COMP3121: Assignment 3 – Q5

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Given a list of chemicals C_i, each of which requiring W_i kilograms, we shall schedule this by weight. More specifically, we sort each of the chemicals in ascending order by their weights and schedule the production in this way. The key to proving the optimality behind this is that the rate at which chemicals are lost is constant – that is, each chemical loses $p\%$ of whatever is left over from the previous day. By scheduling this by their weight, we will begin the proof of optimality, assuming that we can make *just* enough to last the full $N - k$ days where C_k is produced on day k and every chemical is delivered on the N-th day.

Proof of optimality

We will show that this is method is optimal by first proving a few lemmas.

Lemma 0.1. *Let W_i and W_j be the weights of any two chemicals with $W_i < W_j$. Then W_i loses less weight to evaporation than W_j.*

Proof. Let $0 < p < 1$. At any day of production, W_i loses $W_i p$ while W_j loses $W_j p$. Since $W_i < W_j$ and $p > 0$, then we also have that $W_i p < W_j p$. Hence, we arrive at the inequality

$$W_i < W_j \implies W_i - W_i p > W_j - W_j p.$$

□

Lemma 0.2. *Let C_k be a chemical with required weight W_k be produced on day k. Then it is optimal to make just enough to last the full $N - k$ days.*

Proof. We will show this in two parts. Define the optimal amount O with $Q_1 < O < Q_2$. First assume that we produce less than optimal. Then suppose that we produced Q_1 amounts of C_k. Since the chemical evaporates each day for $N - k$ days, this clearly underperforms as less than W_k amounts of C_k will be produced by the N-th day and hence, we will have produced an invalid amount of chemicals. So we will need to have produced *at least* the optimal amount.

On the other hand, suppose that we produced Q_2 amounts of chemicals. Then by the N-th day, we will have more than required. Furthermore, from lemma 0.1, we know that a bigger weight evaporates more than a smaller weight. As a result, it isn't optimal to produce more than what is required.

Hence, it is optimal to produce a quantity O such that it evaporates *just* enough such that at the N-th day, it has a weight of W_k. □

Lemma 0.3. *Let $0 < p < 1$. Additionally, let C_k be a chemical produced on day k. Then amount to make optimally for chemical C_k with required weight W_k is given by*

$$Q_k = \frac{W_k}{(1 - p)^{N-k+1}} - W_k.$$

Proof. We'll proceed with a proof by induction on the number of days between k and N. Since we need to wait a day before chemical W_N is finished, then the minimum day elapsed is 1 day. So it will need to have evaporated by a rate of $p\%$ before delivery. Hence, we require

$$(Q_N - Q_N p) + W_N = 0 \implies Q_N(1-p) = -W_N \implies Q_N = \frac{W_N}{1-p} - W_N.$$

Suppose that chemical C_m is produced on day m. Then, by our assumption, we need to produce $Q_m = \frac{W_m}{(1-p)^{N-m+1}} - W_m$. Consider the $(m+1)$-th day. We notice that, on the m-th day, we had to produce $Q_m = \frac{W_m}{(1-p)^{N-m+1}} - W_m$. This means that on the $(m+1)$-th day, the production of the C_m chemical is simply

$$Q_m(1-p) = \left(\frac{W_m}{(1-p)^{N-m+1}} - W_m\right)(1-p) = \frac{W_m}{(1-p)^{N-m}} - W_m(1-p).$$

Since p is a constant rate, then on the $(m+2)$-th day, the production of chemical C_{m+1} will be

$$\begin{aligned} Q_{m+1}(1-p) = \frac{W_{m+1}}{(1-p)^{N-m-1}} - W_{m+1}(1-p) &\implies Q_{m+1} = \frac{W_{m+1}}{(1-p)^{N-m}} - W_{m+1} \\ &\implies Q_{m+1} = \frac{W_{m+1}}{(1-p)^{N-(m+1)+1}} - W_{m+1}. \end{aligned}$$

Hence, by induction, we have proven the result. □

Using these three lemmas, we have shown that a) if we schedule it in any other way (that is, not necessarily by lightest first), we will get an evaporation that's not optimal from lemma 0.1, b) if we make the production of chemicals on day k, then we need to make it by the amount

$$Q_k = \frac{W_k}{(1-p)^{N-k+1}} - W_k$$

which will optimise evaporation loss since it makes *just* enough to last $N-k$ which is optimal as per lemma 2.

As such, our greedy approach is indeed optimal scheduling it by lightest first.