# KerrGeoPy: A Python Package for Computing Timelike Geodesics in Kerr Spacetime

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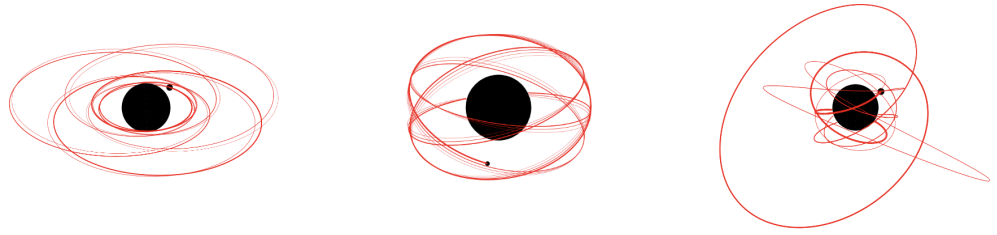
Summary

In general relativity, the motion of a free-falling test particle in a curved spacetime is described by a timelike geodesic - the minimal path between two points in space. Intuitively, geodesics are analogous to straight line paths in flat spacetime. The timelike geodesics of Kerr spacetime are of particular interest in the field of black hole perturbation theory because they describe the zeroth-order motion of a small object moving through the background spacetime of a much more massive spinning black hole. For this reason, computing geodesics is an important step in modeling the gravitational radiation emitted by an extreme-mass-ratio inspiral (EMRI) - an astrophysical binary in which a stellar mass compact object, such as a neutron star or black hole (with mass $10^1 - 10^2 M_\odot$), spirals into a massive black hole (with mass $10^4 - 10^7 M_\odot$).

Kerr spacetime has several nice properties which simplify the problem of computing geodesics. Since it has both time-translation symmetry and rotational symmetry, energy and (the z-component of) angular momentum are conserved quantities. It is also equipped with a higher order symmetry which gives rise to a third constant of motion called the Carter constant. These three constants of motion, along with the spin of the black hole, uniquely define a geodesic up to initial conditions (Schmidt, 2002). Alternatively, geodesics can be identified using a suitably generalized version of the parameters used to define a Keplerian orbit (eccentricity, semi-latus rectum, and inclination angle). Bound geodesics also possess fundamental frequencies since their radial, azimuthal, and polar motions are periodic.

KerrGeoPy is a Python package which computes both stable and plunging timelike geodesics in Kerr spacetime using the analytic solutions to the geodesic equation derived in Fujita & Hikida (2009) and Dyson & Meent (2023). It mirrors and builds upon much of the functionality of the KerrGeodesics (Warburton et al., 2023) Mathematica library. Geodesic solutions are written in terms of Legendre elliptic integrals, which are evaluated using SciPy. Users can construct a geodesic by providing the initial position and four-velocity, or by providing either the constants of motion or the Keplerian parameters described above.

KerrGeoPy provides methods for computing the four-velocity, fundamental frequencies, and constants of motion associated with a given geodesic and also implements the algorithm described in Stein & Warburton (2020) for finding the location of the last stable orbit, known as the separatrix. The package also includes several methods for visualizing and animating geodesics.

KerrGeoPy is a part of the Black Hole Perturbation Toolkit. The source code is hosted on Github and the package is distributed through both PyPI and conda-forge. Automated unit tests are run using Github Actions and comprehensive documentation is available on Read the Docs.

Figure 1: Example of an equatorial (left), spherical (center) and generic (right) orbit computed by KerrGeoPy

Statement of Need

EMRIs are expected to be a major source observable by the Laser Interferometer Space Antenna (LISA), a future space-based gravitational wave observatory consisting of a triangular constellation of three satellites in orbit around the sun. LISA is an ESA-led mission with significant contributions from NASA which is set to launch in the 2030s. It will complement existing ground-based detectors by opening up the millihertz band of the gravitational wave spectrum (Thorpe et al., 2019). Because sources in this band evolve more slowly over time and remain observable for a period of days to years, LISA is expected to detect many overlapping signals at all times. Thus, accurate waveform models are needed in order to identify gravitational wave sources and perform parameter estimation - the process of approximating characteristics of a source.

For most LISA sources, well-developed waveform models based on either numerical relativity or post-Newtonian theory already exist. However, EMRIs are instead more naturally described by black hole perturbation theory, and the EMRI waveform models that currently exist are underdeveloped compared to other LISA sources. In a perturbation theory model, the orbital trajectory is assumed to be a geodesic at leading order. Higher-order corrections are then computed by introducing the gravitational field of the inspiraling object as a perturbation to the background spacetime of the massive black hole, expanded in powers of the mass ratio.

To meet the accuracy requirements for LISA parameter estimation, EMRI waveform models must include both first- and second-order corrections to the orbital trajectory. However, to date, second-order corrections are only available for the most simple systems, quasi-circular inspirals in Schwarzschild (Berry et al., 2019). Open-source tools can aid in rapidly expanding EMRI models to more complicated orbits in Kerr spacetime, but at the moment many tools for modeling EMRIs are only available in Mathematica, which is an expensive and proprietary piece of software. KerrGeoPy is intended to support future development of higher-order waveform models in preparation for LISA by providing a free alternative to the existing KerrGeodesics Mathematica library for other researchers to build on in their own projects.

Although other Python packages (Gourgoulhon et al., 2019) with similar functionality do exist, they mostly rely on numerical integration to compute geodesics. The analytic solutions used by KerrGeoPy have two main advantages over this approach. First, they can be much more numerically stable over long time periods and can be quickly evaluated at any point in time. This is essential for EMRI models, which typically require taking long time-averages over the geodesic motion. Second, they produce several useful intermediate terms which are not calculated by other packages. Therefore, KerrGeoPy, with its analytic solutions and various orbital parametrizations, is specifically tuned to support perturbative models of binary black holes and their gravitational waves.

Software Citations

`KerrGeoPy` has the following dependencies:

- `NumPy` ([Harris et al., 2020](#))
- `SciPy` ([Virtanen et al., 2020](#))
- `Matplotlib` ([Hunter, 2007](#))
- `tqdm` ([Costa-Luis et al., 2023](#))

Acknowledgements

References

Berry, C., Hughes, S., Sopuerta, C., Chua, A., Heffernan, A., Holley-Bockelmann, K., Mihaylov, D., Miller, C., & Sesana, A. (2019). The unique potential of extreme mass-ratio inspirals for gravitational-wave astronomy. *Bulletin of the American Astronomical Society*, *51*, 42. https://doi.org/10.48550/arXiv.1903.03686

Costa-Luis, C. da, Larroque, S. K., Altendorf, K., Mary, H., richardsheridan, Korobov, M., Yorav-Raphael, N., Ivanov, I., Bargull, M., Rodrigues, N., Chen, G., Lee, A., Newey, C., CrazyPython, JC, Zugnoni, M., Pagel, M. D., mjstevens777, Dektyarev, M., … Boyle, M. (2023). *tqdm: A fast, Extensible Progress Bar for Python and CLI* (Version v4.66.1). Zenodo. https://doi.org/10.5281/zenodo.8233425

Dyson, C., & Meent, M. van de. (2023). Kerr-fully diving into the abyss: Analytic solutions to plunging geodesics in Kerr. *Classical and Quantum Gravity*, *40*, 195026. https://doi.org/10.1088/1361-6382/acf552

Fujita, R., & Hikida, W. (2009). Analytical solutions of bound timelike geodesic orbits in Kerr spacetime. *Classical and Quantum Gravity*, *26*, 135002. https://doi.org/10.1088/0264-9381/26/13/135002

Gourgoulhon, E., Le Tiec, A., Vincent, F. H., & Warburton, N. (2019). Gravitational waves from bodies orbiting the Galactic center black hole and their detectability by LISA. *Astronomy and Astrophysics*, *627*, A92. https://doi.org/10.1051/0004-6361/201935406

Harris, C. R., Millman, K. J., Walt, S. J. van der, Gommers, R., Virtanen, P., Cournapeau, D., Wieser, E., Taylor, J., Berg, S., Smith, N. J., Kern, R., Picus, M., Hoyer, S., Kerkwijk, M. H. van, Brett, M., Haldane, A., Río, J. F. del, Wiebe, M., Peterson, P., … Oliphant, T. E. (2020). Array programming with NumPy. *Nature*, *585*(7825), 357–362. https://doi.org/10.1038/s41586-020-2649-2

Hunter, J. D. (2007). Matplotlib: A 2D graphics environment. *Computing in Science & Engineering*, *9*(3), 90–95. https://doi.org/10.1109/MCSE.2007.55

Schmidt, W. (2002). Celestial mechanics in Kerr spacetime. *Classical and Quantum Gravity*, *19*(10), 2743–2764. https://doi.org/10.1088/0264-9381/19/10/314

Stein, L. C., & Warburton, N. (2020). Location of the last stable orbit in Kerr spacetime. *Physical Review D*, *101*, 064007. https://doi.org/10.1103/PhysRevD.101.064007

Thorpe, J. I., Ziemer, J., Thorpe, I., Livas, J., Conklin, J. W., Caldwell, R., Berti, E., McWilliams, S. T., Stebbins, R., Shoemaker, D., Ferrara, E. C., Larson, S. L., Shoemaker, D., Key, J. S., Vallisneri, M., Eracleous, M., Schnittman, J., Kamai, B., Camp, J., … Wass, P. (2019). *The Laser Interferometer Space Antenna: Unveiling the Millihertz Gravitational Wave Sky*. *51*, 77. https://doi.org/10.48550/arXiv.1907.06482

Virtanen, P., Gommers, R., Oliphant, T. E., Haberland, M., Reddy, T., Cournapeau, D., Burovski, E., Peterson, P., Weckesser, W., Bright, J., Walt, S. J. van der, Brett, M., Wilson, J., Millman, K. J., Mayorov, N., Nelson, A. R. J., Jones, E., Kern, R., Larson, E., … SciPy 1.0 Contributors. (2020). SciPy 1.0: Fundamental Algorithms for Scientific Computing in Python. *Nature Methods*, *17*, 261–272. https://doi.org/10.1038/s41592-019-0686-2

Warburton, N., Wardell, B., Long, O., Upton, S., Lynch, P., Nasipak, Z., & Stein, L. C. (2023). *KerrGeodesics*. Zenodo. https://doi.org/10.5281/zenodo.8108265