

FOLIE: Finding Optimal Langevin Inferred Equations

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Summary

This paper introduces FOLIE (Finding Optimal Langevin Inferred Equations), a versatile Python library designed to facilitate the analysis of high-dimensional molecular simulation trajectories in terms of low-dimensional dynamics. FOLIE enables scientists to fit low-dimensional stochastic differential equations (SDEs) to projected high-dimensional data, thereby extracting maximum dynamical insight from the simulations. Key features of FOLIE include robust estimation techniques, comprehensive analysis tools, and simulation capabilities to create synthetic datasets. Its highly modular architecture allows for the implementation of diverse SDEs, time discretization methods, and energy landscapes. By leveraging FOLIE, researchers can effectively extrapolate low-dimensional kinetics from limited simulation data, enhancing their ability to understand and predict complex molecular dynamics.

Statement of need

Quantitative predictions of rare events in molecular and materials simulations suffer from the curse of dimensionality and the prohibitive cost of simulating relevant time scales, starting from microscopic time steps (on the order of 10^{-15}s). Such rare events include chemical reactions in chemistry, drug unbinding in pharmacology, and phase transitions in materials. A strategy to make such predictions tractable is to construct intermediate, low-dimensional kinetic models of the time evolution of the system. Such low-dimensional models are constructed based on a projection of the full dynamics onto a reduced set of collective variables. This set must fulfill two constraints: describe the process of interest, and be informative enough to capture the long-time dynamics of this process.

FOLIE is designed to allow easy and efficient inference of such models from projected molecular simulations. There exists several software packages performing related tasks, but FOLIE differs mainly by its flexibility in the description of the energy landscape and its modular construction for the estimation task. `DeepTime` (previously `pyEmma`) (Hoffmann et al., 2021) is an equivalent for discrete Markovian processes. `pymle` fits continuous SDEs but imposes restraints on the underlying energy landscapes, which are better suited to econometrics and financial markets (Kirkby et al., 2025). `OptLE` (Palacio-Rodriguez & Pietrucci, 2022) is an experimental Fortran package that inspired this work. It focuses on method development and was not designed for flexibility or scalability. `pyOptLE` was our first attempt at improving scalability, with a limited scope. `StochasticForceInference` and `UnderdampedLangevinInference` from the Ronceray group perform the task of fitting continuous SDE, with a focus on biological applications and less flexibility in the underlying energy landscapes.

40 Theoretical background

41 Langevin Models

42 Several types of Langevin equations may be relevant for describing projected dynamicsGirardier
 43 et al. (2023). Projecting high-dimensional dynamics onto a collective variable q leads to the
 44 generalized Langevin equation(Vroylandt, 2022) that features a memory kernel. Assuming that
 45 the timescale of evolution of q is slow with respect to its environment gives the memory-less
 46 (Markovian) Standard Langevin equation. If the inertial effect are quickly damped, a further
 47 approximation can be made, leading to the Overdamped Langevin Equation

$$\dot{q} = -\beta D(q) \frac{\partial A(q)}{\partial q} + \frac{\partial D(q)}{\partial q} + \sqrt{2D(q)}\eta(t) \quad (1)$$

48 where, using $\beta = \frac{1}{k_B T}$, $D(q)$ is a diffusion profile, the effective free energy surface is $A(q) =$
 49 $-k_B T \log(\rho_{eq}(q))$ where ρ_{eq} is the invariant distribution of the dynamics and $\eta(t)$ is a standard
 50 Gaussian noise. This is a first order differential equation, simplifying the mathematical structure.
 51 The current focus of the library is the overdamped case.

52 Kinetic model optimization by a maximum-likelihood approach

53 In order to construct the optimal Langevin model in low dimension, we define $\mathcal{L}(\vec{q}|\theta)$, the
 54 likelihood of observing the trajectory data \vec{q} if they were generated by a Langevin model
 55 parameterized by θ . Here θ represents both the drift $F(q) = -\beta D(q) \frac{\partial A(q)}{\partial q} + \frac{\partial D(q)}{\partial q}$ and
 56 position-dependent diffusion $D(q)$. θ is optimized maximizing the likelihood

$$\hat{\theta} = \underset{\theta}{\operatorname{argmax}} \mathcal{L}(\theta). \quad (2)$$

57 In the overdamped case, due to markovianity, the likelihood is the product of transition
 58 probabilities $p_\theta(q_{i+1}, t_{i+1}|q_i, t_i)$ between consecutive points (Palacio-Rodriguez & Pietrucci,
 59 2022):

$$\mathcal{L}(\vec{q}|\theta) = \prod_{i=0}^{N-1} p_\theta(q_{i+1}, t_{i+1}|q_i, t_i) \quad (3)$$

60 and the log-likelihood is :

$$\log \mathcal{L}(\vec{q}|\theta) = \sum_{i=0}^{N-1} \log [p_\theta(q_{i+1}, t_{i+1}|q_i, t_i)] \quad (4)$$

61 The precise form of the transition density depends on a specific time discretization (closely
 62 related to the choice of an integrator) of the continuous SDE, several of which are implemented
 63 in FOLIE.

64 Implementation

65 Features

66 The library follows a modular structure, allowing users to assemble components into Python
 67 scripts suited to their needs.

- 68 ■ **Model of Overdamped Langevin Dynamics** Implements several forms of Overdamped
 69 Langevin equations, along with particular cases (BrownianMotion, OrnsteinUhlenbeck).
 70 The underdamped case is still under development.

- 71 ▪ **Force and diffusion coefficient functions** Defining the dynamical model requires specifying
72 the drift and space-dependent diffusion coefficient. To that effect, the Function class
73 offers functional forms such as polynomials, splines, etc.
- 74 ▪ **Transition densities** Several approximations for the propagator are implemented([Iacus, 2008](#)). These probability densities are later fed to the Likelihood estimator object,
75 which optimizes the drift and diffusion parameters.
- 76 ▪ **Estimation** The LikelihoodEstimator class performs parameter estimation by maximizing
77 the likelihood of observed trajectories using `scipy.optimize.minimize()` on the negative
78 log-likelihood function of eq. (4).
- 79 ▪ **Simulation** The FOLIE module can also generate simulated trajectories, which is useful
80 for creating synthetic data to be used when developing methods.
81

82 **Initial guess**

83 Optimization of the likelihood requires an initial guess for the drift and diffusion parameter. In
84 FOLIE, Kramers-Moyal estimation provides such a guess([Riskin, 1996](#)). This yields accurate
85 parameters for small timesteps Δt and sufficient trajectories, being then equivalent to maximum-
86 likelihood estimation with Euler discretization. This makes it a strong initial estimate for
87 maximum-likelihood approaches.

88 **Parallel computation**

89 Likelihood calculation scales linearly with data size and model complexity, making optimization
90 potentially slow. Furthermore, Langevin optimization can be integrated in a scheme for
91 optimizing collective variables ([Mouaffac et al., 2023](#)), in which case a large number of model
92 optimizations must be performed before collective variable optimization converges. When
93 running FOLIE in a shared-memory multiprocessor environment, likelihood computation is
94 performed in a data-parallel way over the projected simulation trajectories.

95 **Practical use**

96 **Usage workflow**

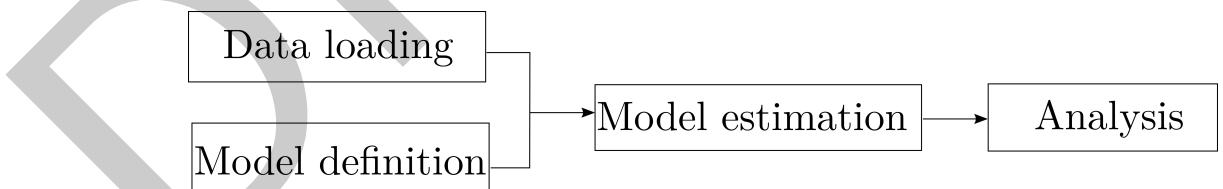


Figure 1: Typical workflow in FOLIE. This basic workflow is illustrated in this [example script in the repository](#). Several options are available for the model definition and estimation.

97 **Perspectives**

98 Further developments are in progress, in particular the more widely applicable underdamped and
99 generalized Langevin dynamics. Thanks to the modular design of FOLIE, these will integrate
100 seamlessly into the workflow.

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