

SLEPLET: Slepian Scale-Discretised Wavelets in Python

Patrick J. Roddy ¹

¹ Advanced Research Computing, University College London, UK

DOI: [10.21105/joss.05221](https://doi.org/10.21105/joss.05221)

Software

- [Review](#) 
- [Repository](#) 
- [Archive](#) 

Editor: [Vissarion Fisikopoulos](#) 

Reviewers:

- [@Saran-nns](#)
- [@klb2](#)

Submitted: 23 February 2023

Published: 20 April 2023

License

Authors of papers retain copyright and release the work under a Creative Commons Attribution 4.0 International License ([CC BY 4.0](#)).

Summary

Wavelets are widely used in various disciplines to analyse signals both in space and scale. Whilst many fields measure data on manifolds (i.e., the sphere), often data are only observed on a partial region of the manifold. Wavelets are a typical approach to data of this form, but the wavelet coefficients that overlap with the boundary become contaminated and must be removed for accurate analysis. Another approach is to estimate the region of missing data and to use existing whole-manifold methods for analysis. However, both approaches introduce uncertainty into any analysis. Slepian wavelets enable one to work directly with only the data present, thus avoiding the problems discussed above. Applications of Slepian wavelets to areas of research measuring data on the partial sphere include gravitational/magnetic fields in geodesy, ground-based measurements in astronomy, measurements of whole-planet properties in planetary science, geomagnetism of the Earth, and cosmic microwave background analyses.

Statement of Need

Many fields in science and engineering measure data that inherently live on non-Euclidean geometries, such as the sphere. Techniques developed in the Euclidean setting must be extended to other geometries. Due to recent interest in geometric deep learning, analogues of Euclidean techniques must also handle general manifolds or graphs. Often, data are only observed over partial regions of manifolds, and thus standard whole-manifold techniques may not yield accurate predictions. Slepian wavelets are designed for datasets like these. Slepian wavelets are built upon the eigenfunctions of the Slepian concentration problem of the manifold ([Landau & Pollak, 1961, 1962](#); [Slepian & Pollak, 1961](#)): a set of bandlimited functions that are maximally concentrated within a given region. Wavelets are constructed through a tiling of the Slepian harmonic line by leveraging the existing scale-discretised framework ([Leistedt, B. et al., 2013](#); [Wiaux et al., 2008](#)). Whilst these wavelets were inspired by spherical datasets, like in cosmology, the wavelet construction may be utilised for manifold or graph data.

To the author's knowledge, there is no public software that allows one to compute Slepian wavelets (or a similar approach) on the sphere or general manifolds/meshes. SHTools ([Wieczorek & Meschede, 2018](#)) is a Python code used for spherical harmonic transforms, which allows one to compute the Slepian functions of the spherical polar cap ([Frederik J. Simons et al., 2006](#)). A series of MATLAB scripts exist in `slepian_alpha` ([Frederik J. Simons et al., 2020](#)), which permits the calculation of the Slepian functions on the sphere. However, these scripts are very specialised and hard to generalise.

SLEPLET ([Roddy, 2023](#)) is a Python package for the construction of Slepian wavelets in the spherical and manifold (via meshes) settings. In contrast to the aforementioned codes, SLEPLET handles any spherical region as well as the general manifold setting. The API is documented and easily extendible, designed in an object-orientated manner. Upon installation, SLEPLET comes with two command line interfaces - `sphere` and `mesh` - that allow one to easily generate plots

on the sphere and a set of meshes using `plotly`. Whilst these scripts are the primary intended use, SLEPLET may be used directly to generate the Slepian coefficients in the spherical/manifold setting and use methods to convert these into real space for visualisation or other intended purposes. The construction of the sifting convolution (Roddy & McEwen, 2021) was required to create Slepian wavelets. As a result, there are also many examples of functions on the sphere in harmonic space (rather than Slepian) that were used to demonstrate its effectiveness. SLEPLET has been used in the development of (Roddy, 2022; Roddy & McEwen, 2021, 2022, 2023).

Whilst Slepian wavelets may be trivially computed from a set of Slepian functions, the computation of the spherical Slepian functions themselves are computationally complex, where the matrix scales as $\mathcal{O}(L^4)$. Although symmetries of this matrix and the spherical harmonics have been exploited, filling in this matrix is inherently slow due to the many integrals performed. The matrix is filled in parallel in Python using `concurrent.futures`, and the spherical harmonic transforms are computed in C using SSHT. This may be sped up further by utilising the new `ducc0` backend for SSHT, which may allow for a multithreaded solution. Ultimately, the eigenproblem must be solved to compute the Slepian functions, requiring sophisticated algorithms to balance speed and accuracy. Therefore, to work with high-resolution data such as these, one requires high-performance computing methods on supercomputers with massive memory and storage. To this end, Slepian wavelets may be exploited at present at low resolutions, but further work is required for them to be fully scalable.

Acknowledgements

The author would like to thank Jason D. McEwen for his advice and guidance on the mathematics behind SLEPLET. Further, the author would like to thank Zubair Khalid for providing his MATLAB implementation to compute the Slepian functions of a polar cap region, as well as the formulation for a limited colatitude-longitude region (Bates et al., 2017). SLEPLET makes use of several libraries the author would like to acknowledge, in particular, `libigl` (Jacobson & Panozzo, 2017), `NumPy` (Harris et al., 2020), `plotly` (Inc., 2015), `SSHT` (McEwen & Wiaux, 2011), `S2LET` (Leistedt, B. et al., 2013).

References

- Bates, A. P., Khalid, Z., & Kennedy, R. A. (2017). Slepian spatial-spectral concentration problem on the sphere: Analytical formulation for limited colatitude-longitude spatial region. *IEEE Transactions on Signal Processing*, 65(6), 1527–1537. <https://doi.org/10.1109/TSP.2016.2646668>
- Harris, C. R., Millman, K. J., Walt, S. J. van der, Gommers, R., Virtanen, P., Cournapeau, D., Wieser, E., Taylor, J., Berg, S., Smith, N. J., Kern, R., Picus, M., Hoyer, S., Kerkwijk, M. H. van, Brett, M., Haldane, A., Río, J. F. del, Wiebe, M., Peterson, P., ... Oliphant, T. E. (2020). Array programming with NumPy. *Nature*, 585(7825), 357–362. <https://doi.org/10.1038/s41586-020-2649-2>
- Inc., P. T. (2015). *Collaborative data science*. <https://plot.ly>
- Jacobson, A., & Panozzo, D. (2017). Libigl: Prototyping geometry processing research in C++. *SIGGRAPH Asia 2017 Courses*. <https://doi.org/10.1145/3134472.3134497>
- Landau, H. J., & Pollak, H. O. (1961). Prolate spheroidal wave functions, Fourier analysis and uncertainty – II. *The Bell System Technical Journal*, 40(1), 65–84. <https://doi.org/10.1002/j.1538-7305.1961.tb03977.x>
- Landau, H. J., & Pollak, H. O. (1962). Prolate spheroidal wave functions, Fourier analysis and uncertainty – III: The dimension of the space of essentially time- and band-limited

- signals. *The Bell System Technical Journal*, 41(4), 1295–1336. <https://doi.org/10.1002/j.1538-7305.1962.tb03279.x>
- Leistedt, B., McEwen, J. D., Vandergheynst, P., & Wiaux, Y. (2013). S2LET: A code to perform fast wavelet analysis on the sphere. *Astronomy & Astrophysics*, 558, A128. <https://doi.org/10.1051/0004-6361/201220729>
- McEwen, J. D., & Wiaux, Y. (2011). A novel sampling theorem on the sphere. *IEEE Transactions on Signal Processing*, 59(12), 5876–5887. <https://doi.org/10.1109/TSP.2011.2166394>
- Roddy, P. J. (2022). *Slepian wavelets for the analysis of incomplete data on manifolds* [PhD thesis, UCL (University College London)]. <https://paddyrodgy.github.io/thesis>
- Roddy, P. J. (2023). *SLEPLET: Slepian scale-discretised wavelets in Python*. <https://doi.org/10.5281/zenodo.7268074>
- Roddy, P. J., & McEwen, J. D. (2021). Sifting convolution on the sphere. *IEEE Signal Processing Letters*, 28, 304–308. <https://doi.org/10.1109/LSP.2021.3050961>
- Roddy, P. J., & McEwen, J. D. (2022). Slepian scale-discretised wavelets on the sphere. *IEEE Transactions on Signal Processing*, 70, 6142–6153. <https://doi.org/10.1109/TSP.2022.3233309>
- Roddy, P. J., & McEwen, J. D. (2023). *Slepian scale-discretised wavelets on manifolds*. arXiv. <https://arxiv.org/abs/2302.06006>
- Simons, Frederik J., Dahlen, F. A., & Wieczorek, M. A. (2006). Spatiospectral concentration on a sphere. *SIAM Review*, 48(3), 504–536. <https://doi.org/10.1137/S0036144504445765>
- Simons, Frederik J., Harig, C., Plattner, A., Hippel, M. von, & Albert. (2020). *slepian_alpha*. <https://doi.org/10.5281/zenodo.4085210>
- Slepian, D., & Pollak, H. O. (1961). Prolate spheroidal wave functions, Fourier analysis and uncertainty – I. *The Bell System Technical Journal*, 40(1), 43–63. <https://doi.org/10.1002/j.1538-7305.1961.tb03976.x>
- Wiaux, Y., McEwen, J. D., Vandergheynst, P., & Blanc, O. (2008). Exact reconstruction with directional wavelets on the sphere. *Monthly Notices of the Royal Astronomical Society*, 388(2), 770–788. <https://doi.org/10.1111/j.1365-2966.2008.13448.x>
- Wieczorek, M. A., & Meschede, M. (2018). SHTools: Tools for working with spherical harmonics. *Geochemistry, Geophysics, Geosystems*, 19(8), 2574–2592. <https://doi.org/10.1029/2018GC007529>