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Introduction to Probability

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Preface

Placeholder

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Fundamentals of Probability and its Axioms

1.1 Combinations and Permutations

Counting plays a very important role in probability. In probability, we often deal with sets, and counting methods such as combinations and permutations help us find the number of elements in a set. Specifically, in probability, we deal with a set called the **sample space** which is the set of all possible outcomes of some random experiment and **events**, which are subsets of the sample space. Typically, the sample space is denoted by S or Ω. Consider the following example:

Example 1.1. Suppose a fair coin is flipped twice.What is the probability of flipping at least one head?

Solution:

The sample space of this experiment can expressed as $S = \{HH, HT, TH, TT\}$. Now, consider the event:

$$A = \{\text{At least one outcome is a head}\}$$

Notice that $A \subset S$ (A is a subset of S). Since $A = \{HH, HT, TH\}$, we can find the probability as follows:

$$P(A) = \frac{\#\text{elements in A}}{\#\text{elements in S}} = \frac{3}{4}.$$

The above method of computing the probability of an event, while very useful, is not always suitable for the following reasons:

1. Each outcome has to be equally likely.

2. The sets must be countable.

From the previous example, it is clear that counting the number of elements of sets is very important when dealing with discrete sets. We will do this more efficiently by utilizing tools from combinations and permutations.

1.1.1 Permutations

We will begin by stating the following crucial theorem.

Theorem 1.1: The Fundamental Principle of Counting

Consider k experiments. Let n_i denote the number of possible outcomes of the ith experiment where $i = 1, 2, \ldots, k$. Then the total of number of possible outcomes is

$$n_1 \times n_2 \times \cdots \times n_k.$$

Example 1.2. A 6-sided die is rolled twice. How many possible outcomes are there?

Solution:

First roll has 6 outcomes and the second roll has 6 outcomes. Therefore, there are $6 \times 6 = 36$ possible outcomes when a dice rolled twice.

In general, if a 6-die is rolled n times, then there are 6^n possible outcomes.

Example 1.3. Suppose a password must contain exactly 4 symbols, where each symbol is either a letter (a-z) or a number (0-9). How many passwords can be created,

1. if the password is not case-sensitive?
2. if the first two symbols must be numbers?
3. if the first two symbols must be numbers and the rest should be letters?

Solution:

1. If the password is not case-sensitive, then there are $36 \times 36 \times 36 \times 36$ different passwords.
2. If the first two symbols must be numbers, then there are $10 \times 10 \times 36 \times 36$ different passwords.
3. If the first two symbols must be numbers and the rest should be letters, then there are $10 \times 10 \times 26 \times 26$ different passwords.

Exercise 1.1. Repeat the above example assuming the password is case-sensitive.

Example 1.4: Suppose a password must contain exactly 4 symbols, where each symbol is either a letter (a-z) or a number (0-9). How many passwords contain at least one letter?

Solution:

Total number of passwords $= 36^4$. Total number of passwords not containing a letter $= 10^4$. Therefore, the number of passwords containing at least one letter is

$$36^4 - 10^4.$$

Imagine three objects labeled A, B, and C. The number of ways we can arrange these objects is called a *permutation.* These objects can be arranged in six different ways:

$$ABC, ACB, BAC, BCA, CAB, CBA.$$

The formula to find the number of permutations of n *distinct* objects is

$$n! = n \times (n-1) \times \cdots \times 2 \times 1.$$

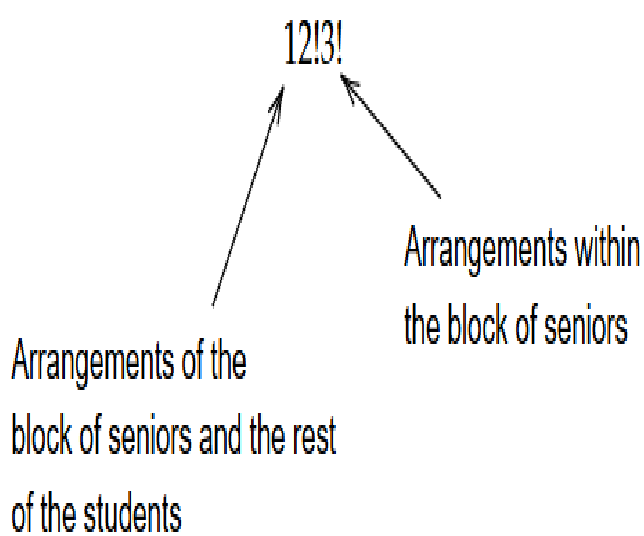
For three distinct objects, the number of permutations is $3! = 6$.

Example 1.5. In a class, there are 4 sophomores, 7 juniors, and 3 seniors.

1. How many ways are these students be arranged in a row?
2. How many ways can these students be arranged if all the seniors should remain together?

Solution:

1. 12!
2. Since the seniors must stand together, we think of the seniors as one "block". Therefore, together with this block, 4 sophomores, and 7 juniors, there are 12 objects to be arranged. We can also arrange the three seniors within the block in 3! ways. Therefore, the total number of permutations is:



Exercise 1.2. How many ways can the students be arranged if students from the same year should remain together?

Exercise 1.3. How many ways can the students be arranged if two particular students do not wish to stand together?

Exercise 1.4. How many ways can the students be arranged if at most two seniors should be together?

Now let's learn how to find the permutations of objects that are non-distinct (some are identical). Consider the permutations of AAB. Since we have two identical objects, the number of permutations has to be less than 3! (since permuting the two "A"s gives us the same arrangement).

Theorem 1.1: Permutations of non-identical objects Consider a set of n objects of which n_1 objects are alike, n_2 objects are alike, ..., n_k objects are alike. Then, the number of permutations of these n objects is given by

$$\frac{n!}{n_1!n_2!\cdots n_k!}$$

Example 1.6. Find the number of ways the following words can be permuted:

1. MATH
2. BEEP
3. COCO
4. SUCCESS

Solution:

1. 4 different letters. Therefore, number of permutations:
$$4!.$$
2. 4 letters word, 2 letters are alike. Therefore, number of permutations:
$$\frac{4!}{2!}.$$
3. 4 letters word, 2 letters are alike and another 2 letters are alike. Therefore, number of permutations:
$$\frac{4!}{2!2!}.$$
4. 7 letters word, 3 letters are alike and another 2 letters are alike. Therefore, number of permutations:
$$\frac{7!}{3!2!}.$$

In each of the above questions, we make an adjustment for overcounting by diving by the permutations of identical objects.

1.1.2 Combinations

Combinations are the number of ways a subset can be chosen from a larger set where the order of the selection does not matter. Consider a set of n distinct objects. The number of ways a subset of $r \leq n$ objects can be chosen can be computed using the formula

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}.$$

The key point here is that, unlike in permutations, the order of the r objects selected is irrelevant.

Example 1.7. Suppose a shelf contains 10 distinct books out of which 4 are math books and the rest are physics books.

1. How many ways can 3 books be chosen where the order does not matter?
2. How many ways can 4 books be chosen so that exactly two are math books?

Solution:

1. $\binom{10}{3}$
2. The number of ways two math books can be chosen = $\binom{4}{2}$\ The number of ways two other books (non-math) can be chosen = $\binom{6}{2}$\

Therefore, the total number of ways exactly two math books can be chosen is

$$\binom{4}{2}\binom{6}{2}.$$

Exercise 1.5. How many ways can at least one math book be chosen?

Exercise 1.6. How many ways can at most one math book be chosen?

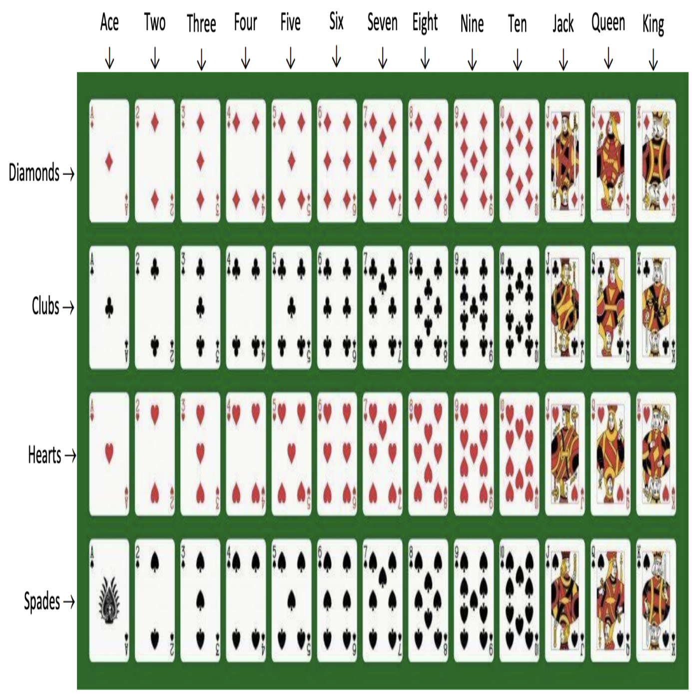


FIGURE 1.1: Illustration of a standard deck of cards. Source: Missouri Western State University (https://intranet.missouriwestern.edu/cas/wp-content/uploads/sites/17/2020/05/Standard-Deck-of-Cards.pdf)

Some questions in this course involves questions based on a deck of cards. This is the composition of a deck of cards:

In the Figure 1.1, the columns represent the different **values** or **denominations** and the rows represent the four **suits.** The last three values, Jack, Queen, and King are called **face cards.**

Example 1.8. Consider a deck of 52 cards.

1. How many ways can 5 cards be chosen?
2. How many ways can 5 cards be chosen if exactly 2 cards must be Queens?
3. How many ways can 5 cards be chosen if exactly 2 Queens and 2 Kings must be chosen?
4. How many ways can we choose 5 cards such that exactly two have the same value?

Solution:

1. $\binom{52}{5}$
2. A deck of cards has 4 Queens. We first choose two Queens from the set of 4 Queens and 3 additional cards are chosen from the remaining 48 cards (since 52-4 = 48). Therefore, the answer is $$\binom{4}{2}\binom{48}{3}.$$
3. $\binom{4}{2}\binom{4}{2}\binom{44}{1}$
4. There are 13 different values in a deck of cards: Ace, 2, 3, ..., Q, K. Each value appears on 4 cards. To choose two cards from the same value, we first pick a value. This can be done in $\binom{13}{1} = 13$ ways. Thereafter, we choose 2 cards from the 4 cards of this chosen value. This can be done in $\binom{4}{2}$ ways. Now we choose 3 additional cards that are different from this value. However, we also must make sure that we do not pick more than one card from another value. Out of the 12 values left, we first choose three values ($\binom{12}{3}$ways), and then from each value, we choose a card

$(\binom{4}{1}^3 = 4^3$ ways). Therefore, the final answer is

$$13 \cdot \binom{4}{2} \cdot \binom{12}{3} \cdot 4^3.$$

Exercise 1.7. How many ways can five cards be chosen such that all the values are unique?

Exercise 1.8. How many ways can five cards be chosen such that two cards are of one value and three cards of another value? (For example, QQAAA,22299,etc.).

1.1.3 Worked Examples

1. How many different ways can 5 cards be chosen from a deck of 52 if exactly two of the cards are of the same value and the other three cards are of different values? (Example: AA5JQ, 66KQ2, etc.)

Solution: A standard deck of cards has 13 different valueS. We need to first choose a value and then pick two cards from that value. Since each value has 4 cards, we can do this in $\binom{13}{1}\binom{4}{2}$ ways. Thereafter, the other three cards must be picked from three other values. Since we have already picked 2 cards from one of the values, there are 12 values left to choose from. We first pick 3 values and then choose a card from each value. This can be done on $\binom{12}{3}\binom{4}{1}\binom{4}{1}\binom{4}{1}$ ways. Therefore, the final answer is

$$\binom{13}{1}\binom{4}{2}\binom{12}{3}\binom{4}{1}\binom{4}{1}\binom{4}{1}.$$

2. How many ways can 5 cards be chosen from a standard 52-card deck if all 5 cards must have different values? (Ex: A2JQ3, 7J2Q4, etc.)

Solution: This is a little bit similar to the previous problem, but here we have to choose 5 distinct values. We first choose 5 values and then from each value, we pick a card. Therefore, the final answer is

$$\binom{13}{5}\binom{4}{1}\binom{4}{1}\binom{4}{1}\binom{4}{1}\binom{4}{1} = \binom{13}{5} \cdot 4^5.$$

3. A class consists of 40 seniors and 60 juniors. The teacher wants to form a committee consisting of 4 seniors and 6 juniors.

a. How many ways can the committee be formed?
b. How many ways can the committee be formed if two specific seniors do not wish to serve on the committee together?
c. Greg is one of the seniors in the class. How many different committees can be formed that include Greg as a member?

Solution:

a. $\binom{40}{4}\binom{60}{6}$
b. We will find the answer to this using an indirect method. Let's first find the number of committees where the two specific students *are* together. The set of seniors consists of these 2 specific students and 38 other seniors. The number of ways we can choose 4 seniors where two specific students are always included in the committee is $\binom{2}{2}\binom{38}{2} = \binom{38}{2}$.
c. Since there are no restrictions for the juniors, the total number of committees where these two specific students are in the committee is $\binom{38}{2}\binom{60}{6}$. Since the total number of committees is $\binom{40}{4}\binom{60}{6}$ (this is what we found in part (a)), we can find the answer to this question by subtracting the number of committees where the two students are together from the total number of committees (this gives us the number of committees where only one of them are

together or none of them are in the committee):

$$\binom{40}{4}\binom{60}{6} - \binom{38}{2}\binom{60}{6}.$$

$$\binom{1}{1}\binom{39}{3}\binom{60}{6} = \binom{39}{3}\binom{60}{6}.$$

4. Ten identical pieces of candy are to be distributed between four students - A,B,C,and D.

a. How many ways can the candy be distributed?

b. How many ways can the candy be distributed if A must receive exactly two pieces?

c. How many ways can the candy be distributed if A must receive at least two pieces?

Solution:

a. Let's use X to represent a piece of candy. We can use three lines (dividers) to divide the candy between the three students. For example

$$X|XX|XXX|XXXX$$

means A gets 1, B gets 2, C gets 3, and D gets 4. Similarly,

$$X||XX|XXXXXXX$$

means A receives 1, B receives none, C receives 2, and D receives 7. Therefore, we can find the total number of distributions by finding the total number of arrangements of the Xs and the lines. Since there are 10 Xs and three lines, but the 10 Xs are identical and the 3 lines are identical (total 13 objects), the total number of ways the candy can be distributed is

$$\frac{13!}{3!10!} = \binom{13}{3}.$$

b.

$$\underbrace{XX}_{A\text{ receives 2}} \quad \underbrace{XXXXXXXX}_{\text{Distribute the rest between B,C,D}} \quad .$$

This problems simplifies to 8 pieces of candy and three students - B,C,D. To distribute between 3 students, we need 2 lines. Therefore, the total number of distributions is

$$\frac{10!}{2!8!} = \binom{10}{2}.$$

c. This is similar to the previous part. Initially, we give A two pieces of candy. However, when we distribute the other 8 pieces of candy, we distribute between all the 4 students. This way, A gets at least 2.

$$\underbrace{XX}_{A\text{ receives 2}} \quad \underbrace{XXXXXXXX}_{\text{Distribute the rest between A,B,C,D}} \quad .$$

Therefore, we have 8 Xs and 3 lines (total 11). The answer is

$$\frac{11!}{3!8!} = \binom{11}{3}.$$

5. All the cards of deck of 52-cards are arranged in a line.

a. How many ways can the cards be arranged such that the colors (red and black) alternate?
b. How many ways can the cards be arranged if not all four Queens are together?

Solution:

a. We can arrange the cards starting with a red card (RBRB...) or with a black card (BRBR...). Since there are 26 red cards and 26 black cards, the answer is

$$26!26! \cdot 2.$$

b. The total number of arrangements without any restrictions is $4!$. The number of ways we arrange the cards keeping *all* the queens together is $4!49!$ (48 other cards + one block of 4 queens = 49 objects). Therefore, the number of arrangements where at least one queen is separated is

$$52! - 49!4!.$$

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Basic Probability