Elementary Concepts in Numerical Analysis

Differentiation

a) Prove that the function $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = x$ is continuous and differentiable. 5

b) Show mathematically why $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = |x|$ is not differentiable at $x_0 = 0$. 3

c) Use the chain rule (amongst others) to calculate the derivative of $2x^3 + x^2 \log 4x$. 1

d) Use R to derive the following expressions symbolically: 3

- $\frac{\mathrm{d}}{\mathrm{d}x}\left(x^4 + 2x^2 + \exp x\right)$
- $\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(x^4 + 2x^2 + \exp x\right)$

Calculate the value of the first derivative at point $x = 3$

e) Use R to derive the following expressions symbolically: 3

- $\frac{\mathrm{d}}{\mathrm{d}x}\left[x^4 + 2x^2 + \exp x\right]$
- $\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left[x^4 + 2x^2 + \exp x\right]$

Calculate the value of the first derivative at point $x = 3$.

f) Approximate the first derivative at point $x = 0.1$ via forward, backward and centered differences for 3

$$f(x) = 3x^4 + 2x \qquad \text{and} \qquad g(x) = \sin\frac{1}{x}.$$

Subsequently, compare the approximated derivatives to their true values. Use $h = 10^{-6}$.

g) Calculate the 2nd order central differences at point $x = 0.01$ for 1

$$g(x) = \sin\frac{1}{x}.$$

and compare to the true value from a symbolic differentiation. Use $h = 10^{-6}$.

h) Derive a formulate to approximate a Hessian matrix of a function $f(x, y)$ using forward differences. Let h be an arbitrary step size. 2

1

i) Write a user-defined function that can approximate a Hessian matrix of $f(x, y)$ using forward differences. Test your function with $f(x, y) = 2x + 3xy^2 + y^3 + 1$ at a point $(x_1, x_2) = (4, 5)$ with a given step size $h = 0.0001$.

Note: You can also pass function names as arguments; see the following example.

```r
f <- function(x) x^2

g <- function(x, func) func(x)+2
g(3, f)

## [1] 11
```

1

j) Consider the following function $f(x, y) = 2x + 3xy^2 + y^3 + 1$. Use the function `optimHess` in R to approximate the Hessian matrix, i. e. with forward differences at a point $(x_1, x_2) = (4, 5)$ with a given step size $h = 0.0001$.

Taylor Approximation

1

k) Define a Taylor series mathematically.

1

l) What is a Maclaurin series?

1

m) What is the Taylor series for e^x with $x_0 = 0$?

1

n) What is the Taylor series for $\ln 1 - x$ with $x_0 = 0$?

1

o) What is the Taylor series for $\ln 1 + x$ with $x_0 = 0$?

1

p) What is the Taylor series for $\frac{1}{1-x}$ with $x_0 = 0$?

4

q) What is the 2-nd order Taylor approximation of a function $f(x, y) = \ln 1 + x + \ln 1 - y$ around $(x_0, y_0) = (0, 0)$?

Hint: a Taylor Series in 2 variables can be written as

$$f(x,y) = f(x_0,y_0) + \left[(x-x_0)\frac{\partial}{\partial x} + (y-y_0)\frac{\partial}{\partial y}\right] f(x_0,y_0) \\ + \frac{1}{2!}\left[(x-x_0)\frac{\partial}{\partial x} + (y-y_0)\frac{\partial}{\partial y}\right]^2 f(x_0,y_0) + \dots$$

4

r) What is the 2-nd order Taylor approximation of a function

$$f(x,y) = \sqrt[5]{x^3 + e^y}$$

around $(x_0, y_0) = (0,0)$?

2

s) Calculate the Taylor approximation of $f(x) = \sin x$ up to degree 4 around $x_0 = 0$. Then evaluate and compare it for $x = 0.1$.

1

t) Visualize the function $f(x) = \log x + 1$ and its Taylor approximation for $x_0 = 0$.

Optimality Conditions

9

u) Find the stationary points of the function $f(x,y) = x^3 + 3y - y^3 - 3x$ and analyze their nature using Sylvester's Rule.

9

v) Consider the function $f(x,y) = \sin x \cdot \cos x$. First of all, plot the function nicely to get an impression of its curvature. Then, consider the points

$$\boldsymbol{p}_1 = \begin{bmatrix} \frac{\pi}{2} \\ 0 \end{bmatrix} \text{ and } \boldsymbol{p}_2 = \begin{bmatrix} 0 \\ \frac{\pi}{2} \end{bmatrix}$$

and check their first and second order optimality conditions using R. What type of stationary points do they belong to?

2

w) Prove that the function $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = x^2$ is convex.