

¹ Sparsity-preserving gradient utility tools for PyTorch

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⁴ Summary

⁵ The `torchsparsegradutils` package provides differentiable sparse linear-algebra utilities for PyTorch ([Paszke et al., 2019](#)) that preserve sparsity for returned gradients during backpropagation. While PyTorch directly supports sparse tensors, its default semantics treat sparse layouts as storage optimisations rather than a mathematical structure that results in optimising directly for that sparse subspace. Gradients resulting from PyTorch native functions are often dense and incompatible with end-to-end training of models that require fixed sparsity patterns (e.g., sparse covariance/precision structures).

¹² To address this limitation, we introduce `torchsparsegradutils`. Key features include: (1) memory-efficient sparse-dense matrix multiplication with sparse gradient preservation, (2) sparse triangular and generic linear system solvers, enabling sparse gradients during backpropagation, and multiple algorithmic backends (BICGSTAB, CG, LSMR, MINRES), (3) cross-platform sparse solver wrappers for CuPy ([Okuta et al., 2017](#)) and JAX ([Bradbury et al., 2018](#)), (4) sparse multivariate normal distributions with LL^T and LDL^T sparse covariance and precision matrix parameterisations with reparameterised sampling methods, and (5) specialised encoders for spatial neighbourhood relationships in N-dimensional data.

²⁰ Statement of need

²¹ Many scientific machine learning models benefit from representing large linear operators (e.g., neighbourhood couplings, precision matrices, sparse Jacobians) using sparse tensors to reduce memory and compute complexity. In high-dimensional settings, such as volumetric medical imaging, dense covariance or precision parameterisations are typically intractable, motivating sparse end-to-end parameterisations.

²⁶ However, learning these models with gradient-based optimisation requires backpropagation through sparse linear algebra (matrix products, triangular solves, and linear system solves). PyTorch's default sparse semantics are not designed to preserve user-imposed sparsity structure during differentiation (PyTorch issue #87448), which can lead to memory blow-ups and prevent end-to-end optimisation of sparse probabilistic models.

³¹ `torchsparsegradutils` addresses this gap by implementing custom autograd functions for key sparse operators that return gradients only for stored nonzeros, enabling practical optimisation of models that rely on fixed sparse structure, such as sparse multivariate normal distributions with sparse covariance/precision factors.

³⁵ State of the field

³⁶ PyTorch ([Paszke et al., 2019](#)) exposes sparse layouts (COO, CSR, and related formats) and implements a growing set of sparse operations. However, PyTorch's design goal is *dense-equivalent semantics* for sparse layouts: a guiding invariant is that applying an operation in

39 sparse form should match applying it in dense form after conversion, including the backward
40 function (PyTorch issue #87448). This makes it difficult to learn parameters that are intended
41 to remain structurally sparse, because gradients may be produced for implicit zeros, or
42 intermediate computations may densify.

43 PyTorch also provides MaskedTensor, distinguishing specified and unspecified elements in
44 tensors and is conceptually closer to the constrained-subspace interpretation of sparsity.
45 However, MaskedTensor remains at prototype stage with incomplete operator coverage, and
46 storing a full boolean mask incurs a significant memory overhead, partially negating the memory
47 benefits of sparse index-based representations for large-scale problems.

48 Other libraries provide efficient sparse kernels but do not directly solve “sparsity-preserving
49 gradients in PyTorch”: SciPy ([Virtanen et al., 2020](#)) provides mature sparse linear algebra
50 but no automatic differentiation; CuPy ([Okuta et al., 2017](#)) and JAX ([Bradbury et al., 2018](#))
51 provide sparse solvers in their respective ecosystems but are not drop-in components for
52 PyTorch autograd/training loops. GPyTorch ([Gardner et al., 2018](#)) targets scalable Gaussian
53 process inference via kernel structure and approximations (e.g., inducing/structured methods)
54 rather than arbitrary user-specified sparse covariance/precision factors. PyTorch Geometric’s
55 `torch_sparse` ([Fey & Lenssen, 2019](#)) focuses on graph message-passing primitives rather than
56 sparse covariance/precision modelling and differentiable sparse solves for probabilistic models.

57 Software design

58 `torchsparsegradutils` is built around `torch.autograd.Function` operators that wrap
59 PyTorch’s forward sparse kernels but override the backward pass to preserve sparsity for
60 selected inputs. This design keeps the user-facing API close to standard PyTorch code while
61 making sparsity preservation an explicit, opt-in choice.

62 Two design trade-offs shaped the implementation. First, the package targets *structure-*
63 *preserving learning* over maximal operator coverage, as only a focused set of operations
64 (sparse matrix products, triangular solves, generic sparse solvers) are implemented, but
65 these are sufficient to support sparse multivariate normal sampling and sparse solver-based
66 models. Second, for broad device/backend compatibility, the package combines native PyTorch
67 implementations (iterative Krylov solvers: CG, BiCGSTAB, LSMR, MINRES) with optional
68 wrappers to external libraries (CuPy, JAX), allowing users to trade off portability versus
69 performance.

70 **Build vs. contribute justification.** PyTorch’s current semantics treat sparse layouts as
71 performance optimisations and prioritise the dense-equivalence invariant (PyTorch issue
72 #87448). In contrast, this package intentionally provides *structure-preserving* backward passes
73 for specific operators to enable learning with fixed sparsity patterns (e.g., sparse triangular
74 factors for covariance/precision). This difference is semantic (not just implementation), so the
75 functionality is better delivered as an opt-in external library rather than changing PyTorch’s
76 default behaviour.

77 Research impact statement

78 This software provides an opt-in path to sparsity-preserving gradients for sparse linear algebra
79 in PyTorch, enabling research prototypes that would otherwise be limited by dense gradients
80 or densification. The package is currently being used in active research projects for medical
81 image segmentation, though publications resulting from this work are still in preparation.

82 The codebase demonstrates community-readiness through comprehensive infrastructure:
83 documentation with quickstart guides and API references, extensive test coverage across
84 all modules, CI/CD pipelines for automated testing, and an open contribution process via
85 GitHub issues and pull requests. The codebase has been developed openly over multiple

⁸⁶ years with public commit history, releases, and issue tracking. Benchmark suites comparing
⁸⁷ solver performance across problem sizes and sparsity patterns provide reproducible reference
⁸⁸ materials.

⁸⁹ Given the broad applicability of sparse structured Gaussians—spanning medical imaging,
⁹⁰ spatial statistics, geostatistics, and large-scale probabilistic modelling, we anticipate growing
⁹¹ adoption as the research community increasingly requires memory-efficient optimisation of
⁹² high-dimensional probabilistic models.

⁹³ Mathematics

⁹⁴ Sparse Matrix Operations

⁹⁵ Sparse Matrix Multiplication

⁹⁶ The package implements sparse-dense matrix multiplication $\mathbf{C} = \mathbf{AB}$ where $\mathbf{A} \in \mathbb{R}^{m \times n}$ is
⁹⁷ sparse and $\mathbf{B} \in \mathbb{R}^{n \times p}$ is dense. The forward pass uses PyTorch's native `torch.sparse.mm`,
⁹⁸ while the backward pass is reimplemented to preserve sparsity patterns in the gradients.

⁹⁹ Given upstream gradient $\frac{\partial \mathcal{L}}{\partial \mathbf{C}} \in \mathbb{R}^{m \times p}$ from some scalar objective function \mathcal{L} , the chain rule
¹⁰⁰ gives:

¹⁰¹ **Gradient wrt \mathbf{B} (dense):**

$$\frac{\partial \mathcal{L}}{\partial \mathbf{B}} = \mathbf{A}^\top \frac{\partial \mathcal{L}}{\partial \mathbf{C}}.$$

¹⁰² **Gradient wrt \mathbf{A} (sparse):** For a nonzero entry \mathbf{A}_{ij} ,

$$\frac{\partial \mathcal{L}}{\partial \mathbf{A}_{ij}} = \sum_{k=1}^p \left(\frac{\partial \mathcal{L}}{\partial \mathbf{C}_{ik}} \right) \mathbf{B}_{jk}.$$

¹⁰³ Sparse Linear System Solvers

¹⁰⁴ The package provides multiple approaches for solving sparse linear systems

$$\mathbf{Ax} = \mathbf{B},$$

¹⁰⁵ where $\mathbf{A} \in \mathbb{R}^{n \times n}$ is sparse, $\mathbf{B} \in \mathbb{R}^{n \times p}$ is dense (with $p = 1$ for a single right-hand side), and
¹⁰⁶ $\mathbf{x} \in \mathbb{R}^{n \times p}$ is the dense solution. We support both direct triangular solves and iterative solvers
¹⁰⁷ (CG, BiCGSTAB, LSMR, MINRES). All are differentiable via the implicit function theorem.

¹⁰⁸ Given upstream gradient $\frac{\partial \mathcal{L}}{\partial \mathbf{x}} \in \mathbb{R}^{n \times p}$:

¹⁰⁹ **Gradient wrt \mathbf{b} (dense):**

$$\frac{\partial \mathcal{L}}{\partial \mathbf{b}} = \mathbf{A}^{-\top} \frac{\partial \mathcal{L}}{\partial \mathbf{x}}.$$

¹¹⁰ **Gradient wrt \mathbf{A} (sparse):** The dense form would be

$$\frac{\partial \mathcal{L}}{\partial \mathbf{A}} = - \left(\mathbf{A}^{-\top} \frac{\partial \mathcal{L}}{\partial \mathbf{x}} \right) \mathbf{x}^\top.$$

¹¹¹ For a nonzero entry \mathbf{A}_{ij} this becomes

$$\frac{\partial \mathcal{L}}{\partial \mathbf{A}_{ij}} = - \sum_{k=1}^p \left(\mathbf{A}^{-\top} \frac{\partial \mathcal{L}}{\partial \mathbf{x}} \right)_{ik} \mathbf{x}_{jk}$$

112 Sparse Multivariate Normal Distributions

113 The package implements sparse multivariate normal distributions $\eta \sim \mathcal{N}(\mu, \Sigma) \equiv \mathcal{N}(\mu, \Omega^{-1})$
 114 with two parameterisations for efficient sampling. Both methods transform standard normal
 115 samples $\epsilon \sim \mathcal{N}(0, I)$ into samples from the desired multivariate normal distribution:

116 **LL^\top Parameterisation:** Sparse lower triangular matrices L with positive diagonals:

$$\eta_{\Sigma} = \mu + L_{\Sigma}\epsilon, \quad \eta_{\Omega} = \mu + L_{\Omega}^{-T}\epsilon$$

117 **LDL^\top Parameterisation:** Sparse unit lower triangular matrices L and diagonal D :

$$\eta_{\Sigma} = \mu + LD^{1/2}\epsilon, \quad \eta_{\Omega} = \mu + L_{\Omega}^{-T}D_{\Omega}^{-1/2}\epsilon$$

118 We store L without its diagonal (strictly lower), and treat it as unit lower-triangular at use
 119 time. The LDL^\top parameterisation provides superior numerical stability for precision matrices
 120 by avoiding strict positive definiteness constraints.

121 AI usage disclosure

122 Generative AI tools were used during development of this software and manuscript. Various large
 123 language models were used to assist with code generation, refactoring, and test scaffolding
 124 for portions of the codebase, and AI assistance was used to draft and edit parts of the
 125 documentation and this manuscript. The repository was initiated prior to widespread AI coding
 126 assistant adoption, with AI tools incorporated during later development phases. All AI-assisted
 127 outputs were reviewed, edited, and validated by the human authors, who take responsibility for
 128 the final software and paper.

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