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ConleyDynamics.jl: A Julia package for multivector dynamics on Lefschetz complexes

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Software

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Summary

Combinatorial topological dynamics is concerned with the qualitative study of dynamical behavior on discrete combinatorial structures. It was originally developed in the context of combinatorial vector fields (Forman, 1998a, 1998b), and has since been extended to combinatorial multivector fields (Lipinski et al., 2023; Mrozek, 2017) on Lefschetz complexes (Lefschetz, 1942). For such systems, one can formulate a complete qualitative theory which includes notions of invariance, attractors, repellers, and connecting orbits. The global dynamical behavior is encoded in a Morse decomposition, and it can be studied further using algebraic topological tools such as the Conley index (Conley, 1978; Mischaikow & Mrozek, 2002) and connection matrices (Franzosa, 1989; Mrozek & Wanner, 2025). If the combinatorial multivector field is generated from a classical flow, one can derive statements about the underlying dyamics of the original system (Mrozek et al., 2022; Thorpe & Wanner, 2025a, 2025b). The Julia (Bezanson et al., 2017) package `ConleyDynamics.jl` provides computational tools for combinatorial topological dynamics, and should be of interest to both researchers and students which are curious about this emerging field.

Statement of need

`ConleyDynamics.jl` provides a full implementation of all aspects of combinatorial topological dynamics within a unified framework. The following form the core of the software:

- General Lefschetz complexes are implemented as the underlying combinatorial structure. They can be formulated over the field of rational numbers, or over any finite field of prime order. Specialized functions for simplicial and cubical complexes are provided.
- To describe the dynamics, general multivector fields are supported, which include Forman combinatorial vector fields. Functions are provided to generate multivector fields from planar and three-dimensional autonomous ordinary differential equations. One can analyze the global dynamics of a combinatorial dynamical system by determining Morse decompositions and Morse intervals.
- In order to implement all major aspects of Conley theory, algebraic topological tools are needed as well. The package provides functions for computing regular and relative homology of Lefschetz complexes, as well as the connection matrix associated with a Morse decomposition.

General Lefschetz complexes (Lefschetz, 1942) have usually been passed over in favor of their two celebrated special subtypes: Simplicial and cubical complexes. Simplicial complexes (Edelsbrunner & Harer, 2010; Munkres, 1984) form the foundation for many computational tools, such as for example triangular meshes. Sometimes cubical complexes (Kaczynski et al., 2004) are more convenient, such as in the analysis of images or voxel data. However, for the analysis of classical dynamics both of these combinatorial structures have disadvantages, as indicated in Boczko et al. (2007). While one suitable generalization could be CW complexes

(Dłotko et al., 2011; Massey, 1991), `ConleyDynamics.jl` provides the most general structure, a Lefschetz complex. To keep the software framework self-contained, special functions for simplicial and cubical complexes have been implemented as well.

Since they were first introduced by Forman, combinatorial vector fields have found numerous applications in areas such as visualization and mesh compression, graph braid groups, homology computation, astronomy, the study of Cech and Delaunay complexes, and many others. See Batko et al. (2020) for detailed references. Despite these applications, there is no general purpose software for working with such combinatorial vector fields. While Forman vector fields have close ties with classical dynamics (Batko et al., 2020; Kaczynski et al., 2016; Mrozek & Wanner, 2021), they are of limited use for the analysis of a given classical flow. For example, in a combinatorial Forman vector field chaos can only be generated through the choice of an appropriate underlying combinatorial structure (Mrozek et al., 2022). For this reason, the notion of combinatorial multivector fields was introduced in Mrozek (2017) and Lipinski et al. (2023), and it allows for the full spectrum of dynamical behavior.

`ConleyDynamics.jl` provides functions which analyze the global dynamics of a combinatorial multivector field, on any underlying Lefschetz complex, in terms of its Morse decomposition. These computations are fast, as they rely on `Graphs.jl` (Fairbanks et al., 2021). Combinatorial multivector fields can be constructed based on the concept of dynamical transitions (Mrozek & Wanner, 2025; Thorpe & Wanner, 2025a, 2025b). Finally, functions are provided that use this approach to create a combinatorial multivector field from a classical planar or three-dimensional ordinary differential equation, on any underlying Lefschetz complex that discretizes the relevant part of phase space. This leads to more flexible discretizations than the ones described in Boczko et al. (2007).

For the necessary homology computations, the classical algorithm described in Edelsbrunner & Harer (2010) is used. Its implementation in `ConleyDynamics.jl` allows for computations over general prime finite fields or over the rationals, and one can work with arbitrary Lefschetz complexes. Note, however, that there is better-performing software available, such as for example, GUDHI Project (2024).

The first algorithm for the computation of connection matrices was described in Harker et al. (2021). Their software, however, is geared towards specific applications. For this reason, `ConleyDynamics.jl` implements the recent method described in Dey et al. (2024). Our implementation extends their method to general fields and Lefschetz complexes.

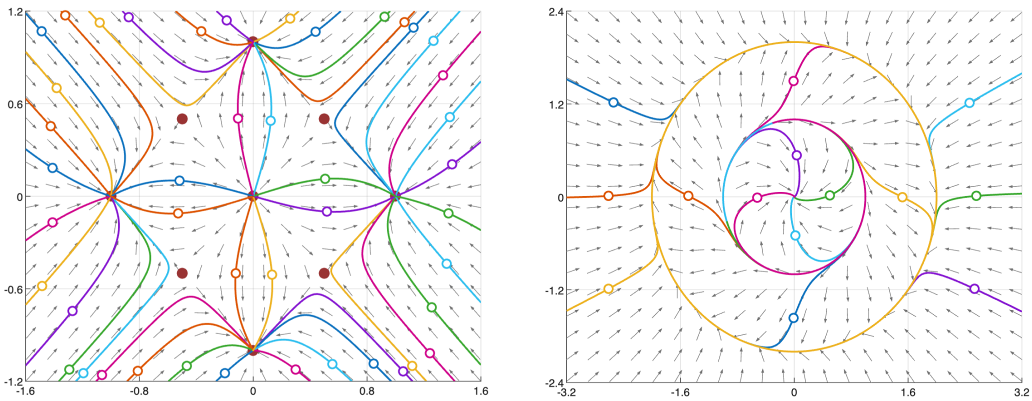

Figure 1: Two sample planar flows. The left image shows a gradient system with nine equilibrium solutions and many connecting orbits betweeen them. The system depicted on the right has one stable equilibrium at the origin and two surrounding periodic orbits.

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Examples

`ConleyDynamics.jl` can determine a combinatorial multivector field which encapsulates the possible dynamical behavior of a classical flow, based on the underlying vector field and a Lefschetz complex discretization of the relevant portion of phase space.

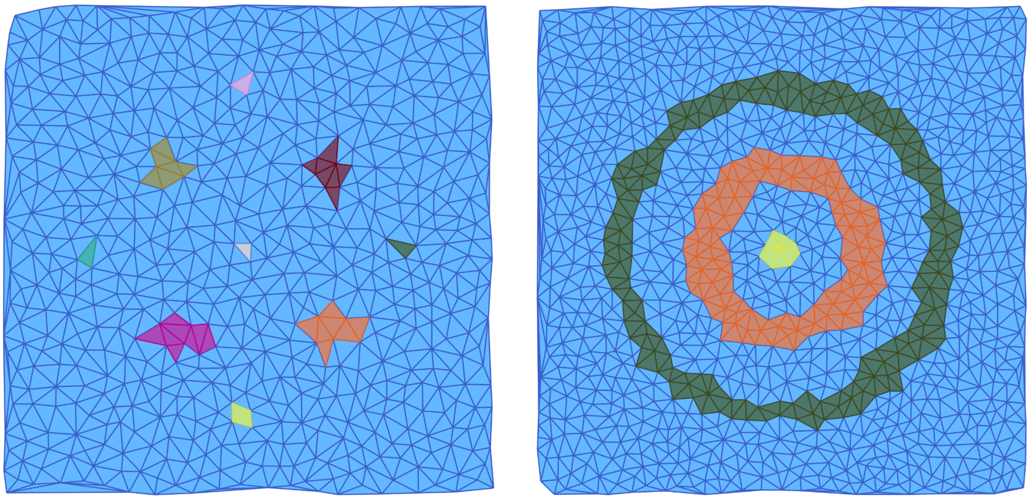

Figure 2: Morse decompositions for the two planar flows shown in the previous figure. In both cases, the underlying Lefschetz complex is a random Delaunay triangulation. The identified Morse sets are shown in different colors.

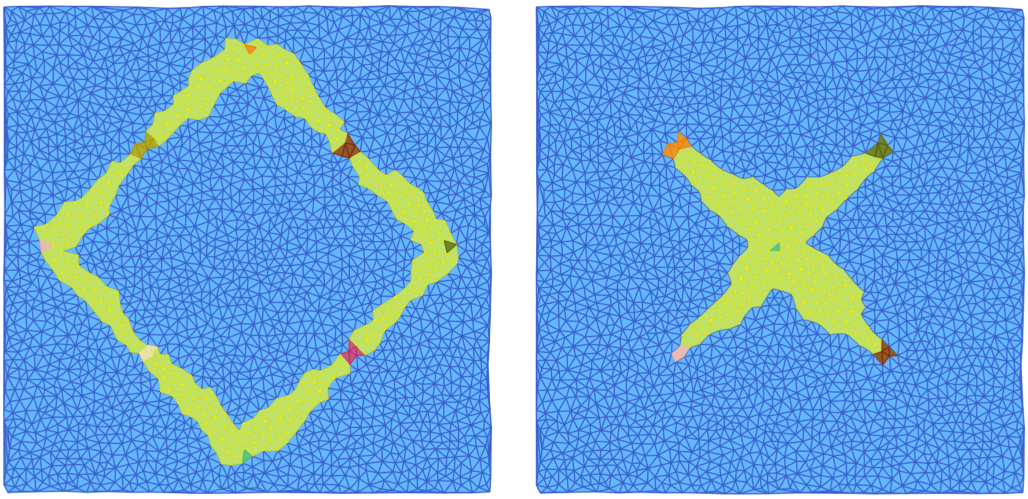

Figure 3: Morse intervals for the planar gradient flow. The yellow regions provide enclosures for heteroclinic orbits in the system.

Consider for example the two planar flows depicted in Figure 1. For these systems, Figure 2 shows Morse decompositions based on underlying random Delaunay triangulations. The left system contains nine equilibrium solutions, within the colored enclosures. The right panel of Figure 2 depicts the Morse decomposition of a planar system with two periodic solutions, which circle around an equilibrium in their center. In both cases, `ConleyDynamics.jl` also provides Conley indices and the connection matrix. In addition, one can determine information on the global dynamics of these systems. For example, Figure 3 visualizes enclosures for the connecting orbits between stationary states, for the system in the left panel of Figure 2. For more details and examples, see the extensive documentation accompanying Wanner (2024), and the two recent papers: Thorpe & Wanner (2025a) and Thorpe & Wanner (2025b). All examples in the book Mrozek & Wanner (2025) were computed using `ConleyDynamics.jl`.

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