Simplex Method

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April 21, 2023

The standard form of LP

$$\max \; z = \sum_{j=1}^{n} c_j x_j$$

$$s.t \begin{cases} \sum_{j=1}^{n} a_{ij} x_j = b_i, i = 1, 2, \ldots, m \\ x_j \geq 0, j = 1, 2, \ldots, n \end{cases}$$

The standard form of LP (matrix form)

$$\max \ z = c^T x$$

$$Ax = b$$

$$x \geq 0$$

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

Assumption

A is linearly independent. In other words, for a matrix $A_{m\times n}$, $m < n$, A has full rank m

This can be check by the process of Gaussian elimination.

Convert to standard form

- $Ax \leq b, Ax \geq b \rightarrow Ax + s = b, Ax - s = b$
  s is the slack variable
- $x_i \leq 0 \rightarrow y_i = -x_i$
- $x_i \in R \rightarrow x_i = x_i^+ - x_i^-$, where $x_i^+, x_i^- \geq 0$

Overview

In standard form, we have m constraint with n variables.

Before the standardization, P is the feasible region (it is a polyhedron). Obviously, P is convex.

$x \in P$ is an extreme point of P, if we can't find two point $y, z \in P$ such that $x = \lambda y + (1 - \lambda z)$, $\lambda \in [0, 1]$

General idea of Simplex Method

In an iteration, we can select m variables as the basic variables, the others are non-basic variables.

Set the non-basic variables as 0, then we have an equation system with m constraint and m variables, the solution can be the extreme point of P or out of P.

How to select and iterate is the issue of simplex method.

Visualization

See 1.html, 2.html.

By the way, the visualization is generated by gilp package in python.

Definition

$$\max \ z = c^T x$$

$$Ax = b$$

$$x \geq 0$$

Assume the basic indices are $B = \{B(1), ...B(m)\}$, then we write

$$A_B = \begin{bmatrix} A_{B(1)} & A_{B(2)} & \dots & A_{B(m)} \end{bmatrix}$$

$$x_B = \begin{bmatrix} x_{B(1)} & x_{B(2)} & \dots & x_{B(m)} \end{bmatrix}^T$$

The non-basic variable's column in A is A_N. The non-basic variable is x_N.

Definition

- Basic solution: The solution x in feasible region P with m feasible variables. (the intersection of m hyperplanes)
- Basic feasible solution: The basic solution x such that $x \geq 0$
- Entering basis & Exiting basis: In one iteration, we select one current non-basic variable to be the basic variable (entering basis), and select one current basic variable to be the non-basic variable (exiting basis).

Optimal

- $N = \{1, \cdots, n\} - B$, the set of non-basic variables's indices.
- Basic variables can be represented by non-basic variables

$$x_i^{'} = b_i + \sum_{j \in N} m_{ij} x_j^{'} (i \in B)$$

where b is a feasible solution of x, i.e. $x_i^{'} = b_i$

- Tips: use the linear transformation

change $B = (3, 4)$ to $(2, 3)$

$$\begin{array}{cc} \mathrm{X} & \begin{matrix} x_1 & x_2 & x_3 & x_4 & b \end{matrix} \\ \begin{matrix} \\ \\ \mathrm{C} \end{matrix} & \begin{bmatrix} 2 & 1 & 1 & 0 & 12 \\ 1 & 2 & 0 & 1 & 9 \\ 1 & 1 & 0 & 0 & z \end{bmatrix} \end{array} \rightarrow \begin{array}{cc} \mathrm{X} & \begin{matrix} x_1 & x_2 & x_3 & x_4 & b \end{matrix} \\ \begin{matrix} \\ \\ \mathrm{C} \end{matrix} & \begin{bmatrix} \frac{3}{2} & 0 & 1 & -\frac{1}{2} & \frac{15}{2} \\ \frac{1}{2} & 1 & 0 & \frac{1}{2} & \frac{9}{2} \\ \frac{1}{2} & 0 & 0 & -\frac{1}{2} & z - \frac{9}{2} \end{bmatrix} \end{array}$$

Optimal

- So, the objective z can be represented by non-basic variables.

$$z = z_0 + \sum_{j \in N} \sigma_j x_j^{'}$$

- When reach the optimal, all $\sigma_j \leq 0$.

Optimal

- If not, $\exists j \in N$, $\sigma_j > 0$, since the objective is the function in respect to non-basic variables.
- Since $x_j' = 0$, we add the value of x_j'. Then the objective will be larger.
- Can we always add the value of x_j' here? No! It exists degenerated case.

Find the initial solution

- We choose $B = (1, 2, ..., m)$. Then, $x_B = A_B^{-1} b$, $x_N = 0$ is a basic feasible solution, if $x_B \geq 0$.
- If x_B has negative term, we should modify the constraint.
- Add x_0 to ensure the new LP has feasible solutions
- $x_1 + x_2 - x_0 \leq -1$
- Then $(x_1, x_2, x_0) = (0, 0, 1)$ is a feasible solution

Find the initial solution

- We define a new LP:

$$\min \ z = x_0$$

$$A'x' \leq b$$

$$x' \geq 0$$

where $x' = [x \ \ x_0]^T$, x is the original variable vector.
$A' = [A \ \ A_0]$, $A_0 = [-1, \cdots, -1]^T$
$x = 0$, $x_0 = -min\{b_i\}$ is a feasible solution. (initial)

Find the initial solution

- Solve the LP by simplex method and see if optimal $z = 0$.
- If not, there is no feasible solution in original LP, otherwise the solution of the new LP is the feasible solution of original LP.

Select the entering basis

- Choose the non-basic variable whose $\sigma_j > 0$. (not random, one way is to use Bland's rule).
- x_j is the entering basis

Bland's Rule

- If we use the smallest index rule for choosing both the entering basis and the exiting basis (when the objective is not changed), then no cycle will occur in the simplex algorithm.

Select the exiting basis

- Assume we have a solution x, and the entering basis x_j. we want to move x to $x + \theta d$, $\theta \geq 0$.
- Since the $x + \theta d$ is still the basic feasible solution. We have

$$A(x + \theta d) = b = Ax$$

$$Ad = 0$$

Select the exiting basis

- We write $d = [d_B \;\; d_N]^T$. $A = [A_B \;\; A_N]$
- In d_N, $d_j = 1$ and $d_{j'} = 0$ for all $j' \in N$, $j' \neq j$

$$[A_B \;\; A_N][d_B \;\; d_N]^T = A_B d_B + A_{.\,j} = 0$$

$$d_B = -A_B^{-1} A_{.\,j}$$

- $d = [d_B \;\; d_N] = [-A_B^{-1} A_{.\,j} \;\; 0 \;\; 0 \;\; \ldots \;\; 0 \;\; 1 \;\; 0 \;\; \cdots \;\; 0]$
- 1 is the at index of x_j

Select the exiting basis

- Feasibility: $x + \theta d \geq 0$
- We choose $\theta^* = \max\{\theta \geq 0 : x + \theta d \geq 0\}$
- If $d \geq 0$, $\theta^* = \infty$. The LP is unbounded.
- Otherwise,

$$\theta^* = \min_{i : d_i < 0} -\frac{x_i}{d_i}$$

- The argmin i of θ^* is the exiting basis.
- When $\theta^* = 0$, it degenerates.

The change of objective

In formal, we have

$$\Delta z = \theta^* c^T d$$

Repeat the iteration until obtain the optimal.