

# <sup>1</sup> GRTresna: An open-source code to solve the initial data constraints in numerical relativity

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Authors of papers retain copyright and release the work under a Creative Commons Attribution 4.0 International License ([CC BY 4.0](#))<sup>33</sup>. Numerical relativity (NR) is a tool for the solution of the Einstein Equations, which describe gravity in strong field regimes. The equations can be expressed as a set of coupled partial differential equations (PDEs) for the 10 metric quantities  $g_{\mu\nu}$  and their time derivatives  $\partial_t g_{\mu\nu}$ . NR is primarily focussed on the hyperbolic PDEs that describe their time evolution from an initial data set, but the initial data itself must satisfy a set of four coupled non-linear elliptic PDEs known as the Hamiltonian and momentum constraints. Whilst these constraints can be solved more straightforwardly by making certain assumptions, this significantly restricts the range of physical scenarios that can be studied. A general solver therefore expands the physics that NR evolutions can be used to probe.

## Summary

In the ADM form of the Einstein Equations ([Arnowitt et al., 2008](#)), we slice the spacetime into 3-dimensional hypersurfaces

$$ds^2 = -(\alpha^2 - \beta_i \beta^i) dt^2 + 2\beta_i dx^i dt + \gamma_{ij} dx^i dx^j$$

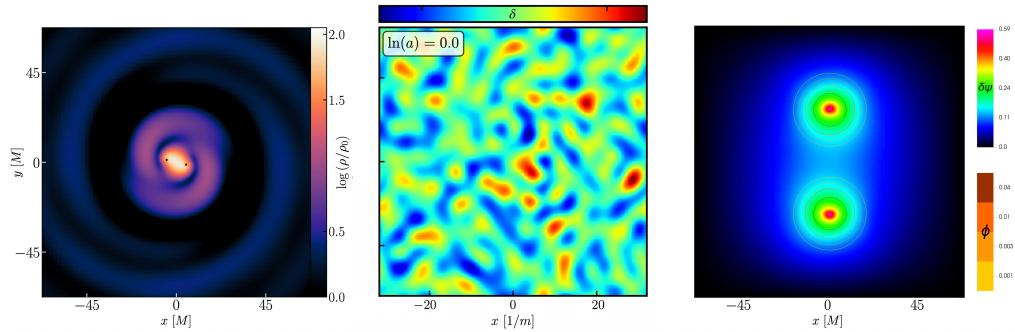
where  $\alpha$  and  $\beta^i$  are the lapse and shift functions, respectively. The Hamiltonian and momentum constraints are expressed as

$$H \equiv R + K^2 - K_{ij} K^{ij} - 16\pi\rho = 0,$$

$$M_i \equiv D^j(K_{ij} - \gamma_{ij}K) - 8\pi S_i = 0.$$

39 Here,  $\gamma_{ij}$  is the 3-metric of the hypersurface,  $R$  is the Ricci scalar associated to this metric,  
40 and  $K_{ij} \sim \partial_t \gamma_{ij}$  is the extrinsic curvature tensor, with  $K = \gamma^{ij} K_{ij}$  its trace. The decomposed  
41 components of the stress-energy tensor of matter  $T^{\mu\nu}$  (measured by normal observers) are  
42 defined as  $\rho = n_\mu n_\nu T^{\mu\nu}$  and  $S_i = -\gamma_{i\mu} n_\nu T^{\mu\nu}$ , where  $n_\mu = (-\alpha, 0, 0, 0)$ . These equations  
43 constitute the set of four PDEs to be solved. There are 16 unknowns: 6 in  $\gamma_{ij}$ , 6 in  $K_{ij}$ ,  
44 1 in  $\rho$  and 3 in  $S_i$ . Usually the matter configuration is set by the physical scenario, which  
45 determines  $\rho$  and  $S_i$ . The constraints only determine 4 quantities, and the remaining 8 (4  
46 of which are physical degrees of freedom, and 4 gauge choices) must be chosen according to  
47 physical principles or knowledge about the system. In this short paper, we introduce GRTresna,  
48 an open-source code to solve these equations.

49 The two main methods for finding initial conditions in numerical relativity are the *conformal*  
50 *transverse-traceless* (CTT) and the *conformal thin sandwich* (CTS) approaches. We  
51 refer the reader to the standard NR texts (Alcubierre, 2008; Baumgarte & Shapiro,  
52 2010, 2021; Gourgoulhon, 2007; Shibata, 2016) for more details about these. GRTresna  
53 implements two variations of the CTT method recently introduced in Aurrekoetxea, Clough  
54 & Lim (Aurrekoetxea, Clough, & Lim, 2023): the CTTK and CTTK-Hybrid methods,  
55 which are particularly well-suited to cases with fundamental fields in the matter content.  
56 Documentation about using and modifying GRTresna can be found in the code wiki  
57 <https://github.com/GRTLCollaboration/GRTresna/wiki>.



58  
59 Some highlights of work using GRTresna to date: (Left:) Dark matter around binary black  
60 holes, from (Aurrekoetxea, Clough, Bamber, et al., 2024; Aurrekoetxea, Marsden, et al., 2024;  
61 Bamber et al., 2023) (Middle:) Evolution of inflationary perturbations during preheating, from  
62 (Aurrekoetxea, Clough, & Muia, 2023). (Right:) Scalar fields around black holes in 4DST  
63 gravity, from (Brady et al., 2023)

## 64 Key features of GRTresna

65 The key features of GRTresna are as follows

- 66     ▪ Flexibility: GRTresna is designed to be extended to various physical scenarios, including  
67       different matter types and gravitational theories beyond GR. It currently supports  
68       cosmological-type periodic spacetimes and a superposition of two boosted and/or spinning  
69       black holes (Bowen-York initial data), with fully general scalar field matter source  
70       configurations and the flexibility to adapt to other setups. While scalar fields are the  
71       only matter sources included in the current version of the code, the templated methods  
72       allow users to easily replace them with other matter types by copying the scalar field  
73       implementation and modifying the methods to compute the corresponding energy and  
74       momentum densities.
- 75     ▪ Methods: GRTresna incorporates the CTTK and CTTK-Hybrid methods to solve the

76        Hamiltonian and momentum constraints. These methods offer several advantages when  
77        dealing with fundamental fields, as discussed in (Aurrekoetxea, Clough, & Lim, 2023).  
78        The method code is also templated, so users can easily implement their preferred methods.

- 79        ▪ Initial conditions: The code supports analytical initial data for the matter fields, as well  
80        as the option to read grids and data from an existing HDF5 file. This functionality  
81        is especially useful when combined with our code that evolves matter on fixed metric  
82        backgrounds GRDzhadzha (Aurrekoetxea, Bamber, et al., 2024), meaning that we can  
83        upgrade the resulting matter configurations to full NR simulations with backreaction.
- 84        ▪ Boundary conditions: The code implements extrapolating, reflective, and periodic  
85        boundary conditions, compatible with those in the NR evolution code GRChombo  
86        (Andrade & others, 2021; Clough et al., 2015).
- 87        ▪ Diagnostics: The code computes the Hamiltonian and momentum constraint errors at  
88        each iteration step, and outputs the norm of these values across the grid to a text file.
- 89        ▪ Compatibility: As GRTresna is developed on top of Chombo, the solver is primarily  
90        designed to be compatible with GRChombo (Andrade & others, 2021; Clough et al.,  
91        2015) and the family of codes developed by the GRTL Collaboration. We provide two  
92        examples that integrate directly with existing examples in the GRChombo evolution code  
93        (via the output of a checkpoint file for restart at  $t = 0$ ), and provide guidance and tools  
94        to validate the results. However, the code outputs data in the standard HDF5 data  
95        format, which should be straightforward to adapt to other NR codes that support HDF5  
96        input or can be accessed using Python.

97        Other features that are inherited from Chombo include

- 98        ▪ C++ class structure: GRTresna is written in the C++ language, and makes heavy use  
99        of object-oriented programming (OOP) and templating.
- 100        ▪ Parallelism: GRTresna uses hybrid OpenMP/MPI parallelism.
- 101        ▪ Adaptive Mesh Refinement: The code inherits the flexible AMR grid structure of Chombo,  
102        with block-structured Berger-Rigoutsos grid generation (Berger & Rigoutsos, 1991). The  
103        tagging of refinement regions is fully flexible and while it is based on the sources of the  
104        elliptic equations by default, other user-defined measures can be defined (Radia et al.,  
105        2022).
- 106        ▪ Fast: The code uses a multigrid method to efficiently reduce errors across a hierarchy  
107        of discretizations, enabling the solver to achieve rapid convergence while minimizing  
108        computational costs. This makes GRTresna highly optimized for handling the demanding  
109        computations of initial data in the presence of AMR.

110        Forthcoming features currently under development include the addition of other methods, in  
111        particular the Extended Conformal Thin Sandwich (XCTS) method, non-conformally flat metric  
112        data, new matter types including vector fields, the modified scalar-tensor gravity formalism  
113        of Brady et. al. (Brady et al., 2023) and dimensional reduction to 2D using the cartoon  
114        formalism (Alcubierre et al., 2001; Cook et al., 2016).

## 115        Statement of Need

116        There are a number of existing initial data solvers for numerical relativity, most of which are  
117        primarily designed to solve for initial conditions in compact object mergers (i.e. neutron stars  
118        and black holes). These include TwoPunctures (Ansorg et al., 2004), SGRID (Tichy, 2009),  
119        BAM (Bruegmann et al., 2008), LORENE (Gourgoulhon et al., 2001; *LORENE Website*, n.d.),  
120        Spells (Pfeiffer et al., 2003), SpECTRE [Vu & others (2022); Vu (2024); deppe\_2023\_8196313]  
121        ((Nee et al., 2025) for modified gravity) COCAL (Tsokaros et al., 2015; Tsokaros & Uryu,  
122        2012; Uryu & Tsokaros, 2012), PCOCAL (Boukas et al., 2024), Elliptica (Rashti et al.,

123 2022), NRPyElliptic (Assumpcao et al., 2022, p. Tootle:2025ikk), KADATH/FUKA ([FUKA](#)  
124 [Website](#), n.d.; Grandclement, 2010; Papenfort et al., 2021}), SPHINCS\_ID (Diener et al.,  
125 2022; Rosswog et al., 2023; [SPHINCS\\_ID Website](#), n.d.), and the solver of East et al. (East et  
126 al., 2012). Many of these codes, particularly those using spectral methods like TwoPunctures  
127 and SpECTRE, provide a higher accuracy in the solution compared to GRTresna, which is  
128 limited to second order accuracy by the multigrid method used. They are therefore better  
129 suited to initial data for waveform generation where precision is key. GRTresna is, however,  
130 designed to be more flexible and general purpose, tackling both cosmological and black hole  
131 spacetimes in a range of scenarios beyond GR and the Standard Model.

132 In particular, to the best of our knowledge, there is no fully general, publicly available initial  
133 condition solver for inhomogeneous cosmological spacetimes. One exception is FLRWSolver,  
134 developed by Macpherson et al. (Macpherson et al., 2017) as part of the Einstein Toolkit  
135 (Loffler & others, 2012), which specializes in initializing data for cosmological perturbations  
136 arising from inflation for studies of late-time cosmology. However, this is limited to only weakly  
137 non-linear initial data. GRTresna aims to provide an open-source tool that not only incorporates  
138 the general features of existing initial data solvers for compact objects in GR but also extends  
139 their capabilities to cosmological spacetimes (see (Aurrekoetxea, Clough, & Lim, 2024) for  
140 a review of the application of numerical relativity in cosmology). GRTresna is particularly  
141 well-suited for fundamental field matter types, such as scalar and vector fields. Its flexible design  
142 allows users to implement new solver methods, additional matter types, or extend the code to  
143 study theories beyond GR. It is fully compatible with the GRTL Collaboration's ecosystem of  
144 codes but can also serve as a complementary tool for generating constraint-satisfying initial  
145 data for other numerical relativity codes.

## 146 Key research projects using GRChombo

147 The code has already been used successfully to study a range of problems in fundamental physics,  
148 including: - The robustness of inflation to inhomogeneities in the scalar field (Aurrekoetxea  
149 et al., 2020; Elley et al., 2025). - The formation of oscillons during inflationary preheating  
150 (Aurrekoetxea, Clough, & Muia, 2023). - Formation of spinning primordial black holes (De  
151 Jong et al., 2023). - The effect of scalar dark matter environments around binary black holes  
152 (Aurrekoetxea, Clough, Bamber, et al., 2024; Aurrekoetxea, Marsden, et al., 2024; Bamber et  
153 al., 2023). - The general relativistic evolution of polarized Proca stars (Wang et al., 2024). -  
154 Solving the initial conditions problem for modified gravity theories (Brady et al., 2023).

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