COMP3121: Assignment 1 – Q3

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This is a classic **divide and conquer** question. We shall begin by describing an algorithm to find the heaviest apple. Begin by lining up all 1024 apples in no particular order. Take the first two apples and weigh them on the balance, whilst recording the result of the heaviest apple out of the two weighed. Then compare the next two apples and record the result of the heaviest apple out of those two weighed. One may repeat this process until all of the apples in the row have been compared. By this point, we have exhausted 512 comparisons.

Next, discard all of the lighter apples and line up all of the heavier apples in a row. We repeat the same process of comparing the two adjacent apples and recording the result of the heavier apple. By this point, we will have exhausted another 256 comparisons.

We repeat this process until we reach the last two apples: note that these are not *necessarily* the heaviest two apples. We only guarantee that one of these two apples will be the heaviest apple. One observation that we make is that the number of comparisons form a geometric series since the number of apples to compare halve each time we discard. We will claim that we will have made 1023 comparisons.

> **Claim**: Let n be a natural number. If there are 2^n apples, then there must have been $2^n - 1$ comparisons before finding the heaviest apple, assuming they are all weighted differently.

Proof. Begin by partitioning the apples into $2^n/2 = 2^{n-1}$ groups of apples, each group containing two apples. We shall compare each of these apples with the other apple in its group and record the results of the weighing. Discard all of the lighter apples. The number of comparisons in this round is simply 2^{n-1}. This will leave us with 2^{n-1} apples remaining, since one apple from each group remains. Repeat the same process but partition the 2^{n-1} apples into 2^{n-2} groups of two apples. The number of comparisons in this round is 2^{n-2}. Repeat the same process until we get down to 2 apples, of which there will only be 1 comparison. Thus, the total number of comparisons is simply

$$\begin{aligned} 1 + 2 + 2^2 + \cdots + 2^{n-2} + 2^{n-1} &= \underbrace{2^0 + 2^1 + 2^2 + \cdots + 2^{n-2} + 2^{n-1}}_{\text{geometric series}} \\ &= \frac{2^n - 1}{2 - 1} \\ &= 2^n - 1. \end{aligned}$$

□

Setting $2^n = 1024$, we claim that there must have been 1023 comparisons made before finding the heaviest apple.

To find the second most heaviest apple, we realise that this apple must have been weighed against the heaviest apple **at some point** along the way. So we need to trace back the path of the heaviest apple; that is, we need to calculate *at what point* did these two apples meet? In fact, they could have met, in the worst case, during the first comparison. This can be calculated by taking $\log_2(1024) = 10$ since, in each iteration there can only be one winner and we have to choose 1 out of the 2 apples to be the heavier one. To find the second heaviest apple, we iterate through each of the apples that was weighed against the heaviest one and compare the two apples, keeping track of which apple is heavier. Since there are a maximum of 10 rounds, there can be *up to* 9 comparisons between any two apples.

We conclude that there must have been *at most* $1023 + 9 = 1032$ comparisons to find the heaviest and second heaviest apple.