High-efficiency DMD operation with DLP TRP pixels

Jessica M. Ullom and Peter T. Brown

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1 Punchlines

- Improved solution for working with DMD mirrors rotated by arbitrary rotation matrix R
- With the DLP TRP pixels, it is not possible to *exactly* satisfy the blaze and DMD diffraction conditions
- The blaze and DMD diffractions can be *nearly* satisfied (to within a few tenths of a degree) using the $(n_x, 0)$ or $(0, n_y)$ diffraction orders for the "on" and "off" states respectively

2 Joint solution of the blaze and diffraction conditions

Suppose we have a DMD with a square pixel array. Define a coordinate system by unit vectors $\hat{\mathbf{e}}_x$ and $\hat{\mathbf{e}}_y$ along the principle directions of the pixel array and $\hat{\mathbf{e}}_z$ normal to the body of the DMD. Let the micromirrors rotate about an arbitrary axis $\hat{\mathbf{n}}$ through angle γ. Let $R(\hat{\mathbf{n}}, \gamma)$ be the matrix describing this rotation. This matrix has components R_{ij}. The only assumption we will make below is that the micromirrors are arranged in a rectangular matrix.

Consider a plane wave incident on the DMD along unit vector $\hat{\mathbf{a}}$. Let $\hat{\mathbf{b}}$ be the direction of a diffracted plane wave. The unit vectors satisfy

$$a_z = -\sqrt{1 - a_x^2 - a_y^2} \tag{1}$$
$$b_z = \sqrt{1 - b_x^2 - b_y^2}. \tag{2}$$

The incoming and outgoing directions must satisfy the diffraction condition

$$b_x - a_x = n_x \frac{\lambda}{d} \tag{3}$$
$$b_y - a_y = n_y \frac{\lambda}{d}. \tag{4}$$

For high efficiency diffraction, the incoming and outgoing directions should also satisfy the blaze condition, which is equivalent to the law of reflection from one micromirror. To simplify our analysis of this condition, we introduce a new basis,

$$\hat{\mathbf{e}}_1 = R(\hat{\mathbf{n}}, \gamma)\hat{\mathbf{e}}_x \tag{5}$$
$$\hat{\mathbf{e}}_2 = R(\hat{\mathbf{n}}, \gamma)\hat{\mathbf{e}}_y \tag{6}$$
$$\hat{\mathbf{e}}_3 = R(\hat{\mathbf{n}}, \gamma)\hat{\mathbf{e}}_z. \tag{7}$$

As usual since R acts on unit vectors, to convert vector $\mathbf{v}$ from the xyz to 123 basis we must use the inverse matrix $\mathbf{v}_{123} = R^{-1}\mathbf{v}_{xyz}$.

In this basis, the blaze condition is

$$b_1 - a_1 = 0 \tag{8}$$
$$b_2 - a_2 = 0 \tag{9}$$
$$b_3 + a_3 = 0. \tag{10}$$

Combining the blaze and diffraction conditions we have six unknowns (the components of $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$) and seven equations (2 diffraction conditions, 3 blaze conditions, and 2 unit vector conditions). The system is overdetermined, and so we will not have any solution in the generic case.

2.1 Exact solutions

However, solving these equations will give us the conditions that R, n_x, n_y must satisfy for solutions to exist. To find these, we rewrite the diffraction condition, eqs. 3 and 4, in the new basis

$$(b_1 - a_1)R_{11} + (b_2 - a_2)R_{12} + (b_3 - a_3)R_{13} = n_x \frac{\lambda}{d} \tag{11}$$

$$(b_1 - a_1)R_{21} + (b_2 - a_2)R_{22} + (b_3 - a_3)R_{23} = n_y \frac{\lambda}{d}. \tag{12}$$

Substituting in the blaze condition relations, we have

$$b_3 R_{13} = n_x \frac{\lambda}{2d} \tag{13}$$

$$b_3 R_{23} = n_y \frac{\lambda}{2d}. \tag{14}$$

These two equations tell use what diffraction orders (n_x, n_y) the blaze condition can be solved for in terms of the the rotation matrix elements. For these equations to have a solution, the ratio R_{13}/R_{23} must be a rational number if both are non-zero.

The rotational matrix elements can be written in terms of the micromirror normal $\hat{\mathbf{e}}_3 = (m_x, m_y, m_z)$ and the micromirror rotation angle γ,

$$R_{13} = m_x m_z (1 - \cos\gamma)\cos\gamma + m_y \sin\gamma \tag{15}$$

$$R_{23} = m_y m_z (1 - \cos\gamma)\cos\gamma - m_x \sin\gamma. \tag{16}$$

2.2 Approximate solutions

We can also take another perspective on this problem. We can regard the diffraction condition equations as constraints and the blaze condition as giving an cost function which we would like to minimize. This approach is more general than our previous attempt — it will provide us an exact solution if one exists, and otherwise it will give us the best compromise solution.

The solution described below is implemented in our simulation code as the function `test()` in simulate_dmd.py.

We choose the cost function

$$F(\hat{\mathbf{b}}, \hat{\mathbf{a}}) = (b_1 - a_1)^2 + (b_2 - a_2)^2 \tag{17}$$

$$= 2 - 2\hat{\mathbf{b}} \cdot \hat{\mathbf{a}} - (b_3 - a_3)^2, \tag{18}$$

which is non-negative and zero if and only if the the blaze condition is satisfied. However, it is not the only function that satisfies those two properties. We could, for example, add $(a_3 + b_3)^2$ to it. but this would add complexity to our system of equations.

We can now solve the problem by the method of Lagrange multipliers, which supplies us with a new set of equations to replace the blaze condition, namely

$$\nabla \left[\alpha \left(b_x - a_x - n_x \frac{\lambda}{d} \right) + \beta \left(b_y - a_y - n_y \frac{\lambda}{d} \right) + F \right] = 0, \tag{19}$$

where the gradient is taken with respect to a_x, a_y, b_x, b_y.

We can evaluate the four components of eq. 19 using the relations

$$b_1 - a_1 = R_{11}(b_x - a_x) + R_{21}(b_y - a_y) + R_{31}(b_z - a_z) \tag{20}$$

$$b_2 - a_2 = R_{12}(b_x - a_x) + R_{22}(b_y - a_y) + R_{32}(b_z - a_z). \tag{21}$$

These are

$$\alpha + 2(b_1 - a_1)\left[R_{11} - R_{31}\frac{b_x}{b_z}\right] + 2(b_2 - a_2)\left[R_{12} - R_{32}\frac{b_x}{b_z}\right] = 0 \quad (22)$$

$$-\alpha + 2(b_1 - a_1)\left[-R_{11} + R_{31}\frac{a_x}{a_z}\right] + 2(b_2 - a_2)\left[-R_{12} + R_{32}\frac{a_x}{a_z}\right] = 0 \quad (23)$$

$$\beta + 2(b_1 - a_1)\left[R_{21} - R_{31}\frac{b_y}{b_z}\right] + 2(b_2 - a_2)\left[R_{22} - R_{32}\frac{b_y}{b_z}\right] = 0 \quad (24)$$

$$-\beta + 2(b_1 - a_1)\left[-R_{21} + R_{31}\frac{a_y}{a_z}\right] + 2(b_2 - a_2)\left[-R_{22} + R_{32}\frac{a_y}{a_z}\right] = 0. \quad (25)$$

Adding eq. 22 and 23 and separately eq. 24 and 25 gives

$$2\left[(b_1 - a_1)R_{31} + (b_2 - a_2)R_{32}\right]\left(\frac{b_x}{b_z} - \frac{a_x}{a_z}\right) = 0 \quad (26)$$

$$2\left[(b_1 - a_1)R_{31} + (b_2 - a_2)R_{32}\right]\left(\frac{b_y}{b_z} - \frac{a_y}{a_z}\right) = 0. \quad (27)$$

This leaves two possibilities, either the term in square brackets is zero or both the terms in round brackets are zero. In the second case, we can easily show that $\hat{\mathbf{b}} = -\hat{\mathbf{a}}$, and the solution directions are fixed by the diffraction condition and do not depend on the rotation matrix R, so that is clearly not the solution we want.

Therefore we must have,

$$(b_1 - a_1)R_{31} + (b_2 - a_2)R_{32} = 0. \quad (28)$$

Substituting eqs. 20 and 21 into eq. 28 we find

$$b_z - a_z = -\frac{\lambda}{d}\frac{n_x(R_{31}R_{11} + R_{32}R_{12}) + n_y(R_{31}R_{21} + R_{32}R_{22})}{R_{31}^2 + R_{32}^2}. \quad (29)$$

This, together with the eqs. 3 and 4 means we know $\hat{\mathbf{b}} - \hat{\mathbf{a}}$. In principle we also know the values of the Lagrange multipliers by reinserting eq. 28 into eqs. 22-25.

We have three parameters left to fix, which are the components of one of the direction unit vectors. But in fact, looking at the form of eq. 18 we see that the cost function is already fixed because the values $b_3 - a_3$ and $\hat{\mathbf{b}} \cdot \hat{\mathbf{a}}$ can be determined from $a^2, b^2, \hat{\mathbf{b}} - \hat{\mathbf{a}}$,

$$\hat{\mathbf{b}} \cdot \hat{\mathbf{a}} = \frac{1}{2}\left[b^2 + a^2 - \left(\hat{\mathbf{b}} - \hat{\mathbf{a}}\right) \cdot \left(\hat{\mathbf{b}} - \hat{\mathbf{a}}\right)\right]. \quad (30)$$

This implies we have full freedom to choose one unit vector component, say a_x, and we will still have a solution to our equations.

Given a value for a_x we can determine all other unit vector components. The only tedious part is to find an equation for a_y starting from the dot product relation

$$b_x a_x + b_y a_y + b_z a_z = \hat{\mathbf{b}} \cdot \hat{\mathbf{a}} \quad (31)$$

$$\left(a_x + n_x\frac{\lambda}{d}\right)a_x + \left(a_y + n_y\frac{\lambda}{d}\right)a_y - \hat{\mathbf{b}} \cdot \hat{\mathbf{a}} = \sqrt{1 - a_x^2 - a_y^2}\sqrt{1 - \left(a_x + n_x\frac{\lambda}{d}\right)^2 - \left(a_y + n_y\frac{\lambda}{d}\right)^2} \quad (32)$$

After squaring both sides and some tedious algebra we find a quadratic equation for a_y

$$a_y^2\left[2(b_x a_x - \hat{\mathbf{b}} \cdot \hat{\mathbf{a}}) + (1 - b_x^2) + (1 - a_x^2)\right] + \quad (33)$$

$$a_y 2 n_y \frac{\lambda}{d}\left[(b_x a_x - \hat{\mathbf{b}} \cdot \hat{\mathbf{a}}) + (1 - a_x^2)\right] + \quad (34)$$

$$\left[\left(b_x a_x - \hat{\mathbf{b}} \cdot \hat{\mathbf{a}}\right)^2 - (1 - a_x^2)(1 - b_x^2) + \left(n_y\frac{\lambda}{d}\right)^2(1 - a_x^2)\right] = 0, \quad (35)$$

which we can easily solve.

Now we have a solution that we can implement on the computer.

2.3 First generation DLP pixels

For the first generation of Texas Instruments (TI) DMD's, including the DLP7000, DLP6500, and DLP9000 we have the rotation matrix parameters

$$\hat{\mathbf{n}}_{\mathrm{on/off}} = \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right) \tag{36}$$

$$\gamma_{\mathrm{on/off}} = \pm 12^\circ \tag{37}$$

$$R = \begin{pmatrix} \frac{1+\cos\gamma}{2} & \frac{1-\cos\gamma}{2} & \frac{\sin\gamma}{\sqrt{2}} \\ \frac{1-\cos\gamma}{2} & \frac{1+\cos\gamma}{2} & -\frac{\sin\gamma}{\sqrt{2}} \\ -\frac{\sin\gamma}{\sqrt{2}} & \frac{\sin\gamma}{\sqrt{2}} & \cos\gamma \end{pmatrix} \tag{38}$$

from which we fined $R_{13} = \sin\gamma/\sqrt{2}$ and $R_{23} = -\sin\gamma/\sqrt{2}$. In particular, this implies that eqs. 13 and 14 only have a solution if $n_x = -n_y$. And in this case $a_3 = n\frac{\lambda}{\sqrt{2}d\sin\gamma}$. This is exactly the solution we found in [1].

Furthermore, from eq. **??** we find

$$b_3 - a_3 = \frac{n_x - n_y}{\sqrt{2}\sin\gamma} \tag{39}$$

which is the value provided by the exact solution.

2.4 DLP TRP pixels

The newer generation of TI DMD's use DLP TRP pixels. The DLP TRP pixel structure uses a mechanism that tilts the mirror first by 12° along on diagonal, then rolls it ±12° along the other diagonal to its "on" and "off" state respectively. The final "on" position of the mirror is tilted approximately 17° along the x-direction, while the "off" position is tilted approximately −17° along the y-direction. Intepreting these two rotations as described in DLPA079, we find

$$R_{\mathrm{on}} = R\left(\frac{\hat{\mathbf{e}}_x + \hat{\mathbf{e}}_y}{\sqrt{2}}, -12^\circ\right) R\left(\frac{\hat{\mathbf{e}}_x - \hat{\mathbf{e}}_y}{\sqrt{2}}, 12^\circ\right) \tag{40}$$

$$= R\left(0.997\hat{\mathbf{e}}_y + 0.074\hat{\mathbf{e}}_z, 16.96^\circ\right) \tag{41}$$

$$R_{\mathrm{off}} = R\left(\frac{\hat{\mathbf{e}}_x + \hat{\mathbf{e}}_y}{\sqrt{2}}, 12^\circ\right) R\left(\frac{\hat{\mathbf{e}}_x - \hat{\mathbf{e}}_y}{\sqrt{2}}, 12^\circ\right) \tag{42}$$

$$= R\left(-0.997\hat{\mathbf{e}}_x - 0.074\hat{\mathbf{e}}_z, 16.96^\circ\right) \tag{43}$$

where we have rewritten the compound rotation as single rotation about a different axis.

In this case, we have

$$R_{13}^{\mathrm{on}} = 0.2908 \tag{44}$$

$$R_{23}^{\mathrm{on}} = 0.0032 \tag{45}$$

$$\frac{R_{13}^{\mathrm{on}}}{R_{23}^{\mathrm{on}}} = 90.523, \tag{46}$$

and from this we see there are no *exact* combined solutions for this type of pixel. Even if the ratio is rational, the large diffraction orders required to satisfy it are not physical.

But it seems clear that the design intent here was to approximate mirrors with normal in the XZ and YZ planes for the on/off states respectively, i.e. $R_{23}^{\mathrm{on}} \approx 0$ and $R_{13}^{\mathrm{off}} \approx 0$.

This suggests we should look at *approximate* solutions for the on states with

$$n_y = 0 \tag{47}$$

$$b_3 = n_x \frac{\lambda}{2dR_{13}^{\mathrm{on}}} \tag{48}$$

or for the off states with

$$n_x = 0 \tag{49}$$

$$b_3 = n_y \frac{\lambda}{2dR_{23}^{\mathrm{off}}} \tag{50}$$

3 DMD diffraction

For the arbitrary rotation matrix R defined in the last section, the analog of eq. 1 from [1] is

$$
\begin{aligned}
E(\hat{\mathbf{b}}) &= \left(\sum_{m_x,m_y} \exp\left[-ikd(m_x,m_y,0)\cdot\left(\hat{\mathbf{b}}-\hat{\mathbf{a}}\right)\right]\right) \\
&\quad \times w^2 \mathrm{sinc}\left(\frac{kw}{2}A_+(R)\right)\mathrm{sinc}\left(\frac{kw}{2}A_-(R)\right) && (51) \\
A_+(R) &= R_{11}\left(b_x-a_x\right)+R_{21}\left(b_y-a_y\right)+R_{31}\left(b_z-a_z\right) \\
&= \hat{\mathbf{e}}_1\cdot\left(\hat{\mathbf{b}}-\hat{\mathbf{a}}\right) && (52) \\
A_-(R) &= R_{12}\left(b_x-a_x\right)+R_{22}\left(b_y-a_y\right)+R_{32}\left(b_z-a_z\right) \\
&= \hat{\mathbf{e}}_2\cdot\left(\hat{\mathbf{b}}-\hat{\mathbf{a}}\right) && (53)
\end{aligned}
$$

where the forward matrix appears because the initial vector $(t,s,0)$ was written in the 123 basis.

References

[1] P. T. Brown, R. Kruithoff, G. J. Seedorf, and D. P. Shepherd. Multicolor structured illumination microscopy and quantitative control of polychromatic light with a digital micromirror device. *Biomedical Optics Express*, 12(6):3700, jun 2021.