Linear Inequalities and Linear Programming

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Contents

Preface

This book covers the fundamentals of linear programming through studying systems of linear inequalities using only basic facts from linear algebra. It is suitable for a crash course on linear programming that emphasizes theoretical aspects of the subject. Discussion on practical solution methods such as the simplex method and interior point methods, though not present in this book, is planned for a future book.

Two excellent references for further study are **?** and **?**.

Notation

The set of real numbers is denoted by $\mathbb{R}$. The set of rational numbers is denoted by $\mathbb{Q}$. The set of integers is denoted by $\mathbb{Z}$.

The set of n-tuples with real entries is denoted by $\mathbb{R}^n$. Similar definitions hold for $\mathbb{Q}^n$ and $\mathbb{Z}^n$.

The set of $m \times n$ matrices (that is, matrices with m rows and n columns) with real entries is denoted $\mathbb{R}^{m\times n}$. Similar definitions hold for $\mathbb{Q}^{m\times n}$ and $\mathbb{Z}^n$.

All n-tuples are written as columns (that is, as $n \times 1$ matrices). An n-tuple is normally represented by a lowercase Roman letter in boldface; for example, $\mathbf{x}$. For an n-tuple $\mathbf{x}$, x_i denotes the ith entry (or component) of $\mathbf{x}$ for $i = 1, \ldots, n$.

Matrices are normally represented by an uppercase Roman letter in boldface; for example, $\mathbf{A}$. The jth column of a matrix $\mathbf{A}$ is denoted by A_j and the (i, j)-entry (that is, the entry in row i and column j) is denoted by a_{ij}.

Scalars are usually represented by lowercase Greek letters; for example, λ, α, β etc.

An n-tuple consisting of all zeros is denoted by $\mathbf{0}$. The dimension of the tuple is inferred from the context.

For a matrix $\mathbf{A}$, $\mathbf{A}^\mathsf{T}$ denotes the transpose of $\mathbf{A}$. For an n-tuple $\mathbf{x}$, $\mathbf{x}^\mathsf{T}$ denotes the transpose of $\mathbf{x}$.

If $\mathbf{A}$ and $\mathbf{B}$ are $m \times n$ matrices, $\mathbf{A} \geq \mathbf{B}$ means $a_{ij} \geq b_{ij}$ for all $i = 1, \ldots, m$, $j = 1, \ldots, n$. Similar definitions hold for $\mathbf{A} \leq \mathbf{B}$, $\mathbf{A} = \mathbf{B}$, $\mathbf{A} < \mathbf{B}$ and $\mathbf{A} > \mathbf{B}$. In particular, if $\mathbf{u}$ and $\mathbf{v}$ are n-tuples, $\mathbf{u} \geq \mathbf{v}$ means $u_i \geq v_i$ for $i = 1, \ldots, n$ and $\mathbf{u} > \mathbf{0}$ means $u_i > 0$ for $i = 1, \ldots, n$.

Superscripts in brackets are used for indexing tuples. For example, we can write $\mathbf{u}^{(1)}, \mathbf{u}^{(2)} \in \mathbb{R}^3$. Then $\mathbf{u}^{(1)}$ and $\mathbf{u}^{(2)}$ are elements of $\mathbb{R}^3$. The second entry of $\mathbf{u}^{(1)}$ is denoted by $u_2^{(1)}$.

Chapter 1

Graphical example

To motivate the subject of linear programming (LP), we begin with a planning problem that can be solved graphically.

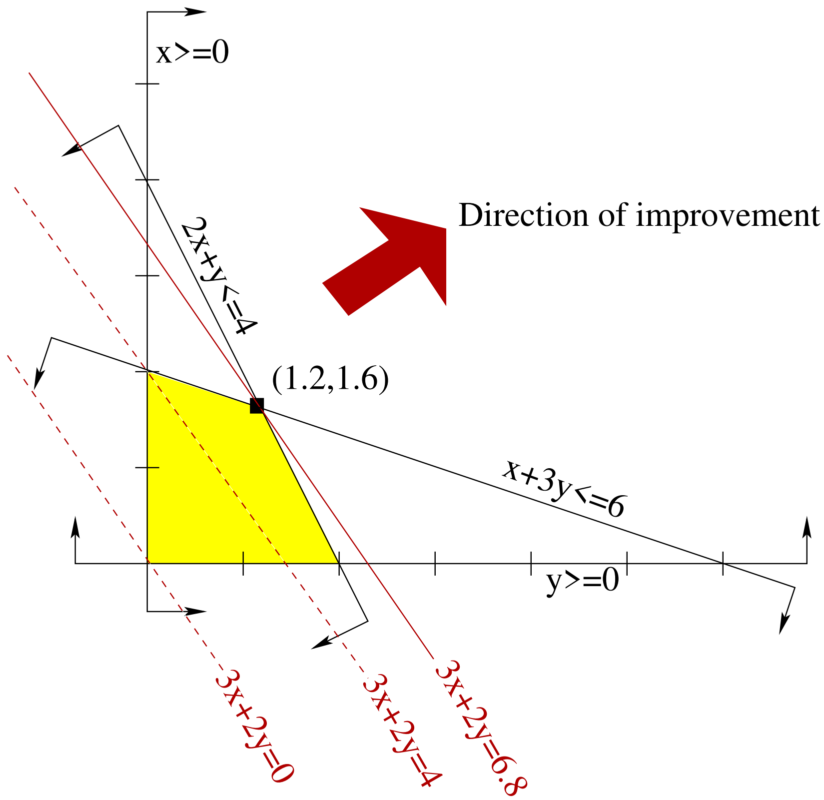
Say you are a vendor of lemonade and lemon juice. Each unit of lemonade requires 1 lemon and 2 litres of water. Each unit of lemon juice requires 3 lemons and 1 litre of water. Each unit of lemonade gives a profit of three dollars. Each unit of lemon juice gives a profit of two dollars. You have 6 lemons and 4 litres of water available. How many units of lemonade and lemon juice should you make to maximize profit?

If we let x denote the number of units of lemonade to be made and let y denote the number of units of lemon juice to be made, then the profit is given by $3x + 2y$ dollars. We call $3x + 2y$ the objective function. Note that there are a number of constraints that x and y must satisfied. First of all, x and y should be nonnegative. The number of lemons needed to make x units of lemonade and y units of lemon juice is $x + 3y$ and cannot exceed 6. The number of litres of water needed to make x units of lemonade and y units of lemon juice is $2x + y$ and cannot exceed 4. Hence, to determine the maximum profit, we need to maximize $3x + 2y$ subject to x and y satisfying the constraints $x + 3y \leq 6$, $2x + y \leq 4$, $x \geq 0$, and $y \geq 0$.

A more compact way to write the problem is as follows:

$$\begin{array}{lrcrcl}
\text{maximize} & 3x & + & 2y & & \\
\text{subject to} & x & + & 3y & \leq & 6 \\
 & 2x & + & y & \leq & 4 \\
 & x & & & \geq & 0 \\
 & & & y & \geq & 0.
\end{array}$$

We can solve this maximizationproblem graphically as follows. We first sketch the set of $\begin{bmatrix} x \\ y \end{bmatrix}$ satisfying the constraints, called the feasible region, on the (x, y)-plane. We then take the objective function $3x + 2y$ and turn it into an equation of a line $3x + 2y = z$ where z is a parameter. Note that as the value of z increases, the line defined by the equation $3x + 2y = z$ moves in the direction of the normal vector $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$. We call this direction the direction of improvement. Determining the maximum value of the objective function, called the optimal value, subject to the contraints amounts to finding the maximum value of z so that the line defined by the equation $3x + 2y = z$ still intersects the feasible region.



In the figure above, the lines with z at 0, 4 and 6.8 have been drawn. From the picture, we can see that if z is greater than 6.8, the line defined by $3x + 2y = z$ will not intersect the feasible region. Hence, the profit cannot exceed 6.8 dollars.

As the line $3x + 2y = 6.8$ does intersect the feasible region, 6.8 is the maximum value for the objective function. Note that there is only one point in the feasible region that intersects the line $3x+2y = 6.8$, namely $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1.2 \\ 1.6 \end{bmatrix}$. In other words, to maximize profit, we want to make 1.2 units of lemonade and 1.6 units of lemon juice.

The above solution method can hardly be regarded as rigorous because we relied on a picture to conclude that $3x + 2y \leq 6.8$ for all $\begin{bmatrix} x \\ y \end{bmatrix}$ satisfying the constraints. But we can actually show this *algebraically*.

Note that multiplying both sides of the constraint $x + 3y \leq 6$ gives $0.2x + 0.6y \leq 1.2$, and multiplying both sides of the constraint $2x + y \leq 4$ gives $2.8x + 1.4y \leq 5.6$. Hence, any $\begin{bmatrix} x \\ y \end{bmatrix}$ that satisfies both $x + 3y \leq 6$ and $2x + y \leq 4$ must also satisfy $(0.2x + 0.6y) + (2.8x + 1.4y) \leq 1.2 + 5.6$, which simplifies to $3x + 2y \leq 6.8$ as desired! (Here, we used the fact that if $a \leq b$ and $c \leq d$, then $a + c \leq b + d$.)

Now, one might ask if it is always possible to find an algebraic proof like the one above for similar problems. If the answer is yes, how does one find such a proof? We will see answers to this question later on.

Before we end this segment, let us consider the following problem:

$$\begin{array}{lrcrcl} \text{minimize} & -2x & + & y & & \\ \text{subject to} & -x & + & y & \leq & 3 \\ & x & - & 2y & \leq & 2 \\ & x & & & \geq & 0 \\ & & & y & \geq & 0. \end{array}$$

Note that for any $t \geq 0$, $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix}$ satisfies all the constraints. The value of the objective function at $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix}$ is $-t$. As $t \to \infty$, the value of the objective function tends to $-\infty$. Therefore, there is no minimum value for the objective function. The problem is said to be unbounded. Later on, we will see how to detect unboundedness algorithmically.

As an exercise, check that unboundedness can also be established by using $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2t+2 \\ t \end{bmatrix}$ for $t \geq 0$.

Exercises

1. Sketch all $\begin{bmatrix} x \\ y \end{bmatrix}$ satisfying

$$x - 2y \leq 2$$

   on the (x, y)-plane.

2. Determine the optimal value of

$$\begin{array}{rl} \text{Minimize} & x + y \\ \text{Subject to} & 2x + y \geq 4 \\ & x + 3y \geq 1. \end{array}$$

3. Show that the problem

$$\begin{array}{rl} \text{Minimize} & -x + y \\ \text{Subject to} & 2x - y \geq 0 \\ & x + 3y \geq 3 \end{array}$$

   is unbounded.

4. Suppose that you are shopping for dietary supplements to satisfy your required daily intake of 0.40mg of nutrient M and 0.30mg of nutrient N. There are three popular products on the market. The costs and the amounts of the two nutrients are given in the following table:

	Product 1	Product 2	Product 3
Cost	\$27	\$31	\$24
Daily amount of M	0.16 mg	0.21 mg	0.11 mg
Daily amount of N	0.19 mg	0.13 mg	0.15 mg

   You want to determine how much of each product you should buy so that the daily intake requirements of the two nutrients are satisfied at minimum cost. Formulate your problem as a linear programming problem, assuming that you can buy a fractional number of each product.

Solutions

1. The points (x, y) satisfying $x - 2y \leq 2$ are precisely those above the line passing through $(2, 0)$ and $(0, -1)$.

2. We want to determine the minimum value z so that $x + y = z$ defines a line that has a nonempty intersection with the feasible region. However, we can avoid referring to a sketch by setting $x = z - y$ and substituting for x in the inequalities to obtain:

$$\begin{aligned} 2(z-y) + y &\geq 4 \\ (z-y) + 3y &\geq 1, \end{aligned}$$

or equivalently,

$$\begin{aligned} z &\geq 2 + \frac{1}{2}y \\ z &\geq 1 - 2y, \end{aligned}$$

Thus, the minimum value for z is $\min\{2 + \frac{1}{2}y, 1 - 2y\}$, which occurs at $y = -\frac{2}{5}$. Hence, the optimal value is $\frac{9}{5}$.

We can verify our work by doing the following. If our calculations above are correct, then an optimal solution is given by $x = \frac{11}{5}$, $y = -\frac{2}{5}$ since $x = z - y$. It is easy to check that this satisfies both inequalities and therefore is a feasible solution.

Now, taking $\frac{2}{5}$ times the first inequality and $\frac{1}{5}$ times the second inequality, we can infer the inequality $x + y \geq \frac{9}{5}$. The left-hand side of this inequality is precisely the objective function. Hence, no feasible solution can have objective function value less than $\frac{9}{5}$. But $x = \frac{11}{5}$, $y = -\frac{2}{5}$ is a feasible solution with objective function value equal to $\frac{9}{5}$. As a result, it must be an optimal solution.

Remark. We have not yet discussed how to obtain the multipliers $\frac{2}{5}$ and $\frac{1}{5}$ for inferring the inequality $x + y \geq \frac{9}{5}$. This is an issue that will be taken up later. In the meantime, think about how one could have obtained these multipliers for this particular exercise.

3. We could glean some insight by first making a sketch on the (x, y)-plane.

The line defined by $-x + y = z$ has x-intercept $-z$. Note that for $z \leq -3$, $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -z \\ 0 \end{bmatrix}$ satisfies both inequalities and the value of the objective function at $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -z \\ 0 \end{bmatrix}$ is z. Hence, there is no lower bound on the value of objective function.

4. Let x_i denote the amount of Product i to buy for $i = 1, 2, 3$. Then, the problem can be formulated as

$$\begin{array}{lrcrcrcl} \text{minimize} & 27x_1 & + & 31x_2 & + & 24x_3 & & \\ \text{subject to} & 0.16x_1 & + & 0.21x_2 & + & 0.11x_3 & \geq & 0.30 \\ & 0.19x_1 & + & 0.13x_2 & + & 0.15x_3 & \geq & 0.40 \\ & x_1 & , & x_2 & , & x_3 & \geq & 0. \end{array}$$

Remark. If one cannot buy fractional amounts of the products, the problem can be formulated as

$$\begin{array}{lrcrcrcl} \text{minimize} & 27x_1 & + & 31x_2 & + & 24x_3 & & \\ \text{subject to} & 0.16x_1 & + & 0.21x_2 & + & 0.11x_3 & \geq & 0.30 \\ & 0.19x_1 & + & 0.13x_2 & + & 0.15x_3 & \geq & 0.40 \\ & x_1 & , & x_2 & , & x_3 & \geq & 0. \\ & x_1 & , & x_2 & , & x_3 & \in & \mathbb{Z}. \end{array}$$

Chapter 2

Definitions

The following is an example of a problem in **linear programming**:

$$\begin{array}{rl} \text{Maximize} & x + y - 2z \\ \text{Subject to} & 2x + y + z \leq 4 \\ & 3x - y + z = 0 \\ & x, y, z \geq 0 \end{array}$$

Solving this problem means finding real values for the **variables** x, y, z satisfying the **constraints** $2x + y + z \leq 4$, $3x - y + z = 0$, and $x, y, z \geq 0$ that gives the maximum possible value (if it exists) for the **objective function** $x + y - 2z$.

For example, $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ satisfies all the constraints and is called a **feasible solution**. Its **objective function value**, obtained by evaluating the objective function at $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$, is $0 + 1 - 2(1) = -1$. The set of feasible solutions to a linear programming problem is called the **feasible region**.

More formally, a linear programming problem is an optimization problem of the following form:

$$\begin{array}{rl} \text{Maximize (or Minimize)} & \displaystyle\sum_{j=1}^{n} c_j x_j \\ \text{Subject to} & P_i(x_1, \ldots, x_n) \quad i = 1, \ldots, m \end{array}$$

where m and n are positive integers, $c_j \in \mathbb{R}$ for $j = 1, \ldots, n$, and for each $i = 1, \ldots, m$, $P_i(x_1, \ldots, x_n)$ is a **linear constraint** on the **(decision) variables** $x_1, \ldots, x_n$ having one of the following forms:

- $a_1 x_1 + \cdots + a_n x_n \geq \beta$
- $a_1 x_1 + \cdots + a_n x_n \leq \beta$
- $a_1 x_1 + \cdots a_n x_n = \beta$

where $\beta, a_1, \ldots, a_n \in \mathbb{R}$. To save writing, the word "Minimize" ("Maximize") is replaced with "$\min$" ("$\max$") and "Subject to" is abbreviated as "s.t.".

A feasible solution $\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ that gives the maximum possible objective function value in the case of a maximization problem is called an **optimal solution** and its objective function value is the **optimal value** of the problem.

The following example shows that it is possible to have multiple optimal solutions:

$$\begin{array}{rl} \max & x + y \\ \text{s.t.} & 2x + 2y \leq 1 \end{array}$$

The constraint says that $x + y$ cannot exceed $\frac{1}{2}$. Now, both $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}$ and $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix}$ are feasible solutions having objective function value $\frac{1}{2}$. Hence, they are both optimal solutions. (In fact, this problem has infinitely many optimal solutions. Can you specify all of them?)

Not all linear programming problems have optimal solutions. For example, a problem can have no feasible solution. Such a problem is said to be **infeasible**. Here is an example of an infeasible problem:

$$\begin{array}{rl} \min & x \\ \text{s.t.} & x \leq 1 \\ & x \geq 2 \end{array}$$

There is no value for x that is at the same time at most 1 and at least 2.

Even if a problem is not infeasible, it might not have an optimal solution as the following example shows:

$$\begin{array}{rl} \min & x \\ \text{s.t.} & x \leq 0 \end{array}$$

Note that now matter what real number M we are given, we can always find a feasible solution whose objective function value is less than M. Such a problem is said to be **unbounded**. (For a maximization problem, it is unbounded if one can find feasible solutions who objective function value is larger than any given real number.)

So far, we have seen that a linear programming problem can have an optimal solution, be infeasible, or be unbounded. Is it possible for a linear programming problem to be not infeasible, not unbounded, and with no optimal solution?
The following optimization problem, though not a linear programming problem, is not infeasible, not unbounded, and has no optimal solution:

$$\begin{array}{rl} \min & 2^x \\ \text{s.t.} & x \leq 0 \end{array}$$

The objective function value is never negative and can get arbitrarily close to 0 but can never attain 0.

A main result in linear programming states that if a linear programming problem is not infeasible and is not unbounded, then it must have an optimal solution. This result is known as the **Fundamental Theorem of Linear Programming** (Theorem 6.1) and we will see a proof of this importan result. In the meantime, we will consider the seemingly easier problem of determining if a system of linear constraints has a solution.

Exercises

1. Determine all values of a such that the problem

$$\begin{aligned} \min \quad & x + y \\ \text{s.t.} \quad & -3x + y \geq a \\ & 2x - y \geq 0 \\ & x + 2y \geq 2 \end{aligned}$$

   is infeasible.

2. Show that the problem

$$\begin{aligned} \min \quad & 2^x \cdot 4^y \\ \text{s.t.} \quad & e^{-3x+y} \geq 1 \\ & |2x - y| \leq 4 \end{aligned}$$

   can be solved by solving a linear programming problem.

Solutions

1. Adding the first two inequalities gives $-x \geq a$. Adding 2 times the second inequality and the third inequality gives $5x \geq 2$, implying that $x \geq \frac{2}{5}$. Hence, if $a > -\frac{2}{5}$, there is no solution.

   Note that if $a \leq -\frac{2}{5}$, then $(x, y) = \left(\frac{2}{5}, \frac{4}{5}\right)$ satisfies all the inequalities. Hence, the problem is infeasible if and only if $a > -\frac{2}{5}$.

2. Note that the constraint $|2x - y| \leq 4$ is equivalent to the constraints $2x - y \leq 4$ and $2x - y \geq -4$ taken together, and the constraint $e^{-3x+y} \geq 1$ is equivalent to $-3x + y \geq 0$. Hence, we can rewrite the problem with linear constraints.

   Finally, minimizing $2^x \cdot 4^y$ is the same as minimizing 2^{x+2y}, which is equivalent to minimizing $x + 2y$.

Chapter 3

Inferring linear constraints

If a, b, c, and d are real numbers such that $a \geq b$ and $c \geq d$, then $a + c \geq b + d$. We say that $a + c \geq b + d$ is **inferred** from $a \geq b$ and $c \geq d$. Casually, we also say that "adding" the two inequalities gives $a + c \geq b + d$.

Note that adding inequalities require that the inequalities to have the same sense; in other words, adding a mixture of $\leq$-inequalities and $\geq$-inequalities is not allowed for obvious reason. However, adding a mixture of inequalities having the same sense and equations is valid. For example, if x and y are real numbers satisfying

$$\begin{aligned} x - 2y &\geq 5 \\ 3x + y &= 7, \end{aligned}$$

then x and y must also satisfy $4x - y \geq 12$.

Going one step further, we can add scalar multiples of inequalities to obtain new inequalities under appropriate conditions. For example, if $a \geq b$, $c \leq d$, $\alpha \geq 0$, and $\beta \leq 0$, then $\alpha a + \beta c \geq \alpha b + \beta d$.

In general, suppose that $\mathbf{x} \in \mathbb{R}^n$ satisfies the system

$$\begin{aligned} \mathbf{P}\mathbf{x} &\geq \mathbf{p} \\ \mathbf{Q}\mathbf{x} &\leq \mathbf{q} \\ \mathbf{R}\mathbf{x} &= \mathbf{r} \end{aligned} \tag{3.1}$$

where $\mathbf{P} \in \mathbb{R}^{m \times n}$, $\mathbf{p} \in \mathbb{R}^m$, $\mathbf{Q} \in \mathbb{R}^{m' \times n}$, $\mathbf{q} \in \mathbb{R}^{m'}$, $\mathbf{R} \in \mathbb{R}^{\bar{m} \times n}$, $\mathbf{r} \in \mathbb{R}^{\bar{m}}$ for some nonnegative integers $m, m', \bar{m}$. If $\mathbf{f} \in \mathbb{R}^m$ with $\mathbf{f} \geq \mathbf{0}$, $\mathbf{g} \in \mathbb{R}^{m'}$ with $\mathbf{g} \leq \mathbf{0}$, and $\mathbf{h} \in \mathbb{R}^{\bar{m}}$, then $\mathbf{x}$ also satisfies

$$\mathbf{c}^\mathsf{T}\mathbf{x} \geq \gamma$$

where $\mathbf{c} = \mathbf{f}^\mathsf{T}\mathbf{P} + \mathbf{g}^\mathsf{T}\mathbf{Q} + \mathbf{h}^\mathsf{T}\mathbf{R}$ and $\gamma = \mathbf{f}^\mathsf{T}\mathbf{p} + \mathbf{g}^\mathsf{T}\mathbf{q} + \mathbf{h}^\mathsf{T}\mathbf{r}$. We say that the inequality $\mathbf{c}^\mathsf{T}\mathbf{x} \geq \gamma$ is *inferred* from the system (3.1). To simplify the language for describing linear constraint inference, we often assign labels to the constraints and write linear combinations of them. For example, say we have the system

$$x_1 + 2x_2 \geq 2 \tag{3.2}$$

$$-x_1 + x_2 \leq 1 \tag{3.3}$$

$$3x_1 - x_2 = -1. \tag{3.4}$$

Then $2\times$ (3.2) + $(-1)\times$ (3.3) + (3.4) refers to the inequality $6x_1 + 2x_2 \geq 2$ since $2(x_1 + 2x_2) + (-1)(-x_1 + x_2) + (3x_1 - x_2)$ gives $6x_1 + 2x_2$ and $2(2) + (-1)(1) + (-1)$ is 2.

Exercises

1. Determine the smallest value of μ such that $x + y \geq \mu$ can be inferred from the system

$$\begin{aligned} 2x + y &\geq 2 \\ x + 3y &\geq 1 \\ 3x + 2y &\geq 6 \end{aligned}$$

2. Show that the inequality $x + 2y \geq 3$ can be inferred from the system

$$\begin{aligned} 2x + y &\geq 2 \\ x + 5y &\geq 7 \\ -x + y &= 1 \end{aligned}$$

in infinitely many ways.

Solutions

1. To infer $x + y \geq \mu$ from the given system, we need $\alpha \geq 0$, $\beta \geq 0$, and $\gamma \geq 0$ such that

$$\begin{aligned} 2\alpha + \beta + 3\gamma &= 1 \\ \alpha + 3\beta + 2\gamma &= 1 \\ 2\alpha + \beta + 6\gamma &= \mu. \end{aligned}$$

Solving gives $\begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} \frac{13}{15} - \frac{7}{15}\mu \\ \frac{4}{15} - \frac{1}{15}\mu \\ -\frac{1}{3} + \frac{1}{3}\mu \end{bmatrix}$. The largest value μ can take so that this tuple has only nonnegative entries is $\frac{13}{7}$.

2. We first label the constraints:

$$2x + y \geq 2 \tag{3.5}$$
$$x + 5y \geq 7 \tag{3.6}$$
$$-x + y = 1. \tag{3.7}$$

Let $C = \left\{ \lambda \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + (1 - \lambda) \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ 0 \end{bmatrix} \ : \ 0 \leq \lambda \leq 1 \right\}$. Note that C has infinitely many elements and that for every $\begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} \in C$, $\alpha\times$ (3.5) + $\beta\times$ (3.6) + $\gamma\times$ (3.7) gives the constraint $x + 2y \geq 3$.

For example, when $\lambda = \frac{1}{2}$, $\lambda \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + (1 - \lambda) \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{2}{3} \\ \frac{1}{6} \\ \frac{1}{2} \end{bmatrix}$.

Chapter 4

Solving systems of linear inequalities

Before we can solve a linear programming problem, we should be able to solve the seemingly simpler problem of finding a feasible solution. We will now consider how one can determine if a system of linear inequalities has a solution.

For the sake of illustrating the principles involved, we limit ourselves to systems consisting of only $\geq$-inequalities. Extending the method to work with any systems of linear constraints is left as an exercise.

Suppose that we want to determine if there exist $x, y \in \mathbb{R}$ satisfying

$$\begin{aligned} x + y &\geq 0 \\ 2x + y &\geq 2 \\ -x + y &\geq 1 \\ -x + 2y &\geq -1. \end{aligned}$$

The key is to take one of the variables and see how it is constrained by the remaining variables. We "isolate" x by rewriting the system to the equivalent system

$$\begin{aligned} x &\geq -y \\ x &\geq 1 - \frac{1}{2}y \\ x &\leq -1 + y \\ x &\leq 1 + 2y. \end{aligned}$$

Hence, x is constrained by the lower bounds $-y$ and $1 - \frac{1}{2}y$ and the upper bounds $-1 + y$ and $1 + 2y$. Therefore, we can find a value for x satisfying these bounds if and only if each of the upper bounds is at least each of the lower bounds; that is,

$$\begin{aligned} -1 + y &\geq -y \\ -1 + y &\geq 1 - \frac{1}{2}y \\ 1 + 2y &\geq -y \\ 1 + 2y &\geq 1 - \frac{1}{2}y. \end{aligned}$$

Simplifying this system gives

$$\begin{aligned}
&2y \geq 1 \\
&\frac{3}{2}y \geq 2 \\
&3y \geq -1 \\
&\frac{5}{2}y \geq 0,
\end{aligned}$$

or more simply,

$$\begin{aligned}
&y \geq \frac{1}{2} \\
&y \geq \frac{4}{3} \\
&y \geq -\frac{1}{3} \\
&y \geq 0.
\end{aligned}$$

Note that this system does not contain the variable x and it has a solution if and only if $y \geq \frac{4}{3}$. Hence, the original system has a solution if and only if $y \geq \frac{4}{3}$. If we set $y = 2$, for example, then x must satisfy

$$\begin{aligned}
&x \geq -2 \\
&x \geq 0 \\
&x \leq 1 \\
&x \leq 5.
\end{aligned}$$

Thus, we can pick x to be any value in the closed interval $[0, 1]$. In particular, $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$ is *one* solution to the given system of linear inequalities. There could be other solutions.

The above example illustrates the process of solving a system of linear inequaltiies by constructing a system that has a reduced number of variables. As the number of variables is finite, the process can be repeated until we obtain a system whose solvability is apparent (as in the one-variable case).

Observe that the pairing of an upper bound constraint of the form $x \leq q$ and a lower bound constraint of the form $x \geq p$ to obtain $q \geq p$ is equivalent to adding the inequalities $-x \geq -q$ and $x \geq p$. This observation leads to the following:

4.1 Fourier-Motzkin Elimination

Given: A system of linear inequalities

$$\sum_{j=1}^{n} a_{ij}x_j \geq b_i, \quad i = 1, \ldots, m$$

where $a_{ij}, b_i \in \mathbb{R}$ for $i = 1, \ldots, m$ and $j = 1, \ldots, n$,

Eliminate x_k for some $k \in \{1, \ldots, n\}$ using the following steps:

1. For each $j \in \{1, \ldots m\}$,
   - if $a_{jk} > 0$, multiply the jth inequality by $\frac{1}{a_{jk}}$,
   - if $a_{jk} < 0$, multiply the jth inequality by $-\frac{1}{a_{jk}}$
2. Form a new system of inequalities as follows:
   - copy down all the inequalities in which the coefficient of x_k is 0
   - for each inequality in which x_k has positive coefficent and for each inequality in which x_k has negative coefficient, obtain a new inequality by adding them together.

Remarks.

1. Step 1 is to ensure that all the nonzero coefficients of x_k are 1 or -1.
2. The new system formed in Step 2 will not contain the variable x_k. Furthermore, if $x_1^*, \ldots, x_n^*$ is a solution to the original system, then $x_1^*, \ldots, x_{k-1}^*, x_{k+1}^*, \ldots, x_n^*$ is a solution to the new system. And if $x_1^*, \ldots, x_{k-1}^*, x_{k+1}^*, \ldots, x_n^*$ is a solution to the new system, then there exists x_k^* such that $x_1^*, \ldots, x_n^*$ is a solution to the original system. (Why?) Hence, the original system has a solution if and only if the new system does.

Now, if we apply Fourier-Motzkin elimination repeatedly, we obtain a system with at most one variable such that it has a solution if and only if the original system does. Since solving systems of linear inequalities with at most one variable is easy, we can conclude whether or not the original system has a solution.

Note that if the coefficients are all rational, the system obtained after eliminating one variable using Fourier-Motzkin elimination will also have only rational coefficients.

Example 4.1. Determine if the following system of inequalities has a solution:

$$\begin{aligned} x_1 + x_2 - 2x_3 &\geq 2 \qquad (1) \\ -x_1 - 3x_2 + x_3 &\geq 0 \qquad (2) \\ x_2 + x_3 &\geq 1 \qquad (3) \end{aligned}$$

We first eliminate x_1. The new system is

$$\begin{aligned} (1) + (2): \quad -2x_2 - x_3 &\geq 2 \qquad (4) \\ x_2 + x_3 &\geq 1 \qquad (3) \end{aligned}$$

We then eliminate x_2. We first normalize the coefficients of x_2:

$$\begin{aligned} \tfrac{1}{2} \times (4) \quad -x_2 - \tfrac{1}{2}x_3 &\geq 1 \qquad (5) \\ x_2 + x_3 &\geq 1 \qquad (3) \end{aligned}$$

So the new system is:

$$(5) + (3): \quad \tfrac{1}{2}x_3 \geq 2$$

So there is a solution. In particular, we can set $x_3 = 4$. Then we must have $x_2 = -3$ and $x_1 = 13$.

Remark. Note that setting x_3 to another value larger than 4 will lead to different solutions to the system. Since there are infinitely many different values that we can set x_3 to, there are infinitely many solutions.

Exercises

1. Use Fourier-Motzkin elimination to determine if there exist $x, y, z \in \mathbb{R}$ satisfying

$$\begin{aligned} x + y + 2z &\geq 1 \\ -x + y + z &\geq 2 \\ x - y + z &\geq 1 \\ -y - 3z &\geq 0. \end{aligned}$$

2. Let $\mathbf{a}^1, \ldots, \mathbf{a}^m \in \mathbb{R}^n$. Let $\beta_1, \ldots, \beta_m \in \mathbb{R}$. Let $\lambda_1, \ldots, \lambda_m \geq 0$. Then the inequality $\left(\sum_{i=1}^m \lambda_i \mathbf{a}^i\right)^{\mathsf{T}} \mathbf{x} \geq \sum_{i=1}^m \lambda_i \beta_i$ is called a nonnegative linear combination of the inequalities ${\mathbf{a}^i}^{\mathsf{T}} \mathbf{x} \geq \beta_i$, $i = 1, \ldots, m$. Show that any new inequality created by Fourier-Motzkin Elimination is a nonnegative linear combination of the original inequalities.

Solutions

1. We use Fourier-Motzkin elimination to eliminate x. We first copy down the inequality $-y - 3z \geq 0$ and then form one new inequality by adding the first two inequalities and another by adding the second and third inequalities. The resulting system is

$$\begin{aligned} -y - 3z &\geq 0 \\ 2y + 3z &\geq 3 \\ 2z &\geq 3. \end{aligned}$$

Note that this system has a solution if and only if the original system does.

We now use Fourier-Motzkin elimination to eliminate y. First we multiply the second inequality by $\frac{1}{2}$ to obtain

$$\begin{aligned} -y - 3z &\geq 0 \\ y + \frac{3}{2}z &\geq \frac{3}{2} \\ 2z &\geq 3. \end{aligned}$$

Eliminating y gives

$$\begin{aligned} 2z &\geq 3 \\ -\frac{3}{2}z &\geq \frac{3}{2}, \end{aligned}$$

or equivalently,

$$\begin{array}{rcl} z & \geq & \frac{3}{2} \\ z & \leq & -1, \end{array}$$

which clearly has no solution. Hence, there is no x, y, z satisfying the original system.

2. First of all, observe that a nonnegative linear combination of $\geq$-inequalites that are themselves nonnegative linear combination of the inequalites in $\mathbf{A}\mathbf{x} \geq \mathbf{b}$ is again a nonnegative linear combination of inequalities in $\mathbf{A}\mathbf{x} \geq \mathbf{b}$.

   It is easy to see that in Step 1 of Fourier-Motzkin Elimination all inequalities are nonnegative linear combinations of the original inequalities. For instance, multiplying ${\mathbf{a}^{i'}}^{\mathsf{T}}\mathbf{x} \geq \beta_{i'}$ by $\alpha > 0$ is the same as taking the nonnegative linear combination $\left(\sum_{i=1}^{m} \lambda_i \mathbf{a}^i\right)^{\mathsf{T}} \mathbf{x} \geq \sum_{i=1}^{m} \lambda_i \beta_i$ with $\lambda_i = 0$ for all $i \neq i'$ and $\lambda_{i'} = \alpha$.

   In Step 2, new inequalities are formed by adding two inequalities from Step 1. Hence, they are nonnegative linear combinations of the inequalities from Step 1. By the observation at the beginning, they are nonnegative linear combinations of the original system.

   Remark. By the observation at the beginning and this result, we see that after repeated applications of Fourier-Motzkin Elimination, all resulting inequalities are nonnegative linear combinations of the original inequalities. This is an important fact that will be exploited later.

Chapter 5

Farkas' Lemma

A well-known result in linear algebra states that a system of linear equations $\mathbf{Ax} = \mathbf{b}$, where $\mathbf{A} \in \mathbb{R}^{m \times n}$, $\mathbf{b} \in \mathbb{R}^m$, and $\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ is a tuple of variables, has no solution if and only if there exists $\mathbf{y} \in \mathbb{R}^m$ such that $\mathbf{y}^\mathsf{T}\mathbf{A} = \mathbf{0}$ and $\mathbf{y}^\mathsf{T}\mathbf{b} \neq 0$.

It is easily seen that if such a $\mathbf{y}$ exists, then the system $\mathbf{Ax} = \mathbf{b}$ cannot have a solution. (Simply multiply both sides of $\mathbf{Ax} = \mathbf{b}$ on the left by $\mathbf{y}^\mathsf{T}$.) However, proving the converse requires a bit of work. A standard elementary proof involves using Gauss-Jordan elimination to reduce the original system to an equivalent system $\mathbf{Qx} = \mathbf{d}$ such that $\mathbf{Q}$ has a row of zero, say in row i, with $\mathbf{d}_i \neq 0$. The process can be captured by a square matrix $\mathbf{M}$ satisfying $\mathbf{MA} = \mathbf{Q}$. We can then take $\mathbf{y}^\mathsf{T}$ to be the ith row of $\mathbf{M}$.

An analogous result holds for systems of linear inequalities. The following result is one of the many variants of a result known as the **Farkas' Lemma**:

Theorem 5.1. *With $\mathbf{A}$, $\mathbf{x}$, and $\mathbf{b}$ as above, the system $\mathbf{Ax} \geq \mathbf{b}$ has no solution if and only if there exists $\mathbf{y} \in \mathbb{R}^m$ such that*

$$\mathbf{y} \geq \mathbf{0},\ \mathbf{y}^\mathsf{T}\mathbf{A} = \mathbf{0},\ \mathbf{y}^\mathsf{T}\mathbf{b} > 0.$$

In other words, the system $\mathbf{Ax} \geq \mathbf{b}$ has no solution if and only if one can infer the inequality $0 \geq \gamma$ for some $\gamma > 0$ by taking a nonnegative linear combination of the inequalities.

This result essentially says that there is always a certificate (the m-tuple $\mathbf{y}$ with the prescribed properties) for the infeasibility of the system $\mathbf{Ax} \geq \mathbf{b}$. This allows third parties to verify the claim of infeasibility without having to solve the system from scratch.

Example 5.1. For the system

$$\begin{aligned} 2x - y + z &\geq 2 \\ -x + y - z &\geq 0 \\ -y + z &\geq 0, \end{aligned}$$

adding two times the second inequality and the third inequality to the first inequality gives $0 \geq 2$. Hence, $\mathbf{y} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ is a certificate of infeasibility for this example.

We now give a proof of Theorem 5.1. It is easy to see that if such a $\mathbf{y}$ exists, then the system $\mathbf{Ax} \geq \mathbf{b}$ has no solution.

Conversely, suppose that the system $\mathbf{Ax} \geq \mathbf{b}$ has no solution. It suffices to show that we can infer the inequality $0 \geq \alpha$ for some postive α by taking nonnegative linear combination of the inequalities in the system $\mathbf{Ax} \geq \mathbf{b}$. If the system already contains an inequality $0 \geq \alpha$ for some positive α, then we are done. Otherwise, we show by induction on n that we can infer such an inequality.

Base case: The system $\mathbf{Ax} \geq \mathbf{b}$ has only one variable.

For the system to have no solution, there must exist two inequalites $ax_1 \geq t$ and $-a'x_1 \geq t'$ such that $a, a' > 0$ and $\frac{t}{a} > \frac{-t'}{a'}$. Adding $\frac{1}{a}$ times the inequality $ax_1 \geq t$ and $\frac{1}{a'}$ times the inequality $-a'x_1 \geq t'$ gives the inequality $0 \geq \frac{t}{a} + \frac{t'}{a'}$ with a positive right-hand side. This establishes the base case.

Induction hypothesis: Let $n \geq 2$ be an integer. Assume that given any system of linear inequalities $\mathbf{A}'\mathbf{x} \geq \mathbf{b}'$ in $n-1$ variables having no solution, one can infer the inequality $0 \geq \alpha'$ for some positive α' by taking a nonnegative linear combination of the inequalities in the system $\mathbf{Px} \geq \mathbf{q}$.

Apply Fourier-Motzkin elimination to eliminate x_n from $\mathbf{Ax} \geq \mathbf{b}$ to obtain the system $\mathbf{Px} \geq \mathbf{q}$. As $\mathbf{Ax} \geq \mathbf{b}$ has no solution, $\mathbf{Px} \geq \mathbf{q}$ also has no solution.

By the induction hypothesis, one can infer the inequality $0 \geq \alpha$ for some positive α by taking a nonnegative linear combination of the inequalities in $\mathbf{Px} \geq \mathbf{q}$. However, each inequality in $\mathbf{Px} \geq \mathbf{q}$ can be obtained from a nonnegative linear combination of the inequalites in $\mathbf{Ax} \geq \mathbf{b}$. Hence, one can infer the inequality $0 \geq \alpha$ by taking a nonnegative linear combination of nonnegative linear cominbations of the inequalities in $\mathbf{Ax} \geq \mathbf{b}$. Since a nonnegative linear combination of nonnegative linear cominbations of the inequalities in $\mathbf{Ax} \geq \mathbf{b}$ is simply a nonnegative linear combination of the inequalities in $\mathbf{Ax} \geq \mathbf{b}$, the result follows.

□

Remark. Notice that in the proof above, if $\mathbf{A}$ and $\mathbf{b}$ have only rational entries, then we can take $\mathbf{y}$ to have only rational entries as well.

Corollary 5.1. Let $\mathbf{A} \in \mathbb{R}^{m \times n}$ and let $\mathbf{b} \in \mathbb{R}^m$. The system

$$\begin{aligned} \mathbf{Ax} &= \mathbf{b} \\ \mathbf{x} &\geq \mathbf{0} \end{aligned}$$

has no solution if and only if there exists $\mathbf{y} \in \mathbb{R}^m$ such that $\mathbf{y}^\mathsf{T}\mathbf{A} \leq \mathbf{0}$ and $\mathbf{y}^\mathsf{T}\mathbf{b} > 0$. Furthermore, if $\mathbf{A}$ and $\mathbf{b}$ are rational, $\mathbf{y}$ can be taken to be rational.

Proof. One can easily check that if such a $\mathbf{y}$ exists, there is no soluton.

We now prove the converse. The system

$$\begin{aligned} \mathbf{Ax} &= \mathbf{b} \\ \mathbf{x} &\geq \mathbf{0} \end{aligned}$$

can be rewritten as

$$\begin{bmatrix} \mathbf{A} \\ -\mathbf{A} \\ \mathbf{I} \end{bmatrix} \mathbf{x} \geq \begin{bmatrix} \mathbf{b} \\ -\mathbf{b} \\ \mathbf{0} \end{bmatrix}$$

where $\mathbf{I}$ is the $n \times n$ identity matrix. Then by Theorem 5.1, if this system has no solution, then there exist $\mathbf{u}, \mathbf{v} \in \mathbb{R}^m$, $\mathbf{w} \in \mathbb{R}^n$ satisfying

$$\mathbf{u}, \mathbf{v}, \mathbf{w} \geq \mathbf{0},\ \mathbf{u}^\mathsf{T}\mathbf{A} - \mathbf{v}^\mathsf{T}\mathbf{A} + \mathbf{w} = \mathbf{0},\ \mathbf{u}^\mathsf{T}\mathbf{b} - \mathbf{v}^\mathsf{T}\mathbf{b} > 0.$$

The result now follows from setting $\mathbf{y} = \mathbf{u} - \mathbf{v}$.

Rationality follows from the remark after the proof of Theorem 5.1.

□

Exercises

1. You are given that the following system has no solution.

$$\begin{aligned} x_1 + x_2 + 2x_3 &\geq 1 \\ -x_1 + x_2 + x_3 &\geq 2 \\ x_1 - x_2 + x_3 &\geq 1 \\ -x_2 - 3x_3 &\geq 0. \end{aligned}$$

Obtain a certificate of infeasibility for the system.

Solutions

1. The system can be written as $\mathbf{A}\mathbf{x} \geq \mathbf{b}$ with $\mathbf{A} = \begin{bmatrix} 1 & 1 & 2 \\ -1 & 1 & 1 \\ 1 & -1 & 1 \\ 0 & -1 & -3 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix}$. So we need to find $\mathbf{y} \geq 0$ such that $\mathbf{y}^\mathsf{T}\mathbf{A} = \mathbf{0}$ and $\mathbf{y}^\mathsf{T}\mathbf{b} > 0$. As the system of equations $\mathbf{y}^\mathsf{T}\mathbf{A} = \mathbf{0}$ is homogeneous, we could without loss of generality fix $\mathbf{y}^\mathsf{T}\mathbf{b} = 1$, thus leading to the system

$$\begin{aligned} \mathbf{y}^\mathsf{T}\mathbf{A} &= \mathbf{0} \\ \mathbf{y}^\mathsf{T}\mathbf{b} &= 1 \\ \mathbf{y} &\geq \mathbf{0} \end{aligned}$$

that we could attempt to solve directly. However, it is possible to obtain a $\mathbf{y}$ using the Fourier-Motzkin Elimination Method.
Let us first label the inequalities:

$$\begin{array}{rcll} x_1+x_2+2x_3 & \geq & 1 & (1) \\ -x_1+x_2+x_3 & \geq & 2 & (2) \\ x_1-x_2+x_3 & \geq & 1 & (3) \\ -x_2-3x_3 & \geq & 0. & (4) \end{array}$$

Eliminating x_1 gives:

$$\begin{array}{rcll} -x_2-3x_3 & \geq & 0 & (4) \\ 2x_2+3x_3 & \geq & 3 & (5) \\ 2x_3 & \geq & 3. & (6) \end{array}$$

Note that (5) is obtained from $(1)+(2)$ and (6) is obtained from $(2)+(3)$.
Multiplying (5) by $\frac{1}{2}$ gives

$$\begin{array}{rcll} -x_2-3x_3 & \geq & 0 & (4) \\ x_2+\dfrac{3}{2}x_3 & \geq & \dfrac{3}{2} & (7) \\ 2x_3 & \geq & 3. & (6) \end{array}$$

Eliminating x_2 gives:

$$\begin{array}{rcll} 2x_3 & \geq & 3 & (6) \\ -\dfrac{3}{2}x_3 & \geq & \dfrac{3}{2} & (8) \end{array}$$

where (8) is obtained from $(4)+(7)$.
Now $\frac{3}{4}\times(6)+(8)$ gives $0\geq\frac{15}{4}$, a contradiction.
To obtain a certificate of infeasibility, we trace back the computations. Note that $\frac{3}{4}(6)+(8)$ is given by $\frac{3}{4}((2)+(3))+(4)+(7)$, which in turn is given by $\frac{3}{4}((2)+(3))+(4)+\frac{1}{2}(5)$, which in turn is given by $\frac{3}{4}((2)+(3))+(4)+\frac{1}{2}((1)+(2))$.
Thus, we can obtain $0\geq\frac{15}{4}$ from the nonnegative linear combination of the original inequalities as follows: $\frac{1}{2}(1)+\frac{5}{4}(2)+\frac{3}{4}(3)+(4)$.
Therefore, $\mathbf{y}=\begin{bmatrix}\frac{1}{2}\\ \frac{5}{4}\\ \frac{3}{4}\\ 1\end{bmatrix}$ is a certificate of infeasibility.
(Check that $\mathbf{y}^\mathsf{T}\mathbf{A}=\mathbf{0}$ and $\mathbf{y}^\mathsf{T}\mathbf{b}>0$.

Chapter 6

Solving linear programming problems

Fourier-Motzkin elimination can actually be used to solve a linear programming problem though the method is not efficient and is almost never used in practice. We illustrate the process with an example.

Consider the following linear programming problem:

$$\begin{array}{rl} \min & x + y \\ \text{s.t.} & x + 2y \geq 2 \\ & 3x + 2y \geq 6. \end{array} \tag{6.1}$$

Observe that (6.1) is equivalent to

$$\begin{array}{rl} \min & z \\ \text{s.t.} & z - x - y = 0 \\ & x + 2y \geq 2 \\ & 3x + 2y \geq 6. \end{array} \tag{6.2}$$

Note that the objective function is replaced with z and z is set to the original objective function in the first constraint of (6.2) since $z = x + y$ if and only if $z - x - y = 0$. Then, solving (6.2) is equivalent to finding among all the solutions to the following system a solution that minimizes z, if it exists.

$$\begin{array}{rl} z - x - y \geq 0 & \quad (1) \\ -z + x + y \geq 0 & \quad (2) \\ x + 2y \geq 2 & \quad (3) \\ 3x + 2y \geq 6 & \quad (4) \end{array}$$

Since we are interested in the minimum possible value for z we use Fourier-Motzking elimination to eliminate the variables x and y.

To eliminate x, we first multiply (4) by $\frac{1}{3}$ to obtain:

$$\begin{array}{rl} z - x - y \geq 0 & \quad (1) \\ -z + x + y \geq 0 & \quad (2) \\ x + 2y \geq 2 & \quad (3) \\ x + \frac{2}{3}y \geq 2 & \quad (5) \end{array}$$

Then eliminate x to obtain

$$\begin{array}{rrl} (1)+(2): & 0 \geq 0 & \\ (1)+(3): & z+y \geq 2 & (6) \\ (1)+(5): & z-\frac{1}{3}y \geq 2 & (7) \end{array}$$

Note that there is no need to keep the first inequality. To eliminate y, we first multiply (7) by 3 to obtain:

$$\begin{array}{rl} z+y \geq 2 & (6) \\ 3z-y \geq 6 & (8) \end{array}$$

Then eliminate y to obtain

$$4z \geq 8 \qquad (9)$$

Multiplying (9) by $\frac{1}{4}$ gives $z \geq 2$. Hence, the minimum possible value for z among all the solutions to the system is 2. So the optimal value of (6.2) is 2. To obtain an optimal solution, set $z = 2$. Then we have no choice but to set $y = 0$ and $x = 2$. One can check that $(x, y) = (2, 0)$ is a feasible solution with objective function value 2.

We can obtain an independent proof that the optimal value is indeed 2 if we trace back the computations. Note that the inequality $z \geq 2$ is given by

$$\begin{aligned} \frac{1}{4}(9) &\Leftarrow \frac{1}{4}(6)+\frac{1}{4}(8) \\ &\Leftarrow \frac{1}{4}(1)+\frac{1}{4}(3)+\frac{3}{4}(7) \\ &\Leftarrow \frac{1}{4}(1)+\frac{1}{4}(3)+\frac{3}{4}(1)+\frac{3}{4}(5) \\ &\Leftarrow (1)+\frac{1}{4}(3)+\frac{1}{4}(4) \end{aligned}$$

This shows that $\frac{1}{4}(3)+\frac{1}{4}(4)$ gives the inequality $x+y \geq 2$. Hence, no feasible solution to (6.1) can have objective function value less than 2. But we have found one feasible solution with objective function value 2. Hence, 2 is the optimal value.

6.1 Fundamental Theorem of Linear Programming

Having used Fourier-Motzkin elimination to solve a linear programming problem, we now will go one step further and use the same technique to prove the following important result.

Theorem 6.1 (Fundamental Theorem of Linear Programming)**.** *For any given linear programming problem, exactly one of the following holds:*

1. *the problem is infeasible;*
2. *the problem is unbounded;*
3. *the problem has an optimal solution.*

Proof. Without loss of generality, we may assume that the linear programming problem is of the form

$$\begin{array}{rl} \min & \mathbf{c}^\mathsf{T}\mathbf{x} \\ \text{s.t.} & \mathbf{A}\mathbf{x} \geq \mathbf{b} \end{array} \qquad (6.3)$$

where m and n are positive integers, $\mathbf{A} \in \mathbb{R}^{m\times n}$, $\mathbf{b} \in \mathbb{R}^m$, $\mathbf{c} \in \mathbb{R}^n$, and $\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ is a tuple of variables. Indeed, any linear programming problem can be converted to a linear programming problem in the form of (6.3) having the same feasible region and optimal solution set. To see this, note that a constraint of the form $\mathbf{a}^\mathsf{T}\mathbf{x} \leq \beta$ can be written as $-\mathbf{a}^\mathsf{T}\mathbf{x} \geq -\beta$; a constraint of the form $\mathbf{a}^\mathsf{T}\mathbf{x} = \beta$ written as a pair of constraints $\mathbf{a}^\mathsf{T}\mathbf{x} \geq \beta$ and $-\mathbf{a}^\mathsf{T}\mathbf{x} \geq -\beta$; and a maximization problem is equivalent to the problem that minimizes the negative of the objective function subject to the same constraints.

Suppose that (6.3) is not infeasible. Form the system

$$\begin{aligned} z - \mathbf{c}^\mathsf{T}\mathbf{x} &\geq 0 \\ -z + \mathbf{c}^\mathsf{T}\mathbf{x} &\geq 0 \\ \mathbf{A}\mathbf{x} &\geq \mathbf{b}. \end{aligned} \tag{6.4}$$

Solving (6.3) is equivalent to finding among all the solutions to (6.4) one that minimizes z, if it exists. Eliminating the variables $x_1, \ldots, x_n$ (in any order) using Fourier-Motzkin elimination gives a system of linear inequalities (S) containing at most the variable z. By scaling, we may assume that the each coefficient of z in (S) is 1, -1, or 0. Note that any z satisfying (S) can be extended to a solution to (6.4) and the z value from any solution to (6.4) must satisfy (S).

That (6.3) is not unbounded implies that (S) must contain an inequality of the form $z \geq \beta$ for some $\beta \in \mathbb{R}$. (Why?) Let all the inequalites in which the coefficient of z is positive be

$$z \geq \beta_i$$

where $\beta_i \in \mathbb{R}$ for $i = 1, \ldots, p$ for some positive integer p. Let $\gamma = \max\{\beta_1, \ldots, \beta_p\}$. Then for any solution x, z to (6.4), z is at least γ. But we can set $z = \gamma$ and extend it to a solution to (6.4). Hence, we obtain an optimal solution for (6.3) and γ is the optimal value. This completes the proof of the theorem.

□

Remark. We can construct multipliers to infer the inequality $\mathbf{c}^\mathsf{T}\mathbf{x} \geq \gamma$ from the system $\mathbf{A}\mathbf{x} \geq \mathbf{b}$. Because we obtained the inequality $z \geq \gamma$ using Fourier-Motzkin elimination, there must exist real numbers $\alpha, \beta, y_1^*, \ldots, y_m^* \geq 0$ such that

$$\begin{bmatrix} \alpha & \beta & y_1^* & \cdots & y_m^* \end{bmatrix} \begin{bmatrix} 1 & -\mathbf{c}^\mathsf{T} \\ -1 & \mathbf{c}^\mathsf{T} \\ 0 & \mathbf{A} \end{bmatrix} \begin{bmatrix} z \\ \mathbf{x} \end{bmatrix} \geq \begin{bmatrix} \alpha & \beta & y_1^* & \cdots & y_m^* \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \mathbf{b} \end{bmatrix}$$

is identically $z \geq \gamma$. Note that we must have $\alpha - \beta = 1$ and

$$\mathbf{y}^* \geq \mathbf{0},\ {\mathbf{y}^*}^\mathsf{T}\mathbf{A} = \mathbf{c}^\mathsf{T},\ \text{and } {\mathbf{y}^*}^\mathsf{T}\mathbf{b} = \gamma$$

where $\mathbf{y}^* = [y_1^*, \ldots, y_m^*]^\mathsf{T}$. Hence, $y_1^*, \ldots, y_m^*$ are the desired multipliers.,

The significance of the fact that we can infer $\mathbf{c}^\mathsf{T}\mathbf{x} \geq \gamma$ where γ will be discussed in more details when we look at duality theory for linear programming.

Exercises

1. Determine the optimal value of the following linear programming problem:

$$\begin{array}{rl} \min & x \\ \text{s.t.} & x + y \geq 2 \\ & x - 2y + z \geq 0 \\ & y - 2z \geq -1. \end{array}$$

2. Determine if the following linear programming problem has an optimal solution:

$$\begin{array}{rl} \min & x_1 + 2x_2 \\ \text{s.t.} & x_1 + 3x_2 \geq 4 \\ & -x_1 + x_2 \geq 0. \end{array}$$

3. A set $S \subset \mathbb{R}^n$ is said to be bounded if there exists a real number $M > 0$ such that for every $\mathbf{x} \in S$, $|x_i| < M$ for all $i = 1, \ldots, n$. Prove that every linear programming problem with a bounded nonempty feasible region has an optimal solution.

Solutions

1. The problem is equivalent to determining the minimum value for x among all x, y, z satisfying

$$\begin{array}{rl} x + y \geq 2 & (1) \\ x - 2y + z \geq 0 & (2) \\ y - 2z \geq -1. & (3) \end{array}$$

We use Fourier-Motzkin Elimination Method to eliminate z. Multiplying (3) by $\frac{1}{2}$, we get

$$\begin{array}{rl} x + y \geq 2 & (1) \\ x - 2y + z \geq 0 & (2) \\ \frac{1}{2}y - z \geq -\frac{1}{2}. & (4) \end{array}$$

Eliminating z, we obtain

$$\begin{array}{rl} x + y \geq 2 & (1) \\ x - \frac{3}{2}y \geq -\frac{1}{2} & (5) \end{array}$$

where (5) is given by $(2) + (4)$.

Multiplying (5) by $\frac{2}{3}$, we get

$$\begin{array}{rl} x + y \geq 2 & (1) \\ \frac{2}{3}x - y \geq -\frac{1}{3} & (6) \end{array}$$

Eliminating y, we get

$$\frac{5}{3}x \geq \frac{5}{3} \qquad (7)$$

where (7) is given by $(1) + (6)$. Multiplying (7) by $\frac{3}{5}$, we obtain $x \geq 1$. Hence, the minimum possible value for x is 1.

Note that setting $x = 1$, the system (1) and (6) forces $y = 1$. And (2) and (3) together force $z = 1$. One can check that $(x, y, z) = (1, 1, 1)$ is a feasible solution.

Remark. Note that the inequality $x \geq 1$ is given by

$$\begin{aligned}
\frac{3}{5}(7) &\Leftarrow \frac{3}{5}(1)+\frac{3}{5}(6)\\
&\Leftarrow \frac{3}{5}(1)+\frac{2}{5}(5)\\
&\Leftarrow \frac{3}{5}(1)+\frac{2}{5}(2)+\frac{2}{5}(4)\\
&\Leftarrow \frac{3}{5}(1)+\frac{2}{5}(2)+\frac{1}{5}(3)
\end{aligned}$$

2. It suffices to determine if there exists a minimum value for z among all the solutions to the system

$$\begin{aligned}
z - x_1 - 2x_2 &\geq 0 & (1)\\
-z + x_1 + 2x_2 &\geq 0 & (2)\\
x_1 + 3x_2 &\geq 4 & (3)\\
-x_1 + x_2 &\geq 0 & (4)
\end{aligned}$$

Using Fourier-Motzkin elimination to eliminate x_1, we obtain:

$$\begin{aligned}
(1)+(2): && 0 &\geq 0 &\\
(1)+(3): && z + x_2 &\geq 4 & (5)\\
(2)+(4): && -z + 3x_2 &\geq 0 & (6)\\
(3)+(4): && 4x_2 &\geq 4 & (7)
\end{aligned}$$

Note that all the coefficients of x_2 is nonnegative. Hence, eliminating x_2 will result in a system with no constraints. Therefore, there is no lower bound on the value of z. In particular, if $z = t$ for $t \leq 0$, then from (5)–(6), we need $x_2 \geq 4 - t$, $3x_2 \geq t$, and $x_2 \geq 1$. Hence, we can set $x_2 = 4 - t$ and $x_1 = -8 + 3t$. This gives a feasible solution for all $t \leq 0$ with objective function value that approaches $-\infty$ as $t \to -\infty$. Hence, the linear programming problem is unbounded.

3. Let (P) denote a linear programming problem with a bounded nonempty feasible region with objective function $\mathbf{c}^\mathsf{T}\mathbf{x}$. By assumption, (P) is not infeasible. Note that (P) is not unbounded because $|\mathbf{c}^\mathsf{T}\mathbf{x}| \leq \sum_i |c_i||x_i| \leq M \sum_i |c_i|$. Thus, by Theorem 6.1, (P) has an optimal solution.

Chapter 7

Linear programming duality

Consider the following problem:

$$\begin{array}{ll} \min & \mathbf{c}^\mathsf{T}\mathbf{x} \\ \text{s.t.} & \mathbf{A}\mathbf{x} \geq \mathbf{b}. \end{array} \tag{7.1}$$

In the remark at the end of Chapter 6, we saw that if (7.1) has an optimal solution, then there exists $\mathbf{y}^* \in \mathbb{R}^m$ such that $\mathbf{y}^* \geq 0$, ${\mathbf{y}^*}^\mathsf{T}\mathbf{A} = \mathbf{c}^\mathsf{T}$, and ${\mathbf{y}^*}^\mathsf{T}\mathbf{b} = \gamma$ where γ denotes the optimal value of (7.1).

Take any $\mathbf{y} \in \mathbb{R}^m$ satisfying $\mathbf{y} \geq \mathbf{0}$ and $\mathbf{y}^\mathsf{T}\mathbf{A} = \mathbf{c}^\mathsf{T}$. Then we can infer from $\mathbf{A}\mathbf{x} \geq \mathbf{b}$ the inequality $\mathbf{y}^\mathsf{T}\mathbf{A}\mathbf{x} \geq \mathbf{y}^\mathsf{T}\mathbf{b}$, or more simply, $\mathbf{c}^\mathsf{T}\mathbf{x} \geq \mathbf{y}^\mathsf{T}\mathbf{b}$. Thus, for any such $\mathbf{y}$, $\mathbf{y}^\mathsf{T}\mathbf{b}$ gives a lower bound for the objective function value of any feasible solution to (7.1). Since γ is the optimal value of (P), we must have $\gamma \geq \mathbf{y}^\mathsf{T}\mathbf{b}$.

As ${\mathbf{y}^*}^\mathsf{T}\mathbf{b} = \gamma$, we see that γ is the optimal value of

$$\begin{array}{ll} \max & \mathbf{y}^\mathsf{T}\mathbf{b} \\ \text{s.t.} & \mathbf{y}^\mathsf{T}\mathbf{A} = \mathbf{c}^\mathsf{T} \\ & \mathbf{y} \geq \mathbf{0}. \end{array} \tag{7.2}$$

Note that (7.2) is a linear programming problem! We call it the **dual problem** of the **primal problem** (7.1). We say that the dual variable y_i is **associated** with the constraint ${\mathbf{a}^{(i)}}^\mathsf{T}\mathbf{x} \geq b_i$ where ${\mathbf{a}^{(i)}}^\mathsf{T}$ denotes the ith row of $\mathbf{A}$.

In other words, we define the dual problem of (7.1) to be the linear programming problem (7.2). In the discussion above, we saw that if the primal problem has an optimal solution, then so does the dual problem and the optimal values of the two problems are equal. Thus, we have the following result:

Theorem 7.1. *Suppose that* (7.1) *has an optimal solution. Then* (7.2) *also has an optimal solution and the optimal values of the two problems are equal.*

At first glance, requiring all the constraints to be $\geq$-inequalities as in (7.1) before forming the dual problem seems a bit restrictive. We now see how the dual problem of a primal problem in general form can be defined. We first make two observations that motivate the definition.

Observation 1

Suppose that our primal problem contains a mixture of all types of linear constraints:

$$\begin{array}{rl} \min & \mathbf{c}^\mathsf{T}\mathbf{x} \\ \text{s.t.} & \mathbf{A}\mathbf{x} \geq \mathbf{b} \\ & \mathbf{A}'\mathbf{x} \leq \mathbf{b}' \\ & \mathbf{A}''\mathbf{x} = \mathbf{b}'' \end{array} \tag{7.3}$$

where $\mathbf{A} \in \mathbb{R}^{m\times n}$, $\mathbf{b} \in \mathbb{R}^m$, $\mathbf{A}' \in \mathbb{R}^{m'\times n}$, $\mathbf{b}' \in \mathbb{R}^{m'}$, $\mathbf{A}'' \in \mathbb{R}^{m''\times n}$, and $\mathbf{b}'' \in \mathbb{R}^{m''}$.

We can of course convert this into an equivalent problem in the form of (7.1) and form its dual.
However, if we take the point of view that the function of the dual is to infer from the constraints of (7.3) an inequality of the form $\mathbf{c}^\mathsf{T}\mathbf{x} \geq \gamma$ with γ as large as possible by taking an appropriate linear combination of the constraints, we are effectively looking for $\mathbf{y} \in \mathbb{R}^m$, $\mathbf{y} \geq \mathbf{0}$, $\mathbf{y}' \in \mathbb{R}^{m'}$, $\mathbf{y}' \leq \mathbf{0}$, and $\mathbf{y}'' \in \mathbb{R}^{m''}$, such that

$$\mathbf{y}^\mathsf{T}\mathbf{A} + \mathbf{y}'^\mathsf{T}\mathbf{A}' + \mathbf{y}''^\mathsf{T}\mathbf{A}'' = \mathbf{c}^\mathsf{T}$$

with $\mathbf{y}^\mathsf{T}\mathbf{b} + \mathbf{y}'^\mathsf{T}\mathbf{b}' + \mathbf{y}''^\mathsf{T}\mathbf{b}''$ to be maximized.

(The reason why we need $\mathbf{y}' \leq \mathbf{0}$ is because inferring a $\geq$-inequality from $\mathbf{A}'\mathbf{x} \leq \mathbf{b}'$ requires nonpositive multipliers. There is no restriction on $\mathbf{y}''$ because the constraints $\mathbf{A}''\mathbf{x} = \mathbf{b}''$ are equalities.)

This leads to the dual problem:

$$\begin{array}{rl} \max & \mathbf{y}^\mathsf{T}\mathbf{b} + \mathbf{y}'^\mathsf{T}\mathbf{b}' + \mathbf{y}''^\mathsf{T}\mathbf{b}'' \\ \text{s.t.} & \mathbf{y}^\mathsf{T}\mathbf{A} + \mathbf{y}'^\mathsf{T}\mathbf{A}' + \mathbf{y}''^\mathsf{T}\mathbf{A}'' = \mathbf{c}^\mathsf{T} \\ & \quad \mathbf{y} \geq \mathbf{0} \\ & \quad \mathbf{y}' \leq \mathbf{0}. \end{array} \tag{7.4}$$

In fact, we could have derived this dual by applying the definition of the dual problem to

$$\begin{array}{rl} \min & \mathbf{c}^\mathsf{T}\mathbf{x} \\ \text{s.t.} & \begin{bmatrix} \mathbf{A} \\ -\mathbf{A}' \\ \mathbf{A}'' \\ -\mathbf{A}'' \end{bmatrix} \mathbf{x} \geq \begin{bmatrix} \mathbf{b} \\ -\mathbf{b}' \\ \mathbf{b}'' \\ -\mathbf{b}'' \end{bmatrix}, \end{array}$$

which is equivalent to (7.3). The details are left as an exercise.

Observation 2

Consider the primal problem of the following form:

$$\begin{array}{rl} \min & \mathbf{c}^\mathsf{T}\mathbf{x} \\ \text{s.t.} & \mathbf{A}\mathbf{x} \geq \mathbf{b} \\ & x_i \geq 0 \quad i \in P \\ & x_i \leq 0 \quad i \in N \end{array} \tag{7.5}$$

where P and N are disjoint subsets of $\{1, \ldots, n\}$. In other words, constraints of the form $x_i \geq 0$ or $x_i \leq 0$ are separated out from the rest of the inequalities.

Forming the dual of (7.5) as defined under Observation 1, we obtain the dual problem

$$\begin{array}{lll}\max & \mathbf{y}^\mathsf{T}\mathbf{b} & \\ \text{s.t.} & \mathbf{y}^\mathsf{T}\mathbf{a}^{(i)} = c_i & i \in \{1,\ldots,n\}\backslash(P\cup N) \\ & \mathbf{y}^\mathsf{T}\mathbf{a}^{(i)} + p_i = c_i & i \in P \\ & \mathbf{y}^\mathsf{T}\mathbf{a}^{(i)} + q_i = c_i & i \in N \\ & p_i \geq 0 & i \in P \\ & q_i \leq 0 & i \in N\end{array} \tag{7.6}$$

where $\mathbf{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix}$. Note that this problem is equivalent to the following without the variables p_i, $i \in P$ and q_i, $i \in N$:

$$\begin{array}{lll}\max & \mathbf{y}^\mathsf{T}\mathbf{b} & \\ \text{s.t.} & \mathbf{y}^\mathsf{T}\mathbf{a}^{(i)} = c_i & i \in \{1,\ldots,n\}\backslash(P\cup N) \\ & \mathbf{y}^\mathsf{T}\mathbf{a}^{(i)} \leq c_i & i \in P \\ & \mathbf{y}^\mathsf{T}\mathbf{a}^{(i)} \geq c_i & i \in N,\end{array} \tag{7.7}$$

which can be taken as the dual problem of (7.5) instead of (7.6). The advantage here is that it has fewer variables than (7.6).

Hence, the dual problem of

$$\begin{array}{ll}\min & \mathbf{c}^\mathsf{T}\mathbf{x} \\ \text{s.t.} & \mathbf{A}\mathbf{x} \geq \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0}\end{array}$$

is simply

$$\begin{array}{ll}\max & \mathbf{y}^\mathsf{T}\mathbf{b} \\ \text{s.t.} & \mathbf{y}^\mathsf{T}\mathbf{A} \leq \mathbf{c}^\mathsf{T} \\ & \mathbf{y} \geq \mathbf{0}.\end{array}$$

As we can see from bove, there is no need to associate dual variables to constraints of the form $x_i \geq 0$ or $x_i \leq 0$ provided we have the appropriate types of constraints in the dual problem. Combining all the observations lead to the definition of the dual problem for a primal problem in general form as discussed next.

7.1 The dual problem

Let $\mathbf{A} \in \mathbb{R}^{m\times n}$, $\mathbf{b} \in \mathbb{R}^m$, $\mathbf{c} \in \mathbb{R}^n$. Let ${\mathbf{a}^{(i)}}^\mathsf{T}$ denote the ith row of $\mathbf{A}$. Let $\mathbf{A}_j$ denote the jth column of $\mathbf{A}$.

Let (P) denote the minimization problem with variables in the tuple $\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ given as follows:

- The objective function to be minimized is $\mathbf{c}^\mathsf{T}\mathbf{x}$

- The constraints are

$$\mathbf{a}^{(i)^\mathsf{T}}\mathbf{x} \sqcup_i b_i$$

  where $\sqcup_i$ is $\leq$, $\geq$, or $=$ for $i = 1, \ldots, m$.

- For each $j \in \{1, \ldots, n\}$, x_j is constrained to be nonnegative, nonpositive, or free (i.e. not constrained to be nonnegative or nonpositive.)

Then the **dual problem** is defined to be the maximization problem with variables in the tuple $\mathbf{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix}$ given as follows:

- The objective function to be maximized is $\mathbf{y}^\mathsf{T}\mathbf{b}$
- For $j = 1, \ldots, n$, the jth constraint is

$$\begin{cases} \mathbf{y}^\mathsf{T}\mathbf{A}_j \leq c_j & \text{if } x_j \text{ is constrained to be nonnegative} \\ \mathbf{y}^\mathsf{T}\mathbf{A}_j \geq c_j & \text{if } x_j \text{ is constrained to be nonpositive} \\ \mathbf{y}^\mathsf{T}\mathbf{A}_j = c_j & \text{if } x_j \text{ is free.} \end{cases}$$

- For each $i \in \{1, \ldots, m\}$, y_i is constrained to be nonnegative if $\sqcup_i$ is $\geq$; y_i is constrained to be nonpositive if $\sqcup_i$ is $\leq$; y_i is free if $\sqcup_i$ is $=$.

The following table can help remember the above.

Primal (min)	Dual (max)
$\geq$ constraint	≥ 0 variable
$\leq$ constraint	≤ 0 variable
$=$ constraint	free variable
≥ 0 variable	$\leq$ constraint
≤ 0 variable	$\geq$ constraint
free variable	$=$ constraint

Below is an example of a primal-dual pair of problems based on the above definition:

Consider the primal problem:

$$\begin{array}{rrcrcrcl} \text{min} & x_1 & - & 2x_2 & + & 3x_3 & & \\ \text{s.t.} & -x_1 & & & + & 4x_3 & = & 5 \\ & 2x_1 & + & 3x_2 & - & 5x_3 & \geq & 6 \\ & & & 7x_2 & & & \leq & 8 \\ & x_1 & & & & & \geq & 0 \\ & & & x_2 & & & & \text{free} \\ & & & & & x_3 & \leq & 0. \end{array}$$

Here, $\mathbf{A} = \begin{bmatrix} -1 & 0 & 4 \\ 2 & 3 & -5 \\ 0 & 7 & 0 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 5 \\ 6 \\ 8 \end{bmatrix}$, and $\mathbf{c} = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$.

The primal problem has three constraints. So the dual problem has three variables. As the first constraint in the primal is an equation, the corresponding variable in the dual is free. As the second constraint in the primal is a $\geq$-inequality, the corresponding variable in the dual is nonnegative. As the third constraint in the primal is a $\leq$-inequality, the corresponding variable in the dual is nonpositive. Now, the primal problem has three variables. So the dual problem has three constraints. As the first variable in the primal is nonnegative, the corresponding constraint in the dual is a $\leq$-inequality. As the second variable in the primal is free, the corresponding constraint in the dual is an equation. As the third variable in the primal is nonpositive, the corresponding constraint in the dual is a $\geq$-inequality. Hence, the dual problem is:

$$\begin{array}{rrcrcrcl} \text{max} & 5y_1 & + & 6y_2 & + & 8y_3 & & \\ \text{s.t.} & -y_1 & + & 2y_2 & & & \leq & 1 \\ & & & 3y_2 & + & 7y_3 & = & -2 \\ & 4y_1 & - & 5y_2 & & & \geq & 3 \\ & y_1 & & & & & & \text{free} \\ & & & y_2 & & & \geq & 0 \\ & & & & & y_3 & \leq & 0. \end{array}$$

Remarks. Note that in some books, the primal problem is always a maximization problem. In that case, what is our primal problem is their dual problem and what is our dual problem is their primal problem.

One can now prove a more general version of Theorem 7.1 as stated below. The details are left as an exercise.

Theorem 7.2 (Duality Theorem for Linear Programming)**.** *Let (P) and (D) denote a primal-dual pair of linear programming problems. If either (P) or (D) has an optimal solution, then so does the other. Furthermore, the optimal values of the two problems are equal.*

Theorem 7.2 is also known informally as **strong duality**.

Exercises

1. Write down the dual problem of

$$\begin{array}{rrcrcl} \min & 4x_1 & - & 2x_2 & & \\ \text{s.t.} & x_1 & + & 2x_2 & \geq & 3 \\ & 3x_1 & - & 4x_2 & = & 0 \\ & & & x_2 & \geq & 0. \end{array}$$

2. Write down the dual problem of the following:

$$\begin{array}{rrcrcrcl} \min & & & 3x_2 & + & x_3 & & \\ \text{s.t.} & x_1 & + & x_2 & + & 2x_3 & = & 1 \\ & x_1 & & & - & 3x_3 & \leq & 0 \\ & x_1 & , & x_2 & , & x_3 & \geq & 0. \end{array}$$

3. Write down the dual problem of the following:

$$\begin{array}{rrcrcrcl}
\min & x_1 & & & - & 9x_3 & & \\
\text{s.t.} & x_1 & - & 3x_2 & + & 2x_3 & = & 1 \\
& x_1 & & & & & \leq & 0 \\
& & & x_2 & & & & \text{free} \\
& & & & & x_3 & \geq & 0.
\end{array}$$

4. Determine all values c_1, c_2 such that the linear programming problem

$$\begin{array}{rl}
\min & c_1x_1 + c_2x_2 \\
\text{s.t.} & 2x_1 + x_2 \geq 2 \\
& x_1 + 3x_2 \geq 1.
\end{array}$$

has an optimal solution. Justify your answer

Solutions

1. The dual is

$$\begin{array}{rrcrcl}
\max & 3y_1 & & & & \\
\text{s.t.} & y_1 & + & 3y_2 & = & 4 \\
& 2y_1 & - & 4y_2 & \leq & -2 \\
& y_1 & & & \geq & 0.
\end{array}$$

2. The dual is

$$\begin{array}{rrcrcl}
\max & y_1 & & & & \\
\text{s.t.} & y_1 & + & y_2 & \leq & 0 \\
& y_1 & & & \leq & 3 \\
& 2y_1 & - & 3y_2 & \leq & 1 \\
& y_1 & & & & \text{free} \\
& & & y_2 & \leq & 0.
\end{array}$$

3. The dual is

$$\begin{array}{rrcl}
\max & y_1 & & \\
\text{s.t.} & y_1 & \geq & 1 \\
& -3y_1 & = & 0 \\
& 2y_1 & \leq & -9 \\
& y_1 & & \text{free.}
\end{array}$$

4. Let (P) denote the given linear programming problem.

Note that $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is a feasible solution to (P). Therefore, by Theorem 6, it suffices to find all values c_1, c_2 such that

(P) is not unbounded. This amounts to finding all values c_1, c_2 such that the dual problem of (P) has a feasible solution.

The dual problem of (P) is

$$\begin{aligned} \max \quad & 2y_1 + y_2 \\ & 2y_1 + y_2 = c_1 \\ & y_1 + 3y_2 = c_2 \\ & y_1, y_2 \geq 0. \end{aligned}$$

The two equality constraints gives $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \frac{3}{5}c_1 - \frac{1}{5}c_2 \\ -\frac{1}{5}c_1 + \frac{2}{5}c_2 \end{bmatrix}$. Thus, the dual problem is feasible if and only if c_1 and c_2 are real numbers satisfying

$$\begin{aligned} \frac{3}{5}c_1 - \frac{1}{5}c_2 \geq & \quad 0 \\ -\frac{1}{5}c_1 + \frac{2}{5}c_2 \geq & \quad 0, \end{aligned}$$

or more simply,

$$\frac{1}{3}c_2 \leq c_1 \leq 2c_2.$$

Chapter 8

Complementary slackness

Theorem 8.1. *Let (P) and (D) denote a primal-dual pair of linear programming problems in generic form as defined previously. Let* $\mathbf{x}^*$ *be a feasible solution to* (P) *and* $\mathbf{y}^*$ *is a feasible solution to (D). Then the following hold:*

1. $\mathbf{c}^\mathsf{T}\mathbf{x}^* \geq {\mathbf{y}^*}^\mathsf{T}\mathbf{b}$.
2. $\mathbf{x}^*$ *and* $\mathbf{y}^*$ *are optimal solutions to the respective problems if and only if the following conditions (known as the* ***complementary slackness conditions****) hold:*

$$\begin{array}{lll} x_j^* = 0 & \textit{or} \quad {\mathbf{y}^*}^\mathsf{T}\mathbf{A}_j = c_j & \textit{for } j = 1, \ldots, n \\ y_i^* = 0 & \textit{or} \quad {\mathbf{a}^{(i)}}^\mathsf{T}\mathbf{x}^* = b_i & \textit{for } i = 1, \ldots, m \end{array}$$

Part 1 of the theorem is known as **weak duality**. Part 2 of the theorem is often called the **Complementary Slackness Theorem**.

Proof of Theorem 8.1

Note that if x_j^* is constrained to be nonnegative, its corresponding dual constraint is ${\mathbf{y}^*}^\mathsf{T}\mathbf{A}_j \leq c_j$. Hence, $(c_j - {\mathbf{y}^*}^\mathsf{T}\mathbf{A}_j)x_j^* \geq 0$ with equality if and only if $x_j^* = 0$ or ${\mathbf{y}^*}^\mathsf{T}\mathbf{A}_j = c_j$ (or both).

If x_j^* is constrained to be nonpositive, its corresponding dual constraint is ${\mathbf{y}^*}^\mathsf{T}\mathbf{A}_j \geq c_j$. Hence, $(c_j - {\mathbf{y}^*}^\mathsf{T}\mathbf{A}_j)x_j^* \geq 0$ with equality if and only if $x_j^* = 0$ or ${\mathbf{y}^*}^\mathsf{T}\mathbf{A}_j = c_j$ (or both).

If x_j^* is free, its corresponding dual constraint is ${\mathbf{y}^*}^\mathsf{T}\mathbf{A}_j = c_j$. Hence, $(c_j - {\mathbf{y}^*}^\mathsf{T}\mathbf{A}_j)x_j^* = 0$.

We can combine these three cases and obtain that $(\mathbf{c}^\mathsf{T} - {\mathbf{y}^*}^\mathsf{T}\mathbf{A})\mathbf{x}^* = \displaystyle\sum_{j=1}^n (c_j - {\mathbf{y}^*}^\mathsf{T}\mathbf{A}_j)x_j^* \geq 0$ with equality if and only if for each $j = 1, \ldots, n$,

$$x_j^* = 0 \text{ or } {\mathbf{y}^*}^\mathsf{T}\mathbf{A}_j = c_j.$$

(Here, the usage of "or" is not exclusive.)

Similarly, ${\mathbf{y}^*}^\mathsf{T}(\mathbf{A}\mathbf{x}^* - \mathbf{b}) = \displaystyle\sum_{i=1}^n y_i^*({\mathbf{a}^{(i)}}^\mathsf{T}\mathbf{x}^* - b_i) \geq 0$ with equality if and only if for each $i = 1, \ldots, n$,

$$y_i^* = 0 \text{ or } {\mathbf{a}^{(i)}}^\mathsf{T}\mathbf{x}^* = b_i.$$

(Again, the usage of "or" is not exclusive.)

Adding the inequalities $(\mathbf{c}^\mathsf{T} - \mathbf{y^*}^\mathsf{T}\mathbf{A})\mathbf{x}^* \geq 0$ and $\mathbf{y^*}^\mathsf{T}(\mathbf{A}\mathbf{x}^* - \mathbf{b}) \geq 0$, we obtain $\mathbf{c}^\mathsf{T}\mathbf{x}^* - \mathbf{y^*}^\mathsf{T}\mathbf{b} \geq 0$ with equality if and only if the complementary slackness conditions hold. By strong duality, $\mathbf{x}^*$ is optimal (P) and $\mathbf{y}^*$ is optimal for (D) if and only if $\mathbf{c}^\mathsf{T}\mathbf{x}^* = \mathbf{y^*}^\mathsf{T}\mathbf{b}$. The result now follows.

□

The complementary slackness conditions give a characterization of optimality which can be useful in solving certain problems as illustrated by the following example.

Example 8.1. Let (P) denote the following linear programming problem:

$$\begin{array}{rrcrcrcl}
\text{min} & 2x_1 & + & 4x_2 & + & 2x_3 & & \\
\text{s.t.} & x_1 & + & x_2 & + & 3x_3 & \leq & 1 \\
& -x_1 & + & 2x_2 & + & x_3 & \geq & 1 \\
& & & 3x_2 & - & 6x_3 & = & 0 \\
& x_1 & , & & & x_3 & \geq & 0 \\
& & & x_2 & & & & \text{free.}
\end{array}$$

Is $\mathbf{x}^* = \begin{bmatrix} 0 \\ \frac{2}{5} \\ \frac{1}{5} \end{bmatrix}$ an optimal solution to (P)?

One could answer this question by solving (P) and then see if the objective function value of $\mathbf{x}^*$, assuming that its feasibiilty has already been verified, is equal to the optimal value. However, there is a way to make use of the given information to save some work.

Let (D) denote the dual problem of (P):

$$\begin{array}{rrcrcrcl}
\max & y_1 & + & y_2 & & & & \\
\text{s.t.} & y_1 & - & y_2 & & & \leq & 2 \\
& y_1 & + & 2y_2 & + & 3y_3 & = & 4 \\
& 3y_1 & + & y_2 & - & 6y_3 & \leq & 2 \\
& y_1 & & & & & \leq & 0 \\
& & & y_2 & & & \geq & 0 \\
& & & & & y_3 & & \text{free.}
\end{array}$$

One can check that $\mathbf{x}^*$ is a feasible solution to (P). If $\mathbf{x}^*$ is optimal, then there must exist a feasible solution $\mathbf{y}^*$ to (D) satisfying together with $\mathbf{x}^*$ the complementary slackness conditions:

$$\begin{array}{rcr}
y_1^* = 0 & \text{or} & x_1^* + x_2^* + 3x_3^* = 1 \\
y_2^* = 0 & \text{or} & -x_1^* + 2x_2^* + x_3^* = 1 \\
y_3^* = 0 & \text{or} & 3x_2^* - 6x_3^* = 0 \\
x_1^* = 0 & \text{or} & y_1^* - y_2^* = 2 \\
x_2^* = 0 & \text{or} & y_1^* + 2y_2^* + 3y_3^* = 4 \\
x_3^* = 0 & \text{or} & 3y_1^* + y_2^* - 6y_3^* = 2.
\end{array}$$

As $x_2^*, x_3^* > 0$, satisfying the above conditions require that

$$\begin{aligned}
y_1^* + 2y_2^* + 3y_3^* &= 4 \\
3y_1^* + y_2^* - 6y_3^* &= 2.
\end{aligned}$$

Solving for y_2^* and y_3^* in terms of y_1^* gives $y_2^* = 2 - y_1^*$, $y_3^* = \frac{1}{3}y_1^*$. To make $\mathbf{y}^*$ feasible to (D), we can set $y_1^* = 0$ to obtain the feasible solution $y_1^* = 0, y_2^* = 2$, $y_3^* = 0$. We can check that this $\mathbf{y}^*$ satisfies the complementary slackness conditions with $\mathbf{x}^*$. Hence, $\mathbf{x}^*$ is an optimal solution to (P) by Theorem 8.1, part 2.

Exercises

1. Let (P) and (D) denote a primal-dual pair of linear programming problems. Prove that if (P) is not infeasible and (D) is infeasible, then (P) is unbounded.

2. Let (P) denote the following linear programming problem:

$$\begin{array}{rrcrcrcl} \min & & & 4x_2 & + & 2x_3 & & \\ \text{s.t.} & x_1 & + & x_2 & + & 3x_3 & \leq & 1 \\ & x_1 & - & 2x_2 & + & x_3 & \geq & 1 \\ & x_1 & + & 3x_2 & - & 6x_3 & = & 0 \\ & x_1 & , & & & x_3 & \geq & 0 \\ & & & x_2 & & & & \text{free.} \end{array}$$

   Determine if $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{3}{5} \\ -\frac{1}{5} \\ 0 \end{bmatrix}$ is an optimal solution to (P).

3. Let (P) denote the following linear programming problem:

$$\begin{array}{rrcrcrcl} \min & x_1 & + & 2x_2 & - & 3x_3 & & \\ \text{s.t.} & x_1 & + & 2x_2 & + & 2x_3 & = & 2 \\ & -x_1 & + & x_2 & + & x_3 & = & 1 \\ & -x_1 & + & x_2 & - & x_3 & \geq & 0 \\ & x_1 & , & x_2 & , & x_3 & \geq & 0 \end{array}$$

   Determine if $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ is an optimal solution to (P).

4. Let m and n be positive integers. Let $\mathbf{A} \in \mathbb{R}^{m \times n}$. Let $\mathbf{b} \in \mathbb{R}^m$. Let $\mathbf{c} \in \mathbb{R}^n$. Let (P) denote the linear programming problem

$$\begin{array}{rl} \min & \mathbf{c}^\mathsf{T}\mathbf{x} \\ \text{s.t.} & \mathbf{A}\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0}. \end{array}$$

   Let (D) denote the dual problem of (P):

$$\begin{array}{rl} \max & \mathbf{y}^\mathsf{T}\mathbf{b} \\ \text{s.t.} & \mathbf{y}^\mathsf{T}\mathbf{A} \leq \mathbf{c}^\mathsf{T}. \end{array}$$

   Suppose that $\mathbf{A}$ has rank m and that (P) has at least one optimal solution. Prove that if $x_j^* = 0$ for *every* optimal solution $\mathbf{x}^*$ to (P), then there exists an optimal solution $\mathbf{y}^*$ to (D) such that ${\mathbf{y}^*}^\mathsf{T}\mathbf{A}_j < c_i$ where $\mathbf{A}_j$ denotes the jth column of $\mathbf{A}$.

Solutions

1. By the Fundamental Theorem of Linear Programming, (P) either is unbounded or has an optimal solution. If it is the latter, then by strong duality, (D) has an optimal solution, which contradicts that (D) is infeasible. Hence, (P) must be unbounded.

2. We show that it is not an optimal solution to (P). First, note that the dual problem of (P) is

$$\begin{array}{rrcrcrcl}
\max & y_1 & + & y_2 & & & & \\
\text{s.t.} & y_1 & + & y_2 & + & y_3 & \leq & 0 \\
 & y_1 & - & 2y_2 & + & 3y_3 & = & 4 \\
 & 3y_1 & + & y_2 & - & 6y_3 & \leq & 2 \\
 & y_1 & & & & & \leq & 0 \\
 & & & y_2 & & & \geq & 0 \\
 & & & & & y_3 & & \text{free.}
\end{array}$$

\end{bmatrix}) were an optimal solution, there would exist $\mathbf{y}^*$ feasible to (D) satisfying the complementary slackness conditions with $\mathbf{x}^*$:

$$\begin{array}{rcr}
y_1^* = 0 & \text{or} & x_1^* + \ x_2^* + 3x_3^* = 1 \\
y_2^* = 0 & \text{or} & x_1^* - 2x_2^* + \ x_3^* = 1 \\
y_3^* = 0 & \text{or} & x_1^* + 3x_2^* - 6x_3^* = 0 \\
x_1^* = 0 & \text{or} & y_1^* + \ \ y_2^* + \ y_3^* = 0 \\
x_2^* = 0 & \text{or} & y_1^* - 2y_2^* + 3y_3^* = 4 \\
x_3^* = 0 & \text{or} & 3y_1^* + \ y_2^* - 6y_3^* = 2.
\end{array}$$

Since $x_1^* + x_2^* + 3x_3^* < 1$, we must have $y_1^* = 0$. Also, x_1^*, x_2^* are both nonzero. Hence,

$$\begin{aligned}
y_1^* + y_2^* + y_3^* &= 0 \\
y_1^* - 2y_2^* + 3y_3^* &= 4,
\end{aligned}$$

implying that

$$\begin{aligned}
y_2^* + y_3^* &= 0 \\
-2y_2^* + 3y_3^* &= 4.
\end{aligned}$$

Solving gives $y_2^* = -\frac{4}{5}$ and $y_3^* = \frac{4}{5}$. But this implies that y^* is not a feasible solution to the dual problem since we need $y_2^* \geq 0$. Hence, $\mathbf{x}^*$ is not an optimal solution to (P).

3. We show that it is not an optimal solution to (P). First, note that the dual problem of (P) is

$$\begin{array}{rrcrcrcl}
\max & 2y_1 & + & y_2 & & & & \\
\text{s.t.} & y_1 & - & y_2 & - & y_3 & \leq & 1 \\
 & 2y_1 & + & y_2 & + & y_3 & \leq & 2 \\
 & 2y_1 & + & y_2 & - & y_3 & \leq & -3 \\
 & y_1 & , & y_2 & & & & \text{free.} \\
 & & & & & y_3 & \geq & 0
\end{array}$$

Note that $\mathbf{x}^* = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ is a feasible solution to (P). If it were an optimal solution to (P), there would exist $\mathbf{y}^*$ feasible to the dual problem (D) satisfying the complementary slackness conditions with $\mathbf{x}^*$:

$$\begin{array}{rcr}
y_1^* = 0 & \text{or} & x_1^* + 2x_2^* + 2x_3^* = 2 \\
y_2^* = 0 & \text{or} & -x_1^* + \ x_2^* + \ x_3^* = 1 \\
y_3^* = 0 & \text{or} & -x_1^* + \ x_2^* - \ x_3^* = 0 \\
x_1^* = 0 & \text{or} & y_1^* - y_2^* - y_3^* = 1 \\
x_2^* = 0 & \text{or} & 2y_1^* + y_2^* + y_3^* = 2 \\
x_3^* = 0 & \text{or} & 2y_1^* + y_2^* - y_3^* = -3.
\end{array}$$

Since $-x_1^* + x_2^* - x_3^* > 0$, we must have $y_3^* = 0$. Also, $x_2^* > 0$ implies that $2y_1^* + y_2^* + y_3^* = 2$. Simplifying gives $y_2^* = 2 - 2y_1^*$.

Hence, for y^* to be feasible to the dual problem, it needs to satisfy the third constraint, $2y_1^* + (2 - 2y_1^*) \leq -3$, which simplifies to the absurdity $2 \leq -3$. Hence, $\mathbf{x}^*$ is not an optimal solution to (P).

4. Let v denote the optimal value of (P). Let (P') denote the problem

$$\begin{array}{rl}
\min & -x_i \\
\text{s.t.} & \mathbf{A}\mathbf{x} = \mathbf{b} \\
 & \mathbf{c}^\mathsf{T}\mathbf{x} \leq v \\
 & \mathbf{x} \geq \mathbf{0}
\end{array}$$

Note that x^* is a feasible solution to (P') if and only if it is an optimal solution to (P). Since $x_i^* = 0$ for every optimal solution to (P), we see that the optimal value of (P') is 0.

Let (D') denote the dual problem of (P'):

$$\begin{array}{rll}
\max & \mathbf{y}^\mathsf{T}\mathbf{b} + vu & \\
\text{s.t.} & \mathbf{y}^\mathsf{T}\mathbf{A}_p + c_p u \leq 0 & \text{for all } p \neq i \\
 & \mathbf{y}^\mathsf{T}\mathbf{A}_i + c_i u \leq -1 & \\
 & u \leq 0. &
\end{array}$$

Suppose that an optimal solution to (D') is given by $\mathbf{y}', u'$. Let $\bar{\mathbf{y}}$ be an optimal solution to (D). We consider two cases.

Case 1: $u' = 0$.

Then ${\mathbf{y}'}^\mathsf{T}\mathbf{b} = 0$. Hence, $\mathbf{y}^* = \bar{\mathbf{y}} + \mathbf{y}'$ is an optimal solution to (D) with ${\mathbf{y}^*}^\mathsf{T}\mathbf{A}_i < c_i$.

Case 2: $u' < 0$.

Then ${\mathbf{y}'}^\mathsf{T}\mathbf{b} + vu' = 0$, implying that $\frac{1}{|u'|}{\mathbf{y}'}^\mathsf{T}\mathbf{b} = v$. Let $\mathbf{y}^* = \frac{1}{|u|}\mathbf{y}'$. Then $\mathbf{y}^*$ is an optimal solution to (D) with ${\mathbf{y}^*}^\mathsf{T}\mathbf{A}_i < c_i$.

Chapter 9

Basic feasible solution

For a linear constraint $\mathbf{a}^\mathsf{T}\mathbf{x} \sqcup \gamma$ where $\sqcup$ is $\geq$, $\leq$, or $=$, we call $\mathbf{a}^\mathsf{T}$ the **coefficient row-vector** of the constraint.

Let S denote a system of linear constraints with n variables and m constraints given by ${\mathbf{a}^{(i)}}^\mathsf{T}\mathbf{x} \sqcup_i b_i$ where $\sqcup_i$ is $\geq$, $\leq$, or $=$ for $i = 1, \ldots, m$.

For $\mathbf{x}' \in \mathbb{R}^n$, let $J(S, \mathbf{x}')$ denote the set $\{i \ : \ {\mathbf{a}^{(i)}}^\mathsf{T}\mathbf{x}' = b_i\}$ and define $\mathbf{A}_{S,\mathbf{x}'}$ to be the matrix whose rows are precisely the coefficient row-vectors of the constraints indexed by $J(S, \mathbf{x}')$.

Example 9.1. Suppose that S is the system

$$\begin{aligned} x_1 + x_2 - x_3 &\geq 2 \\ 3x_1 - x_2 + x_3 &= 2 \\ 2x_1 - x_2 &\leq 1 \end{aligned}$$

If $\mathbf{x}' = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$, then $J(S, \mathbf{x}') = \{1, 2\}$ since $\mathbf{x}'$ satisfies the first two constraints with equality but not the third.

Hence, $\mathbf{A}_{S,\mathbf{x}'} = \begin{bmatrix} 1 & 1 & -1 \\ 3 & -1 & 1 \end{bmatrix}$.

Definition 9.1. A solution $\mathbf{x}^*$ to S is called a **basic feasible solution** if the rank of $\mathbf{A}_{S,\mathbf{x}^*}$ is n.

A basic feasible solution to the system in Example 9.1 is $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$.

It is not difficult to see that in two dimensions, basic feasible solutions correspond to "corner points" of the set of all solutions. Therefore, the notion of a basic feasible solution generalizes the idea of a corner point to higher dimensions.

The following result is the basis for what is commonly known as the **corner method** for solving linear programming problems in two variables.

Theorem 9.1. *Let (P) be a linear programming problem. Suppose that (P) has an optimal solution and there exists a basic feasible solution to its constraints. Then there exists an optimal solution that is a basic feasible solution.*

We first state the following simple fact from linear algebra:

Lemma 9.1. *Let $\mathbf{A} \in \mathbb{R}^{m\times n}$ and $\mathbf{d} \in \mathbb{R}^n$ be such that $\mathbf{A}\mathbf{d} = \mathbf{0}$. If $\mathbf{q} \in \mathbb{R}^n$ satisfies $\mathbf{q}^\mathsf{T}\mathbf{d} \neq 0$ then $\mathbf{q}^T$ is not in the row space of $\mathbf{A}$.*

Proof of Theorem 9.1.
Suppose that the system of constraints in (P), call it S, has m constraints and n variables. Let the objective function be $\mathbf{c}^\mathsf{T}\mathbf{x}$. Let v denote the optimal value.

Let $\mathbf{x}^*$ be an optimal solution to (P) such that the rank of $\mathbf{A}_{S,\mathbf{x}^*}$ is as large as possible. We claim that $\mathbf{x}^*$ must be a basic feasible solution.

To ease notation, let $J = J(S, \mathbf{x}^*)$. Let $N = \{1, \ldots, m\}\backslash J$.

Suppose to the contrary that the rank of $\mathbf{A}_{S,\mathbf{x}^*}$ is less than n. Let $\mathbf{P}\mathbf{x} = \mathbf{q}$ denote the system of equations obtained by setting the constraints indexed by J to equalities. Then $\mathbf{P}\mathbf{x} = \mathbf{A}_{S,\mathbf{x}^*}$. Since $\mathbf{P}$ has n columns and its rank is less than n, there exists a nonzero $\mathbf{d}$ such that $\mathbf{P}\mathbf{d} = \mathbf{0}$.

As $\mathbf{x}^*$ satisfies each constraint indexed by N strictly, for a sufficiently small $\epsilon > 0$, $\mathbf{x}^* + \epsilon\mathbf{d}$ and $\mathbf{x}^* - \epsilon\mathbf{d}$ are solutions to S and therefore are feasible to (P). Thus,

$$\begin{aligned} \mathbf{c}^\mathsf{T}(\mathbf{x}^* + \epsilon\mathbf{d}) &\geq v \\ \mathbf{c}^\mathsf{T}(\mathbf{x}^* - \epsilon\mathbf{d}) &\geq v. \end{aligned} \tag{9.1}$$

Since $\mathbf{x}^*$ is an optimal solution, we have $\mathbf{c}^\mathsf{T}\mathbf{x}^* = v$. Hence, (9.1) simplifies to

$$\begin{aligned} \epsilon\mathbf{c}^\mathsf{T}\mathbf{d} &\geq 0 \\ -\epsilon\mathbf{c}^\mathsf{T}\mathbf{d} &\geq 0, \end{aligned}$$

giving us $\mathbf{c}^\mathsf{T}\mathbf{d} = 0$ since $\epsilon > 0$.

Without loss of generality, assume that the constraints indexed by N are $\mathbf{Q}\mathbf{x} \geq \mathbf{r}$. As (P) does have a basic feasible solution, implying that the rank of $\begin{bmatrix}\mathbf{P}\\ \mathbf{Q}\end{bmatrix}$ is n, at least one row of $\mathbf{Q}$, which we denote by $\mathbf{t}^\mathsf{T}$, must satisfy $\mathbf{t}^\mathsf{T}\mathbf{d} \neq 0$. Without loss of generality, we may assume that $\mathbf{t}^\mathsf{T}\mathbf{d} > 0$, replacing $\mathbf{d}$ with $-\mathbf{d}$ if necessary.

Consider the linear programming problem

$$\begin{array}{ll} \min & \lambda \\ \text{s.t.} & \mathbf{Q}(\mathbf{x}^* + \lambda\mathbf{d}) \geq \mathbf{p} \end{array}$$

Since at least one entry of $\mathbf{Q}\mathbf{d}$ is positive (namely, $\mathbf{t}^\mathsf{T}\mathbf{d}$), this problem must have an optimal solution, say λ'. Setting $\mathbf{x}' = \mathbf{x}^* + \lambda'$, we have that $\mathbf{x}'$ is an optimal solution since $\mathbf{c}^\mathsf{T}\mathbf{x}' = v$.

Now, $\mathbf{x}'$ must satisfy at least one constraint in $\mathbf{Q} \geq \mathbf{p}$ with equality. Let $\mathbf{q}^\mathsf{T}$ be the coefficient row-vector of one such constraint. Then the rows of $\mathbf{A}_{S,\mathbf{x}'}$ must have all the rows of $\mathbf{A}_{S,\mathbf{x}^*}$ and $\mathbf{q}^\mathsf{T}$. Since $\mathbf{q}^\mathsf{T}\mathbf{d} \neq 0$, by Lemma 9.1, the rank of $\mathbf{A}_{S,\mathbf{x}'}$ is larger than rank the rank of $\mathbf{A}_{S,\mathbf{x}^*}$, contradicting our choice of $\mathbf{x}^*$. Thus, $\mathbf{x}^*$ must be a basic feasible solution.

Exercises

1. Find all basic feasible solutions to

$$\begin{aligned} x_1 + 2x_2 - x_3 &\geq 1 \\ x_2 + 2x_3 &\geq 3 \\ -x_1 + 2x_2 + x_3 &\geq 3 \\ -x_1 + x_2 + x_3 &\geq 0. \end{aligned}$$

2. A set $S \subset \mathbb{R}^n$ is said to be bounded if there exists a real number $M > 0$ such that for every $\mathbf{x} \in S$, $|x_i| < M$ for all $i = 1, \ldots, n$. Let $\mathbf{A} \in \mathbb{R}^{m \times n}$ and $\mathbf{b} \in \mathbb{R}^m$. Prove that if $\{\mathbf{x} \ : \ \mathbf{A}\mathbf{x} \geq \mathbf{b}\}$ is nonempty and bounded, then there is a basic feasible solution to $\mathbf{A}\mathbf{x} \geq \mathbf{b}$.

3. Let $\mathbf{A} \in \mathbb{R}^{m \times n}$ and $\mathbf{b} \in \mathbb{R}^m$ where m and n are positive integers with $m \leq n$. Suppose that the rank of $\mathbf{A}$ is m and $\mathbf{x}'$ is a basic feasible solution to

$$\begin{aligned} \mathbf{A}\mathbf{x} &= \mathbf{b} \\ \mathbf{x} &\geq \mathbf{0}. \end{aligned}$$

Let $J = \{i \ : \ x'_i > 0\}$. Prove that the columns of $\mathbf{A}$ indexed by J are linearly independent.

Solutions

1. To obtain all the basic feasible solutions, it suffices to enumerate all subsystems $\mathbf{A}'\mathbf{x} \geq \mathbf{b}'$ of the given system such that the rank of $\mathbf{A}'$ is three and solve $\mathbf{A}'\mathbf{x} = \mathbf{b}'$ for $\mathbf{x}$ and see if is a solution to the system, in which case it is a basic feasible solution. Observe that every basic feasible solution can be discovered in this manner.

We have at most four subsystems to consider.

Setting the first three inequalities to equality gives the unique solution $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ which satisfies the given system..

Hence, $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ is a basic feasible solution.

Setting the first, second, and fourth inequalities to equality gives the unique solution $\begin{bmatrix} \frac{5}{3} \\ \frac{1}{3} \\ \frac{4}{3} \end{bmatrix}$ which violates the third inequality of the given system.

Setting the first, third, and fourth inequalities to equality leads to no solution. (In fact, the coefficient matrix of the system does not have rank 3 and therefore this case can be ignored.)

Setting the last three inequalities to equality gives the unique solution $\begin{bmatrix} 3 \\ 3 \\ 0 \end{bmatrix}$ which satisfies the given system.

Hence, $\begin{bmatrix} 3 \\ 3 \\ 0 \end{bmatrix}$ is a basic feasible solution.

Thus, $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 3 \\ 0 \end{bmatrix}$ are the only basic feasible solutions.

2. Let S denote the system $\mathbf{A}\mathbf{x} \geq \mathbf{b}$. Let $\mathbf{x}'$ be a solution to S such that the rank of $\mathbf{A}_{S,\mathbf{x}'}$ is as large as possible. If the rank is n, then we are done. Otherwise, there exists nonzero $\mathbf{d} \in \mathbb{R}^n$ such $\mathbf{A}_{S,\mathbf{x}'}\mathbf{d} = \mathbf{0}$. Since the set of solutions to S is a bounded set, at least one of the following values is finite:

   - $\max\{\lambda \ : \ \mathbf{A}(\mathbf{x}' + \lambda\mathbf{d}) \geq \mathbf{b}\}$
   - $\min\{\lambda \ : \ \mathbf{A}(\mathbf{x}' + \lambda\mathbf{d}) \geq \mathbf{b}\}$

   Without loss of generality, assume that the maximum is finite and is equal to λ^*. Setting $\mathbf{x}^*$ to $\mathbf{x}' + \lambda^*\mathbf{d}$, we have that the rows of $\mathbf{A}_{S,\mathbf{x}^*}$ contains all the rows of $\mathbf{A}_{S,\mathbf{x}'}$ plus at least one additional row, say $\mathbf{q}^\mathsf{T}$. Since $\mathbf{q}^\mathsf{T}\mathbf{d} \neq 0$, by Lemma 9.1, the rank of $\mathbf{A}_{S,\mathbf{x}^*}$ is larger than the rank of $\mathbf{A}_{S,\mathbf{x}'}$, contradicting our choice of $\mathbf{x}'$.

3. The system of equations obtained from taking all the constraints satisfied with equality by $\mathbf{x}'$ is

$$\begin{aligned} \mathbf{A}\mathbf{x} &= \mathbf{b} \\ x_j &= 0 \quad j \notin J. \end{aligned} \tag{9.2}$$

   Note that the coefficient matrix of this system has rank n if and only if it has a unique solution. Now, (9.2) simplifies to

$$\sum_{j \in J} x_j \mathbf{A}_j = \mathbf{b},$$

   which has a unique solution if and only if the columns of $\mathbf{A}$ indexed by J are linearly independent.

Chapter 10

Using an LP solver

Linear programming problems are rarely solved by hand. Problems from industrial applications often have thousands (and sometimes millions) of variables and constraints. Fortunately, there exist a number of commercial as well as open-source solvers that can handle such large-scale problem. In this chapter, we will take a look at one of them. Before we do so, we introduce the CPLEX LP format which allows us to specify linear programming problems in a way close to how we write them.

10.1 CPLEX LP file format

A description of the CPLEX LP file format can be found here.

However, it is perhaps easiest to illustrate with some examples. Consider

$$\begin{array}{rrcrcrcrcl}
\min & x_1 & - & 2x_2 & + & x_3 & & & & \\
\text{s.t.} & x_1 & - & 2x_2 & + & x_3 & - & x_4 & \geq & 1 \\
& x_1 & + & x_2 & + & 2x_3 & - & 2x_4 & = & 3 \\
& & - & x_2 & + & 3x_3 & - & 5x_4 & \leq & 7 \\
& & & & & x_3 & , & x_4 & \geq & 0.
\end{array}$$

The problem can be specified in LP format as follows:

```
min x_1 - 2x_2 + x_3
st
 x_1 - 2x_2 +  x_3 -  x_4 >= 1
 x_1 +  x_2 + 2x_3 - 2x_4  = 3
     -  x_2 + 3x_3 - 5x_4 <= 7
bounds
 x_1 free
 x_2 free
end
```

Any variable that does not appear in the `bounds` section is automatically assumed to be nonnegative.

10.2 SoPlex

SoPlex is an open-source linear programming solver. It is free for noncommercial use. Binaries for Mac OS X and Windows are readily available for download.

One great feature of SoPlex is that it can return exact rational solutions whereas most other solvers only return solutions as floating-point numbers.

Suppose that the problem for the example at the beginning is saved in LP format in an ASCII file named `eg.lp`. The following is the output of running SoPlex in a macOS command-line terminal:

```
bash-3.2$ ./soplex-2.2.1.darwin.x86_64.gnu.opt -X --solvemode=2 -f=0 -o=0 eg.lp
SoPlex version 2.2.1 [mode: optimized] [precision: 8 byte] [rational: GMP 6.0.0] [githash: 267a44a]
Copyright (c) 1996-2016 Konrad-Zuse-Zentrum fuer Informationstechnik Berlin (ZIB)

int:solvemode = 2
real:feastol = 0
real:opttol = 0

Reading (real) LP file <eg.lp> . . .
Reading took 0.00 seconds.

LP has 3 rows 4 columns and 11 nonzeros.

Initial floating-point solve . . .
Simplifier removed 0 rows, 0 columns, 0 nonzeros, 0 col bounds, 0 row bounds
Reduced LP has 3 rows 4 columns 11 nonzeros
Equilibrium scaling LP

type |   time |   iters | facts |  shift   |violation |    value
  L  |    0.0 |       0 |     1 | 4.00e+00 | 2.00e+00 | 0.00000000e+00
  E  |    0.0 |       1 |     2 | 0.00e+00 | 4.00e+00 | 3.00000000e+00
  E  |    0.0 |       2 |     3 | 0.00e+00 | 0.00e+00 | 1.00000000e+00

Floating-point optimal.
Max. bound violation = 0
Max. row violation = 1/4503599627370496
Max. reduced cost violation = 0
Max. dual violation = 0
Performing rational reconstruction . . .
Tolerances reached.
Solved to optimality.

SoPlex status       : problem is solved [optimal]
Solving time (sec)  : 0.00
Iterations          : 2
Objective value     : 1.00000000e+00


Primal solution (name, value):
x_1              7/3
x_2              2/3
All other variables are zero.
bash-3.2$
```

The option `-X` asks the solver to display the primal rational solution. The options `--solvemode=2` invokes iterative refinement for solving for a rational solution. The options `-f=0 -o=0` set the primal feasibility and dual feasibility tolerances to 0. Without these options, one might get only approximate solutions to the problem. If we remove the last three options and replace `-X` with `-x`, we obtain the following instead:

```
bash-3.2$ ./soplex-2.2.1.darwin.x86_64.gnu.opt -x eg.lp
SoPlex version 2.2.1 [mode: optimized] [precision: 8 byte] [rational: GMP 6.0.0] [githash: 267a44a]
Copyright (c) 1996-2016 Konrad-Zuse-Zentrum fuer Informationstechnik Berlin (ZIB)

Reading (real) LP file <eg.lp> . . .
Reading took 0.00 seconds.

LP has 3 rows 4 columns and 11 nonzeros.

Simplifier removed 0 rows, 0 columns, 0 nonzeros, 0 col bounds, 0 row bounds
Reduced LP has 3 rows 4 columns 11 nonzeros
Equilibrium scaling LP
type |   time |   iters | facts |  shift   |violation |    value
  L  |    0.0 |       0 |     1 | 4.00e+00 | 2.00e+00 | 0.00000000e+00
  E  |    0.0 |       1 |     2 | 0.00e+00 | 4.00e+00 | 3.00000000e+00
  E  |    0.0 |       2 |     3 | 0.00e+00 | 0.00e+00 | 1.00000000e+00
 --- transforming basis into original space
  L  |    0.0 |       0 |     1 | 0.00e+00 | 0.00e+00 | 1.00000000e+00
  L  |    0.0 |       0 |     1 | 0.00e+00 | 0.00e+00 | 1.00000000e+00

SoPlex status       : problem is solved [optimal]
Solving time (sec)  : 0.00
Iterations          : 2
Objective value     : 1.00000000e+00


Primal solution (name, value):
x_1       2.333333333
x_2       0.666666667
All other variables are zero (within 1.0e-16).
bash-3.2$
```

There are many solver options that one can specify. To view the list of all the options, simply run the solver without options and arguments.

10.3 NEOS server for optimization

If one does not want to download and install SoPlex, one can use the NEOS server for optimization. In addition to SoPlex, there are many other solvers to choose from.

To solve a linear programming problem using SoPlex on the NEOS server, one can submit a file in CPLEX LP format here.

Exercises

1. Use SoPlex to obtain the exact optimal value of

$$\begin{array}{lrcrcrcl} \min & 3x & + & 2y & + & 9z & & \\ \text{s.t.} & 53x & + & 20y & + & 96z & \geq & 2 \\ & 13x & - & 7y & + & 6z & \geq & \frac{3}{17} \\ & -x & + & 71y & - & 3z & \geq & \frac{7}{3} \\ & x & , & y & , & z & \geq & 0. \end{array}$$

Solutions

1. The optimal value is $\frac{1882}{11679}$.

Chapter 11

Integer Linear Programming

Recall the problem on lemonade and lemon juice from Chapter 1:

Problem. Say you are a vendor of lemonade and lemon juice. Each unit of lemonade requires 1 lemon and 2 litres of water. Each unit of lemon juice requires 3 lemons and 1 litre of water. Each unit of lemonade gives a profit of $\$3$. Each unit of lemon juice gives a profit of $\$2$. You have 6 lemons and 4 litres of water available. How many units of lemonade and lemon juice should you make to maximize profit?

Letting x denote the number of units of lemonade to be made and letting y denote the number of units of lemon juice to be made, the problem could be formulated as the following linear programming problem:

$$\begin{array}{rrcrcl} \max & 3x & + & 2y & & \\ \text{s.t.} & x & + & 3y & \leq & 6 \\ & 2x & + & y & \leq & 4 \\ & x & & & \geq & 0 \\ & & & y & \geq & 0. \end{array}$$

The problem has a unique optimal solution at $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1.2 \\ 1.6 \end{bmatrix}$ for a profit of 6.8. But this solution requires us to make fractional units of lemonade and lemon juice. What if we require the number of units to be integers? In other words, we want to solve

$$\begin{array}{rrcrcl} \max & 3x & + & 2y & & \\ \text{s.t.} & x & + & 3y & \leq & 6 \\ & 2x & + & y & \leq & 4 \\ & x & & & \geq & 0 \\ & & & y & \geq & 0 \\ & x & , & y & \in & \mathbb{Z}. \end{array}$$

This problem is no longer a linear programming problem. But rather, it is an integer linear programming problem.

A **mixed-integer linear programming problem** is a problem of minimizing or maximizing a linear function subject to finitely many linear constraints such that the number of variables are finite and at least one of which is required to take on integer values.

If all the variables are required to take on integer values, the problem is called a **pure integer linear programming problem** or simply an **integer linear programming problem**. Normally, we assume the problem data to be rational numbers to rule out some pathological cases.

Mixed-integer linear programming problems are in general difficult to solve yet they are too important to ignore because they have a wide range of applications (e.g. transportation planning, crew scheduling, circuit design, resource management etc.) Many solution methods for these problems have been devised and some of them first solve the **linear programming relaxation** of the original problem, which is the problem obtained from the original problem by dropping all the integer requirements on the variables.

Example 11.1. Let (MP) denote the following mixed-integer linear programming problem:

$$\begin{array}{rrcrcrcl}
\text{min} & x_1 & & & + & x_3 & & \\
\text{s.t.} & -x_1 & + & x_2 & + & x_3 & \geq & 1 \\
 & -x_1 & - & x_2 & + & 2x_3 & \geq & 0 \\
 & -x_1 & + & 5x_2 & - & x_3 & = & 3 \\
 & x_1 & , & x_2 & , & x_3 & \geq & 0 \\
 & & & & & x_3 & \in & \mathbb{Z}.
\end{array}$$

The linear programming relaxation of (MP) is:

$$\begin{array}{rrcrcrcl}
\text{min} & x_1 & & & + & x_3 & & \\
\text{s.t.} & -x_1 & + & x_2 & + & x_3 & \geq & 1 \\
 & -x_1 & - & x_2 & + & 2x_3 & \geq & 0 \\
 & -x_1 & + & 5x_2 & - & x_3 & = & 3 \\
 & x_1 & , & x_2 & , & x_3 & \geq & 0.
\end{array}$$

Let (P1) denote the linear programming relaxation of (MP). Observe that the optimal value of (P1) is a lower bound for the optimal value of (MP) since the feasible region of (P1) contains all the feasible solutions to (MP), thus making it possible to find a feasible solution to (P1) with objective function value better than the optimal value of (MP). Hence, if an optimal solution to the linear programming relaxation happens to be a feasible solution to the original problem, then it is also an optimal solution to the original problem. Otherwise, there is an integer variable having a nonintegral value v. What we then do is to create two new subproblems as follows: one requiring the variable to be at most the greatest integer less than v, the other requiring the variable to be at least the smallest integer greater than v. This is the basic idea behind the **branch-and-bound method**. We now illustrate these ideas on (MP).

Solving the linear programming relaxation (P1), we find that $\mathbf{x}' = \begin{bmatrix} 0 \\ \frac{2}{3} \\ \frac{1}{3} \end{bmatrix}$ is an optimal solution to (P1). Note that $\mathbf{x}'$ is not a feasible solution to (MP) because x'_3 is not an integer. We now create two subproblems (P2) and (P3) such that (P2) is obtained from (P1) by adding the constraint $x_3 \leq \lfloor x'_3 \rfloor$ and (P3) is obtained from (P1) by adding the constraint $x_3 \geq \lceil x'_3 \rceil$. (For a number a, $\lfloor a \rfloor$ denotes the greatest integer at most a and $\lceil a \rceil$ denotes the smallest integer at least a.) Hence, (P2) is the problem

$$\begin{array}{rrcrcrcl}
\min & x_1 & & & + & x_3 & & \\
\text{s.t.} & -x_1 & + & x_2 & + & x_3 & \geq & 1 \\
& -x_1 & - & x_2 & + & 2x_3 & \geq & 0 \\
& -x_1 & + & 5x_2 & - & x_3 & = & 3 \\
& & & & & x_3 & \leq & 0 \\
& x_1 & , & x_2 & , & x_3 & \geq & 0,
\end{array}$$

and (P3) is the problem

$$\begin{array}{rrcrcrcl}
\min & x_1 & & & + & x_3 & & \\
\text{s.t.} & -x_1 & + & x_2 & + & x_3 & \geq & 1 \\
& -x_1 & - & x_2 & + & 2x_3 & \geq & 0 \\
& -x_1 & + & 5x_2 & - & x_3 & = & 3 \\
& & & & & x_3 & \geq & 1 \\
& x_1 & , & x_2 & , & x_3 & \geq & 0.
\end{array}$$

Note that any feasible solution to (MP) must be a feasible solution to either (P2) or (P3). Using the help of a solver, one sees that (P2) is infeasible. The problem (P3) has an optimal solution at $\mathbf{x}^* = \begin{bmatrix} 0 \\ \frac{4}{5} \\ 1 \end{bmatrix}$, which is also feasible to (MP). Hence, $\mathbf{x}^* = \begin{bmatrix} 0 \\ \frac{4}{5} \\ 1 \end{bmatrix}$ is an optimal solution to (MP).

We now give a description of the method for a general mixed-integer linear programming problem (MIP). Suppose that (MIP) is a minimization problem and has n variables $x_1, \ldots, x_n$. Let $\mathcal{I} \subseteq \{1, \ldots, n\}$ denote the set of indices i such that x_i is required to be an integer in (MIP).

Branch-and-bound method

Input: The problem (MIP).

Steps:

1. Set $\texttt{bestbound} := \infty$, $\mathbf{x}^*_{\text{best}} := \texttt{N/A}$, $\texttt{activeproblems} := \{(LP)\}$ where (LP) denotes the linear programming relaxation of (MIP).

2. If there is no problem in `activeproblems`, then stop; if $\mathbf{x}^*_{\text{best}} \neq \texttt{N/A}$, then $\mathbf{x}^*_{\text{best}}$ is an optimal solution; otherwise, (MIP) has no optimal solution.

3. Select a problem P from `activeproblems` and remove it from `activeproblems`.

4. Solve P.

- If P is unbounded, then stop and conclude that (MIP) does not have an optimal solution.

- If P is infeasible, go to step 2.

- If P has an optimal solution $\mathbf{x}^*$, then let z denote the objective function value of $\mathbf{x}^*$.

5. If $z \geq \texttt{bestbound}$, go to step 2.

6. If x_i^* is not an integer for some $i \in \mathcal{I}$, then create two subproblems P_1 and P_2 such that P_1 is the problem obtained from P by adding the constraint $x_i \leq \lfloor x_i^* \rfloor$ and P_2 is the problem obtained from P by adding the constraint $x_i \geq \lceil x_i^* \rceil$. Add the problems P_1 and P_2 to `activeproblems` and go to step 2.

7. Set $\mathbf{x}^*_{\text{best}} = \mathbf{x}^*$, `bestbound` $= z$ and go to step 2.

Remarks.

- Throughout the algorithm, `activeproblems` is a set of subproblems remained to be solved. Note that for each problem P in `activeproblems`, P is a linear programming problem and that any feasible solution to P satisfying the integrality requirements is a feasible solution to (MIP).
- x^*_{best} is the feasible solution to (MIP) that has the best objective function value found so far and `bestbound` is its objective function value. It is often called an **incumbent**.
- In practice, how a problem from `activeproblems` is selected in step 3 has an impact on the overall performance. However, there is no general rule for selection that guarantees good performance all the time.
- In step 5, the problem P is discarded since it cannot contain any feasible solution to (MIP) having a better objective function value than x^*_{best}.
- If step 7 is reached, then x^* is a feasible solution to (MIP) having objective function value better than `bestbound`. So it becomes the current best solution.
- It is possible for the algorithm to never terminate. Below is an example for which the algorithm will never stop:
$$\begin{array}{rrcrcrcl} \min & x_1 & & & & & & \\ \text{s.t.} & x_1 & + & 2x_2 & - & 2x_3 & = & 1 \\ & x_1 & , & x_2 & , & x_3 & \geq & 0 \\ & x_1 & , & x_2 & , & x_3 & \in & \mathbb{Z}. \end{array}$$
However, it is easy to see that $\mathbf{x}^* = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is an optimal solution because there is no feasible solution with $x_1 = 0$.

One way to keep track of the progress of the computations is to set up a progress chart with the following headings:

Iter	**solved**	**status**	**branching**	`activeproblems`	$\mathbf{x}^*_{\text{best}}$	`bestbound`

In a given iteration, the entry in the **solved** column denotes the subproblem that has been solved with the result in the **status** column. The **branching** column indicates the subproblems created from the solved subproblem with an optimal solution not feasible to (MIP). The entries in the remaining columns contain the latest information in the given iteration. For the example (MP) above, the chart could look like the following:

Iter	**solved**	**status**	**branching**	`activeproblems`	$\mathbf{x}^*_{\text{best}}$	`bestbound`
1	(P1)	optimal $\mathbf{x}^* = \begin{bmatrix} 0 \\ \frac{2}{3} \\ \frac{1}{3} \end{bmatrix}$	(P2): $x_3 \leq 0$, (P3): $x_3 \geq 1$	(P2), (P3)	N/A	∞

Iter	solved	status	branching	`activeproblems`	$\mathbf{x}^*_{\text{best}}$	`bestbound`
2	(P2)	infeasible	—	(P3)	N/A	∞
3	(P3)	optimal $\mathbf{x}^* = \begin{bmatrix} 0 \\ \frac{4}{5} \\ 1 \end{bmatrix}$	—	—	$\begin{bmatrix} 0 \\ \frac{4}{5} \\ 1 \end{bmatrix}$	1

Exercises

1. Suppose that (MP) in Example 11.1 above has x_2 required to be an integer as well. Continue with the computations and determine an optimal solution to the modified problem.

2. With the help of a solver, determine the optimal value of

$$\begin{array}{rrcrcl} \max & 3x & + & 2y & & \\ \text{s.t.} & x & + & 3y & \leq & 6 \\ & 2x & + & y & \leq & 4 \\ & x & , & y & \geq & 0 \\ & x & , & y & \in & \mathbb{Z}. \end{array}$$

3. Let $\mathbf{A} \in \mathbb{Q}^{m \times n}$ and $\mathbf{b} \in \mathbb{Q}^m$. Let S denote the system

$$\begin{aligned} \mathbf{A}\mathbf{x} &\geq \mathbf{b} \\ \mathbf{x} &\in \mathbb{Z}^n \end{aligned}$$

   a. Suppose that $\mathbf{d} \in \mathbb{Q}^m$ satisfies $\mathbf{d} \geq \mathbf{0}$ and $\mathbf{d}^\mathsf{T}\mathbf{A} \in \mathbb{Z}^n$. Prove that every $\mathbf{x}$ satisfying S also satisfies $\mathbf{d}^T\mathbf{A}\mathbf{x} \geq \lceil \mathbf{d}^\mathsf{T}\mathbf{b} \rceil$. (This inequality is known as a **Chvátal-Gomory cutting plane.**)

   b. Suppose that $\mathbf{A} = \begin{bmatrix} 2 & 3 \\ 5 & 3 \\ 7 & 6 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 2 \\ 1 \\ 8 \end{bmatrix}$. Show that every $\mathbf{x}$ satisfying S also satisfies $x_1 + x_2 \geq 2$.

Solutions

1. An optimal solution to the modified problem is given by $x^* = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

2. An optimal solution is $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$. Thus, the optimal value is 6.

3. a. Since $\mathbf{d} \geq \mathbf{0}$ and $\mathbf{A}\mathbf{x} \geq \mathbf{b}$, we have $\mathbf{d}^T\mathbf{A}\mathbf{x} \geq \mathbf{d}^\mathsf{T}\mathbf{b}$. If $\mathbf{d}^\mathsf{T}\mathbf{b}$ is an integer, the result follows immediately. Otherwise, note that $\mathbf{d}^\mathsf{T}\mathbf{A} \in \mathbb{Z}^n$ and $\mathbf{x} \in \mathbb{Z}^n$ imply that $\mathbf{d}^T\mathbf{A}\mathbf{x}$ is an integer. Thus, $\mathbf{d}^T\mathbf{A}\mathbf{x}$ must be greater than or equal to the least integer greater than $\mathbf{d}^\mathsf{T}\mathbf{b}$.

   b. Take $\mathbf{d} = \begin{bmatrix} \frac{1}{9} \\ 0 \\ \frac{1}{9} \end{bmatrix}$ and apply the result in the previous part.