COMP3121: Assignment 1 – Q5

Gerald Huang

$z5209342$

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a) **Answer**: $f(n) = \Theta(g(n))$.

Proof. Let $f(n) = (\log_2(n))^2$ and $g(n) = \log_2 \left[\left(n^{\log_2 n}\right)^2\right]$. By the basic logarithmic laws, we have

$$g(n) = \log_2 \left[\left(n^{\log_2 n}\right)^2\right] = 2\log_2 \left(n^{\log_2 n}\right) = 2\left(\log_2 n\right)^2 = 2f(n).$$

We will show that $f(n) = \Theta(g(n))$.

We begin the proof by showing that $f(n) = O(g(n))$. So there exist constants $c > 0$ and n_0 such that for all $n > n_0$, $f(n) \leq c \cdot g(n)$. Take $c = 1$ and $n_0 = 1$. Then, for all $n > 1$, it follows that $f(n) \leq 2f(n) = g(n)$ and so $f(n) = O(g(n))$.

Conversely we shall show that $f(n) = \Omega(g(n))$. So there exist positive constants $c > 0$ and n_0 such that for all $n > n_0$, $c \cdot g(n) \leq f(n)$. Take $c = \frac{1}{2}$ and $n_0 = 1$. Then we have, for all $n > 1$,

$$c \cdot g(n) \leq f(n) \implies \frac{1}{2}(2f(n)) \leq f(n).$$

Thus, $f(n) = \Omega(g(n))$. Putting the two implications together, we conclude that $f(n) = \Theta(g(n))$. In other words, $f(n)$ and $g(n)$ grow asymptotically the same. □

b) **Answer**: $f(n) = O(g(n))$.

Proof. Let $f(n) = n^{10}$ and $g(n) = 2^{\sqrt[10]{n}}$. We will show that $f(n) = O(g(n))$. So we need to show that there exists some positive constant c such that, for sufficiently large $n > n_0$, we attain the inequality

$$f(n) \leq c \cdot g(n).$$

Consider the ratio $h(n) = \frac{f(n)}{g(n)} = \frac{n^{10}}{2^{\sqrt[10]{n}}}$. We will show that the limit is finite and as a consequence, there exists a real number M such that

$$\forall n > M, \quad |h(n) - L| < \varepsilon.$$

By considering $n \longrightarrow \infty$, we will apply L'Hôpital's rule to see that while derivatives of $g(n)$ will be multiples of itself, the derivatives of $f(n)$ slowly tends towards 0. By L'Hôpital's rule, if $\lim_{n\to\infty} \frac{f(n)}{g(n)}, \lim_{n\to\infty} \frac{f'(n)}{g'(n)}, \ldots, \lim_{n\to\infty} \frac{f^{(k)}(n)}{g^{(k)}(n)}$ are all indeterminate forms but $\lim_{n\to\infty} \frac{f^{(k+1)}(n)}{g^{(k+1)}(n)}$ is finite, then $\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{f^{(k+1)}(n)}{g^{(k+1)}(n)}$. In this case, we note that the first 10 derivatives of $f(n)$ and $g(n)$ do not converge. However, the 11th derivative of $f(n)$ is 0 and since the 11th derivative of $g(n)$ is still a multiple of itself and its limit is finite as $n \longrightarrow \infty$, then we observe that $h(n) \longrightarrow 0$ as $n \longrightarrow \infty$. Hence, the limit is finite.

So there must exist some real number M and an $\varepsilon > 0$ such that

$$\forall n > M, \quad |h(n) - 0| < \varepsilon \iff \forall n > M, \quad \frac{f(n)}{g(n)} < \varepsilon. \qquad (f(n), g(n) > 0 \text{ for all } n > 0)$$

So take $n_0 = M$ and $c = \varepsilon$ where $c > 0$. Then we have

$$f(n) < \varepsilon \cdot g(n) \quad \text{for all } n > M \implies f(n) = O(g(n)).$$

$\square$

c) **Answer**: Neither.

Proof. Let $f(n) = n^{1+(-1)^n}$ and $g(n) = n$. We will show that it is generally **not** true that either $f(n) = O(g(n))$ or $g(n) = O(f(n))$.

Consider the cases for when n is either odd or even. When n is even, set $n = 2k$ for some integer k. Then we have that $f(n) = (2k)^{1+(-1)^{2k}} = (2k)^2 = n^2$. On the other hand, we have $g(n) = n$. So $g(2k) = O(f(2k))$.

However, consider when n is odd; that is set $n = 2m + 1$ for some integer m. Then we have

$$f(n) = (2m+1)^{1+(-1)^{2m+1}} = (2m+1)^{1+(-1)^{2m}\times(-1)} = (2m+1)^{1-1} = 1$$

and $g(n) = n$. It then follows that $f(2m + 1) = O(g(2m + 1))$. Thus, $g(n)$ is neither a lower bound nor an upper bound of $f(n)$. $\square$