COMP3121: Assignment 2 – Q3

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Suppose that we have a sequence of n binary bits. Define that sequence to be N. So N is composed of n zero's or one's. Further, define N' to be the sequence N but flipped such that $N'_i = N_{n-i-1}$ where $0 \leq i \leq n-1$. Similarly, define M to be the sequence of the number of fish along the shore

$$M = \langle M_0, M_1, \ldots, M_{100n-1} \rangle.$$

The goal is to show that by applying a convolution on N' and M and looking at its peak value, one can attain the maximal number of fish caught in $O(n \log n)$.

To begin, we'll consider what the convolution of two sequences means. Let A and B be two sequences (not necessarily equal in length). As a result, let A be a sequence with m elements and B be a sequence with n elements where $m < n$. Then $A * B$ is a sequence such that

$$[A * B]_k = \sum_{(i,j)\,:\,i+j=k} A_i B_j.$$

The indexing in this problem plays a crucial role in deciding whether to flip the net bits. So consider the convolution between M and N'. Since N' is just the flipped sequence of N, we'll use our sequence N but with indices flipped. As such, we'll consider $N'_i = N_{n-i+1}$ instead. Then the convolution between N' and M is simply

$$N' * M = \left\langle \sum_{(i,j)\,:\,i+j=k} N'_i M_j \right\rangle.$$

We achieve each of the elements in $N' * M$ in $O(n \log n)$ time using the Fast Fourier transform by converting N' and M into their natural polynomials

$$\begin{aligned} N' &= N'_0 + N'_1 x + N'_2 x^2 + \cdots + N_{n-1} x^{n-1} \\ M &= M_0 + M_1 x + M_2 x^2 + \cdots + M_{100n-1} x^{100n-1} \end{aligned}$$

and multiplying them accordingly to achieve a polynomial of order $100n - 1 + (n-1) = 101n - 2$

$$N' * M = \sum_{k=0}^{101n-2} \left(\sum_{(i,j)\,:\,i+j=k} N'_i M_j \right) x^k.$$

Then by considering the coefficient of x^k, we obtain

$$\begin{aligned} \sum_{m=0}^{n-1} N'_m A_{k-m} &= \sum_{m=0}^{n-1} N_{n-m-1} M_{k-m} \\ &= N_{n-1} M_k + N_{n-2} M_1 + \cdots + N_1 M_{k-n+2} + N_0 M_{k-n+1} \\ &= N_0 M_q + N_1 M_{q+1} + \cdots + N_{n-1} M_{q+n-1} && (q = k - n + 1) \\ &= \sum_{p=0}^{n-1} N_p M_{k+p}. \end{aligned}$$

This is the sum of the number of fish caught by the net of length n. Since these coefficients form the elements of N, then we can traverse through the list once to attain the maximum number of fish by comparing the maximum number of fish at any given time with the position of the leftmost position of the net. This can be done in $O(n)$ time and as a result, the entire algorithm can be performed in $O(n \log n) + O(n) = O(n \log n)$ time.