COMP3121: Assignment 2 – Q4

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a) **Answer**: $A * A = \langle 1, \underbrace{0, 0, \dots, 0}_{k}, 2, \underbrace{0, 0, \dots, 0}_{k}, 1 \rangle$.

Let $A = \langle 1, \underbrace{0, 0, \dots, 0}_{k}, 1 \rangle$. Then A in polynomial form can be constructed as $p_A(x) = 1 + x^{k+1}$. Then it follows that

$$A * A = (1 + x^{k+1})^2 = 1 + 2x^{k+1} + x^{2(k+1)}.$$

And hence, we obtain the convolution polynomial of degree $(2k+2)$. Hence we obtain the convolution sequence with $(2k + 3)$ elements of the form

$$A * A = \langle 1, \underbrace{0, 0, \dots, 0}_{k}, 2, \underbrace{0, 0, \dots, 0}_{k}, 1 \rangle.$$

b) **Answer**: $\left\langle 2, 1 + \omega_{k+2}^{k+1}, 1 + \omega_{k+2}^{k}, \dots, 1 + \omega_{k+2}^{2}, 1 + \omega_{k+2} \right\rangle$

Let $A = \langle 1, \underbrace{0, 0, \dots, 0}_{k}, 1 \rangle$. Then, from part a), we convert this into its polynomial form; that is

$$P_A(x) = 1 + x^{k+1}.$$

Now, the **Discrete Fourier transform** is the transform

$$\begin{aligned} A = \langle 1, \underbrace{0, 0, \dots, 0}_{k}, 1 \rangle \iff DFT(A) &= \langle P_A(1), P_A(\omega_{k+2}), P_A(\omega_{k+2}^2), P_A(\omega_{k+2}^3), \dots, P_A(\omega_{k+2}^{k+1}) \rangle \\ &= \left\langle 2, 1 + \omega_{k+2}^{k+1}, 1 + \omega_{k+2}^{2(k+1)}, \dots, 1 + \omega_{k+2}^{(k+1)(k+1)} \right\rangle \\ &= \left\langle 2, 1 + \omega_{k+2}^{k+1}, 1 + \omega_{k+2}^{2(k+1)}, \dots, 1 + \omega_{k+2}^{(k+1)^2} \right\rangle. \end{aligned}$$

But note that since $\omega_{k+2}^{k+2} = 1$, then we can reduce the following roots of unity into roots of unity of degree $k+1$ or less.

Note that $2(k + 1) = 2k + 2 = (k + 2) + k$. So we have

$$\omega_{k+2}^{2(k+1)} = \omega_{k+2}^{(k+2)+k} = \omega_{k+2}^{k+2} \cdot \omega_{k+2}^{k} = \omega_{k+2}^{k}.$$

Similarly, we can observe that $3(k + 1) = 3k + 3 = (k + 2) + (2k + 1) = (k + 2) + (k + 2) + (k - 1)$ and we have

$$\omega_{k+2}^{3(k+1)} = \left(\omega_{k+2}^{k+2}\right)^2 \cdot \omega_{k+2}^{k-1} = \omega_{k+2}^{k-1}.$$

In general, we see that since $m(k + 1) = (m - 1)(k + 2) + (k - m + 2)$, then it follows that

$$\omega_{k+2}^{m(k+1)} = \omega_{k+2}^{(m-1)(k+2)+(k-m+2)} = \left(\omega_{k+2}^{k+2}\right)^{m-1} \cdot \omega_{k+2}^{k-m+2} = \omega_{k+2}^{k-m+2}. \qquad \text{(for } m = 0, 1, 2, \dots, (k + 1))$$

Hence, we see that our original DFT can be simplified down to

$$DFT(A) = \left\langle 2, 1 + \omega_{k+2}^{k+1}, 1 + \omega_{k+2}^{k}, \dots, 1 + \omega_{k+2}^{2}, 1 + \omega_{k+2} \right\rangle .$$