Hull White properties

Start with the Hull-White model writted as:

$$dr_t = (\theta_t - ar_t)dt + \sigma dW_t$$

Now define:

$$G := e^{at} r_t$$

A quick refresher on Ito's lema:

Assume we are given a stochastic differential equation

$$dX_t = a(t)dt + b(t)dW_t$$

Where W_t is a Wiener process and functions a and b are deterministic functions of time.

Now we define a new function $G := f(t, X_t)$

we can calculate the derivative as:

$$dG = \left(\frac{\partial G}{\partial X_t} a(t) + \frac{\partial G}{\partial t} + \frac{1}{2} \frac{\partial G^2}{\partial X_t^2} b(t) \right) dt + \frac{\partial G}{\partial X_t} b(t) dW_t$$

Comming back to our Hull White model, we define:

$$G := e^{at} r_t$$

and we already define the function $dr_t = (\theta_t - ar_t)dt + \sigma dW_t$. We can see that

$$a(t) = (\theta_t - ar_t)$$

$$b(t) = \sigma$$

therefore noting that r_t is our function X:

$$dG = \left(\frac{\partial G}{\partial r_t} a(t) + \frac{\partial G}{\partial t} + \frac{1}{2} \frac{\partial G^2}{\partial r_t^2} b(t) \right) dt + \frac{\partial G}{\partial r_t} b(t) dW_t$$

The partial derivatives of G are

$$\frac{\partial G}{\partial t} = ae^{at} r_t$$

$$\frac{\partial G}{\partial r_t} = e^{at}$$

$$\frac{\partial G}{\partial r_t} = 0$$

Now applying the Ito's lema to the function G brings:

$$dG = \left(e^{at}(\theta_t - ar_t) + ae^{at}r_t + \frac{1}{2}0\right)dt + e^{at}\sigma dW_t =$$

$$= \theta_t e^{at}dt + \sigma e^{at}dW_t$$

To summarize:

$$dG = \theta_t e^{at}dt + \sigma e^{at}dW_t$$

In the next step, we will calculate the integral $\int_0^t dG$ in two different ways, which will lead to the formula for the interest rate r_t

First way of integrating

$$\int_0^t dG = \int_0^t d(e^{au}r_u) = \left[e^{au}r_u\right]_0^t = e^{at}r_t - e^{a0}r_0 = e^{at}r_t - r_0$$

Second way of integrating

$$\int_0^t dG = \int_0^t \theta_u e^{au}du + \int_0^t \sigma e^{au}dW_u = \int_0^t \theta_u e^{au}du + \sigma\int_0^t e^{au}dW_u$$

Since both ways represent the solution of the same integral, we can assume they are equal. Therefore:

$$e^{at}r_t - r_0 = \int_0^t \theta_u e^{au}du + \sigma\int_0^t e^{au}dW_u$$

$$e^{at}r_t = r_0 + \int_0^t \theta_u e^{au}du + \sigma\int_0^t e^{au}dW_u$$

The final formula is the following:

$$r_t = e^{-at}r_0 + \int_0^t \theta_u e^{a(u-t)}du + \sigma\int_0^t e^{a(u-t)}dW_u$$

Expected value

$$E\left[r_t|F_0\right] = E\left[e^{-at}r_0 + \int_0^t \theta_u e^{a(u-t)}du + \sigma\int_0^t e^{a(u-t)}dW_u\right] = e^{-at}r_0 + \int_0^t \theta_u e^{a(u-t)}du$$

Where we used the fact that the expectation of a driftless Wiener process is equal to 0 and the fact that the first two terms are deterministic functions of time therefore the expectation operator can be left out.

Variance

$$V\Big[r_t|F_0\Big] = E\Big[\big(r_t - E[r_t]\big)^2|F_0\Big] =$$

$$= E\Bigg[\bigg(e^{-at}r_0 + \int_0^t \theta_u e^{a(u-t)}du + \sigma\int_0^t e^{a(u-t)}dW_u - e^{-at}r_0 - \int_0^t \theta_u e^{a(u-t)}du\bigg)^2|F_0\Bigg] =$$

$$= E\Bigg[\bigg(\sigma\int_0^t e^{a(u-t)}dW_u\bigg)^2|F_0\Bigg] = E\Bigg[\sigma^2\bigg(\int_0^t e^{a(u-t)}dW_u\bigg)^2|F_0\Bigg]$$

Applying the Ito isometry to the last expression we obtain

$$E\Bigg[\sigma^2\bigg(\int_0^t e^{a(u-t)}dW_u\bigg)^2|F_0\Bigg] = E\Bigg[\sigma^2\int_0^t e^{2a(u-t)}du|F_0\Bigg] = E\Bigg[\sigma^2 e^{-2at}\int_0^t e^{2au}du|F_0\Bigg] = *$$

Using the integration formula to avoid some lines of equations:

$$\int e^{kx} = \frac{1}{k}e^{kx} + C$$

We can calculate separately

$$\int_0^t e^{2au}du = \bigg[\frac{1}{2a}e^{2au}\bigg]_0^t = \frac{e^{2at}-1}{2a}$$

Continuing from the "*"

$$* = E\Bigg[\sigma^2 e^{-2at}\frac{e^{2at}-1}{2a}|F_0\Bigg] = E\Bigg[\sigma^2\frac{1-e^{-2at}}{2a}|F_0\Bigg] = \frac{\sigma^2}{2a}(1-e^{-2at})$$

To summarize:

$$V\Big[r_t|F_0\Big] = \frac{\sigma^2}{2a}(1-e^{-2at})$$

Expectation if θ is a piecewise constant function

Assume that θ is a piecewise constant parameter that changes values at times $\{0 = t_0 < t_1 < \cdots < t_n = t\}$.

The parameter holds the values $\{\theta_0, \theta_1, \ldots, \theta_n\}$.

The expectation of the Hull-White model would require us to calculate the integral

$$E\Big[r_t|F_0\Big] = e^{-at}r_0 + \int_0^t \theta_u e^{a(u-t)}du$$

Using the fact that θ is a piecewise constant function, this expresion can be further simplified:

$$e^{-at}r_0 + \int_0^t \theta_u e^{a(u-t)}du = e^{-at}r_0 + \sum_{i=1}^{n}\int_{t_i}^{t_{i-1}} \theta_i e^{a(u-t)}du =$$

$$= e^{-at} r_0 + \sum_{i=1}^{n} \theta_i \int_{t_i}^{t_{i-1}} e^{au} e^{-at} du = e^{-at} r_0 + e^{-at} \sum_{i=1}^{n} \theta_i \int_{t_i}^{t_{i-1}} e^{au} du =$$

$$= e^{-at} r_0 + e^{-at} \sum_{i=1}^{n} \theta_i \left[\frac{1}{a} e^{au} \right]_{t_{i-1}}^{t_i} = e^{-at} r_0 + \frac{e^{-at}}{a} \sum_{i=1}^{n} \theta_i (e^{at_i} - e^{at_{i-1}})$$

In [ ]: