Reyna: A minimal overhead polytopal discontinuous Galerkin finite element library

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Summary

Partial differential equations (PDEs) underpin much of engineering, physics and applied science, providing the basis for simulations across a wide range of physical and biological systems. Many high-performance numerical schemes are implemented in compiled languages, which deliver speed but at the cost of accessibility. These implementations are often difficult to adapt or extend, limiting rapid experimentation, the development of new methods and their use in training.

Reyna is a Python package for solving second-order PDEs with non-negative characteristic form using the polytopal discontinuous Galerkin finite element method (DGFEM). It offers a flexible and approachable workflow while avoiding the complexity and steep learning curve of HPC-focused codebases.

Reyna is designed with a modular structure and an optimised implementation aiming at balancing clarity with performance. It provides a practical platform for developing and testing finite element methods and for producing concise instructional examples, lowering the barrier to experimentation whilst retaining state-of-the-art numerical capability.

Statement of Need

DGFEMs offer flexibility and high-order accuracy, and polytopal elements extend this adaptability further through general meshing and refined discretisations. Yet most available implementations are embedded in complex C++ or Fortran frameworks, see MFEM by Anderson et al. (2021), Feel++ by Prud'Homme et al. (2012), Dune-FEM-DG by Dedner et al. (2017) (now supporting polygonal meshes), deal.II by Bangerth et al. (2007) and FEMPAR by Badia et al. (2018), for example, which require substantial technical overhead to adapt or extend. Packages such as FEniCS by Alnæs et al. (2015) and Firedrake by Rathgeber et al. (2016) exist with Python interfaces, but extensive knowledge is required, limiting their use for prototyping and teaching. Quail by Ching et al. (2022) provides a Python DGFEM package aimed at this purpose, but it is restricted to triangular and quadrilateral meshes.

Reyna provides a lightweight, vectorised Python framework that makes polytopal DGFEMs accessible without sacrificing efficiency. It enables rapid experimentation, straightforward extension and practical use in both research and education by leveraging established scientific Python libraries and its modular architecture.

Description

Reyna is a lightweight, vectorised Python framework that makes polytopal DGFEMs accessible without sacrificing efficiency. It gives researchers and educators a tool that makes advanced techniques easy to prototype, test and share. The package supports rapid exploration of new ideas while retaining the rigour needed for scientific computation with a particular emphasis on clarity and minimal ramp-up time.

The architecture of Reyna (Figure 1) is intentionally modular, reflecting the natural workflow of finite element methods. It consists of three modules: polymesher, geometry, and DGFEM. The polymesher module provides an interface for defining polytopal meshes, whether generated internally or supplied by custom methods, enabling flexible handling of general domains (see Talischi et al. (2012) for methodology and Calloo et al. (2025) for an early application in neutronics). The geometry module builds on this mesh to compute the quantities required for discretisation (see Dong et al. (2020)), while the DGFEM module assembles and solves the global system (see Cangiani et al. (2014) and Cangiani et al. (2016) for mathematical details). This separation of concerns makes the code easier to extend; for example, different meshing algorithms or geometry formulations can be swapped in without altering the rest of the pipeline.

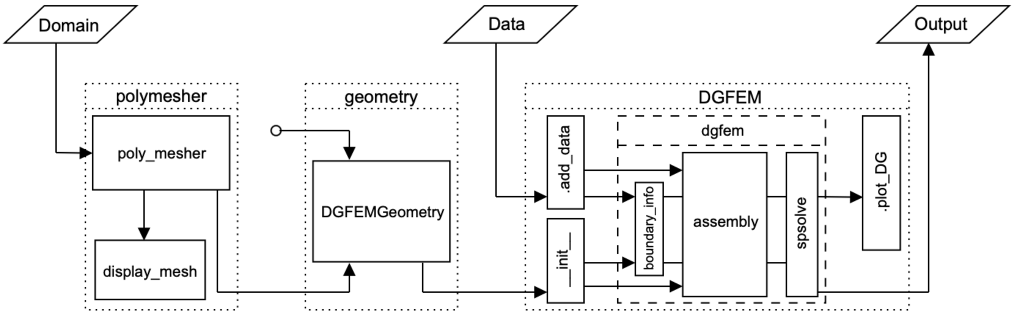

Figure 1: Architechture flow diagram of Reyna.

Reyna provides vectorised assembly routines for constructing the global stiffness matrix and forcing vector. These routines adapt automatically to the geometry of each polytopal element without significant computational overhead. Reyna combines efficiency with transparency, producing compact code that closely follows the underlying mathematics through accessible NumPy array operations. Simple visualisation tools based on Matplotlib are also included, allowing meshes and solutions to be rendered directly in scripts or notebooks (Figure 2). For further examples, Reyna ships with Jupyter notebooks as an interactive learning resource.

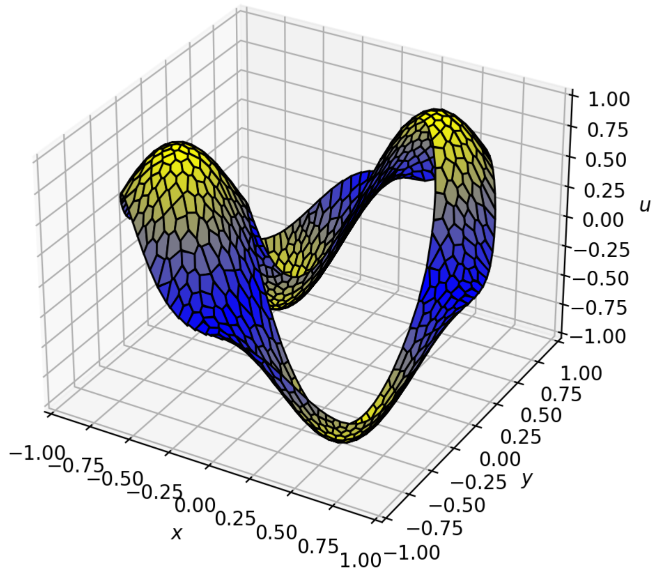

Figure 2: DGFEM solution plotted with Reyna.

The design philosophy is clarity first, a focused codebase with minimal dependencies (NumPy, SciPy, Shapely, Numba and Matplotlib) written so that users can readily inspect, modify, and extend the implementation. The combination of polytopal DGFEMs with Python accessibility makes Reyna a practical platform for researchers developing new applications and for educators introducing advanced numerical methods.

Example: Minimal Implementation of DGFEM

A key strength of Reyna is the ability to run DGFEM simulations in a compact and readable Python style. As an illustration, we provide a minimal example, inspired by Alberty et al. (1999) and Sutton (2017). Despite its brevity, the implementation captures the complete computational workflow of the package.

We consider the exact solution $u(x, y) = \sin(\pi x)\sin(\pi y)$ on the two-dimensional `CircleCircleDomain` eccentric annular domain. The code below solves this problem with diffusion, advection and reaction coefficients, together with a forcing term chosen appropriately for the exact solution.

```
import numpy as np

from reyna.polymesher.two_dimensional.domains import CircleCircleDomain
from reyna.polymesher.two_dimensional.main import poly_mesher

from reyna.geometry.two_dimensional.DGFEM import DGFEMGeometry
from reyna.DGFEM.two_dimensional.main import DGFEM
```

```python
# Defining the Domain

dom = CircleCircleDomain()

# Defining the Coefficients

def diffusion(x):
    out = np.zeros((x.shape[0], 2, 2), dtype=np.float64)
    for i in range(x.shape[0]):
        out[i, 0, 0] = 1.0
        out[i, 1, 1] = 1.0
    return out


advection = lambda x: np.ones(x.shape, dtype=float)
reaction = lambda x: np.pi ** 2 * np.ones(x.shape[0], dtype=float)
forcing = lambda x: (np.pi * (np.cos(np.pi * x[:, 0]) *
                              np.sin(np.pi * x[:, 1]) +
                              np.sin(np.pi * x[:, 0]) *
                              np.cos(np.pi * x[:, 1])) +
                     3.0 * np.pi ** 2 *
                     np.sin(np.pi * x[:, 0]) *
                     np.sin(np.pi * x[:, 1]))

bcs = lambda x: np.sin(np.pi * x[:, 0]) * np.sin(np.pi * x[:, 1])

# Generating the Mesh and Geometry

poly_mesh = poly_mesher(dom, max_iterations=10, n_points=1024, cleaned=True)
geometry = DGFEMGeometry(poly_mesh)

# Solving the PDE

dg = DGFEM(geometry, polynomial_degree=1)
dg.add_data(
    diffusion=diffusion,
    advection=advection,
    reaction=reaction,
    dirichlet_bcs=bcs,
    forcing=forcing
)

dg.dgfem(solve=True)
```

This minimal example is fully functional and illustrates the flexibility and simplicity of the package. Plotting the solution with `dg.plot_DG()` produces Figure 2.

The brevity of this example highlights two advantages of Reyna. First, the vectorised design produces concise code that mirrors the underlying mathematics, reducing complexity and making it straightforward to experiment with new numerical schemes. Second, the example is entirely self-contained, which makes it well-suited to research demonstrations and teaching. The full workflow is visible, easy to follow, and simple to adapt without low-level setup. While the example is kept deliberately short, the implementation remains faithful to the full mathematical formulation, and users can explore these details directly in the code. These qualities make Reyna a practical and approachable tool for both researchers developing new methods and educators introducing advanced numerical techniques.

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References

Alberty, J., Carstensen, C., & Funken, S. A. (1999). Remarks around 50 lines of Matlab: Short finite element implementation. *Numerical Algorithms*, *20*(2), 117–137. https://doi.org/10.1023/A:1019155918070

Alnæs, M., Blechta, J., Hake, J., Johansson, A., Kehlet, B., Logg, A., Richardson, C., Ring, J., Rognes, M. E., & Wells, G. N. (2015). The FEniCS project version 1.5. *Archive of Numerical Software*, *3*(100). https://doi.org/10.11588/ans.2015.100.20553

Anderson, R., Andrej, J., Barker, A., Bramwell, J., Camier, J.-S., Cerveny, J., Dobrev, V., Dudouit, Y., Fisher, A., Kolev, T., & others. (2021). MFEM: A modular finite element methods library. *Computers & Mathematics with Applications*, *81*, 42–74. https://doi.org/10.1016/j.camwa.2020.06.009

Badia, S., Martín, A. F., & Principe, J. (2018). FEMPAR: An object-oriented parallel finite element framework. *Archives of Computational Methods in Engineering*, *25*(2), 195–271. https://doi.org/10.1007/s11831-017-9244-1

Bangerth, W., Hartmann, R., & Kanschat, G. (2007). Deal.II—a general-purpose object-oriented finite element library. *ACM Transactions on Mathematical Software (TOMS)*, *33*(4), 24–es. https://doi.org/10.1145/1268776.1268779

Calloo, A., Evans, M., Lockyer, H., Madiot, F., Pryer, T., & Zanetti, L. (2025). Cycle-free polytopal mesh sweeping for Boltzmann transport. *Numerical Algorithms*, 1–24. https://doi.org/10.1007/s11075-025-02081-1

Cangiani, A., Dong, Z., Georgoulis, E. H., & Houston, P. (2016). hp-version discontinuous Galerkin methods for advection-diffusion-reaction problems on polytopic meshes. *ESAIM: Mathematical Modelling and Numerical Analysis*, *50*(3), 699–725. https://doi.org/10.1051/m2an/2015059

Cangiani, A., Georgoulis, E. H., & Houston, P. (2014). hp-version discontinuous Galerkin methods on polygonal and polyhedral meshes. *Mathematical Models and Methods in Applied Sciences*, *24*(10), 2009–2041. https://doi.org/10.1142/S0218202514500146

Ching, E. J., Bornhoft, B., Lasemi, A., & Ihme, M. (2022). Quail: A lightweight open-source discontinuous Galerkin code in Python for teaching and prototyping. *SoftwareX*, *17*, 100982. https://doi.org/10.1016/j.softx.2022.100982

Dedner, A., Girke, S., Klöfkorn, R., & Malkmus, T. (2017). The DUNE-FEM-DG module. *Archive of Numerical Software*, *5*(1), 21–61. https://doi.org/10.11588/ans.2017.1.28602

Dong, Z., Georgoulis, E. H., & Pryer, T. (2020). Recovered finite element methods on polygonal and polyhedral meshes. *ESAIM: Mathematical Modelling and Numerical Analysis*, *54*(4), 1309–1337. https://doi.org/10.1051/m2an/2019047

Prud'Homme, C., Chabannes, V., Doyeux, V., Ismail, M., Samake, A., & Pena, G. (2012). Feel++: A computational framework for Galerkin methods and advanced numerical methods. *ESAIM: Proceedings*, *38*, 429–455. https://doi.org/10.1051/proc/201238024

Rathgeber, F., Ham, D. A., Mitchell, L., Lange, M., Luporini, F., McRae, A. T., Bercea, G.-T., Markall, G. R., & Kelly, P. H. (2016). Firedrake: Automating the finite element method by composing abstractions. *ACM Transactions on Mathematical Software (TOMS)*, *43*(3), 1–27. https://doi.org/10.1145/2998441

Sutton, O. J. (2017). The virtual element method in 50 lines of MATLAB. *Numerical Algorithms*, *75*(4), 1141–1159. https://doi.org/10.1007/s11075-016-0235-3

Talischi, C., Paulino, G. H., Pereira, A., & Menezes, I. F. (2012). PolyMesher: A general-purpose mesh generator for polygonal elements written in Matlab. *Structural and Multidisciplinary Optimization*, *45*(3), 309–328. https://doi.org/10.1007/s00158-011-0706-z