

¹ TikhonovFenichelReductions.jl: A systematic approach ² to geometric singular perturbation theory

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Software

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⁵ Summary

⁶ Singular perturbation theory is a mathematical toolbox that allows dimensionality reduction of⁷ ODE systems whose components evolve on different time scales emanating from the presence⁸ of a small parameter ε . More precisely, we obtain the *reduced system* as the limit

$$\begin{aligned} \dot{u} &= g(u, v) \\ \varepsilon \dot{v} &= h(u, v) \end{aligned} \xrightarrow{\varepsilon \rightarrow 0} \begin{aligned} \dot{u} &= g(u, v) \\ 0 &= h(u, v) \end{aligned} \quad (1)$$

⁹ as stated in Tikhonov's theorem (1952; Theorem 1.1 in Verhulst, 2007). For autonomous¹⁰ systems, Fenichel (1979) established a geometric singular perturbation theory (GSPT) in¹¹ coordinate-free settings (see Wechselberger, 2020). Convergence properties then follow from¹² the slow manifold $M_0 = \{(u, v) \mid h(u, v) = 0\}$, on which the reduced system is defined.

¹³ The algebraic approach to GSPT recently developed by Goeke and Walcher (and colleagues)¹⁴ allows to systematically find all critical parameters admitting a reduction for polynomial or¹⁵ rational ODE systems of the form

$$\dot{x} = f(x, \pi) = f^{(0)}(x) + \varepsilon f^{(1)}(x) + \mathcal{O}(\varepsilon^2), \quad x \in \mathbb{R}^n, \pi \in \mathbb{R}^m, \quad (2)$$

¹⁶ i.e. we obtain all reductions for a system with a slow-fast separation of processes instead of¹⁷ components as in the standard form (1), which renders this a coordinate-free approach. This¹⁸ can be achieved by evaluating necessary conditions for the existence of a reduction for a system¹⁹ as in (2) (Goeke, 2013; Goeke et al., 2015; Goeke & Walcher, 2013, 2014).

²⁰ TikhonovFenichelReductions.jl is a Julia (Bezanson et al., 2017) package implementing²¹ this approach for polynomial ODE systems. Apelt & Liebscher (2025) provide a showcasing²² example and more detailed explanations.

²³ Statement of need

²⁴ The ad-hoc approach to singular perturbation theory requires prior knowledge about a suitable²⁵ time scale separation and substantial mathematical effort to compute the reduction. The²⁶ algebraic approach yields algorithmically accessible conditions for the existence of a reduction,²⁷ which allows to find *all* reductions of a given polynomial ODE system using methods from²⁸ computational algebra, and simplifies the computation of reduced systems (Goeke et al., 2015).²⁹ TikhonovFenichelReductions.jl makes the required computations easily accessible (even for³⁰ non-expert users) by utilizing Oscar.jl (Decker et al., 2025; The OSCAR Team, 2025).

³¹ The author is not aware of any publicly available implementation of the theory by Goeke and³² Walcher, but there exist an implementation for a related approach for computing invariant³³ manifolds by Roberts (n.d., 1997).

34 Software design

35 TikhonovFenichelReductions.jl is implemented in Julia due to its flexibility, the use of
 36 multiple dispatch and the availability of the feature-rich computer algebra system Oscar.jl
 37 ([Decker et al., 2025](#); [The OSCAR Team, 2025](#)).
 38 Core features are the search for critical parameters admitting a reduction, so-called *Tikhonov-*
Fenichel Parameter Values (TFPVs), and the computation of the corresponding reduced
 39 systems. Crucially, this requires various computations with multiple symbolic representations
 40 (i.e. different polynomial rings, rational function fields and matrix spaces) and therefore parsing
 41 of data between the corresponding types in Oscar.jl, which TikhonovFenichelReductions.jl
 42 performs hidden away from the user. Thus, the user mostly works in an object-oriented manner
 43 with the types and methods provided.
 44

45 The package is essentially designed around two types: ReductionProblem, which constructs
 46 all symbolic data types needed for the search of TFPVs, and Reduction, which holds all
 47 relevant information for the reduced system and the steps required to compute it. The latter
 48 also contains the reduced system and other information parsed to the appropriate types from
 49 Oscar.jl, that can be further used, e.g. for a symbolic analysis.

50 Features

51 Detailed explanations and examples are provided by [Apelt & Liebscher \(2025\)](#) and in the
 52 [documentation](#).

53 Finding TFPVs

54 The package provides a method to obtain all possible TFPVs implicitly by computing a Gröbner
 55 Basis and one to find *slow-fast separations of rates*, i.e. TFPVs with some parameters set to
 56 zero. Although the former method is an extensive search, the latter is usually better suited in
 57 practice as it yields the TFPVs one is typically interested in explicitly, is more efficient, and
 58 directly outputs the corresponding slow manifolds (implicitly as affine varieties in phase space).

59 Computing reductions

60 Computing a reduction for a slow-fast separation of rates as in Theorem 1 in [Goeke & Walcher](#)
 61 ([2014](#)) requires essentially two steps. First, we need to provide a parametric representation of
 62 the slow manifold, which is given as an irreducible component of the affine variety $\mathcal{V}(f^{(0)})$.
 63 Then, we need to find a product decomposition $f^{(0)} = P\psi$, where P is a $n \times r$ matrix of
 64 rational functions and ψ is a vector of polynomials locally satisfying $\mathcal{V}(\psi) = \mathcal{V}(f^{(0)})$ and
 65 $\text{rank } P = \text{rank } D\psi = r$.

66 With this, the reduced system in the sense of Tikhonov is given by

$$\dot{x} = [1_n - P(x)(D\psi(x)P(x))^{-1}D\psi(x)] f^{(1)}(x).$$

67 For convenience, the packages provides multiple heuristics, which automate the steps required
 68 to compute a reduction and allows bulk computation of multiple reductions at once.

69 Integration with the Julia ecosystem

70 The input system can be given as a reaction network defined with Catalyst.jl ([Loman et al.,](#)
 71 [2023](#)). Because the reduced systems are represented using types from Oscar.jl, the latter's
 72 functions can be used to aid the symbolic analysis. Julia's support for metaprogramming
 73 allows to perform further tasks such as a numerical analysis without having to copy or parse
 74 code (see e.g. [TFRSimulations.jl](#)). For convenience, there are several methods for displaying
 75 the output, including printing as LATEX source code via [Latexify.jl](#).

76 Research impact statement

77 Time scale separations are widely used in various areas of mathematical modelling
78 ([Wechselberger, 2020](#)). However, as far as the author is aware, the systematic approach
79 due to Goeke and Walcher seems to be scarcely adopted, even though it comes with many
80 advantages compared to the ad-hoc approach. This package aims to make the theory more
81 accessible and convenient to use.

82 In the field of mathematical ecology (which the author is most familiar with), the theory was
83 successfully used by Kruff et al. ([2019](#)) and more recently by Apelt & Liebscher ([2025](#)). For
84 the model introduced in the former, the application was relatively straightforward, but more
85 complex models as in the latter require some more work. In particular, the method for finding
86 TFPVs that relies on the computation of a Gröbner Basis may fail for complex systems. In
87 this case, `TikhonovFenichelReductions.jl` enables and simplifies the search for the most
88 common TFPVs. Given the ubiquity of time scale separation techniques in this field alone
89 (see e.g. [Abbott et al., 2020](#); [Poggiale & Auger, 2004](#); [Revilla, 2015](#)), the package can be a
90 potentially useful tool for modellers.

91 AI usage disclosure

92 No generative AI tools were used in the development of this software, the writing of this
93 manuscript, or the preparation of supporting materials.

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