Sum Sensitivity Proofs

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Definition 1. *A sum query on database x of size n is defined to be*

$$s(x) = \sum_{i=1}^{n} x_i.$$

1 NEIGHBORING DEFINITION: CHANGE ONE

1.1 ℓ_1-sensitivity

Theorem 1. *Say the space of datapoints $\mathcal{X}$ is bounded above by M and bounded below by m. Then s over $\mathcal{X}^n$ has ℓ_1-sensitivity in the change-one model bounded above by $M - m$.*

Proof. Say X and X' are neighboring databases which differ at data-point x_j. Then

$$
\begin{aligned}
\Delta s &= \max_{X,X'} \left| s(X) - s(X)' \right| \\
&= \max_{X,X'} \left| \left(\sum_{\{i \in [n] | i \neq j\}} x_i \right) + x_j - \left(\sum_{\{i \in [n] | i \neq j\}} x_i' \right) + x_j' \right| \\
&= \max_{X,X'} \left| x_j - x_j' \right| \\
&\leq M - m.
\end{aligned}
$$

$\square$

1.2 ℓ_2-sensitivity

Theorem 2. *Say the space of datapoints $\mathcal{X}$ is bounded above by M and bounded below by m. Then s over $\mathcal{X}^n$ has ℓ_1-sensitivity in the change-one model bounded above by $M - m$.*

Proof. This follows the logic of the above proof. $\square$

2 Neighboring Definition: Add/Drop One

2.1 ℓ_1-sensitivity

Theorem 3. *Say the space of datapoints $\mathcal{X}$ is bounded above by M and bounded below by m. Then s has ℓ_1-sensitivity in the add/drop-one model bounded above by $\max(|m|, |M|)$.*

Proof. For notational ease, let n always refer to the size of database x. We must consider both adding and removing an element from x. First, consider adding a point:

Let $X' = X \cup x$. Without loss of generality, assume the point added is the $(n+1)^{\text{th}}$ element of database x'. Note that

$$
\begin{aligned}
\left| s(X) - s(X') \right| &= \left| \sum_{i=1}^{n} x_i - \sum_{i=1}^{n+1} x_i \right| \\
&= \left| \sum_{i=1}^{n} x_i - \left(\sum_{i=1}^{n} x_i \right) - x \right| \\
&= x \\
&\leq \max(|m|, |M|).
\end{aligned}
$$

Second, consider removing a point:
Let $X' = X \setminus \{x\}$. Without loss of generality assume that the point subtracted is the nth element of database x.

$$
\begin{aligned}
\left| s(X) - s(X') \right| &= \left| \sum_{i=1}^{n} x_i - \sum_{i=1}^{n-1} x_i \right| \\
&= \left| \left(\sum_{i=1}^{n-1} x_i \right) + x - \sum_{i=1}^{n-1} x_i \right| \\
&= x \\
&\leq \max(|m|, |M|).
\end{aligned}
$$

$\square$

2.2 ℓ_2-sensitivity

Theorem 4. *Say the space of datapoints $\mathcal{X}$ is bounded above by M and bounded below by m. Then s has ℓ_2-sensitivity in the add/drop-one model bounded above by $\max(|m|, |M|)$.*

Proof. This follows the same logic as the above proof. $\square$