_\beta & 0 & S_\beta & 0 \\ 0 & 1 & 0 & 0 \\ -S_\beta & 0 & C_\beta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} C_\alpha & -S_\alpha & 0 & 0 \\ S_\alpha & C_\alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} C_{-\alpha}C_{-\beta} & -S_{-\alpha} & C_{-\alpha}S_{-\beta} & 0 \\ S_{-\alpha}C_{-\beta} & C_{-\alpha} & S_{-\alpha}S_{-\beta} & 0 \\ -S_{-\beta} & 0 & C_{-\beta} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & C_\theta & -S_\theta & 0 \\ 0 & S_\theta & C_\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$
$$\begin{pmatrix} C_\beta C_\alpha & -S_\alpha C_\beta & S_\beta & 0 \\ S_\alpha & C_\alpha & 0 & 0 \\ -S_\beta C_\alpha & S_\beta S_\alpha & C_\beta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} C_{-\alpha}C_{-\beta} & -S_{-\alpha}C_\theta + C_{-\alpha}S_{-\beta}S_\theta & S_{-\alpha}S_\theta + C_{-\alpha}S_{-\beta}C_\theta & 0 \\ S_{-\alpha}C_{-\beta} & C_{-\alpha}C_\theta + S_{-\alpha}S_{-\beta}S_\theta & -S_\theta C_{-\alpha} + S_{-\alpha}S_{-\beta}C_\theta & 0 \\ -S_{-\beta} & C_{-\beta}S_\theta & C_{-\beta}C_\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$
$$\begin{pmatrix} C_\beta C_\alpha & -S_\alpha C_\beta & S_\beta & 0 \\ S_\alpha & C_\alpha & 0 & 0 \\ -S_\beta C_\alpha & S_\beta S_\alpha & C_\beta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} C_{-\alpha}C_{-\beta} & -S_{-\alpha}C_\theta + C_{-\alpha}S_{-\beta}S_\theta & S_{-\alpha}S_\theta + C_{-\alpha}S_{-\beta}C_\theta & 0 \\ S_{-\alpha}C_{-\beta} & C_{-\alpha}C_\theta + S_{-\alpha}S_{-\beta}S_\theta & -S_\theta C_{-\alpha} + S_{-\alpha}S_{-\beta}C_\theta & 0 \\ -S_{-\beta} & C_{-\beta}S_\theta & C_{-\beta}C_\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$
$$\begin{pmatrix} C_\beta C_\alpha & -S_\alpha C_\beta & S_\beta & 0 \\ S_\alpha & C_\alpha & 0 & 0 \\ -S_\beta C_\alpha & S_\beta S_\alpha & C_\beta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} a_x & b_x & c_x & 0 \\ a_y & b_y & c_y & 0 \\ a_z & b_z & c_z & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

with

$$
\begin{aligned}
a_x &= C_{-\alpha}C_{-\beta}C_\beta C_\alpha - S_{-\alpha}C_\theta S_\alpha + C_{-\alpha}S_{-\beta}S_\theta S_\alpha - S_\beta C_\alpha S_{-\alpha}S_\theta - S_\beta C_\alpha C_{-\alpha}S_{-\beta}C_\theta \\
&= C_{-\alpha}C_{-\beta}C_\beta C_\alpha - S_{-\alpha}C_\theta S_\alpha - S_\beta C_\alpha C_{-\alpha}S_{-\beta}C_\theta \\
&= C_\alpha C_\beta C_\beta C_\alpha + S_\alpha C_\theta S_\alpha + S_\beta C_\alpha C_\alpha S_\beta C_\theta \\
&= C_\alpha^2 C_\beta^2 + S_\alpha^2 C_\theta + S_\beta^2 C_\alpha^2 C_\theta \\
a_y &= S_{-\alpha}C_{-\beta}C_\beta C_\alpha + C_{-\alpha}C_\theta S_\alpha + S_{-\alpha}S_{-\beta}S_\theta S_\alpha + S_\beta C_\alpha S_\theta C_{-\alpha} - S_\beta C_\alpha S_{-\alpha}S_{-\beta}C_\theta \\
&= -S_\alpha C_\beta C_\beta C_\alpha + C_\alpha C_\theta S_\alpha + S_\alpha S_\beta S_\theta S_\alpha + S_\beta C_\alpha S_\theta C_\alpha - S_\beta C_\alpha S_\alpha S_\beta C_\theta \\
&= -S_\alpha C_\beta^2 C_\alpha + C_\alpha C_\theta S_\alpha + S_\alpha^2 S_\beta S_\theta + S_\beta C_\alpha^2 S_\theta - S_\beta^2 C_\alpha S_\alpha C_\theta \\
a_z &= -S_{-\beta}C_\beta C_\alpha + C_{-\beta}S_\theta S_\alpha - S_\beta C_\alpha C_{-\beta}C_\theta \\
&= S_\beta C_\beta C_\alpha + C_\beta S_\theta S_\alpha - S_\beta C_\alpha C_\beta C_\theta \\
b_x &= -S_\alpha C_{-\alpha}C_{-\beta}C_\beta - S_{-\alpha}C_\theta C_\alpha + C_{-\alpha}S_{-\beta}S_\theta C_\alpha + S_\beta S_\alpha S_{-\alpha}S_\theta + S_\beta S_\alpha C_{-\alpha}S_{-\beta}C_\theta \\
&= -S_\alpha C_\alpha C_\beta C_\beta + S_\alpha C_\theta C_\alpha - C_\alpha S_\beta S_\theta C_\alpha - S_\beta S_\alpha S_\alpha S_\theta - S_\beta S_\alpha C_\alpha S_\beta C_\theta \\
&= -S_\alpha C_\alpha C_\beta^2 + S_\alpha C_\theta C_\alpha - C_\alpha^2 S_\beta S_\theta - S_\beta S_\alpha^2 S_\theta - S_\beta^2 S_\alpha C_\alpha C_\theta \\
b_y &= -S_\alpha C_\beta S_{-\alpha}C_{-\beta} + C_\alpha C_{-\alpha}C_\theta + C_\alpha S_{-\alpha}S_{-\beta}S_\theta - S_\theta C_{-\alpha}S_\beta S_\alpha + S_{-\alpha}S_{-\beta}C_\theta S_\beta S_\alpha \\
&= S_\alpha C_\beta S_\alpha C_\beta + C_\alpha C_\alpha C_\theta + C_\alpha S_\alpha S_\beta S_\theta - S_\theta C_\alpha S_\beta S_\alpha + S_\alpha S_\beta C_\theta S_\beta S_\alpha \\
&= S_\alpha^2 C_\beta^2 + C_\alpha^2 C_\theta + C_\alpha S_\alpha S_\beta S_\theta - S_\theta C_\alpha S_\beta S_\alpha + S_\alpha^2 S_\beta^2 C_\theta \\
b_z &= S_{-\beta}S_\alpha C_\beta + C_{-\beta}S_\theta C_\alpha + C_{-\beta}C_\theta S_\beta S_\alpha \\
&= -S_\beta S_\alpha C_\beta + C_\beta S_\theta C_\alpha + C_\beta C_\theta S_\beta S_\alpha \\
c_x &= C_{-\alpha}C_{-\beta}S_\beta + C_\beta S_{-\alpha}S_\theta + C_\beta C_{-\alpha}S_{-\beta}C_\theta \\
&= C_\alpha C_\beta S_\beta - C_\beta S_\alpha S_\theta - C_\beta C_\alpha S_\beta C_\theta \\
c_y &= S_{-\alpha}C_{-\beta}S_\beta - S_\theta C_{-\alpha}C_\beta + S_{-\alpha}S_{-\beta}C_\theta C_\beta \\
&= -S_\alpha C_\beta S_\beta - S_\theta C_\alpha C_\beta + S_\alpha S_\beta C_\theta C_\beta \\
c_z &= -S_{-\beta}S_\beta + C_{-\beta}C_\theta C_\beta \\
&= S_\beta S_\beta + C_\beta C_\theta C_\beta \\
&= S_\beta^2 + C_\beta^2 C_\theta
\end{aligned}
$$

which is correct, because

$$S_{-\alpha} = -S_\alpha, \quad S_{-\beta} = -S_\beta, \quad C_{-\alpha} = C_\alpha, \quad C_{-\beta} = C_\beta$$

We observe α and β and get that

$$S_\alpha = \frac{k_y}{\sqrt{k_x^2 + k_y^2}}, \quad S_\beta = \frac{k_z}{\sqrt{k_x^2 + k_y^2 + k_z^2}}, \quad C_\alpha = \frac{k_x}{\sqrt{k_x^2 + k_y^2}}, \quad C_\beta = \frac{\sqrt{k_x^2 + k_y^2}}{\sqrt{k_x^2 + k_y^2 + k_z^2}}$$

because of the definitions of sine and cosine. Now, we can put these equations into the entries of the matrix:

$$a_x = C_\alpha^2 C_\beta^2 + S_\alpha^2 C_\theta + S_\beta^2 C_\alpha^2 C_\theta$$

$$
\begin{aligned}
&= \frac{k_x^2}{k_x^2+k_y^2}\cdot\frac{k_x^2+k_y^2}{k_x^2+k_y^2+k_z^2}+\frac{k_y^2}{k_x^2+k_y^2}C_\theta+\frac{k_z^2}{k_x^2+k_y^2+k_z^2}\cdot\frac{k_x^2}{k_x^2+k_y^2}C_\theta\\
&= \frac{k_x^2}{k_x^2+k_y^2+k_z^2}+\frac{k_y^2}{k_x^2+k_y^2}C_\theta+\frac{k_z^2}{k_x^2+k_y^2+k_z^2}\cdot\frac{k_x^2}{k_x^2+k_y^2}C_\theta\\
a_y &= -S_\alpha C_\beta^2 C_\alpha + C_\alpha C_\theta S_\alpha + S_\alpha^2 S_\beta S_\theta + S_\beta C_\alpha^2 S_\theta - S_\beta^2 C_\alpha S_\alpha C_\theta\\
a_z &= S_\beta C_\beta C_\alpha + C_\beta S_\theta S_\alpha - S_\beta C_\alpha C_\beta C_\theta\\
b_x &= -S_\alpha C_\alpha C_\beta^2 + S_\alpha C_\theta C_\alpha - C_\alpha^2 S_\beta S_\theta - S_\beta S_\alpha^2 S_\theta - S_\beta^2 S_\alpha C_\alpha C_\theta\\
b_y &= S_\alpha^2 C_\beta^2 + C_\alpha^2 C_\theta + C_\alpha S_\alpha S_\beta S_\theta - S_\theta C_\alpha S_\beta S_\alpha + S_\alpha^2 S_\beta^2 C_\theta\\
b_z &= -S_\beta S_\alpha C_\beta + C_\beta S_\theta C_\alpha + C_\beta C_\theta S_\beta S_\alpha\\
c_x &= C_\alpha C_\beta S_\beta - C_\beta S_\alpha S_\theta - C_\beta C_\alpha S_\beta C_\theta\\
c_y &= -S_\alpha C_\beta S_\beta - S_\theta C_\alpha C_\beta + S_\alpha S_\beta C_\theta C_\beta\\
c_z &= S_\beta^2 + C_\beta^2 C_\theta\\
&= \frac{k_z^2}{k_x^2+k_y^2+k_z^2}+\frac{k_x^2+k_y^2}{k_x^2+k_y^2+k_z^2}C_\theta\\
&= \frac{k_z^2+(k_x^2+k_y^2)C_\theta}{k_x^2+k_y^2+k_z^2}
\end{aligned}
$$

This did not lead us to the solution, but we nonetheless like the approach. There must be some obvious error in our calculations or something we missed :(

Task 4.4. **Version 2:** We try a second time using a similar approach, but doing some calculations beforehand, under the assumption that our rotation axis k is indeed a unit vector.

(a) Rotate the vector into xz plane (around global x axis)
(b) Rotate the vector onto z axis (around global y axis)
(c) Rotate around z axis by θ
(d) Reverse (b)
(e) Reverse (a)

There is no need for us to find out the angles to rotate about. Instead, we define a helping vector $\vec{d}$, that is the projection of k onto the yz plane.

For simplicity, we assume $k = (x, y, z)$ and use $c = \cos(\theta)$ and $s = \sin(\theta)$. The vector $\vec{d} = (0, y, z)$ has the length

$$d = |\vec{d}| = \sqrt{x^2 + y^2}$$

We now use these formulas for the angle α between two vectors $\vec{a}$ and $\vec{b}$:

$$\cos(\alpha) = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\cdot|\vec{b}|} \qquad \sin(\alpha) = \frac{|\vec{a}\times\vec{b}|}{|\vec{a}|\cdot|\vec{b}|}$$

For the first rotation, we get

$$\cos(\alpha_1) = \frac{(0,0,z)\cdot(0,y,z)}{z\cdot d} = \frac{z^2}{z\cdot d} = \frac{z}{d}$$

$$\sin(\alpha_1) = \frac{|(0,0,z) \times (0,y,z)|}{z \cdot d} = \frac{zy}{zd} = \frac{y}{d}$$

For the second rotation, we calculate the angles using $\vec{k}$ and $\vec{d}$, so

$$\cos(\alpha_2) = \frac{(x,y,z) \cdot (0,y,z)}{1 \cdot d} = \frac{y^2+z^2}{d} = d$$
$$\sin(\alpha_2) = \frac{|(x,y,z) \times (0,y,z)|}{1 \cdot d} = \frac{\sqrt{x^2y^2+x^2z^2}}{\sqrt{y^2z^2}} = x$$

Now we have the cosine and sine form of both rotation angles, so we can build our rotation matrices (using 3x3 matrices since we only care about the rotations):

$$Rot_{k,\theta} = Rot_x^{-1}(\alpha_1) Rot_y^{-1}(\alpha_2) Rot_z(\theta) Rot_y(\alpha_2) Rot_x(\alpha_1)$$

With

$$Rot_x(\alpha_1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{z}{d} & -\frac{y}{d} \\ 0 & \frac{y}{d} & \frac{z}{d} \end{pmatrix} \qquad Rot_x^{-1}(\alpha_1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{z}{d} & \frac{y}{d} \\ 0 & -\frac{y}{d} & \frac{z}{d} \end{pmatrix}$$

$$Rot_y(\alpha_2) = \begin{pmatrix} d & 0 & -x \\ 0 & 1 & 0 \\ x & 0 & d \end{pmatrix} \qquad Rot_y^{-1}(\alpha_2) = \begin{pmatrix} d & 0 & x \\ 0 & 1 & 0 \\ -x & 0 & d \end{pmatrix}$$

$$Rot_z(\theta) = \begin{pmatrix} c & -s & 0 \\ s & c & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The result of above matrix multiplication is not very helpful, as it is still a monstrous matrix very far from the desired result:

$$\begin{pmatrix} x^2 + cd^2 & (-c+1)\,yx - sz & (-c+1)\,zx + sy \\ (-c+1)\,yx + sz & \frac{cy^2}{d^2}x^2 + \left(\frac{c}{d^2}z^2 + y^2\right) & \frac{cy}{d^2}zx^2 + \left(\frac{-s}{d^2}z^2 - \frac{-sy^2}{d^2}\right)x + \left(\frac{yd^2 - cy}{d^2}\right)z \\ (-c+1)\,zx - sy & \frac{cy}{d^2}zx^2 + \left(\frac{s}{d^2}z^2 + \frac{sy^2}{d^2}\right)x + \left(\frac{yd^2-cy}{d^2}\right)z & \frac{c}{d^2}z^2x^2 + \left(z^2 + \frac{cy^2}{d^2}\right) \end{pmatrix}$$

Yes, I did use a matrix multiplier and made a screenshot. Yes, I also did it by hand and had *similar* results. Sadly, we don't know where to go from here.