# 2

# Structure of the Intertemporal General Equilibrium Model

## 2.1. Introduction

The Intertemporal General Equilibrium Model (IGEM) presented in chapter 1 is a comprehensive model of the US economy. The defining characteristic of a general equilibrium model is that prices are determined together with quantities through the interactions between supply and demand. In this chapter we describe the production sector that is central to the supply side of the model, and the household sector that forms the core of the demand side.

In the factor markets the household sector supplies capital and labor services, while the production sector demands these services as inputs. The model is completed by market-clearing conditions that determine supplies and demands for all commodities and all factors along with the corresponding prices. In chapter 5 below we describe the role of investment demand, government demand and demand for exports in the demand side of the model. We also consider the role of imports from the rest of the world in the supply side of the model.

The intertemporal equilibrium defined by IGEM requires that market-clearing conditions be satisfied for each commodity and each factor at every point of time. In addition, the markets are linked by investments in capital goods and asset prices. Asset pricing imparts forward-looking dynamics to IGEM, since the price of an asset is equal to the discounted value of the prices of capital services over the asset's future lifetime. Capital services are generated by stocks of assets accumulated through past investments, so that capital accumulation provides backward-looking dynamics for the model.

A distinguishing feature of IGEM is that unknown parameters of the behavioral equations are estimated econometrically, rather than calibrated. This is essential in capturing the heterogeneity of the industries and households that make up the US economy, as well as providing confidence intervals for model outcomes, as in chapter 9. We defer a description of the econometrics of the demand side of the model to chapter 3. Similarly, we defer the estimation of the producer model and import demands to chapters 4 and 5, respectively.

The model determines the flows of goods and factor services among the four main sectors of the economy – production, household, government, and rest of the world. The flows of payments among these sectors determine the expenditure patterns for the economy as a whole, including expenditures on consumption, investment, government, exports, and imports. In this chapter we show how the demands and supplies interact in markets to arrive at the intertemporal equilibrium prices. We also describe the solution algorithm for determining the equilibrium prices.

A complete list of the equations in IGEMis given in appendix A with all the distinguishing subscripts.Section A.8 of that appendix provides a glossary of the symbols we have used. In describing the model in this chapter we have simplified the notation in order to avoid a proliferation of symbols. For example, where this can be done without creating confusion, we suppress the variables that index industries, commodities, and households.

## 2.2 Producer Behavior and Technical Change

Market-based policies, such as energy and environmental taxes and tradable permits, insert tax wedges between supply and demand prices and generate government revenue. The supply and demand for tradable permits can be modeled along with demands and supplies for commodities. The costs associated with market-based policies are determined through the price responses to changes in policy.

The key to analyzing the economic impacts of energy and environmental policy is the substitutability among productive inputs, especially energy inputs, in response to price changes induced by policy. While production patterns reflect substitutability among inputs in response to price changes, these patterns also depend on changes in technology. Our model of producer behavior incorporates both substitution among inputs and technical change.

In the long run the material well-being of the population depends on the rate of technical change. Part of technical change is induced by changes in relative prices of inputs, while another part is autonomous with respect to these price changes. In addition, the relative demands for inputs may be altered by biased technical change. For example, energy use may decline in intensity due to energy-saving changes in technology, as well as substitution away from higher-priced energy.

Estimates for the parameters that describe substitutability and technical change for the industrial sectors that comprise the US economy are not available in the literature. Since the changes in the patterns of production are complex, specification and estimation of models of production suitable for the analysis of energy and environmental policy requires econometric methods. In chapter 4 we discuss the issues that arise in specifying and estimating the parameters that characterize substitution and technical change.

We subdivide the business sector into the 35 industries listed in Table 2.1. Households also purchase energy inputs but are treated separately from the business sector.  Six of the industries are energy producers – Oil Mining (industry 2), Gas Mining (3), Coal Mining (4), Electric Utilities (6), Gas Utilities(7) and Petroleum Refining (23). We have chosen the classification of "non-energy" industries to distinguish sectors that differ in the intensity of utilization of different inputs, especially energy, and exhibit different patterns of technical change.

**[Table 2.1 about here]**

The output of the production sector is divided among 36 commodities, each the primary product of one of the 36 industries. For example, steel is a primary product of the Primary Metals industry, while brokerage services are included in the Finance and Insurance industry. Many industries produce secondary products as well. For example, Petroleum Refining produces commodities that are the primary outputs of the Chemicals industry. The model permits joint production for all sectors and all commodities. We model joint production together with substitution and technical change for each industry.

The parameters of our production model are estimated econometrically from a historical data base covering the period 1960-2010. This data base is described in appendix B, and includes a time series of input-output tables in current and constant prices, as well as data on the prices and quantities of capital and labor services. These data comprise the industry-level production account of the “new architecture” for the US national accounts developed by Jorgenson (2009) and Jorgenson and Landefeld (2006). The methodology for constructing the data is presented in Jorgenson, Ho and Stiroh (2005).

The input-output tables consist of *use* and *make* matrices for each year. The use matrix gives the inputs used by each industry – intermediate inputs supplied by other industries, non-comparable imports, capital services and labor services. This matrix also gives commodity use by each category of final demand – consumption, investment, government, exports, and imports.

The use matrix is illustrated in figure 2.1. The rows of this table correspond to commodities, while the columns correspond to industries. Each entry in the table is the amount of a given commodity used by a particular industry. The sum of all entries in a row is equal to the total demand for the commodity, while the sum of all entries in a column is the value of all the inputs used in a given industry.

**[Figure 2.1 about here]**

**[Figure 2.2 about here]**

The make, or supply, matrix is illustrated in figure 2.2. This matrix describes an essential part of the technology of the US economy. The rows of the make matrix correspond to industries, while the columns correspond to commodities. Each entry in the table is the amount of a commodity supplied or “made” by a given industry. The domestic supply of the commodity is the sum of all the entries in the corresponding column, while the sum of all the entries in a row represents the value of the commodities produced by a given industry.

In short, the inter-industry accounts of the system of US national accounts are critical components of the historical data base for IGEM. The inter-industry accounts in current and constant prices are discussed in detail in appendix B.1. The use matrix includes the values of capital and labor inputs for each industry. These values are divided into prices and quantities, like the values of the inter-industry flows.

The value of capital services consists of all property-type income – profits and other operating surplus, depreciation, and taxes on property and property-type income. The price of capital services consists of the price of the corresponding asset, multiplied by an annualization factor that we denote as the *cost of capital*. The cost of capital consists of the rate of return plus the rate of depreciation, less capital gains or plus capital losses, all adjusted for taxes. The price of capital services and the cost of capital are discussed in appendix B.3, which is based on the detailed development in Jorgenson and Yun (2001, 2012).

The quantity of capital services is the annual flow from a given asset. The assets included in our historical data base are plant, equipment, inventories, and land. Plant and equipment are sub-divided into detailed sub-categories. For example, equipment includes items ranging from motor vehicles and construction equipment to computers and software. We aggregate over the prices and quantities of capital services from these asset types to obtain a price and a quantity index of capital services.

To obtain a quantity index each type of capital stock is weighted by the rental price of capital services. Similarly, we aggregate over asset types to obtain a price and quantity index of capital stock. Each type of capital stock is weighted by the asset price to obtain a quantity index of capital stock. The price and quantity of capital services and capital stocks are described in more detail in appendix B.2.

Similarly, the value of labor services includes all labor-type income – wages and salaries, labor income from self-employment, supplements such as contributions to social insurance, and taxes such as payroll taxes. The quantity of labor input for each demographic category is hours worked. We construct a price and a quantity index of labor services by weighting the hours worked for each demographic group by labor compensation per hour. These groups range from young workers with only secondary education to mature workers with advanced degrees. The price and quantity of labor services are described in more detail in appendix B.2.

Our methods for aggregating over detailed categories of capital and labor services are an essential feature of the historical data set. Arithmetic aggregates consisting of sums of hours worked or capital stocks would not accurately represent the substitution possibilities within each aggregate. For example, a simple sum of computers and software with industrial buildings would not reflect the impact of a shift in composition toward information technology equipment and software on capital input. Similarly, a simple sum of hours worked over college-educated and non-college workers would not reveal the impact on labor input of increases in educational attainment of the US work force.

### *2.2.1 Notation*

To describe our sub-model for producer behavior we begin with notation. The general system is to use P for prices, Q for quantities, and Greek letters for parameters.

| | |
|---|---|
| $QI_j$ | quantity of output of industry *j* |
| $PO_j$ | price of output to producers in industry *j* |
| $PI_j$ | price of output to purchasers from industry *j* |
| $QP_{ij}^j$ | quantity of commodity input *i* into industry *j* |
| $PS_i$ | price of commodity *i* to buyers |
| $KD_j$ | quantity of capital input into *j* |
| $LD_j$ | quantity of labor input into *j* |
| $E_j$ | index of energy intermediate input into *j* |
| $M_j$ | index of total non-energy intermediate input into *j* |
| $P_{E,j}$ | price of energy intermediate input into *j* |
| $P_{M,j}$ | price of total non-energy intermediate input into *j* |
| $PKD_j$ | price of total capital input to industry *j* |
| $PLD_j$ | price of total labor input to industry *j* |
| $v$ | value shares |
| $QC_i$ | quantity of domestically produced commodity *i* |
| $PC_i$ | price of domestically produced commodity *i* |
| $M_{j,i}$ | MAKE matrix; value of commodity *i* made by industry *j* |

### *2.2.2 Top Tier Production Functions with Technical Change*

The production function represents output as a function of capital services, labor services, and intermediate inputs. Output also depends on the level of technology *t*, so that the output of industry takes the form:

(2.1) $QI_j = f(KD_j, LD_j, QP_1^j, QP_2^j, ...QP_m^j, QP_{36}^j, t)$, ( j=1,2,...36)

The dimensionality of this form of the production function is too great to be tractable.

To reduce the dimensionality of the production function we assume that this function is separable in energy and materials inputs. Output depends on inputs of energy and non-energy materials, as well as inputs of capital and labor services:

(2.2) $$\begin{aligned} &QI_j = f(KD_j, LD_j, E_j, M_j, t); \\ &E_j = E(QP_3^j ...); \qquad M_j = M(QP_1^j, ...) \end{aligned}$$

In the second stage of the production model the energy and non-energy inputs depend on the components of each of these aggregates. For example, energy input depends on inputs of coal, crude oil, natural gas, electricity and refined petroleum products. These are primary outputs of Coal Mining, Oil Mining, Gas Mining, Electric Utilities, Gas Utilities and Petroleum Refining, industries. Similarly, non-energy input depends on inputs of all the non-energy commodities listed in Table 2.1. The tier structure of the production model is given in table 2.2.

**[Table 2.2 about here]**

We assume constant returns to scale and competitive markets, so that the production function (2.2) is homogeneous of degree one and the value of output is equal to the sum of the values of all inputs:

$$(2.3)\quad PO_{jt}QI_{jt} = PKD_{jt}KD_{jt} + PLD_{jt}LD_{jt} + P_{Ejt}E_{jt} + P_{Mjt}M_{jt}$$

$$P_{Ejt}E_{jt} = PS_{2t}QP_{2t}^{j} + PS_{3t}QP_{3t}^{j} + .. + PS_{23t}QP_{23t}^{j}$$

$$P_{Mjt}M_{jt} = PS_{1t}QP_{1t}^{j} + PS_{5t}QP_{5t}^{j} + .. + PS_{36,t}QP_{36,t}^{j}$$

Under constant returns to scale the price function expresses the price of output as a function of the input prices and technology, so that for industry *j*:

$$(2.4)\quad PO_{j} = p(PKD_{j}, PLD_{j}, P_{Ej}, P_{Mj}, t)$$

Since we assume that all producers within a sector face the same prices, the aggregate price function can be represented as a function of these prices. In order to characterize substitution and technical change we find it convenient to work with the price function, rather than the production function (2.1)[1].

In order to represent substitutability among inputs and technical change in a flexible manner we have chosen the translog form of the price function, following Jin and Jorgenson (2010):

$$(2.5)\quad \ln PO_{t} = \alpha_{0} + \sum_{i}\alpha_{i}\ln p_{it} + \tfrac{1}{2}\sum_{i,k}\beta_{ik}\ln p_{it}\ln p_{kt} + \sum_{i}\ln p_{it} f_{it}^{p} + f_{t}^{p}$$

$$p_{i},, p_{k}=\{PKD,PLD,P_{E},P_{M}\}$$

For simplicity we have dropped the industry *j* subscript. The parameters -- $\alpha_i$, $\beta_{ik}$ and $\alpha_0$ -- are estimated separately for each industry.

The vector of latent variables

$$\xi_{t} = (1, f_{Kt}^{p}, f_{Lt}^{p}, f_{Et}^{p}, f_{Mt}^{p}, \Delta f_{t}^{p})'$$

is generated by a first-order vector autoregressive scheme:

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[1] The price function contains the same information about technology as the production function. For further details, see Jorgenson (1986).

(2.6) $\xi_t = F\xi_{t-1} + v_t$

The *p* superscript denotes that these are latent variables for the production sector. We introduce additional latent variables for the consumption, investment and import functions. We estimate these latent variables by means of the Kalman filter, following Jin and Jorgenson (2010).

The production sub-model (2.5) and (2.6) achieves considerable flexibility in the representation of substitution and technical change. An important advantage of this model is that it generates equations for the input shares that are linear in the logarithms of the prices and the latent variables. Differentiating equation (2.5) with respect to the logarithms of the prices, we obtain equations for the shares of inputs. For example, if we differentiate with respect to the price of capital services, we obtain the share of capital input:

(2.7) $v_K = \dfrac{PKD_t KD_t}{PO_t QI_t} = \alpha_K + \sum_k \beta_{Kk} \ln P_k + f_{Kt}^p$

The parameters { $\beta_{ik}$ } are share elasticities, giving the change in the share of the *i*th input in the value of output with respect to a proportional change in the price of the *k*th input. These parameters represent the degree of substitutability among the capital (K), labor (L), energy (E), and non-energy (M) inputs. If the share elasticity is positive, the value share increases with a change in the price of the input; if the share elasticity is negative, the share decreases with a change in the price. A zero share elasticity implies that the value share is constant, as in the linear-logarithmic or Cobb-Douglas specification of technology used by Johansen.

The price function is homogeneous of degree one, so that a doubling of input prices results in a doubling of the output price. This implies that the row and column sums of the matrix of share elasticities must be equal to zero:

(2.8) $\sum_i \beta_{ik} = 0$ for each k; $\sum_k \beta_{ik} = 0$ for each i.

Symmetry of the price effects implies that the matrix of share elasticities is symmetric. Monotonicity and concavity restrictions on the price function are discussed in chapter 4.

The level of technology $f_t^p$ and the biases of technical change { $f_{it}^p$ } evolve according to equation (2.6). The latent variables { $f_{it}^p$ } describe the biases of technical change. For example, if the energy share declines, holding prices of all inputs constant, the bias with respect to energy is negative and we say that technical change is energy-saving. Similarly, a positive bias implies that technical change is energy-using. Note that while the parameters describing substitution are constant, reflecting responses to varying price changes, the biases of technical change may vary from time to time, since historical patterns involve both energy-using and energy-saving technical change.

The latent variable $f_t^p$ represents the level of technology. The first difference of the level of technology takes the form:

(2.9) $$\Delta f_t^p = F_{p1} + F_{pK} f_{K,t-1}^p + F_{pL} f_{L,t-1}^p + F_{pE} f_{E,t-1}^p + F_{pM} f_{M,t-1}^p + F_{pp} \Delta f_{t-1}^p + v_{pt}$$

A more detailed description of the production sub-model, including the price function, is presented in chapter 4.

### *2.2.3 Lower Tier Production Functions for Intermediate Inputs*

In modeling producer behavior in section 2.1.2 we have introduced multi-stage allocation in order to avoid an intractable specification of the production function (2.1). In the lower tiers of the model energy and non-energy materials inputs are allocated to individual commodities, as summarized in table 2.2. The energy and materials sub-models are represented as:

$$(2.10)\quad E_j = E(QP_3^j, QP_4^j, QP_{16}^j, QP_{30}^j, QP_{31}^j); \qquad M_j = M(QP_1^j, \ldots QP_{NCI}^j)$$

As in the top tier, we work with the price function instead of the production function. To illustrate the elements of the tier structure we consider the translog price function for energy input:

$$(2.11)\quad \ln P_{Et} = \alpha_0 + \sum_{i \in energy} \alpha_i \ln P_{it}^{P,E} + \tfrac{1}{2}\sum_{i,k} \beta_{ik} \ln P_{it}^{P,E} \ln P_{kt}^{P,E} + \sum_{i \in energy} f_{it}^{node=E} \ln P_{it}^{P,E}$$

$$P_i^{P,E} \in \{PS_3, PS_4, PS_{16}, PS_{30}, PS_{31}\}$$

The share equations are obtained by differentiating with respect to the logarithms of the prices. For coal, the first input, the share in the value of energy inputs is:

$$(2.12)\quad v_3 = \frac{PS_3 QP_3}{P_E E} = \alpha_3 + \sum_{k \in energy} \beta_{3k} \ln P_k^{P,E} + f_{3t}^{node=E}$$

The other four energy input demands correspond to crude petroleum, refined petroleum products, electricity, and natural gas. As before, the parameters { $\beta_{ik}$ } are share elasticities and represent the degree of substitutability among the five types of energy commodities. The latent variables { $f_{it}^{p}$ } represent biases of technical change.

The components of the non-energy materials (M) input include 30 commodities in table 2.1and non-comparable imports (NCI) ,commodities not produced domestically and denoted $X_{NCI}$ in figure 2.1. We model the demand for individual commodities within the non-energy materials aggregate for each industry *j* by means of a hierarchical tier structure of translog price functions. The set of nodes for the production sector is denoted as $I_{PNODE}$ and the full set of price functions is given in appendix A.

The price functions for the sub-tiers (2.11) differ from the price function for the top tier (2.5), since the sub-tiers do not include a latent variable representing the level of technology. This reflects the fact that the price of energy for the sub-tier is constructed from the prices of the individual components, while the price of output in the top tier is measured separately from the prices of capital, labor, energy, and non-energy materials inputs. The price of output would fall, relative to the input prices, as productivity rises.

The latent variables of the sub-tier { $f_{it}^{node}$ } represent the biases of technical change. For example, an increase in the latent variable $f_{30t}^{node=E}$ implies that the electricity share of total energy input is increasing, so that technical change is electricity-using, while a decrease in this latent variable implies that technical change is electricity-saving. The latent variables are generated by a vector autoregression, as in (2.6).

The non-energy materials input consists of five sub-aggregates – construction, agriculture materials, metal materials, non-metal materials, and services. Each of these sub-aggregates in turn is a function of sub-tiers until all the 31 non-energy commodities are included. Each node to the tier structure employs a translog price function like equation (2.11) and includes latent variables that represent the biases of technical change.

### *2.2.4 Commodities, Industries and Output Taxes*

Production or sales taxes may be proportional to the value of output or expressed as a tax per quantity unit. We refer to taxes proportional to the value of output as ad valorem taxes. In the base case we represent all taxes on production as ad valorem taxes. In the policy simulations we introduce additional ad valorem taxes in order to represent energy and environmental taxes and tradable permits. These taxes are included for all 35 sectors of the production model and introduce wedges between the prices faced by sellers and buyers.

Denoting the purchaser's price of the output of industry *j* by $PI_j$, we have:

(2.13) $PI_j = (1 + tt_j^{full})PO_j$,

where $PO_j$ is the seller's price given in (2.4). The full superscript on the output tax $tt_j^{full}$ indicates that this is the sum of various output taxes that are described below. The value of industry *j*'s output is denoted:

(2.14) $VT_j^{QI} = PI_j QI_j$.

We have noted above that each industry makes a primary product and many industries make secondary products that are the primary outputs of other industries. We denote the price, quantity, and value of commodity *i* by $PC_i$, $QC_i$ and $V_i^{QC}$ respectively, all from the purchasers' point of view. For column *i* in the make matrix, let the shares contributed by the various industries to that commodity in the base year *T* be denoted:

(2.15) $m_{ji} = \dfrac{M_{ji,t=T}}{V_{i,t=T}^{QC}}$; $\qquad \sum_j m_{ji} = 1$

For row *j* we denote the shares of the output of industry *j* be allocated to the various commodities as follows:

$$(2.16)\quad m_{ji}^{row} = \frac{M_{ji,t=T}}{VT_{j,t=T}^{QI}}; \qquad \sum_i m_{ji}^{row} = 1$$

The shares (2.15) and (2.16) are fixed for all periods after the base year. We assume that the production function for each commodity is a linear logarithmic or Cobb-Douglas aggregate of the outputs from the various industries. The weights are the base-year shares, that is, we write the price of commodity *i* as a linear logarithmic function of the component industry inputs:

$$(2.17)\quad PC_i = PI_1^{m_{1i}} \ldots PI_m^{m_{mi}} \qquad \text{for i=1,2,....35}$$

The values and quantities of commodity *i* are given by:

$$(2.18)\quad V_{it}^{QC} = \sum_j m_{ji}^{row} PI_{jt} QI_{jt} \qquad \text{for i=1,2,....35}$$

$$(2.19)\quad QC_i = \frac{V_i^{QC}}{PC_i}$$

**2.3 Consumer Behavior and Demography**

Policies that affect energy prices have different impacts on different households, so that econometric methods are essential for capturing the heterogeneity observed in micro-economic data. On average, households in warmer regions have larger electricity bills for cooling, elderly persons drive less, and households with children drive more. To capture these differences we subdivide the household sector into demographic groups. We treat each household as a consuming unit, that is, a unit with preferences over commodities and leisure.

$C_t$

Our household model has three stages. In the first stage lifetime full income is allocated between consumption and savings. Full income includes leisure as well as income from the supply of capital and labor services. Consumption consists of commodities and leisure and we refer to this as full consumption. In the second stage full consumption is allocated to leisure and three commodity groups – nondurables, capital services, and services. In the third stage the three commodity groups are allocated to the 36 commodities, including the five types of energy.

### *2.3.1 Notation*

We next describe the three stages of the household model, beginning with the notation:

| | |
|---|---|
| $A_k$ | vector of demographic characteristics of household $k$ |
| $C_i^X$ | quantity of consumption of commodity $i$ |
| $C_{ik}$ | quantity of consumption of commodity $i$ by household $k$ |
| $F_t$ | quantity of full consumption |
| $R_t$ | quantity of aggregate leisure |
| $R_k^m$ | quantity of leisure |
| $LS_t$ | quantity of aggregate labor supply |
| $m_k$ | value of full expenditures of household $k$ |
| $KS_t$ | quantity of aggregate capital stock at end of period $t$ |
| $n_t$ | growth rate of population |
| $PF_t$ | price of $F_t$ |

| | |
|---|---|
| $P_t^L$ | price of labor to employer, economy average |
| $P_t^K$ | rental price of capital, economy average |
| $r_t$ | rate of return between *t-1* and *t* |
| $Y_t$ | household disposable income |
| $S_t$ | household savings |

### *2.3.2. First Stage: Intertemporal Optimization*

Let $V_{kt}$ denote the utility of household *k* derived from consuming goods and leisure during period *t*. In the first stage household *k* maximizes an additively separable intertemporal utility function:

$$(2.20) \quad \max_{F_{kt}} U_k = E_t\{\sum_{t=1}^{T}(1+\rho)^{-(t-1)}[\frac{V_{kt}^{(1-\sigma)}}{(1-\sigma)}]\}$$

subject to the lifetime budget constraint:

$$(2.21) \quad \sum_{t=1}^{T}(1+r_t)^{-(t-1)} PF_{kt}F_{kt} \le W_k$$

Where $F_{kt}$ is the full consumption in period *t*, $PF_{kt}$ is its price, $r_t$ is the nominal interest rate, and $W_k$ is the "full wealth" at time 0. The parameter σ represents intertemporal curvature and $\rho$ is the subjective rate of time preference. The within-period utility function is logarithmic if σ is equal to one:

$$(2.22) \quad \max_{F_{kt}} U_k = E_t\{\sum_{t=1}^{T}(1+\rho)^{-(t-1)} \ln V_{kt}\}$$

The term full wealth refers to the present value of future earnings from the supply of tangible assets and labor, plus transfers from the government, and imputations for the value of leisure. Tangible assets include domestic capital, government bonds and net foreign assets. Equations (2.20) and (2.21) are standard in economic growth models.

The first-order condition for optimality is expressed in the Euler equation:

$$(2.23)\quad \Delta \ln PF_{k,t+1}F_{k,t+1} = (1-\sigma)\Delta \ln V_{k,t+1} + \Delta \ln(-D(p_{k,t+1})) + \ln(1+r_{t+1}) - \ln(1+\rho) + \eta_{kt}$$

where $D(p_{kt})$ is a function of the prices of goods and leisure and $\eta_{kt}$ is an expectational error. In chapter 3 we describe in more detail how the Euler equation is estimated from data for synthetic cohorts obtained by adding over all households in each cohort. From these household Euler equations we derive an aggregate Euler equation. This Euler equation is forward-looking, so that the current level of full consumption incorporates expectations about all future prices and discount rates.

In the simulations reported in chapter 8 we use a simplified version of the aggregate Euler equation with the curvature parameter σ equal to one. The aggregate version of the intertemporal optimization problem is written as:

$$(2.24)\quad U = E_t \sum_{t=1}^{\infty} N_0 \prod_{s=1}^{t} \left( \frac{N_t^{eq}}{1+\rho} \right) \ln F_t$$

The aggregate Euler equation derived from this is simply:

$$(2.25)\quad \frac{F_t}{F_{t-1}} = \frac{(1+n_t)(1+r_t)}{1+\rho} \frac{PF_{t-1}}{PF_t} + \eta_t$$

Where $n_t$ is the rate of growth of population. This is discussed in more detail in chapter 3, section 3.3.1.

### *2.3.3Second Stage: Goods and Leisure*

In the second stage of the household model, full consumption is divided between the value of leisure time and personal consumption expenditures on commodities. Given the time endowment of the household sector, the choice of leisure time also determines the supply of labor. The allocation of full consumption employs a very detailed household demand model that incorporates demographic characteristics of the population. The data base for this model includes the Consumer Expenditure Survey (CEX) and Personal Consumption Expenditures (PCE) from the US National Income and Product Accounts (NIPAs).

Conceptually, we determine the consumption $C_{ik}^X$ of commodity *i* for household *k* by maximizing a utility function $U(C_{1k}^X, ..C_{ik}^X .., C_{Rk}^X; A_k)$, where $C_{Rk}^X$ is leisure and $A_k$ denotes the demographic characteristics of household *k*, such as the number of children and the number of adults. Summation over all households gives the total demand for commodity *i*:

(2.26) $$PC_{it}^X C_{it}^X = \sum_k P_{ikt}^{CX} C_{ikt}^X \qquad \text{i=1, 2,…, R}$$

The price $P_{ik}^{CX}$ is the price of good *i* faced by household *k*. The superscript *X* denotes that this is a CEX measure that must be distinguished from measures based on the NIPAs and Inter-industry Transactions Accounts discussed below. Similarly, total leisure demand is the sum over all households' leisure demands ($\sum_k P_{Rkt}^{CX} C_{Rkt}^X$); and the sum of goods and leisure gives the full consumption determined in stage 1:

(2.27) $$PF_t F_t = \sum_i PC_{it}^X C_{it}^X + PC_R C_R^X$$

The list of commodities included in the household model is presented in table 2.3 along with the values in 2005. These are defined in terms of categories of Personal Consumption Expenditures (PCE) and the last column of the table gives the precise definitions in the NIPAs. One major difference between our classification system and the PCE is the treatment of consumers' durables. Purchases of new housing are included in investment in the NIPAs, while only the annual rental value of housing is included in the PCE.

**[Table 2.3 about here]**

Purchases of consumer durables such as automobiles are treated as consumption expenditures in the PCE. In the new architecture for the US national accounts discussed by Jorgenson (2009), these purchases are included in investment, while rental values are treated as consumption. This has the advantage of achieving symmetry in the treatment of housing and consumers' durables. The annual flow of capital services from these household assets is given as item 35 in table 2.3.

The dimensionality of a utility function written as $U(C_{1k}^{X},...,C_{Rk}^{X};A_k)$ is intractable. Accordingly, we impose a tier structure much like the production model of section 2.1. At the top tier, utility function depends on nondurables, capital services, services, and leisure:

$$U = U(C_{ND,k}, C_{K,k}, C_{SV,k}, C_{R,k}; A_k) \tag{2.28}$$

$$C_{ND} = C(C_1, C_2, ...C_{16}); \qquad C_{SV} = C(C_{17}, ...C_{NCI})$$

Consumer nondurables ($C_{ND}$) and services ($C_{SV}$) are further allocated to the 36 commodities in the third stage of the household model. For the remainder of this subsection we focus on the top tier. We first describe how the parameters are estimated from CEX data. We then indicate how the model for individual households is aggregated to obtain the model of the household sector in IGEM. The full details of the household model are given in chapter 3.

In order to characterize substitutability among leisure and the commodity groups, we find it convenient to derive household *k*'s demands from a translog indirect utility function $V(p_k, m_k; A_k)$, following Jorgenson and Slesnick (2008):

$$(2.29) \quad -\ln V_k = \alpha_0 + \alpha^H \ln \frac{p_k}{m_k} + \tfrac{1}{2} \ln \frac{p_k}{m_k}' B^H \ln \frac{p_k}{m_k} + \ln \frac{p_k}{m_k}' B_A A_k$$

where $p_k$ is a vector of prices faced by household *k*, $m_k$ is full expenditure of household *k*, $\alpha^H$ is a vector of parameters, $B^H$ and $B_A$ are matrices of parameters that describe price, total expenditure, and demographic effects, and $A_k$ is a vector of variables that describe the demographic characteristics of household *k*.[2]

The value of full expenditure on leisure and the three commodity groups is:

$$(2.30) \quad m_k = P^C_{ND} C_{NDk} + P^C_K C_{Kk} + P^C_{SV} C_{SVk} + P^C_R C_{Rk}$$

In (2.29) the demands are allowed to be non-homothetic, so that full expenditure elasticities are not constrained to unity. The commodity groups in (2.30)represent consumption of these commodities by household *k*.

The leisure time consumed by household *k* takes into account the opportunity costs of time of the individual members of the household. These opportunity costs are reflected in the after-tax wage $p^m_R$ for each worker. We assume that the effective quantity

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[2] The aggregation properties of this indirect utility function are discussed in Jorgenson and Slesnick (2008).

of leisure of person $m$ ( $R_k^m$ ) is non-work hours multiplied by the after-tax wage, relative to the base wage: $q_k^m = p_R^m / p_R^0$ .

We assume a time endowment of 14 hours per day for each adult. The annual leisure of person is the time endowment less hours worked:

$$(2.31) \quad R_k^m = q_k^m (\bar{H}_k^m - LS_k^m) = q_k^m (14 * 365 - \text{hours worked}_k^m) .$$

The quantity of leisure for household $k$ is the sum over all adult members:

$$(2.32) \quad C_{Rk} = \sum_m R_k^m$$

and the value of household leisure is:

$$(2.33) \quad P_R^C C_{Rk} = p_R^0 \sum_m R_k^m = \sum_m p_R^m (\bar{H}_k^m - LS_k^m)$$

The demand functions for commodities and leisure are derived from the indirect utility function (2.29) by applying Roy's Identity:

$$(2.34) \quad \mathbf{w}_k = \frac{1}{D(p_k)} (\alpha^H + B^H \ln p_k - \iota' B^H \ln m_k + B_A A_k)$$

where $\mathbf{w}_k$ is the vector of shares of full consumption, $\iota$ is a vector of ones, and $D(p_k) = -1 + \iota' B^H \ln p_k$ . For example, the demand for consumer nondurables is:

$$(2.35) \quad w_{ND,k} = \frac{1}{D(p_k)} (\alpha_{ND}^H + B_{ND\bullet}^H \ln p_k - \iota B^H \ln m_k + B_{A,ND\bullet} A_k)$$

where $B_{ND\bullet}^H$ denotes the top row of the $B^H$ matrix of share elasticities.

We require that the indirect utility function obey the restrictions:

$$(2.36) \quad B^H = B^{H\prime} ; \qquad \iota' B^H \iota = 0 , \qquad \iota' B_A = 0 , \qquad \iota' \alpha^H = -1 ,$$

where $B^H$ is the matrix of share elasticities representing the price effects and must be symmetric, while $\iota' B^H$ represents the full expenditure effect and must sum to zero. The matrix $B_A$ determines how the expenditure shares differ among demographic groupsand must sum to zero.

The restrictions in (2.36) are implied by the theory of individual consumer behavior and the requirement that individual demand functions can be aggregated exactly to obtain the aggregate demand functions used in the model. The restrictions are discussed in greater detail in chapter 3.The estimation of the parameters describing consumer demand from household survey data is described in chapter 3 as well.

The demographic characteristics employed in the model include the number of children, the four Census regions, and race, sex and three age groups for the head of household. Since it is infeasible to include demand functions for individual households in IGEM, we derive an aggregate version of the household demand functions (2.34). Let $n_k$ be the number of households of type $k$. Then the vector of demand shares for the US economy, $w = (\frac{P_{ND}^{CX} C_{ND}^X}{MF^X}, \frac{P_K^{CX} C_K^X}{MF^X}, \frac{P_{SV}^{CX} C_{SV}^X}{MF^X}, \frac{P_R^{CX} C_R^X}{MF^X})'$, is obtained by aggregating over all types of households:

$$(2.37) \quad w = \frac{\sum_k n_k m_k \mathbf{w}_k}{\sum_k n_k m_k}$$

$$= \frac{1}{D(p)} \left[ \alpha^H + B^H \ln p - \mathbf{\iota} B^H \xi^d + B_A \xi^L \right]$$

where the distribution terms are:

$$(2.38) \qquad \xi^d = \sum_k n_k m_k \ln m_k / M \; ; \qquad M = \sum_k n_k m_k$$

$$(2.39) \qquad \xi^L = \sum_k n_k m_k A_k / M$$

For example, the nondurables component of the aggregate share vector is:

$$(2.40)\quad w_{ND} = \frac{P_{ND}^{CX} C_{ND}^{X}}{MF^{X}}$$

where $MF^X$ denote the national value of full consumption expenditures in CEX units:

$$(2.41)\quad MF^{X} = \sum_k n_k m_k = P_{ND}^{CX} C_{ND}^{X} + P_{K}^{CX} C_{K}^{X} + P_{SV}^{CX} C_{SV}^{X} + P_{R}^{CX} C_{R}^{X}$$

By constructing the model of consumer demand through exact aggregation over individual demands, we incorporate the restrictions implied by the theory of individual consumer behavior. In addition, we incorporate demographic information through the distribution terms (2.38) and (2.39). For the sample period we observe the actual values of these terms. For the period beyond the sample we project the distribution terms, using projections of the population by sex and race.

### *2.3.4 Linking Survey Data to the National Accounts*

The top tier of the household model is estimated from data in the CEX. This is the only source of microeconomic data that includes the information necessary to characterize household demands for commodities and leisure in a consistent way. The CEX also includes the information on demographic characteristics of individual households that is required to capture the enormous heterogeneity of US households.

Personal Consumption Expenditures (PCE) in the NIPAs includes many items omitted from the CEX.Since we base IGEMon the NIPAs, we must reconcile the CEX-based estimates and the NIPAs-based estimates. We denote the quantity of consumption of item $i$ in the NIPAs by $N_i$, and the price by $PN_i$, $i$=1, 2,…35, R, as listed in table 2.3. Corresponding to the CEX-based commodity groups for nondurables, $C_{ND}$, and services,

$C_{SV}$, we have the quantities from the NIPAs, $N^{ND}$ and $N^{SV}$, and their prices,$PN^{ND}$ and $PN^{SV}$. The price of leisure $PN^{R}$ is derived by aggregation over the population.

The share equations (2.37) allocate full consumption among the shares of commodities and leisure ($w_{ND}$, $w_K$, $w_{SV}$, and $w_R$). We denote the consumption shares based on the CEX as $SC_i^X = w_i$. We need to reconcile these to the shares based on NIPAs:

$$(2.42) \quad SC^N \equiv \left( \frac{PN^{ND}N^{ND}}{MF^N}, \frac{PN^K N^K}{MF^N}, \frac{PN^{CS}N^{CS}}{MF^N}, \frac{PN^R N^R}{MF^N} \right)'$$

where $MF^N$ is full consumption.

Our reconciliation of the CEX and NIPAs data on consumption is accomplished by expressing the differences between the two sets of shares as an autoregressive process:

$$(2.43) \quad \Delta SC_{it} = SC_{it}^N - SC_{it}^X \qquad i=\{\text{ND,K,CS,R}\}$$

$$(2.44) \quad \Delta SC_{it} = \alpha + \beta \Delta SC_{it-1} + \varepsilon_{it} \qquad \varepsilon_{it} = \rho \varepsilon_{it-1} + u_{it}$$

We estimate (2.44) from sample period data and then project the differences forward. This provides an exogenous projection of the difference between the two sets of shares.

The value of full consumption in CEX units can be rewritten as the sum of the value of leisure and total expenditures on commodities:

$$(2.45) \quad \begin{aligned} MF^X &= P_{ND}^{CX} C_{ND}^X + P_K^{CX} C_K^X + P_{SV}^{CX} C_{SV}^X + P_R^{CX} C_R^X \\ &= P^{CC,X} CC^X + P_R^{CX} C_R^X \end{aligned}$$

After rescaling to NIPA units the value of full consumption is:

$$(2.46) \quad \begin{aligned} PF_t F_t &= P_t^{CC} CC_t + PN^R N^R \\ &= \sum_i PN_{it} N_{it} + PN^R N^R \end{aligned}$$

where $P_t^{CC} CC_t$ denotes the value of aggregate tangible consumption. This is the value that is matched to the Euler equation(2.23).

*__2.3.5 Third Stage: Demands for Commodities__*

In the third and final stage of the household model we allocate the quantities of nondurables, capital services, and other services ($N^{ND}$, $N^{K}$ and $N^{CS}$) to the 35 commodities, non-comparable imports and capital services. We do not employ demographic information for this allocation, but utilize a hierarchical model like the one employed for production in section 2.2. We impose homotheticity on each of the sub-models.

There is a total of 34 commodity groups, one type of capital services, and one type of leisure, as listed in table 2.3. These are arranged in 17 nodes, as shown in table 2.4. This set of nodes is denoted $I_{CNODE}$. At each node *m* we represent the demand by a translog indirect utility function, $V^m(P^{Hm}, m_m; t)$:

$$(2.47) \quad -\ln V^m = \alpha_0 + \alpha^{Hm} \ln \frac{P^{Hm}}{m_m} + \tfrac{1}{2} \ln \frac{P^{Hm}}{m_m}' B^{Hm} \ln \frac{P^{Hm}}{m_m} + f^{Hm} \ln \frac{P^{Hm}}{m_m} \qquad m \in I_{CNODE}$$

$$\ln P^{Hm} \equiv (\ln PN_{m1}, ..., \ln PN_{mi}, ..., \ln PN_{m,im})' \qquad i \in I_{\text{CNODE}m}$$

The value of expenditures at node *m* is:

$$(2.48) \quad m_m = PN_{m1} N_{m1} + ... + PN_{m,im} N_{m,im}$$

[Table 2.4 about here]

The shares of full consumption derived from (2.47) exclude demographic variables and include latent variables representing changes in preferences $f_t^{Hm}$. When we impose homotheticity, by requiring that $\iota' B^{Hm} = 0$, the share demands simplify to an expression that is independent of the level of expenditures ($m_m$):

$$(2.49)\quad SN^m = \begin{bmatrix} PN_{m1}N_{m1} / PN^m N^m \\ \cdots \\ PN_{m,im}N_{m,im} / PN^m N^m \end{bmatrix} = \alpha^{Hm} + B^{Hm} \ln PN^{Hm} \ + f^{Hm}$$

Note that with homotheticity, the indirect utility function reduces to:

$$(2.50)\quad -\ln V^m = \alpha^{Hm} \ln P^{Hm} + \tfrac{1}{2} \ln P^{Hm}{}' B^{Hm} \ln P^{Hm} + f^{Hm} \ln P^{Hm} - \ln m_m$$

The first three terms in (2.50) are analogous to the price function in the production model. We can define the price of the $m^{\text{th}}$ commodity group as:

$$(2.51)\quad \ln PN^m = \alpha^{Hm} \ln P^{Hm} + \tfrac{1}{2} \ln P^{Hm}{}' B^{Hm} \ln P^{Hm} + f^{Hm} \ln P^{Hm}$$

Next, we express the value of expenditures as the price (2.51), multiplied by the corresponding quantity:

$$(2.52)\quad m_m = PN^m N^m$$

Substituting (2.51) and (2.52) into (2.50) we see that the utility index is the quantity of the $m^{\text{th}}$ commodity group, $V^m = N^m$.

As an example, in the $m$=5 node the energy goods aggregate is a function of $N_3$ (gasoline), and $N_4$ (household fuel), so that $N^{EN} = N^{EN}(N_6, N_4)$. The demand shares are functions of the prices of these two components and the state variables representing the biases:

$$(2.53)\quad SN^{m=5} = \begin{bmatrix} PN_3 N_3 / PN^{m=5} N^{m=5} \\ PN_4 N_4 / PN^{m=5} N^{m=5} \end{bmatrix} = \alpha^{H5} + B^{H5} \ln PN^{H5} \ + f^{H5}$$

The value of energy purchases that appears in the next higher node for nondurables ($m=2$) is:

$$(2.54)\quad PN^{EN} N^{EN} = PN_3 N_3 + PN_4 N_4$$

### *2.3.6 Linking the National Accountsto the Interindustry Transactions Accounts*

The categories of PCE from the NIPAs are given in table 2.3. The expenditures are in purchasers' prices, which include the trade and transportation margins. These prices must be linked to the supply side of the model, where expenditures are in producers' prices. In the official input-output tables this link is provided by a bridge table[3], for example, the PCE expenditures of $20.2 billion in 2002 for "other video equipment" is comprised of the following commodity groups from the input-output tables: $12.1 billion from IT equipment, $0.45 billion from Electrical machinery, $0.15 billion from transportation and $7.5 billion from Wholesale and Retail trade.

We denote the bridge matrix by **H**, where $H_{ij}$ is the share of input-output commodity $i$ in PCE item $j$. The value of total demand by households for commodity $i$ is:

$$VC_i = \sum_j H_{ij} PN_j N_j \tag{2.55}$$

The prices from the NIPAs are also linked to input-output prices through this bridge matrix.

In section 2.1 we denoted the supply price of input $i$ to the producing industries by $PS_i$, where we assumed that all industries pay the same price. We allow the sectors represented in the final demands to pay a different price $PS_i^C$; in particular, households may be charged a consumption tax. The price of PCE item $j$ is thus expressed in terms of the commodity prices, and the transpose of the bridge matrix:

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[3] For the 1992 Benchmark in the *Survey of Current Business*, November 1997, this is given in Table D, Input-Output Commodity Composition of NIPA Personal Consumption Expenditure Categories.

(2.56) $PN_j = \sum_i H_{ij}^T PS_i^C$

The quantity of commodity *i* consumed by the household sector is:

(2.57) $C_i = VC_i / PS_i^C \qquad i \in I_{COM}$

The value of total personal consumption expenditures is the sum over all commodities in either definition:

(2.58) $PCC_t CC_t = \sum_i VC_i = \sum_i PN_i N_i$

We emphasize again that the consumption expenditures in IGEM exclude the purchases of new consumer durables but include the service flow from the stock of durables. Purchases of new durables are treated as investment in order to preserve symmetry between housing and consumers' durables.

### *2.3.7 Accounting for Leisure, the Time Endowment, and Full Income*

The demand for leisure in CEX units ($C_R^X$) is given by the fourth element of the share vector in (2.35). The aggregate demand for leisure in NIPA units ($N^R$) is obtained from $C_R^X$ by applying the exogenous CEX-NIPA difference from (2.43). Individual leisure is related to hours supplied to the labor market.

We define the aggregate time endowment $LH_t$ as an index number of the population, where individuals are distinguished by gender, age, and educational attainment. Let $POP_{kt}$ denote the number of people in group *k* at time *t*, and the price of time is the after-tax hourly wage of person *k*, $(1-tl_t^m)P_{kt}^L$. The value of the aggregate time endowment with 14 hours per day to each person is:

(2.59) $P_t^h LH_t = VLH_t = \sum_k (1 - tl_t^m) P_{kt}^L * 14 * 365 * POP_{kt}$

We express the value of time endowment as the product of the quantity *LH* and the price $P^h$. The Tornqvist index for the quantity of the time endowment is:

(2.60) $d \ln LH_t = \sum_k \tfrac{1}{2}(v_{kt}^L + v_{kt-1}^L) d \ln(14 * 365 * POP_{kt})$

$$v_{kt}^L = \frac{(1 - tl_t^m) P_{kt}^L * 14 * 365 * POP_{kt}}{VLH_t} \qquad \text{k=\{gender, age, education\}}$$

The price of aggregate time endowment is the value divided by this quantity index:

(2.61) $P_t^h = \dfrac{VLH_t}{LH_t}; \qquad P_{baseyear}^h \equiv 1$

In a similar manner, we define the quantity of aggregate leisure by aggregating over all population groups, where the annual hours of leisure for a person in group *k* is denoted by $H_{kt}^R$:

(2.62) $d \ln N_t^R = \sum_k \tfrac{1}{2}(v_{kt}^R + v_{kt-1}^R) d \ln(H_{kt}^R * POP_{kt})$

$$v_{kt}^R = \frac{(1 - tl_t^m) P_{kt}^L * H_{kt}^R * POP_{kt}}{VR_t} \qquad k\text{=\{gender, age, education\}}$$

$$VR_t = \sum_k (1 - tl_t^m) P_{kt}^L H_{kt}^R POP_{kt}$$

The leisure hours of group *k* are derived from the accounts for hours worked used in estimating the quantity of industry labor input in section 2.1 above. The details of these labor accounts are given in appendix B.2. Individual leisure is equal to the annual time endowment, less the average hours worked for individuals in group *k:*[4]

[4] The average hours worked for people of type *k* are derived from the data that correspond to the $h_{scaej}$ variable in appendix B, equation (B.29).

(2.63) $H_{kt}^{R} = 14*365 - h_{kt}$ .

The value of aggregate leisure is:

(2.64) $VR_{t} = \sum_{k}(1 - tl_{t}^{m})P_{kt}^{L} * H_{kt}^{R} * POP_{kt}$ .

The price of aggregate leisure is this value divided by the quantity index:

(2.65) $PN_{t}^{R} = \frac{VR_{t}}{N_{t}^{R}}$ ; $PN_{baseyear}^{R} \equiv 1$.

The price of aggregate time endowment ( $P_{t}^{h}$ ) is not the same as the price of aggregate leisure even though they are the same at the level of the individual in group *k*. This is due to the differences in weights. We relate the two with a price aggregation coefficient:

(2.66) $PN_{t}^{R} = \psi_{Ct}^{R} P_{t}^{h}$

Taking the aggregation coefficient (2.66) into account, we define aggregate labor supply as time endowment, less adjusted leisure:

(2.67) $LS = LH - \psi_{C}^{R} N^{R}$

This implies that the price of labor supply is identical to the price of time endowment and the values are related as:

(2.68) $P^{h}LH = P^{h}LS + PN^{R}N^{R}$

The value of labor supply is the gross payments by employers less the marginal tax on labor income:

(2.69) $P^{h}LS = (1 - tl^{m})\sum_{j} PLD_{j}LD_{j}$

Labor income is the main source of household income. We can now describe the household financial accounts. In the lifetime budget constraint, $W_{0}^{F}$ represents the present value of the stream of household full income, that is, tangible income plus the imputed value of leisure. Household tangible income, $Y_t$, is the sum of after-tax capital income ($YK^{net}$), labor income ($YL$), and transfers from the government ($G^{TRAN}$):

(2.70) $Y_t = YK_t^{net} + YL_t + G_t^{TRAN} - TLUMP_t - twW_{t-1}$

The term $twW_{t-1}$ represents taxes on wealth, and $TLUMP_t$ represents lump-sum taxes that are zero in the base case but may be used in the policy cases[5].

Labor income after taxes is:

(2.71) $YL = P^h LS \frac{1\text{-}tl^a}{1\text{-}tl^m} = (1 - tl^a)\sum_j PLD_j LD_j$

We distinguish between marginal tax rates and average tax rates. The price of the time endowment and leisure refers to the marginal price – the wage rate reduced by the marginal tax rate–while income is defined in terms of the wage rate less average taxes.

Capital income is the sum of dividend income from the private stock of physical assets and financial assets in the form of claims on the government and rest-of-the-world:

(2.72) $YK_t^{net} = DIV - YK^{gov} + (1 - tk)\left(GINT_t + Y_t^{row}\right)$

The components of capital income are explained in more detail in section 2.4 on capital accounts, section 2.5 on government accounts, and section 2.6 on foreign accounts.

Full income includes the value of the time endowment and is equal to household tangible income $Y_t$ plus the value of leisure:

(2.73) $YF_t = YK_t^{net} + YL_t + PN_t^N N_t^R + G_t^{TRAN} - TLUMP_t - twW_{t-1}$

Private household savings is income less consumption, non-tax payments to the government ( $R_t^N$ ), and transfers to rest-of-the-world ( $H^{row}$ ) :

[5]Taxes on wealth are estate taxes and are described more fully in section 2.4.1.

(2.74) $S_t = YF_t - PF_t F_t - H_t^{row} - R_t^N + tcVCC^{exempt}$

$$= Y_t - P_t^{CC} CC_t - H_t^{row} - R_t^N + tcVCC^{exempt}$$

## 2.4 Investment and Cost of Capital

The primary factors of production are capital and labor services. Capital in IGEM includes structures, producer's durable equipment, land, inventories, and consumers' durables. Capital stocks are rented to producers and the annual rental value of capital services is included in the capital income of households. We focus on capital owned by the private sector in this section; government-owned capital is not part of the capital market and will be discussed in chapter 5.

We assume that the supply of capital is determined by past investments. We also assume that there are no installation or adjustment costs in converting new investment goods into capital stocks. With these simplifying assumptions the savings decision by households is identical to the investment decision. We analyze the savings-investment decision in more detail in order to clarify the role of the cost of capital, a key equation of IGEM that links investment, capital stock, and capital services.

The owner of the stock of capital chooses the time path of investment by maximizing the present value of the stream of after-tax capital income, subject to a capital accumulation constraint:

(2.75) Max $\sum_{t=u}^{\infty} \frac{(1\text{-}tk)(PKD_t \psi^K K_{t\text{-}1}\text{-}tpPK_{t\text{-}1})\text{-}(1\text{-}t^{ITC})PII_t I_t^a}{\prod_{s=u}^{t} 1+r_s}$

(2.76) s.t. $K_t = (1-\delta)K_{t-1} + \psi^I I_t^a$

After-tax capital income $(1\text{-}tk)(PKD_t\psi^K K_{t\text{-}1}\text{-}tpPK_{t\text{-}1})$ is related to the capital income of households $YK^{net}$ in (2.72). The discount rate $r_s$ is the same as in the Euler equation for the household sector and the rental price of capital services is $PKD_t$. The stock of capital available at the end of the period is $K_t$.. The remaining terms are the property tax rate *tp*, the capital income tax rate *tk*, and the price of the capital assets *PK*. We introduce the aggregation coefficient $\psi^K$ to convert the capital stock measure to the flow of capital services.[6] Finally, $I_t^a$ is the quantity of aggregate investment and $(1-t^{ITC})PII_t$ is the price of investment, net of the investment tax credit. To simplify this model we have ignored details such as depreciation allowances and the distinction between debt and equity.

Aggregate investment includes commodities ranging from computers to structures. Capital stock is also an aggregate of these commodities, but with different weights. At the level of the individual commodity, the capital accumulation equation is $K_{kjt} = (1-\delta_k)K_{kj,t-1} + I_{kjt}$. Aggregation over all the commodities results in the aggregate capital accumulation equation (2.76). The $\psi_t^I$ aggregation coefficient converts the investment to capital stock and the aggregate depreciation rate is denoted $\delta$. The issues involved in aggregation of capital services, capital stocks, and investment flows are discussed in more detail in appendix B.

The maximization of the present value of capital income after taxes results in the Euler equation[7]:

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[6] These concepts are explained in appendix B.3.1.5 describing the construction of historical data for aggregate investment and capital.
[7] The Hamiltonian for the maximization problem is given in appendix A, equation A.3.3.

(2.77) $$(1+r_t)\frac{(1-t^{ITC})PII_{t-1}}{\psi_{t-1}^I} = (1-tk)(PKD_t\psi_t^K - tpPK_{t-1}) + (1-\delta)\frac{(1-t^{ITC})PII_t}{\psi_t^I}$$

This equation describes the consequences of arbitrage between the market for investment goods and the market for capital services. If we were to put $(1-t^{ITC})PII_{t-1}$ dollars in a bank in period *t-1* we would earn a gross return of $(1+r_t)(1-t^{ITC})PII_{t-1}$ at time *t*. On the other hand, if we used those dollars to buy one unit of investment goods ($=\psi^I$ units of capital) we would collect a rental for one period, pay taxes, and the depreciated capital would be worth $(1-\delta)(1-t^{ITC})PII_t$ in period *t* prices. The Euler equation requires that these two returns are the same.

Our simplifying assumption of no installation costs implies that new investment goods are perfectly substitutable for existing capital in the capital accumulation equation. This implies that the price of capital stock is linked to the price of aggregate investment:

(2.78) $$PK_t = \psi_t^{PK} PII_t(1-t^{ITC})$$

The aggregation coefficient $\psi_t^{PK}$ is used to transform asset prices to investment goods prices and plays a symmetrical role to $\psi_t^I$ in the accumulation equation. The aggregation coefficients are taken to be exogenous in our policy simulations.

In equilibrium the price of one unit of capital stock (*PK*) is the present value of the discounted stream of rental payments (*PKD*). Capital rental prices, asset prices, prices of capital stock, rates of return, and interest rates for each period are related by the Euler equation (2.77). This incorporates the forward-looking dynamics of asset pricing into our model of intertemporal equilibrium. The asset accumulation equation (2.76) imparts backward-looking dynamics.

Combining the link between the prices of capital assets and investment goods (2.78) and the Euler equation (2.77), we obtain the well-known cost of capital equation:[8]

$$(2.79) \quad PKD_t = \frac{1}{(1-tk)}[(r_t - \pi_t) + \delta(1+\pi_t) + tp]PK_{t-1}$$

where $\pi_t = (PK_t - PK_{t-1}) / PK_{t-1}$ is the inflation rate for the asset price. The rental price of capital *PKD* equates the demands for capital by the 35 industries and households with the aggregate supply given by capital stock at the beginning of the period $K_{t-1}$.

The rental payment by industry *j* for capital services is $PKD_j\, KD_j$; the sum over all industries is the gross capital income in the objective function (2.75):

$$(2.80) \quad PKD_t \psi_t^K K_{t-1} = \sum_j PKD_{jt} KD_{jt}$$

We denote the after-tax payments to capital by *DIV*. This notation evokes the notion of dividends, but the payments also include retained earnings. The payment to households is gross capital income less the capital income tax, the property tax, and taxes on owner-occupied housing $RK^{hh}$:

$$(2.81) \quad DIV = (1-tk)[\sum_j PKD_j KD_j - RK^{hh} - tpPK_t K_t]$$

These after-tax payments are a major component of household capital income $YK^{net}$ in (2.72) above.

## 2.5 Intertemporal Equilibrium

### *2.5.1 Market Clearing Conditions*

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[8] See Jorgenson (1963).

We have now described the principal sources of supply and demand in the model. The other source of supply of commodities is imports and that is described in chapter 5. Government, investment and export demand at the commodity level is also given in chapter 5. For the factors, the supply of labor is given in the household model above while the supply of capital is given in (2.80). We next characterize intertemporal equilibrium where demand and supply are balanced by an intertemporal price system. This is essential for an understanding of the algorithm for solving the model presented in the following section.

The Cass-Koopmans neo-classical model of economic growth has a saddle-path property: Given the initial value of the state variable there is a unique value of the costate variable for which the model will converge to a steady state that satisfies a transversality condition. In this model there is only one state variable, the capital stock. The path of this stock is governed by the Euler equation derived by maximizing the household's objective function. We treat full consumption as the costate variable. Given the initial stock of capital, there is only one initial value of full consumption on the saddle path.

Additional state variables in the system include the government debt, claims on foreigners, and the latent variables in the behavioral equations. However, these state variables are not governed by optimizing behavior. They are set exogenously and do not have associated costate variables.[9] We first describe the equilibrium within each period, given the inherited capital stock and a guess of full consumption for that period. We then

[9] A model with portfolio choice allocating savings between capital and government debt is presented by McKibbin and Wilcoxen (1998, 2012) . This requires an additional costate variable for the price of the debt.

describe the intertemporal equilibrium where the Euler equation links full consumption across time periods.

With constant returns to scale and factor mobility the equilibrium prices clear all markets for each period. In the commodity markets, the demand side of the economy consists of intermediate demands by producers, household consumption, investor demand, government demand and exports. The supply comes from domestic producers and imports as explained in chapter 5, equation 5.31. In the equilibrium for each period the industry output prices $PO_j$ equate supply and demand, so that for each commodity *i*:

(2.82) $$PQ_i\, QS_i = \sum_j PS_i\, QP_{ij} + PS_i (C_i + I_i + G_i) + PC_i X_i$$

In capital market equilibrium, the demand for capital input from all industries and households is equal to the supply from the stock of inherited capital. In section 2.4 above we have been careful to stress the distinction between the stock and flow of capital. The stock measures are related to the flow of services for each of the asset types and these stock and flow measures are independently aggregated. Capital income is equal to the capital service price multiplied by the service flow, which in turn is given by the capital stock multiplied by the aggregation coefficient $\psi_t^K$ . The equilibrium condition in value terms is:

(2.83) $$PKD_t\, \psi_t^K K_{t-1} = \sum_j PKD_{jt} KD_{jt}$$

Since we assume that capital is mobile across sectors, only one capital rental price is needed to clear this market. However, we observe different rates of return in our historical database for the period 1960-2005 presented in appendix B. To reconcile the actual movement in historical prices with our simplifying assumption of capital mobility, we treat the industry rental price as a constant times the economy-wide rental price:

$$PKD_{jt} = \psi_{jt}^{K} PKD_{t} \tag{2.84}$$

In the sample period we calculate the aggregation coefficients $\psi_{jt}^{K}$ from the actual data on industry costs of capital and the aggregate cost of capital. For the projection period we set these coefficients equal to the last sample observation, so that the ratios of marginal products of capital across industries are constant in the projection period. With these industry-specific adjustments, the economy-wide price $PKD_t$ equates supply and demand for capital services:

$$\sum_{j=1}^{C} \psi_{jt}^{K} KD_{jt} = KD_{t} = \psi_{t}^{K} K_{t-1} \tag{2.85}$$

An exception to the above treatment is the Petroleum and Gas Mining industry, industry number 4. The capital stock measure $K_t$ includes land. Given the non-reproducible nature of this resource-based industry we have allowed for two possible closures of the market for $KD_{4t}$ : one is to treat it symmetrically with all other industries; and two is to assume that the stock of capital in this sector is fixed (no investment and no depreciation). In the second option, we have an endogenous rental price of this fixed stock of capital $PKD_{t}^{oil}$ such that the demand for capital input is equal to the fixed supply:

$$KD_{j=4,t} = \overline{KD}^{oil} \tag{2.86}$$

Turning to the labor market, the supply comes from the household demand for leisure and the time endowment. The demand is the sum over the demands from the 35 industries and government. The equilibrium condition in value terms is:

(2.87) $$P_t^h LS_t = P_t^h \left( LH_t - \psi_{Ct}^R N_t^R \right) = (1 - tl_t^m) \sum_j PLD_{jt} LD_{jt}$$

The aggregation coefficient $\psi_C^R$ links the time endowment to aggregate leisure.

Similarly, time trends in the industry labor prices are also different among industries. To reconcile the historical movements of these prices with the simplifying assumption of labor mobility, we first set the economy-wide wage rate equal to the price of the time endowment, adjusted for the marginal labor tax. We then use fixed constants to scale the industry wage rates to the economy-wide wage rate:

(2.88) $$PLD_j = \psi_j^L \frac{P^h}{\left(1 - tl^m\right)}$$

The price of aggregate time endowment $P^h$ clears the market for labor:

(2.89) $$LS_t = LH_t - \psi_{Ct}^R N_t^R = \sum_j \psi_{jt}^L LD_{jt}$$

Our model of producer behavior is estimated over the sample period 1960-2005. In simulating the model beyond the sample period, projecting the hours available for work raises an issue about the treatment of business cycles. In particular, the actual data for 2008 and 2009 shows a sharp recession and fall in hours worked. The near-term projections are for a slow recovery with a period of above-average unemployment rates.

A simulation that begins in 2010 that ignores unemployment would overstate work, output, and energy use by a substantial margin. In order to project a baseline level of energy consumption that is consistent with the widely held views of below-trend output we introduce a simple adjustment for the time endowment:

(2.90) $$LH_t^{adj} = (1 - u_t^{adj}) LH_t$$

The construction of the unemployment adjustment ratio for above-trend unemployment is described in chapter 6, section 6.1.

Three additional equations must hold in equilibrium. The first is the exogenous government deficit, which is satisfied by the endogenous spending on goods. This is described in section 5.2.2, equation 5.10. The second is the exogenous current account surplus (5.35), which is satisfied by the world relative price $e_t$. The third item is the savings and investment balance:

(2.91) $$S_t = P_t^I I_t^a + \Delta G_t + CA_t$$

Household saving is first allocated to the two exogenous items – lending to the government to finance the public deficit ( $\Delta G$ ) and lending to the rest of the world (*CA*). The remainder is allocated to investment in domestic private capital. As we have explained in section 2.4, there are no separate savings and investment decisions in IGEM and (2.91) holds as a result of household intertemporal optimization.

### *2.5.2 Steady State and Transition Path*

Our projections of the exogenous variables converge to long-run values. These include stocks of debt, the population, the latent variables in the price functions, and the aggregation coefficients. The steady state equilibrium is reached when all the conditions characterizing the market balance hold and, in addition, two further conditions are met:

(2.92) $$K_t = K_{t-1}$$

(2.93) $$F_t = F_{t-1}$$

The capital accumulation equation implies that investment exactly covers depreciation in the steady state:

(2.94) $$\delta K_{ss} = \psi^{I}_{ss} I^{a}_{ss} ,$$

The Euler equation implies that the steady state interest rate equals the rate of time preference:

(2.95) $$r_{ss} = \rho .$$

The saddle path feature of the Cass-Koopmans model is well known as illustrated in figure 2.3. The locus of $\dot{K}_t = 0$ and $\dot{F}_t = 0$ intersect at the steady state values, ($K_{ss}$, $F_{ss}$). The saddle path is given by the dashed line going through ($K_{ss}$, $F_{ss}$). Given an initial capital stock $K_0$, the unique costate variable value that lies on the saddle path is $F_0$, this is the value of full consumption in the first period that is on an intertemporal equilibrium path that obeys the transversality condition.

Along the transition path from the first period to the steady state, the following equations must hold: (1) the capital accumulation equation, (2) the Euler equation linking full consumption between adjacent periods, and (3) the cost of capital equation linking the marginal product of capital with the rate of return and capital gains.

### *2.5.3 Numeraire and Walras Law*

The model is homogenous in prices, so that doubling all prices will leave the equilibrium unchanged. We are free to choose a normalization for the price system and we use the price of time endowment $P^h_t$ as the numeraire. For each period $t$, $P^h_t$ is set to an exogenous value; for the sample period we set to the actual data and for the projection period we set to the value in the last year of the sample. Excess demands must sum to zero, that is, one of the market clearing equations is implied by the other equations and Walras Law. In our solution algorithm we drop the labor market clearing equation.

### 2.6Solution Algorithm

There are more than 4000 endogenous variables in IGEM in each time period, as summarized above and in chapter 5. We approximate the steady state at *T*=120 periods after the initial shock. Roughly half a million values of the unknown variables must be determined. It would be difficult, if not impossible, to solve this system of equations for all the unknown variables simultaneously. The structure of our solution algorithm makes it possible to do this in a sequence of computational steps. The algorithm is implemented by us in Fortran and C programming languages. This is in contrast to most models that use a modeling package such as GAMS or GEMPACK.

Conceptually, the model consists of three main components: (1) an intratemporal module; (2) a steady-state module; and (3) an intertemporal module. The intratemporal module computes a complete equilibrium for any given year *t* conditional on that year's exogenous variables and the values of two intertemporal variables: the capital stock available at the beginning of the period ($K_{t-1}$), and the value of aggregate full-consumption ($PFF_t = PF_tF_t$).This intratemporal equilibrium is discussed in section 2.5.1.

The steady-state, as described in section 2.5.2, contains all the equations of the intratemporal equilibrium, plus two additional conditions. Thus, in the steady state module we iterate over values of the state variable $K_{ss}$ and the costate variable $PFF_{ss}$, and call the intratemporal module repeatedly until the accumulation and Euler equations (2.94 and 2.95) are satisfied.

Finally, the intertemporal module iterates over complete intertemporal trajectories of $\{K_t\}$ and $\{PFF_t\}$, $t$=1,2,…$T$, invoking the intratemporal module for every period until it finds a set that satisfy the model's accumulation and Euler conditions (2.76, 2.79, 2.25). In addition, the intertemporal module ensures that the trajectories satisfy two boundary conditions: (1) the initial capital stock matches the value of the capital stock in the model's data set, and (2) the value of full consumption in the final period of the simulation matches its steady state value.

The solution algorithm is structured to solve the three modules efficiently. The process is illustrated in table 2.5. In the first step we solve the steady state module by using an enhanced version of Newton's Method described in appendix D. A subroutine iterates over the $\{K_{ss}, PFF_{ss}\}$ pair until equations (2.94) and (2.95) are satisfied. For each guess of $\{K_{ss}, PFF_{ss}\}$, this subroutine invokes the intratemporal module in each time period $t$=$T$ to solve for all the other endogenous variables. The exogenous variables are set to their steady state values.

After the steady state has been determined the second step of the intertemporal algorithm is to determine the trajectories of $\{K_t\}_{t=1}^{T}$ and $\{PFF_t\}_{t=1}^{T}$, using the hybrid intertemporal algorithm described in appendix D.3. The algorithm is implemented in a subroutine and a second subroutine is called to evaluate each trajectory at each iteration. The procedure is illustrated in figure 2.4. We begin with an initial guess of the state and costate paths, say, $\{K_t^g\}_{t=1}^T$ and $\{PFF_t^g\}_{t=1}^T$. In figure 2.4 we only illustrate the path of the guess of full consumption, $\{PFF_t^g\}_{t=1}^T$.

For each period, given $K_{t-1}^g$ and $PFF_t^g$, a subroutine calls the intratemporal module to solve for all the other endogenous variables. The intratemporal equilibria for the periods *t* and *t+1* determines the implied capital stock for the beginning of *t+1*:

(2.96) $K_t = (1-\delta)K_{t-1}^g + \psi^I I_t^a$,

and the interest rate $r_{t+1}$ using the cost of capital equation:

(2.97) $PKD_{t+1} = \frac{1}{(1-tk)}[(r_{t+1} - \pi_{t+1}) + \delta(1+\pi_{t+1}) + tp]PK_t$ ,

and, finally, the implied full consumption using the Euler equation:

(2.98) $PFF_{t+1} = \frac{(1+n_{t+1})(1+r_{t+1})}{1+\rho} PFF_t^g$; $\quad t=1,2,\ldots,T\text{-}1$

The implied values of the full consumption costate variable are marked by the red arrows in figure 2.4. The difference between the guesses and the implied values are labeled as "miss" in the figure:

(2.99) $miss_t^{PFF} = PFF_t - PFF_t^g$; $\quad miss_t^K = K_t - K_t^g$

These *miss* values are used to update the guesses using a hybrid intertemporal algorithm that generalizes Fair and Taylor (1983) and employs certain features of the "multiple shooting" procedure[10] as described in appendix D. A straightforward implementation of Fair-Taylor will simply use a weighted average as the new guess:

(2.100) $PFF_t^{g+1} = \omega PFF_t + (1-\omega)PFF_t^g$

[10] See Lipton, et al. (1982).

The hybrid algorithm introduced in Wilcoxen (1988) uses the *miss* values to compute a Jacobian that generates revised guesses that converge much faster.

The guess for the costate variable in the final period for each iteration is the steady state value: $PFF_T^g = PFF_{ss}$. The terminal period $T$ is chosen such that the implied value of $PFF_T$ after convergence differs from the steady state values by less than a chosen tolerance level for $T$=120.

During both the steady state calculation and intertemporal calculation, the intratemporal module is invoked repeatedly. For computational efficiency, that module consists of a two-tiered suite of enhanced Newton's Method algorithms as summarized in table 2.5. The outer loop is implemented in subroutine *Newton_FP* and iterates over a vector of factor prices and other variables. The corresponding miss distances are computed by subroutine *Intra_miss*. For each iteration of *Newton_FP*, an inner algorithm is called to determine industry output prices conditional on the guessed vector of factor prices. The inner algorithm is implemented in *Newton_PO* and the miss distances are calculated by *PO_miss*.

The design of the nested loops requires careful organization of the solution algorithm, rather a brute force approach for solving all of the equations at the same time. The result is an algorithm that solves the model quickly and is relatively easy to debug. After the base case transition path is obtained, the solutions for alternative policy cases usually require only seconds to solve.

## 2.7. Summary and Conclusion.

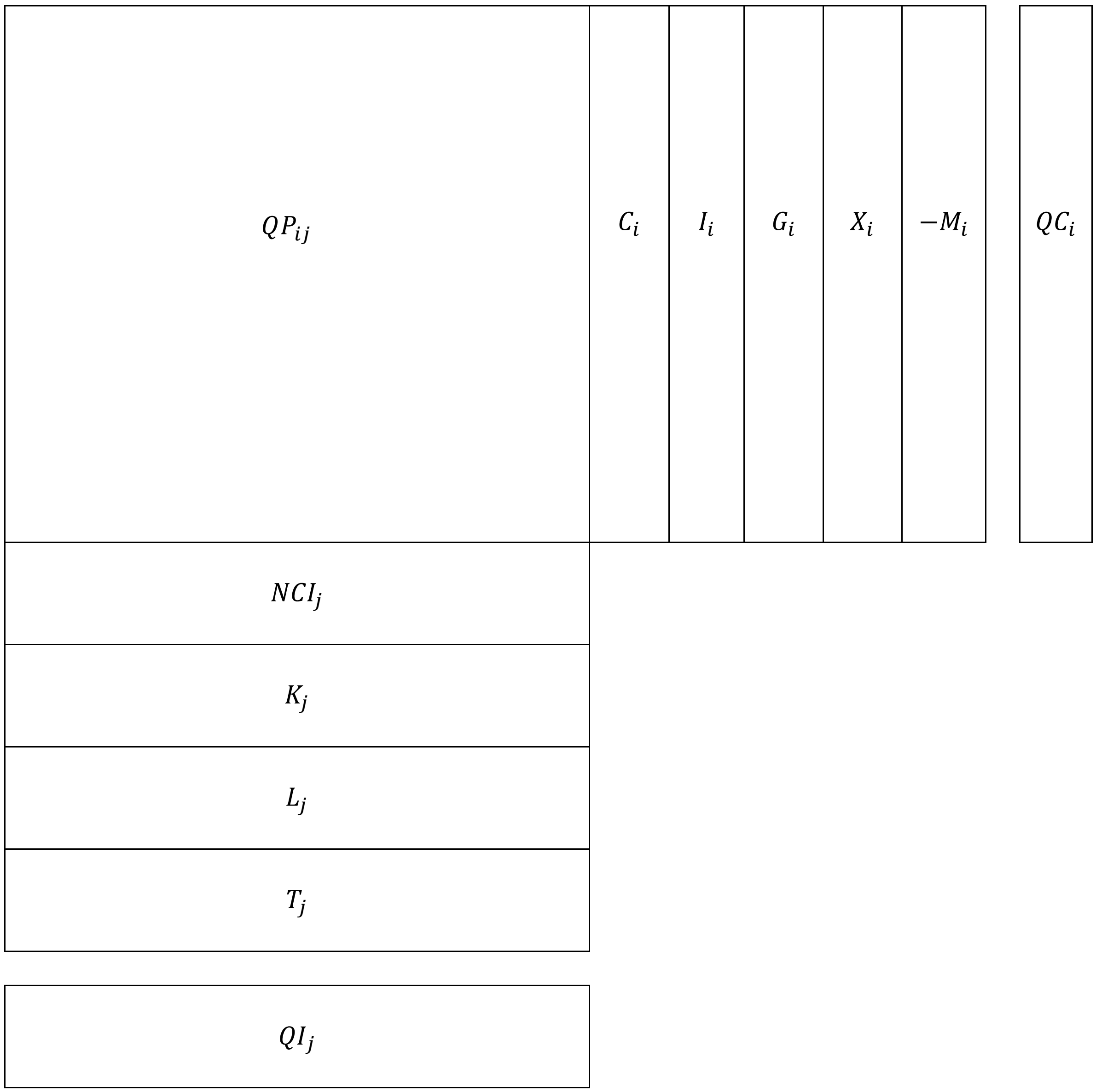
In this chapter we have presented the Intertemporal General Equilibrium Model of the United States. The core of the demand side of the model presented in section 2.3 is a model of consumer behavior that consists of three stages. In the first stage full wealth is allocated to full consumption in each time period by means of an Euler equation. In the second stage full consumption is allocated among goods and services and leisure time. In the third and final stage the consumption of goods and services is allocated to the specific commodity groups that appear in IGEM.

Consumer demands, including leisure, depend on prices and full consumption as well as the demographic characteristics of individual households. These characteristics enable us to capture the heterogeneity of the households that make up the US population. Combining the time endowment for each household with the demand for leisure, we obtain the supply of labor. We sum over the demands and supplies of individual households to obtain the model of aggregate consumer behavior that appears in IGEM. We discuss the household model in more detail in chapter 3.

The core of the supply side of IGEM is the models of producer behavior for the 35 industrial sectors presented in section 2.2. Each model determines the supply price for the industry, as well as the allocation of the value of output to the inputs of capital and labor services and intermediate inputs of energy and materials. The model also allocates energy among five energy sources and materials among 30 types of non-energy goods and services. We obtain total intermediate demands for each type of energy and materials as well and capital and labor services by summing over the 35 industrial sectors.

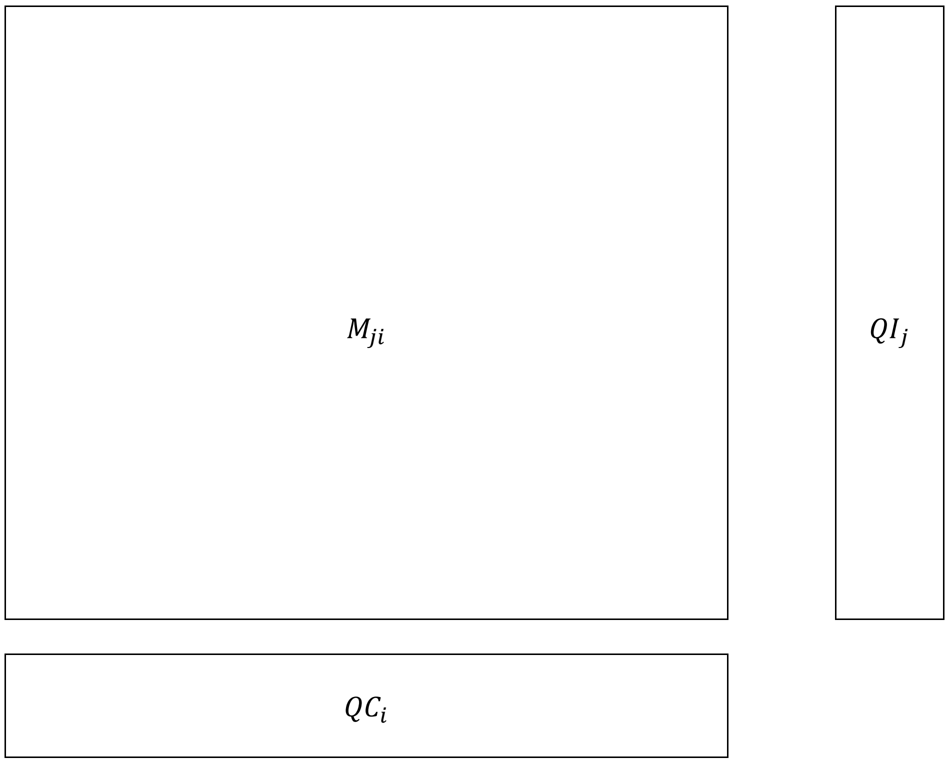
For each investment good included in the model we include investment, capital stock, and capital services and the corresponding prices. The price of investment goods is linked to the demand for capital services through the cost of capital. The prices determined in markets for investment goods and capital services in each time period summarize the information about expectations of future prices that is relevant for current decisions. We discuss the model of producer behavior in more detail in chapter 4.

Each solution of IGEM determines an intertemporal equilibrium, consisting of the balance between demands and supplies for all commodities in every time period. This balance is achieved by the intertemporal price system, consisting of current prices as well as expectations of future prices. The effects of energy taxes work through markets for both energy and non-energy commodities and through current and expected future prices determined in these markets. We consider the solution of the model in greater detail in appendix D.




*Notes:*
$QI_j$: industry *j* output
$QC_i$: quantity of domestic commodity *i*
$K_j$: capital input
$L_j$: labor input
$T_j$: sales tax
$NCI_j$: noncomparable imports
$QP_{ij}$ : quantity of intermediate input *i* into *j*
$C_i$, $I_i$, $G_i$, $X_i$, $M_i$: final demands for commodity *i*
$M_{ji}$: quantity of commodity *i* made by industry *j*

**Figure 2.1:**
Input output USE table




*Notes:*
$QI_j$: industry *j* output
$QC_i$: quantity of domestic commodity *i*
$M_{ji}$: quantity of commodity *i* made by industry *j*

**Figure 2.2:**
Input output MAKE table

**Figure 2.3**

Phase diagram for state and costate variables (capital and full consumption)

**Figure 2.4**

Solving for the intertemporal equilibrium path

**Table 2.1**
Industry Output and inputs, year 2010

| | Industry Name | Output | Input | | |
|---|---|---|---|---|---|
| | | | Energy | Material | Value-Added |
| 1 | Agriculture | 378.8 | 27.0 | 187.1 | 164.7 |
| 2 | Oil mining | 94.4 | 10.5 | 25.9 | 58.0 |
| 3 | Gas mining | 129.7 | 14.1 | 35.0 | 80.6 |
| 4 | Coal mining | 32.9 | 4.4 | 10.3 | 18.2 |
| 5 | Non-energy mining & support | 155.3 | 8.7 | 75.4 | 71.2 |
| 6 | Electric utilities (pvt+govt) | 260.6 | 14.6 | 19.4 | 226.6 |
| 7 | Natural gas distribution | 115.0 | 60.1 | 16.4 | 38.6 |
| 8 | Water and sewage | 11.2 | 0.2 | 3.0 | 8.0 |
| 9 | Construction | 985.7 | 57.8 | 419.5 | 508.4 |
| 10 | Wood and paper products | 235.3 | 12.7 | 144.3 | 78.4 |
| 11 | Nonmetallic mineral products | 89.9 | 5.7 | 49.4 | 34.8 |
| 12 | Primary metals | 217.9 | 21.9 | 153.3 | 42.8 |
| 13 | Fabricated metal products | 275.7 | 3.7 | 152.6 | 119.4 |
| 14 | Machinery | 306.3 | 2.8 | 164.3 | 139.1 |
| 15 | Information technology equip | 231.0 | 0.7 | 67.1 | 163.2 |
| 16 | Electrical equipment | 259.3 | 1.6 | 112.3 | 145.4 |
| 17 | Motor vehicles and parts | 359.9 | 3.0 | 302.5 | 54.4 |
| 18 | Other transportation equip | 228.1 | 1.7 | 155.1 | 71.3 |
| 19 | Miscellaneous manufacturing | 211.4 | 1.3 | 92.5 | 117.6 |
| 20 | Food, beverage and tobacco | 773.3 | 17.1 | 577.4 | 178.8 |
| 21 | Textile Apparel Leather | 68.3 | 1.4 | 39.6 | 27.3 |
| 22 | Printing and related activities | 87.7 | 2.3 | 53.8 | 31.5 |
| 23 | Petroleum and coal products | 643.4 | 439.5 | 33.0 | 170.9 |
| 24 | Chemicals, rubber and plastic | 807.7 | 54.3 | 463.8 | 289.6 |
| 25 | Wholesale Trade | 1066.5 | 22.8 | 411.8 | 631.9 |
| 26 | Retail Trade | 1326.1 | 17.2 | 433.4 | 875.5 |
| 27 | Transportation & warehousing | 763.7 | 112.3 | 261.0 | 390.3 |
| 28 | Publishing, Broadcasting, Telecom | 841.5 | 6.3 | 385.4 | 449.8 |
| 29 | Software, information tech svcs | 599.8 | 2.2 | 275.1 | 322.5 |
| 30 | Finance & Insurance | 2305.3 | 6.8 | 1083.9 | 1214.6 |
| 31 | Real Estate (ex OOH) and Leasing | 1740.7 | 14.0 | 748.7 | 978.0 |
| 32 | Business Services | 2298.0 | 39.2 | 683.9 | 1574.9 |
| 33 | Educational services (pvt+gov) | 1240.1 | 40.6 | 381.5 | 818.1 |
| 34 | Health care, social assist. (pvt+gov) | 2091.7 | 22.6 | 766.3 | 1302.8 |
| 35 | Accomodation and Other services | 1480.9 | 26.9 | 613.4 | 840.7 |
| 36 | Government (ex elect, health, educ) | 1896.5 | 69.1 | 740.9 | 1086.5 |
| | Household (housing+durable flow) | 2391.5 | 0.0 | 0.0 | 2391.5 |

**Table 2.2**
Tier structure of industry production function.

| | Sym | Node name | Components |
|---|---|---|---|
| 1 | Q | Gross output | capital, labor, energy, materials<br>Q=f(K,L,E,M) |
| 2 | E | Energy | E=f(electricity , fossil fuel)<br>E=f(X6, FF) |
| 3 | M | Nonenergy intermediates | M=f(agriculture, metals, nonmetals, services)<br>M=f(MA,MM,MN,MS) |
| 4 | FF | Fossil fuel | FF=f(coal, OIL, GAS)<br>FF=f(X4, OIL, GAS) |
| 5 | OIL | oil products | OIL=f(oil mining, refining)<br>OIL=f(X2, X23) |
| 6 | GAS | gas products | GAS=f(gas mining, gas utilities)<br>GAS=f(X3, X7) |
| 7 | MA | Agri intermed. | MA=f(agriculture, wood-paper, food mfg, textile)<br>MA=f(X1, X10, X20, X21) |
| 8 | MM | Metallic intermed. | MM=f(nonenergy mine, pri metal, fab metal, Equip.)<br>MM=f(X5,X12, X13, EQ) |
| 9 | MN | Nonmetallic intermed. | MN=f(nmm, misc mfg, printing, chemicals)<br>MN=f(X11, X19, X22, X24) |
| 10 | MS | Services and margins | MS=f(bus svc, TT, SV)<br>MS=f(X32, TT, SV) |
| 11 | EQ | Equipment | EQ=f(mach, IT eq, elect eq, TR)<br>EQ=f(X14, X15, X16, TR) |
| 12 | TR | Transp. Equip | TR=f(motor veh, other trnsp eq)<br>TR=f(X17,X18) |
| 13 | TT | Trade & Transp | TT=f(Wholesale, Retail, Transportation)<br>TT=f(X25, X26, X27) |
| 14 | SV | Services | SV=f(BL, OB, OS)<br>SV=f(BL, OB, OS) |
| 15 | BL | Building svc | BL=f(water, construction, realestate)<br>BL=f(X8, X9, X31) |
| 16 | OB | Other business svc | OB=f(publishing, software, finance)<br>OB=f(X28,X29, X30) |
| 17 | OS | Other services | OS=f(33,34,35,36)<br>OS=f(Educ, Health, Accom, Gov) |

**Table 2.3**
Personal Consumption Expenditures and leisure, IGEM categories, 2010.

| | IGEM categories | Consumption ($bil) | NIPA PCE category |
|---|---|---|---|
| 1 | Food & Tobacco | 860.7 | 27, 28,29,44 |
| 2 | Clothing & Footwear | 334.3 | 32, 33,34,35 |
| 3 | Gasoline | 331.4 | 37 |
| 4 | Fuel-household | 22.7 | 38 |
| 5 | Pharmaceuticals | 330.6 | 40 |
| 6 | Recreational goods | 141.4 | 41 |
| 7 | Household goods | 114.1 | 42 |
| 8 | Personal care products | 95.2 | 43 |
| 9 | Reading materials | 64.6 | 45 |
| 10 | Rental | 360.5 | 51, 54 |
| 11 | Water | 85.7 | 56 |
| 12 | Electricity | 168.4 | 58 |
| 13 | Gas | 55.4 | 59 |
| 14 | Health care | 1667.4 | 62, 63,64,66,67 |
| 15 | Own transportation | 211.6 | 70, 71 |
| 16 | Transportation | 83.9 | 73, 74,75 |
| 17 | Sports, theaters | 180.4 | 77, 80 |
| 18 | Video, IT svcs | 102.7 | 78 |
| 19 | Gambling | 99.6 | 79 |
| 20 | Meals | 547.4 | 83, 84 |
| 21 | Hotels | 90.6 | 85 |
| 22 | Financial services | 514.5 | 88, 89 |
| 23 | Life insurance | 85.8 | 91, 92 |
| 24 | Health Insurance | 119.9 | 93 |
| 25 | Vehicle insurance | 60.1 | 94 |
| 26 | Telecommunications | 214.9 | 97, 99 |
| 27 | Postal and delivery | 8.4 | 98 |
| 28 | Education | 235.9 | 101, 102,103 |
| 29 | Business services | 163.3 | 104 |
| 30 | Personal services | 112.2 | 105 |
| 31 | Social services | 154.1 | 106 |
| 32 | Household services | 55.2 | 107 |
| 33 | Foreign Travel | 121.9 | 46, 109 |
| 34 | Nonprofit consumption | 280.2 | 111 |
| 35 | Owner maintenance | 197.0 | our imputation |
| | HH capital flow | 2574.8 | our imputation |
| | Leisure | 17207.5 | our imputation |

NIPA PCE category refers to the line number in Table 2.4.5 of SCB 2011.

**Table 2.4**
Tier structure of consumption function.

| | Sym | Name | Components |
|---|---|---|---|
| 1 | F | Full consumption | Nondurables, capital, Consumer Services, leisure<br>$F = F(N^{ND}, N^{KS}, N^{CS}, N_R)$ |
| 2 | ND | Nondurables | food, pharmaceuticals, Energy, Consumer Goods<br>$N^{ND} = N^{ND}(N_1, N_5, N^{EN}, N^{CG})$ |
| 3 | KS | Capital services | rental, durable services<br>$N^{KS} = N^{KS}(N_{10}, N_{36})$ |
| 4 | CS | Consumer Services | Medical, Financial & Business services, Household Operation, Recreation<br>$N^{CS} = N^{CS}(N^{MD}, N^{FB}, N^{HO}, N^{RC})$ |
| 5 | EN | Energy Goods | gasoline, household fuel<br>$N^{EN} = N^{EN}(N_3, N_4)$ |
| 6 | CG | Consumer goods | clothing-footwear, recreational goods, reading materials, Household & personal goods<br>$N^{CG} = N^{CG}(N_2, N_6, N_9, N^{HPG})$ |
| 7 | MD | Medical | health care, health Insurance<br>$N^{MD} = N^{MD}(N_{14}, N_{24})$ |
| 8 | FB | Financial & business services | services, life insurance, business services<br>$N^{FB} = N^{FB}(N_{22}, N_{23}, N_{29})$ |
| 9 | HO | Household operation | owner maintenance, Education & Nonprofit, Utilities & personal svc, Transportation & Commun.<br>$N^{HO} = N^{HO}(N_{35}, N^{EDN}, N^{UPS}, N^{TRC})$ |
| 10 | RC | Recreation | meals, Recreation services, Travel<br>$N^{RC} = N^{RC}(N_{20}, N^{RCS}, N^{TRV})$ |
| 11 | HPG | Household & personal goods | household goods, personal care products<br>$N^{HPG} = N^{HPG}(N_7, N_8)$ |
| 12 | EDN | Education & Nonprofit | education, nonprofit consumption<br>$N^{EDN} = N^{EDN}(N_{28}, N_{34})$ |
| 13 | UPS | Utilities & personal services | water, energy services, household & personal svc<br>$N^{UPS} = N^{UPS}(N_{11}, N^{ENS}, N^{HPS})$ |
| 14 | TRC | Transportation & Communication | own transportation, transportation, vehicle insurance, communications<br>$N^{TRC} = N^{TRC}(N_5, N_{16}, N_{25}, N^{COM})$ |
| 15 | RCS | Recreation services | sports & theaters, video & IT services, gambling<br>$N^{RCS} = N^{RCS}(N_{17}, N_{18}, N_{19})$ |
| 16 | TRV | Travel | hotels, foreign travel<br>$N^{TRV} = N^{TRV}(N_{21}, N_{33})$ |
| 17 | ENS | Energy services | electricity, gas utilities<br>$N^{ENS} = N^{ENS}(N_{12}, N_{13})$ |
| 18 | HPS | Household & | personal services, social services, household services |

| | | | |
|---|---|---|---|
| | | personal services | $N^{HPS} = N^{HPS}(N_{30},N_{31},N_{32})$ |
| 19 | COM | Communication | telecommunications, postal & delivery<br>$N^{COM} = N^{COM}(N_{26},N_{27})$ |

Note: Subscripts 1-36 refers to list of PCE items given in Table 2-3;
in the description column, these 36 items are in lower case and the bundles are capitalized

**Table 2.5**
Algorithm to solve IGEM by triangulating the system

Solving the Steady State Module

Outer Loop

A subroutine *Newton_SS* iterates over $\{K_{ss}, PFF_{ss}\}$ to solve the two steady state conditions (2.86) and (2.87)

Inner Loop

For each trail pair of {Kss, PFFss}, call the Intratemporal Module to solve all other endogenous variables

Solving the Intratemporal Module in period t, given a value for Kt-1 and PFFt

Outer loop

A subroutine *Newton_FP* iterates over factor prices and other variables,
{PKD, PKD_oil, e, VII, VGG} ,
to solve 5 equations {2.76, 2.79, 5.42, 2.84, 5.10}.
The 5 "miss" distances are calculated in *Intra_miss*.

Inner Loop

For each iteration of Newton_FP, i.e. for each guess of the factor prices, call a subroutine *Newton_PO* that iterates over 35 industry prices to solve 35 industry cost equations {2.5}. The 35 "miss" distances are computed in subroutine *PO_miss*

With these 35 commodity prices, and guess of the outer loop variables, we can derive all demands and supplies with all intratemporal equations holding, except for the 5 listed.

Solving the Intertemporal Module

Outer loop

A subroutine *FAIR_TAYLOR* iterates over the whole time path of
$\{K_{t-1}, PFF_t\}$, t=1,2,…, T
until the intertemporal equations (2.25, 5.22) are satisfied.

Inner loop

For each guess of the path, $\{K_t^g, PFF_t^g\}_{t=1}^T$
a subroutine PATH is called to compute the implied values for t+1.
See Figure 2.4. PATH calls the Intratemporal Module to compute the equilibrium for each t.

The miss distance between the guess and the values implied by the intertemporal equations is used to revise the guess.
(See Figure 2.4.)

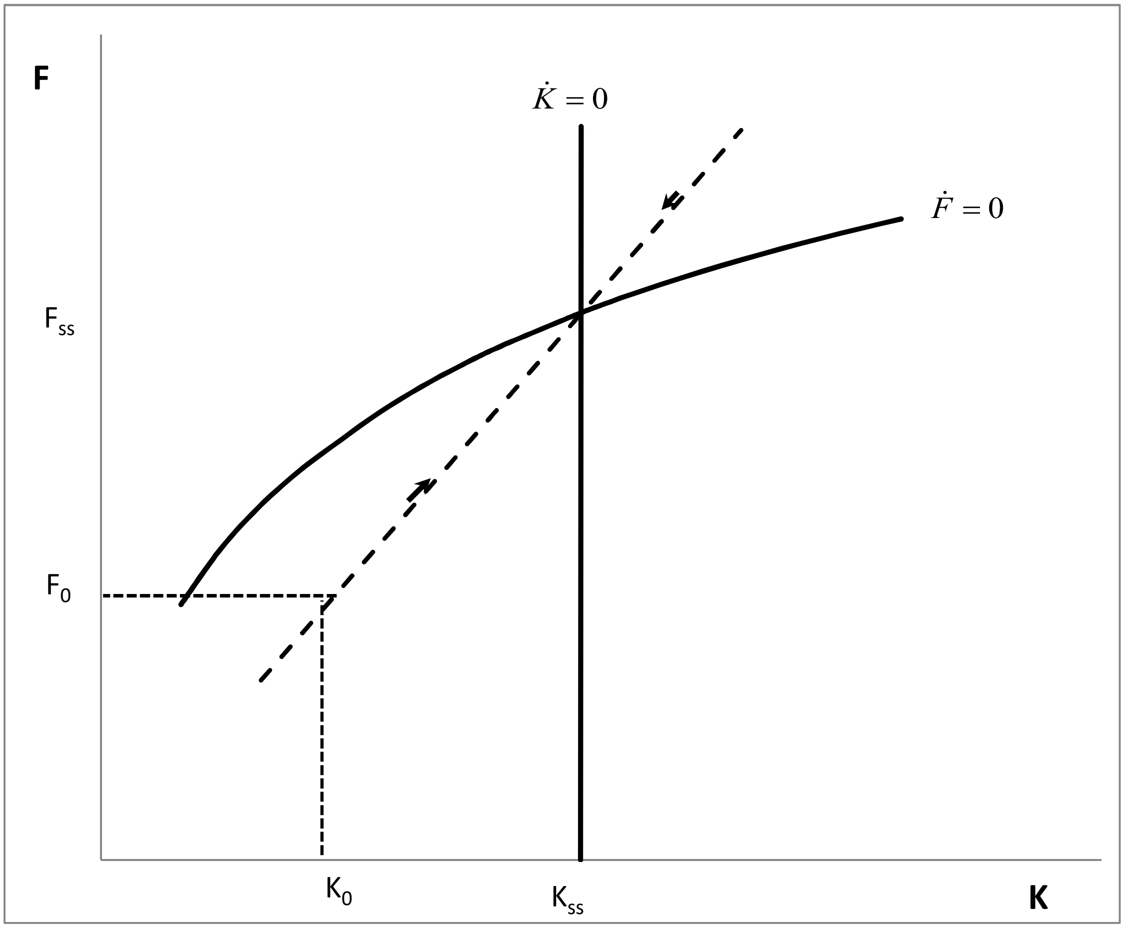



**Figure 2.3**
Phase diagram for state and costate variables (capital and full consumption)

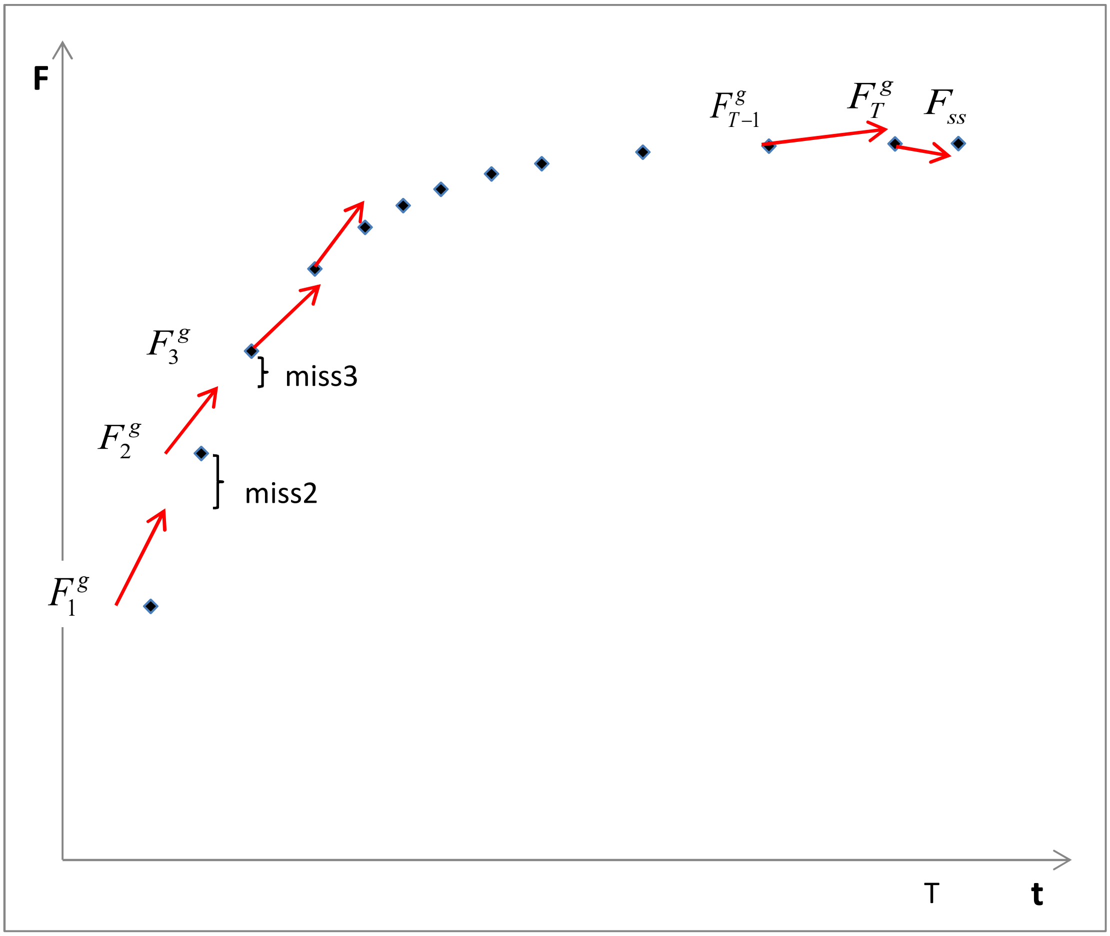



**Figure 2.4**
Solving for the intertemporal equilibrium path