QMCPy: A Python Framework for (Quasi-)Monte Carlo Algorithms

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Summary

Monte Carlo (MC) methods estimate high-dimensional integrals by computing sample averages at independent and identically distributed (IID) random points. Quasi-Monte Carlo (QMC) methods replace IID samples with low-discrepancy (LD) sequences which more uniformly cover the integration domain, leading to faster convergence and reduced computational requirements. Figure 1 visualizes IID and LD sequences.

`QMCPy` (https://qmcsoftware.github.io/QMCSoftware) (Choi et al., 2026) is our Python package for high-dimensional numerical integration using MC and QMC methods, collectively "(Q)MC." Its object-oriented design enables researchers to easily implement novel (Q)MC algorithms. The framework offers user-friendly APIs, diverse (Q)MC algorithms, adaptive error estimation techniques, and integration with scientific libraries following reproducible research practices (Choi et al., 2022; Choi, 2014). Compared to previous versions, `QMCPy` v2.2 (which is easily installed with `pip install -U qmcpy`) includes

- improved documentation,
- strengthened tests and demos, and
- expanded support for randomized LD sequences.

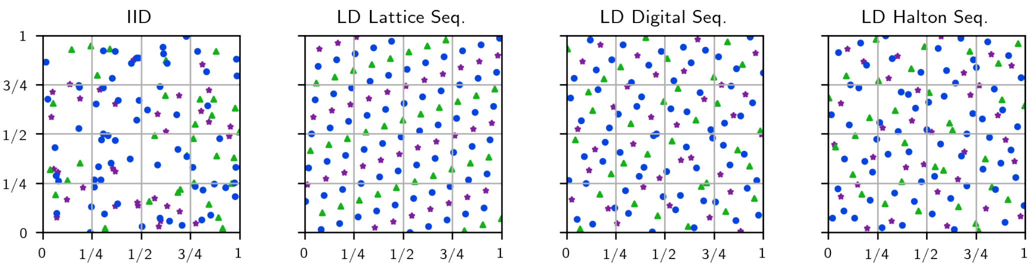

Figure 1: An IID sequence with gaps and clusters alongside LD sequences with more uniform coverage. Each sequence contains purple stars (initial 32 points), green triangles (next 32), and blue circles (subsequent 64).

Statement of Need

(Q)MC methods are essential for computational finance (Giles & Waterhouse, 2009; Lemieux, 2004; X. Wang & Sloan, 2005; Zhang et al., 2021), uncertainty quantification (Kaarnioja et al., 2021; Marzouk et al., 2016; Parno et al., 2014, 2021; Seelinger et al., 2023), machine learning (Chen et al., 2018; Dick & Feischl, 2021), and physics (Albert & Barabási, 2002; Bernhard et al., 2015; Landau & Binder, 2014). While (Q)MC methods are well established (Dick et al., 2013; Dick & Pillichshammer, 2010), practical implementation demands numerical and algorithmic expertise. `QMCPy` follows MATLAB's Guaranteed Automatic Integration Library (GAIL) (Choi et al., 2021; Tong et al., 2022) in consolidating a broad range of cutting-edge (Q)MC algorithms into a unified framework (Choi et al., 2022, 2024; Hickernell et al., 2026; Sorokin, 2025; Sorokin & Rathinavel, 2022). `QMCPy` features

- **intuitive APIs** for (Q)MC components,
- **flexible integrations** with `NumPy` (Harris et al., 2020), `SciPy` (Virtanen et al., 2020), and `PyTorch` (Paszke et al., 2019),
- **robust and adaptive sampling** with theoretically grounded error estimation, and
- **extensible components** enabling researchers to implement and test new algorithms.

While popular modules like `scipy.stats.qmc` (Roy et al., 2023) and `torch.quasirandom` (Paszke et al., 2019) provide basic (Q)MC sequences such as Sobol' and Halton, `QMCPy` provides (Q)MC researchers and practitioners an end-to-end framework with additional capabilities to enable state-of-the-art (Q)MC techniques. Advanced features unique to `QMCPy` include

- customizable LD sequences with diverse randomization techniques,
- efficient generators of LD sequences with multiple independent randomizations,
- automatic variable transformations for (Q)MC compatibility, and
- rigorous adaptive error estimation algorithms.

Components

(Q)MC methods approximate the multivariate integral

$$\mu := \mathbb{E}[g(\mathbf{T})] = \int_{\mathcal{T}} g(\mathbf{t})\, \lambda(\mathbf{t})\, d\mathbf{t}, \qquad \mathbf{T} \sim \lambda, \tag{1}$$

where g is the **integrand** and λ is the probability density of a random variable $\mathbf{T}$ whose distribution we call the **true measure**. To accommodate LD samples (approximately uniform on $[0,1]^d$), a transformation $\boldsymbol{\psi}$ is performed to rewrite μ as

$$\mu = \mathbb{E}[f(\mathbf{X})] = \int_{[0,1]^d} f(\mathbf{x})\, d\mathbf{x}, \qquad \mathbf{X} \sim \mathcal{U}[0,1]^d. \tag{2}$$

If $\mathbf{T} \sim \boldsymbol{\psi}(\mathbf{X})$, then $f = g \circ \boldsymbol{\psi}$.

(Q)MC methods estimate the population mean μ in (2) via the sample mean

$$\hat{\mu} := \frac{1}{n} \sum_{i=1}^{n} f(\mathbf{X}_i). \tag{3}$$

MC methods use IID $\mathbf{X}_1, \dots, \mathbf{X}_n$ and have error $|\hat{\mu} - \mu|$ like $\mathcal{O}(n^{-1/2})$ (Niederreiter, 1978). QMC methods choose dependent LD nodes that fill $[0,1]^d$ more evenly, i.e., the discrepancy between the **discrete distribution** of $\mathbf{X}_1, \dots, \mathbf{X}_n$ and the uniform distribution is small. QMC methods can achieve errors like $\mathcal{O}(n^{-1+\delta})$ where $\delta > 0$ is arbitrarily small (Hickernell & Wang, 2002; X. Wang, 2003). A key feature of `QMCPy` is **stopping criteria** that automatically determine n so $|\mu - \hat{\mu}| \le \varepsilon$ for a user-specified tolerance $\varepsilon > 0$, deterministically or with high probability.

`QMCPy` contains four main abstract classes:

1. **Discrete Distributions** generate IID or randomized LD sequences (Sorokin, 2025) including
    - **Lattices** with random shifts (Coveyou & MacPherson, 1967; Cranley & Patterson, 1976; Hickernell et al., 2005; Richtmyer, 1951; Y. Wang & Hickernell, 2002).
    - **Digital Sequences** (including Sobol' and Faure constructions) with digital shifts (DS), linear matrix scrambling (LMS), or nested uniform scrambling (NUS, also

called Owen scrambling) (Dick, 2011; Dick & Pillichshammer, 2005; Dick & Pillichshammer, 2010; Matoušek, 1998; Niederreiter, 1987, 1992; Owen, 1995, 2003; Sobol', 1967). Higher-order digital sequences are available to enable QMC convergence like $\mathcal{O}(n^{-\alpha+\delta})$ when f has α degrees of smoothness (Dick, 2011).
- **Halton Sequences** with digital permutations, DS, LMS, or NUS (Halton, 1960; Matoušek, 1998; Morokoff & Caflisch, 1994; Owen & Pan, 2024; X. Wang & Hickernell, 2000).

Internally, `QMCPy`'s LD generators call our C package `QMCToolsCL` (Sorokin, 2026). We also integrate with the `LDData` repository (Sorokin et al., 2025) which collects lattice generating vectors and digital sequence generating matrices from Kuo's websites (Cools et al., 2006; Joe & Kuo, 2003, 2010; Joe & Kuo, 2008; Kuo, 2007; Nuyens & Cools, 2006), the `Magic Point Shop` (Kuo & Nuyens, 2016), and `LatNet Builder` (L'Ecuyer et al., 2022).

2. **True Measures** come with default transformations ψ satisfying $\psi(\mathbf{X}) \sim \mathbf{T}$. For example, if $\mathbf{T} \sim \mathcal{N}(\mathbf{m}, \Sigma = \mathbf{A}\mathbf{A}^T)$ is a d-dimensional Gaussian, then $\psi(\mathbf{X}) = \mathbf{A}\Phi^{-1}(\mathbf{X}) + \mathbf{m}$ where Φ^{-1} is the inverse Gaussian distribution function applied elementwise. We support the broad range of measures included in `scipy.stats` (Virtanen et al., 2020).

3. **Integrands** g, given a transformation ψ, automatically set $f = g \circ \psi$ so that $\mu = \mathbb{E}[g(\mathbf{T})] = \mathbb{E}[f(\mathbf{X})]$.

4. **Stopping Criteria (SC)** adaptively increase the sample size n until (Q)MC estimates satisfy user-defined error tolerances (Hickernell, Choi, et al., 2018; Owen, 2024; Tong et al., 2022). SC include guaranteed MC algorithms (Hickernell et al., 2013) and QMC algorithms based on:

   - multiple randomizations of LD sequences (L'Ecuyer et al., 2023),
   - quickly tracking the decay of Fourier coefficients (Ding et al., 2020; Hickernell, Jiménez Rugama, et al., 2018; Hickernell & Jiménez Rugama, 2016; Jiménez Rugama & Hickernell, 2016), or
   - fast Bayesian cubature (Rathinavel, 2019; Rathinavel & Hickernell, 2019, 2022).

`QMCPy` is also capable of simultaneously approximating functions of multiple integrands (Sorokin & Rathinavel, 2022), and we are actively expanding support for multilevel (Q)MC algorithms following Julia's `MultilevelEstimators.jl` (Robbe, 2024).

Figure 2 compares (Q)MC SC for Asian option pricing with 100 independent trials per error tolerance ε. The left and middle plots show median lines and shaded regions for 10%–90% quantiles. While MC SC require $n = \mathcal{O}(1/\varepsilon^2)$ samples (and time), QMC SC require only $n = \mathcal{O}(1/\varepsilon)$. (Q)MC SC consistently meet tolerances, with the right plot showing distributions of errors for a single error tolerance.

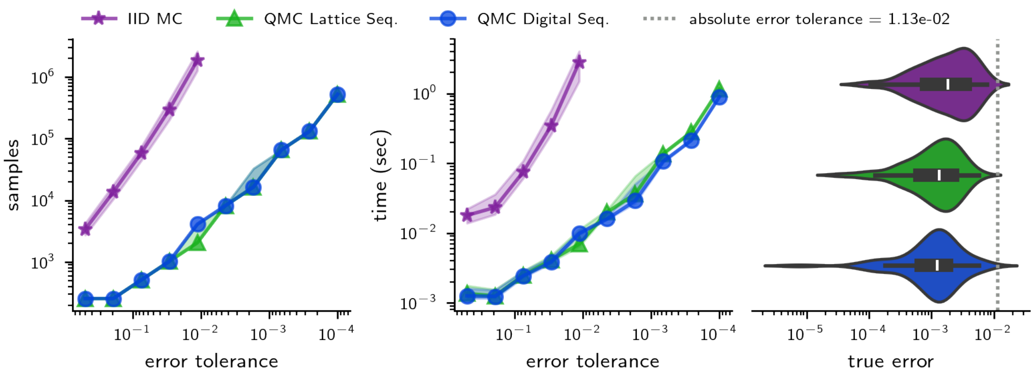

Figure 2: (Q)MC SC for Asian option pricing.

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