

qnm: A Python package for calculating Kerr quasinormal modes, separation constants, and spherical-spheroidal mixing coefficients

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Software

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Background

Black holes can be characterized from far away by their spectroscopic gravitational-wave “fingerprints,” in analogy to electromagnetic spectroscopy of atoms, ions, and molecules. The idea of using the quasi-normal modes (QNMs) of black holes (BHs) for gravitational-wave (GW) spectroscopy was first made explicit by Detweiler (1980). QNMs of rotating Kerr BHs in general relativity (GR) depend only on the mass and spin of the BH. Thus GWs containing QNMs can be used to infer the remnant BH properties in a binary merger, or as a test of GR by checking the consistency between the inspiral and ringdown portions of a GW signal (Abbott & others, 2016; Isi, Giesler, Farr, Scheel, & Teukolsky, 2019).

For a review of QNMs see Berti, Cardoso, & Starinets (2009). A Kerr BH’s QNMs are the homogeneous (source-free) solutions to the Teukolsky equation (Teukolsky, 1973) subject to certain physical conditions. The Teukolsky equation can apply to different physical fields based on their spin-weight s ; for gravitational perturbations, we are interested in $s = -2$ (describing the Newman-Penrose scalar ψ_4). The physical conditions for a QNM are quasi-periodicity in time, of the form $\propto e^{-i\omega t}$ with complex ω ; conditions of regularity, and that the solution has waves that are only going down the horizon and out at spatial infinity. Separating the radial/angular Teukolsky equations and imposing these conditions gives an eigenvalue problem where the frequency ω and separation constant A must be found simultaneously. This eigenvalue problem has a countably infinite, discrete spectrum labeled by angular harmonic numbers (ℓ, m) with $\ell \geq 2$ (or $\ell \geq |s|$ for fields of other spin weight), $-\ell \leq m \leq +\ell$, and overtone number $n \geq 0$.

There are several analytic techniques, e.g. one presented by Dolan & Ottewill (2009), to approximate the desired complex frequency and separation constant $(\omega_{\ell, m, n}(a), A_{\ell, m, n}(a))$ as a function of spin parameter $0 \leq a < M$ (we follow the convention of using units where the total mass is $M = 1$). These analytic techniques are useful as starting guesses before applying the numerical method of Leaver (1985) for root-polishing. Leaver’s method uses Frobenius expansions of the radial and angular Teukolsky equations to find 3-term recurrence relations that must be satisfied at a complex frequency ω and separation constant A . The recurrence relations are made numerically stable to find so-called minimal solutions by being turned into infinite continued fractions. In Leaver’s approach, there are thus two “error” functions $E_r(\omega, A)$ and $E_a(\omega, A)$ (each depending on a, ℓ, m, n) which are given as infinite continued fractions, and the goal is to find a pair of complex numbers (ω, A) which are simultaneous roots of both functions. This is typically accomplished by complex root-polishing, alternating between the radial and angular continued fractions.

A refinement of this method was put forth by Cook & Zalutskiy (2014) (see also Appendix A of Hughes (2000)). Instead of solving the angular Teukolsky equation “from the endpoint” using Leaver’s approach, one can use a spectral expansion with a good choice of basis functions.

The solutions to the angular problem are the *spin-weighted spheroidal harmonics*, and the appropriate spectral basis are the spin-weighted *spherical harmonics*. This expansion is written as (spheroidal on the left, sphericals on the right):

$${}_sY_{\ell m}(\theta, \phi; a\omega) = \sum_{\ell'=\ell_{\min}(s,m)}^{\ell_{\max}} C_{\ell'\ell m}(a\omega) {}_sY_{\ell' m}(\theta, \phi),$$

where $\ell_{\min} = \max(|m|, |s|)$, and the coefficients $C_{\ell'\ell m}(a\omega)$ are called the spherical-spheroidal mixing coefficients (we follow the conventions of Cook & Zalutskiy (2014), but compare Berti & Klein (2014)). When recast in this spectral form, the angular equation becomes very easy to solve via standard matrix eigenvector routines, see Cook & Zalutskiy (2014) for details. If one picks values for (s, ℓ, m, a, ω) , then the separation constant $A(a\omega)$ is returned as an eigenvalue, and a vector of mixing coefficients $C_{\ell'\ell m}(a\omega)$ are returned as an eigenvector. From this new point of view there is now only one error function to root-polish, $E_r(\omega) = E_r(\omega, A(\omega))$ where the angular separation constant is found from the matrix method at any value of ω . Polishing roots of E_r proceeds via any standard 2-dimensional root-finding or optimization method.

The main advantage of the spectral approach is rapid convergence, and getting the spherical-spheroidal mixing coefficients “for free” since they are found in the process of solving the spectral angular eigenvalue problem.

Summary

`qnm` is an open-source Python package for computing the Kerr QNM frequencies, angular separation constants, and spherical-spheroidal mixing coefficients, for given values of (ℓ, m, n) and spin a . There are several QNM codes available, but some (London, 2017) implement either analytic fitting formulae (which only exist for a range of s, ℓ, m, n) or interpolation from tabulated data (so the user can not root-polish); others (Berti, 2010) are in proprietary languages such as Mathematica. We are not aware of any packages that provide spherical-spheroidal mixing coefficients, which are necessary for multi-mode ringdown GW modeling.

The `qnm` package includes a Leaver solver with the Cook-Zalutskiy spectral approach to the angular sector, thus providing mixing coefficients. We also include a caching mechanism to avoid repeating calculations. When the user wants to solve at a new value of a , the cached data is used to interpolate a good initial guess for root-polishing. We provide a large cache of low ℓ, m, n modes so the user can start interpolating right away, and this precomputed cache can be downloaded and installed with a single function call. We have adapted the core algorithms so that `numba` (Lam, Pitrou, & Seibert, 2015) can just-in-time compile them to optimized, machine-speed code. We rely on `numpy` (Walt, Colbert, & Varoquaux, 2011) for common operations such as solving the angular eigenvalue problem, and we rely on `scipy` (Jones, Oliphant, Peterson, & others, 2001) for two-dimensional root-polishing, and interpolating from the cache before root-polishing.

This package should enable researchers to perform ringdown modeling of gravitational-wave data in Python, without having to interpolate into precomputed tables or write their own Leaver solver. The author and collaborators are already using this package for multiple active research projects. By creating a self-documented, open-source code, we hope to alleviate the high frequency of re-implementation of Leaver’s method, and instead focus efforts on making a single robust, fast, high-precision, and easy-to-use code for the whole community. In the future, this code can be extended to incorporate new features (like special handling of algebraically special modes) or to apply to more general BH solutions (e.g., solving for QNMs of Kerr-Newman or Kerr-de Sitter).

Development of `qnm` is hosted on [GitHub](#) and distributed through [PyPI](#); it can be installed with the single command `pip install qnm`. Documentation is automatically built on [Read](#)

the [Docs](#), and can be accessed interactively via Python docstrings. Automated testing is run on [Travis CI](#). The `qnm` package is part of the [Black Hole Perturbation Theory Toolkit](#).

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