




tukey_hsd: An Accurate Implementation of the Tukey Honestly Significant Difference Test in Python

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Summary

In a world awash with data and computers, it is tempting to automate the process of scientific discovery by performing comparisons between many pairs of variables in the hope of finding correlations. When frequentist hypothesis tests are performed at a fixed confidence level, increasing the number of tests increases the probability of observing a “statistically significant” result, even when the null hypothesis is actually true. Carefully-designed tests, such as Tukey’s honestly significant difference (HSD) test ([Tukey, 1949](#)), protect against this practice of “p-hacking” by producing p-values and confidence intervals that account for the number of comparisons performed. Several such tests rely on the studentized range distribution ([Lund & Lund, 1983](#)), which models the range (i.e., the difference between maximum and minimum values) of the means of samples from a normally distributed population. Although there are already implementations of these tests available in the scientific Python ecosystem, all of them rely on approximations of the studentized range distribution, which are not accurate outside the range of inputs for which they are designed. Here we present a very accurate and sufficiently fast implementation of the studentized range distribution and a function for performing Tukey’s HSD test. Both of these features are available in SciPy 1.8.0.

Performance

The most computationally-challenging part of implementing Tukey’s HSD test is the evaluation of the cumulative distribution function of the studentized range distribution, which is given by

$$F(q; k, \nu) = \frac{k\nu^{\nu/2}}{\Gamma(\nu/2)2^{\nu/2-1}} \int_0^\infty \int_{-\infty}^\infty s^{\nu-1} e^{-\nu s^2/2} \phi(z) [\Phi(sq + z) - \Phi(z)]^{k-1} dz ds$$

where q is the studentized range, k is the number of groups, ν is the number of degrees of freedom used to determine the pooled sample variance, and $\phi(z)$ and $\Phi(z)$ represent the normal probability density function and normal cumulative distribution function, respectively. There is no closed-form expression for this integral, and numerical integration requires care, as naive evaluation of the integrand results in overflow even for modest values of the parameters. Consequently, other packages in the open-source scientific Python ecosystem, such as statsmodels ([Seabold & Perktold, 2010](#)) and Pingouin ([Vallat, 2018](#)), have relied on interpolation between tabulated values. To satisfy the need for a more accurate implementation of this integral, we contributed `scipy.stats.studentized_range` ([Chmiel et al., 2021](#)), a class that evaluates the cumulative distribution function and many other functions of the distribution.

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To address numerical difficulties, ratios in the integrand are implemented by exponentiating the difference in the logarithms of the individual terms. Consequently, `scipy.stats.studentized_range` works well for ν through at least 10^5 . The integrand is written in Cython and integrated using `scipy.integrate.nquad`, resulting in evaluation times on the order of 5 ms on modern computers. Compared to a reference implementation evaluated using arbitrary precision arithmetic, the relative error in the cumulative distribution function is typically on the order of 10^{-14} : six orders of magnitude better than a commonly used implementation in R, and ten orders of magnitude better than the approximation provided by `statsmodels`. A thorough assessment of the methods, accuracy, and speed of the underlying calculations is available in (Wallan et al., 2021a), and an extensive test suite included in SciPy guards against regressions.

Statement of need

When analysis of variance (ANOVA) indicates that there is a statistically significant difference between at least one pair of groups in an experiment, researchers are often interested in *which* of the differences is statistically significant. Researchers use “post-hoc tests” to study these pairwise differences while controlling the experiment-wise error rate. Until recently, no post-hoc tests were available in SciPy (Virtanen et al., 2020), the de-facto standard library of fundamental algorithms for scientific computing in Python. To fill this gap, we contributed `scipy.stats.tukey_hsd` (Wallan et al., 2021b), a function for performing Tukey’s HSD test. Also, there was no accurate implementation of the underlying studentized range distribution in Python, so we contributed `scipy.stats.studentized_range`. Both `statsmodels` and `Pingouin` have since adopted this class to perform studentized range distribution calculations.

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