

HULL-WHITE AND EIOPA RISK-FREE CURVE TERM STRUCTURE COVERAGE AND VARIANCE CHECK

The Hull-White model is a very popular choice when modeling interest rates. The simulated risk-free curve is one of the principal inputs into such economic scenario generator. This notebook has 4 sections. The first section uses the EIOPA RFR calibration to produce the yield curve. In the second section, this yield curve is used to produce a number of stochastic scenarios using a HW model. The third section checks how close is the average interest rate from ESG to the term structure provided as input. The final section calculates the closed form variance and compares it to the simulated variance. There are two forms used in the literature. The Hull-White model used in this script describes the short rate using the following differential equation:

$$dr(t) = (\theta(t) - ar(t))dt + \sigma dW(t)$$

Where:

- t is time.
- r(t) is the short rate at time t.
- θ is the time-dependent parameter theta.
- a is the mean reversion speed parameter of the Hull-White model.
- σ is the volatility parameter of the Hull-White mode.
- $W(t)$ is the Brownian motion.

Summary

The goal of this script is to implement two checks:

- Verifies if the company is using a sufficiently large number of stochastic scenarios to adequately cover the term structure.
- Verifies if the parameters used in the simulation are correct by comparing the closed-form calculated volatility with the volatility of the actual simulation.

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Note on Smith & Wilson algorithm

The calculation of RFR rate was first performed in another script that checks the correct calibration of the EIOPA's RFR output. This example uses a modified Smith&Wilson implementation (The original implementation is available on [GitHub](#):

- [Python](#)
- [Matlab](#)
- [JavaScript](#)

Note on EIOPA RFR curve

The recalculation of the RFR curve from EIOPA's parameters is available on:

- [Jupyter notebook](#)

Limitations of the implementation

This script covers a synthetic dataset of ESG scenarios. It also simplifies the day-count convention with the assumption that each month represents 1/12-th of a year. The discretisation and numeric integration uses the simplest Euler scheme and the rectangular rule respectively.

Data requirements

This script contains the EIOPA risk-free rate publication for March 2023. The publication can be found on the [EIOPA RFR website](#).

The observed maturities `m_obs` and the calibrated vector `Qb` can be found in the Excel sheet `EIOPA_RFR_20230331_Qb_SW.xlsx`.

For this example, the curve without the volatility adjustment (VA) is used. It can be found in the sheet `SW_Qb_no_VA`. This example is focused on the EUR curve, but this example can be easily modified for any other curve.

The target maturities (`t_obs`), the additional parameters (`UFR` and `alpha`), and the given

curve can be found in the Excel *EIOPA_RFR_20230331_Term_Structures.xlsx*. Sheet *RFR_spot_no_VA* if the test looks at the curve without the Volatility Adjustment and the sheet *RFR_spot_with_VA* if it includes the Volatility Adjustment.

Some example parameters for the Hull-White model are used:

In [1]:

```
NoOfPaths = 20000 # Number of stochastic scenarios
NoOfSteps = 250 # Number of equidistant discrete modelling points ( 50*12 = 600)
T = 50.0          # Time horizon in years
a = 0.02          # Hull-White mean reversion parameter a
sigma = 0.02       # Hull-White volatility parameter sigma
epsilon = 0.01     # Incremental distance used to calculate for numerical approximation
# of for example the instantaneous spot rate
```

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External dependencies

In [2]:

```
import numpy as np
import pandas as pd
import matplotlib.pyplot as plt
```

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Importing data

In [3]:

```
selected_param_file = 'Param_no_VA.csv'
selected_curves_file = 'Curves_no_VA.csv'
```

In [4]:

```
param_raw = pd.read_csv(selected_param_file, sep=',', index_col=0)
```

The country selected is:

In [5]:

```
country = "Slovenia"
```

In [6]:

```
maturities_country_raw = param_raw.loc[:,country+"_Maturities"].iloc[6:]
param_country_raw = param_raw.loc[:,country + "_Values"].iloc[6:]
extra_param = param_raw.loc[:,country + "_Values"].iloc[:6]
```

```
In [7]: relevant_positions = pd.notna(maturities_country_raw.values)

In [8]: maturities_country = maturities_country_raw.iloc[relevant_positions]

In [9]: Qb = param_country_raw.iloc[relevant_positions]

In [10]: curve_raw = pd.read_csv(selected_curves_file, sep=',', index_col=0)

In [11]: curve_country = curve_raw.loc[:,country]
```

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Calibration parameters and calibration vector provided by EIOPA

```
In [12]: # Maturity of observations:
m_obs = np.transpose(np.array(maturities_country.values))

# Ultimate forward rate ufr represents the rate to which the rate curve will
# converge as time increases:
ufr = extra_param.iloc[3]/100

# Convergence speed parameter alpha controls the speed at which the curve
# converges towards the ufr from the last liquid point:
alpha = extra_param.iloc[4]

# For which maturities do we want the SW algorithm to calculate the rates.
# In this case, for every year up to 150:
M_Target = np.transpose(np.arange(1,151))

# Qb calibration vector published by EIOPA for the curve calibration:
Qb = np.transpose(np.array(Qb.values))
```

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Smith & Wilson calculation functions

In this step, an independent version of the Smith&Wilson algorithm is implemented. To do this, two functions are taken from the publicly available repository and modified to accept the

product of Q^*b instead of the calibration vector b .

In [13]:

```

def sw_extrapolate(m_target, m_obs, Qb, ufr, alpha, epsilon = 0.00001):
    """
    Interpolate or extrapolate rates for targeted maturities using a
    Smith-Wilson algorithm.
    sw_extrapolate(m_target, m_obs, Qb, ufr, alpha) calculates the rates for
    maturities specified in M_Target using the calibration vector b.
    Args:
        m_target (ndarray): k x 1 array of targeted bond maturities.
        m_obs (ndarray): n x 1 array of observed bond maturities.
        Qb (ndarray): n x 1 array. Calibration vector calculated on bonds.
        ufr (float): Ultimate forward rate.
        alpha (float): Convergence speed parameter.
        epsilon (float): Increment to calculate the instantaneous spot rate.

    Returns:
        ndarray: k x 1 array of targeted rates for zero-coupon bonds with
        maturity from m_target.

    For more information see
    https://www.eiopa.europa.eu/sites/default/files/risk_free_interest_rate
    /12092019-technical_documentation.pdf
    """

def sw_heart(u, v, alpha):
    """
    Calculate the heart of the Wilson function. sw_heart(u, v, alpha)
    calculates the matrix H (Heart of the Wilson function) for maturities
    specified by vectors u and v. The formula is taken from the EIOPA technical
    specifications paragraph 132.

    Args:
        u (ndarray): n_1 x 1 vector of maturities.
        v (ndarray): n_2 x 1 vector of maturities.
        alpha (float): Convergence speed parameter.

    Returns:
        ndarray: n_1 x n_2 matrix representing the Heart of the Wilson function.
    """
    u_mat = np.tile(u, [v.size, 1]).transpose()
    v_mat = np.tile(v, [u.size, 1])
    return 0.5 * (alpha * (u_mat + v_mat) + np.exp(-alpha * (u_mat + v_mat)))
        - alpha * np.absolute(u_mat - v_mat)
        - np.exp(-alpha * np.absolute(u_mat - v_mat)))

# Heart of the Wilson function from paragraph 132
h = sw_heart(m_target, m_obs, alpha)

# Discount pricing function for targeted maturities from paragraph 147
p = np.exp(-np.log(1 + ufr) * m_target) + np.diag(np.exp(-np.log(1 + ufr)
    * m_target)) @ h @ Qb

# If the first element of m_target is zero, replace it with time "epsilon"
# to avoid division by zero error.
m_target[0] = epsilon if m_target[0] == 0 else m_target[0]

return p ** (-1 / m_target) - 1

```

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Hull-White interest rate simulation functions

In this step, the functions that calculate the necessary Hull-White outputs are defined. The calculation of a discounted ZCB bond, calculation of a forward rate, the parameter theta and finally the generation of multiple sample paths.

The function P0t_f calculates the price of a zero-coupon bond (ZCB) based on the given yield and maturity.

$$P0t = e^{-y0t*t}$$

Where:

- t is the time of interest expressed as a year fraction (Ex. for 18 months, $t = 1.5$).
- $y0t$ is the yield obtained by applying the Smith-Wilson calibration to the term structure for a single payoff at time t .
- $P0t$ is the price of a zero coupon bond with a notional amount of 1 and maturity at time t .

In [14]:

```
def P0t_f(t, m_obs, Qb, ufr, alpha):
    """
    P0t_f function calculates the price of a zero-coupon bond issued at time 0,
    for a given maturity 't', using the Smith-Wilson extrapolation technique.

    Args:
        t (float or ndarray): vector (or a single number) of maturities represented
            as time fraction (Ex. for 18 months; t=1.5).
        m_obs (ndarray): n x 1 array of observed bond maturities used for
            calibration.
        Qb (ndarray): n x 1 calibration vector of the Smith-Wilson algorithm
            calculated on observed bonds.
        ufr (float): Ultimate forward rate parameter for the Smith-Wilson algorithm.
        alpha (float): Convergence speed parameter for the Smith-Wilson algorithm.

    Returns:
        ndarray: n x 1 the price of zero-coupon bonds with a notional amount of 1
            and maturity t.

    Example of use
    m_obs = np.array([1, 2, 3, 5, 7, 10, 15, 20, 30])
    Qb = np.array([0.02474805, 0.02763133, 0.02926931, 0.0302894, 0.03061605,
                  0.03068016, 0.03038397, 0.02999401, 0.02926168])
    ufr = 0.042
    alpha = 0.05

    # For a single maturity
    t = 5
    price = P0t_f(t, m_obs, Qb, ufr, alpha)
    print(f"Price of zero-coupon bond with maturity {t} years is: {price}")

    # For multiple maturities
    t = [1, 3, 5, 10]
    prices = P0t_f(t, m_obs, Qb, ufr, alpha)
    print("Prices of zero-coupon bonds with maturities", t, "years are:")
    print(prices)

    Implemented by Gregor Fabjan from Open-Source Modelling on 29/07/2023
    """

    if isinstance(t, np.ndarray): # If the input is a numpy array
        y0t = sw_extrapolate(np.transpose(t), m_obs, Qb, ufr, alpha)
        out = np.exp(-y0t*np.transpose(t))
    else:# If the input is a single maturity given as a number
        y0t = sw_extrapolate(np.transpose([t]), m_obs, Qb, ufr, alpha)
        out = np.exp(-y0t*[t])
    return out
```

The function f0t calculates the instantaneous forward rate at time t using a numerical approximation with a step size given by the parameter ϵ . The calculation of the instantaneous forward rate relies on the difference between two zero-coupon bond prices. One maturing at time $t + \epsilon$ and another at time $t - \epsilon$.

The centered finite difference method estimates the instantaneous forward rate using the following approximation:

$$f(0, t) \approx -\frac{\log(P(0, t + \epsilon)) - \log(P(0, t - \epsilon))}{2\epsilon}$$

In [15]:

```
def f0t(t, P0t, epsilon):
    """
    Calculates the instantaneous forward rate for time t given the zero-coupon
    bond price function P0t, using the centered finite difference method.

    Args:
        t (float): Time at which the instantaneous forward rate is calculated.
        P0t (function): Function that takes a float argument `t` and
                         returns the price of a zero-coupon bond with maturity `t` and
                         notional amount 1.
        epsilon (float): Step size for the centered finite difference method.

    Returns:
        float: The instantaneous forward rate at time t, calculated using the
               centered finite difference method.
    """
    p_plus = P0t(t + epsilon)
    p_minus = P0t(t - epsilon)
    return -(np.log(p_plus) - np.log(p_minus)) / (2 * epsilon)
```

The function HW_theta calculates the time depened θ . In this implementation, this parameter is calibrated to the term stucture obtained using the EIOPA RFR. The term structure is specified by the P0t function that is passed as argument.

The parameter `theta` is calibrated using the following relation:

$$\theta(t) = \frac{\partial f(0, t)}{\partial t} + af(0, t) + \frac{\sigma^2}{2a}(1 - e^{-2at})$$

Where:

- t is the time at which we wish to calibrate θ to the term structure. (Ex. 18 months means $t = 1.5$).
- a is the a parameter in the Hull-White model.
- σ is the volatility parameter σ of the Hull-White model.
- $f(0, t)$ is the instantaneous forward rate at time t estimated at time 0.
- $\theta(t)$ is the time dependet parameter θ of the Hull-White model.

In [16]:

```
def HW_theta(a, sigma, P0t, eps):
    """
    Calculates the theta value for the Hull-White model
    using a numeric approximation of the instantaneous forward rate
    and the spot rate.

    Args:
        a (float): Mean reversion rate parameter a.
        sigma (float): Volatility parameter sigma.
        P0t (function handle): Function that calculates the price of a
            zero-coupon bond as a function of time.
        eps (float): Increment of time used in the numeric calculation of the
            derivative of the instantaneous forward rate.

    Returns:
        theta (function): Function that returns the parameter theta of
            Hull-White model at the time t implied by the calibration.
    """

    def theta(t):
        insta_forward_term = (f0t(t+eps, P0t, eps)
                              - f0t(t-eps, P0t, eps))/(2.0*eps)

        forward_term = a*f0t(t, P0t, eps)
        variance_term = sigma**2/(2.0*a)*(1.0-np.exp(-2.0*a*t))
        return insta_forward_term + forward_term + variance_term
    return theta
```

In [17]:

```
def Paths(NoOfPaths, NoOfSteps, T, P0t, a, sigma, epsilon):
    """
    Simulates a series of stochastic interest rate paths using the Hull-White model.

    Args:
        NoOfPaths (int): number of paths to simulate.
        NoOfSteps (int): number of time steps per path.
        T (float): end of the modelling window (in years).
            (Ex. a modelling window of 50 years means T=50).
        P0t (function): function that calculates the price of a
            zero coupon bond issued at time 0 that matures at time t, with a
            notional amount 1 and discounted using the assumed term structure.
        a (float): mean reversion speed parameter a of
            the Hull-White model.
        sigma (float): volatility parameter sigma of the Hull-White model.
        epsilon (float): size of the increment used for finite
            difference approximation.
```

Returns:

```
dict: A dictionary containing arrays with time steps, interest rate paths,
and bond prices.
time (array): array of time steps.
R (array): array of interest rate paths with
shape (NoOfPaths, NoOfSteps+1).
M (array): array of bond prices with
shape (NoOfPaths, NoOfSteps+1).
```

Implemented by Gregor Fabjan from Open-Source Modelling on 29/07/2023.

Original inspiration: <https://www.youtube.com/watch?v=BIZdwUDbnDo>

```
# Initial instantaneous forward rate at time t-> 0 (also spot rate at time 0).
# r(0) = f(0,0) = - partial derivative of log(P_mkt(0, epsilon)) w.r.t epsilon)
r0 = f0t(epsilon, P0t, epsilon)

# Calculation of theta = 1/a * partial derivative of f(0,t) w.r.t. t
# + f(0,t) + sigma^2/(2 a^2)* (1-exp(-2*a*t)).
theta = HW_theta(a, sigma, P0t, epsilon)

# Generate the single source of random noise.
Z = np.random.normal(0.0, 1.0, [NoOfPaths, NoOfSteps])

# Initialize arrays

# Vector of time moments. Difference between two time moments is equal
# to the time increment.
time = np.linspace(0, T, NoOfSteps+1)

W = np.zeros([NoOfPaths, NoOfSteps+1])

# Initialize vector with interest rate increments
R = np.zeros([NoOfPaths, NoOfSteps+1])

# First interest rate equals the instantaneous forward (spot)
# rate at time 0.
R[:, 0] = r0
dt = T/float(NoOfSteps) # Size of increments between two steps
```

```
for iTime in range(1, NoOfSteps+1): # For each time increment
    # Making sure the samples from the normal distribution have a mean of 0
    # and variance 1
    if NoOfPaths > 1:
        Z[:,iTime-1] = (Z[:,iTime-1]-np.mean(Z[:,iTime-1]))/np.std(Z[:,iTime-1])

    # Apply the Euler-Maruyama discretisation scheme for the Hull-White model
    # at each time increment.
    W[:,iTime] = W[:,iTime-1] + np.power(dt,0.5)*Z[:, iTime-1]
    noise_term = sigma* (W[:,iTime]-W[:,iTime-1])
    rate_term = (theta(time[iTime-1])-a*R[:,iTime-1])*dt
    R[:,iTime] = R[:,iTime-1] + rate_term + noise_term

    # Vectorized numeric integration using the Euler integration method .
    M = np.exp(0.5 * (R[:, :-1] + R[:, 1:])) * dt
    M = np.insert(M, 0, 1, axis=1).cumprod(axis=1)

    # Output is a dataframe with time moment, the interest rate path and the price
    # of a zero coupon bond issued at time 0 that matures at the selected time
    # moment with a notional value of 1.
    paths = {"time":time, "R":R, "M":M}
return paths
```

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Comparison between average ESG output and the assumed term structure

In this final section, the ESG output is generated and visually compared to the assumed term structure.

In [18]:

```
def mainCalculation(NoOfPaths, NoOfSteps, T, a, sigma, P0t, epsilon):
    """
    Calculates and plots the prices of zero-coupon bonds (ZCB) calculated
    using the Hull-White model's analytical formula and the Monte Carlo simulation.

    Args:
        NoOfPaths (int): number of Monte Carlo simulation paths.
        NoOfSteps (int): number of time steps per path.
        T (float): length in years of the modelling window (Ex. 50 years means t=50)
        a (float): mean reversion rate parameter a of the Hull-White model.
        sigma (float): volatility parameter sigma of the Hull-White model.
        P0t (function): function that calculates the price of a zero coupon bond issued
            at time 0 that matures at time t, with a notional amount 1 and discounted
            the assumed term structure.
        epsilon (float): the size of the increment used for finite difference approximations.

    Returns:
        Nothing.
    """

paths = Paths(NoOfPaths, NoOfSteps, T, P0t, a, sigma, epsilon)
M = paths["M"]
t = paths["time"]

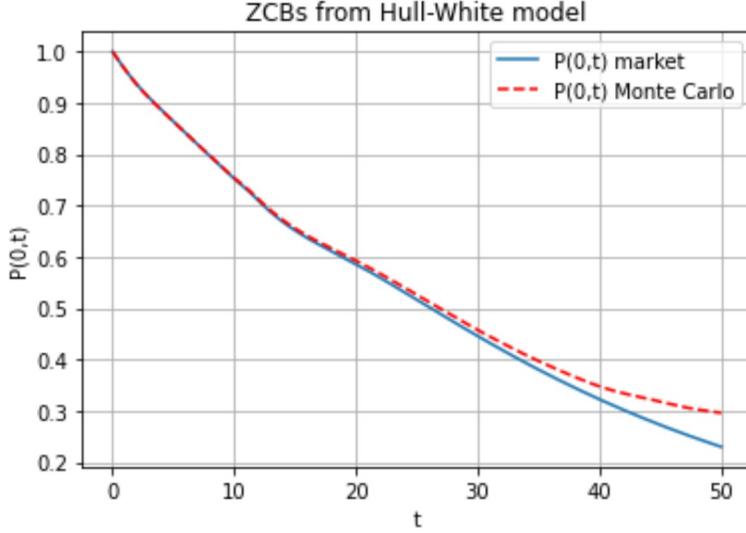
# Compare the price of an option on a ZCB from Monte Carlo and the analytical expression
P = np.zeros([NoOfSteps+1])
for i in range(0, NoOfSteps+1):
    P[i] = np.mean(1.0/M[:, i])
plt.figure(1)
plt.grid()
plt.xlabel("t")
plt.ylabel("P(0,t)")
plt.plot(t,P0t(t))
plt.plot(t,P,"--r")
plt.legend(["P(0,t) market", "P(0,t) Monte Carlo"])
plt.title("ZCBs from Hull-White model")
```

In [19]:

```
# Zero coupon bond prices calculated using the assumed term structure
P0t = lambda t: P0t_f(t, m_obs, Qb, ufr, alpha)
```

In [20]:

```
# Final comparison
mainCalculation(NoOfPaths, NoOfSteps, T, a, sigma, P0t, epsilon)
```



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Comparison between simulated an calculated volatility

The Hull-White model is one of the few widely used short rate models that has a closed-form formula for the calculation of the volatility. The formula is:

$$V[r(t) \mid F_0] = \frac{\sigma^2}{2a} (1 - e^{-2at})$$

This fact makes it possible to compare the volatility calculated from the parameters with the simulated volatility.

```
In [21]: paths = Paths(NoOfPaths, NoOfSteps, T, P0t, a, sigma, epsilon)
R = paths["R"]
t = paths["time"]
```

```
In [22]: time = np.linspace(start=0, stop=T, num=NoOfSteps+1, endpoint=True)
```

```
In [23]: var_series = np.var(R, axis=0)
```

```
In [24]: vol = sigma**2/(2*a)*(1-np.exp(-2*a*time))
```

```
In [25]: plt.figure(1)
plt.grid()
plt.xlabel("t")
plt.ylabel("V(0,t)")
plt.plot(t,var_series)
plt.plot(t,vol,"--r")
plt.legend(["V(0,t) simulated", "V(0,T) closed formula"])
plt.title("Variance function from Hull-White model")
```

Out[25]: Text(0.5, 1.0, 'Variance function from Hull-White model')

