6 Rotation and orientation

6.1 Introduction

The orientation of an object in 3D Euclidean space specifies how that object is aligned with respect to a reference configuration of that object. The reference configuration is a conceptual copy of the object that is positioned with respect to a particular orthonormal frame (see 4.3). The orientation of the object may be specified by a length-preserving transformation that would make the reference configuration congruent to the object. Only the rotational components of this transformation are essential for the specification, as translation operations do not affect alignment.

For computational purposes, an orthonormal frame is conceptually attached to the object. This orthonormal frame is termed the *object-frame*. Another orthonormal frame is conceptually attached in the same manner to the corresponding position of the reference configuration of the object. This other orthonormal frame is termed the *reference-frame*. An orientation specification is a rotation operation that would bring the reference frame into alignment with the corresponding object frame. Only a single rotation is required for such a specification, since, as a consequence of Euler's rotation theorem (see 6.5.2), a series of rotations about various axes is equivalent to one rotation about a single axis.

Alternatively, an orientation specification is the change of coordinate basis operator that changes the coordinate of a position-vector from the object-frame basis to the reference-frame basis (assuming a common origin for the two frames). These two ways of specifying the orientation of an object with respect to a reference frame, a rotation or a change of basis, are closely related operations on 3D Euclidean space (see 6.5.3).

Rotation operator concepts and various mathematical representations of rotations have been in wide use from before the time of Euler's work on the subject. As a result, there are many different treatments in the literature, using similar terms with different meanings and different notational conventions. For this reason, rotation terms and notation used in this International Standard are fully defined.

The specification of an ORM (see 7.4.4) depends on a similarity transformation (see 7.3.2) for which a rotation operator is a key component. Converting the representation of such rotation operators to and from the Matrix representation (see 6.7.2) is required for some change of SRF operations (see 10.3.2 and 10.4.5). This includes the ability to convert an object's orientation represented with respect to one SRF to its equivalent with respect to another SRF.Rotation operators are also important in some of the application domains that fall within the scope of this International Standard.

The remainder of this clause deals with 3D rotations, rotations with respect to frames, change of basis operations and associated rotations, the specification of orientation, data representations of rotations and the inter-conversion of data representations.

6.2 Preliminaries and conventions

6.2.1 Euclidean space, vector space, and orthonormal frames

In this clause, the terms Euclidean space, vector space, and orthonormal frame are used extensively. These terms are defined in 4.3.

Euclidean space forms a vector space once an origin point has been selected. Each point in Euclidean space may then be associated with the position vector that extends from the origin to the point and has length equal to the Euclidean distance between the origin and the point. Thus, points in space and position vectors with respect to a selected origin may be treated as equivalent concepts. An n-dimensional vector space with inner product (see A.2) may also be represented by the vector space of n-tuples of scalars, provided an orthonormal basis of n position vectors is selected. The result of selecting an origin and an ordered orthonormal basis is termed an *orthonormal frame*. Every point in n-dimensional Euclidean space is thus uniquely represented in an orthonormal frame by the linear combination of the basis vectors that is equal to the position vector associated with the point. The n scalars in the linear combination are represented by a corresponding n-tuple of scalars in the corresponding basis order.

A 3D orthonormal frame is termed right-handed if the vertices of the triangle formed by the basis vector unit points are in clockwise order when viewed from the origin, as defined in ISO 80000-2. In this International Standard, all 3D orthonormal frames shall be right-handed.

6.2.2 Right-hand rule

6.2.3 Angular measurements

In this International Standard, the unit of angular measure is the radian (see 4.12). However, in some cases, arc degree or arc second is used to support common usage or to prevent loss of precision in data specification.

In general, when measuring angles, only the magnitude is considered. However, if the angular measurement is made in the context of a rotation operation on a position-vector, the rotation angle is measured from the state of the input position-vector of the rotation operation to the state of the output position-vector of the rotation operation.

A rotation angle symbol, i.e., theta, always represents a signed value. When a rotation angle symbol is preceded by a minus sign (“-“), it indicates a rotation angle that is the additive inverse of the rotation angle represented by the symbol alone.

If a rotation axis is assigned a direction, the angle of rotation can be specified as a positive angle or a negative angle using the *right-hand rule*: conceptually, if the right-hand holds the axis with thumb pointing in the axis direction, the fingers curl in the positive angle direction.

6.2.4 Labelling conventions

It is neither necessary nor desirable to explicitly represent time in this clause. However, in discussing use case scenarios involving multiple rotation operations, referencing sequences of states (and transitions from one state to another) is unavoidable. The labels and notation used for objects and their states need to satisfy multiple goals:

1) The persistent identity of the object, as it changes state, must be apparent, so that different states of the same object are readily identifiable. This is normally accomplished by labelling the object with a lower-case letter (i.e., point/vector $\boldsymbol{p}$).

2) The frame with respect to which the state of the object is represented. This is normally accomplished by labelling the frame with an upper-case letter (i.e., frame $\boldsymbol{E}$). When a frame is closely associated with a particular object, the label of the frame should reflect that association (i.e., frame $\boldsymbol{P}$ associated with object $\boldsymbol{p}$.)

    a) The frame is also uniquely identified by its set of three basis vectors. These can be identified by:

        i) Uniquely labeling the basis vectors, i.e., $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$, or

        ii) Numbering the basis vectors, i.e., $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{e}_3$; this is particularly useful in the context of a matrix, where the subscript corresponds to the matrix row or column.

3) The state(s) of the object, as the object passes through a sequence of states, so that it is easy to understand how the state of the object changes as various operations are performed on it.

    a) Identity. This is normally accomplished by adding zero or more primes to the label of the object (i.e, $\boldsymbol{p}$, $\boldsymbol{p}'$, $\boldsymbol{p}''$, ...). In addition, it should be clear when the states of two (or more) objects correspond to one another. This is normally accomplished by adding the same number of primes to the labels of both objects.

b) Position. This is specified by the vector-coordinate component values of an object, with respect to a particular orthonormal frame, and/or to the basis vectors of that frame:
   i) Identifying individual vector-coordinate components with individual basis vectors:
      1) Using basis vector label subscripts, i.e., p_x, p_y, p_z or
      2) Using basis vector numerical subscripts, i.e., p_1, p_2, p_3
   ii) Identifying the set of vector-coordinate components using the label of the orthonormal frame:
      1) Set of three components, i.e., $(p_x, p_y, p_z)_E$ or $(p_1, p_2, p_3)_E$,
      2) Shorthand for the set of all three components, i.e., $\boldsymbol{p}_E$
c) Orientation. This can be specified in several ways:
   i) Axis-angle – $\boldsymbol{n}, \theta$
   ii) Matrix
   iii) Euler angles

Subscripts – it may be useful to enlarge subscripts. The subscript that designates a rotation axis is three increments larger than the normal subscript size.

6.3 Rotation

6.3.1 Introduction

Rotation plays a critical role in the representation of motion, dynamics, and orientation in a number of application domains, including mechanics, aviation, and astronomy. Rotation can be interpreted in terms of the physical movement of objects, or a physical change of state. These actions may be mathematically modelled in terms of abstract geometry and/or as vector space transformations.

As illustrated in Figure 6.1, a rotation operation rotates one or more points about a given directed axis of rotation through an angle θ.

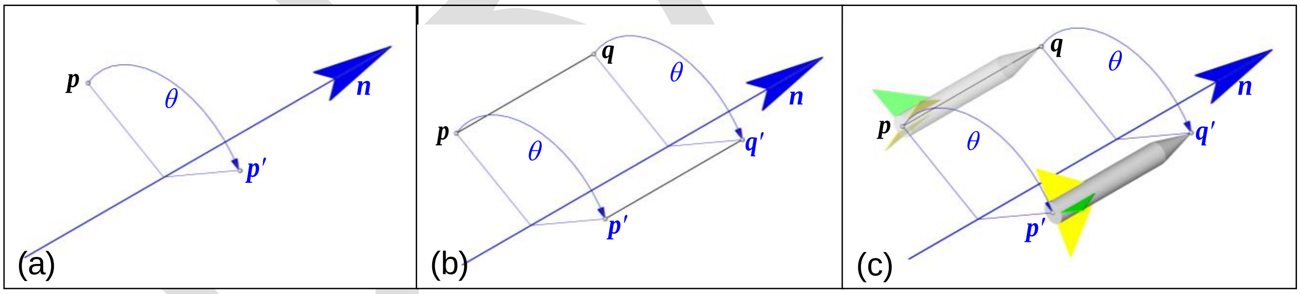

Figure 6.1 — Rotation operator applied to: (a) a point, (b) a line segment, and (c) a 3D object

As shown in Figure 6.1(a), the rotation operation can be applied to a single point $\boldsymbol{p}$, producing the rotated point $\boldsymbol{p'}$. As shown in Figure 6.1(b), the same rotation operation can be applied to any geometric primitive, such as the line segment with endpoints $\boldsymbol{p}$ and $\boldsymbol{q}$, producing the rotated line segment defined by the rotated endpoints $\boldsymbol{p'}$ and $\boldsymbol{q'}$. Furthermore, as shown in Figure 6.1(c), this rotation operation can be applied to any rigid three-dimensional object. Each of the infinite set of points that make up the object is rotated about the axis in exactly the same manner.

6.3.2 Origin-fixed rotations

In this clause, all rotation operations are performed within a 3D Euclidean space with a designated origin, forming a vector space. The notion of rotation is translation independent.

Assuming the availability of translation operations, rotation about an arbitrary axis in space is equivalent to an *origin-fixed rotation*, that is, a rotation about an axis that passes through a designated origin. This is illustrated in Figure 6.2. Given an arbitrary rotation axis, a point $\boldsymbol{p}$ to be rotated about that axis by an angle θ, and an origin, as shown in Figure 6.2(a), select a point on the arbitrary rotation axis and designate the position-vector from the origin to that point as $\boldsymbol{t}$. Translate the arbitrary rotation axis by $-\boldsymbol{t}$, so that the translated axis passes through the origin. Normalize the translated axis to yield the unit vector $\boldsymbol{n}$; similarly translate the point to be rotated yielding the point $(\boldsymbol{p}-\boldsymbol{t})$, as shown in Figure 6.2(b). Perform the rotation about the axis $\boldsymbol{n}$, yielding the rotated point $\boldsymbol{R_n}\langle\theta\rangle(\boldsymbol{p}-\boldsymbol{t})$, as shown in Figure 6.2(c). Finally, translate the rotated point back by $\boldsymbol{t}$, yielding the point $\boldsymbol{R_n}\langle\theta\rangle(\boldsymbol{p}-\boldsymbol{t})+\boldsymbol{t}$, as shown in Figure 6.2(d). This sequence of operations produces the same result as the rotation about the arbitrary axis.

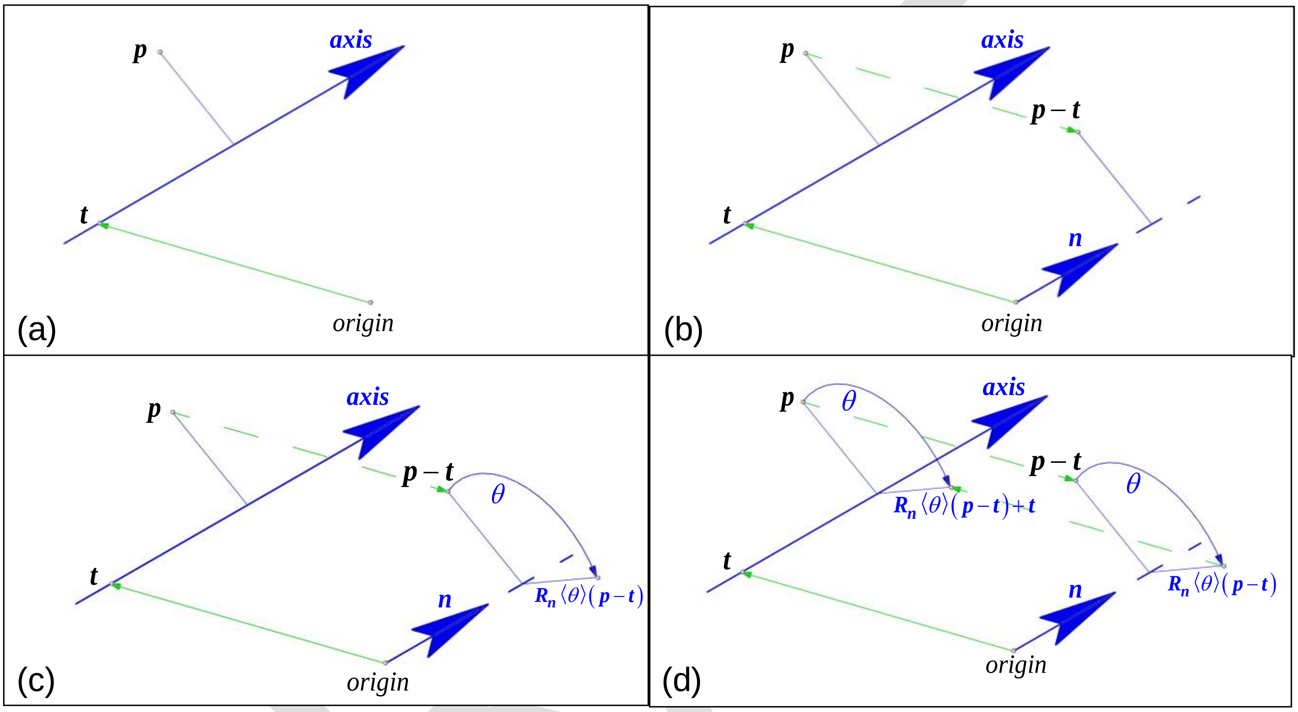

Figure 6.2 — Origin-fixed rotation: (a) initial state; (b) translation to origin; (c) rotation; (d) translation back to axis

An origin-fixed rotation is specified by a non-zero position-vector and a signed rotation angle. The span of the position-vector specifies the axis of the rotation. The axis is then a set of points forming a straight line and the signed angle value specifies the amount of positive or negative rotation about that axis. The position-vector direction from the origin gives the axis direction and the direction of rotation about the axis is given by the *right-hand rule*: conceptually, if the right-hand holds the axis with thumb pointing in the axis direction, the fingers curl in the positive angle direction.

Without loss of generality, it may be assumed that the position-vector is a unit vector. In that case, $\boldsymbol{R_n}\langle\theta\rangle$ shall denote the origin-fixed rotation by angle θ about the unit vector $\boldsymbol{n}$. The axis of rotation is the span of $\boldsymbol{n}$, that is, the set $\{\alpha\boldsymbol{n} | \alpha \in \mathbb{R}\}$ with the forward direction given as the direction of increasing scalar values α.

An origin-fixed rotation is a coordinate-independent operation, since only an origin is required, rather than a complete orthonormal basis.

In the remainder of this clause, all rotation operations are origin-fixed rotation operations, unless otherwise indicated.

6.3.3 Rodrigues' rotation formula

The action performed by an origin-fixed rotation $\boldsymbol{R}_{\boldsymbol{n}}\langle\theta\rangle$ on an arbitrary position-vector $\boldsymbol{p}$ may be computed usiing *Rodrigues' rotation formula* (see [BERN]):

$$\boldsymbol{R}_{\boldsymbol{n}}\langle\theta\rangle(\boldsymbol{p}) = \cos(\theta)\,\boldsymbol{p} + (1-\cos(\theta))(\boldsymbol{p}\bullet\boldsymbol{n})\boldsymbol{n} + \sin(\theta)\,\boldsymbol{n}\times\boldsymbol{p} \tag{0.1}$$

Using Lagrange's formula, $\boldsymbol{n}\times(\boldsymbol{n}\times\boldsymbol{r}) = (\boldsymbol{r}\bullet\boldsymbol{n})\boldsymbol{n} - (\boldsymbol{n}\bullet\boldsymbol{n})\boldsymbol{r}$ with $(\boldsymbol{n}\bullet\boldsymbol{n}) = 1$, the formula terms may be rearranged to a useful alternate form (see 6.x.x):

$$\boldsymbol{R}_{\boldsymbol{n}}\langle\theta\rangle(\boldsymbol{p}) = \boldsymbol{p} + (1-\cos(\theta))\boldsymbol{n}\times(\boldsymbol{n}\times\boldsymbol{p}) + \sin(\theta)\,\boldsymbol{n}\times\boldsymbol{p} \tag{0.2}$$

As is evident from the sine and cosine terms, $\boldsymbol{R}_{\boldsymbol{n}}\langle\theta\rangle = \boldsymbol{R}_{\boldsymbol{n}}\langle\theta+2\pi\rangle$, where k is any positive or negative integer value. Rotations greater than one full revolution are important in some applications. However, in this International Standard rotation angles shall be considered equivalent modulo 2π.

NOTE Rodrigues' rotation formula is a coordinate-free specification of the action of a rotation operator on a position-vector. That is to say, the formula does not use coordinate components from any basis for the position-vector terms appearing in the formulation. (See A.2 Notes 2 and 3 for coordinate-free expressions of the vector dot and cross products.)

6.3.4 Rotation properties

An origin-fixed rotation operator $\boldsymbol{R}_{\boldsymbol{n}}\langle\theta\rangle$ is linear and length-preserving (see A.2). That is, given a scalar α and any two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$, the following hold:

$$\begin{aligned} &\boldsymbol{R}_{\boldsymbol{n}}\langle\theta\rangle(a\boldsymbol{u}+\boldsymbol{v}) = a\boldsymbol{R}_{\boldsymbol{n}}\langle\theta\rangle(\boldsymbol{u}) + \boldsymbol{R}_{\boldsymbol{n}}\langle\theta\rangle(\boldsymbol{v}) && \text{linearity} \\ &\|\boldsymbol{R}_{\boldsymbol{n}}\langle\theta\rangle(\boldsymbol{v}) - \boldsymbol{R}_{\boldsymbol{n}}\langle\theta\rangle(\boldsymbol{u})\| = \|\boldsymbol{v}-\boldsymbol{u}\| && \text{length-preserving.} \end{aligned} \tag{0.3}$$

Since the Pythagorean theorem holds in Euclidean space, the length-preserving property implies the angle preserving property: The angle between two vectors is preserved when they are rotated together.

In the general case of a rotation about an arbitrary axis, the operation is not linear. Consider an axis passing through the position-vector $\boldsymbol{t}\neq 0$ in a direction parallel to a unit position-vector $\boldsymbol{n}$ (and assume that $\boldsymbol{n}$ and $\boldsymbol{t}$ are linearly independent). The action on a position-vector $\boldsymbol{p}$ performed by the rotation operator $\boldsymbol{R}$ through angle θ about the directed axis $\{\boldsymbol{t}+\alpha\boldsymbol{n}|\alpha\in\mathbb{R}\}$ may be computed in terms of two translations and an origin-fixed rotation using the equality given by:

$$\boldsymbol{R}(\boldsymbol{p}) = \boldsymbol{R}_{\boldsymbol{n}}\langle\theta\rangle(\boldsymbol{p}-\boldsymbol{t}) + \boldsymbol{t} = \boldsymbol{R}_{\boldsymbol{n}}\langle\theta\rangle(\boldsymbol{p}) + \big(\boldsymbol{t} - \boldsymbol{R}_{\boldsymbol{n}}\langle\theta\rangle(\boldsymbol{t})\big).$$

Thus, in the general case of a non-origin-fixed rotation $\boldsymbol{R}$, the rotation operator is an affine operator, that is, it is the sum of a linear operator $\boldsymbol{R}_{\boldsymbol{n}}\langle\theta\rangle$ and a translation by $\big(\boldsymbol{t} - \boldsymbol{R}_{\boldsymbol{n}}\langle\theta\rangle(\boldsymbol{t})\big)$. Consequently, $\boldsymbol{R}$ is not linear, but is length preserving.

An origin-fixed rotation operator $\boldsymbol{R}_{\boldsymbol{n}}\langle\theta\rangle$ is invertible with inverse $\boldsymbol{R}_{\boldsymbol{n}}\langle\theta\rangle^{-1} = \boldsymbol{R}_{\boldsymbol{n}}\langle-\theta\rangle = \boldsymbol{R}_{-\boldsymbol{n}}\langle\theta\rangle$. The expression $-\theta$ denotes the angle of rotation that is the additive inverse of the signed quantity θ, and $-\boldsymbol{n}$ denotes that the direction of the rotation axis is the reverse of the axis spanned by $\boldsymbol{n}$.

NOTE The fact that the rotation operator $\boldsymbol{R}_{\boldsymbol{n}}\langle\theta\rangle$ can be inverted in multiple ways, i.e., by reversing the sign of the rotation angle, or by reversing the direction of the rotation axis, is a common source of confusion and errors with working with rotation operations. In this International Standard, $\boldsymbol{R}_{\boldsymbol{n}}\langle\theta\rangle^{-1}$ is used to denote the inverse unless it is necessary to specify the manner in which the operator is being inverted.

6.3.5 Consecutive rotations

In some applications, two or more consecutive rotation operations are used to produce a desired end state for an object of interest. This sequence of rotation operations is a functional composition (see A.5), and is also a rotation operation.

In general, the composition of rotation operators is not commutative. That is, given two origin-fixed rotation operators, $\boldsymbol{R_n}\langle\theta\rangle$ and $\boldsymbol{R_m}\langle\varphi\rangle$, applied sequentially to a rigid body representing an object of interest:

$$\boldsymbol{R_m}\langle\varphi\rangle \circ \boldsymbol{R_n}\langle\theta\rangle \neq \boldsymbol{R_n}\langle\theta\rangle \circ \boldsymbol{R_m}\langle\varphi\rangle.$$

However, in the special case that the two rotation axes are co-linear, that is when $\boldsymbol{m} = \pm\boldsymbol{n}$, the rotations are commutative:

$$\boldsymbol{R_n}\langle\theta\rangle \circ \boldsymbol{R_m}\langle\varphi\rangle = \boldsymbol{R_m}\langle\varphi\rangle \circ \boldsymbol{R_n}\langle\theta\rangle = \boldsymbol{R_n}\langle\theta \pm \varphi\rangle, \text{ when } \boldsymbol{m} = \pm\boldsymbol{n}, \text{ with matching signs.}$$

The composition of rotation operators is associative:

$$\boldsymbol{R_k}\langle\gamma\rangle \circ (\boldsymbol{R_m}\langle\varphi\rangle \circ \boldsymbol{R_n}\langle\theta\rangle) = (\boldsymbol{R_k}\langle\gamma\rangle \circ \boldsymbol{R_m}\langle\varphi\rangle) \circ \boldsymbol{R_n}\langle\theta\rangle = \boldsymbol{R_k}\langle\gamma\rangle \circ \boldsymbol{R_m}\langle\varphi\rangle \circ \boldsymbol{R_n}\langle\theta\rangle.$$

Example. Figures 6.3 and 6.4 illustrate two different sequences of consecutive rotations of a rigid body. The different sequences of consecutive rotations produce different end states. Two consecutive rotation operations, $\boldsymbol{R_n}\langle\theta\rangle$ and $\boldsymbol{R_m}\langle\varphi\rangle$ are applied to the same 3D object. In this example, the rotation axis $\boldsymbol{n}$ is parallel to the long axis of the object, and the rotation axis $\boldsymbol{m}$ is perpendicular to the axis $\boldsymbol{n}$. However, these conditions are not significant. The only relevant constraint on the two axes is that they are not co-linear.

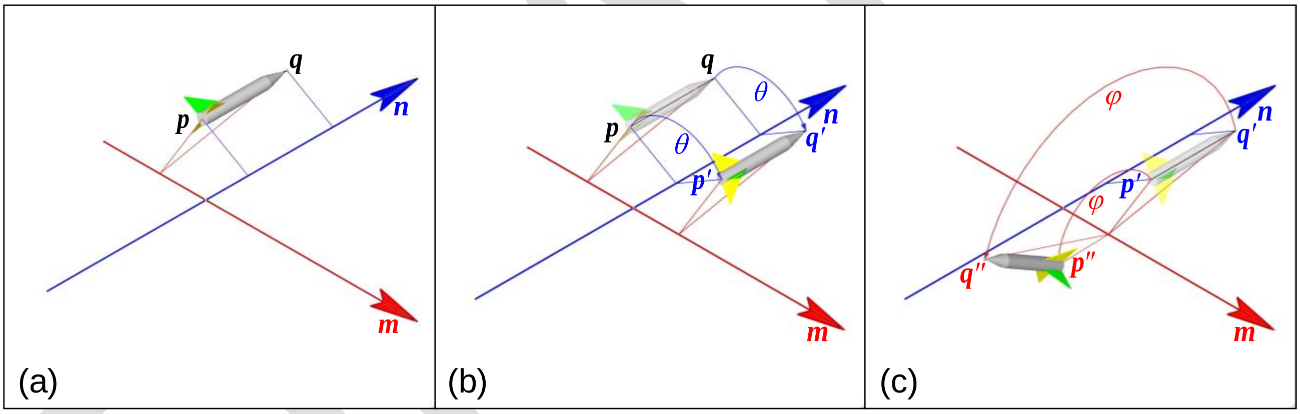

Figure 6.3 — Consecutive rotation operations $R_m\langle\varphi\rangle \circ R_n\langle\theta\rangle$: (a) initial configuration; (b) after first rotation; (c) after second rotation

In Figure 6.3, the first rotation operation performed is $\boldsymbol{R_n}\langle\theta\rangle$, and the second rotation operation is $\boldsymbol{R_m}\langle\varphi\rangle$. Figure 6.3(a) shows the initial configuration, Figure 6.3(b) shows the result of the first rotation operation $\boldsymbol{R_n}\langle\theta\rangle$, and Figure 6.3(c) shows the result of the second rotation operation $\boldsymbol{R_m}\langle\varphi\rangle$. Using composition notation in right-to-left order, this can be written as $\boldsymbol{R_n}\langle\theta\rangle$.

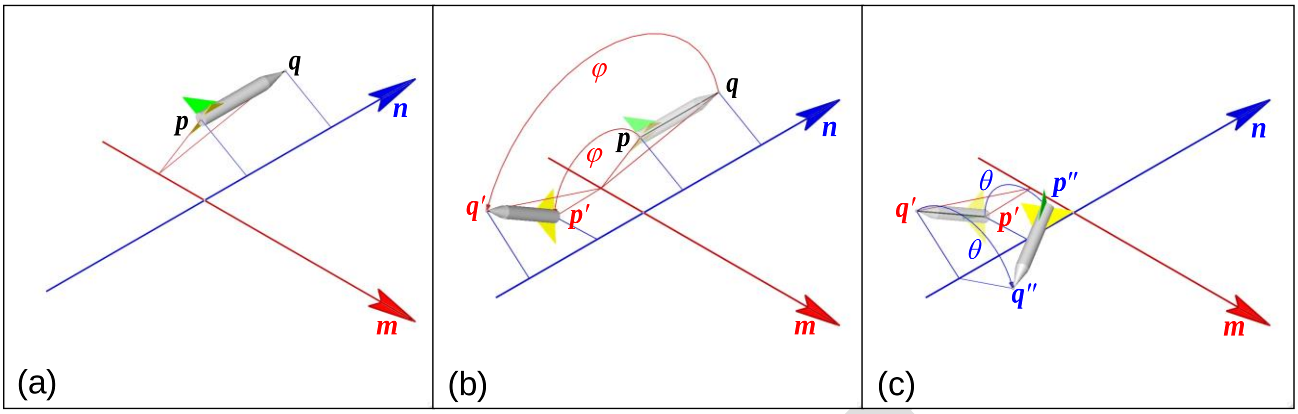

Figure 6.4 — Consecutive rotation operations $\boldsymbol{R_n}\langle\boldsymbol{\theta}\rangle \circ \boldsymbol{R_m}\langle\boldsymbol{\varphi}\rangle$: (a) initial configuration; (b) after first rotation; (c) after second rotation

In Figure 6.4, the order of the rotation operations is reversed. The first rotation operation performed is $\boldsymbol{R_m}\langle\varphi\rangle$, and the second rotation operation performed is $\boldsymbol{R_n}\langle\theta\rangle$. Figure 6.4(a) shows the initial configuration, Figure 6.4(b) shows the result of the first rotation operation $\boldsymbol{R_m}\langle\varphi\rangle$, and Figure 6.4(c) shows the result of the second rotation operation $\boldsymbol{R_n}\langle\theta\rangle$. Using composition notation in right-to-left order, this can be written as $\boldsymbol{R_n}\langle\theta\rangle \circ \boldsymbol{R_m}\langle\varphi\rangle$. The end states of the 3D object in Figures 6.3(c) and 6.4(c) are different from each other.

6.3.6 Consecutive rotation conventions

In some applications, when two or more consecutive rotation operations are used to produce a desired end state for an object of interest, the rotation axes themselves may be rotated along with the object. Given two origin-fixed rotations $\boldsymbol{R_n}\langle\theta\rangle$ and $\boldsymbol{R_m}\langle\varphi\rangle$ applied sequentially to an object of interest, the result depends on whether the rotation axes $\boldsymbol{n}$ and/or $\boldsymbol{m}$ are rotated with the body or remain stationary. This choice is determined by the intent of the rotation operations in the context of the application. This International Standard uses the terms *body-fixed* convention and *space-fixed* convention to distinguish between these two cases. Numerous other sets of distinguishing terms are also found in the literature.

In the space-fixed convention, as in 6.3.5, the axes $\boldsymbol{n}$ and $\boldsymbol{m}$ remain stationary, and the rotations are applied only to the object. The resulting composite operation, in right-to-left operator composition order, is given by:

$\boldsymbol{R_m}\langle\varphi\rangle \circ \boldsymbol{R_n}\langle\theta\rangle$ space-fixed convention.

In the body-fixed convention, the rotations are also applied to the axes $\boldsymbol{n}$ and $\boldsymbol{m}$, so that the axes rotate together with the object. The first rotation $\boldsymbol{R_n}\langle\theta\rangle$ does not affect axis $\boldsymbol{n}$, $\boldsymbol{n} = \boldsymbol{R_n}\langle\theta\rangle(\boldsymbol{n})$, but rotates axis $\boldsymbol{m}$ to a new state $\boldsymbol{m}'$, $\boldsymbol{m}' = \boldsymbol{R_n}\langle\theta\rangle(\boldsymbol{m})$. The second rotation in this convention uses the rotation axis in its new state $\boldsymbol{m}'$. Hence, the resulting composite operation, in right-to-left operator composition order, is given by:

$\boldsymbol{R_{m'}}\langle\varphi\rangle \circ \boldsymbol{R_n}\langle\theta\rangle$ body-fixed convention.

In typical applications, the axis $\boldsymbol{m}$ is known, but additional computation would be required to determine $\boldsymbol{m}'$. However, it can be shown that: $\boldsymbol{R_{m'}}\langle\varphi\rangle \circ \boldsymbol{R_n}\langle\theta\rangle = \boldsymbol{R_n}\langle\theta\rangle \circ \boldsymbol{R_m}\langle\varphi\rangle$ (see A.11). Thus the two cases are simply expressed as:

$$\begin{aligned} &\boldsymbol{R_m}\langle\varphi\rangle \circ \boldsymbol{R_n}\langle\theta\rangle && \text{space-fixed convention, and} \\ &\boldsymbol{R_{m'}}\langle\varphi\rangle \circ \boldsymbol{R_n}\langle\theta\rangle = \boldsymbol{R_n}\langle\theta\rangle \circ \boldsymbol{R_m}\langle\varphi\rangle && \text{body-fixed convention.} \end{aligned} \quad (0.4)$$

This relationship between the space-fixed and body-fixed conventions is also true in the general case of non-origin-fixed arbitrary rotation axes (see A.x).

The body-fixed equivalence expressed in Equation (0.4) may be generalized to more than two rotation operators. Given a third origin-fixed rotation $\boldsymbol{R}_{\boldsymbol{k}}\langle\gamma\rangle$, let $\boldsymbol{k}' = \boldsymbol{R}_{\boldsymbol{n}}\langle\theta\rangle(\boldsymbol{k})$ and let $\boldsymbol{k}" = \boldsymbol{R}_{\boldsymbol{m}'}\langle\varphi\rangle(\boldsymbol{k}') = \boldsymbol{R}_{\boldsymbol{m}}\langle\varphi\rangle \circ \boldsymbol{R}_{\boldsymbol{n}}\langle\theta\rangle(\boldsymbol{k})$, then:

$$\boldsymbol{R}_{\boldsymbol{k}"}\langle\gamma\rangle \circ (\boldsymbol{R}_{\boldsymbol{m}'}\langle\varphi\rangle \circ \boldsymbol{R}_{\boldsymbol{n}}\langle\theta\rangle) = (\boldsymbol{R}_{\boldsymbol{m}'}\langle\varphi\rangle \circ \boldsymbol{R}_{\boldsymbol{n}}\langle\theta\rangle) \circ \boldsymbol{R}_{\boldsymbol{k}}\langle\gamma\rangle = (\boldsymbol{R}_{\boldsymbol{n}}\langle\theta\rangle \circ \boldsymbol{R}_{\boldsymbol{m}}\langle\varphi\rangle) \circ \boldsymbol{R}_{\boldsymbol{k}}\langle\gamma\rangle, \text{ or:}$$
$$\boldsymbol{R}_{\boldsymbol{k}"}\langle\gamma\rangle \circ \boldsymbol{R}_{\boldsymbol{m}'}\langle\varphi\rangle \circ \boldsymbol{R}_{\boldsymbol{n}}\langle\theta\rangle = \boldsymbol{R}_{\boldsymbol{n}}\langle\theta\rangle \circ \boldsymbol{R}_{\boldsymbol{m}}\langle\varphi\rangle \circ \boldsymbol{R}_{\boldsymbol{k}}\langle\gamma\rangle \qquad \text{body-fixed convention.} \tag{6.5}$$

NOTE Other terminology used for the space-fixed and body-fixed concepts include extrinsic and intrinsic rotations; and fixed-frame and moving-frame rotations.

Example. Figures 6.5 through 6.7 illustrate the difference between the space-fixed and body-fixed conventions. Starting with the same initial configuration shown in Figures 6.5(a), 6.6(a), and 6.7(a), two consecutive origin-fixed rotation operations, $\boldsymbol{R}_{\boldsymbol{n}}\langle\theta\rangle$ and $\boldsymbol{R}_{\boldsymbol{m}}\langle\varphi\rangle$ are applied to the same 3D object.

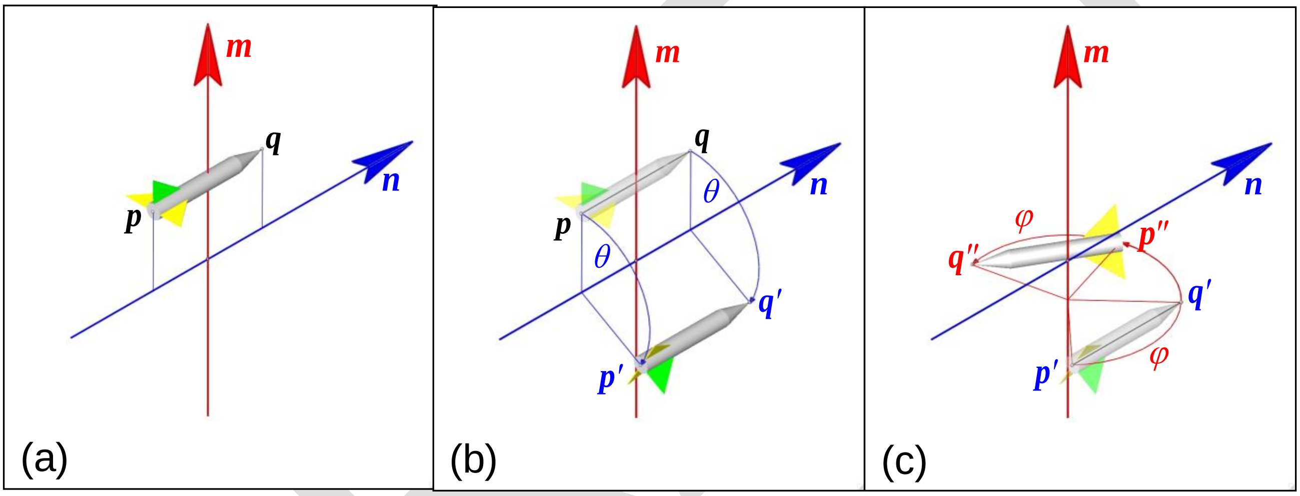

Figure 6.5 — Space-fixed rotation composition convention $\boldsymbol{R}_{\boldsymbol{m}}\langle\boldsymbol{\varphi}\rangle \circ \boldsymbol{R}_{\boldsymbol{n}}\langle\boldsymbol{\theta}\rangle$: (a) initial configuration; (b) after first rotation; (c) after second rotation

Figure 6.5 illustrates the space-fixed convention. Figure 6.5(b) shows the result of the first rotation operation $\boldsymbol{R}_{\boldsymbol{n}}\langle\theta\rangle$, using the space-fixed convention. The rotated points are labeled $\boldsymbol{p}'$ and $\boldsymbol{q}'$ to distinguish them from the original states of the points $\boldsymbol{p}$ and $\boldsymbol{q}$. The rotation operation is applied only to the object. Figure 6.5(c) shows the result of the second rotation operation $\boldsymbol{R}_{\boldsymbol{m}}\langle\varphi\rangle$. The points $\boldsymbol{p}"$ and $\boldsymbol{q}"$ are the final states of the original $\boldsymbol{p}$ and $\boldsymbol{q}$.

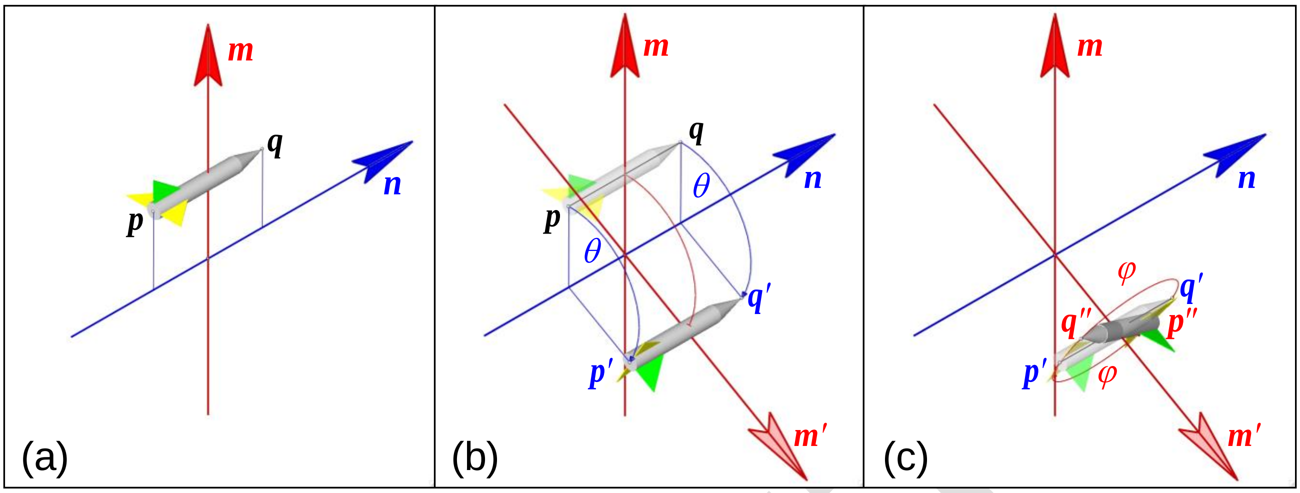

Figure 6.6 — Body-fixed rotation composition convention $\boldsymbol{R}_{\boldsymbol{m}'}\langle\boldsymbol{\varphi}\rangle \circ \boldsymbol{R}_{\boldsymbol{n}}\langle\boldsymbol{\theta}\rangle$: (a) initial configuration; (b) after first rotation; (c) after second rotation

Figure 6.6 illustrates the body-fixed convention. Figure 6.6(b) shows the result of the first rotation operation $\boldsymbol{R}_{\boldsymbol{n}}\langle\theta\rangle$, using the body-fixed convention. The rotation operation is applied to the axis $\boldsymbol{m}$ as well as to the object. The resulting rotated axis is labelled $\boldsymbol{m}'$ to distinguish it from the original state of axis $\boldsymbol{m}$. Figure 6.6(c) shows the result of the second rotation operation $\boldsymbol{R}_{\boldsymbol{m}'}\langle\varphi\rangle$. In general, the body-fixed convention results in a different final state of the object than the space-fixed convention.

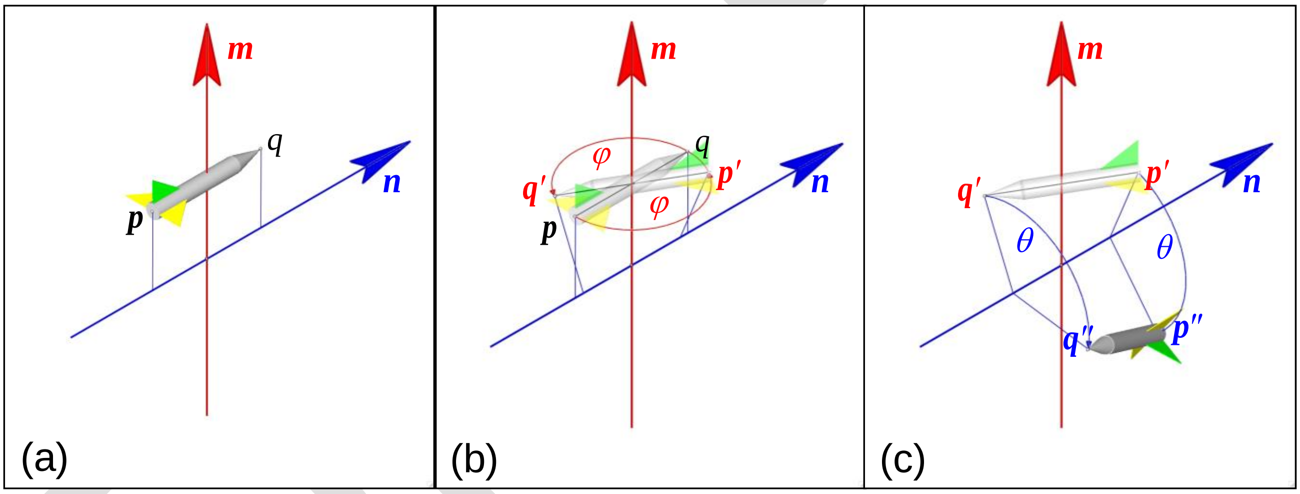

Figure 6.7 — Equivalence of body-fixed $\boldsymbol{R}_{\boldsymbol{m}'}\langle\boldsymbol{\varphi}\rangle \circ \boldsymbol{R}_{\boldsymbol{n}}\langle\boldsymbol{\theta}\rangle$ and space-fixed $\boldsymbol{R}_{\boldsymbol{n}}\langle\boldsymbol{\theta}\rangle \circ \boldsymbol{R}_{\boldsymbol{m}}\langle\boldsymbol{\varphi}\rangle$: (a) initial configuration; (b) after first rotation; (c) after second rotation

Figure 6.7 illustrates the result of applying the two rotation operations in the opposite order, using the space-fixed convention. Figure 6.7(b) shows the result of the first rotation operation $\boldsymbol{R}_{\boldsymbol{m}}\langle\varphi\rangle$, using the space-fixed convention. The rotation operation is applied only to the object. Figure 6.7(c) shows the result of the second rotation operation $\boldsymbol{R}_{\boldsymbol{n}}\langle\theta\rangle$. Although the intermediate states shown in Figures 6.6(b) and 6.7(b) are different, the final states of the object shown in Figures 6.6(c) and 6.7(c) are exactly the same, illustrating that reversing the order of the space-fixed convention rotation operations is equivalent to the body-fixed convention.

NOTE Confusion between the space-fixed and body-fixed conventions is a common source of errors when working with rotation operations.

6.4 Rotation in an orthonormal frame

6.4.1 Rotation of a position-vector

In the earlier subclauses, rotation operators were treated without requiring a coordinate-system or orthonormal frame, specifying only an origin. In this regard, they have been treated as coordinate-frame-independent operations. However, in order to explicitly represent the vector-coordinates of position-vectors in rotation operations, it is necessary to choose an orthonormal frame by specifying a set of basis vectors (see 4.3).

Designating an orthonormal frame based at a given origin allows position-vectors to be represented by vector-coordinate tuples, and allows linear operations to be represented by matrix multiplications of vector-coordinate tuples. This reduction to tuples and matrices is important in many application domains.

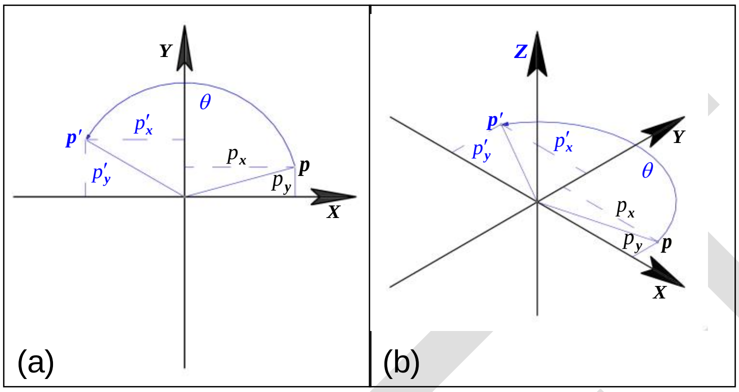

Figure 6.8 — Rotation of a position-vector within an orthonormal frame: (a) top view; (b) isometric view

Given an orthonormal frame $\boldsymbol{E}$ with basis $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$, a position-vector $\boldsymbol{p}$ is represented by the scalar triple $(p_x, p_y, p_z)_{\boldsymbol{E}}$ where the scalars satisfy the equation $\boldsymbol{p} = p_x\boldsymbol{x} + p_y\boldsymbol{y} + p_z\boldsymbol{z}$. Since $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$ is an orthonormal basis, the solution is unique and is given by: $p_x = \boldsymbol{p} \bullet \boldsymbol{x},\ \ p_y = \boldsymbol{p} \bullet \boldsymbol{y},\ \ p_z = \boldsymbol{p} \bullet \boldsymbol{z}$. Thus for any position-vector $\boldsymbol{p}$ and orthonormal basis $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$:

$$\boldsymbol{p} = p_x\boldsymbol{x} + p_y\boldsymbol{y} + p_z\boldsymbol{z} = (\boldsymbol{p} \bullet \boldsymbol{x})\boldsymbol{x} + (\boldsymbol{p} \bullet \boldsymbol{y})\boldsymbol{y} + (\boldsymbol{p} \bullet \boldsymbol{z})\boldsymbol{z}. \tag{6.6}$$

Using Equation **Error! Reference source not found.**, the result of any linear operator $\boldsymbol{L}$ acting on an arbitrary position-vector $\boldsymbol{p}$ is completely determined by the three values that the operator assigns to the basis position vectors:

$$\boldsymbol{L}(\boldsymbol{p}) = p_x\boldsymbol{L}(\boldsymbol{x}) + p_y\boldsymbol{L}(\boldsymbol{y}) + p_z\boldsymbol{L}(\boldsymbol{z}).$$

Thus any linear operator may be represented as a matrix multiplication of vector-coordinates. Vector-coordinates are also necessary for other representations of rotation operations (see 6.x.x) and are otherwise important in many application domains.

The notation $[\boldsymbol{L}]_{\boldsymbol{E}}$ shall denote the matrix representation of the linear operator $\boldsymbol{L}$ operating by matrix multiplication of vector-coordinates in orthonormal frame $\boldsymbol{E}$. The notional subscript is omitted when the relevant frame is otherwise indicated (see 6.x.x).

Given an orthonormal frame $\boldsymbol{E}$ with basis $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$, the origin-fixed rotation operator $\boldsymbol{R_n}\langle\theta\rangle$ applied to a position-vector $\boldsymbol{p}$ produces a rotated position-vector $\boldsymbol{p}'$. The rotated position-vector $\boldsymbol{p}'$ has vector-coordinates $[p_x', p_y', p_z']_{\boldsymbol{E}}$ in frame $\boldsymbol{E}$. Figure 6.8(a) illustrates the rotation of the point $\boldsymbol{p}$ through an angle θ about the $\boldsymbol{z}$-axis (which points out of the page), yielding the rotated point $\boldsymbol{p}'$. Figure 6.8(b) shows an isometric view of the same rotation operation.

The rotation operation $\boldsymbol{R_n}\langle\theta\rangle$, where the rotation axis $\boldsymbol{n}$ has vector-coordinates $[n_x, n_y, n_z]_{\boldsymbol{E}}$ in the orthonormal frame $\boldsymbol{E}$, can be represented as a rotation matrix multiplication $\boldsymbol{p}_{\boldsymbol{E}}' = [\boldsymbol{R_n}\langle\theta\rangle]_{\boldsymbol{E}}\boldsymbol{p}_{\boldsymbol{E}}$. The matrix form of Rodrigues' rotation formula (Equation (6.2)) is:

(6.7)

$$= [\cos(\theta)\, \boldsymbol{I}_{3\times3} + (1-\cos(\theta))\boldsymbol{n} \otimes \boldsymbol{n} + \sin(\theta)\, \boldsymbol{S_n}]$$

where:

$\boldsymbol{S_n} = \begin{bmatrix} 0 & -n_z & n_y \\ n_z & 0 & -n_x \\ -n_y & n_x & 0 \end{bmatrix}$ is the skew-symmetric matrix associated with $\boldsymbol{n} = [n_x \quad n_y \quad n_z]^{\mathrm{T}}$ and

$\boldsymbol{n} \otimes \boldsymbol{n} = \begin{bmatrix} n_x n_x & n_x n_y & n_x n_z \\ n_y n_x & n_y n_y & n_y n_z \\ n_z n_x & n_z n_y & n_z n_z \end{bmatrix}$ is the outer-product of $\boldsymbol{n}$ with $\boldsymbol{n}$.

Expanding the matrix elements yields:

$$[\boldsymbol{R_n}\langle\theta\rangle] = \cos\theta \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + (1-\cos\theta) \begin{bmatrix} n_x n_x & n_x n_y & n_x n_z \\ n_y n_x & n_y n_y & n_y n_z \\ n_z n_x & n_z n_y & n_z n_z \end{bmatrix} + \sin\theta \begin{bmatrix} 0 & -n_z & n_y \\ n_z & 0 & -n_x \\ -n_y & n_x & 0 \end{bmatrix}$$

$$= \begin{bmatrix} (1-\cos\theta)n_x^2 + \cos\theta & (1-\cos\theta)n_x n_y - n_z \sin\theta & (1-\cos\theta)n_x n_z + n_y \sin\theta \\ (1-\cos\theta)n_y n_x + n_z \sin\theta & (1-\cos\theta)n_y^2 + \cos\theta & (1-\cos\theta)n_y n_z - n_x \sin\theta \\ (1-\cos\theta)n_z n_x - n_y \sin\theta & (1-\cos\theta)n_z n_y + n_x \sin\theta & (1-\cos\theta)n_z^2 + \cos\theta \end{bmatrix} \quad (6.8)$$

NOTE In general, the matrix representation of a linear operator such as $\boldsymbol{R_n}\langle\theta\rangle$ depends on the selection of a basis. In the case of Equation (6.8) the matrix coefficient values depend on the vector-coordinate values of the rotation axis $\boldsymbol{n}$. Replacing orthonormal frame $\boldsymbol{E}$ with a different orthonormal frame that uses the same origin will yield different vector-coordinate values for the rotation axis $\boldsymbol{n}$, and therefore a different matrix.

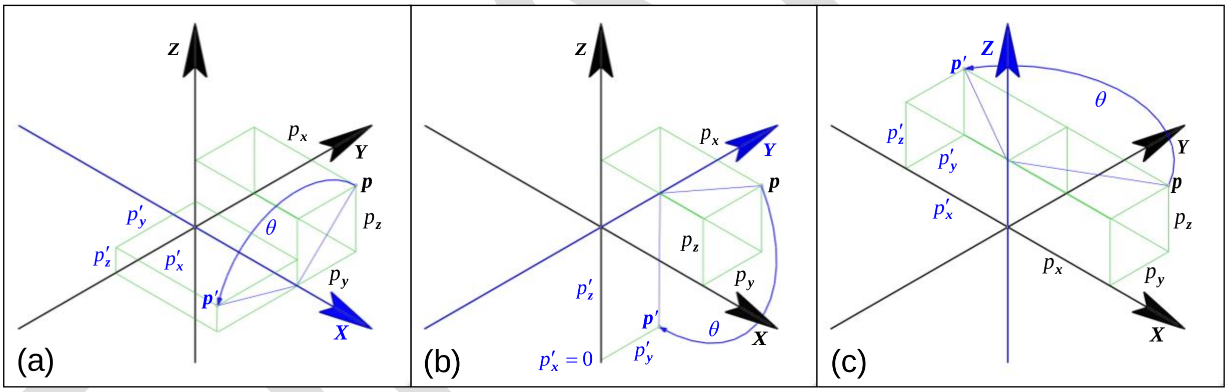

Figure 6.9 — Principal axis rotations: (a) about x-axis; (b) about y-axis; (c) about z-axis

EXAMPLE 1 Given an orthonormal frame $\boldsymbol{E}$ with basis $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$, the matrix for a rotation about the $\boldsymbol{z}$-axis $\boldsymbol{R_z}\langle\theta\rangle$ as shown in Figure 6.9(c) may be determined by replacing $(n_x, n_y, n_z)_{\boldsymbol{E}}$ in Equation (6.8) with the vector-coordinates of the $\boldsymbol{z}$-axis basis vector, i.e., $(0, 0, 1)$, yielding:

$$[\boldsymbol{R_z}\langle\theta\rangle] = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and so } \begin{bmatrix} p_x' \\ p_y' \\ p_z' \end{bmatrix} = \begin{bmatrix} p_x \cos\theta - p_y \sin\theta \\ p_x \sin\theta + p_y \cos\theta \\ p_z \end{bmatrix}$$

This is the 3D form of the familiar 2D rotation matrix. Similarly, the matrices for rotations about the $\boldsymbol{x}$-axis and $\boldsymbol{y}$-axis, i.e, $\boldsymbol{R_x}\langle\theta\rangle$ and $\boldsymbol{R_y}\langle\theta\rangle$ as shown in Figure 6.9(a) and (b) respectively may be determined by replacing $(n_x, n_y, n_z)_{\boldsymbol{E}}$ in Equation (6.8) with the vector-coordinates of the $\boldsymbol{x}$-axis basis vector $(1, 0, 0)$ and the vector-coordinates of the $\boldsymbol{y}$-axis basis vector $(0, 1, 0)$, yielding:

$$[\boldsymbol{R_x}\langle\theta\rangle] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \text{ and so } \begin{bmatrix} p_x' \\ p_y' \\ p_z' \end{bmatrix} = \begin{bmatrix} p_x \\ p_y \cos\theta - p_z \sin\theta \\ p_y \sin\theta + p_z \cos\theta \end{bmatrix}$$

$$[\boldsymbol{R}_{\boldsymbol{y}}\langle\theta\rangle] = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \text{ and so } \begin{bmatrix} p_x^{'} \\ p_y^{'} \\ p_z^{'} \end{bmatrix} = \begin{bmatrix} p_x\cos\theta + p_z\sin\theta \\ p_y \\ -p_x\sin\theta + p_z\cos\theta \end{bmatrix}$$

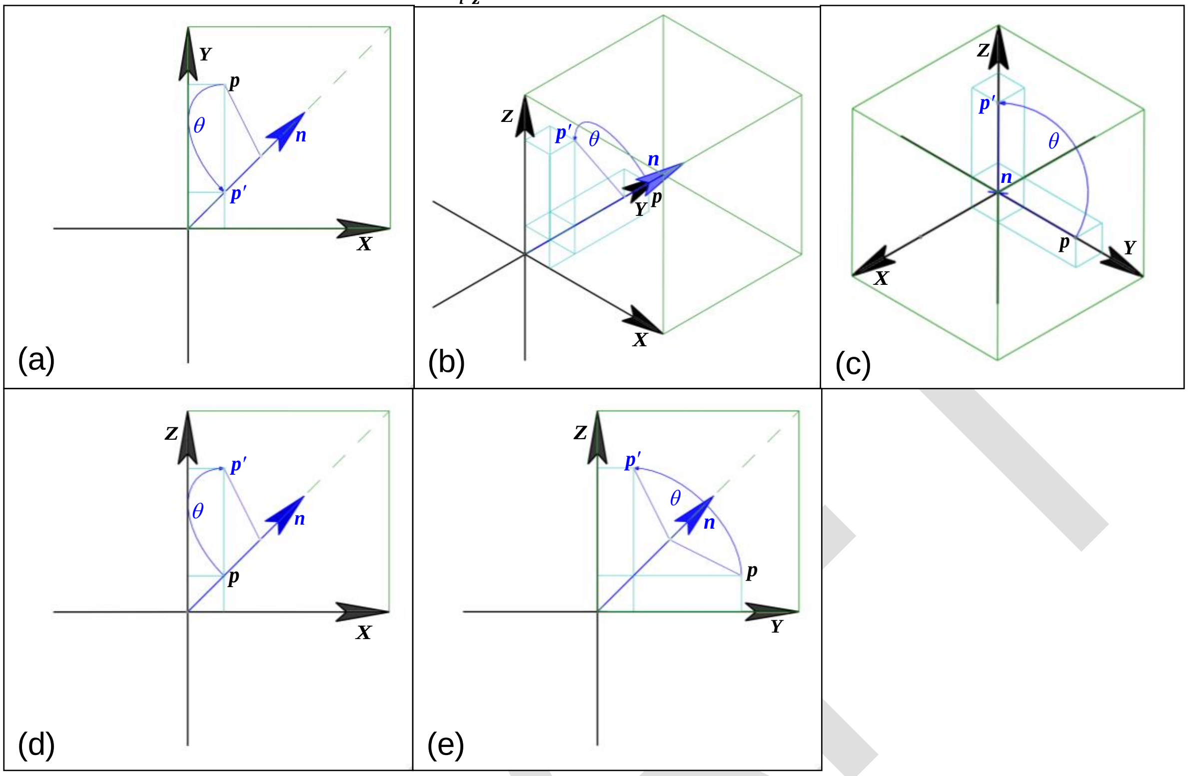

Figure 6.10 — Rotation of a position-vector about the unit cube diagonal: (a) top, (b) isometric, (c) along rotation axis toward origin, (d) front and (e) side views

EXAMPLE 2 Figure 6.10 illustrates a rotation operation where the rotation axis is along the diagonal of the unit cube of the orthonormal frame $\boldsymbol{E}$, i.e., $\boldsymbol{n}_{\boldsymbol{E}} = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix}^{\mathrm{T}}$, and the rotation angle is $\theta = 120°$. Therefore $\sin\theta = \frac{\sqrt{3}}{2}$ and $\cos\theta = \frac{-1}{2}$, and the rotation matrix is:

$$[\boldsymbol{R}_{\boldsymbol{n}}\langle\theta\rangle] = \begin{bmatrix} \left(\frac{3}{2}\right)\left(\frac{1}{3}\right) + \left(\frac{-1}{2}\right) & \left(\frac{3}{2}\right)\left(\frac{1}{3}\right) - \left(\frac{1}{\sqrt{3}}\right)\left(\frac{\sqrt{3}}{2}\right) & \left(\frac{3}{2}\right)\left(\frac{1}{3}\right) + \left(\frac{1}{\sqrt{3}}\right)\left(\frac{\sqrt{3}}{2}\right) \\ \left(\frac{3}{2}\right)\left(\frac{1}{3}\right) + \left(\frac{1}{\sqrt{3}}\right)\left(\frac{\sqrt{3}}{2}\right) & \left(\frac{3}{2}\right)\left(\frac{1}{3}\right) + \left(\frac{-1}{2}\right) & \left(\frac{3}{2}\right)\left(\frac{1}{3}\right) - \left(\frac{1}{\sqrt{3}}\right)\left(\frac{\sqrt{3}}{2}\right) \\ \left(\frac{3}{2}\right)\left(\frac{1}{3}\right) - \left(\frac{1}{\sqrt{3}}\right)\left(\frac{\sqrt{3}}{2}\right) & \left(\frac{3}{2}\right)\left(\frac{1}{3}\right) + \left(\frac{1}{\sqrt{3}}\right)\left(\frac{\sqrt{3}}{2}\right) & \left(\frac{3}{2}\right)\left(\frac{1}{3}\right) + \left(\frac{-1}{2}\right) \end{bmatrix}, \text{ or}$$

$$[\boldsymbol{R}_{\boldsymbol{n}}\langle\theta\rangle] = \begin{bmatrix} \left(\frac{1}{2}\right) + \left(\frac{-1}{2}\right) & \left(\frac{1}{2}\right) - \left(\frac{1}{2}\right) & \left(\frac{1}{2}\right) + \left(\frac{1}{2}\right) \\ \left(\frac{1}{2}\right) + \left(\frac{1}{2}\right) & \left(\frac{1}{2}\right) + \left(\frac{-1}{2}\right) & \left(\frac{1}{2}\right) - \left(\frac{1}{2}\right) \\ \left(\frac{1}{2}\right) - \left(\frac{1}{2}\right) & \left(\frac{1}{2}\right) + \left(\frac{1}{2}\right) & \left(\frac{1}{2}\right) + \left(\frac{-1}{2}\right) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \text{ and so}$$

$$\begin{pmatrix} p_x^{'} \\ p_y^{'} \\ p_z^{'} \end{pmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix} \text{ or } \begin{pmatrix} p_x^{'} \\ p_y^{'} \\ p_z^{'} \end{pmatrix} = \begin{pmatrix} p_z \\ p_x \\ p_y \end{pmatrix}$$

This operation "rotates" the coordinate components of the position-vector $\boldsymbol{p}$, so that the $\boldsymbol{x}$-component of $\boldsymbol{p}$ becomes the $\boldsymbol{y}$-component of $\boldsymbol{p}'$, the $\boldsymbol{y}$-component of $\boldsymbol{p}$ becomes the $\boldsymbol{z}$-component of $\boldsymbol{p}'$, and the $\boldsymbol{z}$-component of $\boldsymbol{p}$ becomes the $\boldsymbol{x}$-component of $\boldsymbol{p}'$.

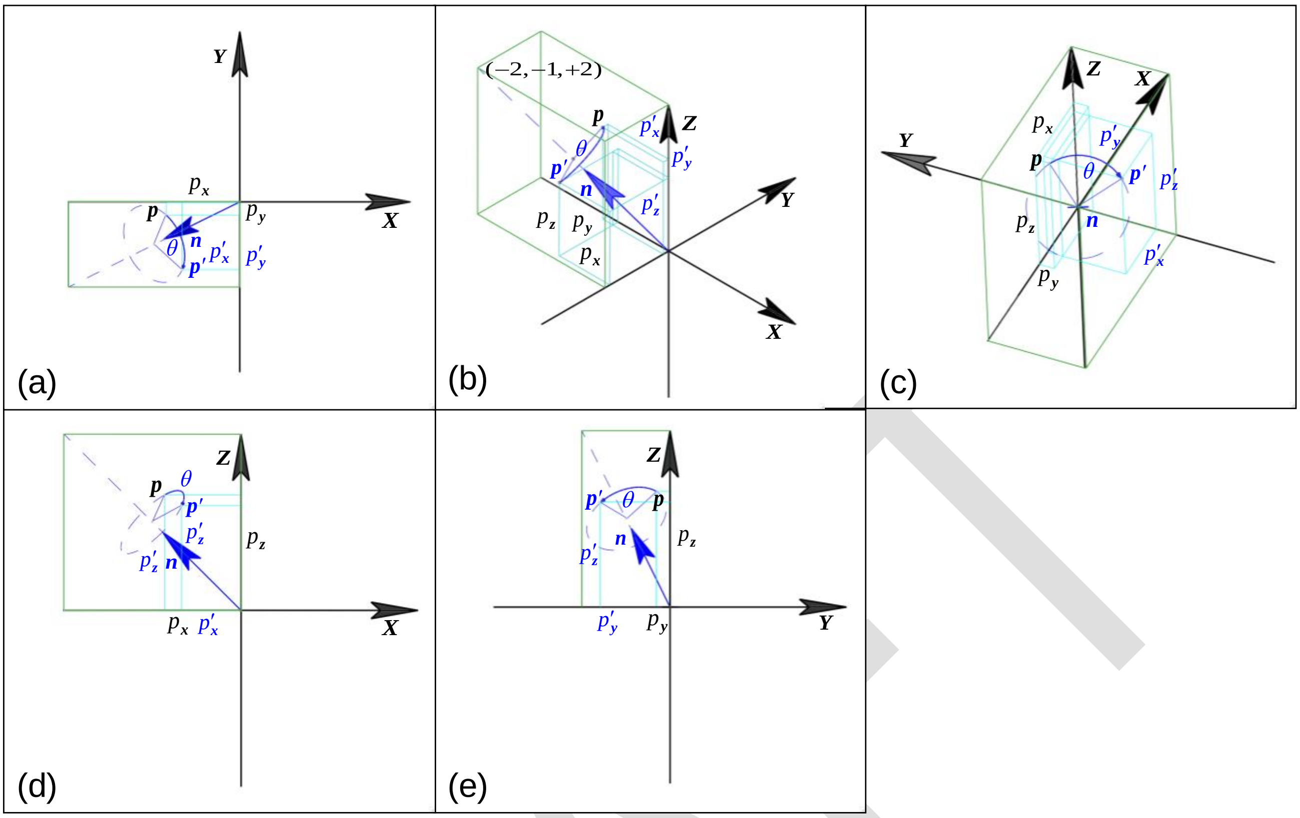

Figure 6.11 — Rotation of a position-vector about an asymmetric axis: (a) top, (b) isometric, (c) along rotation axis toward origin, (d) front and (e) side views

EXAMPLE 3 Figure 6.11 illustrates a rotation operation where the rotation axis passes through the point (-2, -1, 2) of the orthonormal frame $\boldsymbol{E}$, i.e., $\boldsymbol{n_E} = \begin{pmatrix} \frac{-2}{3} & \frac{-1}{3} & \frac{2}{3} \end{pmatrix}^{\mathrm{T}}$, and the rotation angle is $\theta = -90°$. Therefore $\sin\theta = -1$ and $\cos\theta = 0$, and the rotation matrix is:

$$[\boldsymbol{R_n}\langle\theta\rangle] = \begin{bmatrix} (1)\left(\frac{-2}{3}\right)\left(\frac{-2}{3}\right) + 0 & (1)\left(\frac{-2}{3}\right)\left(\frac{-1}{3}\right) - \left(\frac{2}{3}\right)(-1) & (1)\left(\frac{-2}{3}\right)\left(\frac{2}{3}\right) + \left(\frac{-1}{3}\right)(-1) \\ (1)\left(\frac{-1}{3}\right)\left(\frac{-2}{3}\right) + \left(\frac{2}{3}\right)(-1) & (1)\left(\frac{-1}{3}\right)\left(\frac{-1}{3}\right) + 0 & (1)\left(\frac{-1}{3}\right)\left(\frac{2}{3}\right) - \left(\frac{-2}{3}\right)(-1) \\ (1)\left(\frac{2}{3}\right)\left(\frac{-2}{3}\right) - \left(\frac{-1}{3}\right)(-1) & (1)\left(\frac{2}{3}\right)\left(\frac{-1}{3}\right) + \left(\frac{-2}{3}\right)(-1) & (1)\left(\frac{2}{3}\right)\left(\frac{2}{3}\right) + 0 \end{bmatrix}$$

$$[\boldsymbol{R_n}\langle\theta\rangle] = \begin{bmatrix} \left(\frac{4}{9}\right) & \left(\frac{2}{9}\right) - \left(\frac{-2}{3}\right) & \left(\frac{-4}{9}\right) + \left(\frac{1}{3}\right) \\ \left(\frac{2}{9}\right) + \left(\frac{-2}{3}\right) & \left(\frac{1}{9}\right) & \left(\frac{-2}{9}\right) - \left(\frac{2}{3}\right) \\ \left(\frac{-4}{9}\right) - \left(\frac{1}{3}\right) & \left(\frac{-2}{9}\right) + \left(\frac{2}{3}\right) & \left(\frac{4}{9}\right) \end{bmatrix} = \begin{bmatrix} \left(\frac{4}{9}\right) & \left(\frac{8}{9}\right) & \left(\frac{-1}{9}\right) \\ \left(\frac{-4}{9}\right) & \left(\frac{1}{9}\right) & \left(\frac{-8}{9}\right) \\ \left(\frac{-7}{9}\right) & \left(\frac{4}{9}\right) & \left(\frac{4}{9}\right) \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 4 & 8 & -1 \\ -4 & 1 & -8 \\ -7 & 4 & 4 \end{bmatrix}$$

$$\begin{pmatrix} p_x' \\ p_y' \\ p_z' \end{pmatrix} = \frac{1}{9}\begin{bmatrix} 4 & 8 & -1 \\ -4 & 1 & -8 \\ -7 & 4 & 4 \end{bmatrix}\begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix}$$

6.4.2 Rotation of basis vectors

Given an orthonormal frame $\boldsymbol{E}$ with basis $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$, the origin-fixed rotation operator $\boldsymbol{R_n}\langle\theta\rangle$ may be applied to each of its basis vectors, i.e.:

$$\boldsymbol{x}' = \boldsymbol{R_n}\langle\theta\rangle(\boldsymbol{x}),$$
$$\boldsymbol{y}' = \boldsymbol{R_n}\langle\theta\rangle(\boldsymbol{y}), \quad (6.9)$$

$z' = R_n\langle\theta\rangle(z)$. and

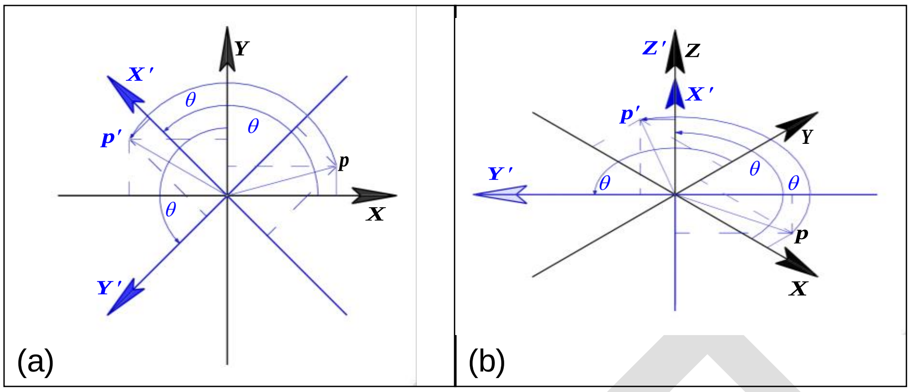

Figure 6.12 — Rotation of the basis vectors in the same direction as the position-vector: (a) top view; (b) isometric view

Figure 6.12(a) and (b) illustrate the x- and y-axes of the orthonormal frame E being rotated about the z-axis through the same angle θ, in the same direction, as the position-vector p, yielding the rotated basis vectors $x' = R_z\langle\theta\rangle(x)$, $y' = R_z\langle\theta\rangle(y)$, and $z' = z = R_z\langle\theta\rangle(z)$.

Applying the matrix multiplication of Equation 6.8 to the basis vectors of the frame E yields the coordinate components of the rotated basis vectors with respect to frame E.

$$x'_E = [R_n\langle\theta\rangle]x_E$$

$$= \begin{bmatrix} (1-\cos\theta)n_x^2 + \cos\theta & (1-\cos\theta)n_x n_y - n_z\sin\theta & (1-\cos\theta)n_x n_z + n_y\sin\theta \\ (1-\cos\theta)n_y n_x + n_z\sin\theta & (1-\cos\theta)n_y^2 + \cos\theta & (1-\cos\theta)n_y n_z - n_x\sin\theta \\ (1-\cos\theta)n_z n_x - n_y\sin\theta & (1-\cos\theta)n_z n_y + n_x\sin\theta & (1-\cos\theta)n_z^2 + \cos\theta \end{bmatrix}\begin{bmatrix}1\\0\\0\end{bmatrix}$$

$$x'_E = \begin{bmatrix} (1-\cos\theta)n_x^2 + \cos\theta \\ (1-\cos\theta)n_y n_x + n_z\sin\theta \\ (1-\cos\theta)n_z n_x - n_y\sin\theta \end{bmatrix}$$

The rotated basis vector x' is represented by the first column of the matrix $[R_n\langle\theta\rangle]$. Similarly, the rotated basis vectors y' and z' are represented by the second and third columns of the matrix $[R_n\langle\theta\rangle]$, respectively.

$$y'_E = [R_n\langle\theta\rangle]y_E$$

$$= \begin{bmatrix} (1-\cos\theta)n_x^2 + \cos\theta & (1-\cos\theta)n_x n_y - n_z\sin\theta & (1-\cos\theta)n_x n_z + n_y\sin\theta \\ (1-\cos\theta)n_y n_x + n_z\sin\theta & (1-\cos\theta)n_y^2 + \cos\theta & (1-\cos\theta)n_y n_z - n_x\sin \\ (1-\cos\theta)n_z n_x - n_y\sin\theta & (1-\cos\theta)n_z n_y + n_x\sin\theta & (1-\cos\theta)n_z^2 + \cos\theta \end{bmatrix}\begin{bmatrix}0\\1\\0\end{bmatrix}$$

$$y'_E = \begin{bmatrix} (1-\cos\theta)n_x n_y - n_z\sin\theta \\ (1-\cos\theta)n_y^2 + \cos\theta \\ (1-\cos\theta)n_z n_y + n_x\sin\theta \end{bmatrix}$$

$$z'_E = [R_n\langle\theta\rangle]z_E$$

$$\boldsymbol{z}'_{\boldsymbol{E}} = \begin{bmatrix} (1-\cos\theta)n_x n_z + n_y \sin\theta \\ (1-\cos\theta)n_y n_z - n_x \sin\theta \\ (1-\cos\theta)n_z^2 + \cos\theta \end{bmatrix}$$

The original position-vector $\boldsymbol{p}$ and the rotated position-vector $\boldsymbol{p}'$ each can be represented by vector-coordinates in terms of the original basis vectors $\boldsymbol{x}$, $\boldsymbol{y}$, $\boldsymbol{z}$, and can also be represented by different vector-coordinates with respect to the rotated basis vectors $\boldsymbol{x}'$, $\boldsymbol{y}'$, $\boldsymbol{z}'$. The original position-vector $\boldsymbol{p}$ has vector-coordinate $(p_x, p_y, p_z)_{\boldsymbol{E}}$, in terms of the original basis vectors $\boldsymbol{x}$, $\boldsymbol{y}$, $\boldsymbol{z}$, with respect to frame $\boldsymbol{E}$, and also has vector-coordinate $\left(p_{x'}, p_{y'}, p_{z'}\right)_{\boldsymbol{E}}$, in terms of the rotated basis vectors $\boldsymbol{x}'$, $\boldsymbol{y}'$, $\boldsymbol{z}'$, also with respect to frame $\boldsymbol{E}$. The rotated position-vector $\boldsymbol{p}'$ has vector-coordinate $(p'_x, p'_y, p'_z)_{\boldsymbol{E}}$, in terms of the original basis vectors $\boldsymbol{x}$, $\boldsymbol{y}$, $\boldsymbol{z}$, with respect to frame $\boldsymbol{E}$, and also has vector-coordinate $\left(p'_{x'}, p'_{y'}, p'_{z'}\right)_{\boldsymbol{E}}$, in terms of the rotated basis vectors $\boldsymbol{x}'$, $\boldsymbol{y}'$, $\boldsymbol{z}'$, also with respect to frame $\boldsymbol{E}$.

Because $\boldsymbol{p}$ is a position-vector, the sum of the products of each of its coordinate-components with the corresponding basis vectors is invariant. Thus, $\boldsymbol{p} = p_x\boldsymbol{x} + p_y\boldsymbol{y} + p_z\boldsymbol{z} = p_{x'}\boldsymbol{x}' + p_{y'}\boldsymbol{y}' + p_{z'}\boldsymbol{z}'$. Similarly, because $\boldsymbol{p}'$ is a position-vector, $\boldsymbol{p}' = p'_x\boldsymbol{x} + p'_y\boldsymbol{y} + p'_z\boldsymbol{z} = p'_{x'}\boldsymbol{x}' + p'_{y'}\boldsymbol{y}' + p'_{z'}\boldsymbol{z}'$.

Given $\boldsymbol{p} = p_x\boldsymbol{x} + p_y\boldsymbol{y} + p_z\boldsymbol{z}$, then

$$\boldsymbol{p}' = \boldsymbol{R_n}\langle\theta\rangle(\boldsymbol{p}) = p_x\boldsymbol{R_n}\langle\theta\rangle(\boldsymbol{x}) + p_y\boldsymbol{R_n}\langle\theta\rangle(\boldsymbol{y}) + p_z\boldsymbol{R_n}\langle\theta\rangle(\boldsymbol{z}) = p_x\boldsymbol{x}' + p_y\boldsymbol{y}' + p_z\boldsymbol{z}'.$$

The rotated position-vector $\boldsymbol{p}'$ has the same vector-coordinate component values $\left(p'_{x'}, p'_{y'}, p'_{z'}\right)_{\boldsymbol{E}}$ in terms of the rotated basis vectors $\boldsymbol{x}'$, $\boldsymbol{y}'$, $\boldsymbol{z}'$, as the original position-vector $\boldsymbol{p}$ has in terms of the original basis vectors $\boldsymbol{x}$, $\boldsymbol{y}$, $\boldsymbol{z}$, i.e., $(p_x, p_y, p_z)_{\boldsymbol{E}}$. Thus, $p'_{x'} = p_x$, $p'_{y'} = p_{y'}$ and $p'_{z'} = p_z$.

Using Equation **Error! Reference source not found.**, and substituting the expression above for $\boldsymbol{p}'$, the vector-coordinate components of $\boldsymbol{p}'$ are:

$$p'_x = \boldsymbol{p}' \bullet \boldsymbol{x} = \left(p_x\boldsymbol{x}' + p_y\boldsymbol{y}' + p_z\boldsymbol{z}'\right) \bullet \boldsymbol{x} = p_x\boldsymbol{x}' \bullet \boldsymbol{x} + p_y\boldsymbol{y}' \bullet \boldsymbol{x} + p_z\boldsymbol{z}' \bullet \boldsymbol{x},$$

$$p'_y = \boldsymbol{p}' \bullet \boldsymbol{y} = \left(p_x\boldsymbol{x}' + p_y\boldsymbol{y}' + p_z\boldsymbol{z}'\right) \bullet \boldsymbol{y} = p_x\boldsymbol{x}' \bullet \boldsymbol{y} + p_y\boldsymbol{y}' \bullet \boldsymbol{y} + p_z\boldsymbol{z}' \bullet \boldsymbol{y},$$

$$p'_z = \boldsymbol{p}' \bullet \boldsymbol{z} = \left(p_x\boldsymbol{x}' + p_y\boldsymbol{y}' + p_z\boldsymbol{z}'\right) \bullet \boldsymbol{z} = p_x\boldsymbol{x}' \bullet \boldsymbol{z} + p_y\boldsymbol{y}' \bullet \boldsymbol{z} + p_z\boldsymbol{z}' \bullet \boldsymbol{z}.$$

The matrix form of these three equations is:

$$\begin{bmatrix} p'_x \\ p'_y \\ p'_z \end{bmatrix}_{\boldsymbol{E}} = [\boldsymbol{R}]_{\boldsymbol{E}} \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix}_{\boldsymbol{E}} \quad \text{where:} \quad [\boldsymbol{R}]_{\boldsymbol{E}} = \begin{bmatrix} \boldsymbol{x}' \bullet \boldsymbol{x} & \boldsymbol{y}' \bullet \boldsymbol{x} & \boldsymbol{z}' \bullet \boldsymbol{x} \\ \boldsymbol{x}' \bullet \boldsymbol{y} & \boldsymbol{y}' \bullet \boldsymbol{y} & \boldsymbol{z}' \bullet \boldsymbol{y} \\ \boldsymbol{x}' \bullet \boldsymbol{z} & \boldsymbol{y}' \bullet \boldsymbol{z} & \boldsymbol{z}' \bullet \boldsymbol{z} \end{bmatrix} \tag{6.10}$$

The matrix $[\boldsymbol{R}]_{\boldsymbol{E}}$ is a representation of the rotation operator $\boldsymbol{R_n}\langle\theta\rangle$ with respect to orthonormal frame $\boldsymbol{E}$ with basis $\boldsymbol{x}$, $\boldsymbol{y}$, $\boldsymbol{z}$, expressed in terms of the relationships between the original basis vectors $\boldsymbol{x}$, $\boldsymbol{y}$, $\boldsymbol{z}$ and the rotated basis vectors $\boldsymbol{x}'$, $\boldsymbol{y}'$, $\boldsymbol{z}'$.

Evaluating the dot products in each of the elements of this matrix yields:

$$[\boldsymbol{R}]_{\boldsymbol{E}} = \begin{bmatrix} \begin{bmatrix} (1-\cos\theta)n_x^2+\cos\theta \\ (1-\cos\theta)n_yn_x+n_z\sin\theta \\ (1-\cos\theta)n_zn_x-n_y\sin\theta \end{bmatrix} \bullet \begin{bmatrix}1\\0\\0\end{bmatrix} & \begin{bmatrix} (1-\cos\theta)n_xn_y-n_z\sin\theta \\ (1-\cos\theta)n_y^2+\cos\theta \\ (1-\cos\theta)n_zn_y+n_x\sin\theta \end{bmatrix} \bullet \begin{bmatrix}1\\0\\0\end{bmatrix} & \begin{bmatrix} (1-\cos\theta)n_xn_z+n_y\sin\theta \\ (1-\cos\theta)n_yn_z-n_x\sin\theta \\ (1-\cos\theta)n_z^2+\cos\theta \end{bmatrix} \bullet \begin{bmatrix}1\\0\\0\end{bmatrix} \\ \begin{bmatrix} (1-\cos\theta)n_x^2+\cos\theta \\ (1-\cos\theta)n_yn_x+n_z\sin\theta \\ (1-\cos\theta)n_zn_x-n_y\sin\theta \end{bmatrix} \bullet \begin{bmatrix}0\\1\\0\end{bmatrix} & \begin{bmatrix} (1-\cos\theta)n_xn_y-n_z\sin\theta \\ (1-\cos\theta)n_y^2+\cos\theta \\ (1-\cos\theta)n_zn_y+n_x\sin\theta \end{bmatrix} \bullet \begin{bmatrix}0\\1\\0\end{bmatrix} & \begin{bmatrix} (1-\cos\theta)n_xn_z+n_y\sin\theta \\ (1-\cos\theta)n_yn_z-n_x\sin\theta \\ (1-\cos\theta)n_z^2+\cos\theta \end{bmatrix} \bullet \begin{bmatrix}0\\1\\0\end{bmatrix} \\ \begin{bmatrix} (1-\cos\theta)n_x^2+\cos\theta \\ (1-\cos\theta)n_yn_x+n_z\sin \\ (1-\cos\theta)n_zn_x-n_y\sin\theta \end{bmatrix} \bullet \begin{bmatrix}0\\0\\1\end{bmatrix} & \begin{bmatrix} (1-\cos\theta)n_xn_y-n_z\sin\theta \\ (1-\cos\theta)n_y^2+\cos\theta \\ (1-\cos\theta)n_zn_y+n_x\sin\theta \end{bmatrix} \bullet \begin{bmatrix}0\\0\\1\end{bmatrix} & \begin{bmatrix} (1-\cos\theta)n_xn_z+n_y\sin\theta \\ (1-\cos\theta)n_yn_z-n_x\sin\theta \\ (1-\cos\theta)n_z^2+\cos\theta \end{bmatrix} \bullet \begin{bmatrix}0\\0\\1\end{bmatrix} \end{bmatrix}$$

$$[\boldsymbol{R}]_{\boldsymbol{E}} = \begin{bmatrix} (1-\cos\theta)n_x^2+\cos\theta & (1-\cos\theta)n_xn_y-n_z\sin\theta & (1-\cos\theta)n_xn_z+n_y\sin\theta \\ (1-\cos\theta)n_yn_x+n_z\sin\theta & (1-\cos\theta)n_y^2+\cos\theta & (1-\cos\theta)n_yn_z-n_x\sin\theta \\ (1-\cos\theta)n_zn_x-n_y\sin\theta & (1-\cos\theta)n_zn_y+n_x\sin\theta & (1-\cos\theta)n_z^2+\cos\theta \end{bmatrix} = [\boldsymbol{R}_{\boldsymbol{n}}\langle\theta\rangle]$$

Thus, the rotation operator $\boldsymbol{R}_{\boldsymbol{n}}\langle\theta\rangle$, where the rotation axis $\boldsymbol{n}$ has vector-coordinates $(n_1, n_2, n_3)_{\boldsymbol{E}}$ in the orthonormal frame $\boldsymbol{E}$ with basis $\boldsymbol{x}$, $\boldsymbol{y}$, $\boldsymbol{z}$, can be represented as a matrix multiplication $\boldsymbol{p}_{\boldsymbol{E}}^{'} = [\boldsymbol{R}]_{\boldsymbol{E}}\boldsymbol{p}_{\boldsymbol{E}}$, where $[\boldsymbol{R}]_{\boldsymbol{E}} = [\boldsymbol{R}_{\boldsymbol{n}}\langle\theta\rangle]$. The matrix $[\boldsymbol{R}]_{\boldsymbol{E}}$ represents the rotation relationships between the original basis vectors $\boldsymbol{x}$, $\boldsymbol{y}$, $\boldsymbol{z}$ of the frame $\boldsymbol{E}$ and the rotated basis vectors $\boldsymbol{x}'$, $\boldsymbol{y}'$, $\boldsymbol{z}'$. Each element of the matrix $[\boldsymbol{R}]_{\boldsymbol{E}}$ represents the rotation angle from the original basis vector that corresponds to the matrix row to the rotated basis vector that corresponds to the matrix column.

EXAMPLE 1 Figure 6.12 illustrates a rotation operation where the rotation axis is the $\boldsymbol{z}$-axis the orthonormal frame $\boldsymbol{E}$, i.e., $\boldsymbol{n}_{\boldsymbol{E}} = [0 \quad 0 \quad 1]^{\mathrm{T}}$, and the rotation angle is $\theta = 135°$. Therefore $\sin\theta = \frac{1}{\sqrt{2}}$ and $\cos\theta = \frac{-1}{\sqrt{2}}$, and the rotation matrix is:

$$[\boldsymbol{R}_{\boldsymbol{n}}\langle\theta\rangle] = \begin{bmatrix} (1-\cos\theta)n_x^2+\cos\theta & (1-\cos\theta)n_xn_y-n_z\sin\theta & (1-\cos\theta)n_xn_z+n_y\sin\theta \\ (1-\cos\theta)n_yn_x+n_z\sin\theta & (1-\cos\theta)n_y^2+\cos\theta & (1-\cos\theta)n_yn_z-n_x\sin\theta \\ (1-\cos\theta)n_zn_x-n_y\sin\theta & (1-\cos\theta)n_zn_y+n_x\sin\theta & (1-\cos\theta)n_z^2+\cos\theta \end{bmatrix} = \begin{bmatrix} \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Applying this rotation operation to the basis vectors $\boldsymbol{x}$, $\boldsymbol{y}$, $\boldsymbol{z}$ of the frame $\boldsymbol{E}$ yields:

$$\boldsymbol{x}_{\boldsymbol{E}}^{'} = \begin{bmatrix} (1-\cos\theta)n_x^2+\cos\theta \\ (1-\cos\theta)n_yn_x+n_z\sin\theta \\ (1-\cos\theta)n_zn_x-n_y\sin\theta \end{bmatrix} = \begin{bmatrix} \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}$$

$$\boldsymbol{y}_{\boldsymbol{E}}^{'} = \begin{bmatrix} (1-\cos\theta)n_xn_y-n_z\sin\theta \\ (1-\cos\theta)n_y^2+\cos\theta \\ (1-\cos\theta)n_zn_y+n_x\sin\theta \end{bmatrix} = \begin{bmatrix} \frac{-1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \\ 0 \end{bmatrix}$$

$$\boldsymbol{z}_{\boldsymbol{E}}^{'} = \begin{bmatrix} (1-\cos\theta)n_xn_z+n_y\sin\theta \\ (1-\cos\theta)n_yn_z-n_x\sin\theta \\ (1-\cos\theta)n_z^2+\cos\theta \end{bmatrix} = \begin{bmatrix}0\\0\\1\end{bmatrix} = \boldsymbol{z}_{\boldsymbol{E}}$$

$$[\boldsymbol{R}]_E = \begin{bmatrix} \boldsymbol{x}_E' \bullet \boldsymbol{x}_E & \boldsymbol{y}_E' \bullet \boldsymbol{x}_E & \boldsymbol{z}_E' \bullet \boldsymbol{x}_E \\ \boldsymbol{x}_E' \bullet \boldsymbol{y}_E & \boldsymbol{y}_E' \bullet \boldsymbol{y}_E & \boldsymbol{z}_E' \bullet \boldsymbol{y}_E \\ \boldsymbol{x}_E' \bullet \boldsymbol{z}_E & \boldsymbol{y}_E' \bullet \boldsymbol{z}_E & \boldsymbol{z}_E' \bullet \boldsymbol{z}_E \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix} \bullet \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} & \begin{bmatrix} \frac{-1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \\ 0 \end{bmatrix} \bullet \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \bullet \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix} \bullet \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} & \begin{bmatrix} \frac{-1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \\ 0 \end{bmatrix} \bullet \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \bullet \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \\ \begin{bmatrix} \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix} \bullet \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} & \begin{bmatrix} \frac{-1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \\ 0 \end{bmatrix} \bullet \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \bullet \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Evaluating the matrix $[\boldsymbol{R}]_E$ shows that it is identical to the matrix $[\boldsymbol{R}_{\boldsymbol{n}}\langle\theta\rangle]$. Furthermore, each element of the matrix is the cosine of the angle between one of the original basis vectors, corresponding to the matrix row, and one of the rotated basis vectors, corresponding to the matrix column. This indicates that the angle from the vector $\boldsymbol{z}$ to the vector $\boldsymbol{z}'$ is zero (since the cosine of this angle is equal to 1). The angles from the vector $\boldsymbol{x}$ to the vector $\boldsymbol{x}'$, and from the vector $\boldsymbol{y}$ to the vector $\boldsymbol{y}'$ are $135°$, while the angle from the vector $\boldsymbol{x}$ to the vector $\boldsymbol{y}'$ is $-135°$ (the cosines of these angles are all equal to $\frac{-1}{\sqrt{2}}$). The angle from the vector $\boldsymbol{y}$ to the vector $\boldsymbol{x}'$ is $45°$ (the cosine of this angle is $\frac{1}{\sqrt{2}}$). The angles for all other pairs of original and rotated basis vectors are $90°$ (since the cosines of these angles are equal to zero).

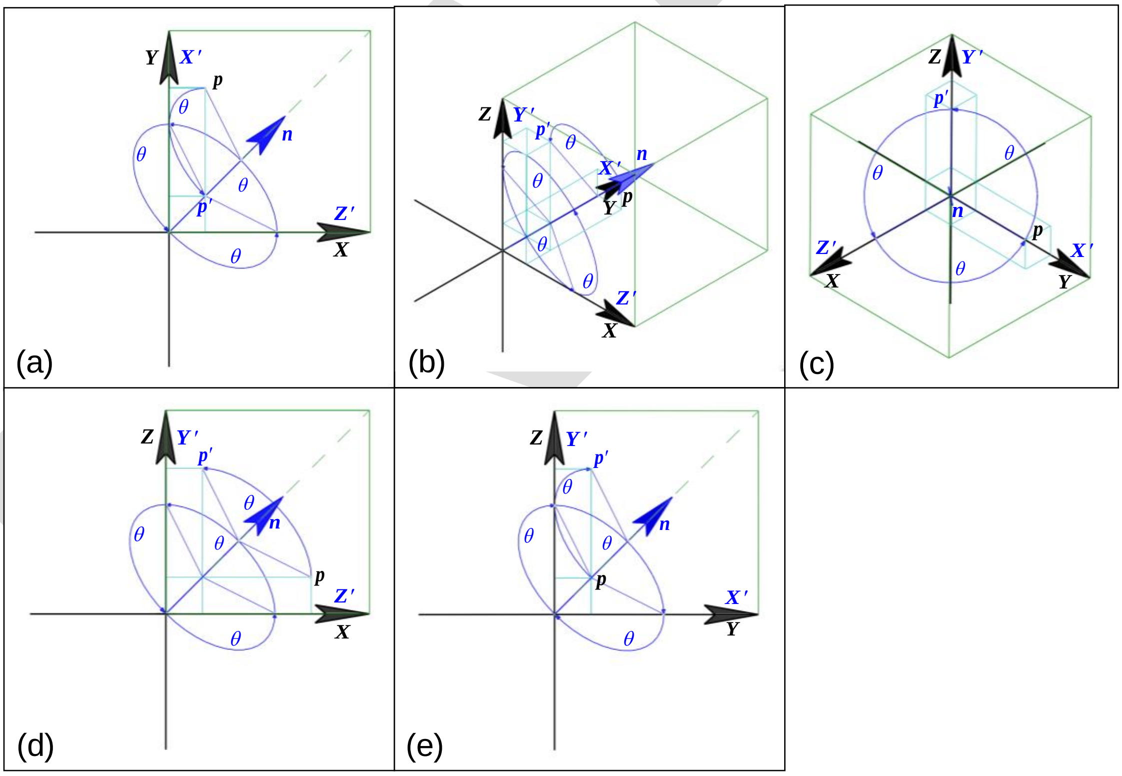

Figure 6.13 — Rotation of the basis vectors about the unit cube diagonal in the same direction as the position-vector: (a) top, (b) isometric, (c) along rotation axis toward origin, (d) front and (e) side views

EXAMPLE 2 Figure 6.13 illustrates a rotation operation where the rotation axis is along the diagonal of the unit cube of the orthonormal frame $\boldsymbol{E}$, i.e., $\boldsymbol{n}_E = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix}^{\mathrm{T}}$, and the rotation angle is $\theta = 120°$. Therefore $\sin\theta = \frac{\sqrt{3}}{2}$ and $\cos\theta = \frac{-1}{2}$, and the rotation matrix is:

$$[\boldsymbol{R_n}\langle\theta\rangle] = \begin{bmatrix} (1-\cos\theta)n_x^2+\cos\theta & (1-\cos\theta)n_xn_y - n_z\sin\theta & (1-\cos\theta)n_xn_z + n_y\sin\theta \\ (1-\cos\theta)n_yn_x + n_z\sin\theta & (1-\cos\theta)n_y^2+\cos\theta & (1-\cos\theta)n_yn_z - n_x\sin\theta \\ (1-\cos\theta)n_zn_x - n_y\sin\theta & (1-\cos\theta)n_zn_y + n_x\sin\theta & (1-\cos\theta)n_z^2+\cos\theta \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Applying this rotation operation to the basis vectors $\boldsymbol{x}$, $\boldsymbol{y}$, $\boldsymbol{z}$ of the frame $\boldsymbol{E}$ yields:

$$\boldsymbol{x}_{\boldsymbol{E}}' = \begin{bmatrix} (1-\cos\theta)n_x^2+\cos\theta \\ (1-\cos\theta)n_yn_x + n_z\sin\theta \\ (1-\cos\theta)n_zn_x - n_y\sin\theta \end{bmatrix} = \begin{bmatrix} \left(\frac{3}{2}\right)\left(\frac{1}{3}\right) + (-1/2) \\ \left(\frac{3}{2}\right)\left(\frac{1}{3}\right) + \left(\frac{1}{\sqrt{3}}\right)\left(\frac{\sqrt{3}}{2}\right) \\ \left(\frac{3}{2}\right)\left(\frac{1}{3}\right) - \left(\frac{1}{\sqrt{3}}\right)\left(\frac{\sqrt{3}}{2}\right) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \boldsymbol{y}_{\boldsymbol{E}}$$

$$\boldsymbol{y}_{\boldsymbol{E}}' = \begin{bmatrix} (1-\cos\theta)n_xn_y - n_z\sin\theta \\ (1-\cos\theta)n_y^2+\cos\theta \\ (1-\cos\theta)n_zn_y + n_x\sin\theta \end{bmatrix} = \begin{bmatrix} \left(\frac{3}{2}\right)\left(\frac{1}{3}\right) - \left(\frac{1}{\sqrt{3}}\right)\left(\frac{\sqrt{3}}{2}\right) \\ \left(\frac{3}{2}\right)\left(\frac{1}{3}\right) + (-1/2) \\ \left(\frac{3}{2}\right)\left(\frac{1}{3}\right) + \left(\frac{1}{\sqrt{3}}\right)\left(\frac{\sqrt{3}}{2}\right) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \boldsymbol{z}_{\boldsymbol{E}}$$

$$\boldsymbol{z}_{\boldsymbol{E}}' = \begin{bmatrix} (1-\cos\theta)n_xn_z + n_y\sin\theta \\ (1-\cos\theta)n_yn_z - n_x\sin\theta \\ (1-\cos\theta)n_z^2+\cos\theta \end{bmatrix} = \begin{bmatrix} \left(\frac{3}{2}\right)\left(\frac{1}{3}\right) + \left(\frac{1}{\sqrt{3}}\right)\left(\frac{\sqrt{3}}{2}\right) \\ \left(\frac{3}{2}\right)\left(\frac{1}{3}\right) - \left(\frac{1}{\sqrt{3}}\right)\left(\frac{\sqrt{3}}{2}\right) \\ \left(\frac{3}{2}\right)\left(\frac{1}{3}\right) + (-1/2) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \boldsymbol{x}_{\boldsymbol{E}}$$

This operation “rotates” the basis vectors, so that the x',-axis rotates onto the $\boldsymbol{y}$-axis, the $\boldsymbol{y}'$-axis rotates onto the $\boldsymbol{z}$-axis, and the $\boldsymbol{z}$-axis rotates onto the $\boldsymbol{x}$-axis.

$$[\boldsymbol{R}]_{\boldsymbol{E}} = \begin{bmatrix} \boldsymbol{x}_{\boldsymbol{E}}' \bullet \boldsymbol{x}_{\boldsymbol{E}} & \boldsymbol{y}_{\boldsymbol{E}}' \bullet \boldsymbol{x}_{\boldsymbol{E}} & \boldsymbol{z}_{\boldsymbol{E}}' \bullet \boldsymbol{x}_{\boldsymbol{E}} \\ \boldsymbol{x}_{\boldsymbol{E}}' \bullet \boldsymbol{y}_{\boldsymbol{E}} & \boldsymbol{y}_{\boldsymbol{E}}' \bullet \boldsymbol{y}_{\boldsymbol{E}} & \boldsymbol{z}_{\boldsymbol{E}}' \bullet \boldsymbol{y}_{\boldsymbol{E}} \\ \boldsymbol{x}_{\boldsymbol{E}}' \bullet \boldsymbol{z}_{\boldsymbol{E}} & \boldsymbol{y}_{\boldsymbol{E}}' \bullet \boldsymbol{z}_{\boldsymbol{E}} & \boldsymbol{z}_{\boldsymbol{E}}' \bullet \boldsymbol{z}_{\boldsymbol{E}} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix}0\\1\\0\end{bmatrix} \bullet \begin{bmatrix}1\\0\\0\end{bmatrix} & \begin{bmatrix}0\\0\\1\end{bmatrix} \bullet \begin{bmatrix}1\\0\\0\end{bmatrix} & \begin{bmatrix}1\\0\\0\end{bmatrix} \bullet \begin{bmatrix}1\\0\\0\end{bmatrix} \\ \begin{bmatrix}0\\1\\0\end{bmatrix} \bullet \begin{bmatrix}0\\1\\0\end{bmatrix} & \begin{bmatrix}0\\0\\1\end{bmatrix} \bullet \begin{bmatrix}0\\1\\0\end{bmatrix} & \begin{bmatrix}1\\0\\0\end{bmatrix} \bullet \begin{bmatrix}0\\1\\0\end{bmatrix} \\ \begin{bmatrix}0\\1\\0\end{bmatrix} \bullet \begin{bmatrix}0\\0\\1\end{bmatrix} & \begin{bmatrix}0\\0\\1\end{bmatrix} \bullet \begin{bmatrix}0\\0\\1\end{bmatrix} & \begin{bmatrix}1\\0\\0\end{bmatrix} \bullet \begin{bmatrix}0\\0\\1\end{bmatrix} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Evaluating the matrix $[\boldsymbol{R}]_{\boldsymbol{E}}$ shows that it is identical to the matrix $[\boldsymbol{R_n}\langle\theta\rangle]$. Furthermore, each element of the matrix is the cosine of the angle between one of the original basis vectors, corresponding to the matrix row, and one of the rotated basis vectors, corresponding to the matrix column. This indicates that the angles from the vector $\boldsymbol{y}$ to the vector $\boldsymbol{x}'$, from the vector $\boldsymbol{z}$ to the vector $\boldsymbol{y}'$, and from the vector $\boldsymbol{x}$ to the vector $\boldsymbol{z}'$ are all equal to zero (since the cosines of these angles are equal to 1), while the angles for all other pairs of original and rotated basis vectors are equal to 90° (since the cosines of these angles are equal to zero).

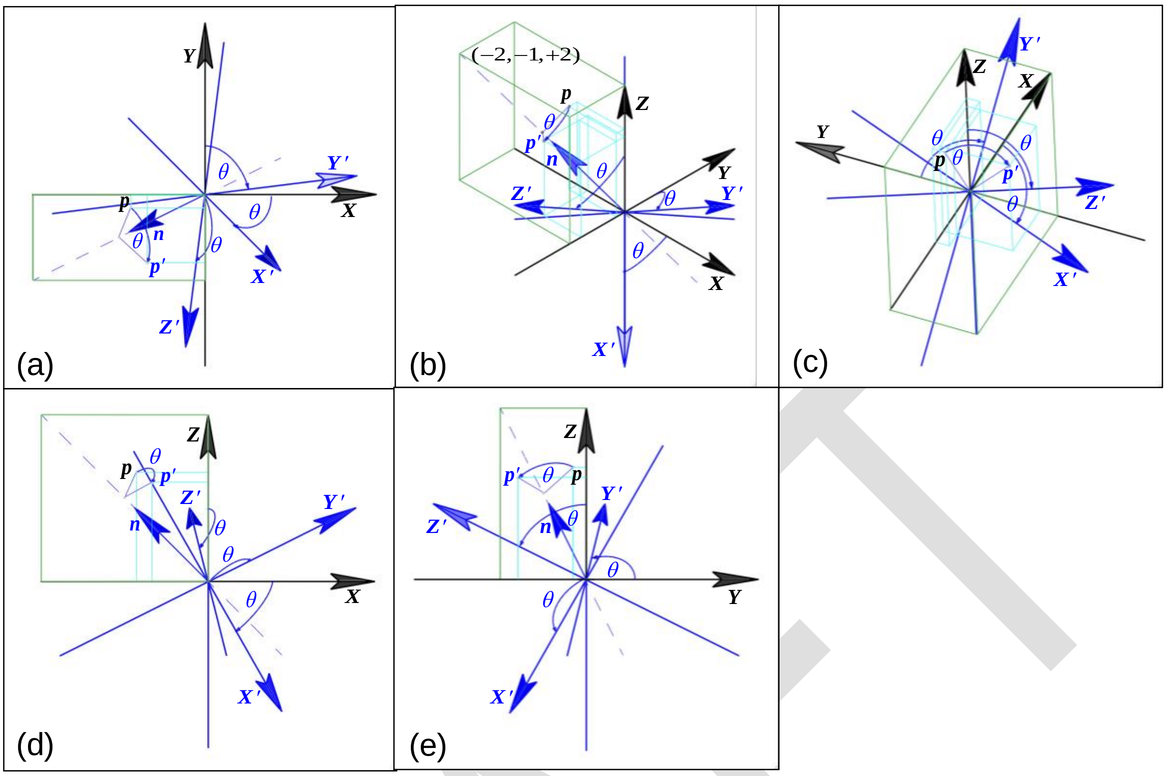

Figure 6.14 — Rotation of the basis vectors about an asymmetric axis in the same direction as the position-vector: (a) top, (b) isometric, (c) along rotation axis toward origin, (d) front and (e) side views

EXAMPLE 3 Figure 6.14 illustrates a rotation operation where the rotation axis passes through the point (-2, -1, 2) of the orthonormal frame $\boldsymbol{E}$, i.e., $\boldsymbol{n_E} = \begin{pmatrix} \frac{-2}{3} & \frac{-1}{3} & \frac{2}{3} \end{pmatrix}^{\mathrm{T}}$, and the rotation angle is $\theta = -90°$. Therefore $\sin\theta = -1$ and $\cos\theta = 0$, and the rotation matrix is:

$$[\boldsymbol{R_n}\langle\theta\rangle] = \begin{bmatrix} (1-\cos\theta)n_x^2 + \cos\theta & (1-\cos\theta)n_x n_y - n_z\sin\theta & (1-\cos\theta)n_x n_z + n_y\sin\theta \\ (1-\cos\theta)n_y n_x + n_z\sin\theta & (1-\cos\theta)n_y^2 + \cos\theta & (1-\cos\theta)n_y n_z - n_x\sin\theta \\ (1-\cos\theta)n_z n_x - n_y\sin\theta & (1-\cos\theta)n_z n_y + n_x\sin\theta & (1-\cos\theta)n_z^2 + \cos\theta \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 4 & 8 & -1 \\ -4 & 1 & -8 \\ -7 & 4 & 4 \end{bmatrix}$$

Applying this rotation operation to the basis vectors $\boldsymbol{x}$, $\boldsymbol{y}$, $\boldsymbol{z}$ of the frame $\boldsymbol{E}$ yields:

$$\boldsymbol{x}_{\boldsymbol{E}}' = \begin{bmatrix} (1-\cos\theta)n_x^2 + \cos\theta \\ (1-\cos\theta)n_y n_x + n_z\sin\theta \\ (1-\cos\theta)n_z n_x - n_y\sin\theta \end{bmatrix} = \begin{bmatrix} (1)\left(\frac{-2}{3}\right)\left(\frac{-2}{3}\right) + (0) \\ (1)\left(\frac{-1}{3}\right)\left(\frac{-2}{3}\right) + \left(\frac{2}{3}\right)(-1) \\ (1)\left(\frac{2}{3}\right)\left(\frac{-2}{3}\right) - \left(\frac{-1}{3}\right)(-1) \end{bmatrix} = \begin{bmatrix} \left(\frac{4}{9}\right) + (0) \\ \left(\frac{2}{9}\right) + \left(\frac{-2}{3}\right) \\ \left(\frac{-4}{9}\right) - \left(\frac{1}{3}\right) \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 4 \\ -4 \\ -7 \end{bmatrix} = \boldsymbol{y_E}$$

$$\boldsymbol{y}_{\boldsymbol{E}}' = \begin{bmatrix} (1-\cos\theta)n_x n_y - n_z\sin\theta \\ (1-\cos\theta)n_y^2 + \cos\theta \\ (1-\cos\theta)n_z n_y + n_x\sin\theta \end{bmatrix} = \begin{bmatrix} (1)\left(\frac{-2}{3}\right)\left(\frac{-1}{3}\right) - \left(\frac{2}{3}\right)(-1) \\ (1)\left(\frac{-1}{3}\right)\left(\frac{-1}{3}\right) + (0) \\ (1)\left(\frac{2}{3}\right)\left(\frac{-1}{3}\right) + \left(\frac{-2}{3}\right)(-1) \end{bmatrix} = \begin{bmatrix} \left(\frac{2}{9}\right) - \left(\frac{-2}{3}\right) \\ \left(\frac{1}{9}\right) + (0) \\ \left(\frac{-2}{9}\right) + \left(\frac{2}{3}\right) \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 8 \\ 1 \\ 4 \end{bmatrix} = \boldsymbol{z_E}$$

$$\boldsymbol{z}_{\boldsymbol{E}}' = \begin{bmatrix} (1-\cos\theta)n_x n_z + n_y\sin\theta \\ (1-\cos\theta)n_y n_z - n_x\sin\theta \\ (1-\cos\theta)n_z^2 + \cos\theta \end{bmatrix} = \begin{bmatrix} (1)\left(\frac{-2}{3}\right)\left(\frac{2}{3}\right) + \left(\frac{-1}{3}\right)(-1) \\ (1)\left(\frac{-1}{3}\right)\left(\frac{2}{3}\right) - \left(\frac{-2}{3}\right)(-1) \\ (1)\left(\frac{2}{3}\right)\left(\frac{2}{3}\right) + (0) \end{bmatrix} = \begin{bmatrix} \left(\frac{-4}{9}\right) + \left(\frac{1}{3}\right) \\ \left(\frac{-2}{9}\right) - \left(\frac{2}{3}\right) \\ \left(\frac{4}{9}\right) + (0) \end{bmatrix} = \frac{1}{9}\begin{bmatrix} -1 \\ -8 \\ 4 \end{bmatrix} = \boldsymbol{x_E}$$

$$[\boldsymbol{R}]_E = \begin{bmatrix} x'_E \bullet x_E & y'_E \bullet x_E & z'_E \bullet x_E \\ x'_E \bullet y_E & y'_E \bullet y_E & z'_E \bullet y_E \\ x'_E \bullet z_E & y'_E \bullet z_E & z'_E \bullet z_E \end{bmatrix} = \begin{bmatrix} \frac{1}{9}\begin{bmatrix}4\\-4\\-7\end{bmatrix} \bullet \begin{bmatrix}1\\0\\0\end{bmatrix} & \frac{1}{9}\begin{bmatrix}8\\1\\4\end{bmatrix} \bullet \begin{bmatrix}1\\0\\0\end{bmatrix} & \frac{1}{9}\begin{bmatrix}-1\\-8\\4\end{bmatrix} \bullet \begin{bmatrix}1\\0\\0\end{bmatrix} \\ \frac{1}{9}\begin{bmatrix}4\\-4\\-7\end{bmatrix} \bullet \begin{bmatrix}0\\1\\0\end{bmatrix} & \frac{1}{9}\begin{bmatrix}8\\1\\4\end{bmatrix} \bullet \begin{bmatrix}0\\1\\0\end{bmatrix} & \frac{1}{9}\begin{bmatrix}-1\\-8\\4\end{bmatrix} \bullet \begin{bmatrix}0\\1\\0\end{bmatrix} \\ \frac{1}{9}\begin{bmatrix}4\\-4\\-7\end{bmatrix} \bullet \begin{bmatrix}0\\0\\1\end{bmatrix} & \frac{1}{9}\begin{bmatrix}8\\1\\4\end{bmatrix} \bullet \begin{bmatrix}0\\0\\1\end{bmatrix} & \frac{1}{9}\begin{bmatrix}-1\\-8\\4\end{bmatrix} \bullet \begin{bmatrix}0\\0\\1\end{bmatrix} \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 4 & 8 & -1 \\ -4 & 1 & -8 \\ -7 & 4 & 4 \end{bmatrix}$$

Evaluating the matrix $[\boldsymbol{R}]_E$ shows that it is identical to the matrix $[\boldsymbol{R}_{\boldsymbol{n}}\langle\theta\rangle]$. Furthermore, each element of the matrix is the cosine of the angle between one of the original basis vectors, corresponding to the matrix row, and one of the rotated basis vectors, corresponding to the matrix column. For example, the angle from the original basis vector $\boldsymbol{x}$ to the rotated basis vector $\boldsymbol{x}'$ has a cosine value of $4/9$, and therefore is approximately equal to $63.6°$.

6.4.3 Rotation of the orthonormal frame

Given an orthonormal frame $\boldsymbol{E}$ with basis $\boldsymbol{x}$, $\boldsymbol{y}$, $\boldsymbol{z}$, the inverse rotation operator $\boldsymbol{R}_{\boldsymbol{n}}\langle\theta\rangle^{-1}$ may also be applied to each of its basis vectors, i.e.:

$$\begin{aligned} \boldsymbol{x}'' &= \boldsymbol{R}_{\boldsymbol{n}}\langle\theta\rangle^{-1}(\boldsymbol{x}), \\ \boldsymbol{y}'' &= \boldsymbol{R}_{\boldsymbol{n}}\langle\theta\rangle^{-1}(\boldsymbol{y}), \text{ and} \\ \boldsymbol{z}'' &= \boldsymbol{R}_{\boldsymbol{n}}\langle\theta\rangle^{-1}(\boldsymbol{z}). \end{aligned} \tag{6. 11}$$

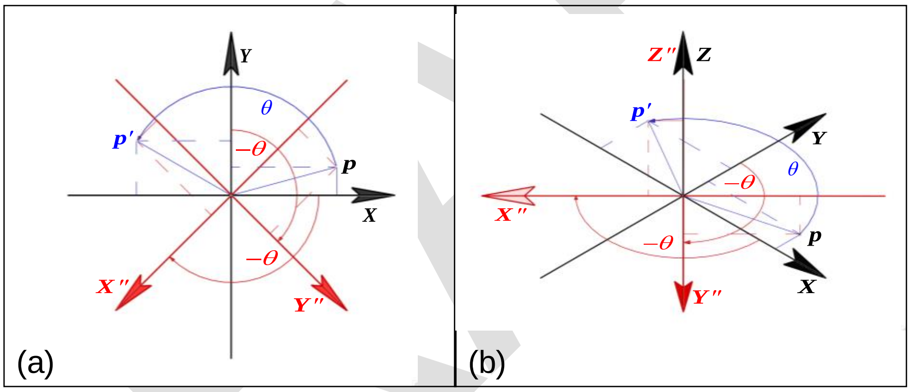

Figure 6.15 — Rotation of the basic vectors in the opposite direction from the position-vector: (a) top view; (b) isometric view

Figure 6.15(a) and (b) show the $\boldsymbol{x}$- and $\boldsymbol{y}$-axes of the orthonormal frame $\boldsymbol{E}$ being rotated about the $\boldsymbol{z}$-axis through the angle $-\theta$, which is the additive inverse of the angle θ through which the position-vector $\boldsymbol{p}$ is rotated, regardless of whether the value of θ is less than, equal to, or greater than zero. This implements the inverse rotation operator $\boldsymbol{R}_{\boldsymbol{z}}\langle\theta\rangle^{-1}$ as $\boldsymbol{R}_{\boldsymbol{z}}\langle\theta\rangle$. Thus the basis vectors are rotated in the opposite direction from the position-vector $\boldsymbol{p}$, yielding the rotated basis vectors $\boldsymbol{x}'' = \boldsymbol{R}_{\boldsymbol{z}}\langle-\theta\rangle(\boldsymbol{x})$, $\boldsymbol{y}'' = \boldsymbol{R}_{\boldsymbol{z}}\langle-\theta\rangle(\boldsymbol{y})$, and $\boldsymbol{z}'' = \boldsymbol{z} = \boldsymbol{R}_{\boldsymbol{z}}\langle-\theta\rangle(\boldsymbol{z})$.

Replacing θ with $-\theta$ in Equation 6.8, and applying the trigonometric identities $\sin(-\theta) = -\sin\theta$ and $\cos(-\theta) = \cos\theta$ yields:

$$[\boldsymbol{R}_{\boldsymbol{n}}\langle-\theta\rangle] = \begin{bmatrix} (1-\cos(-\theta))n_x^2 + \cos(-\theta) & (1-\cos(-\theta))n_x n_y - n_z\sin(-\theta) & (1-\cos(-\theta))n_x n_z + n_y\sin(-\theta) \\ (1-\cos(-\theta))n_y n_x + n_z\sin(-\theta) & (1-\cos(-\theta))n_y^2 + \cos(-\theta) & (1-\cos(-\theta))n_y n_z - n_x\sin(-\theta) \\ (1-\cos(-\theta))n_z n_x - n_y\sin(-\theta) & (1-\cos(-\theta))n_z n_y + n_x\sin(-\theta) & (1-\cos(-\theta))n_z^2 + \cos(-\theta) \end{bmatrix}$$

$$[\boldsymbol{R}_{\boldsymbol{n}}\langle-\theta\rangle] = \begin{bmatrix} (1-\cos\theta)n_x^2 + \cos\theta & (1-\cos\theta)n_x n_y + n_z\sin\theta & (1-\cos\theta)n_x n_z - n_y\sin\theta \\ (1-\cos\theta)n_y n_x - n_z\sin\theta & (1-\cos\theta)n_y^2 + \cos\theta & (1-\cos\theta)n_y n_z + n_x\sin\theta \\ (1-\cos\theta)n_z n_x + n_y\sin\theta & (1-\cos\theta)n_z n_y - n_x\sin\theta & (1-\cos\theta)n_z^2 + \cos\theta \end{bmatrix} \tag{6.12}$$

The matrix $[\boldsymbol{R_n}\langle -\theta \rangle]$ is the transpose, and therefore the inverse, of the matrix $[\boldsymbol{R_n}\langle \theta \rangle]$, and so implements the inverse rotation operator $\boldsymbol{R_n}\langle \theta \rangle^{\boldsymbol{-1}}$.

The inverse rotation operator $\boldsymbol{R_n}\langle \theta \rangle^{\boldsymbol{-1}}$ can also be implemented by reversing the direction of the rotation axis. Replacing $\boldsymbol{n} = (n_x \quad n_y \quad n_z)^{\mathrm{T}}$ with $-\boldsymbol{n} = (-n_x \quad -n_y \quad -n_z)^{\mathrm{T}}$ in Equation 6.8, yields:

$$[\boldsymbol{R_{-n}}\langle \theta \rangle] = \begin{bmatrix} (1-\cos\theta)(-n_x)^2 + \cos\theta & (1-\cos\theta)(-n_x)(-n_y) - (-n_z)\sin\theta & (1-\cos\theta)(-n_x)(-n_z) + (-n_y)\sin\theta \\ (1-\cos\theta)(-n_y)(-n_x) + (-n_z)\sin\theta & (1-\cos\theta)(-n_y)^2 + \cos\theta & (1-\cos\theta)(-n_y)(-n_z) - (-n_x)\sin\theta \\ (1-\cos\theta)(-n_z)(-n_x) - (-n_y)\sin\theta & (1-\cos\theta)(-n_z)(-n_y) + (-n_x)\sin\theta & (1-\cos\theta) - (-n_z)^2 + \cos\theta \end{bmatrix}$$

$$[\boldsymbol{R_{-n}}\langle \theta \rangle] = \begin{bmatrix} (1-\cos\theta)n_x^2 + \cos\theta & (1-\cos\theta)n_x n_y + n_z\sin\theta & (1-\cos\theta)n_x n_z - n_y\sin\theta \\ (1-\cos\theta)n_y n_x - n_z\sin\theta & (1-\cos\theta)n_y^2 + \cos\theta & (1-\cos\theta)n_y n_z + n_x\sin\theta \\ (1-\cos\theta)n_z n_x + n_y\sin\theta & (1-\cos\theta)n_z n_y - n_x\sin\theta & (1-\cos\theta)n_z^2 + \cos\theta \end{bmatrix}$$

Thus $[\boldsymbol{R_n}\langle \theta \rangle^{\boldsymbol{-1}}] = [\boldsymbol{R_{-n}}\langle \theta \rangle] = [\boldsymbol{R_n}\langle -\theta \rangle]$.

Applying the inverse rotation operation $\boldsymbol{R_n}\langle \theta \rangle^{\boldsymbol{-1}}$ to the basis vectors of the frame $\boldsymbol{E}$ yields the coordinate components of the rotated basis vectors with respect to frame $\boldsymbol{E}$.

$$\begin{aligned} \boldsymbol{x}''_{\boldsymbol{E}} &= [\boldsymbol{R_n}\langle \theta \rangle^{\boldsymbol{-1}}]\boldsymbol{x_E} \\ &= \begin{bmatrix} (1-\cos\theta)n_x^2 + \cos\theta & (1-\cos\theta)n_x n_y + n_z\sin\theta & (1-\cos\theta)n_x n_z - n_y\sin\theta \\ (1-\cos\theta)n_y n_x - n_z\sin\theta & (1-\cos\theta)n_y^2 + \cos\theta & (1-\cos\theta)n_y n_z + n_x\sin\theta \\ (1-\cos\theta)n_z n_x + n_y\sin\theta & (1-\cos\theta)n_z n_y - n_x\sin\theta & (1-\cos\theta)n_z^2 + \cos\theta \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \\ \boldsymbol{x}''_{\boldsymbol{E}} &= \begin{bmatrix} (1-\cos\theta)n_x^2 + \cos\theta \\ (1-\cos\theta)n_y n_x - n_z\sin\theta \\ (1-\cos\theta)n_z n_x + n_y\sin\theta \end{bmatrix} \end{aligned}$$

The rotated basis vector $\boldsymbol{x}''$ is represented by the first column of the matrix $[\boldsymbol{R_n}\langle \theta \rangle^{\boldsymbol{-1}}]$. Similarly, the rotated basis vectors $\mathbf{y}''$ and $\mathbf{z}''$ are represented by the second and third columns of the matrix $[\boldsymbol{R_n}\langle \theta \rangle^{\boldsymbol{-1}}]$, respectively.

$$\begin{aligned} \boldsymbol{y}''_{\boldsymbol{E}} &= [\boldsymbol{R_n}\langle \theta \rangle^{\boldsymbol{-1}}]\boldsymbol{y_E} \\ &= \begin{bmatrix} (1-\cos\theta)n_x^2 + \cos\theta & (1-\cos\theta)n_x n_y + n_z\sin\theta & (1-\cos\theta)n_x n_z - n_y\sin\theta \\ (1-\cos\theta)n_y n_x - n_z\sin\theta & (1-\cos\theta)n_y^2 + \cos\theta & (1-\cos\theta)n_y n_z + n_x\sin\theta \\ (1-\cos\theta)n_z n_x + n_y\sin\theta & (1-\cos\theta)n_z n_y - n_x\sin\theta & (1-\cos\theta)n_z^2 + \cos\theta \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \\ \boldsymbol{y}''_{\boldsymbol{E}} &= \begin{bmatrix} (1-\cos\theta)n_x n_y + n_z\sin\theta \\ (1-\cos\theta)n_y^2 + \cos\theta \\ (1-\cos\theta)n_z n_y - n_x\sin\theta \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \boldsymbol{z}''_{\boldsymbol{E}} &= [\boldsymbol{R_n}\langle \theta \rangle^{\boldsymbol{-1}}]\boldsymbol{z_E} \\ &= \begin{bmatrix} (1-\cos\theta)n_x^2 + \cos\theta & (1-\cos\theta)n_x n_y + n_z\sin\theta & (1-\cos\theta)n_x n_z - n_y\sin\theta \\ (1-\cos\theta)n_y n_x - n_z\sin\theta & (1-\cos\theta)n_y^2 + \cos\theta & (1-\cos\theta)n_y n_z + n_x\sin\theta \\ (1-\cos\theta)n_z n_x + n_y\sin\theta & (1-\cos\theta)n_z n_y - n_x\sin\theta & (1-\cos\theta)n_z^2 + \cos\theta \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \\ \boldsymbol{z}''_{\boldsymbol{E_E}} &= \begin{bmatrix} (1-\cos\theta)n_x n_z - n_y\sin\theta \\ (1-\cos\theta)n_y n_z + n_x\sin\theta \\ (1-\cos\theta)n_z^2 + \cos\theta \end{bmatrix} \end{aligned}$$

The original position-vector $\boldsymbol{p}$ and the rotated position-vector $\boldsymbol{p'}$ each can be represented by vector-coordinates with respect to the original basic vectors $\boldsymbol{x}$, $\boldsymbol{y}$, $\boldsymbol{z}$, and can also be represented by different vector-coordinates with respect to the rotated basic vectors $\boldsymbol{x}'', \boldsymbol{y}'', \boldsymbol{z}''$. The original position-vector $\boldsymbol{p}$ has vector-coordinate

$(p_x, p_y, p_z)_{\boldsymbol{E}}$, in terms of the original basis vectors $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$, with respect to frame $\boldsymbol{E}$, and also has vector-coordinate $\left(p_{x''}, p_{y''}, p_{z''}\right)_{\boldsymbol{E}}$, in terms of the rotated basis vectors $\boldsymbol{x}'', \boldsymbol{y}'', \boldsymbol{z}''$, also with respect to frame $\boldsymbol{E}$. The rotated position-vector $\boldsymbol{p}'$ has vector-coordinate $(p_x', p_y', p_z')_{\boldsymbol{E}}$, in terms of the original basis vectors $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$, with respect to frame $\boldsymbol{E}$, and also has vector-coordinate $\left(p_{x''}', p_{y''}', p_{z''}'\right)_{\boldsymbol{E}}$, in terms of the rotated basis vectors $\boldsymbol{x}'', \boldsymbol{y}'', \boldsymbol{z}''$, also with respect to frame $\boldsymbol{E}$.

Because $\boldsymbol{p}$ is a position-vector, the sum of the products of each of its coordinate-components with the corresponding basis vectors is invariant across all orthonormal frames. Thus, $\boldsymbol{p} = p_x\boldsymbol{x} + p_y\boldsymbol{y} + p_z\boldsymbol{z} = p_{x''}\boldsymbol{x}'' + p_{y''}\boldsymbol{y}'' + p_{z''}\boldsymbol{z}''$. Similarly, because $\boldsymbol{p}'$ is also a position-vector, $\boldsymbol{p}' = p_x'\boldsymbol{x} + p_y'\boldsymbol{y} + p_z'\boldsymbol{z} = p_{x''}'\boldsymbol{x}'' + p_{y''}'\boldsymbol{y}'' + p_{z''}'\boldsymbol{z}''$.

The original position-vector $\boldsymbol{p}$ has the same coordinate component values $\left(p_{x''}, p_{y''}, p_{z''}\right)_{\boldsymbol{E}}$, in terms of the rotated basis vectors $\boldsymbol{x}'', \boldsymbol{y}'', \boldsymbol{z}''$, as the rotated position-vector $\boldsymbol{p}'$ has in terms of the original basis vectors $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$, i.e., $p_x' = p_{x''}$, $p_y' = p_{y''}$ and $p_z' = p_{z''}$.

Given $\boldsymbol{p} = p_x\boldsymbol{x} + p_y\boldsymbol{y} + p_z\boldsymbol{z}$, then

$$\boldsymbol{p}'' = \boldsymbol{R_n}\langle\theta\rangle^{-1}(\boldsymbol{p}) = p_x\boldsymbol{R_n}\langle\theta\rangle^{-1}(\boldsymbol{x}) + p_y\boldsymbol{R_n}\langle\theta\rangle^{-1}(\boldsymbol{y}) + p_z\boldsymbol{R_n}\langle\theta\rangle^{-1}(\boldsymbol{z}) = p_x\boldsymbol{x}'' + p_y\boldsymbol{y}'' + p_z\boldsymbol{z}''.$$

Using Equation **Error! Reference source not found.**, and substituting the expression above for $\boldsymbol{p}''$, the vector-coordinate components of $\boldsymbol{p}''$ are:

$$p_x'' = \boldsymbol{p}'' \bullet \boldsymbol{x} = \left(p_x\boldsymbol{x}'' + p_y\boldsymbol{y}'' + p_z\boldsymbol{z}''\right) \bullet \boldsymbol{x} = p_x\boldsymbol{x}'' \bullet \boldsymbol{x} + p_y\boldsymbol{y}'' \bullet \boldsymbol{x} + p_z\boldsymbol{z}'' \bullet \boldsymbol{x},$$

$$p_y'' = \boldsymbol{p}'' \bullet \boldsymbol{y} = \left(p_x\boldsymbol{x}'' + p_y\boldsymbol{y}'' + p_z\boldsymbol{z}''\right) \bullet \boldsymbol{y} = p_x\boldsymbol{x}'' \bullet \boldsymbol{y} + p_y\boldsymbol{y}'' \bullet \boldsymbol{y} + p_z\boldsymbol{z}'' \bullet \boldsymbol{y},$$

$$p_z'' = \boldsymbol{p}'' \bullet \boldsymbol{z} = \left(p_x\boldsymbol{x}'' + p_y\boldsymbol{y}'' + p_z\boldsymbol{z}''\right) \bullet \boldsymbol{z} = p_x\boldsymbol{x}'' \bullet \boldsymbol{z} + p_y\boldsymbol{y}'' \bullet \boldsymbol{z} + p_z\boldsymbol{z}'' \bullet \boldsymbol{z}.$$

The matrix form of these three equations is:

$$\begin{bmatrix} p_x'' \\ p_y'' \\ p_z'' \end{bmatrix}_{\boldsymbol{E}} = [\boldsymbol{R}^{-1}]_{\boldsymbol{E}} \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix}_{\boldsymbol{E}} \quad \text{where:} \quad [\boldsymbol{R}^{-1}]_{\boldsymbol{E}} = \begin{pmatrix} \boldsymbol{x}'' \bullet \boldsymbol{x} & \boldsymbol{y}'' \bullet \boldsymbol{x} & \boldsymbol{z}'' \bullet \boldsymbol{x} \\ \boldsymbol{x}'' \bullet \boldsymbol{y} & \boldsymbol{y}'' \bullet \boldsymbol{y} & \boldsymbol{z}'' \bullet \boldsymbol{y} \\ \boldsymbol{x}'' \bullet \boldsymbol{z} & \boldsymbol{y}'' \bullet \boldsymbol{z} & \boldsymbol{z}'' \bullet \boldsymbol{z} \end{pmatrix} \qquad (6.13)$$

The matrix $[\boldsymbol{R}^{-1}]_{\boldsymbol{E}}$ is a matrix representation of the inverse rotation operator $\boldsymbol{R_n}\langle\theta\rangle^{-1}$ with respect to orthonormal frame $\boldsymbol{E}$ with basis $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$, expressed in terms of the relationships between the original basis vectors $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$ and the rotated basis vectors $\boldsymbol{x}'', \boldsymbol{y}'', \boldsymbol{z}''$.

Evaluating the dot products in each of the elements of this matrix yields:

$$[\boldsymbol{R}^{-1}]_{\boldsymbol{E}} = \begin{bmatrix}
\begin{bmatrix} (1-\cos\theta)n_x^2 + \cos\theta \\ (1-\cos\theta)n_yn_x - n_z\sin\theta \\ (1-\cos\theta)n_zn_x + n_y\sin\theta \end{bmatrix} \bullet \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} &
\begin{bmatrix} (1-\cos\theta)n_xn_y + n_z\sin\theta \\ (1-\cos\theta)n_y^2 + \cos\theta \\ (1-\cos\theta)n_zn_y - n_x\sin\theta \end{bmatrix} \bullet \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} &
\begin{bmatrix} (1-\cos\theta)n_xn_z - n_y\sin\theta \\ (1-\cos\theta)n_yn_z + n_x\sin\theta \\ (1-\cos\theta)n_z^2 + \cos\theta \end{bmatrix} \bullet \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \\
\begin{bmatrix} (1-\cos\theta)n_x^2 + \cos\theta \\ (1-\cos\theta)n_yn_x - n_z\sin\theta \\ (1-\cos\theta)n_zn_x + n_y\sin\theta \end{bmatrix} \bullet \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} &
\begin{bmatrix} (1-\cos\theta)n_xn_y + n_z\sin\theta \\ (1-\cos\theta)n_y^2 + \cos\theta \\ (1-\cos\theta)n_zn_y - n_x\sin\theta \end{bmatrix} \bullet \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} &
\begin{bmatrix} (1-\cos\theta)n_xn_z - n_y\sin\theta \\ (1-\cos\theta)n_yn_z + n_x\sin\theta \\ (1-\cos\theta)n_z^2 + \cos\theta \end{bmatrix} \bullet \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \\
\begin{bmatrix} (1-\cos\theta)n_x^2 + \cos\theta \\ (1-\cos\theta)n_yn_x - n_z\sin\theta \\ (1-\cos\theta)n_zn_x + n_y\sin\theta \end{bmatrix} \bullet \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} &
\begin{bmatrix} (1-\cos\theta)n_xn_y + n_z\sin\theta \\ (1-\cos\theta)n_y^2 + \cos\theta \\ (1-\cos\theta)n_zn_y - n_x\sin\theta \end{bmatrix} \bullet \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} &
\begin{bmatrix} (1-\cos\theta)n_xn_z - n_y\sin\theta \\ (1-\cos\theta)n_yn_z + n_x\sin\theta \\ (1-\cos\theta)n_z^2 + \cos\theta \end{bmatrix} \bullet \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}
\end{bmatrix}$$

$$[\boldsymbol{R}^{-1}]_{\boldsymbol{E}} = \begin{bmatrix} (1-\cos\theta)n_x^2 + \cos\theta & (1-\cos\theta)n_xn_y + n_z\sin\theta & (1-\cos\theta)n_xn_z - n_y\sin\theta \\ (1-\cos\theta)n_yn_x - n_z\sin\theta & (1-\cos\theta)n_y^2 + \cos\theta & (1-\cos\theta)n_yn_z + n_x\sin\theta \\ (1-\cos\theta)n_zn_x + n_y\sin\theta & (1-\cos\theta)n_zn_y - n_x\sin\theta & (1-\cos\theta)n_z^2 + \cos\theta \end{bmatrix} = [\boldsymbol{R_n}\langle\theta\rangle^{-1}]$$

Thus, the inverse rotation operator $\boldsymbol{R_n}\langle\theta\rangle^{-1}$, where the rotation axis $\boldsymbol{n}$ has vector-coordinates $(n_1, n_2, n_3)_{\boldsymbol{E}}$ in the orthonormal frame $\boldsymbol{E}$ with basis $\boldsymbol{x}$, $\boldsymbol{y}$, $\boldsymbol{z}$, can be represented as a matrix multiplication $\boldsymbol{p}''_{\boldsymbol{E}} = [\boldsymbol{R}^{-1}]_{\boldsymbol{E}}\boldsymbol{p}_{\boldsymbol{E}}$, where $[\boldsymbol{R}^{-1}]_{\boldsymbol{E}} = [\boldsymbol{R_n}\langle\theta\rangle^{-1}]$. The matrix $[\boldsymbol{R}^{-1}]_{\boldsymbol{E}}$ is the inverse of the matrix $[\boldsymbol{R}]_{\boldsymbol{E}}$. The matrix $[\boldsymbol{R}^{-1}]_{\boldsymbol{E}}$ represents the rotation relationships between the original basis vectors $\boldsymbol{x}$, $\boldsymbol{y}$, $\boldsymbol{z}$ of the frame $\boldsymbol{E}$ and the rotated basis vectors $\boldsymbol{x}'', \boldsymbol{y}'', \boldsymbol{z}''$. Each element of the matrix $[\boldsymbol{R}^{-1}]_{\boldsymbol{E}}$ represents the rotation angle from the original basis vector that corresponds to the matrix row to the rotated basis vector that corresponds to the matrix column.

Thus, if the position-vector $\boldsymbol{p}$ is rotated through an angle θ about an axis, or the basis vectors $\boldsymbol{x}$, $\boldsymbol{y}$, $\boldsymbol{z}$ are rotated through an angle of the same magnitude but in the opposite direction, i.e., $-\theta$, about the same axis, the resulting relationship between the rotated position-vector and the original basis vectors is the same as the relationship between the original position-vector and the rotated basis vectors.

The length and angle preserving properties of the rotation operator imply that the rotated basis vectors $\boldsymbol{x}'', \boldsymbol{y}'', \boldsymbol{z}''$ define an orthonormal basis with the same origin as $\boldsymbol{E}$. Thus, an orthonormal frame $\boldsymbol{E}$ with basis $\boldsymbol{x}$, $\boldsymbol{y}$, $\boldsymbol{z}$ may be rotated by an inverse rotation operator $\boldsymbol{R_n}\langle\theta\rangle^{-1}$ to create another orthonormal frame with the same origin and basis $\boldsymbol{x}'', \boldsymbol{y}'', \boldsymbol{z}''$. Call this frame $\boldsymbol{E}''$. The relationship between the original frame $\boldsymbol{E}$ with basis $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$ and the rotated frame $\boldsymbol{E}''$ with basis $\boldsymbol{x}'', \boldsymbol{y}'', \boldsymbol{z}''$ is given by the matrix $[\boldsymbol{R}^{-1}]_{\boldsymbol{E}}$. This matrix representation is expressed solely in terms of the original frame $\boldsymbol{E}$.

The relationship between the rotated position-vector and the original frame is the same as the relationship between the original position-vector and the rotated frame. This relative equivalence between rotating a position-vector about an axis, and rotating the orthonormal frame about the same axis by an equal amount in the opposite direction is termed the *rotation duality*.

NOTE The relationship between rotating a position-vector and rotating the orthonormal frame through the same angle in the opposite direction is another common source of confusion and errors when working with rotation operations. Furthermore, this relationship has resulted in confusing terminology that is used in different communities to describe a) whether the position-vector is rotated or the coordinate frame is rotated, b) the sign convention for measuring rotation angles in coordinate transformations, or c) the direction of a coordinate transformation.

EXAMPLE 1 Figure 6.15 illustrates a rotation operation where the rotation axis is the $\boldsymbol{z}$-axis the orthonormal frame $\boldsymbol{E}$, i.e., $\boldsymbol{n_E} = (0 \quad 0 \quad 1)^{\mathrm{T}}$, and the rotation angle is $\theta = -135°$. Therefore $\sin\theta = \frac{-1}{\sqrt{2}}$ and $\cos\theta = \frac{-1}{\sqrt{2}}$, and the rotation matrix is:

$$[\boldsymbol{R_n}\langle\theta\rangle] = \begin{bmatrix} (1-\cos\theta)n_x^2 + \cos\theta & (1-\cos\theta)n_x n_y - n_z\sin\theta & (1-\cos\theta)n_x n_z + n_y\sin\theta \\ (1-\cos\theta)n_y n_x + n_z\sin\theta & (1-\cos\theta)n_y^2 + \cos\theta & (1-\cos\theta)n_y n_z - n_x\sin\theta \\ (1-\cos\theta)n_z n_x - n_y\sin\theta & (1-\cos\theta)n_z n_y + n_x\sin\theta & (1-\cos\theta)n_z^2 + \cos\theta \end{bmatrix} = \begin{bmatrix} \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Applying this rotation operation to the basis vectors $\boldsymbol{x}$, $\boldsymbol{y}$, $\boldsymbol{z}$ of the frame $\boldsymbol{E}$ yields:

$$\boldsymbol{x}''_{\boldsymbol{E}} = \begin{bmatrix} (1-\cos\theta)n_x^2 + \cos\theta \\ (1-\cos\theta)n_y n_x + n_z\sin\theta \\ (1-\cos\theta)n_z n_x - n_y\sin\theta \end{bmatrix} = \begin{bmatrix} \frac{-1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \\ 0 \end{bmatrix}$$

$$\boldsymbol{y}''_{\boldsymbol{E}} = \begin{bmatrix} (1-\cos\theta)n_x n_y - n_z\sin\theta \\ (1-\cos\theta)n_y^2 + \cos\theta \\ (1-\cos\theta)n_z n_y + n_x\sin\theta \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \\ 0 \end{bmatrix}$$

$$\boldsymbol{z}''_{\boldsymbol{E}} = \begin{bmatrix} (1-\cos\theta)n_x n_z + n_y\sin\theta \\ (1-\cos\theta)n_y n_z - n_x\sin\theta \\ (1-\cos\theta)n_z^2 + \cos\theta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \boldsymbol{z_E}$$

$$[\boldsymbol{R}]_E = \begin{bmatrix} \boldsymbol{x}''_E \bullet \boldsymbol{x}_E & \boldsymbol{y}''_E \bullet \boldsymbol{x}_E & \boldsymbol{z}''_E \bullet \boldsymbol{x}_E \\ \boldsymbol{x}''_E \bullet \boldsymbol{y}_E & \boldsymbol{y}''_E \bullet \boldsymbol{y}_E & \boldsymbol{z}''_E \bullet \boldsymbol{y}_E \\ \boldsymbol{x}'' \bullet \boldsymbol{z}_E & \boldsymbol{y}''_E \bullet \boldsymbol{z}_E & \boldsymbol{z}''_E \bullet \boldsymbol{z}_E \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} \frac{-1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \\ 0 \end{bmatrix} \bullet \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} & \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \\ 0 \end{bmatrix} \bullet \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \bullet \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} \frac{-1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \\ 0 \end{bmatrix} \bullet \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} & \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \\ 0 \end{bmatrix} \bullet \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \bullet \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \\ \begin{bmatrix} \frac{-1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \\ 0 \end{bmatrix} \bullet \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} & \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \\ 0 \end{bmatrix} \bullet \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \bullet \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Evaluating the matrix $[\boldsymbol{R}]_E$ shows that it is identical to the matrix $[\boldsymbol{R_n}\langle\theta\rangle]$. Furthermore, each element of the matrix is the cosine of the angle between one of the original basis vectors, corresponding to the matrix row, and one of the rotated basis vectors, corresponding to the matrix column. This indicates that the angle from the vector $\boldsymbol{z}$ to the vector $\boldsymbol{z}''$ is zero (since the cosine of this angle is equal to 1). The angles from the vector $\boldsymbol{x}$ to the vector $\boldsymbol{x}''$, and from the vector $\boldsymbol{y}$ to the vector $\boldsymbol{y}''$ are $-135°$, while the angle from the vector $\boldsymbol{y}$ to the vector $\boldsymbol{x}''$ is $135°$ (the cosines of these angles are all equal to $\frac{-1}{\sqrt{2}}$). The angle from the vector $\boldsymbol{x}$ to the vector $\boldsymbol{y}''$ is $-45°$ (the cosine of this angle is $\frac{1}{\sqrt{2}}$). The angles for all other pairs of original and rotated basis vectors are $90°$ (since the cosines of these angles are equal to zero).

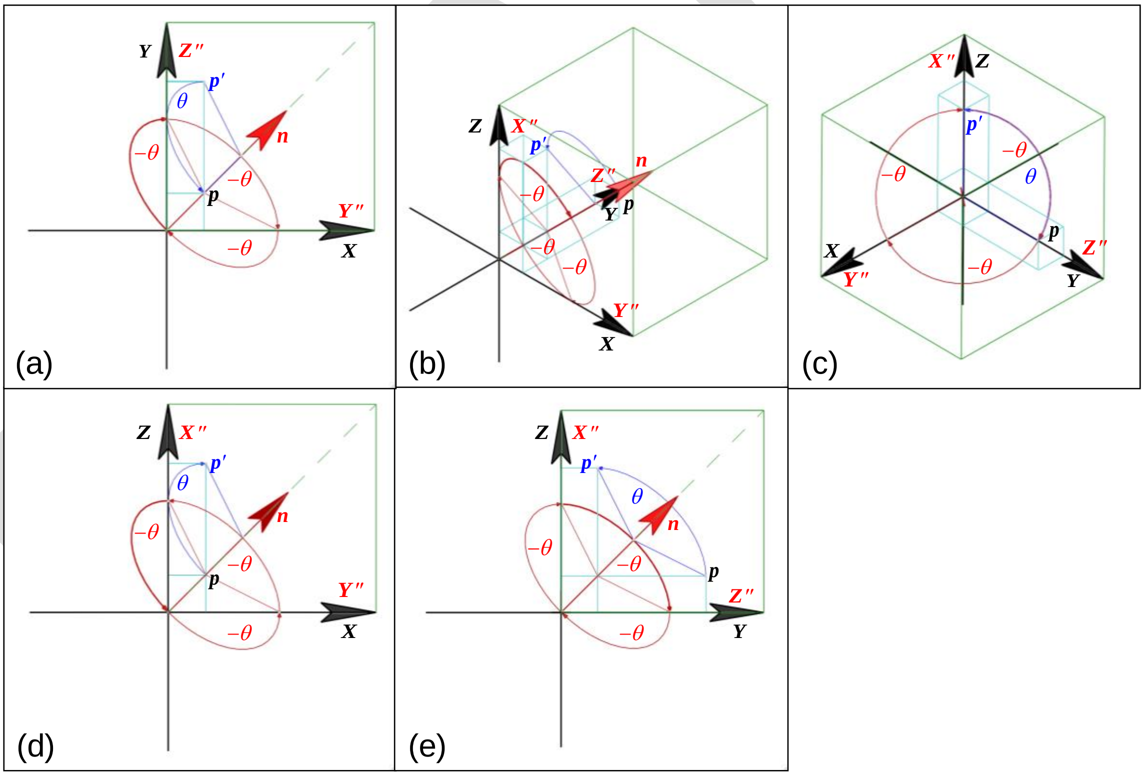

Figure 6.16 — Rotation of the basis vectors about the unit cube diagonal axis in the opposite direction from the position-vector: (a) top, (b) isometric, (c) along rotation axis toward origin, (d) front and (e) side views

EXAMPLE 2 Figure 6.16 illustrates a rotation operation where the rotation axis is along the diagonal of the unit cube of the orthonormal frame $\boldsymbol{E}$, i.e., $\boldsymbol{n}_E = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix}^{\mathrm{T}}$, and the rotation angle is $\theta = 120°$. Therefore $\sin\theta = \frac{\sqrt{3}}{2}$ and $\cos\theta = \frac{-1}{2}$, and the rotation matrix is:

$$[\boldsymbol{R_n}\langle\theta\rangle^{-1}] = \begin{bmatrix} \left(\frac{3}{2}\right)\left(\frac{1}{3}\right) + \left(\frac{-1}{2}\right) & \left(\frac{3}{2}\right)\left(\frac{1}{3}\right) + \left(\frac{1}{\sqrt{3}}\right)\left(\frac{\sqrt{3}}{2}\right) & \left(\frac{3}{2}\right)\left(\frac{1}{3}\right) - \left(\frac{1}{\sqrt{3}}\right)\left(\frac{\sqrt{3}}{2}\right) \\ \left(\frac{3}{2}\right)\left(\frac{1}{3}\right) - \left(\frac{1}{\sqrt{3}}\right)\left(\frac{\sqrt{3}}{2}\right) & \left(\frac{3}{2}\right)\left(\frac{1}{3}\right) + \left(\frac{-1}{2}\right) & \left(\frac{3}{2}\right)\left(\frac{1}{3}\right) + \left(\frac{1}{\sqrt{3}}\right)\left(\frac{\sqrt{3}}{2}\right) \\ \left(\frac{3}{2}\right)\left(\frac{1}{3}\right) + \left(\frac{1}{\sqrt{3}}\right)\left(\frac{\sqrt{3}}{2}\right) & \left(\frac{3}{2}\right)\left(\frac{1}{3}\right) - \left(\frac{1}{\sqrt{3}}\right)\left(\frac{\sqrt{3}}{2}\right) & \left(\frac{3}{2}\right)\left(\frac{1}{3}\right) + \left(\frac{-1}{2}\right) \end{bmatrix}, \text{ or}$$

$$[\boldsymbol{R_n}\langle\theta\rangle^{-1}] = \begin{bmatrix} \left(\frac{1}{2}\right) + \left(\frac{-1}{2}\right) & \left(\frac{1}{2}\right) + \left(\frac{1}{2}\right) & \left(\frac{1}{2}\right) - \left(\frac{1}{2}\right) \\ \left(\frac{1}{2}\right) - \left(\frac{1}{2}\right) & \left(\frac{1}{2}\right) + \left(\frac{-1}{2}\right) & \left(\frac{1}{2}\right) + \left(\frac{1}{2}\right) \\ \left(\frac{1}{2}\right) + \left(\frac{1}{2}\right) & \left(\frac{1}{2}\right) - \left(\frac{1}{2}\right) & \left(\frac{1}{2}\right) + \left(\frac{-1}{2}\right) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

This operation "rotates" the basis vectors, so that the $\boldsymbol{x}''$-axis rotates onto the $\boldsymbol{z}$-axis, the $\boldsymbol{y}''$-axis rotates onto the $\boldsymbol{x}$-axis, and the $\boldsymbol{z}''$-axis rotates onto the $\boldsymbol{y}$-axis.

The matrix $[\boldsymbol{R_n}\langle\theta\rangle^{-1}]$ is the transpose, and therefore the inverse, of the matrix $[\boldsymbol{R_n}\langle\theta\rangle]$ from Example 2 in 6.4.2.

$$[\boldsymbol{R_n}\langle\theta\rangle^{-1}][\boldsymbol{R_n}\langle\theta\rangle] = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \bullet \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Applying this rotation operation to the basis vectors $\boldsymbol{x}$, $\boldsymbol{y}$, $\boldsymbol{z}$ of the frame $\boldsymbol{E}$ yields:

$$\boldsymbol{x}''_{\boldsymbol{E}} = \begin{bmatrix} (1-\cos\theta)n_x^2 + \cos\theta \\ (1-\cos\theta)n_y n_x - n_z \sin\theta \\ (1-\cos\theta)n_z n_x + n_y \sin\theta \end{bmatrix} = \begin{bmatrix} \left(\frac{3}{2}\right)\left(\frac{1}{3}\right) + (-1/2) \\ \left(\frac{3}{2}\right)\left(\frac{1}{3}\right) - \left(\frac{1}{\sqrt{3}}\right)\left(\frac{\sqrt{3}}{2}\right) \\ \left(\frac{3}{2}\right)\left(\frac{1}{3}\right) + \left(\frac{1}{\sqrt{3}}\right)\left(\frac{\sqrt{3}}{2}\right) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \boldsymbol{z_E}$$

$$\boldsymbol{y}''_{\boldsymbol{E}} = \begin{bmatrix} (1-\cos\theta)n_x n_y + n_z \sin\theta \\ (1-\cos\theta)n_y^2 + \cos\theta \\ (1-\cos\theta)n_z n_y - n_x \sin\theta \end{bmatrix} = \begin{bmatrix} \left(\frac{3}{2}\right)\left(\frac{1}{3}\right) + \left(\frac{1}{\sqrt{3}}\right)\left(\frac{\sqrt{3}}{2}\right) \\ \left(\frac{3}{2}\right)\left(\frac{1}{3}\right) + (-1/2) \\ \left(\frac{3}{2}\right)\left(\frac{1}{3}\right) - \left(\frac{1}{\sqrt{3}}\right)\left(\frac{\sqrt{3}}{2}\right) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \boldsymbol{x_E}$$

$$\boldsymbol{z}''_{\boldsymbol{E}} = \begin{bmatrix} (1-\cos\theta)n_x n_z - n_y \sin\theta \\ (1-\cos\theta)n_y n_z + n_x \sin\theta \\ (1-\cos\theta)n_z^2 + \cos\theta \end{bmatrix} = \begin{bmatrix} \left(\frac{3}{2}\right)\left(\frac{1}{3}\right) - \left(\frac{1}{\sqrt{3}}\right)\left(\frac{\sqrt{3}}{2}\right) \\ \left(\frac{3}{2}\right)\left(\frac{1}{3}\right) + \left(\frac{1}{\sqrt{3}}\right)\left(\frac{\sqrt{3}}{2}\right) \\ \left(\frac{3}{2}\right)\left(\frac{1}{3}\right) + (-1/2) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \boldsymbol{y_E}$$

$$[\boldsymbol{R}^{-1}]_{\boldsymbol{E}} = \begin{bmatrix} \boldsymbol{x}''_{\boldsymbol{E}} \bullet \boldsymbol{x_E} & \boldsymbol{y}''_{\boldsymbol{E}} \bullet \boldsymbol{x_E} & \boldsymbol{z}''_{\boldsymbol{E}} \bullet \boldsymbol{x_E} \\ \boldsymbol{x}''_{\boldsymbol{E}} \bullet \boldsymbol{y_E} & \boldsymbol{y}''_{\boldsymbol{E}} \bullet \boldsymbol{y_E} & \boldsymbol{z}'' \bullet \boldsymbol{y_E} \\ \boldsymbol{x}''_{\boldsymbol{E}} \bullet \boldsymbol{z_E} & \boldsymbol{y}''_{\boldsymbol{E}\boldsymbol{E}} \bullet \boldsymbol{z_E} & \boldsymbol{z}'' \bullet \boldsymbol{z_E} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \bullet \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} & \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \bullet \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \bullet \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \bullet \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} & \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \bullet \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \bullet \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \\ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \bullet \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} & \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \bullet \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} & \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \bullet \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

The matrix $[\boldsymbol{R}^{-1}]_{\boldsymbol{E}}$ is the transpose, and therefore the inverse, of the matrix $[\boldsymbol{R}]_{\boldsymbol{E}}$ from Example 2 in 6.4.2.

$$[\boldsymbol{R}^{-1}]_{\boldsymbol{E}}[\boldsymbol{R}]_{\boldsymbol{E}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \bullet \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

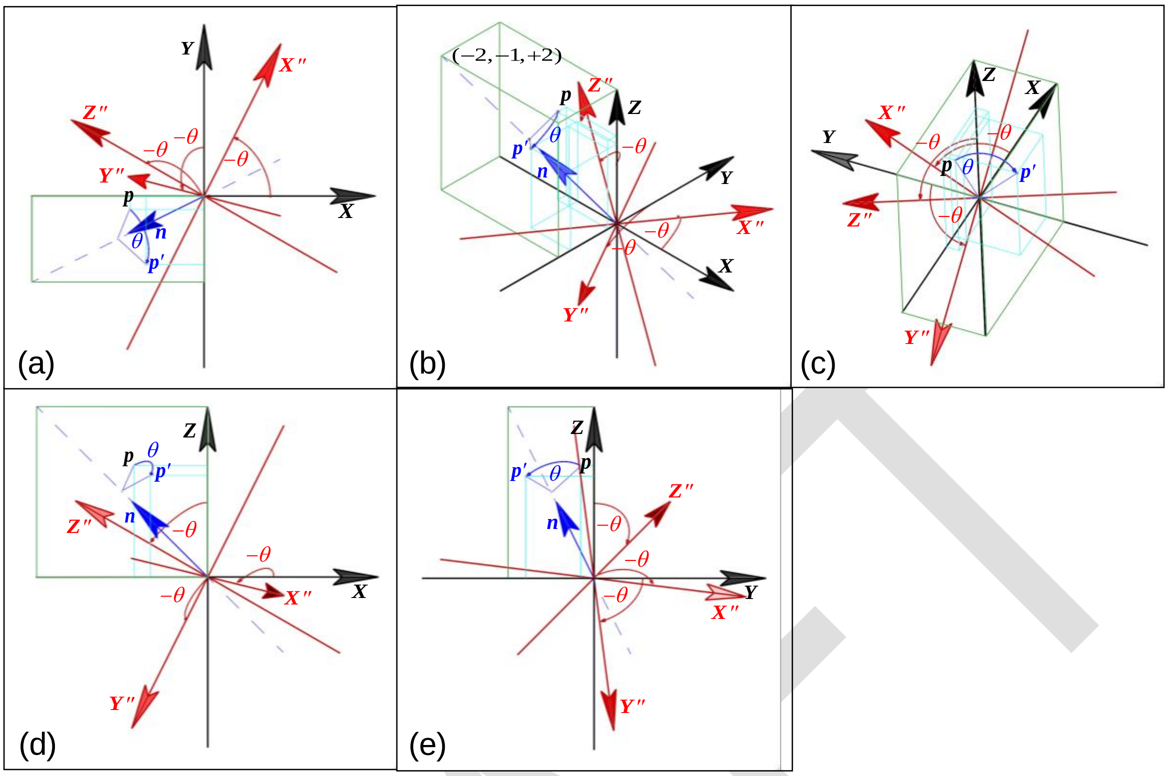

Figure 6.17 — Rotation of the basis vectors about an asymmetric axis in the opposite direction from the position-vector: (a) top, (b) isometric, (c) along rotation axis toward origin, (d) front and (e) side views

EXAMPLE 3 Figure 6.17 illustrates a rotation operation where the rotation axis passes through the point $(\text{-}2, \text{-}1, 2)$ of the orthonormal frame $\boldsymbol{E}$, i.e., $\boldsymbol{n}_{\boldsymbol{E}} = \begin{pmatrix} \frac{-2}{3} & \frac{-1}{3} & \frac{2}{3} \end{pmatrix}^{\mathrm{T}}$, and the rotation angle is $\theta = -90°$. Therefore $\sin\theta = -1$ and $\cos\theta = 0$, and the rotation matrix is:

$$[\boldsymbol{R}_{\boldsymbol{n}}\langle\theta\rangle^{-1}] = \begin{bmatrix} (1)\left(\frac{-2}{3}\right)\left(\frac{-2}{3}\right) + (0) & (1)\left(\frac{-2}{3}\right)\left(\frac{-1}{3}\right) + \left(\frac{2}{3}\right)(-1) & (1)\left(\frac{-2}{3}\right)\left(\frac{2}{3}\right) - \left(\frac{-1}{3}\right)(-1) \\ (1)\left(\frac{-1}{3}\right)\left(\frac{-2}{3}\right) - \left(\frac{2}{3}\right)(-1) & (1)\left(\frac{-1}{3}\right)\left(\frac{-1}{3}\right) + (0) & (1)\left(\frac{-1}{3}\right)\left(\frac{2}{3}\right) + \left(\frac{-2}{3}\right)(-1) \\ (1)\left(\frac{2}{3}\right)\left(\frac{-2}{3}\right) + \left(\frac{-1}{3}\right)(-1) & (1)\left(\frac{2}{3}\right)\left(\frac{-1}{3}\right) - \left(\frac{-2}{3}\right)(-1) & (1)\left(\frac{2}{3}\right)\left(\frac{2}{3}\right) + (0) \end{bmatrix}, \text{ or}$$

$$[\boldsymbol{R}_{\boldsymbol{n}}\langle\theta\rangle^{-1}] = \begin{bmatrix} \left(\frac{4}{9}\right) + (0) & \left(\frac{2}{9}\right) + \left(\frac{-2}{3}\right) & \left(\frac{-4}{9}\right) - \left(\frac{1}{3}\right) \\ \left(\frac{2}{9}\right) - \left(\frac{-2}{3}\right) & \left(\frac{1}{9}\right) + (0) & \left(\frac{-2}{9}\right) + \left(\frac{2}{3}\right) \\ \left(\frac{-4}{9}\right) + \left(\frac{1}{3}\right) & \left(\frac{-2}{9}\right) - \left(\frac{2}{3}\right) & \left(\frac{4}{9}\right) + (0) \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 4 & -4 & -7 \\ 8 & 1 & 4 \\ -1 & -8 & 4 \end{bmatrix}$$

The matrix $[\boldsymbol{R}_{\boldsymbol{n}}\langle\theta\rangle^{-1}]$ is the transpose, and therefore the inverse, of the matrix $[\boldsymbol{R}_{\boldsymbol{n}}\langle\theta\rangle]$ from Example 3 in 6.4.2.

$$[\boldsymbol{R}_{\boldsymbol{n}}\langle\theta\rangle^{-1}][\boldsymbol{R}_{\boldsymbol{n}}\langle\theta\rangle] = \frac{1}{9}\begin{bmatrix} 4 & -4 & -7 \\ 8 & 1 & 4 \\ -1 & -8 & 4 \end{bmatrix} \bullet \frac{1}{9}\begin{bmatrix} 4 & 8 & -1 \\ -4 & 1 & -8 \\ -7 & 4 & 4 \end{bmatrix} = \frac{1}{81}\begin{bmatrix} 16+16+49 & 32-4-28 & -4+32-28 \\ 32-4-28 & 64+1+16 & -8-8+16 \\ -4+32-28 & -8-8+16 & 1+64+16 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Applying this rotation operation to the basis vectors $\boldsymbol{x}$, $\boldsymbol{y}$, $\boldsymbol{z}$ of the frame $\boldsymbol{E}$ yields:

$$\boldsymbol{x}_{\boldsymbol{E}}'' = \begin{bmatrix} (1-\cos\theta)n_x^2 + \cos\theta \\ (1-\cos\theta)n_y n_x - n_z \sin\theta \\ (1-\cos\theta)n_z n_x + n_y \sin\theta \end{bmatrix} = \begin{bmatrix} (1)\left(\frac{-2}{3}\right)\left(\frac{-2}{3}\right) + (0) \\ (1)\left(\frac{-1}{3}\right)\left(\frac{-2}{3}\right) - \left(\frac{2}{3}\right)(-1) \\ (1)\left(\frac{2}{3}\right)\left(\frac{-2}{3}\right) + \left(\frac{-1}{3}\right)(-1) \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 4 \\ 8 \\ -1 \end{bmatrix}$$

$$\boldsymbol{y}_{\boldsymbol{E}}'' = \begin{bmatrix} (1-\cos\theta)n_x n_y + n_z \sin\theta \\ (1-\cos\theta)n_y^2 + \cos\theta \\ (1-\cos\theta)n_z n_y - n_x \sin\theta \end{bmatrix} = \begin{bmatrix} (1)\left(\frac{-2}{3}\right)\left(\frac{-1}{3}\right) + \left(\frac{2}{3}\right)(-1) \\ (1)\left(\frac{-1}{3}\right)\left(\frac{-1}{3}\right) + (0) \\ (1)\left(\frac{2}{3}\right)\left(\frac{-1}{3}\right) - \left(\frac{-2}{3}\right)(-1) \end{bmatrix} = \frac{1}{9}\begin{bmatrix} -4 \\ 1 \\ -8 \end{bmatrix}$$

$$\boldsymbol{z}_{\boldsymbol{E}}'' = \begin{bmatrix} (1-\cos\theta)n_x n_z - n_y \sin\theta \\ (1-\cos\theta)n_y n_z + n_x \sin\theta \\ (1-\cos\theta)n_z^2 + \cos\theta \end{bmatrix} = \begin{bmatrix} (1)\left(\frac{-2}{3}\right)\left(\frac{2}{3}\right) - \left(\frac{-1}{3}\right)(-1) \\ (1)\left(\frac{-1}{3}\right)\left(\frac{2}{3}\right) + \left(\frac{-2}{3}\right)(-1) \\ (1)\left(\frac{2}{3}\right)\left(\frac{2}{3}\right) + (0) \end{bmatrix} = \frac{1}{9}\begin{bmatrix} -7 \\ 4 \\ 4 \end{bmatrix}$$

$$[\boldsymbol{R}^{-1}]_{\boldsymbol{E}} = \begin{bmatrix} \boldsymbol{x}_{\boldsymbol{E}}'' \bullet \boldsymbol{x}_{\boldsymbol{E}} & \boldsymbol{y}_{\boldsymbol{E}}'' \bullet \boldsymbol{x}_{\boldsymbol{E}} & \boldsymbol{z}_{\boldsymbol{E}}'' \bullet \boldsymbol{x}_{\boldsymbol{E}} \\ \boldsymbol{x}_{\boldsymbol{E}}'' \bullet \boldsymbol{y}_{\boldsymbol{E}} & \boldsymbol{y}_{\boldsymbol{E}}'' \bullet \boldsymbol{y}_{\boldsymbol{E}} & \boldsymbol{z}'' \bullet \boldsymbol{y}_{\boldsymbol{E}} \\ \boldsymbol{x}_{\boldsymbol{E}}'' \bullet \boldsymbol{z}_{\boldsymbol{E}} & \boldsymbol{y}_{\boldsymbol{E}_{\boldsymbol{E}}}'' \bullet \boldsymbol{z}_{\boldsymbol{E}} & \boldsymbol{z}'' \bullet \boldsymbol{z}_{\boldsymbol{E}} \end{bmatrix} = \begin{bmatrix} \frac{1}{9}\begin{bmatrix} 4 \\ 8 \\ -1 \end{bmatrix} \bullet \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} & \frac{1}{9}\begin{bmatrix} -4 \\ 1 \\ -8 \end{bmatrix} \bullet \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} & \frac{1}{9}\begin{bmatrix} -7 \\ 4 \\ 4 \end{bmatrix} \bullet \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \\ \frac{1}{9}\begin{bmatrix} 4 \\ 8 \\ -1 \end{bmatrix} \bullet \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} & \frac{1}{9}\begin{bmatrix} -4 \\ 1 \\ -8 \end{bmatrix} \bullet \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} & \frac{1}{9}\begin{bmatrix} -7 \\ 4 \\ 4 \end{bmatrix} \bullet \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \\ \frac{1}{9}\begin{bmatrix} 4 \\ 8 \\ -1 \end{bmatrix} \bullet \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} & \frac{1}{9}\begin{bmatrix} -4 \\ 1 \\ -8 \end{bmatrix} \bullet \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} & \frac{1}{9}\begin{bmatrix} -7 \\ 4 \\ 4 \end{bmatrix} \bullet \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 4 & -4 & -7 \\ 8 & 1 & 3 \\ -1 & -8 & 3 \end{bmatrix}$$

The matrix $[\boldsymbol{R}^{-1}]_{\boldsymbol{E}}$ is the transpose, and therefore the inverse, of the matrix $[\boldsymbol{R}]_{\boldsymbol{E}}$ from Example 3 in 6.4.2.

$$[\boldsymbol{R}^{-1}]_{\boldsymbol{E}}[\boldsymbol{R}]_{\boldsymbol{E}} = \frac{1}{9}\begin{bmatrix} 4 & -4 & -7 \\ 8 & 1 & 4 \\ -1 & -8 & 4 \end{bmatrix} \bullet \frac{1}{9}\begin{bmatrix} 4 & 8 & -1 \\ -4 & 1 & -8 \\ -7 & 4 & 4 \end{bmatrix} = \frac{1}{81}\begin{bmatrix} 16+16+49 & 32-4-28 & -4+32-28 \\ 32-4-28 & 64+1+16 & -8-8+16 \\ -4+32-28 & -8-8+16 & 1+64+16 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

6.5 Change of basis

6.5.1 Change of basis operator

If $\boldsymbol{E}$ and $\boldsymbol{F}$ are two orthonormal frames with common origin and respective bases $\boldsymbol{x}$, $\boldsymbol{y}$, $\boldsymbol{z}$ and $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$, the operation that converts the vector-coordinate $(p_{\boldsymbol{u}}, p_v, p_{\boldsymbol{w}})_{\boldsymbol{F}}$ of a position-vector $\boldsymbol{p}$ with respect to frame $\boldsymbol{F}$ to its vector-coordinate $\left(p_x, p_y, p_z\right)_{\boldsymbol{E}}$ with respect to frame $\boldsymbol{E}$ shall be denoted by $\boldsymbol{\Omega}_{\boldsymbol{E}\leftarrow\boldsymbol{F}}$. Thus: $\left(p_x, p_y, p_z\right)_{\boldsymbol{E}} = \boldsymbol{\Omega}_{\boldsymbol{E}\leftarrow\boldsymbol{F}}((p_{\boldsymbol{u}}, p_v, p_{\boldsymbol{w}})_{\boldsymbol{F}})$. Figure 6.18 illustrates the case when $\boldsymbol{E}$ and $\boldsymbol{F}$ have a common third basis vector, $\boldsymbol{z} = \boldsymbol{w}$.

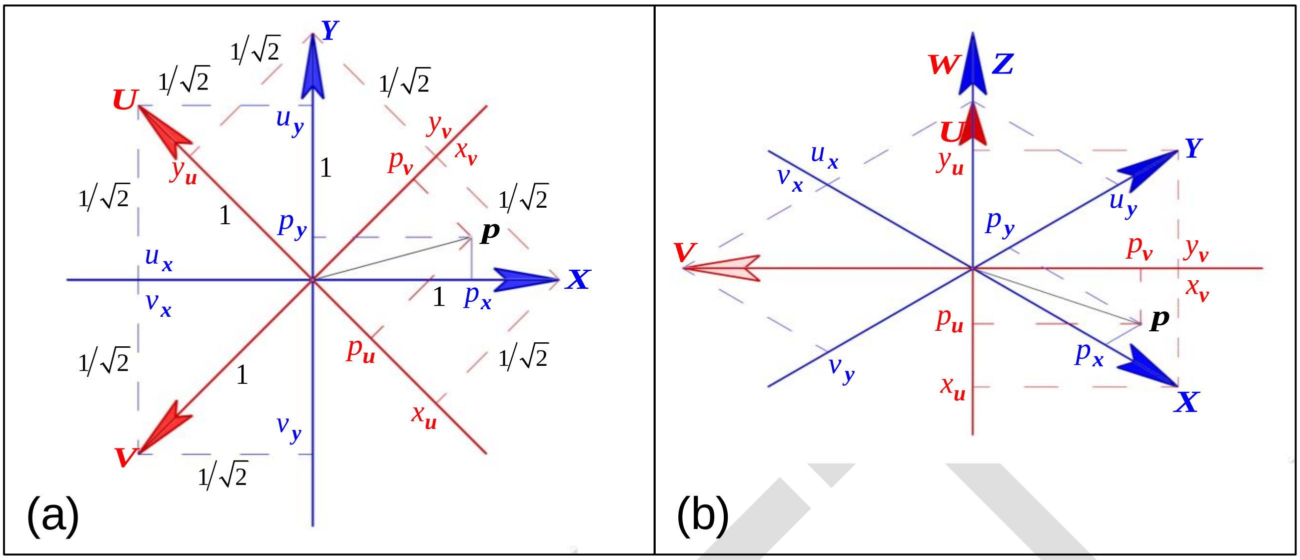

Figure 6.18 – Change of basis operation: a) top view; b) isometric view.

Using Equation **Error! Reference source not found.** for vector-coordinate components of $\boldsymbol{p}$ with respect to $\boldsymbol{x}$, $\boldsymbol{y}$, $\boldsymbol{z}$ and subsequently with respect to $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$ yields:

$$p_x = \boldsymbol{p} \bullet \boldsymbol{x} = (p_u\boldsymbol{u} + p_v\boldsymbol{v} + p_w\boldsymbol{w}) \bullet \boldsymbol{x} = p_u(\boldsymbol{u} \bullet \boldsymbol{x}) + p_v(\boldsymbol{v} \bullet \boldsymbol{x}) + p_w(\boldsymbol{w} \bullet \boldsymbol{x}),$$

$$p_y = \boldsymbol{p} \bullet \boldsymbol{y} = (p_u\boldsymbol{u} + p_v\boldsymbol{v} + p_w\boldsymbol{w}) \bullet \boldsymbol{y} = p_u(\boldsymbol{u} \bullet \boldsymbol{y}) + p_v(\boldsymbol{v} \bullet \boldsymbol{y}) + p_w(\boldsymbol{w} \bullet \boldsymbol{y}),$$

$$p_z = \boldsymbol{p} \bullet \boldsymbol{z} = (p_u\boldsymbol{u} + p_v\boldsymbol{v} + p_w\boldsymbol{w}) \bullet \boldsymbol{z} = p_u(\boldsymbol{u} \bullet \boldsymbol{z}) + p_v(\boldsymbol{v} \bullet \boldsymbol{z}) + p_w(\boldsymbol{w} \bullet \boldsymbol{z}).$$

The matrix form of these three equations is:

$$\begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix}_{\boldsymbol{E}} = [\boldsymbol{\Omega}_{\boldsymbol{E}\leftarrow\boldsymbol{F}}] \begin{bmatrix} p_u \\ p_v \\ p_w \end{bmatrix}_{\boldsymbol{F}} \quad \text{where: } [\boldsymbol{\Omega}_{\boldsymbol{E}\leftarrow\boldsymbol{F}}] = \begin{bmatrix} \boldsymbol{u} \bullet \boldsymbol{x} & \boldsymbol{v} \bullet \boldsymbol{x} & \boldsymbol{w} \bullet \boldsymbol{x} \\ \boldsymbol{u} \bullet \boldsymbol{y} & \boldsymbol{v} \bullet \boldsymbol{y} & \boldsymbol{w} \bullet \boldsymbol{y} \\ \boldsymbol{u} \bullet \boldsymbol{z} & \boldsymbol{v} \bullet \boldsymbol{z} & \boldsymbol{w} \bullet \boldsymbol{z} \end{bmatrix} \tag{6.14}$$

The matrix $[\boldsymbol{\Omega}_{\boldsymbol{E}\leftarrow\boldsymbol{F}}]$ is thus a representation of the change of basis operator $\boldsymbol{\Omega}_{\boldsymbol{E}\leftarrow\boldsymbol{F}}$, which converts vector-coordinates with respect to the frame $\boldsymbol{F}$ into vector-coordinates with respect to the frame $\boldsymbol{E}$.

Since basis vectors are unit vectors, each dot product in Equation (6.14) is the cosine of the angle between the two basis vectors (see A.2). This matrix is thus termed the *direction cosine matrix*. The direction cosine matrix represents the directed relationship between the two orthonormal frames, from frame $\boldsymbol{F}$ to frame $\boldsymbol{E}$. Each element in the direction cosine matrix represents the angle between a specific pair of basis vectors, one basis vector from frame $\boldsymbol{F}$, and one basis vector from frame $\boldsymbol{E}$.

The columns of the direction cosine matrix are the basis vectors $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$ expressed in terms of the basis vectors $\boldsymbol{x}$, $\boldsymbol{y}$, $\boldsymbol{z}$; the rows of the direction cosine matrix are the basis vectors $\boldsymbol{x}$, $\boldsymbol{y}$, $\boldsymbol{z}$ expressed in terms of the basis vectors $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$.

The matrix $[\boldsymbol{\Omega}_{\boldsymbol{E}\leftarrow\boldsymbol{F}}]$ is independent of any specific orthonormal frame. The basis vectors in the direction cosine matrix may be represented in terms of any orthonormal frame. In particular, they may be represented in terms of frame $\boldsymbol{E}$, they may be represented in terms of frame $\boldsymbol{F}$, or they may be represented in terms of any other frame. It is only when it becomes necessary to assign specific numerical values to the basis vector components, in order to perform computations, that an orthonormal frame must be chosen. A single orthonormal frame must be chosen in which to represent both sets of basis vectors.

EXAMPLE 1 Given two orthonormal frames with a common origin: frame $\boldsymbol{E}$ with basis $\boldsymbol{x}$, $\boldsymbol{y}$, $\boldsymbol{z}$; and frame $\boldsymbol{F}$ with basis $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$, where $\boldsymbol{z} = \boldsymbol{w}$, as shown in Figure 6.18, the matrix for the change of basis of operation $\boldsymbol{\Omega}_{\boldsymbol{E}\leftarrow\boldsymbol{F}}$ may be constructed by evaluating both sets of basis vectors with respect to a selected frame.

If the basis vectors are represented in frame $\boldsymbol{E}$, $\boldsymbol{x_E} = (1 \quad 0 \quad 0)^{\mathrm{T}}$, $\boldsymbol{y_E} = (0 \quad 1 \quad 0)^{\mathrm{T}}$, and $\boldsymbol{z_E} = (0 \quad 0 \quad 1)^{\mathrm{T}}$. As can be seen in Figure 6.18, the basis vectors of frame $\boldsymbol{F}$ represented in frame $\boldsymbol{E}$ are $\boldsymbol{u_E} = \left(-\frac{1}{\sqrt{2}} \quad \frac{+1}{\sqrt{2}} \quad 0\right)^{\mathrm{T}}$, $\boldsymbol{v_E} = \left(-\frac{1}{\sqrt{2}} \quad \frac{-1}{\sqrt{2}} \quad 0\right)^{\mathrm{T}}$, and $\boldsymbol{w_E} = \boldsymbol{z_E} = (0 \quad 0 \quad 1)^{\mathrm{T}}$. The direction cosine matrix in Equation 6.14 is then:

$$[\boldsymbol{\Omega}_{\boldsymbol{E}\leftarrow\boldsymbol{F}}] = \begin{bmatrix} \boldsymbol{u_E}\bullet\boldsymbol{x_E} & \boldsymbol{v_E}\bullet\boldsymbol{x_E} & \boldsymbol{w_E}\bullet\boldsymbol{x_E} \\ \boldsymbol{u_E}\bullet\boldsymbol{y_E} & \boldsymbol{v_E}\bullet\boldsymbol{y_E} & \boldsymbol{w_E}\bullet\boldsymbol{y_E} \\ \boldsymbol{u_E}\bullet\boldsymbol{z_E} & \boldsymbol{v_E}\bullet\boldsymbol{z_E} & \boldsymbol{w_E}\bullet\boldsymbol{z_E} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ +\frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}\bullet\begin{bmatrix}1\\0\\0\end{bmatrix} & \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}\bullet\begin{bmatrix}1\\0\\0\end{bmatrix} & \begin{bmatrix}0\\0\\1\end{bmatrix}\bullet\begin{bmatrix}1\\0\\0\end{bmatrix} \\ \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ +\frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}\bullet\begin{bmatrix}0\\1\\0\end{bmatrix} & \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}\bullet\begin{bmatrix}0\\1\\0\end{bmatrix} & \begin{bmatrix}0\\0\\1\end{bmatrix}\bullet\begin{bmatrix}0\\1\\0\end{bmatrix} \\ \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ +\frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}\bullet\begin{bmatrix}0\\0\\1\end{bmatrix} & \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}\bullet\begin{bmatrix}0\\0\\1\end{bmatrix} & \begin{bmatrix}0\\0\\1\end{bmatrix}\bullet\begin{bmatrix}0\\0\\1\end{bmatrix} \end{bmatrix} = \begin{bmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ +\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

If the basis vectors are represented in frame $\boldsymbol{F}$, $\boldsymbol{u_F} = (1 \quad 0 \quad 0)^{\mathrm{T}}$, $\boldsymbol{v_F} = (0 \quad 1 \quad 0)^{\mathrm{T}}$, and $\mathbf{w_F} = (0 \quad 0 \quad 1)^{\mathrm{T}}$. Again, as can be seen in Figure 6.18, the basis vectors of frame $\boldsymbol{E}$ represented in frame $\boldsymbol{F}$ are $\boldsymbol{x_F} = \left(-\frac{1}{\sqrt{2}} \quad \frac{-1}{\sqrt{2}} \quad 0\right)^{\mathrm{T}}$, $\boldsymbol{y_F} = \left(+\frac{1}{\sqrt{2}} \quad \frac{-1}{\sqrt{2}} \quad 0\right)^{\mathrm{T}}$, and $\boldsymbol{z_F} = \boldsymbol{w_F} = (0 \quad 0 \quad 1)^{\mathrm{T}}$. The direction cosine matrix in Equation 6.14 is then:

$$[\boldsymbol{\Omega}_{\boldsymbol{E}\leftarrow\boldsymbol{F}}] = \begin{bmatrix} \boldsymbol{u_F}\bullet\boldsymbol{x_F} & \boldsymbol{v_F}\bullet\boldsymbol{x_F} & \boldsymbol{w_F}\bullet\boldsymbol{x_F} \\ \boldsymbol{u_F}\bullet\boldsymbol{y_F} & \boldsymbol{v_F}\bullet\boldsymbol{y_F} & \boldsymbol{w_F}\bullet\boldsymbol{y_F} \\ \boldsymbol{u_F}\bullet\boldsymbol{z_F} & \boldsymbol{v_F}\bullet\boldsymbol{z_F} & \boldsymbol{w_F}\bullet\boldsymbol{z_F} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix}1\\0\\0\end{bmatrix}\bullet\begin{bmatrix} -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{bmatrix} & \begin{bmatrix}0\\1\\0\end{bmatrix}\bullet\begin{bmatrix} -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{bmatrix} & \begin{bmatrix}0\\0\\1\end{bmatrix}\bullet\begin{bmatrix} -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{bmatrix} \\ \begin{bmatrix}1\\0\\0\end{bmatrix}\bullet\begin{bmatrix} +\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{bmatrix} & \begin{bmatrix}0\\1\\0\end{bmatrix}\bullet\begin{bmatrix} +\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{bmatrix} & \begin{bmatrix}0\\0\\1\end{bmatrix}\bullet\begin{bmatrix} +\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{bmatrix} \\ \begin{bmatrix}1\\0\\0\end{bmatrix}\bullet\begin{bmatrix}0\\0\\1\end{bmatrix} & \begin{bmatrix}0\\1\\0\end{bmatrix}\bullet\begin{bmatrix}0\\0\\1\end{bmatrix} & \begin{bmatrix}0\\0\\1\end{bmatrix}\bullet\begin{bmatrix}0\\0\\1\end{bmatrix} \end{bmatrix} = \begin{bmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ +\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The columns of the matrix are the basis vectors $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$ expressed in terms of the basis vectors $\boldsymbol{x}$, $\boldsymbol{y}$, $\boldsymbol{z}$, i.e., $\boldsymbol{u_E}$, $\boldsymbol{v_E}$, $\boldsymbol{w_E}$. The rows of the matrix are the basis vectors $\boldsymbol{x}$, $\boldsymbol{y}$, $\boldsymbol{z}$ expressed in terms of the basis vectors $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$, i.e., $\boldsymbol{x_F}$, $\boldsymbol{y_F}$, $\boldsymbol{_Fz}$.

EXAMPLE 2 Two frames related by a circular mapping among their basis vectors.

EXAMPLE 3 Two frames with an arbitrary relationship.

6.5.2 Bidirectional change of basis

If $\boldsymbol{E}$ and $\boldsymbol{F}$ are two orthonormal frames with common origin and respective bases $\boldsymbol{x}$, $\boldsymbol{y}$, $\boldsymbol{z}$ and $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$, the operation that converts the vector-coordinate $(p_x, p_y, p_z)_{\boldsymbol{E}}$of a position-vector $\boldsymbol{p}$ with respect to frame $\boldsymbol{E}$ to its vector-coordinate $(p_u, p_v, p_w)_{\boldsymbol{F}}$ with respect to frame $\boldsymbol{F}$ shall be denoted by $\boldsymbol{\Omega}_{\boldsymbol{F}\leftarrow\boldsymbol{E}}$. Thus: $(p_u, p_v, p_w)_{\boldsymbol{F}} = \boldsymbol{\Omega}_{\boldsymbol{F}\leftarrow\boldsymbol{E}}\left((p_x, p_y, p_z)_{\boldsymbol{E}}\right)$.

Using Equation **Error! Reference source not found.** for vector-coordinate components of $\boldsymbol{p}$ with respect to $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$ and subsequently with respect to $\boldsymbol{x}$, $\boldsymbol{y}$, $\boldsymbol{z}$ yields:

$$p_u = \boldsymbol{p}\bullet\boldsymbol{u} = (p_x\boldsymbol{x} + p_y\boldsymbol{y} + p_z\boldsymbol{z})\bullet\boldsymbol{u} = p_x(\boldsymbol{x}\bullet\boldsymbol{u}) + p_y(\boldsymbol{y}\bullet\boldsymbol{u}) + p_z(\boldsymbol{z}\bullet\boldsymbol{u}),$$

$$p_v = \boldsymbol{p}\bullet\boldsymbol{v} = (p_x\boldsymbol{x} + p_y\boldsymbol{y} + p_z\boldsymbol{z})\bullet\boldsymbol{v} = p_x(\boldsymbol{x}\bullet\boldsymbol{v}) + p_y(\boldsymbol{y}\bullet\boldsymbol{v}) + p_z(\boldsymbol{z}\bullet\boldsymbol{v}),$$

$$p_w = \boldsymbol{p}\bullet\boldsymbol{w} = (p_x\boldsymbol{x} + p_y\boldsymbol{y} + p_z\boldsymbol{z})\bullet\boldsymbol{w} = p_x(\boldsymbol{x}\bullet\boldsymbol{w}) + p_y(\boldsymbol{y}\bullet\boldsymbol{w}) + p_z(\boldsymbol{z}\bullet\boldsymbol{w}).$$

The matrix form of these three equations is:

$$\begin{bmatrix} p_u \\ p_v \\ p_w \end{bmatrix}_{\boldsymbol{F}} = [\boldsymbol{\Omega}_{\boldsymbol{F}\leftarrow\boldsymbol{E}}] \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix}_{\boldsymbol{E}} \quad \text{where: } [\boldsymbol{\Omega}_{\boldsymbol{F}\leftarrow\boldsymbol{E}}] = \begin{bmatrix} \boldsymbol{x}\bullet\boldsymbol{u} & \boldsymbol{y}\bullet\boldsymbol{u} & \boldsymbol{z}\bullet\boldsymbol{u} \\ \boldsymbol{x}\bullet\boldsymbol{v} & \boldsymbol{y}\bullet\boldsymbol{v} & \boldsymbol{z}\bullet\boldsymbol{v} \\ \boldsymbol{x}\bullet\boldsymbol{w} & \boldsymbol{y}\bullet\boldsymbol{w} & \boldsymbol{z}\bullet\boldsymbol{w} \end{bmatrix} \tag{6.15}$$

The matrix $[\boldsymbol{\Omega}_{\boldsymbol{F}\leftarrow\boldsymbol{E}}]$ is thus a representation of the inverse change of basis operator $\boldsymbol{\Omega}_{\boldsymbol{F}\leftarrow\boldsymbol{E}}$, which converts vector-coordinates with respect to the frame $\boldsymbol{E}$ into vector-coordinates with respect to the frame $\boldsymbol{F}$.

The matrix $[\boldsymbol{\Omega}_{\boldsymbol{F}\leftarrow\boldsymbol{E}}]$ is the transpose, and the inverse, of the matrix $[\boldsymbol{\Omega}_{\boldsymbol{E}\leftarrow\boldsymbol{F}}]$. Thus:

$$[\boldsymbol{\Omega}_{\boldsymbol{F}\leftarrow\boldsymbol{E}}] = [\boldsymbol{\Omega}_{\boldsymbol{E}\leftarrow\boldsymbol{F}}]^{\mathrm{T}} = [\boldsymbol{\Omega}_{\boldsymbol{E}\leftarrow\boldsymbol{F}}]^{-1}$$

The matrix $[\boldsymbol{\Omega}_{\boldsymbol{F}\leftarrow\boldsymbol{E}}]$ is also a direction cosine matrix, and thus represents the directed relationship between the two orthonormal frames, from frame $\boldsymbol{E}$ to frame $\boldsymbol{F}$. Each element in the direction cosine matrix represents the angle between a specific pair of basis vectors, one basis vector from frame $\boldsymbol{E}$, and one basis vector from frame $\boldsymbol{F}$.

The columns of the direction cosine matrix are the basis vectors $\boldsymbol{x}$, $\boldsymbol{y}$, $\boldsymbol{z}$ expressed in terms of the basis vectors $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$; the rows of the direction cosine matrix are the basis vectors $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$ expressed in terms of the basis vectors $\boldsymbol{x}$, $\boldsymbol{y}$, $\boldsymbol{z}$.

The matrix $[\boldsymbol{\Omega}_{\boldsymbol{F}\leftarrow\boldsymbol{E}}]$ is independent of any specific orthonormal frame. The basis vectors in the direction cosine matrix may be represented in terms of any orthonormal frame. In particular, they may be represented in terms of frame $\boldsymbol{E}$, they may be represented in terms of frame $\boldsymbol{F}$, or they may be represented in terms of any other frame. It is only when it becomes necessary to assign specific numerical values to the basis vector components, in order to perform computations, that an orthonormal frame must be chosen. A single orthonormal frame must be chosen in which to represent both sets of basis vectors.

EXAMPLE 1 Given two orthonormal frames with a common origin: frame $\boldsymbol{E}$ with basis $\boldsymbol{x}$, $\boldsymbol{y}$, $\boldsymbol{z}$; and frame $\boldsymbol{F}$ with basis $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$, where $\boldsymbol{z} = \boldsymbol{w}$, as shown in Figure 6.18, the matrix for the change of basis of operation $\boldsymbol{\Omega}_{\boldsymbol{F}\leftarrow\boldsymbol{E}}$ may be constructed by evaluating both sets of basis vectors with respect to a selected frame.

If the basis vectors are represented in frame $\boldsymbol{E}$, $\boldsymbol{x}_{\boldsymbol{E}} = (1 \quad 0 \quad 0)^{\mathrm{T}}$, $\boldsymbol{y}_{\boldsymbol{E}} = (0 \quad 1 \quad 0)^{\mathrm{T}}$, and $\boldsymbol{z}_{\boldsymbol{E}} = (0 \quad 0 \quad 1)^{\mathrm{T}}$. As can be seen in Figure 6.18, the basis vectors of frame $\boldsymbol{F}$ represented in frame $\boldsymbol{E}$ are $\boldsymbol{u}_{\boldsymbol{E}} = \left(-\frac{1}{\sqrt{2}} \quad \frac{+1}{\sqrt{2}} \quad 0\right)^{\mathrm{T}}$, $\boldsymbol{v}_{\boldsymbol{E}} = \left(-\frac{1}{\sqrt{2}} \quad \frac{-1}{\sqrt{2}} \quad 0\right)^{\mathrm{T}}$, and $\boldsymbol{w}_{\boldsymbol{E}} = \boldsymbol{z}_{\boldsymbol{E}} = (0 \quad 0 \quad 1)^{\mathrm{T}}$. The direction cosine matrix in Equation 6.15 is then:

$$[\boldsymbol{\Omega}_{\boldsymbol{F}\leftarrow\boldsymbol{E}}] = \begin{bmatrix} \boldsymbol{x}_{\boldsymbol{E}}\bullet\boldsymbol{u}_{\boldsymbol{E}} & \boldsymbol{y}_{\boldsymbol{E}}\bullet\boldsymbol{u}_{\boldsymbol{E}} & \boldsymbol{z}_{\boldsymbol{E}}\bullet\boldsymbol{u}_{\boldsymbol{E}} \\ \boldsymbol{x}_{\boldsymbol{E}}\bullet\boldsymbol{v}_{\boldsymbol{E}} & \boldsymbol{y}_{\boldsymbol{E}}\bullet\boldsymbol{v}_{\boldsymbol{E}} & \boldsymbol{z}_{\boldsymbol{E}}\bullet\boldsymbol{v}_{\boldsymbol{E}} \\ \boldsymbol{x}_{\boldsymbol{E}}\bullet\boldsymbol{w}_{\boldsymbol{E}} & \boldsymbol{y}_{\boldsymbol{E}}\bullet\boldsymbol{w}_{\boldsymbol{E}} & \boldsymbol{z}_{\boldsymbol{E}}\bullet\boldsymbol{w}_{\boldsymbol{E}} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix}1\\0\\0\end{bmatrix}\bullet\begin{bmatrix}-\frac{1}{\sqrt{2}}\\+\frac{1}{\sqrt{2}}\\0\end{bmatrix} & \begin{bmatrix}0\\1\\0\end{bmatrix}\bullet\begin{bmatrix}-\frac{1}{\sqrt{2}}\\+\frac{1}{\sqrt{2}}\\0\end{bmatrix} & \begin{bmatrix}0\\0\\1\end{bmatrix}\bullet\begin{bmatrix}-\frac{1}{\sqrt{2}}\\+\frac{1}{\sqrt{2}}\\0\end{bmatrix} \\ \begin{bmatrix}1\\0\\0\end{bmatrix}\bullet\begin{bmatrix}-\frac{1}{\sqrt{2}}\\-\frac{1}{\sqrt{2}}\\0\end{bmatrix} & \begin{bmatrix}0\\1\\0\end{bmatrix}\bullet\begin{bmatrix}-\frac{1}{\sqrt{2}}\\-\frac{1}{\sqrt{2}}\\0\end{bmatrix} & \begin{bmatrix}0\\0\\1\end{bmatrix}\bullet\begin{bmatrix}-\frac{1}{\sqrt{2}}\\-\frac{1}{\sqrt{2}}\\0\end{bmatrix} \\ \begin{bmatrix}1\\0\\0\end{bmatrix}\bullet\begin{bmatrix}0\\0\\1\end{bmatrix} & \begin{bmatrix}0\\1\\0\end{bmatrix}\bullet\begin{bmatrix}0\\0\\1\end{bmatrix} & \begin{bmatrix}0\\0\\1\end{bmatrix}\bullet\begin{bmatrix}0\\0\\1\end{bmatrix} \end{bmatrix} = \begin{bmatrix} -\frac{1}{\sqrt{2}} & +\frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

If the basis vectors are represented in frame $\boldsymbol{F}$, $\boldsymbol{u}_{\boldsymbol{F}} = (1 \quad 0 \quad 0)^{\mathrm{T}}$, $\boldsymbol{v}_{\boldsymbol{F}} = (0 \quad 1 \quad 0)^{\mathrm{T}}$, and $\boldsymbol{w}_{\boldsymbol{F}} = (0 \quad 0 \quad 1)^{\mathrm{T}}$. Again, as can be seen in Figure 6.18, the basis vectors of frame $\boldsymbol{E}$ represented in frame $\boldsymbol{F}$ are $\boldsymbol{x}_{\boldsymbol{F}} = \left(-\frac{1}{\sqrt{2}} \quad \frac{-1}{\sqrt{2}} \quad 0\right)^{T}$, $\boldsymbol{y}_{\boldsymbol{F}} = \left(+\frac{1}{\sqrt{2}} \quad \frac{-1}{\sqrt{2}} \quad 0\right)^{\mathrm{T}}$, and $\boldsymbol{z}_{\boldsymbol{F}} = \boldsymbol{w}_{\boldsymbol{F}} = (0 \quad 0 \quad 1)^{\mathrm{T}}$. The direction cosine matrix in Equation 6.15 is then:

$$[\boldsymbol{\Omega}_{F\leftarrow E}] = \begin{bmatrix} x_F \bullet u_F & y_F \bullet u_F & z_F \bullet u_F \\ x_F \bullet v_F & y_F \bullet v_F & z_F \bullet v_F \\ x_F \bullet w_F & y_F \bullet w_F & z_F \bullet w_F \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{bmatrix} \bullet \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} & \begin{bmatrix} +\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{bmatrix} \bullet \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \bullet \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{bmatrix} \bullet \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} & \begin{bmatrix} +\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{bmatrix} \bullet \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \bullet \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \\ \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{bmatrix} \bullet \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} & \begin{bmatrix} +\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{bmatrix} \bullet \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \bullet \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} -\frac{1}{\sqrt{2}} & +\frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The columns of the matrix are the basis vectors $\boldsymbol{x}$, $\boldsymbol{y}$, $\boldsymbol{z}$ expressed in terms of the basis vectors $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$, i.e., $\boldsymbol{x}_F$, $\boldsymbol{y}_F$, $\boldsymbol{z}_F$. The rows of the matrix are the basis vectors $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$ expressed in terms of the basis vectors $\boldsymbol{x}$, $\boldsymbol{y}$, $\boldsymbol{z}$, i.e., $\boldsymbol{u}_E$, $\boldsymbol{v}_E$, $\boldsymbol{w}_E$.

EXAMPLE 2 Two frames related by a circular mapping among their basis vectors.

EXAMPLE 3 Two frames with an arbitrary relationship.

6.5.3 Change of basis and Euler's rotation theorem

Euler's rotation theorem states that any length-preserving transformation of 3D space that has at least one point fixed under the transformation is equivalent to a single rotation about an axis through that point. The theorem is applicable to the transformation $\boldsymbol{\Omega}_{E\leftarrow F}$ with the origin as the fixed point, thus there exists an equivalent operator in the form $\boldsymbol{R}_{\boldsymbol{n}}\langle\theta\rangle$. The unit position-vector $\boldsymbol{n}$ is an eigenvector for $\boldsymbol{\Omega}_{E\leftarrow F}$with three eigenvalues: 1, $e^{+i\theta}$, and $e^{-i\theta}$. Conversely, if $\boldsymbol{n}$ is a unit eigenvector of a length-preserving transformation of 3D space with eigenvalues: 1, $e^{+i\theta}$, and $e^{-i\theta}$, then the transformation is equivalent to $\boldsymbol{R}_{\boldsymbol{n}}\langle-\theta\rangle$ with the sign of θ to be determined.

An algorithm to compute θ and $(n_x, n_y, n_z)_E$, the vector-coordinate of $\boldsymbol{n}$, using the numeric elements of the direction cosine matrix for $\boldsymbol{\Omega}_{E\leftarrow F}$ is specified in 6.XX.

Comparing $[\boldsymbol{\Omega}_{E\leftarrow F}]$ in Equation (6.14) to $[\boldsymbol{R}]_{\boldsymbol{E}}$ in Equation (6.10), with the substitutions of Equation (6.11?), it is seen that $\boldsymbol{R}_{\boldsymbol{n}}\langle\theta\rangle$ is the rotation operator that rotates the orthonormal frame $\boldsymbol{E}$ to form orthonormal frame $\boldsymbol{F}$.

$$\begin{aligned} \boldsymbol{x}' &= \boldsymbol{R}_{\boldsymbol{n}}\langle\theta\rangle(\boldsymbol{x}), \\ \boldsymbol{y}' &= \boldsymbol{R}_{\boldsymbol{n}}\langle\theta\rangle(\boldsymbol{y}), \\ \boldsymbol{z}' &= \boldsymbol{R}_{\boldsymbol{n}}\langle\theta\rangle(\boldsymbol{z}). \end{aligned} \tag{6.9}$$

$$[\boldsymbol{R}]_{\boldsymbol{E}} = \begin{bmatrix} \boldsymbol{x}' \bullet \boldsymbol{x} & \boldsymbol{y}' \bullet \boldsymbol{x} & \boldsymbol{z}' \bullet \boldsymbol{x} \\ \boldsymbol{x}' \bullet \boldsymbol{y} & \boldsymbol{y}' \bullet \boldsymbol{y} & \boldsymbol{z}' \bullet \boldsymbol{y} \\ \boldsymbol{x}' \bullet \boldsymbol{z} & \boldsymbol{y}' \bullet \boldsymbol{z} & \boldsymbol{z}' \bullet \boldsymbol{z} \end{bmatrix} \tag{6.10}$$

$$[\boldsymbol{\Omega}_{E\leftarrow F}] = \begin{bmatrix} \boldsymbol{u} \bullet \boldsymbol{x} & \boldsymbol{v} \bullet \boldsymbol{x} & \boldsymbol{w} \bullet \boldsymbol{x} \\ \boldsymbol{u} \bullet \boldsymbol{y} & \boldsymbol{v} \bullet \boldsymbol{y} & \boldsymbol{w} \bullet \boldsymbol{y} \\ \boldsymbol{u} \bullet \boldsymbol{z} & \boldsymbol{v} \bullet \boldsymbol{z} & \boldsymbol{w} \bullet \boldsymbol{z} \end{bmatrix} \tag{6.14}$$

$$\boldsymbol{u} = \boldsymbol{R}(\boldsymbol{x}),\ \boldsymbol{v} = \boldsymbol{R}(\boldsymbol{y}),\ \boldsymbol{w} = \boldsymbol{R}(\boldsymbol{z})$$

These two types of coordinate transformations may be viewed as:

$(r_1, r_2, r_3)_{\boldsymbol{E}} \mapsto (r_1', r_2', r_3')_{\boldsymbol{E}}$, a rotation with respect to fixed frame $\boldsymbol{E}$

$(r_1, r_2, r_3)_{\boldsymbol{F}} \mapsto (\tilde{r}_1, \tilde{r}_2, \tilde{r}_3)_{\boldsymbol{F}}$, a rotation with respect to rotated frame $\boldsymbol{F}$

The rotation with respect to a fixed frame $\boldsymbol{E}$ is equivalent to $\boldsymbol{R_n}\langle\theta\rangle$. The rotation with respect to the rotated frame $\boldsymbol{F}$ shall be denoted by $\boldsymbol{\Omega_n}\langle\theta\rangle$. The latter case may also be viewed as a change of coordinate operator $(\tilde{r}_1, \tilde{r}_2, \tilde{r}_3)_{\boldsymbol{F}} = \boldsymbol{\Omega_{F \leftarrow E}}((r_1, r_2, r_3)_{\boldsymbol{E}})$. Thus it is seen that the operators $\boldsymbol{R_n}\langle\theta\rangle$ and $\boldsymbol{\Omega_n}\langle\theta\rangle$ are inverses of each other, that is $\boldsymbol{\Omega_n}\langle\theta\rangle = \boldsymbol{R_n}\langle-\theta\rangle$.

Without other context, there is no preference as to which orthonormal frame is the fixed frame and which is the rotated frame. If it is required to represent the axis vector $\boldsymbol{n}$ with a vector-coordinate, $\boldsymbol{E}$ serves as the reference frame and $\boldsymbol{F}$ is the frame rotated by $\boldsymbol{R_n}\langle\theta\rangle$. However, $\boldsymbol{n}$ is fixed under the rotation so that $\boldsymbol{F}$ may also serve as the reference frame with $\boldsymbol{E}$ as the frame rotated by $\boldsymbol{R_n}\langle-\theta\rangle$.

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6.5.4 Change of coordinate basis and rotations

The notion of orientation is translation independent so in this clause, without loss of generality, two or more orthonormal frames may be assumed to have a common origin point. Similarly, a rotation of space about an arbitrary axis line in space is, with translations, equivalent to a rotation about an axis that passes through a designated origin. Select a point on the arbitrary axis and set $\boldsymbol{p}$ to the vector from the origin to the selected point. Translate the axis by $-\boldsymbol{p}$, rotate about the translated axis (which passes through the origin) and finally translate back by $\boldsymbol{p}$. This sequence of operations produces the same result as the rotation about the arbitrary axis.

If $\boldsymbol{E}$ with basis $\boldsymbol{x}$, $\boldsymbol{y}$, $\boldsymbol{z}$, and $\boldsymbol{F}$ with basis $\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}, \tilde{\boldsymbol{z}}$, are two orthonormal frames with common origin, the coordinate representations of a point $\boldsymbol{r}$ in each frame is given by:

$(r_1, r_2, r_3)_{\boldsymbol{E}}$, where $\boldsymbol{r} = r_1\boldsymbol{x} + r_2\boldsymbol{y} + r_3\boldsymbol{z}$, and

$(\tilde{r}_1, \tilde{r}_2, \tilde{r}_3)_{\boldsymbol{F}}$, where $\boldsymbol{r} = \tilde{r}_1\tilde{\boldsymbol{x}} + \tilde{r}_2\tilde{\boldsymbol{y}} + \tilde{r}_3\tilde{\boldsymbol{z}}$.

Since each basis is orthonormal, the coordinate-component scalars may be computed as the dot product of $\boldsymbol{r}$ with each corresponding basis vector:

$r_1\boldsymbol{x} + r_2\boldsymbol{y} + r_3\boldsymbol{z} = (\boldsymbol{r} \bullet \boldsymbol{x})\boldsymbol{x} + (\boldsymbol{r} \bullet \boldsymbol{y})\boldsymbol{y} + (\boldsymbol{r} \bullet \boldsymbol{z})\boldsymbol{z}$, and

$\tilde{r}_1\tilde{\boldsymbol{x}} + \tilde{r}_2\tilde{\boldsymbol{y}} + \tilde{r}_3\tilde{\boldsymbol{z}} = (\boldsymbol{r} \bullet \tilde{\boldsymbol{x}})\tilde{\boldsymbol{x}} + (\boldsymbol{r} \bullet \tilde{\boldsymbol{y}})\tilde{\boldsymbol{y}} + (\boldsymbol{r} \bullet \tilde{\boldsymbol{z}})\tilde{\boldsymbol{z}}$.

The change coordinate basis operation taking an $\boldsymbol{E}$ coordinate $(r_1, r_2, r_3)_{\boldsymbol{E}}$ to an $\boldsymbol{F}$ coordinate $(\tilde{r}_1, \tilde{r}_2, \tilde{r}_3)_{\boldsymbol{F}}$ shall be denoted by $\boldsymbol{\Omega_{F \leftarrow E}}$. This operation, $\boldsymbol{\Omega_{F \leftarrow E}}$, is a linear transformation and can thus be realized as a matrix multiplication of coordinate column vectors:

$(\tilde{r}_1, \tilde{r}_2, \tilde{r}_3)_{\boldsymbol{F}} = \boldsymbol{\Omega_{F \leftarrow E}}((r_1, r_2, r_3)_{\boldsymbol{E}})$

where:

$$\begin{bmatrix}\tilde{r}_1\\ \tilde{r}_2\\ \tilde{r}_3\end{bmatrix} = \boldsymbol{M}\begin{bmatrix}r_1\\ r_2\\ r_3\end{bmatrix}, \text{ and}$$

$$\boldsymbol{M} = \begin{bmatrix}\boldsymbol{x} \bullet \tilde{\boldsymbol{x}} & \boldsymbol{y} \bullet \tilde{\boldsymbol{x}} & \boldsymbol{z} \bullet \tilde{\boldsymbol{x}}\\ \boldsymbol{x} \bullet \tilde{\boldsymbol{y}} & \boldsymbol{y} \bullet \tilde{\boldsymbol{y}} & \boldsymbol{z} \bullet \tilde{\boldsymbol{y}}\\ \boldsymbol{x} \bullet \tilde{\boldsymbol{z}} & \boldsymbol{y} \bullet \tilde{\boldsymbol{z}} & \boldsymbol{z} \bullet \tilde{\boldsymbol{z}}\end{bmatrix} \quad (6.1)$$

Since basis vectors are unit vectors, each dot product in Equation (6.1) is the cosine of the angle between the two vectors (see A.2). This matrix is thus termed the *direction cosine matrix*. Note that the columns of the matrix are the $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$ basis vectors in $\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}, \tilde{\boldsymbol{z}}$ coordinate representation while the rows (or columns of the transpose matrix) are the $\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}, \tilde{\boldsymbol{z}}$ basis vectors in $\boldsymbol{x}$, $\boldsymbol{y}$, $\boldsymbol{z}$ coordinate representation. In particular, the transpose $\boldsymbol{M}^T$ is the matrix for the inverse change of coordinate basis operation $(r_1, r_2, r_3)_{\boldsymbol{E}} = \boldsymbol{\Omega_{E \leftarrow F}}((\tilde{r}_1, \tilde{r}_2, \tilde{r}_3)_{\boldsymbol{F}})$. Thus $\boldsymbol{M}^{-1} = \boldsymbol{M}^{\mathrm{T}}$.

Euler's rotation theorem states that any length-preserving transformation of 3D space that has at least one point fixed under the transformation is equivalent to a single rotation about an axis through that point. If the axis is assigned a direction, the angle of rotation can be specified as a positive angle or a negative angle using the *right-hand rule*: conceptually, if the right-hand holds the axis with thumb pointing in the axis direction, the fingers curl in the positive angle direction.

Euler's rotation theorem applies to any linear transformation that is length-preserving, and thus, the transformation has a unit eigenvector $\boldsymbol{n}$ and three eigenvalues: 1, $e^{+i\theta}$, and $e^{-i\theta}$. The line passing through the origin and $\boldsymbol{n}$ is fixed under the transformation and represents the axis of rotation. The angle of rotation is given by $\pm\theta$. $\boldsymbol{R}_{\boldsymbol{n}}\langle\theta\rangle$ shall denote the rotation about rotation axis $\boldsymbol{n}$ through angle θ. The axis direction is from the origin towards $\boldsymbol{n}$.

There are two conventions in use for specifying the angle of rotation. Either the angle is measured from the starting position of a point to its rotated position, or it is measured from its rotated position to its starting position. The first convention is the *position vector rotation* (PVR) convention, and the second convention is the *coordinate frame rotation* (CFR) convention. Figure 6.1 illustrates the two conventions for a point $\boldsymbol{r}$ that is rotated to a new position $\boldsymbol{r}'$ about an axis that is perpendicular to the plane of the figure. Thus, $\boldsymbol{r}' = \boldsymbol{R}_{\boldsymbol{n}}\langle\theta_{\mathrm{PVR}}\rangle(\boldsymbol{r}) = \boldsymbol{R}_{\boldsymbol{n}}\langle\theta_{\mathrm{CFR}}\rangle(\boldsymbol{r})$. When an angle convention is not specified, the PVR convention is assumed.

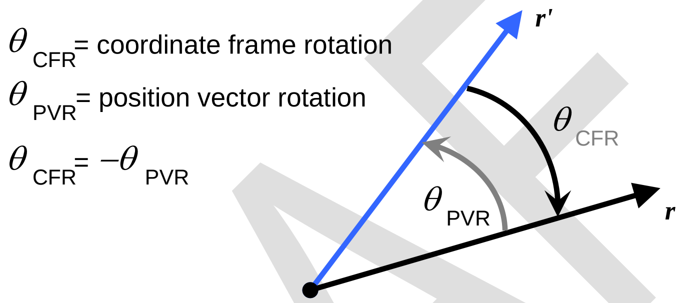


Figure 6.1 — Rotation between $\boldsymbol{r}$ and $\boldsymbol{r}'$ in two conventions

In the case of a change of coordinate basis operator $\boldsymbol{\Omega}_{F\leftarrow E}$, the direction cosine matrix $\boldsymbol{M}$ operating on coordinate three-tuples is length-preserving. Thus, Euler's rotation theorem associates a rotation operation with a change of coordinate basis operation. In particular, the associated rotation $\boldsymbol{R}_{\boldsymbol{n}}\langle\theta_{\mathrm{PVR}}\rangle$ rotates the basis vectors $\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}, \tilde{\boldsymbol{z}}$ to coincide with corresponding basis vectors $\boldsymbol{x}$, $\boldsymbol{y}$, $\boldsymbol{z}$. That is,

$\boldsymbol{x} = \boldsymbol{R}_{\boldsymbol{n}}\langle\theta_{\mathrm{PVR}}\rangle(\tilde{\boldsymbol{x}})$

$\boldsymbol{y} = \boldsymbol{R}_{\boldsymbol{n}}\langle\theta_{\mathrm{PVR}}\rangle(\tilde{\boldsymbol{y}})$

$\boldsymbol{z} = \boldsymbol{R}_{\boldsymbol{n}}\langle\theta_{\mathrm{PVR}}\rangle(\tilde{\boldsymbol{z}})$, or equivalently $\tilde{\boldsymbol{x}} = \boldsymbol{R}_{\boldsymbol{n}}\langle\theta_{\mathrm{CFR}}\rangle(\boldsymbol{x})$

$\tilde{\boldsymbol{y}} = \boldsymbol{R}_{\boldsymbol{n}}\langle\theta_{\mathrm{CFR}}\rangle(\boldsymbol{y})$

$\tilde{\boldsymbol{z}} = \boldsymbol{R}_{\boldsymbol{n}}\langle\theta_{\mathrm{CFR}}\rangle(\boldsymbol{z})$.

The two ways of viewing the vector space operation represented by the direction cosine matrix, either as a rotation or as a change of coordinate basis, are illustrated in Figure 6.2.

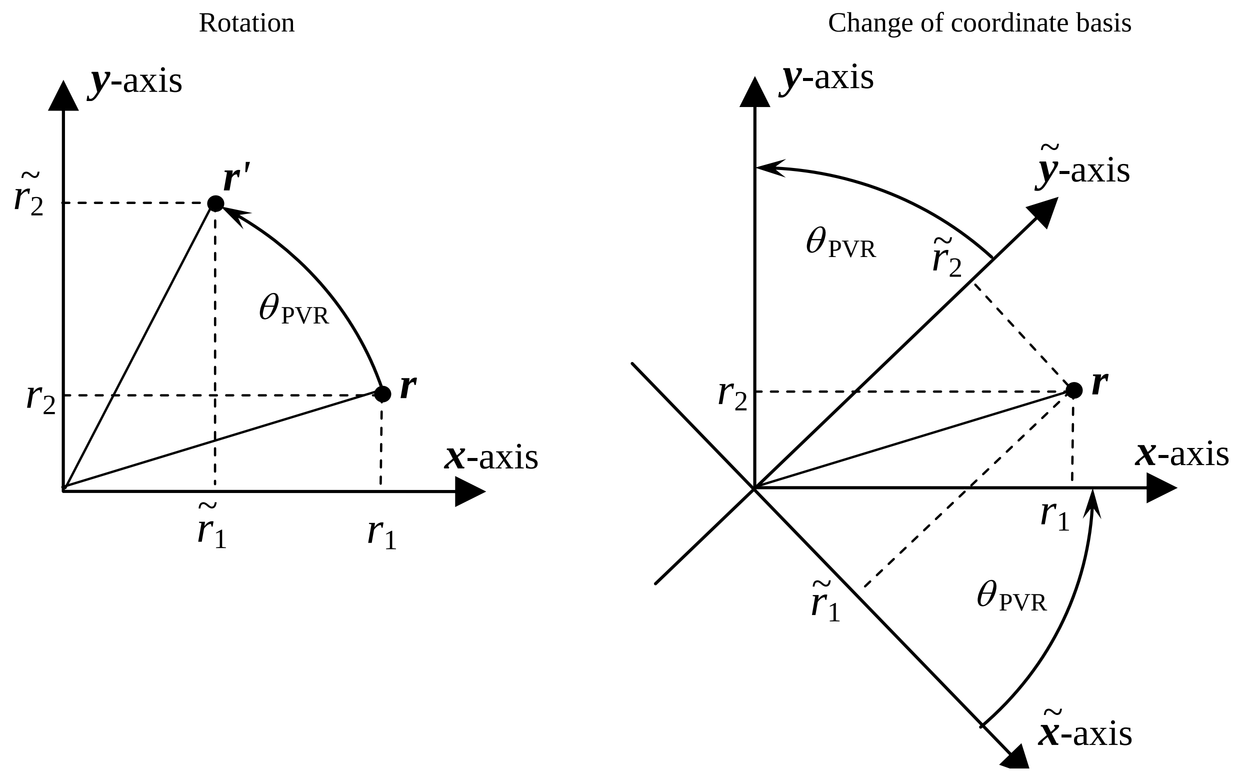


Figure 6.2 — Vector space operations

6.5.5 Consecutive change of basis operations

If $\boldsymbol{E}$, $\boldsymbol{F}$, $\boldsymbol{G}$ denotes three orthonormal frames with common origin, and if

> $\boldsymbol{\Omega}_{\boldsymbol{F}\leftarrow\boldsymbol{E}}$is the change of coordinate basis operator from $\boldsymbol{E}$ to $\boldsymbol{F}$, and
>
> $\boldsymbol{\Omega}_{\boldsymbol{G}\leftarrow\boldsymbol{F}}$ is the change of coordinate basis operator from $\boldsymbol{F}$ to $\boldsymbol{G}$,

then the change of coordinate basis operator from $\boldsymbol{E}$ to $\boldsymbol{G}$ is given by the composition of operators in right-to-left operator order:

$$\boldsymbol{\Omega}_{\boldsymbol{G}\leftarrow\boldsymbol{E}} = \boldsymbol{\Omega}_{\boldsymbol{G}\leftarrow\boldsymbol{F}} \circ \boldsymbol{\Omega}_{\boldsymbol{F}\leftarrow\boldsymbol{E}}.$$

If an origin is designated for 3D Euclidean space, and $\boldsymbol{R}_{\boldsymbol{m}}\langle\theta\rangle$ and $\boldsymbol{R}_{\boldsymbol{n}}\langle\varphi\rangle$ are two rotation operators with respect to that origin, then the combined effect of the sequential combination of operation $\boldsymbol{R}_{\boldsymbol{m}}\langle\theta\rangle$ followed by operation $\boldsymbol{R}_{\boldsymbol{n}}\langle\varphi\rangle$ is ambiguous as stated. This is because the first operation $\boldsymbol{R}_{\boldsymbol{m}}\langle\theta\rangle$ will rotate the axis $\boldsymbol{n}$ to a new direction $\boldsymbol{n}' = \boldsymbol{R}_{\boldsymbol{m}}\langle\theta\rangle(\boldsymbol{n})$ and the sequential combination of these operations may be taken to mean that the first rotation is followed either by $\boldsymbol{R}_{\boldsymbol{n}}\langle\varphi\rangle$ or by $\boldsymbol{R}_{\boldsymbol{n}'}\langle\varphi\rangle$ and, in general, the final result will differ accordingly. Both choices are useful and important and need to be clearly distinguished. This International standard uses the term *space-fixed* for the un-rotated second axis case $\boldsymbol{n}$ and the term *body-fixed* for the rotated second axis case $\boldsymbol{n}'$. The possible results of the composition of two consecutive rotations in right-to-left operator order are:

$\boldsymbol{R}_{\boldsymbol{n}}\langle\varphi\rangle \circ \boldsymbol{R}_{\boldsymbol{m}}\langle\theta\rangle$	space-fixed composition, and
$\boldsymbol{R}_{\boldsymbol{n}'}\langle\varphi\rangle \circ \boldsymbol{R}_{\boldsymbol{m}}\langle\theta\rangle$	body-fixed composition.

In terms of coordinate computations, $\boldsymbol{n}$ and $\boldsymbol{m}$ have coordinates $(n_1,\ n_2,\ n_3)$ and $(m_1,\ m_2,\ m_3)$ in an orthonormal frame $\boldsymbol{S}$. The first rotation rotates the basis vectors to new directions resulting in a different basis. Denote the orthonormal frame of the second basis as $\boldsymbol{B}$. (It is useful to think of a copy of the basis vectors of $\boldsymbol{S}$ attached to a rigid body. The body with attached vectors is rotated by $\boldsymbol{R_m}\langle\theta\rangle$ to form an orthonormal frame $\boldsymbol{B}$.) In the space-fixed case, $\boldsymbol{n}$ is interpreted as a coordinate $(n_1,\ n_2,\ n_3)_{\boldsymbol{S}}$ in $\boldsymbol{S}$. In the body-fixed case, $\boldsymbol{n}$ is interpreted as a coordinate $(n_1,\ n_2,\ n_3)_{\boldsymbol{B}}$ in $\boldsymbol{B}$. With respect to $\boldsymbol{S}$, $\left(n_1, n_2, n_3\right)_{\boldsymbol{B}}$ is a different vector. It is $\boldsymbol{n}'$, the result of rotating $\boldsymbol{n}$, and its coordinate representation in $\boldsymbol{S}$ may be computed by $\boldsymbol{n}' = \left(n_1',\ n_2',\ n_3'\right)_{\boldsymbol{S}} = \boldsymbol{\Omega}_{\boldsymbol{S}\leftarrow\boldsymbol{B}}((n_1,\ n_2,\ n_3)_{\boldsymbol{B}})$.

Change of coordinate basis operators may be utilized to compute the rotation operator $\boldsymbol{R}_{\boldsymbol{n}'}\langle\varphi\rangle$ in terms of $\boldsymbol{R_n}\langle\varphi\rangle$:

$$\boldsymbol{R}_{\boldsymbol{n}'}\langle\varphi\rangle = \boldsymbol{\Omega}_{\boldsymbol{B}\leftarrow\boldsymbol{S}} \circ \boldsymbol{R_n}\langle\varphi\rangle \circ \boldsymbol{\Omega}_{\boldsymbol{S}\leftarrow\boldsymbol{B}}.$$

Since the operator $\boldsymbol{\Omega}_{\boldsymbol{S}\leftarrow\boldsymbol{B}}^{-1} = \boldsymbol{\Omega}_{\boldsymbol{B}\leftarrow\boldsymbol{S}} = \boldsymbol{R_m}\langle\theta\rangle$, the body-fixed case may be simplified:

$$\boldsymbol{R}_{\boldsymbol{n}'}\langle\varphi\rangle \circ \boldsymbol{R_m}\langle\theta\rangle = (\boldsymbol{\Omega}_{\boldsymbol{S}\leftarrow\boldsymbol{B}}^{-1} \circ \boldsymbol{R_n}\langle\varphi\rangle \circ \boldsymbol{\Omega}_{\boldsymbol{S}\leftarrow\boldsymbol{B}}) \circ \boldsymbol{R_m}\langle\theta\rangle = \boldsymbol{R_m}\langle\theta\rangle \circ \boldsymbol{R_n}\langle\varphi\rangle \circ (\boldsymbol{\Omega}_{\boldsymbol{S}\leftarrow\boldsymbol{B}} \circ \boldsymbol{\Omega}_{\boldsymbol{S}\leftarrow\boldsymbol{B}}^{-1}) = \boldsymbol{R_m}\langle\theta\rangle \circ \boldsymbol{R_n}\langle\varphi\rangle.$$

Thus the two cases are simply expressed as:

$$\begin{array}{ll} \boldsymbol{R_n}\langle\varphi\rangle \circ \boldsymbol{R_m}\langle\theta\rangle & \text{space-fixed composition, and} \\ \boldsymbol{R}_{\boldsymbol{n}'}\langle\varphi\rangle \circ \boldsymbol{R_m}\langle\theta\rangle = \boldsymbol{R_m}\langle\theta\rangle \circ \boldsymbol{R_n}\langle\varphi\rangle & \text{body-fixed composition.} \end{array} \tag{6.2}$$

NOTE Other terminology used for the space-fixed and body-fixed concepts include: extrinsic and intrinsic rotations; and fixed-frame and moving-frame.

Finale: Roadmap through the hall of mirrors!!!

6.6 Orientation specification

An *orientation specification* for an object-frame $\boldsymbol{F}$ with respect to a reference-frame $\boldsymbol{E}$ shall be specified by either:

a) The change of coordinate basis operator $\boldsymbol{\Omega}_{\boldsymbol{E}\leftarrow\boldsymbol{F}}$ that converts a vector-coordinate in object-frame $\boldsymbol{F}$ to a corresponding vector-coordinate in reference-frame $\boldsymbol{E}$, or

b) The rotation operator $\boldsymbol{R_n}\langle\theta\rangle$ that would rotate the reference-frame $\boldsymbol{E}$ to align with the object-frame $\boldsymbol{F}$.

The rotation (b) that specifies the orientation of object-frame $\boldsymbol{F}$ with respect to reference-frame $\boldsymbol{E}$ shall be denoted by $\boldsymbol{R}_{\boldsymbol{E}\rightarrow\boldsymbol{F}}$. The direction cosine matrix $\boldsymbol{M}_{\boldsymbol{E}\leftarrow\boldsymbol{F}}$ corresponding to (a) and the rotation matrix $\boldsymbol{M_E}$ corresponding to (b) are the same matrix of values. In that sense, the two ways of specifying an orientation are equivalent.

The transpose of the matrix associated with (a) and (b) is the direction cosine matrix corresponding to $\boldsymbol{\Omega}_{\boldsymbol{F}\leftarrow\boldsymbol{E}}$ and is also the rotation matrix corresponding to $\boldsymbol{R}_{\boldsymbol{F}\rightarrow\boldsymbol{E}}$. Since the matrix transpose is easily computed, these latter two operators (the inverses of (a) and (b)) are equally suitable for use in the definition of an orientation specification. To avoid the ambiguity and/or confusion inherent in using both operators and their inverses in defining a concept, this International Standard uses two equivalent operators ((a) and (b)) out of the four options. This choice of two over the remaining two (equivalent) operators corresponds to popular usage in the modelling and simulation user domain.

The operator in an orientation specification may be represented in any one of the forms delineated in 6.X.

EXAMPLE In IEEE Std 1278.1-1995 (DIS) the orientation of the body coordinate system of an entity (object-frame) $\boldsymbol{b}$ with respect to the world coordinate system (reference-frame) $\boldsymbol{w}$ is defined as Euler angle $\boldsymbol{z}$-$\boldsymbol{y}$-$\boldsymbol{x}$ convention representation (see **Error! Reference source not found.**) of the rotation $\boldsymbol{R}_{w\rightarrow b}$ that would align reference-frame $\boldsymbol{w}$ with object-frame frame $\boldsymbol{b}$.

6.7 Change of orientation reference frame

Given the specification of the orientation of an object orthonormal frame $\boldsymbol{F}$ with respect to one reference orthonormal frame $\boldsymbol{E}$, the orientation of the same object with respect to a second reference orthonormal frame $\boldsymbol{D}$ may be calculated directly if the orientation specification of the first reference frame $\boldsymbol{E}$ with respect to the second $\boldsymbol{D}$ is known.

In terms of change of coordinate basis orientation specifications:

$\boldsymbol{\Omega}_{\boldsymbol{E}\leftarrow\boldsymbol{F}}$ denotes the change of basis from $\boldsymbol{F}$ to $\boldsymbol{E}$,

$\boldsymbol{\Omega}_{\boldsymbol{D}\leftarrow\boldsymbol{E}}$ denotes the change of basis from $\boldsymbol{E}$ to $\boldsymbol{D}$, and

$\boldsymbol{\Omega}_{\boldsymbol{D}\leftarrow\boldsymbol{F}}$ denotes the change of basis from $\boldsymbol{F}$ to $\boldsymbol{D}$.

$\boldsymbol{\Omega}_{\boldsymbol{F}\leftarrow\boldsymbol{D}}$ may be computed by: $\boldsymbol{\Omega}_{\boldsymbol{F}\leftarrow\boldsymbol{D}} = \boldsymbol{\Omega}_{\boldsymbol{F}\leftarrow\boldsymbol{E}} \circ \boldsymbol{\Omega}_{\boldsymbol{E}\leftarrow\boldsymbol{D}}$.

In terms of rotation orientation specifications:

$\boldsymbol{R}_{\boldsymbol{E}\rightarrow\boldsymbol{F}}$ denotes the rotation of the reference-frame $\boldsymbol{E}$ to align with the object-frame $\boldsymbol{F}$,

$\boldsymbol{R}_{\boldsymbol{D}\rightarrow\boldsymbol{E}}$ denotes the rotation of the reference-frame $\boldsymbol{D}$ to align with the object-frame $\boldsymbol{E}$, and

$\boldsymbol{R}_{\boldsymbol{D}\rightarrow\boldsymbol{F}}$ denotes the rotation of the reference-frame $\boldsymbol{D}$ to align with the object-frame $\boldsymbol{F}$.

$\boldsymbol{R}_{\boldsymbol{D}\rightarrow\boldsymbol{F}}$ may be computed by: $\boldsymbol{R}_{\boldsymbol{D}\rightarrow\boldsymbol{F}} = \boldsymbol{R}_{\boldsymbol{D}\rightarrow\boldsymbol{E}} \circ \boldsymbol{R}_{\boldsymbol{E}\rightarrow\boldsymbol{F}}$.

6.8 Representations of Rotations

6.8.1 Representation degrees of freedom and computational complexity

A consequence of Euler's rotation theorem is that any rotation operation on 3D Euclidean space has three degrees of freedom and may be specified by three scalar numbers. That is explicitly the case with Euler angle conventions (see 6.7.4).

Other less compact specifications using four or more scalar parameters together with constraint rules are commonly used because they are more amenable to some computations, such as performing a rotation operation on a point, composing rotations, interpolating rotations, and other operations, and/or because these parameters can be measured or modelled directly. The Matrix representation (see 6.7.2) and the Quaternion representation (see 6.7.5) are in common use because the rotation of a point and the composition of rotations are directly computable as matrix or quaternion multiplications. Computing the composition of rotations in the Axis-angle representation (see 6.7.3) or in an Euler angle convention (see 6.7.4) usually require conversion to and from Matrix or Quaternion forms. All rotation representations defined in the remainder of this clause tacitly require an orthonormal basis for the coordinate representation of position-vectors.

The various representation methods in prevalent use present different tradeoffs with respect to storage size, computational complexity, speed, and error control. Thus the best representation is dependent on the requirements and computational environment of a user application. For this reason, different representations are in use and interoperability becomes an issue. This issue is compounded by the non-standard meaning of terms in prevalent use. To support interoperability and SRM operations, this International Standard defines these terms and identifies several representation methods as well as algorithms for key operations on and inter-conversions between the representation methods.

6.8.2 Axis-angle representation

The *axis-angle* representation $(n_1, n_2, n_3,\ \theta)$, for a given orthonormal frame, is a representation of an origin-fixed rotation $\boldsymbol{R}_{\boldsymbol{n}}\langle\theta\rangle$ consisting of the vector-coordinates of a unit position-vector $\boldsymbol{n} = (n_1 \quad n_2 \quad n_3)^{\mathrm{T}}$ in the frame and a rotation angle θ in radians. This representation uses four scalar parameters $n_1,\ n_2,\ n_3$ and θ. The unit vector constraint $\|\boldsymbol{n}\| = 1$ reduces the degrees of freedom to three. The axis-angle representation is not unique. In particular, the axis-angle pairs $(n_1, n_2, n_3,\ \theta)$ and $(-n_1, -n_2, -n_3,\ -\theta)$ represent the same rotation. When $\theta = 0$, $\boldsymbol{n}$ may be any unit vector or the zero vector.

NOTE A three parameter version in the form $(a_1, a_2, a_3) = (\theta n_1, \theta n_2, \theta n_3) = \theta \boldsymbol{n}$ is also in use. In this form, θ is non-negative and is computed as $\theta = \|(a_1, a_2, a_3)\|$ and $\boldsymbol{n} = \frac{1}{\theta}(a_1, a_2, a_3)$ when $\theta \neq 0$.

The operation of an axis-angle rotation $(n_1, n_2, n_3,\ \theta)$ on 3D Euclidean space is given by Rodrigues' rotation formula (Equation (6.3)). There is no direct computational formulation of the composition of two axis-angle rotations in axis-angle form.

6.8.3 Matrix representation

A 3x3 matrix $\boldsymbol{M}$ is a rotation matrix, if it satisfies these properties:

$$\det(\boldsymbol{M}) = 1$$
$$\boldsymbol{M}^{-1} = \boldsymbol{M}^{T} \tag{6.3}$$

Matrices satisfying these properties form an algebraic group with respect to matrix multiplication. This group is known as the *special orthogonal group* of degree 3, SO(3). In particular, the product of any two rotation matrices is itself a rotation matrix.

For a given orthonormal frame, the operation of left matrix multiplication by $\boldsymbol{M}$ corresponds to an origin-fixed rotation $\boldsymbol{R}_{\boldsymbol{n}}\langle\theta\rangle$. The axis-angle parameters $(n_1, n_2, n_3,\ \theta)$ for this rotation in the given frame are algorithmically determined as follows:

$$\text{If } \boldsymbol{M} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \theta_{\mathrm{PVR}} = \arccos\left(\left(\frac{\mathrm{Trace}(\boldsymbol{M})-1}{2}\right)\right) = \arccos\left(\left(\frac{(a_{11}+a_{22}+a_{33})-1}{2}\right)\right),\quad 0 \leq \theta_{\mathrm{PVR}} \leq \pi.$$

There are three cases for the computation of $\boldsymbol{n} = (n_1, n_2, n_3)^{\mathrm{T}}$ that depend on the value of θ.

Case $\theta_{\mathrm{PVR}} = 0$: There is no rotation so $\boldsymbol{n}$ is indeterminate.

Case $0 < \theta_{\mathrm{PVR}} < \pi$: Let $\boldsymbol{n} = \boldsymbol{v}/\|\boldsymbol{v}\|$, where:

$\boldsymbol{v} = \begin{bmatrix} a_{32} - a_{23} \\ a_{13} - a_{31} \\ a_{21} - a_{12} \end{bmatrix}$. In this case, $\|\boldsymbol{v}\| = 2|\sin(\theta_{\mathrm{PVR}})|$.

Case: $\theta = \pi$: First find the maximum diagonal element $a_{11},\ a_{22}$, or a_{33} of $\boldsymbol{M}$. Then:

Sub-case: a_{11} is the maximum and $\boldsymbol{v} = (a_{11} + 1,\ a_{12},\ a_{13})^{\mathrm{T}}$.

Sub-case: a_{22} is the maximum and $\boldsymbol{v} = (a_{21},\ a_{22} + 1,\ a_{23})^{\mathrm{T}}$.

Sub-case: a_{33} is the maximum and $\boldsymbol{v} = (a_{31},\ a_{32},\ a_{33} + 1)^{\mathrm{T}}$.

Finally $\boldsymbol{n} = \boldsymbol{v}/\|\boldsymbol{v}\|$.

In all cases $(-n_1, -n_2, -n_3, -\ \theta)$ is also a solution.

NOTE 1 Matrix multiplication is generally not commutative.

NOTE 2 The matrix has nine parameters; however the constraints on the determinant and the transpose reduce the degrees of freedom to three.

A special case of a rotation matrix is the direction cosine matrix that arises from a change of coordinate basis operation (see **Error! Reference source not found.**). That matrix is also the matrix representation of the r otation specification $\boldsymbol{R}_{\boldsymbol{E}\to\boldsymbol{F}}$ of the orientation of object-frame $\boldsymbol{F}$ with respect to reference-frame $\boldsymbol{E}$.

A special case of a rotation matrix arises from a change of coordinate basis operation. If $\boldsymbol{E}$ and $\boldsymbol{F}$ are two orthonormal frames with common origin and respective bases $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{e}_3$, and $\boldsymbol{f}_1$, $\boldsymbol{f}_2$, $\boldsymbol{f}_3$, the matrix $\boldsymbol{M}$ corresponding to the coordinate basis operation $\boldsymbol{\Omega}_{\boldsymbol{F}\leftarrow\boldsymbol{E}}$ by the direction cosine matrix (see 6.2):

$$\boldsymbol{M} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} \boldsymbol{e}_1 \bullet \boldsymbol{f}_1 & \boldsymbol{e}_2 \bullet \boldsymbol{f}_1 & \boldsymbol{e}_3 \bullet \boldsymbol{f}_1 \\ \boldsymbol{e}_1 \bullet \boldsymbol{f}_2 & \boldsymbol{e}_2 \bullet \boldsymbol{f}_2 & \boldsymbol{e}_3 \bullet \boldsymbol{f}_2 \\ \boldsymbol{e}_1 \bullet \boldsymbol{f}_3 & \boldsymbol{e}_2 \bullet \boldsymbol{f}_3 & \boldsymbol{e}_3 \bullet \boldsymbol{f}_3 \end{bmatrix} \tag{6.4}$$

$\boldsymbol{M}$ is also the matrix representation of the rotation specification $\boldsymbol{R}_{\boldsymbol{F}\to\boldsymbol{E}}$ of the orientation of orthonormal frame $\boldsymbol{E}$ with respect to reference frame $\boldsymbol{F}$.

6.8.4 Principal rotations and Euler angle conventions

6.8.4.1 Principal rotations

Principal rotations are defined with respect to a given orthonormal frame. Each vector in the frame basis, $\boldsymbol{x}$, $\boldsymbol{y}$, $\boldsymbol{z}$, is a unit vector and, as an axis of rotation, each of these vectors is termed a *principal axis* of rotation. A rotation about a principal axis is termed a *principal rotation*. Some authors refer to these rotations as *elementary rotations*. The vector space operators: $\boldsymbol{R}_x\langle\alpha\rangle$, $\boldsymbol{R}_y\langle\beta\rangle$, and $\boldsymbol{R}_z\langle\gamma\rangle$ denote the three principal rotations through the respective angles α, β, and γ. The axis-angle representation of the principal rotations in the given frame are, respectively, $(1,\ 0,\ 0,\ \alpha)$, $(0,\ 1,\ 0,\ \beta)$, and $(0,\ 0,\ 1,\ \gamma)$.

6.8.4.2 Euler angles

Euler angles are a specification of a rotation obtained by the composition of three consecutive principal rotations in the body-fixed convention (see 6.X). Allowing for repeated axes, there are twelve distinct ways to select a sequence of three principal axes and apply the principal rotations (24 if left-handed axes are considered)[19]. Each such ordered selection of axes is termed an *Euler angle convention*. There is little agreement among authors on names or notations for these conventions. There are numerous Euler angle conventions in use and many are named inconsistently. Some authors use a left-handed coordinate system. All orthonormal frames in this International Standard are right-handed. The special case of Euler angle conventions that use all tree principal axes are termed *Tait-Bryan angles*.

This International Standard adopts the following convention and notation for Euler angles: Given a 3-tuple of Euler angles (α, β, γ) the Euler convention specification shall be specified by a character string denoting the sequence of principal axes in the form A_1– A_2 – A_3 where each symbol A_1, A_2, A_3 is one of the axis letters $\boldsymbol{x}$, $\boldsymbol{y}$, or $\boldsymbol{z}$. Thus (α, β, γ) in the $\boldsymbol{z}$-$\boldsymbol{x}$-$\boldsymbol{z}$ Euler convention is the body-fixed convention of the rotation sequence consisting of a principal rotation by angle α about the $\boldsymbol{z}$-axis first, by angle β about $\boldsymbol{x}'$, the once rotated $\boldsymbol{x}$-axis, second, and by angle γ about $\boldsymbol{z}''$, the twice rotated $\boldsymbol{z}$-axis, for the third rotation. The resulting body-fixed convention sequence in right-to-left operator order is $\boldsymbol{R}_{z''}\langle\gamma\rangle \circ \boldsymbol{R}_{x'}\langle\beta\rangle \circ \boldsymbol{R}_z\langle\alpha\rangle$. Using Equation **Error! Reference source not found.** the same result in un-rotated axes is $\boldsymbol{R}_z\langle\alpha\rangle \circ \boldsymbol{R}_x\langle\beta\rangle \circ \boldsymbol{R}_z\langle\gamma\rangle$. In general, the equivalent expressions in rotated and non-rotated principal axes are:

[19] There cannot be two consecutive rotations on the same axis as they would combine to a single rotation (see 6.4). Thus, among right-handed axis systems, there are 3 choices for the first rotation axis, 2 choices each for the second and third rotation axes to avoid repeating the preceding axis choice (3x2x2=12).

$$\boldsymbol{R}_{A''_3}\langle\gamma\rangle \circ \boldsymbol{R}_{A'_2}\langle\beta\rangle \circ \boldsymbol{R}_{A_1}\langle\alpha\rangle = \boldsymbol{R}_{A_1}\langle\alpha\rangle \circ \boldsymbol{R}_{A_2}\langle\beta\rangle \circ \boldsymbol{R}_{A_3}\langle\gamma\rangle$$

where the second and third axes sequentially are rotated:

$A'_2 = \boldsymbol{R}_{A_1}\langle\alpha\rangle(A_2)$, and

$$A''_3 = \boldsymbol{R}_{A'_2}\langle\beta\rangle \circ \boldsymbol{R}_{A_1}\langle\alpha\rangle(A_3). \qquad (6.5)$$

The three angles representing a rotation in a given Euler angle convention are not necessarily unique modulo 2π. The conditions that result in non-unique angle 3-tuples are given in Table 6.4 for the $\boldsymbol{z}$-$\boldsymbol{x}$-$\boldsymbol{z}$ Euler angle convention and in Table 6.7 for the $\boldsymbol{x}$-$\boldsymbol{y}$-$\boldsymbol{z}$ and $\boldsymbol{z}$-$\boldsymbol{y}$-$\boldsymbol{x}$ Euler angle conventions (see also 6.7.4.5).

EXAMPLE Substituting in Equation (6.7), the Euler sequence (ψ, θ, φ) in the Euler $\boldsymbol{z}$-$\boldsymbol{y}$-$\boldsymbol{x}$ convention is $\boldsymbol{R}_{x''}\langle\varphi\rangle \circ \boldsymbol{R}_{y'}\langle\theta\rangle \circ \boldsymbol{R}_{z}\langle\psi\rangle$ or equivalently $\boldsymbol{R}_{z}\langle\psi\rangle \circ \boldsymbol{R}_{y}\langle\theta\rangle \circ \boldsymbol{R}_{x}\langle\varphi\rangle$.

There are no direct computational formulations for the operation of an Euler angle rotation on 3D Euclidean space or for representing the composition of two Euler angle rotations as a single Euler angle rotation. For these computations, the principal rotation sequence is commonly realized as a product of matrices or quaternions.

6.8.4.3 The z-x-z convention

In the $\boldsymbol{z}$-$\boldsymbol{x}$-$\boldsymbol{z}$ Euler convention, the initial $\boldsymbol{xy}$-plane and the final rotated $\boldsymbol{x}''\boldsymbol{y}''$-plane generally intersect in a line. This line is termed the *line of nodes* for this convention. The Euler angles in the $\boldsymbol{z}$-$\boldsymbol{x}$-$\boldsymbol{z}$ convention are the three angles defined as follows:

α is the angle between the line of nodes and the $\boldsymbol{x}''$-axis,

β is the angle between the $\boldsymbol{z}$-axis and the $\boldsymbol{z}''$-axis, and

γ is the angle between the $\boldsymbol{x}$-axis and the line of nodes.

In the case that the initial $\boldsymbol{xy}$-plane lies in the final rotated $\boldsymbol{x}''\boldsymbol{y}''$-plane, $\beta = 0$ or $\beta = \pi$ (see 6.7.4.5).

In some contexts $\alpha,\ \beta,\ \gamma$ are known, respectively, as the *spin* angle, the *nutation* angle, and the *precession* angle. These three angles specify the principal rotation angles the body-fixed composition of the $\boldsymbol{z}$-axis principal rotation followed by (the rotated) $\boldsymbol{x}'$-axis principal rotation followed by the (twice rotated) $\boldsymbol{z}''$-axis principal rotation. The sequence of body-fixed rotations is illustrated in Figure 6.2. The resulting composite rotation is $\boldsymbol{R}_{z''}\langle\gamma\rangle \circ \boldsymbol{R}_{x'}\langle\beta\rangle \circ \boldsymbol{R}_{z}\langle\alpha\rangle = \boldsymbol{R}_{z}\langle\alpha\rangle \circ \boldsymbol{R}_{x}\langle\beta\rangle \circ \boldsymbol{R}_{z}\langle\gamma\rangle$.

6.8.4.4 Tait-Bryan angles

Euler angle conventions that use all three principal axes are sometimes referred to as *Tait-Bryan angles*. In particular, the angles in the $\boldsymbol{x}$-$\boldsymbol{y}$-$\boldsymbol{z}$ and $\boldsymbol{z}$-$\boldsymbol{y}$-$\boldsymbol{x}$ Euler conventions are variously termed Tait-Bryan angles, *Cardano angles*, or *nautical angles*. The various names given to these angle symbols include:

φ roll or bank or tilt,

θ pitch or elevation, and

ψ yaw or heading or azimuth (see Figure 6.4).

In the $\boldsymbol{x}$-$\boldsymbol{y}$-$\boldsymbol{z}$ Euler convention the line of nodes is the intersection of the $\boldsymbol{xy}$-plane and the final rotated $\boldsymbol{y}''\boldsymbol{z}''$-plane. The Euler angles in this convention are defined as follows:

φ is the angle between the line of nodes and the $\boldsymbol{y}''$-axis,

θ is the angle between $\boldsymbol{x}''$-axis and the $\boldsymbol{xy}$-plane, (equivalently, the $\boldsymbol{z}$-axis and the $\boldsymbol{y}''\boldsymbol{z}''$-plane), and

ψ is the angle between the $\boldsymbol{y}$-axis and the line of nodes.

These three angles (φ, θ, ψ) specify the following body-fixed composition of consecutive principal rotations:

$$\boldsymbol{R}_{\boldsymbol{x}''}\langle\psi\rangle \circ \boldsymbol{R}_{\boldsymbol{y}'}\langle\theta\rangle \circ \boldsymbol{R}_{\boldsymbol{z}}\langle\varphi\rangle = \boldsymbol{R}_{\boldsymbol{z}}\langle\varphi\rangle \circ \boldsymbol{R}_{\boldsymbol{y}}\langle\theta\rangle \circ \boldsymbol{R}_{\boldsymbol{x}}\langle\psi\rangle \quad \boldsymbol{x}\text{-}\boldsymbol{y}\text{-}\boldsymbol{z} \text{ Euler convention.}$$

In the $\boldsymbol{z}$-$\boldsymbol{y}$-$\boldsymbol{x}$ Euler convention the line of nodes is the intersection of the $\boldsymbol{yz}$-plane and the final rotated $\boldsymbol{x}''\boldsymbol{y}''$-plane. The Euler angles in this convention are defined as follows:

φ is the angle between the line of nodes and the $\boldsymbol{y}$-axis,

θ is the angle between $\boldsymbol{x}$-axis and the $\boldsymbol{x}''\boldsymbol{y}''$-plane, (equivalently, the $\boldsymbol{z}''$-axis and the $\boldsymbol{yz}$-plane), and

ψ is the angle between the $\boldsymbol{y}''$-axis and the line of nodes.

These three angles (ψ, θ, φ) specify the following body-fixed composition of consecutive principal rotations:

$$\boldsymbol{R}_{\boldsymbol{x}''}\langle\varphi\rangle \circ \boldsymbol{R}_{\boldsymbol{y}'}\langle\theta\rangle \circ \boldsymbol{R}_{\boldsymbol{z}}\langle\psi\rangle = \boldsymbol{R}_{\boldsymbol{z}}\langle\psi\rangle \circ \boldsymbol{R}_{\boldsymbol{y}}\langle\theta\rangle \circ \boldsymbol{R}_{\boldsymbol{x}}\langle\varphi\rangle \quad \boldsymbol{z}\text{-}\boldsymbol{y}\text{-}\boldsymbol{x} \text{ Euler convention.}$$

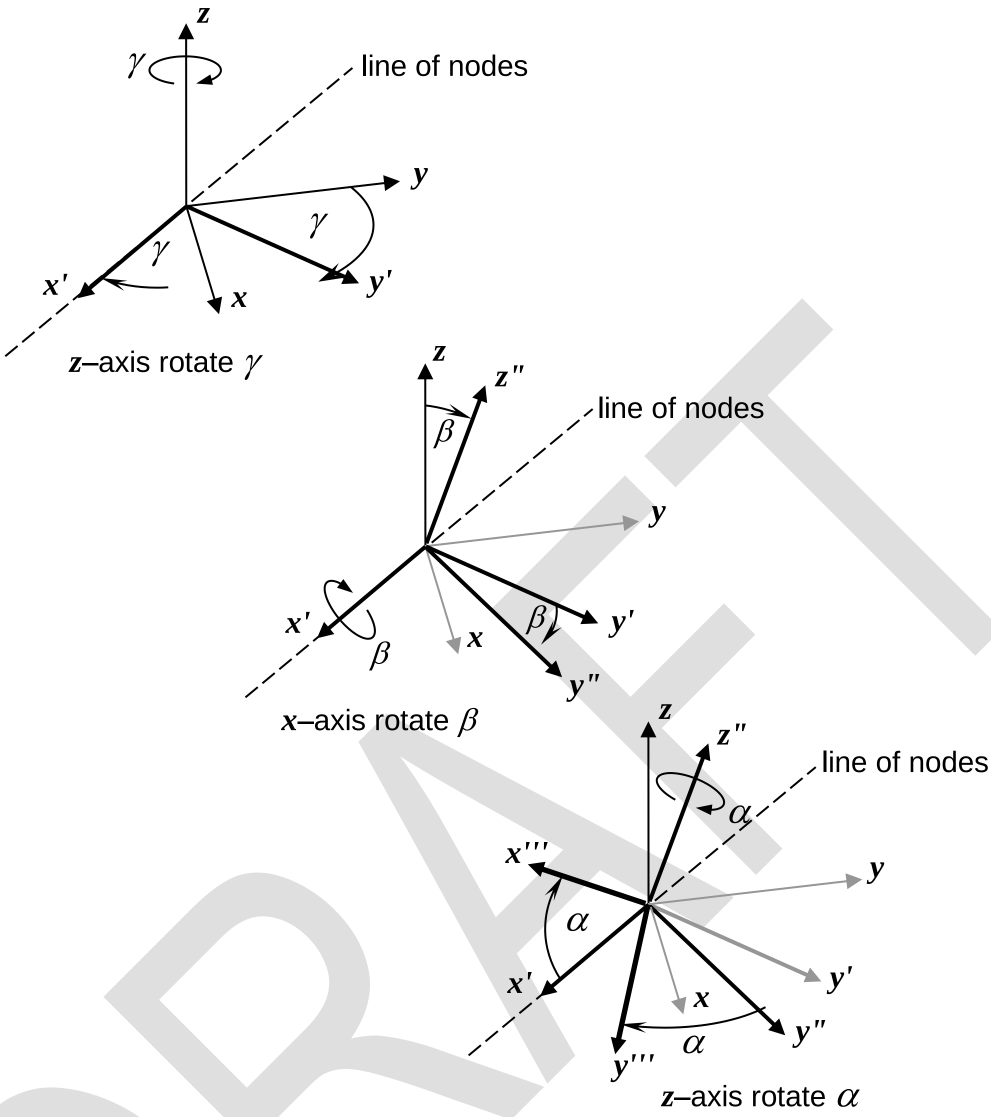
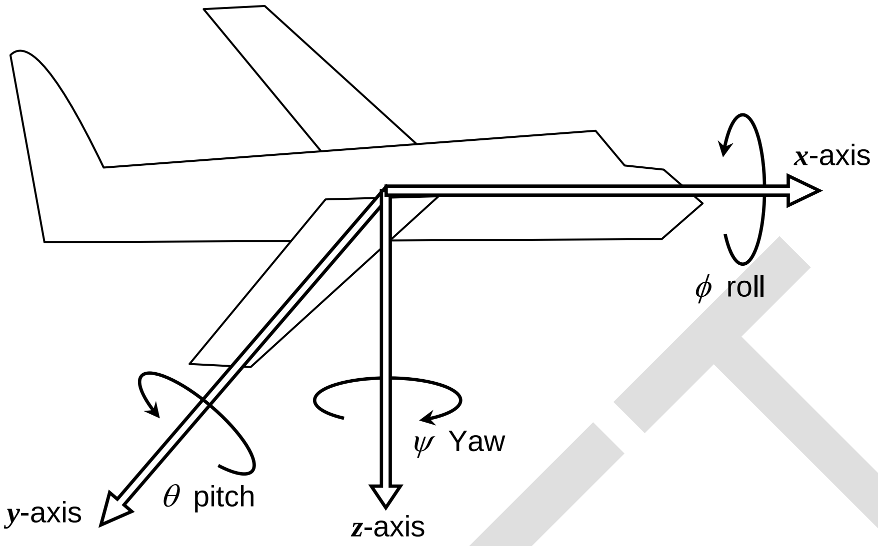


Figure 6.3 — Euler *z*-*x*-*z* convention

Figure 6.4 — Tait-Bryan angles



6.8.4.5 Gimbal lock

The term *gimbal lock* refers to a gyroscope mounted in three nested gimbals to provide three degrees of rotational freedom. Each mounting scheme corresponds to an Euler angle convention. In any such mounting scheme, there exist critical angles for the middle gimbal that reduce the rotational degrees of freedom from three to two. In those critical configurations, the gimbals lie in a single plane and rotation within that plane is figuratively "locked out" by the gimbal mechanism. This loss of a degree of freedom is termed "gimbal lock".

In the case of the Euler angle $\boldsymbol{z}$-$\boldsymbol{x}$-$\boldsymbol{z}$ rotation convention, it is assumed that the $\boldsymbol{xy}$-plane and $\boldsymbol{x}''\boldsymbol{y}''$-plane intersect in a line (the line on nodes). That assumption is met when (modulo 2π) $\beta \neq 0$ and $\beta \neq \pi$. If $\beta = 0$, $\boldsymbol{R}_x\langle 0\rangle$ is the identity operator and has no effect. If $\beta = \pi$, $\boldsymbol{R}_x\langle \pi\rangle$ reverses the direction of the preceding $\boldsymbol{z}$-axis rotation so that $\boldsymbol{R}_x\langle \pi\rangle \circ \boldsymbol{R}_z\langle \gamma\rangle = \boldsymbol{R}_z\langle -\gamma\rangle \circ \boldsymbol{R}_x\langle \pi\rangle$. In either case, the consecutive rotations collapse down to a single principal rotation:

$$\begin{aligned}\beta = 0&: \boldsymbol{R}_z\langle\alpha\rangle \circ \boldsymbol{R}_x\langle 0\rangle \circ \boldsymbol{R}_z\langle\gamma\rangle = \boldsymbol{R}_z\langle\alpha\rangle \circ \boldsymbol{R}_z\langle\gamma\rangle = \boldsymbol{R}_z\langle\alpha+\gamma\rangle\\ \beta = \pi&: \boldsymbol{R}_z\langle\alpha\rangle \circ \boldsymbol{R}_x\langle \pi\rangle \circ \boldsymbol{R}_z\langle\gamma\rangle = \boldsymbol{R}_z\langle\alpha\rangle \circ \boldsymbol{R}_z\langle-\gamma\rangle \circ \boldsymbol{R}_x\langle\pi\rangle = \boldsymbol{R}_z\langle\alpha-\gamma\rangle \circ \boldsymbol{R}_x\langle\pi\rangle.\end{aligned} \tag{6.6}$$

NOTE 1 This situation is illustrated by a spinning table top. The top spins on its spin-axis and precesses about the precession-axis. The angle between the spin- and precession-axes is the nutation angle. When the spin-axis is perfectly vertical (either upright or upside down), the nutation angle is 0 or π and the spin- and precession-axes become indistinguishable from each other as indicated in Equation (6.8).

In the case of the Euler angle $\boldsymbol{x}$-$\boldsymbol{y}$-$\boldsymbol{z}$ convention (Tait-Bryan) it is assumed that the $\boldsymbol{xy}$-plane and $\tilde{\boldsymbol{y}}\tilde{\boldsymbol{z}}$-plane intersect in a line (the line of nodes). That assumption is met when $\theta \neq \pm\pi/2$ modulo 2π. If not, $\theta = \pm\pi/2$ and the $\tilde{x}$-axis becomes parallel to the $\boldsymbol{z}$-axis and the consecutive rotations collapse down to a single principal rotation:

$$\begin{aligned}\theta &= +\pi/2: \boldsymbol{R}_x\langle\varphi\rangle \circ \boldsymbol{R}_y\langle+\pi/2\rangle \circ \boldsymbol{R}_z\langle\psi\rangle = \boldsymbol{R}_x\langle\varphi+\psi\rangle \circ \boldsymbol{R}_y\langle+\pi/2\rangle\\ \theta &= -\pi/2: \boldsymbol{R}_x\langle\varphi\rangle \circ \boldsymbol{R}_y\langle-\pi/2\rangle \circ \boldsymbol{R}_z\langle\psi\rangle = \boldsymbol{R}_x\langle\varphi-\psi\rangle \circ \boldsymbol{R}_y\langle-\pi/2\rangle.\end{aligned} \tag{6.7}$$

$$\begin{aligned}\theta &= +\pi/2: \boldsymbol{R}_x\langle\varphi\rangle \circ \boldsymbol{R}_y\langle+\pi/2\rangle \circ \boldsymbol{R}_z\langle\psi\rangle = \boldsymbol{R}_x\langle\psi+\varphi\rangle \circ \boldsymbol{R}_y\langle+\pi/2\rangle\\ \theta &= -\pi/2: \boldsymbol{R}_x\langle\psi\rangle \circ \boldsymbol{R}_y\langle-\pi/2\rangle \circ \boldsymbol{R}_z\langle\varphi\rangle = \boldsymbol{R}_x\langle\psi-\varphi\rangle \circ \boldsymbol{R}_y\langle-\pi/2\rangle.\end{aligned} \tag{6.8}$$

The case of the Euler angle $\boldsymbol{z}$-$\boldsymbol{y}$-$\boldsymbol{x}$ convention has a similar result:

$$\begin{aligned}\theta = +\pi/2: \boldsymbol{R}_z\langle\psi\rangle \circ \boldsymbol{R}_y\langle+\pi/2\rangle \circ \boldsymbol{R}_x\langle\varphi\rangle = \boldsymbol{R}_z\langle\psi+\varphi\rangle \circ \boldsymbol{R}_y\langle+\pi/2\rangle \\ \theta = -\pi/2: \boldsymbol{R}_z\langle\psi\rangle \circ \boldsymbol{R}_y\langle-\pi/2\rangle \circ \boldsymbol{R}_x\langle\varphi\rangle = \boldsymbol{R}_z\langle\psi-\varphi\rangle \circ \boldsymbol{R}_y\langle-\pi/2\rangle\end{aligned} \quad (6.9)$$

NOTE 2 This situation is illustrated by an aircraft as in Figure 6.4. When the aircraft either climbs vertically, or dives vertically, roll-rotation cannot be distinguished from (plus or minus) yaw-rotation. This occurs at critical pitch angles of $\theta = \pm\pi/2$ as indicated in Equation (6.9).

6.8.5 Quaternion representation

6.8.5.1 Quaternion notations and conventions

The quaternion system is a 4-dimensional vector space together with a vector multiplication operation that forms a non-commutative associative algebra. In analogy to complex numbers that are written as $a + \boldsymbol{i}b$, $\boldsymbol{i}^2 = -1$, quaternion axes $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$ are defined with the following relationships: $\boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = \boldsymbol{ijk} = -1$. There are several notational conventions in use including the three termed in this International Standard as the *Hamilton form*, the *4-tuple form*, and the *scalar vector form*. In these notation forms a quaternion $\boldsymbol{q}$ is denoted as follows:

$\boldsymbol{q} = e_0 + e_1\boldsymbol{i} + e_2\boldsymbol{j} + e_3\boldsymbol{k}$	Hamilton form
$\boldsymbol{q} = (e_0, e_1, e_2, e_3)$	4-tuple form
$\boldsymbol{q} = (e_0, \boldsymbol{e})$, $\boldsymbol{e} = (e_1 \quad e_2 \quad e_3)^{\mathrm{T}}$	scalar vector form

where e_0, e_1, e_2, e_3 are scalar values.

The e_0 value is termed the *real* (or "scalar") part of $\boldsymbol{q}$ and (e_1, e_2, e_3) is termed the *imaginary* (or "vector") part of $\boldsymbol{q}$. The remainder of this clause uses the scalar vector form.

NOTE 1 In the literature, the component order of the scalar vector form is sometimes reversed: $\boldsymbol{q} = (\boldsymbol{e}, e_0)$.

NOTE 2 A unit quaternion (see below) in 4-tuple form is also termed the *Euler parameters* (or the *Euler-Rodrigues parameters*) of a rotation. In the literature, the real part of the 4-tuple form is sometimes placed last: $\boldsymbol{q} = (e_1, e_2, e_3, e_4)$ where $e_4 = e_0$.

6.8.5.2 Quaternion algebra

Quaternion multiplication and other operations are defined in Annex A in all the three notational forms. Given quaternions $\boldsymbol{q} = (e_0, \boldsymbol{e})$ and $\boldsymbol{p} = (d_0, \boldsymbol{d})$, A.10 defines:

the product: $\boldsymbol{pq} = \big((d_0 e_0 - \boldsymbol{d} \bullet \boldsymbol{e}),\ (e_0\boldsymbol{d} + d_0\boldsymbol{e} + \boldsymbol{d} \times \boldsymbol{e})\big)$,

the conjugate: $\boldsymbol{q}^* = (e_0, -\boldsymbol{e})$, and

the modulus: $|\boldsymbol{q}| = \sqrt{\boldsymbol{qq}^*} = \sqrt{e_0^2 + e_1^2 + e_2^2 + e_3^2}$,

where $\boldsymbol{qq}^* = \boldsymbol{q}^*\boldsymbol{q} = (e_0^2 + e_1^2 + e_2^2 + e_3^2,\ \ 0)$.

A quaternion $\boldsymbol{q}$ is a *unit quaternion* if $|\boldsymbol{q}| = 1$. In that case $\boldsymbol{qq}^* = \boldsymbol{q}^*\boldsymbol{q} = (1,\ 0)$ which is the multiplicative identity so that, for a unit quaternion, its conjugate is its multiplicative inverse $\boldsymbol{q}^{-1} = \boldsymbol{q}^*$. Any unit quaternion may be expressed in the form:

$$\boldsymbol{q} = (\cos(\theta/2),\ \sin(\theta/2)\,\boldsymbol{n}) \quad (6.10)$$

where:

$\boldsymbol{n} = \frac{1}{\|\boldsymbol{e}\|}\boldsymbol{e}$ is a unit vector in 3D space.

$\theta = 2 \cdot \text{arctan2}\left(\sqrt{e_1^2 + e_2^2 + e_3^2},\ e_0\right)$.

NOTE The two argument arctangent function arctan2() is defined in Annex A.

6.8.5.3 Quaternion operators on 3D Euclidean space

Each quaternion $\boldsymbol{q}$ corresponds to a transformation of 3D Euclidean space as follows. If $\boldsymbol{r}$ is a position-vector in 3D Euclidean space, the corresponding quaternion is formed by using 0 for the real part and $\boldsymbol{r}$ for the imaginary part $(0, \boldsymbol{r})$. A unit quaternion $\boldsymbol{q}$ operates on $(0, \boldsymbol{r})$ by left multiplying with $\boldsymbol{q}$ and right multiplying with its conjugate $\boldsymbol{q}^*$. The real part of the product, $\boldsymbol{q}(0, \boldsymbol{r})\boldsymbol{q}^* = (r_0', \boldsymbol{r}')$, is 0. Thus, $\boldsymbol{q}(0, \boldsymbol{r})\boldsymbol{q}^* = (0, \boldsymbol{r}')$ is pure imaginary and the quaternion $\boldsymbol{q}$ associates $\boldsymbol{r}'$ with $\boldsymbol{r}$. Symbolically the operation on $\boldsymbol{r}$ is:

$$\boldsymbol{r} \mapsto \boldsymbol{r}' = \text{imaginary part}\{\boldsymbol{q}(0, \boldsymbol{r})\boldsymbol{q}^*\}. \tag{6.11}$$

This is equivalent to:

$$\boldsymbol{r}' = (e_0^2 - \boldsymbol{e} \bullet \boldsymbol{e})\boldsymbol{r} + 2(\boldsymbol{e} \bullet \boldsymbol{r})\boldsymbol{e} + 2e_0\boldsymbol{e} \times \boldsymbol{r}. \tag{6.12}$$

$-\boldsymbol{q} = (-e_0, -\boldsymbol{e})$ produces the same $\boldsymbol{r}'$ so that $\boldsymbol{q}$ and $-\boldsymbol{q}$ produce equivalent rotations.

If $\boldsymbol{q} = \left(\cos\left(\frac{\theta}{2}\right),\ \sin\left(\frac{\theta}{2}\right)\boldsymbol{n}\right)$ is a unit quaternion, Equation (6.13) reduces to the Rodrigues rotation formula for a clockwise rotation about $\boldsymbol{n}$ through angle θ:

$$\boldsymbol{r}' = \cos(\theta)\,\boldsymbol{r} + (1 - \cos(\theta))(\boldsymbol{n} \bullet \boldsymbol{r})\boldsymbol{n} + \sin(\theta)\,\boldsymbol{n} \times \boldsymbol{r}.$$

A non-zero quaternion $\boldsymbol{p}$ and its corresponding unit quaternion $\boldsymbol{q} = \boldsymbol{p}/|\boldsymbol{p}|$ perform the same rotation $\boldsymbol{p}(0, \boldsymbol{r})\boldsymbol{p}^{-1} = \boldsymbol{q}(0, \boldsymbol{r})\boldsymbol{q}^*$.

For this reason, some authors use $\boldsymbol{p}(0, \boldsymbol{r})\boldsymbol{p}^{-1}$ operations for any non-zero quaternion while others use the $\boldsymbol{q}(0, \boldsymbol{r})\boldsymbol{q}^*$ operator and restrict operations only to unit quaternions.

The quaternion representation of rotation facilitates the computation of the composition of two rotations.

If $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$ are two unit quaternions, the composite rotation on $\boldsymbol{r}$ that is obtained by first rotating with the rotation induced by $\boldsymbol{q}_1$ and then rotating the result with the rotation induced by $\boldsymbol{q}_2$ is the same as the single rotation induced by the product $\boldsymbol{q}_2\boldsymbol{q}_1$ since $\boldsymbol{q}_2\{\boldsymbol{q}_1(0, \boldsymbol{r})\boldsymbol{q}_1^*\}\boldsymbol{q}_2^* = \boldsymbol{q}_2\boldsymbol{q}_1(0, \boldsymbol{r})\boldsymbol{q}_1^*\boldsymbol{q}_2^* = \{\boldsymbol{q}_2\boldsymbol{q}_1\}(0, \boldsymbol{r})\{\boldsymbol{q}_2\boldsymbol{q}_1\}^*$.

6.8.6 Representation summary

Some important attributes of the representations in this section are summarized in Table 6.1.

Represen-tation type	Data compo-nents	Data constraints	Ambiguities (modulo 2π)	Composition	Inverse
Axis-angle (n_1, n_2, n_3, θ)	4	$\|\boldsymbol{n}\| = 1$ $\boldsymbol{n} = (n_1 \quad n_2 \quad n_3)^{\mathrm{T}}$	(n_1, n_2, n_3, θ) is equivalent to $(-n_1, -n_2, -n_3, -\theta)$. If $\theta = 0$, $\boldsymbol{n}$ is indeterminate	Convert to/from another representation for the operation	$(n_1, n_2, n_3, -\theta)$ or $(-n_1, -n_2, -n_3, \theta)$

Represen-tation type	Data compo-nents	Data constraints	Ambiguities (modulo 2π)	Composition	Inverse
Matrix $\boldsymbol{M}$	9	$\det(\boldsymbol{M}) = 1$ $\boldsymbol{M}^{\mathrm{T}} = \boldsymbol{M}^{-1}$	None	Matrix multiplication	$\boldsymbol{M}^{\mathrm{T}}$
Euler angle conventions	3	None	2 or more $\boldsymbol{z}$-$\boldsymbol{x}$-$\boldsymbol{z}$ convention: see Table 6.4 Tait-Bryan $\boldsymbol{z}$-$\boldsymbol{y}$-$\boldsymbol{x}$ or $\boldsymbol{x}$-$\boldsymbol{y}$-$\boldsymbol{z}$ angles: see Table 6.5 and Table 6.6	Convert to/from another representation for the operation (see Note 2)	See Note 1
Unit quaternion $\boldsymbol{q}$	4	unit constraint: $\boldsymbol{q}\boldsymbol{q}^* = 1$	$\boldsymbol{q}$ is equivalent to $-\boldsymbol{q}$ (see Note 3)	Quaternion multiplication	$\boldsymbol{q}^*$ or $-\boldsymbol{q}^*$

NOTE 1 The inverse in the Euler angle $\boldsymbol{z}$-$\boldsymbol{x}$-$\boldsymbol{z}$ convention is:

$$[\boldsymbol{R}_z\langle\alpha\rangle \circ \boldsymbol{R}_x\langle\beta\rangle \circ \boldsymbol{R}_z\langle\gamma\rangle]^{-1} = \boldsymbol{R}_z\langle-\gamma\rangle \circ \boldsymbol{R}_x\langle-\beta\rangle \circ \boldsymbol{R}_z\langle-\alpha\rangle.$$

The inverse in the Euler angle $\boldsymbol{x}$-$\boldsymbol{y}$-$\boldsymbol{z}$ and $\boldsymbol{z}$-$\boldsymbol{y}$-$\boldsymbol{x}$ conventions (Tait-Bryan angles) are

$$[\boldsymbol{R}_x\langle\phi\rangle \circ \boldsymbol{R}_y\langle\theta\rangle \circ \boldsymbol{R}_z\langle\psi\rangle]^{-1} = \boldsymbol{R}_z\langle-\psi\rangle \circ \boldsymbol{R}_y\langle-\theta\rangle \circ \boldsymbol{R}_x\langle-\phi\rangle$$
$$[\boldsymbol{R}_z\langle\psi\rangle \circ \boldsymbol{R}_y\langle\theta\rangle \circ \boldsymbol{R}_x\langle\phi\rangle]^{-1} = \boldsymbol{R}_x\langle-\phi\rangle \circ \boldsymbol{R}_y\langle-\theta\rangle \circ \boldsymbol{R}_z\langle-\psi\rangle$$

NOTE 2 The composition of Euler angle operations may also be performed in a "direct" method that involves lengthy expressions combining forward and inverse trigonometric functions.

NOTE 3 Formulae such as Equation (6.13) require the unit quaternion constraint. Other useful relationships such as Equation (6.12) do not have that requirement. For that reason, some applications do not enforce the unit constraint. In the unconstrained case, every non-zero scalar multiple of a given quaternion is rotationally equivalent to it.

6.9 Inter-converting between rotations representations

6.9.1 Euler angle conventions and matrix representation

6.9.1.1 Matrix forms of principal rotations

For notation convenience, given a principal axis $\boldsymbol{a}$ ($\boldsymbol{a} = \boldsymbol{x}$ or $\boldsymbol{y}$ or $\boldsymbol{z}$), $\boldsymbol{R}_a\langle\omega\rangle$ shall denote a principal rotation with angle specification in the PVR convention and $\boldsymbol{\Omega}_a\langle\omega\rangle$ shall denote a principal rotation with angle specification in the CFR convention. In particular, $\boldsymbol{\Omega}_a\langle\omega\rangle = \boldsymbol{R}_a\langle-\omega\rangle$. The matrix representations of principal rotations in this notation are given in Table 6.2.

Table 6.1 — Principal rotations as matrix operators

Name	Notation	Matrix operator (left multiplication)
$\boldsymbol{x}$-axis principal rotation CFR convention	$\boldsymbol{\Omega}_x\langle\omega_1\rangle$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\omega_1) & \sin(\omega_1) \\ 0 & -\sin(\omega_1) & \cos(\omega_1) \end{bmatrix}$, where ω_1 is the angle of rotation.

Name	Notation	Matrix operator (left multiplication)
x-axis principal rotation PVR convention	$\boldsymbol{R}_x\langle\omega_1\rangle$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\omega_1) & -\sin(\omega_1) \\ 0 & \sin(\omega_1) & \cos(\omega_1) \end{bmatrix}$, where ω_1 is the angle of rotation.
y-axis principal rotation CFR convention	$\boldsymbol{\Omega}_y\langle\omega_2\rangle$	$\begin{bmatrix} \cos(\omega_2) & 0 & -\sin(\omega_2) \\ 0 & 1 & 0 \\ \sin(\omega_2) & 0 & \cos(\omega_2) \end{bmatrix}$, where ω_2 is the angle of rotation.
y-axis principal rotation PVR convention	$\boldsymbol{R}_y\langle\omega_2\rangle$	$\begin{bmatrix} \cos(\omega_2) & 0 & \sin(\omega_2) \\ 0 & 1 & 0 \\ -\sin(\omega_2) & 0 & \cos(\omega_2) \end{bmatrix}$, where ω_2 is the angle of rotation.
z-axis principal rotation CFR convention	$\boldsymbol{\Omega}_z\langle\omega_3\rangle$	$\begin{bmatrix} \cos(\omega_3) & \sin(\omega_3) & 0 \\ -\sin(\omega_3) & \cos(\omega_3) & 0 \\ 0 & 0 & 1 \end{bmatrix}$, where ω_3 is the angle of rotation.
z-axis principal rotation PVR convention	$\boldsymbol{R}_z\langle\omega_3\rangle$	$\begin{bmatrix} \cos(\omega_3) & -\sin(\omega_3) & 0 \\ \sin(\omega_3) & \cos(\omega_3) & 0 \\ 0 & 0 & 1 \end{bmatrix}$, where ω_3 is the angle of rotation.

6.9.1.2 The z-x-z Euler angle convention

The angle sequence (α, β, γ) in the Euler $\boldsymbol{z}$-$\boldsymbol{x}$-$\boldsymbol{z}$ convention is converted to a matrix $\boldsymbol{M}$ by forming the matrix product of the corresponding three principal rotation matrices specified in Table 6.2 corresponding to the consecutive rotations $\boldsymbol{R}_z\langle\alpha\rangle \circ \boldsymbol{R}_x\langle\beta\rangle \circ \boldsymbol{R}_z\langle\gamma\rangle$. The resulting matrix is given in Equation (6.14).

$$\boldsymbol{M} = \begin{pmatrix} \cos\alpha\cos\gamma - \cos\beta\sin\alpha\sin\gamma & -\sin\gamma\cos\alpha - \cos\cos\gamma\sin\alpha & \sin\beta\sin\alpha \\ \cos\beta\sin\gamma\cos\alpha + \cos\gamma\sin\alpha & \cos\beta\cos\alpha\cos\gamma - \sin\alpha\sin\gamma & -\sin\beta\cos\alpha \\ \sin\beta\sin\gamma & \sin\beta\cos\gamma & \cos\beta \end{pmatrix} \tag{6.13}$$

Conversely, given a matrix $\boldsymbol{M}$ with elements a_{ij}, the equation may solved for the principal rotation factors $\boldsymbol{R}_z\langle\alpha\rangle \circ \boldsymbol{R}_x\langle\beta\rangle \circ \boldsymbol{R}_z\langle\gamma\rangle$, and therefore solved for angles (α, β, γ). The solution is given in Table 6.3.

Table 6.2 — Principal factors for the Euler z-x-z convention

Case	Principal factors for rotation $\boldsymbol{R}_{z''}\langle\gamma\rangle \circ \boldsymbol{R}_{x'}\langle\beta\rangle \circ \boldsymbol{R}_z\langle\alpha\rangle = \boldsymbol{R}_z\langle\alpha\rangle \circ \boldsymbol{R}_x\langle\beta\rangle \circ \boldsymbol{R}_z\langle\gamma\rangle$ (all angles modulo 2π, $\boldsymbol{M}=[a_{ij}]$)		
$a_{33} \neq \pm 1$	$\beta = \arccos(a_{33})$ [principal value] $0 < \beta < \pi$	$\gamma = \text{arctan2}(a_{31},\ a_{32})$	$\alpha = \text{arctan2}(a_{13},\ -a_{23})$
	$\beta = \arccos(a_{33})$ [2π − principal value] $\pi < \beta < 2\pi$	$\gamma = \text{arctan2}(-a_{31},\ -a_{32})$	$\alpha = \text{arctan2}(-a_{13},\ a_{23})$
$a_{33} = -1$	$\beta = \pi$	any value of γ	$\alpha = \text{arctan2}(a_{21},\ a_{11}) + \gamma$
$a_{33} = +1$	$\beta = 0$	any value of γ	$\alpha = \text{arctan2}(a_{21},\ a_{11}) - \gamma$

In the case $a_{3_3} \neq \pm 1$, $\arccos()$ is multi-valued so that there are two valid solution sets depending on the quadrants selected for arccosine values[20]. The principal value solution is the commonly used one. The two argument arctangent function $\mathrm{arctan2}()$ is defined in Annex A.

The case $a_{3_3} = -1$ corresponds to $\boldsymbol{R}_z\langle\alpha\rangle \circ \boldsymbol{R}_x\langle\pi\rangle \circ \boldsymbol{R}_z\langle\gamma\rangle$. Using trigonometric identities, the matrix expression in this case reduces to:

$$\begin{bmatrix} \cos(\alpha-\gamma) & \sin(\alpha-\gamma) & 0 \\ \sin(\alpha-\gamma) & -\cos(\alpha-\gamma) & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

For this reason, only the difference of the other two angles can be determined by using $\alpha - \gamma = \mathrm{arctan2}(a_{21},\ a_{11})$. Therefore, all values are valid for α if $\gamma = \mathrm{arctan2}(a_{21},\ a_{11}) + \alpha$. The case $a_{3_1} = +1$ is similar to the previous case with the sum of the angles determined by using $\gamma + \alpha = \mathrm{arctan2}(a_{21},\ a_{11})$. These two cases correspond to the gimbal lock Equation (6.8).

As seen in the preceding tables, the three angle sequence corresponding to a given rotation or orientation operator is not unique modulo 2π. Two sequences, $(\alpha_1, \beta_1, \gamma_1)$ and $(\alpha_2, \beta_2, \gamma_2)$ of $\boldsymbol{z}$-$\boldsymbol{x}$-$\boldsymbol{z}$ principal factors specify the same operator if they satisfy one the criteria specified in Table 6.4.

Table 6.3 — Equivalence of z-x-z principal factor sequences

Case (equality modulo 2π)	**Criteria for the equivalence of angle sequences $(\alpha_1, \beta_1, \gamma_1)$ and $(\alpha_2, \beta_2, \gamma_2)$ for principal factor z-x-z sequences**	
$\beta_1 = \beta_2$	$\alpha_1 = \alpha_2,\ \gamma_1 = \gamma_2 [\beta_1, \beta_2 \neq 0 \text{ or } \pi]$	(in)equalities modulo 2π
$\lvert\beta_1 + \beta_2\rvert = 2\pi$	$\lvert\alpha_2 - \alpha_1\rvert = \pi, \lvert\gamma_2 - \gamma_1\rvert = \pi [\beta_1, \beta_2 \neq 0 \text{ or } \pi]$	(in)equalities modulo 2π
$\beta_1 = \beta_2 = \pi$	$\alpha_1 - \gamma_1 = \alpha_2 - \gamma_2$	equality modulo 2π
$\beta_1 = \beta_2 = 0$	$\alpha_1 + \gamma_1 = \alpha_2 + \gamma_2$	equality modulo 2π

6.9.1.3 The Tait-Bryan conventions

The Euler angle sequences (φ, θ, ψ) in convention $\boldsymbol{x}$-$\boldsymbol{y}$-$\boldsymbol{z}$ and (ψ, θ, φ) in convention $\boldsymbol{z}$-$\boldsymbol{y}$-$\boldsymbol{x}$ are converted to respective matrices $\boldsymbol{M}$ by forming the matrix product of the corresponding three principal rotation matrices specified in Table 6.2. The resulting matrices are given in Equation (6.15).

Conversely, given matrix $\boldsymbol{M}$ with elements a_{ij}, the equation may solved for the principal rotation factors $\boldsymbol{R}_z\langle\psi\rangle \circ \boldsymbol{R}_y\langle\theta\rangle \circ \boldsymbol{R}_x\langle\varphi\rangle$, and therefore solved for angles (φ, θ, ψ). The solution is given in Table 6.5.

(6.14)

Corresponding to: $\boldsymbol{R}_{z''}\langle\psi\rangle \circ \boldsymbol{R}_y\langle\theta\rangle \circ \boldsymbol{R}_x\langle\varphi\rangle = \boldsymbol{R}_x\langle\varphi\rangle \circ \boldsymbol{R}_y\langle\theta\rangle \circ \boldsymbol{R}_z\langle\psi\rangle$

[20] Computer library functions such as $\mathrm{acos}()$ return the principal value only. The second solution for β may be obtained by subtracting the principal value from 2π.

Corresponding to: $\boldsymbol{R}_{x''}\langle\varphi\rangle \circ \boldsymbol{R}_{y'}\langle\theta\rangle \circ \boldsymbol{R}_{z}\langle\psi\rangle = \boldsymbol{R}_{z}\langle\psi\rangle \circ \boldsymbol{R}_{y}\langle\theta\rangle \circ \boldsymbol{R}_{x}\langle\varphi\rangle$

$$\boldsymbol{M} = \begin{bmatrix} \cos\psi\cos\theta & \cos\psi\sin\theta\sin\varphi - \sin\psi\cos\varphi & \cos\psi\sin\theta\cos\varphi + \sin\psi\sin\varphi \\ \sin\psi\cos\theta & \sin\psi\sin\theta\sin\varphi + \cos\psi\cos\varphi & \sin\psi\sin\theta\cos\varphi - \cos\psi\sin\varphi \\ -\sin\theta & \cos\theta\sin\varphi & \cos\theta\cos\varphi \end{bmatrix}$$

Conversely, given matrix $\boldsymbol{M}$ with elements a_{ij}, the equation may solved for the angle sequence of the principal rotation factors. The solution for the x-y-z case $\boldsymbol{R}_{x}\langle\varphi\rangle \circ \boldsymbol{R}_{y}\langle\theta\rangle \circ \boldsymbol{R}_{z}\langle\psi\rangle$ is given in Table 6.5, and the solution for the z-y-x case $\boldsymbol{R}_{z}\langle\psi\rangle \circ \boldsymbol{R}_{y}\langle\theta\rangle \circ \boldsymbol{R}_{x}\langle\varphi\rangle$ is given in Table 6.6.

Table 6.4 — Principal factors for the Euler x-y-z convention (Tait-Bryan)

Case	Principal factors for x-y-z Euler rotation $\boldsymbol{R}_{z''}\langle\psi\rangle \circ \boldsymbol{R}_{y}\langle\theta\rangle \circ \boldsymbol{R}_{x}\langle\varphi\rangle = \boldsymbol{R}_{x}\langle\varphi\rangle \circ \boldsymbol{R}_{y}\langle\theta\rangle \circ \boldsymbol{R}_{z}\langle\psi\rangle$ (all angles modulo 2π, $\boldsymbol{M}=[a_{ij}]$)		
$a_{13} \neq \pm 1$	$\theta = \arcsin(a_{13})$ [principal value] $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$\varphi = \operatorname{arctan2}(-a_{23},\ a_{33})$	$\psi = \operatorname{arctan2}(-a_{12},\ a_{11})$
	$\theta = \arcsin(a_{13})$ [π − principal value] $\frac{\pi}{2} < \theta < \frac{3\pi}{2}$	$\varphi = \operatorname{arctan2}(a_{23},\ -a_{33})$	$\psi = \operatorname{arctan2}(a_{12},\ -a_{11})$
$a_{13} = +1$	$\theta = \frac{\pi}{2}$	$\varphi = \operatorname{arctan2}(a_{21},\ -a_{31}) - \psi$	any value of ψ
$a_{13} = -1$	$\theta = \frac{-\pi}{2}$	$\varphi = \operatorname{arctan2}(a_{21},\ a_{31}) + \psi$	any value of ψ

In the case $a_{13} \neq \pm 1$, arcsin() is multi-valued so that there are two valid solution sets depending on the quadrant selected for arcsine values[21]. The principal value solution is the commonly used one.

The case $a_{13} = +1$ corresponds to $\boldsymbol{R}_{x}\langle\varphi\rangle \circ \boldsymbol{R}_{y}\langle\pi/2\rangle \circ \boldsymbol{R}_{z}\langle\psi\rangle$. Using the trigonometric identities for the difference of angles and substituting $\sin\theta = 1$ and $\cos\theta = 0$, the matrix in this case reduces to:

$$\begin{bmatrix} 0 & 0 & 1 \\ \sin(\varphi+\psi) & \cos(\varphi+\psi) & 0 \\ -\cos(\varphi+\psi) & \sin(\varphi+\psi) & 0 \end{bmatrix}.$$

For this reason only the sum of the other two angles is determined as $\varphi + \psi = \operatorname{arctan2}(a_{21},\ -a_{31})$. Therefore, all values are valid for ψ if we set $\varphi = \operatorname{arctan2}(a_{21},\ -a_{31}) - \psi$. The case $a_{13} = -1$ is similar to the previous case with the difference of the angles determined by $\varphi - \psi = \operatorname{arctan2}(a_{21},\ a_{31})$. These two cases correspond to Equation (6.9) and are the gimbal lock cases.

[21] Computer library functions such as asin() return the principal value only. The second solution for θ may be obtained by subtracting the principal value from π.

Table 6.5 — Principal factors for the Euler *z*-*y*-*x* convention (Tait-Bryan)

Case	Principal factors for *z*-*y*-*x* Euler rotation $\boldsymbol{R}_{x''}\langle\varphi\rangle \circ \boldsymbol{R}_{y'}\langle\theta\rangle \circ \boldsymbol{R}_{z}\langle\psi\rangle = \boldsymbol{R}_{z}\langle\psi\rangle \circ \boldsymbol{R}_{y}\langle\theta\rangle \circ \boldsymbol{R}_{x}\langle\varphi\rangle$ (all angles modulo 2π, $\boldsymbol{M}=[a_{ij}]$)		
$a_{31} \neq \pm 1$	$\theta = \arcsin(-a_{31})$ [principal value] $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$\varphi = \operatorname{arctan2}(a_{32},\ a_{33})$	$\psi = \operatorname{arctan2}(a_{21},\ a_{11})$
	$\theta = \arcsin(-a_{31})$ [$\pi -$ principal value] $\frac{\pi}{2} < \theta < \frac{3\pi}{2}$	$\varphi = \operatorname{arctan2}(-a_{32},\ -a_{33})$	$\psi = \operatorname{arctan2}(-a_{21},\ -a_{11})$
$a_{31} = -1$	$\theta = \frac{\pi}{2}$	$\varphi = \operatorname{arctan2}(a_{12},\ a_{13}) + \psi$	any value of ψ
$a_{31} = +1$	$\theta = \frac{-\pi}{2}$	$\varphi = \operatorname{arctan2}(-a_{12},\ -a_{13}) - \psi$	any value of ψ

In the case $a_{3_1} = -1$, using the trigonometric identities for the difference of angles and substituting $\sin\theta = 1$ and $\cos\theta = 0$, the matrix reduces to:

$$\boldsymbol{R}_{z}\langle\psi\rangle \circ \boldsymbol{R}_{y}\langle\pi/2\rangle \circ \boldsymbol{R}_{x}\langle\varphi\rangle = \begin{bmatrix} 0 & \sin(\varphi-\psi) & \cos(\varphi-\psi) \\ 0 & \cos(\varphi-\psi) & -\sin(\varphi-\psi) \\ -1 & 0 & 0 \end{bmatrix}.$$

For this reason only the difference of the other two angles is determined as $\varphi - \psi = \operatorname{arctan2}(a_{12},\ a_{13})$. Therefore, all values are valid for ψ if we set $\varphi = \operatorname{arctan2}(a_{12},\ a_{13}) + \psi$. The case $a_{3_1} = +1$ is similar to the previous case with the sum of the angles determined by $\varphi + \psi = \operatorname{arctan2}(-a_{12},\ -a_{13})$. These two cases correspond to Equation (6.9) and are the gimbal lock cases.

As seen in the preceding tables, the three angle sequence corresponding to a given rotation or orientation operator is not unique modulo 2π. Two sequences, $(\varphi_1, \theta_1, \psi_1)$ and $(\varphi_2, \theta_2, \psi_2)$ of ***x*-*y*-*z*** principal factors specify the same operator if they satisfy one the criteria specified in Table 6.6.

Table 6.6 — Equivalence of *x*-*y*-*z* or *z*-*y*-*x* principal factor sequences

Case (equality modulo 2π)	**Criteria for the equivalence of angle sequences $(\varphi_1, \theta_1, \psi_1)$ and $(\varphi_2, \theta_2, \psi_2)$ for principal factor *z*-*y*-*x* or *x*-*y*-*z* sequences**
$\theta_1 = \theta_2$	$\varphi_1 = \varphi_2,\ \ \psi_1 = \psi_2 \left[\theta_1 \neq \pm\frac{\pi}{2} \neq \theta_2\right]$ (in)equalities modulo 2π
$\lvert\theta_1 + \theta_2\rvert = \pi$	$\lvert\varphi_2 - \varphi_1\rvert = \pi,\ \ \lvert\psi_2 - \psi_1\rvert = \pi \left[\theta_1 \neq \pm\frac{\pi}{2} \neq \theta_2\right]$ (in)equalities modulo 2π

Case (equality modulo 2π)	Criteria for the equivalence of angle sequences $(\varphi_1, \theta_1, \psi_1)$ and $(\varphi_2, \theta_2, \psi_2)$ for principal factor z-y-x or x-y-z sequences
$\theta_1 = \theta_2 = \dfrac{\pi}{2}$	$\varphi_1 + \psi_1 = \varphi_2 + \psi_2$ x-y-z case, equality modulo 2π $\varphi_1 - \psi_1 = \varphi_2 - \psi_2$ z-y-x case, equality modulo 2π
$\theta_1 = \theta_2 = -\dfrac{\pi}{2}$	$\varphi_1 - \psi_1 = \varphi_2 - \psi_2$ x-y-z case, equality modulo 2π $\varphi_1 + \psi_1 = \varphi_2 + \psi_2$ z-y-x case, equality modulo 2π

6.9.2 Matrix and axis-angle

Given a rotation matrix $\boldsymbol{M} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, the corresponding axis-angle representation (n_1, n_2, n_3, θ) is determined using the procedure in 6.7.2.

An axis-angle rotation (n_1, n_2, n_3, θ), is converted to rotation matrix $\boldsymbol{M}$, using the matrix form of Rodrigues' rotation formula (Equation (6.3)).

$$\begin{aligned} \boldsymbol{M} &= [\boldsymbol{I}_{3\times 3} + \sin(\theta)\,\boldsymbol{S_n} + (1-\cos(\theta))\boldsymbol{S_n^2}] \\ &= [\cos(\theta)\,\boldsymbol{I}_{3\times 3} + (1-\cos(\theta))\boldsymbol{n}\otimes\boldsymbol{n} + \sin(\theta)\,\boldsymbol{S_n}] \end{aligned}$$

where:

$$\boldsymbol{S_n} = \begin{bmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{bmatrix} \text{ is the skew-symmetric matrix associated with} \tag{6.15}$$

$\boldsymbol{n} = (n_1 \quad n_2 \quad n_3)^{\mathrm{T}}$ and

$$\boldsymbol{n}\otimes\boldsymbol{n} = \begin{bmatrix} n_1n_1 & n_1n_2 & n_1n_3 \\ n_2n_1 & n_2n_2 & n_2n_3 \\ n_3n_1 & n_3n_2 & n_3n_3 \end{bmatrix} \text{ is the outer-product of } \boldsymbol{n} \text{ with } \boldsymbol{n}.$$

The equation expands to yield matrix elements:

$$\boldsymbol{M} = \begin{bmatrix} (1-\cos\theta)n_1^2 + \cos\theta & (1-\cos\theta)n_1n_2 - n_3\sin\theta & (1-\cos\theta)n_1n_3 + n_2\sin\theta \\ (1-\cos\theta)n_2n_1 + n_3\sin\theta & (1-\cos\theta)n_2^2 + \cos\theta & (1-\cos\theta)n_2n_3 - n_1\sin\theta \\ (1-\cos\theta)n_3n_1 - n_2\sin\theta & (1-\cos\theta)n_3n_2 + n_1\sin\theta & (1-\cos\theta)n_3^2 + \cos\theta \end{bmatrix} \tag{6.16}$$

6.9.3 Axis-angle and quaternion

A rotation in axis-angle form (n_1, n_2, n_3, θ) corresponds to unit quaternion $\boldsymbol{q} = (\cos(\theta/2),\ \sin(\theta/2)\,\boldsymbol{n})$ with $\boldsymbol{n} = (n_1, n_2, n_3)$.

A unit quaternion corresponds to axis-angle form (n_1, n_2, n_3, θ) computed as in Equation (6.11).

6.9.4 Matrix and quaternion

The matrix $\boldsymbol{M}$ corresponding to a unit quaternion $\boldsymbol{q} = (e_0, \boldsymbol{e})$, $\boldsymbol{e} = (e_1, e_2, e_3)^T$ is

$$\boldsymbol{M} = \begin{bmatrix} 1-2(e_2^2+e_3^2) & 2(e_1e_2-e_0e_3) & 2(e_1e_3+e_0e_2) \\ 2(e_1e_2+e_0e_3) & 1-2(e_1^2+e_3^2) & 2(e_2e_3-e_0e_1) \\ 2(e_1e_3-e_0e_2) & 2(e_2e_3+e_0e_1) & 1-2(e_1^2+e_2^2) \end{bmatrix} \tag{6.17}$$

The quaternion $\boldsymbol{q}$ corresponding to a rotation matrix $\boldsymbol{M} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ is computed as follows:

$e_0^2 = \frac{1}{4}\left(1+\mathrm{Trace}(\boldsymbol{M})\right) = \frac{1}{4}(1+a_{11}+a_{22}+a_{33})$

if $e_0^2 > 0$,

$\boldsymbol{e} = \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = \frac{1}{4e_0}\begin{bmatrix} a_{32}-a_{23} \\ a_{13}-a_{31} \\ a_{21}-a_{12} \end{bmatrix}$,

else

$e_0 = 0, \quad e_1^2 = -\frac{1}{2}(a_{22}+a_{33})$,

if $e_1^2 > 0$,

$e_2 = \frac{a_{12}}{2e_1}, \quad e_3 = \frac{a_{13}}{2e_1}$,

else

$e_1 = 0, \quad e_2^2 = \frac{1}{2}(1-a_{33})$,

if $e_2^2 > 0$,

$e_3 = \frac{a_{23}}{2e_2}$

else

$e_2 = 0, \quad e_3 = 1$.

A rotationally equivalent quaternion is $-\boldsymbol{q}$.

6.9.5 Euler angle conventions and quaternions

The principal rotations (see 6.7.4.1) correspond to the following quaternions:

$$\boldsymbol{R_x}\langle\alpha\rangle \leftrightarrow (\cos(\alpha/2),\ \sin(\alpha/2)\,\boldsymbol{x})$$

$$\boldsymbol{R_y}\langle\beta\rangle \leftrightarrow (\cos(\beta/2),\ \sin(\beta/2)\,\boldsymbol{y})$$

$$\boldsymbol{R_z}\langle\gamma\rangle \leftrightarrow (\cos(\gamma/2),\ \sin(\gamma/2)\,\boldsymbol{z})$$

For each Euler angle convention, multiply the corresponding quaternions in body-fixed composition order. Terms in the resulting product may be simplified using the orthonormal property of the vector set $\boldsymbol{x}$, $\boldsymbol{y}$, $\boldsymbol{z}$, and various trigonometric identities.

For the Euler angle $z\text{-}x\text{-}z$ convention, the quaternion $\boldsymbol{q}$ corresponding to $\boldsymbol{R_{z''}}\langle\gamma\rangle \circ \boldsymbol{R_{x'}}\langle\beta\rangle \circ \boldsymbol{R_z}\langle\alpha\rangle = \boldsymbol{R_z}\langle\alpha\rangle \circ \boldsymbol{R_x}\langle\beta\rangle \circ \boldsymbol{R_z}\langle\gamma\rangle$ is:

$$\boldsymbol{q} = (\cos(\alpha/2),\ \sin(\alpha/2)\,\boldsymbol{z})\ (\cos(\beta/2),\ \sin(\beta/2)\,\boldsymbol{x})\ (\cos(\gamma/2),\ \sin(\gamma/2)\,\boldsymbol{z}).$$

Multiplied out, the expression reduces to:

$\boldsymbol{q} = (e_0,\ \boldsymbol{e})$

where:

$e_0 = \cos((\alpha+\gamma)/2)\cos(\beta/2)$,

$\boldsymbol{e} = (\cos((\alpha-\gamma)/2)\sin(\beta/2),\ \ \sin((\alpha-\gamma)/2)\sin(\beta/2),\ \ \sin((\alpha+\gamma)/2)\cos(\beta/2))$

For the Euler angle $\boldsymbol{x}$-$\boldsymbol{y}$-$\boldsymbol{z}$ convention (Tait-Bryan angles), the quaternion $\boldsymbol{q}$ corresponding to $\boldsymbol{R}_{z''}\langle\psi\rangle \circ \boldsymbol{R}_{y'}\langle\theta\rangle \circ \boldsymbol{R}_{x}\langle\varphi\rangle = \boldsymbol{R}_{x}\langle\varphi\rangle \circ \boldsymbol{R}_{y}\langle\theta\rangle \circ \boldsymbol{R}_{z}\langle\psi\rangle$ is:

$$\boldsymbol{q} = \ (\cos(\varphi/2),\ \sin(\varphi/2)\,\boldsymbol{x})\ (\cos(\theta/2),\ \sin(\theta/2)\,\boldsymbol{y})\ (\cos(\psi/2),\ \sin(\psi/2)\,\boldsymbol{z}).$$

Multiplied out, the expression reduces to:

$$\boldsymbol{q} = (e_0,\ \boldsymbol{e}) = (e_0,\ e_1,\ e_2,\ e_3)$$

where:

$$e_0 = \cos(\varphi/2)\cos(\theta/2)\cos(\psi/2) - \sin(\varphi/2)\sin(\theta/2)\sin(\psi/2)$$

$$e_1 = \cos(\varphi/2)\sin(\theta/2)\sin(\psi/2) + \sin(\varphi/2)\cos(\theta/2)\cos(\psi/2)$$

$$e_2 = \cos(\varphi/2)\sin(\theta/2)\cos(\psi/2) - \sin(\varphi/2)\cos(\theta/2)\sin(\psi/2)$$

$$e_3 = \cos(\varphi/2)\cos(\theta/2)\sin(\psi/2) + \sin(\varphi/2)\sin(\theta/2)\cos(\psi/2)$$

For the Euler angle $\boldsymbol{z}$-$\boldsymbol{y}$-$\boldsymbol{x}$ convention (Tait-Bryan angles), the quaternion $\boldsymbol{q}$ corresponding to $\boldsymbol{R}_{x''}\langle\varphi\rangle \circ \boldsymbol{R}_{y'}\langle\theta\rangle \circ \boldsymbol{R}_{z}\langle\psi\rangle = \boldsymbol{R}_{z}\langle\psi\rangle \circ \boldsymbol{R}_{y}\langle\theta\rangle \circ \boldsymbol{R}_{x}\langle\varphi\rangle$ is:

$$\boldsymbol{q} = (\cos(\psi/2),\ \sin(\psi/2)\,\boldsymbol{z})\ (\cos(\theta/2),\ \sin(\theta/2)\,\boldsymbol{y})\ (\cos(\varphi/2),\ \sin(\varphi/2)\,\boldsymbol{x}).$$

Multiplied out, the expression reduces to:

$$\boldsymbol{q} = (e_0,\ \boldsymbol{e}) = (e_0,\ e_1,\ e_2,\ e_3)$$

where:

$$e_0 = \cos(\psi/2)\cos(\theta/2)\cos(\varphi/2) + \sin(\psi/2)\sin(\theta/2)\sin(\varphi/2)$$

$$e_1 = \cos(\psi/2)\cos(\theta/2)\sin(\varphi/2) - \sin(\psi/2)\sin(\theta/2)\cos(\varphi/2)$$

$$e_2 = \cos(\psi/2)\sin(\theta/2)\cos(\varphi/2) + \sin(\psi/2)\cos(\theta/2)\sin(\varphi/2)$$

$$e_3 = \sin(\psi/2)\cos(\theta/2)\cos(\varphi/2) - \cos(\psi/2)\sin(\theta/2)\sin(\varphi/2)$$

To convert a unit quaternion $\boldsymbol{q} = (e_0,\ \boldsymbol{e}) = (e_0,\ e_1,\ e_2,\ e_3)$ to the Euler angle $\boldsymbol{z}$–$\boldsymbol{x}$–$\boldsymbol{z}$ convention $\boldsymbol{R}_{z}\langle\alpha\rangle \circ \boldsymbol{R}_{x}\langle\beta\rangle \circ \boldsymbol{R}_{z}\langle\gamma\rangle$, compute as follows:

if $0 < (e_1^2 + e_2^2) < 1$:

$$\alpha = \operatorname{arctan2}\big((e_1 e_3 + e_0 e_2),\ -(e_2 e_3 - e_0 e_1)\big)$$

$$\beta = \arccos\big(1 - 2(e_1^2 + e_2^2)\big) \qquad \text{principal value: } 0 < \beta < \pi$$

$$\gamma = \operatorname{arctan2}\big((e_1 e_3 - e_0 e_2),\ (e_2 e_3 + e_0 e_1)\big)$$

if $(e_1^2 + e_2^2) = 0$:

$\beta = 0$ and $\alpha + \gamma = \operatorname{arctan2}\left((e_1 e_2 - e_0 e_3),\ \frac{1}{2} - (e_2^2 + e_3^2)\right)$.

if $(e_1^2 + e_2^2) = 1$:

$\beta = \pi$ and $\alpha - \gamma = \operatorname{arctan2}\left((e_1 e_2 - e_0 e_3),\ \frac{1}{2} - (e_2^2 + e_3^2)\right)$.

The solution in the first case is not unique, see Table 6.4. The last two cases are Euler angle gimbal lock cases.

To convert a unit quaternion $\boldsymbol{q} = (e_0,\ \boldsymbol{e}) = (e_0,\ e_1,\ e_2,\ e_3)$ to the Euler angle $\boldsymbol{x}$-$\boldsymbol{y}$-$\boldsymbol{z}$ convention (Tait-Bryan angles) $\boldsymbol{R}_{x}\langle\varphi\rangle \circ \boldsymbol{R}_{y}\langle\theta\rangle \circ \boldsymbol{R}_{z}\langle\psi\rangle$, compute as follows.

If $2(e_1e_3 + e_0e_2) \neq \pm 1$:

$$\varphi = \mathrm{arctan2}\left((e_2e_3 - e_0e_1),\ \frac{1}{2} - (e_1^2 + e_2^2)\right)$$

$$\theta = \arcsin\big(2(e_1e_3 + e_0e_2)\big)\ \text{principal value: } -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

$$\psi = \mathrm{arctan2}\left(-(e_1e_2 - e_0e_3),\ \frac{1}{2} - (e_2^2 + e_3^2)\right)$$

If $2(e_1e_3 + e_0e_2) = +1$:

$\theta = \frac{-\pi}{2}$ and $\varphi + \psi = \mathrm{arctan2}\big((e_1e_2 + e_0e_3),\ -(e_1e_3 - e_0e_2)\big)$.

If $2(e_1e_3 + e_0e_2) = -1$:

$\theta = \frac{\pi}{2}$ and $\varphi - \psi = \mathrm{arctan2}\big((e_1e_2 + e_0e_3),\ (e_1e_3 - e_0e_2)\big)$.

The solution in the first case is not unique, see Table 6.7. The last two cases are Euler angle gimbal lock cases.

To convert a unit quaternion $\boldsymbol{q} = (e_0,\ \boldsymbol{e}) = (e_0,\ e_1,\ e_2,\ e_3)$ to the Euler angle $\boldsymbol{z}$-$\boldsymbol{y}$-$\boldsymbol{x}$ convention (Tait-Bryan angles) $\boldsymbol{R_z}\langle\psi\rangle \circ \boldsymbol{R_y}\langle\theta\rangle \circ \boldsymbol{R_x}\langle\phi\rangle$, compute as follows.

If $2(e_1e_3 - e_0e_2) \neq \pm 1$:

$$\varphi = \mathrm{arctan2}\left((e_2e_3 + e_0e_1),\ \frac{1}{2} - (e_1^2 + e_2^2)\right)$$

$$\theta = \arcsin\big(-2(e_1e_3 - e_0e_2)\big)\ \text{principal value: } -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

$$\psi = \mathrm{arctan2}\left((e_1e_2 + e_0e_3),\ \frac{1}{2} - (e_2^2 + e_3^2)\right)$$

If $2(e_1e_3 - e_0e_2) = +1$:

$\theta = \frac{-\pi}{2}$ and $\varphi + \psi = \mathrm{arctan2}\big((e_1e_2 - e_0e_3),\ (e_1e_3 + e_0e_2)\big)$.

If $2(e_1e_3 - e_0e_2) = -1$:

$\theta = \frac{\pi}{2}$ and $\varphi - \psi = \mathrm{arctan2}\big((e_1e_2 - e_0e_3),\ (e_1e_3 + e_0e_2)\big)$.

The solution in the first case is not unique, see Table 6.7. The last two cases are Euler angle gimbal lock cases.

http://standards.iso.org/ittf/PubliclyAvailableStandards/