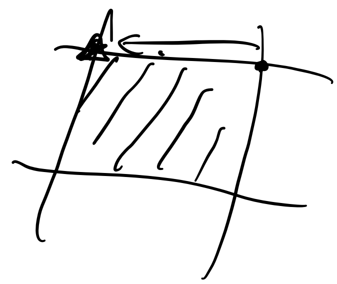
Simplex Method.

$\min_{x} c^T x$ s.t. $Ax=b$. $x \geq 0$. $A \in R^{m \times n}$. full rank.



1. Basic Idea.

Moving from a BFS to another.

(1). Why BFS?

- BFS $\Leftrightarrow$ extreme point $\Leftrightarrow$ vertex.
- Fundamental LP Thm. opt. soln $\Rightarrow$ BFS.

(2). How to define "another"?

- "Neighborhood" of a BFS.

$$x = (x_B, x_N). \quad x_B \in R^m, \quad x_N \in R^{n-m}.$$

$$B(i) \Leftrightarrow N(j).$$

$$x' = (x_B', x_N').$$

(3). How can we guarantee x' is still a BFS?

① Basic Soln. $A = (A_{B'}, A_{N'})$ column vectors of $A_{B'}$ are linear independent. and $x_{N'} = 0$.

② feasible. $Ax' = b$. $x' \geq 0$

2. Derivation.

Given a BFS $x = (x_B, x_N)$. we want to obtain its neighbor $x' = (x_B, x_N) = x + \theta d = (x_B + \theta d_B, x_N + \theta d_N)$ $\theta \geq 0$.

$B(i) \leftrightarrow N(j)$

$$x = (x_{B(1)}, \dots x_{B(i)}, \dots x_{B(m)}, \boxed{x_{N(1)}, \dots, x_{N(j)} \dots x_{N(n-m)}}) \; 0$$

$$x' = (x_{B(1)}, \dots, x_{N(j)}, \dots x_{B(m)}, x_{N(1)} \dots x_{B(i)} \dots x_{N(n-m)})$$

set $\boxed{d_N = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \to N(j)}$. we want x' to satisfy:

① . BFS $\begin{cases} \text{basic} \checkmark \\ \text{feasible} . \checkmark \end{cases}$

② . $\bar{c} = c^T x' - c^T x = \theta c^T d . < 0$. (reduced cost)

Now lets do this by construction.

– feasible:

(1). $Ax' = b$. $\Rightarrow$. $Ax + \theta A d = b$. $\Rightarrow$ $Ad = 0$.

$(A_B, A_N)\begin{pmatrix} d_B \\ d_N \end{pmatrix} = 0$. $\Rightarrow$. $A_B d_B + A_N d_N = 0$.

$\Rightarrow$. $A_B d_B + A_{N(j)} = 0$. $\Rightarrow$. $d_B = -A_B^{-1} A_{N(j)}$.

$$\boxed{d = \begin{pmatrix} d_B \\ d_N \end{pmatrix} = \begin{pmatrix} -A_B^{-1} A_{N(j)} \\ 0 \\ \vdots \\ 1 \end{pmatrix}}.$$

$x' = x + \theta d = \begin{pmatrix} x_B + \theta d_B \\ x_N + \theta d_N \end{pmatrix} \geq 0.$ $x_B + \theta d_B \geq 0$.

0. ≥ 0.

by setting θ proper.

- Reduced cost. $\bar{c} < 0$.

$$\bar{c} = \theta \cdot c^T d = \theta (c_B^T, c_N^T) \begin{pmatrix} d_B \\ d_N \end{pmatrix} = \theta (c_B^T d_B + c_N^T d_N).$$

$$= \theta (c_{N(j)} - c_B^T A_B^{-1} A_{N(j)}).$$

(1). $\theta \geq 0$. $c_{N(j)} - c_B^T A_B^{-1} A_{N(j)} < 0$. for $j \in N$.

If $\exists ! j \in N$. s.t. $c_{N(j)} - c_B^T A_B^{-1} A_{N(j)} < 0$. then we can say x is optimal. $\boxed{\text{stopping rule}}$

(2). $\exists j \in N$, s.t. $c_{N(j)} - c_B^T A_B^{-1} A_{N(j)} < 0$. then.

$\theta \uparrow$, $\bar{c} \downarrow$. obj val $\downarrow$.

$$\theta^* = \max_{\theta} \{ \theta : \underline{x_B + \theta d_B \geq 0} \}.$$

$$= \max_{\theta} \{ \theta : x_{B(k)} + \theta d_{B(k)} \geq 0. \ \forall k \}.$$

$$= \max_{\theta} \{ \theta : \underline{x_{B(k)} + \theta d_{B(k)} \geq 0}. \ \forall k. \text{ satisfies } \underline{d_{B(k)} < 0} \}$$

$$= \max_{\theta} \{ \theta : \theta \leq -\frac{x_{B(k)}}{d_{B(k)}} \quad \forall k \ldots \}.$$

$$= \min_{\{k \in B,\, d_{B(k)} < 0\}} \{ -\frac{x_{B(k)}}{d_{B(k)}} \}.$$

Notice that if $\forall k$. $d_{B(k)} \geq 0$. it means that

$$x_B + \theta d_B \geq 0.$$

always holds. so we can set $\theta \uparrow (+\infty)$. obj val $\downarrow -\infty$.

- Basic.

$$A = (A_{B(1)}, \dots A_{B(i)}, \dots, A_{B(m)}, A_{N(1)} \dots A_{N(j)} \dots, A_{N(n-m)}).$$

$$\Downarrow$$

$$A' = (A_{B(1)} \dots A_{N(j)}, \dots, A_{B(m)}, A_{N(1)}, \dots A_{B(i)}, \dots A_{N(n-m)}).$$

$B' = \{B(1) \dots N(j) \dots B(m)\}$. we need to check:

$A_{B'}$ has full rank.

$$A_B^{-1} A_{B'} = (\underbrace{A_B^{-1} A_{B(1)}} \dots A_B^{-1} A_{N(j)}, \dots, A_B^{-1} A_{B(m)}).$$

For $\forall K$. $A_B^{-1} A_{B(K)} = A_B^{-1} A_B \cdot \underline{1_K} = 1_K = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \to B(K).$

So $A_B^{-1} A_{B'}$

$$= \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & & A_B^{-1}A_{N(j)} & \\ & & & & \ddots \\ & & & & & 1 \end{pmatrix} = \begin{pmatrix} 1 & & -d_{B(1)} & & \\ & \ddots & \vdots & & \\ & & 1 & & \\ & & -d_{B(i)} & \cdots & \\ & & \vdots & 1 & \\ & & \vdots & & \ddots \\ & & -d_{B(m)} & & & 1 \end{pmatrix}.$$

if $d_B(i)$ is non-zero. $A_B^{-1} A_{B'}$ has full rank.

$$\begin{array}{l} x = (x_{B(1)} \dots \boxed{x_{B(i)}}^{\text{pos}} \dots x_{B(m)}, x_{N(1)} \dots \boxed{x_{N(j)}} \dots x_{N(n-m)}). \\ + \quad (\dots \dots \boxed{\theta d_{B(i)}} \dots \dots \dots \boxed{\theta d_{N(j)}} \text{ positive.} \quad). \\ \qquad\qquad\qquad \downarrow 0. \quad \| \qquad\qquad \| \\ x' = (\qquad 0 \qquad\qquad\qquad \theta. \qquad) \end{array}$$

switch.

$d_{B(i)} \neq 0$. so. $A_{B'}$ has full rank. because. $A_B^{-1} A_{B'}$ has full rank.

So. x' is a BFS, and can make $\bar{c} < 0$. when simplex method doesn't stop.

Two Issues remaining.

1. Degeneracy.

$x' = x + \theta d$. $x \longrightarrow x'$. by simplex method

but. $c^T x' = c^T x$. To handle this, we need to set specific rule to pick index. (pivoting rule.)

2. How to obtain the initial BFS?.

- Two-phase method.
- Big-M method.