STACK-SORTING SIMPLICES: GEOMETRY AND LATTICE-POINT ENUMERATION

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ABSTRACT. We study the polytopes that arise from the convex hulls of stack-sorting on particular permutations. We show that they are simplices and proceed to study their geometry and lattice-point enumeration. First, we prove some enumerative results on Ln1 permutations, i.e., permutations of length n whose penultimate and last entries are n and 1, respectively. Additionally, we then focus on a specific permutation, which we call L'n1, and show that the convex hull of all its iterations through the stack-sorting algorithm share the same lattice-point enumerator as that of the (n-1)-dimensional unit cube and lecture-hall simplex. Lastly, we detail some results on the real lattice-point enumerator for variations of the simplices arising from stack-sorting L'n1 permutations. This then allows us to show that L'n1 simplices are Gorenstein of index 2.

1. Introduction

The study of sorting a permutation using stacks was first introduced by Knuth in the 1960s [7]. Classically, the aim of the stack-sorting problem is to sort a permutation using a last-in/first-out algorithm. In its simplest form, a stack is used to rearrange a permutation $\pi = \pi_1 \pi_2 \cdots \pi_{n-1} \pi_n$ as follows. The elements of π are pushed onto an originally empty stack and an output permutation is formed by popping elements from the stack. In this paper, we consider the convex hull of all the permutations that arise in each step of the stack-sorting algorithm, given a particular kind of input permutation. This paper is organized as follows. All undefined terms or notation will be introduced in the relevant sections.

In Section 2, we present necessary preliminaries and background. Subsections 2.1 and 2.2 present the stack-sorting algorithm and a brief overview of polyhedral geometry and Ehrhart theory, respectively. We continue with background on integral and unimodular equivalence and lecture-hall simplices, in Subsections 2.3 and 2.4, repsectively. We conclude this section with a notation table in Subsection 2.5.

Section 3 presents some enumerative and structural results on Ln1 permutations. For S_n , the group of all permutations on [n], we label a permutation $\pi \in S_n$ as a Ln1 permutation if it has the form Ln1, where L is a permutation of $\{2, 3, ..., n-1\}$. We denote the set of all permutations in S_n of this form as Ln1. The main theorem of this section is the following:

Theorem 3.12. A permutation $\pi \in \mathcal{L}n1$ if and only if π is exactly (n-1)-stack sortable.

In Section 4, we initiate the study of the geometry of Ln1 families. We take a Ln1 family, \mathcal{F}_{π} , to be the set of n permutations obtained from a fixed $\pi \in \mathcal{L}n1$ and all its iterations under the stack-sorting algorithm. We denote the special $\mathcal{L}n1$ permutation where $L=23\dots(n-1)$ as L'n1. The main results of this section are the following proposition and theorem:

Proposition 4.2. The convex hull of any Ln1 family forms a (n-1)-simplex in \mathbb{R}^n .

Theorem 4.3. The L'n1 simplex is hollow. Specifically, all non-vertex integer points in \triangle_n lie on the facet formed from the convex hull of $\mathcal{F}_{L'n1} - \{12 \dots n\}$.

Lastly, Section 5 deals with the Ehrhart theory of L'n1 simplices. The main result of this section is the following:

Theorem 5.2. The L'n1 simplex \triangle_n and (n-1)-dimensional lecture hall simplex P_{n-1} are integrally equivalent. In particular,

$$L_{\mathbb{Z}}(\triangle_n;t)=L_{\mathbb{Z}}(P_{n-1};t)=(t+1)^{n-1}.$$

We conclude by exploring the following additional results, including developing a recursive relationship and proving the Gorenstein property for translates of the L'n1 simplex. In what follows, allow π_n to denote L'n1.

Theorem 5.9. For all $\lambda \in \mathbb{R}_{>0}$,

$$L_{\mathbb{R}}(\triangle_{n+1}-\pi_{n+1};\lambda)=\sum_{k=0}^{\lfloor n\lambda\rfloor}L_{\mathbb{R}}\left(\triangle_{n}-\pi_{n};\frac{k}{n}\right).$$

Theorem 5.12. For all $\lambda \in \mathbb{R}_{>0}$,

$$L_{\mathbb{R}}(\triangle_n - \pi_n; \lambda) = L_{\mathbb{R}}((\triangle_n - \pi_n)^{\circ}; \lambda + 2).$$

Hence, any integer translate of $\triangle_n - \pi_n$, in particular \triangle_n , is Gorenstein of index 2.

2. Background & Preliminaries

2.1. The stack-sorting algorithm.

The following algorithm for sorting an input sequence was first outlined in Donald Knuth's influential work, *The Art of Computer Programming* [7].

Definition 2.1. Given a permutation π on an ordered set, the *stack-sorting algorithm* is defined as follows:

- (1) Initialize an empty stack (i.e., a last-in, first-out collection of elements).
- (2) For each input value $x \in \pi$, starting with the first element, π_1 :
 - If the stack is non-empty and x is greater than the top (most-recently added) element y of the stack, then $pop\ y$ from the stack (i.e., move y to the output) and repeat.
 - Otherwise, push x onto the stack (move x to the top of the stack).
- (3) Once every element has been pushed and there is no input left to consider, pop all remaining in the stack from last-in to first-in.

For a permutation π , we denote the output of the stack-sorting algorithm as $s(\pi)$.

Example 2.2. The algorithm would sort the permutation $\pi = 213$ as follows:

- (1) We begin with an empty stack $S = \{\}$.
- (2) The first element in the permutation is 2. The stack is empty; thus, we push 2 onto the stack and obtain $S = \{2\}$.

 \Diamond

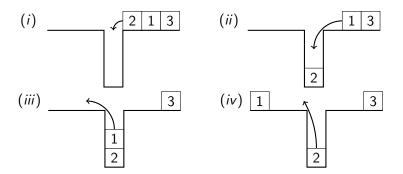


FIGURE 1. A visualization of step 2 of the stack-sorting algorithm as presented in Example 2.2.

- The next element is 1. The stack is non-empty but 1 is not greater the top of the stack, 2. Therefore, we push 1 onto the stack and obtain $S = \{2, 1\}$.
- The last element is 3. The stack is non-empty and 3 is greater than the top, 1. Thus, we pop 1 from the stack and obtain the first element of $s(\pi)$ to be 1. We repeat and the exact same thing occurs with 2. We deduce that $s(\pi)$ begins with 12.
- (3) We push 3 onto an empty stack and then pop it to obtain that $s(\pi) = 123$.

Definition 2.3. A permutation π on $[n] := \{1, 2, ..., n\}$ is t-stack-sortable if t iterations of the stack-sorting algorithm yields the identity permutation, i.e., $s^t(\pi) = 12 \cdots n$. Furthermore, π is exactly t-stack-sortable if $s^t(\pi) = 12 \cdots n$ and m < t implies $s^m(\pi) \neq 12 \cdots n$.

The identity permutation itself is defined to be 0-stack-sortable, and $s(12\cdots n)=12\cdots n$ for all n. All permutations on [n] will reach the identity after at most n-1 iterations of the algorithm. If π is exactly (n-1)-stack-sortable, then π is said to be maximal with respect to the algorithm. Thus, revisiting Example 2.2, $\pi=213$ is exactly 1-stack-sortable, is also t-stack-sortable for any $t\geq 1$, and thus is not maximal.

2.2. Polyhedral geometry & Ehrhart theory.

In this paper, we investigate the properties of a family of *convex polytopes*, in particular, a family of lattice simplices. We present some preliminaries on polyhedral geometry and Ehrhart theory, i.e., the study of lattice-points in dilations of polytopes.

A convex polytope P is the convex hull of a set of points $C = \{x_1, ..., x_k\}$, that is,

(1)
$$P = \operatorname{conv}(C) := \left\{ \sum_{i=1}^k \lambda_i \mathbf{x}_i : \lambda_i \in \mathbb{R}_{\geq 0} \text{ and } \sum_{i=1}^k \lambda_i = 1 \right\}.$$

Any expression of the form $\sum_{i=1}^{k} \lambda_i \mathbf{x}_i$, with the same conditions as imposed on the λ_i in (1), is called a *convex combination* of the points \mathbf{x}_i . The convex hull of C is the set of all convex combinations of points in C.

An *affine subspace* or affine space is a linear subspace with a possible translation by a fixed vector to its elements. In other words, it is not required that the origin be part of a set for it to be an affine subspace. Every linear subspace is an affine subspace.

Example 2.4. Any line in \mathbb{R}^2 is an affine subspace of \mathbb{R}^2 . The same applies for any plane in \mathbb{R}^3 . Notice these objects partition their respective spaces into two parts, those on one side of the line (plane) and those on the other. A line in \mathbb{R}^2 is defined by a linear equation ax + by + c = 0 and a plane in \mathbb{R}^3 is defined by ax + by + cz + d = 0, where $a, b, c, d \in \mathbb{R}$ and x, y, z are variables. At least one coefficient attached to a variable must be nonzero.

The concept from Example 2.4 can be generalized to define a hyperplane in \mathbb{R}^n as an (n-1)-dimensional affine subspace determined by the solution space of a linear equation

$$a_0+a_1x_1+\cdots+a_nx_n=0,$$

where each $a_i \in \mathbb{R}$ and at least one of a_1 through a_n is nonzero. Any hyperplane of \mathbb{R}^n partitions \mathbb{R}^n into two parts called *halfspaces*; namely, without loss of generality,

$$a_0 + a_1 x_1 + \dots + a_n x_n \ge 0$$
 and $a_0 + a_1 x_1 + \dots + a_n x_n < 0$.

A polytope P can also be defined as the bounded intersection of finitely many halfspaces; equivalently, this is the bounded solution set of a system of linear inequalities.

A polytope P is contained in the affine hull of C, denoted by aff(C), which is similar to (1), but with the looser condition that $\lambda_i \in \mathbb{R}$. We can characterize aff(C) as the smallest affine subspace containing conv(C). Moreover, the set $C = \{\mathbf{x}_1, \dots, \mathbf{x}_k\}$ is affinely independent if aff(C) is a (k-1)-dimensional space. Equivalently, C is affinely independent if and only if vectors,

$$\mathbf{x}_1 - \mathbf{x}_i, \dots, \mathbf{x}_{i-1} - \mathbf{x}_i, \mathbf{x}_{i+1} - \mathbf{x}_i, \dots, \mathbf{x}_k - \mathbf{x}_i \in \mathbb{R}^n$$

are linearly independent in \mathbb{R}^n for any $i \in [k]$.

The dimension of a polytope P, denoted $\dim(P)$, is the dimension of its affine hull. We say a polytope $P \subset \mathbb{R}^n$ is full-dimensional if $\dim(P) = n$. The convex hull of n+1 affinely independent points forms a special polytope called a simplex of dimension n. A simplex generalizes the concept of a triangle in \mathbb{R}^2 or the tetrahedron in \mathbb{R}^3 .

Changing every inequality to strict inequalities in the minimal halfspace description of P yields the *interior* of P, denoted P° . Changing every inequality to an equality in a minimal halfspace description yields the *boundary* of P, denoted ∂P . Note that $P = P^{\circ} \uplus \partial P$, where \uplus denotes a disjoint union. A polytope is *hollow* if all its integer points lie on its boundary or, equivalently, it has no integer points in its interior.

A polytope is said to be a *lattice polytope* if all of its vertices have integral coordinates. The *Ehrhart function* (or *lattice-point enumerator* or *discrete volume*) of any polytope P is defined as

$$L_{\mathbb{Z}}(P;t) := |tP \cap \mathbb{Z}^n|,$$

where $tP = \{t\mathbf{x} : \mathbf{x} \in P\}$ and t is an integer. When P is a lattice polytope, the Ehrhart function is a polynomial in t of degree equal to the dimension of P, which we refer to as the *Ehrhart polynomial*. We can encode the information of an Ehrhart polynomial in a generating series to obtain the

Ehrhart series of a polytope:

$$\mathsf{Ehr}_{\mathbb{Z}}(P;z) := 1 + \sum_{n \in \mathbb{Z}_+} L_{\mathbb{Z}}(P;n) z^n = \frac{h_{\mathbb{Z}}^*(P;z)}{(1-z)^{n+1}},$$

where n is the dimension of P and $h_{\mathbb{Z}}^*(P;z) = 1 + h_1^*z + \cdots + h_n^*z^n$ is a polynomial in z of degree at most n called the h^* -polynomial of P. The coefficients h_i^* of this polynomial are nonnegative integers and have relevant combinatorial interpretations. The interested reader can consult [2] for more information.

A fundamental result in Ehrhart theory is one of reciprocity, known as Ehrhart-Macdonald reciprocity. For any convex lattice polytope P of dimension d:

$$L_{\mathbb{Z}}(P;-t)=(-1)^nL_{\mathbb{Z}}(P^\circ;t).$$

This not only provides a relationship between a polytope and its interior, but also provides an interpretation for negative values of t. Evaluating an Ehrhart polynomial at -t gives the lattice count for the tth dilate of its interior up to a sign.

A lattice polytope $P \subset \mathbb{R}^d$ is said to be *Gorenstein of index k* if there exists a positive integer k such that

(2)
$$L_{\mathbb{Z}}(P^{\circ}; k-1) = 0, \quad L_{\mathbb{Z}}(P^{\circ}; k) = 1, \quad \text{and } L_{\mathbb{Z}}(P^{\circ}; t) = L_{\mathbb{Z}}(P; t-k)$$

for all integers t > k [2].

More generally, for any convex polytope $P \subseteq \mathbb{R}^d$ (i.e., P has any vertices in \mathbb{R}^n), the real Ehrhart counting function for all dilates $\lambda \in \mathbb{R}_{>0}$ is defined as:

$$L_{\mathbb{R}}(P;\lambda) := |\lambda P \cap \mathbb{Z}^n|.$$

For more on Ehrhart theory, one can consult [2] for an in-depth treatment of the material. A larger survey on the combinatorics of polytopes can be found in [8]. Additionally, one can learn more about rational Ehrhart theory from [1].

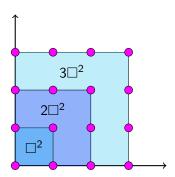


FIGURE 2. The first three integral dilates of the two-dimensional unit cube \square^2 .

Example 2.5. We define the *n*-dimensional unit cube as the convex hull over all binary strings of length n, i.e., the binary strings form the vertex set. The two-dimensional unit cube \Box^2 contains 4 points in its first dilate, 9 in its second, 16 in its third, etc. Without much difficulty, one can show directly or inductively that:

$$L_{\mathbb{Z}}(\square^2;t)=\left|t\square^2\cap\mathbb{Z}^2\right|=(t+1)^2.$$

Analogous argumentation can be used to show $L_{\mathbb{Z}}(\square^3;t)=(t+1)^3$; and, in fact, that

$$L_{\mathbb{Z}}(\square^n;t)=(t+1)^n.$$

We can encode the lattice-point enumerator in a generating function to obtain the Ehrhart series of \square^n :

$$\sum_{n\in\mathbb{N}} L_{\mathbb{Z}}(\square^n; t) z^n = \frac{\sum_{k=0}^{n-1} A(n, k) z^k}{(1-z)^{n-1}},$$

where A(n, k) denotes the Eulerian number, i.e., the number of permutations of 1 through n having exactly k descents. Further, by reciprocity, we have:

$$L_{\mathbb{Z}}(\square^n;-t)=(1-t)^n=(-1)^n(t-1)^n=(-1)^nL_{\mathbb{Z}}(\square^{n\circ};t),$$

and can conclude the interior of the t-dilate has exactly as many lattice points as the entirety of the (t-2)-dilate. For example, the interior of $3\square^2$ has 4 lattice points, the same number as all of \square^2 . We also notice that the interior of the first dilate of \square^n is empty and the interior of the second dilate has exactly one integer point. These are the conditions from (2) for k=2, which implies \square^n is Gorenstein of index 2, an important property that will appear again later in the paper. \diamondsuit

2.3. Integral and unimodular equivalence.

We recall the definition of a unimodular transformation.

Definition 2.6. A unimodular transformation in \mathbb{R}^n is a linear transformation U, i.e., a n by n matrix, with coefficients in \mathbb{Z} such that $det(U) = \pm 1$.

The following result from [6] provides a heuristic for determining whether two polytopes have the same lattice-point count.

Proposition 2.7. If a linear transformation on a lattice polytope P is unimodular, then it preserves the lattice.

In other words, the resulting polytope will have the same integer point count as P. However, not all polytopes that have equivalent lattice counts have a unimodular transformation between them. For example, if the two polytopes have square matrix representations (vertices as columns of a matrix), they must have the same determinant up to a sign for a unimodular transformation to exist between them.

Definition 2.8. Two lattice polytopes $P \subset \mathbb{R}^m$ and $Q \subset \mathbb{R}^n$ are integrally equivalent if there exists an affine transformation $\varphi : \mathbb{R}^m \to \mathbb{R}^n$ whose restriction to P preserves the lattice. In other words, φ is a bijection

$$P \cap \mathbb{Z}^m \longleftrightarrow Q \cap \mathbb{Z}^n$$
.

In particular, $L_{\mathbb{Z}}(P;t) = L_{\mathbb{Z}}(Q;t)$ for all $t \in \mathbb{N}$.

We now have an effective method to show two n-polytopes are integrally equivalent: we can search for a unimodular transformation from the lattice points of one polytope to the other.

2.4. Lecture-hall simplices.

We begin this subsection by recalling the definition of lecture-hall partitions, which were studied in [5]. They were further studied in the context of lecture-hall simplices in [4].

Definition 2.9. A lecture-hall partition of length n is a sequence $\{\alpha_1, \dots, \alpha_n\} \in \mathbb{Z}^n$ such that

$$0 \le \alpha_1 \le \frac{\alpha_2}{2} \le \dots \le \frac{\alpha_n}{n}.$$

We can construct the *lecture-hall simplex* of dimension n as

$$P_n := \operatorname{conv}\left\{ \boldsymbol{\alpha} \in \mathbb{Z}^n : 0 \le \alpha_1 \le \frac{\alpha_2}{2} \le \cdots \le \frac{\alpha_n}{n} \le 1 \right\}.$$

Dilations of P_n yields the result $tP_n = \text{conv}\left\{\alpha \in \mathbb{Z}^n : 0 \le \alpha_1 \le \frac{\alpha_2}{2} \le \cdots \le \frac{\alpha_n}{n} \le t\right\}$. We also have that $P_n \subset 2P_n \subset \cdots \subset tP_n$ because $(0,0,\ldots,0) \in P_n$. This fixes a vertex and thus avoids any translations of the polytope away from previous dilates, an observation that will become useful later on.

Example 2.10. For n = 4, we must have $0 \le \alpha_1 \le \cdots \le \frac{\alpha_4}{4} \le 1$. The first points we can identify in this simplex P_n are (0,0,0,0) as well as (1,2,3,4). We can also take the four points (0,0,0,n) for $n \in [4]$. We have (0,0,1,2), (0,0,1,3), and (0,0,1,4); (0,0,2,3) and (0,0,2,4); as well as (0,0,3,4). Finally, we can take (0,1,2,3), (0,1,2,4), (0,1,3,4), and (0,2,3,4).

This consists of a total of 16 lattice points and these are the only points in the simplex in the first dilate. Note, not all of these points are vertices, for example, (0,0,0,2) will be on the line between the origin and (0,0,0,4). This gives $L_{\mathbb{Z}}(P_4;1) = 16 = 2^4 = (1+1)^4$. In fact, we will later find the distribution of points in the lecture-hall simplex and our L'n1 simplex to be remarkably similar.

In general, the vertex set for P_n consists of exactly the n+1 affinely independent points:

$$(0,0,\ldots,0),(0,0,\ldots,n),\ldots(0,2,\ldots,n),$$
 and $(1,2,\ldots,n).$

More information about P_n , including the proof of the following proposition can be found in [4].

Proposition 2.11 (Theorems 1 and 2.2, [4]). Let P_n be the lecture-hall simplex of dimension n.

$$L_{\mathbb{Z}}(P_n;t)=(t+1)^n.$$

Observe that the lattice-point enumerator is equivalent for both the lecture-hall simplex P_n and the n-dimensional unit cube \square^n from Example 2.5. We will show that our L'n1 simplices (defined in Section 4) are also in bijection with these polytopes, and hence have the same lattice-point-enumerator.

2.5. Notation.

We conclude our preliminaries with a notation table for reference. Note that it includes notation for terms that will be defined in later sections.

Symbol	Meaning
[n]	the set $\{1, 2, \dots, n\}$
S_n	the set of all permutations on $[n]$
$s(\pi)$	the output from stack-sorting a permutation π
conv	convex hull
aff	affine hull
P°	the interior of the polytope P
∂P	the boundary of the polytope P
$L_{\mathbb{Z}}(P;t)$	the number of lattice points in tP where $t \in \mathbb{Z}$
$Ehr_{\mathbb{Z}}(P;z)$	the ordinary generating function for $L_{\mathbb{Z}}(P;t)$
$h_{\mathbb{Z}}^{*}(P;z)$	the polynomial numerator in the rational form of $Ehr_{\mathbb{Z}}(P;z)$
Пи	the <i>n</i> -dimensional unit cube
P_n	the n -dimensional lecture-hall simplex
Ln1	the set of permutations on $[n]$ ending in $n1$
$L'n1$ or π_n	the permutation $(2,3,,n,1)$
$\mathcal{F}_{L'n1}$	the $L'n1$ family
\triangle_n	the $(n-1)$ -dimensional $L'n1$ simplex
е	the permutation $(1, 2,, n)$
P'	the projection of $P \subset \mathbb{R}^n$ onto the hyperplane defined by $x_n = 0$
P	the projection of $P \subset \mathbb{R}^n$ into \mathbb{R}^{n-1} by removing each point's last coordinate
Ê	the lift of $P \subset \mathbb{R}^n$ into \mathbb{R}^{n+1} by appending a last coordinate of 0 to each point

Table 1. Notation Table

3. STACK-SORTING ON Ln1 PERMUTATIONS

In this section we explore the behavior of the stack-sorting algorithm on a special family of permutations, one we will find to act in an interesting way with respect to the algorithm.

Definition 3.1. A permutation $\pi \in S_n$ is a Ln1 permutation if π has the form Ln1, such that L is a permutation of $\{2, 3, ..., n-1\}$, i.e., a permutation that exactly ends with n and then 1. We denote the set of all permutations of such form as Ln1 and write $\pi \in Ln1$ if it takes the appropriate form.

Notice that there are (n-2)! distinct permutations in $\mathcal{L}n1$ as they can be counted by the number of different possibilities for L, the number of permutations on a set of n-2 elements. We denote the special Ln1 permutation where L=23...(n-1), i.e., where L is already sorted in ascending order, as L'n1.

Example 3.2. For n = 4, 2341 and 3241 are the two permutations that comprise the set $\mathcal{L}41$. $\mathcal{L}'n1$ will always refer to a unique permutation; as an example, $\mathcal{L}'51 = 23451$. We have that $\mathcal{L}'51$ is one of (5-2)! = 6 permutations in the set $\mathcal{L}51$.

Lemma 3.3. Let $\pi = \pi_1 \pi_2 \dots \pi_n$ be a permutation of n distinct ordered elements and let $x = \max\{\pi_1, \pi_2, \dots, \pi_n\}$. If L and R are the sub-permutations of π such that $\pi = LxR$, then the stack-sorting algorithm yields $s(\pi) = s(L)s(R)x$.

This result is a generalization of [3, Lemma 1.2]. In other words, the algorithm sorts everything to the left of x (the maximum element) first, regardless of the contents to its right. Then, everything to the right of x is sorted and x takes the nth and final place in the output permutation.

Proof. Consider an arbitrary $\pi_i \in L$. Since $x \notin L$ and thus $\pi_i \neq x$, we must have $\pi_i < x$. This implies that when all of L has been pushed on the stack and x is taken as input by the algorithm, all of L will be popped before x is pushed on the stack. After all of L has been popped, x will be pushed onto an empty stack.

Similarly, it can be determined that if $\pi_j \in R$, then $\pi_j < x$. Because there is no element greater than x that can pop it from the stack, all of R will be pushed and then subsequently popped in some order before x (at the bottom) is finally popped. Hence, the algorithm will sort R entirely into s(R) before popping x; by combining these two results, we obtain that $s(\pi) = s(L)s(R)x$. \square

As a consequence of this lemma, we obtain the following two corollaries.

Corollary 3.4. If $\pi \in \mathcal{L}n1$, then $s(\pi) = s(L)1n$.

Corollary 3.5. If $\pi \in \mathcal{L}n1$, then $s(\pi)$ exactly ends with n-1, 1, then n.

Note that Corollary 3.5 is simply an extra application of Lemma 3.3 to Corollary 3.4, specifically to s(L). Both corollaries can be readily extended to a biconditional.

Theorem 3.6. Let $\pi \in \mathcal{L}n1$. For all $i \in [n-2]$, $s^i(\pi)$ exactly ends with

$$(n-i)1(n-i+1)...(n-2)(n-1)n.$$

Proof. The base case when i=1 is the statement of Corollary 3.5. Assume the theorem holds for $s^{i-1}(\pi)$, i.e., it begins with some permutation ρ of $\{2,3,\ldots,n-i\}$ and ends exactly with $(n-i+1)1(n-i+2)\ldots(n-1)n$.

Consider another application of the stack-sorting algorithm to $s^{i-1}(\pi)$. Notice that (n-i+1) is greater than all elements in ρ . Thus, when (n-i+1) is considered by the algorithm, all elements in ρ will be popped before (n-i+1) is pushed. By Lemma 3.3, we know the final element in $s(\rho)$ will be n-i. Now, the algorithm sorts $(n-i+1)1(n-i+2)\dots(n-1)n$ into $1(n-i+1)(n-i+2)\dots(n-1)n$.

Therefore, we obtain that $s^i(\pi)$ exactly ends with $(n-i)1(n-i+1)\dots(n-1)n$ as desired. Further, note that when i=n-1 we finally obtain the identity permutation after the maximal number of iterations.

Example 3.7. For n = 6, Theorem 3.6 tells us that $\pi \in \mathcal{L}n1$ ends with 61, as expected; $s(\pi)$ ends with 516, as stated in Corollary 3.5. Further, $s^2(\pi)$ ends with 4156, $s^3(\pi)$ ends with 31456, and thus $s^3(\pi) = 231456$. We conclude $s^4(\pi) = 213456$ and $s^5(\pi) = 123456$.

Remark 3.8. For all $n \ge 4$, if $\pi \in \mathcal{L}n1$, then $s^{n-3}(\pi) = 2314 \cdots n$, $s^{n-2}(\pi) = 213 \cdots n$, and of course $s^{n-1}(\pi) = 123 \cdots n$.

This tells us that regardless of what $\pi \in \mathcal{L}n1$ is, the final three iterations of the algorithm are all the same. The permutations "meet" at this point. However, before those last few iterations, we cannot explicitly describe what $s^i(\pi)$ will start with without specifying L. Specifying L = L' yields the following useful result.

Corollary 3.9. Let $\pi = L' n1$. For $i \in [n-1]$,

$$s^{i}(\pi) = 23 \cdots (n-i)1(n-i+1) \cdots (n-1)n.$$

Inputting any sequence in ascending order will have the stack-sorting algorithm output the original sequence. Thus, the initial segment ρ of the permutation will always be $23 \dots (n-i-1)$. We effectively learn, not only about the maximality of Ln1, but also enumerate the general form of each algorithm iteration in the special case of L'n1.

Definition 3.10. We define the L'n1 family to be the set of n permutations obtained from L'n1 and its iterations under the stack-sorting algorithm. We denote this family by $\mathcal{F}_{L'n1}$. Generally, any Ln1 permutation with its iterations up to and including the identity form a Ln1 family.

Example 3.11. Consider L'51 = 23451. By Corollary 3.9 we obtain that s(L'51) = 23415, $s^2(L'51) = 23145$, $s^3(L'51) = 21345$, and $s^4(L'51) = 12345$. Notice that L'51 is maximal and

$$\mathcal{F}_{L'51} = \{23451, 23415, 23145, 21345, 12345\}.$$

We are essentially just moving the '1' one place to the left each time.

The following theorem tells us that maximal permutations and Ln1 permutations are equivalent.

Theorem 3.12. A permutation $\pi \in \mathcal{L}$ n1 if and only if π is exactly (n-1)-stack sortable.

Proof. If $\pi \in \mathcal{L}n1$, the desired consequence is a result of Theorem 3.6, which dictates that $s^{n-2}(\pi) = 213 \cdots n$. It follows $s^{n-1}(\pi) = 123 \cdots n$, therefore π is exactly (n-1)-stack-sortable.

If $\pi \notin \mathcal{L}n1$, then 1 is not in the last place of π or some k < n is directly left of 1 in π . We consider these two cases.

Case 1. If 1 is not the last digit in π , then 1 will reach the first position of the output in less than n-1 iterations of the algorithm. This is because the 1 will always move at least one place to the left if it is not already leftmost; the top of the stack will always be greater than 1, so we will always push it onto the stack where it will be immediately popped before the previous top element. The remaining n-1 elements will also have been sorted in less than n-1 iterations because the maximal possible number of iterations it takes to sort the remaining sequence of n-1 elements is n-2. Thus, π can be stack-sorted in less than n-1 iterations and is not exactly (n-1)-stack-sortable.

Case 2. If $\pi = Lk1$ such that k < n and L is a permutation of $\{1, 2, ..., k-1, k+1, ..., n\}$, then 1 will move more than one space left after the first iteration of the algorithm. This is because n will remain on the bottom of the stack (possibly more elements) while k and 1 are both pushed and then cleared first, therefore $s(\pi)$ will contain 1 to the left of both k and n. Similar to the reasoning in the prior case, 1 will reach the first position of the output in less than n-2 more iterations, while the remaining elements will also be sorted in ascending order in less than n-2 more iterations by maximality. As before, π can be stack-sorted in less than n-1 iterations and is not exactly (n-1)-stack-sortable.

4. Geometry of *Ln*1 families

In this section we interpret our permutations as points in \mathbb{R}^n , which leads us to ask: What is the geometry arising by taking the convex hull of stack-sorting iterations?

Proposition 4.1. Every Ln1 family forms an affinely independent set in \mathbb{R}^n .

Proof. Consider $\pi \in \mathcal{L}n1$. Let $\mathbf{e} = 123 \cdots n = s^{n-1}(\pi)$. Subtracting \mathbf{e} component-wise from each of the other family members and utilizing Theorem 3.6, we deduce that $s^i(\pi) - \mathbf{e}$ must end with 1(1-n+i)0...000, where the 1 is in the $(n-i-1)^{\text{th}}$ position.

Note that

$$s^{n-2}(\pi) - \mathbf{e} = 1(-1)00 \cdots 00 \text{ and } s^{n-3}(\pi) - \mathbf{e} = 11(-2)0 \cdots 00$$

by Remark 3.8. As vectors in \mathbb{R}^n , these two are linearly independent as no scalar multiple of one vector can produce the other – there is no way to obtain a nonzero number in the third coordinate of $s^{n-2}(\pi) - \mathbf{e}$.

Assume $S = \{s^{n-2}(\pi) - \mathbf{e}, s^{n-3}(\pi) - \mathbf{e}, \dots, s^{n-k+1}(\pi) - \mathbf{e}\}$ is a linearly independent set in \mathbb{R}^n and consider $S \cup \{s^{n-k}(\pi) - \mathbf{e}\}$. We know that $s^{n-k}(\pi) - \mathbf{e}$ is the only vector in this set with a nonzero number in the k^{th} coordinate. Therefore, no linear combination of the vectors in S yields $s^{n-k}(\pi) - \mathbf{e}$ and $S \cup \{s^{n-k}(\pi) - \mathbf{e}\}$ is linearly independent.

We can conclude the original Ln1 family is an affinely independent set in \mathbb{R}^n .

As a corollary we obtain the following proposition.

Proposition 4.2. The convex hull of any Ln1 family forms a (n-1)-simplex in \mathbb{R}^n .

When the family is $\mathcal{F}_{L'n1}$, we will denote its convex hull as \triangle_n and refer to it as the L'n1 simplex. Notice that the simplex formed is one dimension less than the dimension of its ambient space. This is not a trivial detail, it will necessitate some caution as we proceed to develop the Ehrhart theory of \triangle_n , especially when comparing it to the unit cube and lecture-hall simplex which are both full-dimensional.

Theorem 4.3. The L'n1 simplex is hollow. Specifically, all non-vertex integer points in \triangle_n lie on the facet formed from the convex hull of $\mathcal{F}_{L'n1} \setminus \{e\}$.

Proof. Note that all elements of $\mathcal{F}_{L'n1}$, treated as points in \mathbb{R}^n , have 2 as their fist coordinate entry except for $\mathbf{e} = (1, 2, ..., n)$. This implies that if we take a convex combination of points in $\mathcal{F}_{L'n1}$, the first coordinate entry is of the form

$$\lambda_1 + 2(\lambda_2 + \cdots + \lambda_n) = k,$$

where k is a real number and $0 < k \le 2$ because of the constraints on λ from Equation (1). Further note that $\lambda_2 + \cdots + \lambda_n = 1 - \lambda_1$. Hence, $2 - \lambda_1 = k$, which we can rewrite as $\lambda_1 = 2 - k$. For our first-coordinate entry to be an integer, we can only consider k = 1 and 2. If k = 1, then $\lambda_1 = 1$ and we obtain the vertex point \mathbf{e} . If k = 2, then $\lambda_1 = 0$ and the integer point exists on the facet containing all vertices except \mathbf{e} .

Hollowness is not a property special to \triangle_n . The unit cube and lecture-hall simplex also satisfy this property. Because we know their Ehrhart polynomials, we can argue this much more efficiently

using reciprocity:

$$(-1)^d L_{\mathbb{Z}}(P_n^{\circ}; 1) = L_{\mathbb{Z}}(P_n; -1) = (-1+1)^n = 0.$$

Remark 4.4. The lecture-hall simplex is even more so like \triangle_n because it is hollow in the exact same way, all non-vertex points are on the facet formed from the convex hull of its vertices excluding (1, 2, ..., n). An almost identical argument to Theorem 4.3 can show this to be the case.

Remark 4.5. A lattice polytope P is hollow if and only if -1 is a root of its Ehrhart polynomial. More generally, the dilate tP is hollow if and only if -t is a root of the Ehrhart polynomial of P.

5. Ehrhart theory of L'n1 simplices

We now develop the Ehrhart theory of our family of L'n1 simplices. There is a slight caveat that prevents us from constructing unimodular transformations directly between \triangle_n and \square^{n-1} or P_{n-1} because \triangle_n , unlike the other objects being full-dimensional in \mathbb{R}^{n-1} , is (n-1)-dimensional in \mathbb{R}^n . This means a matrix representation of \triangle_n will be n by n, while that of say P_{n-1} will be n-1 by n or \square^{n-1} will be n-1 by n-1. Any transformation between them is a non-square matrix and thus cannot be unimodular.

We transform from \triangle_n to P_{n-1} instead of \square^{n-1} because the former is also a simplex of the same dimension, which implies they both have n vertices. We then need to make a correction for the difference in their ambient spaces.

One option, the one we will proceed with, is to lift P_{n-1} to \mathbb{R}^n by appending a new last coordinate that does not affect the lattice point count. We will establish a unimodular transformation from this new construction to \triangle_n . Another option is to project \triangle_n onto a hyperplane, ensuring the lattice count remains preserved, and then transform this construction in \mathbb{R}^{n-1} to P_{n-1} . Regardless, to construct a unimodular transformation we need to compare two polytopes living in the same ambient space.

Before we proceed, a quick observation about the recursive structure of the lecture-hall simplex will be insightful. Let $\mathbf{0}_n$ be the zero vector in \mathbb{R}^n .

Proposition 5.1. Let V_n denote the vertex set of P_n and $Q_n := \text{conv}(V_n \setminus \{\mathbf{0}_n\})$. Then

$$L_{\mathbb{Z}}(P_n;t) = L_{\mathbb{Z}}(Q_{n+1};t).$$

In other terms, the polytopes $conv(V_n)$ and $conv(V_{n+1} \setminus \{\mathbf{0}_{n+1}\})$ are integrally equivalent.

Proof. Consider V_n , which consists of the points $\mathbf{0}_n$, $(0,0,\ldots,n)$, ..., $(0,2,\ldots,n)$, $(1,2,\ldots,n)$. Append a last coordinate of n+1 to each vertex to obtain a polytope in \mathbb{R}^{n+1} that is integrally equivalent to the original. Notice this new set is V_{n+1} without $\mathbf{0}_{n+1}$.

Theorem 5.2. The L'n1 simplex \triangle_n and (n-1)-dimensional lecture hall simplex P_{n-1} are integrally equivalent. In particular,

$$L_{\mathbb{Z}}(\triangle_n;t)=L_{\mathbb{Z}}(P_{n-1};t).$$

We employ the notation $S + \mathbf{v}$ when appropriate to denote adding the point \mathbf{v} to each point in the set S. If M is a matrix, $M + \mathbf{v}$ adds \mathbf{v} to each column of M.

Proof. Let M be the n by n matrix representation of \triangle_n with column i as $s^{n-i}(L'n1)$. Further, let L be the n by n lower-triangular matrix of -1's, let $v_k = \sum_{i=2}^k i$, and let $\mathbf{v}_k = (v_2, v_3, \dots, v_{k+1})$.

We have that $L \cdot M + \mathbf{v}_n$ has columns (1, 2, ..., n), (0, 2, ..., n), ... (0, 0, ..., n). This is the vertex set V_n of P_n without $\mathbf{0}_n$. Recall from Proposition 5.1 that $Q_n = \operatorname{conv}(V_n \setminus \{\mathbf{0}_n\})$ yields a polytope integrally equivalent to P_{n-1} . We now have a unimodular transformation from Δ_n to an integrally equivalent polytope to P_{n-1} , from which we obtain:

$$L_{\mathbb{Z}}(\triangle_n;t) = L_{\mathbb{Z}}(Q_n - \mathbf{v};t) = L_{\mathbb{Z}}(Q_n;t) = L_{\mathbb{Z}}(P_{n-1};t).$$

Example 5.3. Observe in Figure 3 that the three polytopes, the 2341 simplex $\triangle_4 \subseteq \mathbb{R}^4$, the 3-unit cube \square^3 , and the 3-dimensional lecture-hall simplex P_3 , all share the same lattice-point count in their first dilate. Also note that \triangle_4 is a 3-dimensional polytope in \mathbb{R}^4 , whereas the other two polytopes are full-dimensional in their ambient space. Notice the similarity between how points are distributed in P_3 and \triangle_3 .

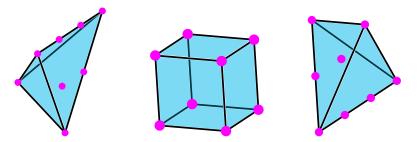


FIGURE 3. From Left to Right: the 2341-simplex $\triangle_4 \subseteq \mathbb{R}^4$, the 3-unit cube \square^3 , and the 3-dimensional lecture-hall simplex P_3 .

Since the Ehrhart polynomial for the unit cube and the lecture-hall simplex is known, we quickly obtain the following result as a corollary.

Corollary 5.4. The lattice point enumerator of the L'n1 simplex is

$$L_{\mathbb{Z}}(\triangle_n;t)=(t+1)^{n-1}.$$

Corollary 5.5. The Ehrhart series of the L'n1 simplex is

$$\mathsf{Ehr}_{\mathbb{Z}}(\triangle_n;z) = \frac{\sum_{k=0}^{n-1} A(n,k)z^k}{(1-z)^{n-1}}$$

where A(n, k) is the number of permutations of [n] with k descents.

Remark 5.6. By Ehrhart-Macdonald reciprocity, the following holds:

$$L_{\mathbb{Z}}(\triangle_{n}^{\circ}; t+2) = (-1)^{n-1}L_{\mathbb{Z}}(\triangle_{n}; -t-2)$$

= $(-1)^{n-1}(-t-1)^{n-1} = (t+1)^{n-1} = L_{\mathbb{Z}}(\triangle_{n}; t).$

Also note that $L_{\mathbb{Z}}(\triangle_n^{\circ}; 1) = 0$ and $L_{\mathbb{Z}}(\triangle_n^{\circ}; 2) = 1$. Therefore, \triangle_n is Gorenstein of index 2.

5.1. Using real lattice-point enumerators for recursive structures. Initial efforts to obtain the lattice-point enumerator for L'n1 simplices resulted in differing techniques than those used previously in the section. While a less direct method, the approach used in the following results yielded some connections to real lattice-point enumerators of various translations of our simplices which we present now.

Note that an integral equivalence of two polytopes, as defined in Definition 2.8, does not imply that the two polytopes have the same *real* lattice-point count. For example, \triangle_n and $\triangle_n - \pi_n$ are integrally equivalent, but have different real-lattice point enumerators, as we will later see. This is in notable contrast to the fact integer translations are lattice invariant for lattice polytopes.

Taking the last coordinate in each lattice point of a polytope $P \subset \mathbb{R}^n$ and setting it equal to 0 is the *projection* of P to the hyperplane defined by $x_n = 0$. We denote this operation as $\operatorname{proj}_{x_n=0}(P)$ and use an apostrophe P' as shorthand.

Lemma 5.7. Let $\mathbf{p} = (p_1, ..., p_n) \in \operatorname{aff}(\triangle_n)$, and $\mathbf{p}' = (p_1, ..., p_{n-1}, 0)$. Consider the L'n1 simplex \triangle_n and let $\triangle'_n := \operatorname{proj}_{(x_n=0)}(\triangle_n)$. Then, for any $\lambda \in \mathbb{R}_{\geq 0}$:

$$L_{\mathbb{R}}(\triangle_n - \mathbf{p}; \lambda) = L_{\mathbb{R}}(\triangle'_n - \mathbf{p}'; \lambda).$$

We shift \triangle_n by \mathbf{p} (and thus \triangle'_n by \mathbf{p}') as a slight extra step in order to show the much stronger result that these two shifted polytopes are in bijection for *all* real dilates λ instead of just integer dilates. Our desired result for the unshifted polytopes will come as a brief corollary.

Proof. Let
$$X := \lambda(\triangle_n - \mathbf{p}) \cap \mathbb{Z}^n$$
 and $X' := \lambda(\triangle'_n - \mathbf{p}') \cap \mathbb{Z}^n$. Define $\varphi : X \to X'$ such that $\varphi(x_1, \dots, x_{n-1}, x_n) = (x_1, \dots, x_{n-1}, 0)$.

This map is well-defined because of our construction of X' from X; it is immediate that if $(x_1, ..., x_n) \in X$ then $(x_1, ..., x_{n-1}, 0) \in X'$ because X' is the projection of each point in X to the hyperplane $x_n = 0$. To prove injectivity, suppose $\mathbf{x}, \mathbf{y} \in X$ such that $\varphi(\mathbf{x}) = \varphi(\mathbf{y})$. Thus,

$$(x_1, \dots, x_{n-1}, 0) = (y_1, \dots, y_{n-1}, 0)$$
 and $x_i = y_i$ for $i \in [n-1]$.

Since the vertices of \triangle_n are permutations of [n], points $\mathbf{z} = (z_1, \dots, z_n)$ in \triangle_n exist on the hyperplane defined by

$$\sum_{k=1}^{n} z_k = \sum_{k=1}^{n} k = \frac{n(n+1)}{2}.$$

Therefore, any point \mathbf{z} in $\triangle_n - \mathbf{p}'$ lives in $\sum_{k=1}^n z_k = 0$, as does any nonnegative real dilate $\sum_{k=1}^n \lambda z_k = \lambda \sum_{k=1}^n z_k = 0$. Hence, $\sum_{k=1}^n x_k = \sum_{k=1}^n y_k$ and further $x_n = y_n$. We conclude that $\mathbf{x} = \mathbf{y}$ and φ is injective.

For surjectivity, consider $\mathbf{x}' = (x_1, \dots, x_{n-1}, 0) \in X'$. We know there exists a unique convex combination of vertices of $\lambda(\triangle'_n - \mathbf{p}')$ such that $\sum_{k=1}^n c_k \mathbf{v}'_k = \mathbf{x}'$, where each $c_i \in \mathbb{R}_{\geq 0}$ and $\sum_{k=1}^n c_k = 1$. An integral point in $\lambda(\triangle_n - \mathbf{p})$ is a convex combination of the vertices of $\lambda(\triangle_n - \mathbf{p})$ and similarly for $\lambda(\triangle'_n - \mathbf{p}')$. Note that the corresponding vertices of both polytopes share the exact same first n-1 coordinates; thus, the convex combination will yield x_1, \dots, x_{n-1} as the first n-1 coordinates of the resulting point $\mathbf{x} \in \lambda(\triangle_n - \mathbf{p})$. Further, this point lies in the affine span of $\lambda(\triangle_n - \mathbf{p})$ and thus we know $\sum_{i=1}^n x_i = 0$. This implies $x_n = -\sum_{i=1}^{n-1} x_i$ and therefore $x_n \in \mathbb{Z}$ because $x_1, \dots, x_{n-1} \in \mathbb{Z}$. We conclude $\mathbf{x} \in X$ and φ is surjective.

Hence, we have established a bijection between X and X' for all nonnegative real dilates λ and can now deduce the polytopes $\lambda(\triangle_n - \mathbf{p})$ and $\lambda(\triangle'_n - \mathbf{p}')$ will always have the same number of lattice points.

Lemma 5.8. Let \triangle_n be the L'n1 simplex, $\pi_n = L'n1$, and $\overline{\pi}_n = L'n = (\pi_1, ..., \pi_{n-1})$. For all $t \in \mathbb{N}$, $L_{\mathbb{R}}\left(\triangle_n - \overline{\pi_{n+1}}; \frac{t}{n}\right) = L_{\mathbb{R}}\left(\triangle_n - \pi; \frac{t}{n}\right)$.

Proof. Recall integer translations preserve lattice count and note that $\pi_n = \overline{\pi_{n+1}} - (0, ..., 0, n)$. Thus, translating $\Delta_n - \pi_n$ by (0, ..., 0, n) yields $\Delta_n - \overline{\pi_{n+1}}$. We must conclude they have equivalent lattice counts for all integer dilates t. Further, translation by $(0, ..., 0, \lambda n)$ will remain integral after dilation for all rational dilates with form $\lambda = \frac{t}{n}$, making it lattice-preserving in these special cases as well.

The following result presents a recursive relationship that allows for the enumeration of lattice points of real dilates of \triangle_{n+1} from rational dilates of \triangle_n .

Theorem 5.9. For all $\lambda \in \mathbb{R}_{>0}$,

$$\begin{split} L_{\mathbb{R}}(\triangle_{n+1} - \pi_{n+1}; \lambda) &= \left| \lambda ((\triangle_{n+1} - \pi_{n+1}) \cap \mathbb{Z}^{n+1}) \right| \\ &= \sum_{k=0}^{\lfloor n\lambda \rfloor} \left| \frac{k}{n} (\triangle_n - \pi_n) \cap \mathbb{Z}^n \right| = \sum_{k=0}^{\lfloor n\lambda \rfloor} L_{\mathbb{R}} \left(\triangle_n - \pi_n; \frac{k}{n} \right). \end{split}$$

Before we present the proof of this result, we provide an example of the theorem.

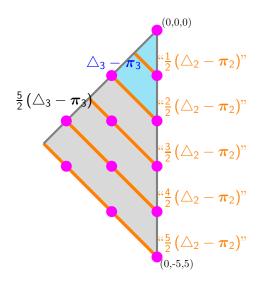


FIGURE 4. A lattice translate of the 231-simplex, $\triangle_3 - \pi_3 \subset \mathbb{R}^3$, is in blue, lattice points are in magenta, $\frac{5}{2}(\triangle_3 - \pi_3)$ is in gray, and the (lifted) five rational dilates of $\triangle_2 - \pi_2$ are in orange.

Example 5.10. Take n=2, then $\pi_3=231$ and $\pi_2=21$ and consider $\lambda=\frac{5}{2}$. Note that

$$\left|\frac{5}{2}(\triangle_3-\pi_3)\cap\mathbb{Z}^3\right|=12,$$

which we can directly count from the schematic in Figure 4. Observe that $\triangle_3 - \pi_3$ lives in \mathbb{R}^3 , but is a 2-dimensional simplex. By Theorem 5.9, we can alternatively count the number of lattice points of $\frac{5}{2}(\triangle_3 - \pi_3)$ by counting the number of lattice points in the five dilates of $\frac{k}{2}(\triangle_2 - \pi_2)$ for k = 1, 2, 3, 4, and 5. In the schematic, consider the polytopes on their affine span and we can treat the polytopes as being projected onto the hyperplane $x_3 = 0$ (hence, the quotations in some of the figure labels). Counting the number of lattice points in the five dilates of $\frac{k}{2}(\triangle_2 - \pi_2)$ for k = 1, 2, 3, 4, and 5 as shown in orange, we also obtain 12 lattice points.

Proof of Theorem 5.9. Let $\triangle'_n = \operatorname{proj}_{(x_n=0)}(\triangle_n)$, and $\pi'_n = (\pi_1, \dots, \pi_{n-1}, 0)$ from $\pi_n = (\pi_1, \dots, \pi_n) = L'n1$. By Lemma 5.7,

$$\left|\lambda(\triangle_{n+1}-\pi_{n+1})\cap\mathbb{Z}^{n+1}\right|=\left|\lambda(\triangle'_{n+1}-\pi'_{n+1})\cap\mathbb{Z}^{n+1}\right|.$$

Obtain $\hat{\triangle}_n$ by lifting \triangle_n to \mathbb{R}^{n+1} by appending 0 to the end of each lattice point of \triangle_n . By Claim 6.3, for all $\lambda \in \mathbb{R}_{>0}$,

$$\lambda(\triangle'_{n+1}-\pi'_{n+1})\cap\mathbb{Z}^{n+1}=iguplus_{k=0}^{\lfloor\lambda n
floor}\left(rac{k}{n}(\hat{\triangle}_n-\pi'_{n+1})\cap\mathbb{Z}^{n+1}
ight),$$

that is, the lattice points in the real dilates of $\triangle'_{n+1} - \pi'_{n+1}$ correspond the disjoint union of rational dilates of $\hat{\triangle}_n - \pi'_{n+1}$. Thus,

$$ig|\lambda(riangle'_{n+1}-\pi'_{n+1})\cap\mathbb{Z}^{n+1}ig|=igg|igg|_{k=0}^{\lfloor\lambda n
floor}igg(rac{k}{n}(\hat{ riangle}_n-\pi'_{n+1})\cap\mathbb{Z}^{n+1}igg)igg| = \sum_{k=0}^{\lfloor\lambda n
floor}igg|rac{k}{n}(\hat{ riangle}_n-\pi'_{n+1})\cap\mathbb{Z}^{n+1}igg|.$$

Further, we have that

$$\left|\lambda(\triangle_{n+1}-\pi_{n+1})\cap\mathbb{Z}^{n+1}\right|=\sum_{k=0}^{\lfloor n\lambda\rfloor}\left|\frac{k}{n}(\triangle_n-\overline{\pi_{n+1}})\cap\mathbb{Z}^n\right|,$$

and by Lemma 5.8,

$$=\sum_{k=0}^{\lfloor n\lambda\rfloor}\left|\frac{k}{n}(\triangle_n-\pi_n)\cap\mathbb{Z}^n\right|.$$

We continue by presenting a recursive relationship that deals with the enumeration of relative interior lattice points of real dilates of \triangle_{n+1} from rational dilates of \triangle_n .

Proposition 5.11. For all $\lambda \in \mathbb{R}_{>0}$,

$$egin{aligned} L_{\mathbb{R}}((riangle_{n+1}-oldsymbol{\pi}_{n+1})^{\circ};\lambda) &= \left| (\lambda(riangle_{n+1}-oldsymbol{\pi}_{n+1}))^{\circ} \cap \mathbb{Z}^{n+1}
ight| \ &= \sum_{k=1}^{\lceil n\lambda
ceil-1} \left| \left(rac{k}{n} (riangle_{n}-oldsymbol{\pi}_{n})
ight)^{\circ} \cap \mathbb{Z}^{n}
ight| = \sum_{k=1}^{\lceil n\lambda
ceil-1} L_{\mathbb{R}} \left((riangle_{n}-oldsymbol{\pi}_{n})^{\circ}; rac{k}{n}
ight). \end{aligned}$$

Proof. The proof follows similar arguments to that of the proof of Theorem 5.9; modifying that proof the claim holds. \Box

We conclude this section by making use of the similarities in the recurrence relations of Theorem 5.9 and Proposition 5.11 to prove the following result by induction. The result shows that the real lattice-point count for the λ -th dilate of $\Delta_n - \pi_n$ coincides with the relative interior lattice-point count for the $(\lambda + 2)$ -th dilate of $\Delta_n - \pi_n$ for every real number λ . Furthermore, the result shows that Δ_n is Gorenstein of index 2. This alternative proof is included because it allows for the Gorenstein result without knowledge of the entire lattice-point enumerator.

Theorem 5.12. For $\lambda \in \mathbb{R}$,

$$L_{\mathbb{R}}(\triangle_n - \pi_n; \lambda) = L_{\mathbb{R}}((\triangle_n - \pi_n)^{\circ}; \lambda + 2).$$

Hence, any integer translate of $\triangle_n - \pi_n$, in particular \triangle_n , is Gorenstein of index 2.

Proof. First, we prove the claim: for all $t \in \mathbb{Z}$,

$$\left|\left(\left(\frac{t}{n}+\frac{2n-1}{n}\right)(\triangle_n-\pi_n)\right)^{\circ}\cap\mathbb{Z}^n\right|=\left|\frac{t}{n}(\triangle_n-\pi_n)\cap\mathbb{Z}^n\right|.$$

In words, the relative interior lattice-point count for the $(\frac{t}{n} + \frac{2n-1}{n})$ -dilate of $\triangle_n - \pi_n$ matches the lattice-point count of the $\frac{t}{n}$ -dilate of $\triangle_n - \pi_n$. We proceed by induction on n, which relies on manipulations of summations.

Consider n=2. Observe that $\triangle_2-\pi_2$ is simply a line segment from (0,0) to (-1,1), $(\frac{t}{2}+\frac{3}{2})(\triangle_2-\pi_2)$ is a line segment from (0,0) to $(-\frac{t}{2}-\frac{3}{2},\frac{t}{2}+\frac{3}{2})$, and $\frac{t}{2}(\triangle_2-\pi_2)$ is a line segment from (0,0) to $(-\frac{t}{2},\frac{t}{2})$. Notice that all the interior lattice points of the line segment from (0,0) to $(-\frac{t}{2}-\frac{3}{2},\frac{t}{2}+\frac{3}{2})$ lie on the line segment from (0,0) to $(-\frac{t}{2},\frac{t}{2})$ giving the following relation:

$$\mathcal{L}_{\mathbb{R}}\left((\triangle_2-oldsymbol{\pi}_2)^\circ;rac{t}{2}+rac{3}{2}
ight)=\mathcal{L}_{\mathbb{R}}\left(\triangle_2-oldsymbol{\pi}_2;rac{t}{2}
ight)$$

for all $t \in \mathbb{Z}$.

Assume the statement holds for some $m \in \mathbb{Z}_{\geq 2}$. We then have:

$$\left| \frac{t}{m+1} (\triangle_{m+1} - \pi_{m+1}) \cap \mathbb{Z}^{m+1} \right| = \sum_{k=0}^{\lfloor m \cdot \frac{t}{m+1} \rfloor} \left| \frac{k}{m} (\triangle_m - \pi_m) \cap \mathbb{Z}^m \right|$$

$$= \underbrace{\sum_{k=0}^{\lfloor m \cdot \frac{t}{m+1} \rfloor}}_{(\bigstar)} \left| \left(\frac{k}{m} + \frac{2m-1}{m} (\triangle_m - \pi_m) \right)^{\circ} \cap \mathbb{Z}^m \right|.$$

By Claim 6.4, we can derive the following:

$$\begin{split} (\bigstar) &= \sum_{k=2m-1}^{\lceil m \cdot \frac{t}{m+1} + \frac{2m+1}{m+1} \rceil - 1} \left| \left(\frac{k}{m} (\triangle_m - \pi_m) \right)^{\circ} \cap \mathbb{Z}^m \right| \\ &= \sum_{k=0}^{\lceil m \cdot \frac{t}{m+1} + \frac{2m+1}{m+1} \rceil - 1} \left| \left(\frac{k}{m} (\triangle_m - \pi_m) \right)^{\circ} \cap \mathbb{Z}^m \right| - \sum_{k=0}^{2m-2} \left| \left(\frac{k}{m} (\triangle_m - \pi_m) \right)^{\circ} \cap \mathbb{Z}^m \right| \\ &= \left| \left(\left(\frac{t}{m+1} + \frac{2m+1}{m+1} \right) (\triangle_{m+1} - \pi_{m+1}) \right)^{\circ} \cap \mathbb{Z}^{m+1} \right| - \sum_{k=0}^{2m-2} \left| \left(\frac{k}{m} (\triangle_m - \pi_m) \right)^{\circ} \cap \mathbb{Z}^m \right| . \end{split}$$

Here, note that by Proposition 5.11,

$$\sum_{k=0}^{2m-2} \left| \left(\frac{k}{m} (\triangle_m - \pi_m) \right)^{\circ} \cap \mathbb{Z}^m \right| = \left| \left(\frac{2m-1}{m} (\triangle_{m+1} - \pi_{m+1}) \right)^{\circ} \cap \mathbb{Z}^{m+1} \right|$$
$$= \left| (\triangle_{m+1} - \pi_{m+1})^{\circ} \cap \mathbb{Z}^{m+1} \right|,$$
$$= 0.$$

where the last equality holds by Theorem 4.3. Hence,

$$(\bigstar) = \left| \left(\left(\frac{t}{m+1} + \frac{2m+1}{m+1} \right) (\triangle_{m+1} - \pi_{m+1}) \right)^{\circ} \cap \mathbb{Z}^{m+1} \right|.$$

Therefore, by induction, we have proved that for all $t \in \mathbb{Z}_{\geq 0}$, $n \in \mathbb{Z}_{\geq 2}$,

$$\left|\left(\left(\frac{t}{n}+\frac{2n-1}{n}\right)(\triangle_n-\pi_n)\right)^{\circ}\cap\mathbb{Z}^n\right|=\left|\frac{t}{n}(\triangle_n-\pi_n)\cap\mathbb{Z}^n\right|.$$

Now, by Theorem 5.9, for all $n \in \mathbb{Z}_{\geq 3}$

$$L_{\mathbb{R}}(\triangle_{n}-\pi_{n};\lambda)=\sum_{k=0}^{\lfloor (n-1)\lambda\rfloor}L_{\mathbb{R}}\left(\triangle_{n-1}-\pi_{n-1};\frac{k}{n}\right).$$

By our recently proven claim,

$$= \sum_{k=0}^{\lfloor (n-1)\lambda\rfloor} \left| \left(\left(\frac{k}{n-1} + \frac{2n-3}{n-1} \right) (\triangle_{n-1} - \pi_{n-1}) \right)^{\circ} \cap \mathbb{Z}^{n-1} \right|$$

$$= \sum_{k=0}^{\lceil (n-1)(\lambda+2)\rceil-1} \left| \left(\left(\frac{k}{n-1} \right) (\triangle_{n-1} - \pi_{n-1}) \right)^{\circ} \cap \mathbb{Z}^{n-1} \right|$$

$$- \sum_{k=0}^{2n-4} \left| \left(\left(\frac{k}{n-1} \right) (\triangle_{n-1} - \pi_{n-1}) \right)^{\circ} \cap \mathbb{Z}^{n-1} \right|,$$

and by Theorem 4.3,

$$= |((\lambda + 2)(\triangle_n - \pi_n))^{\circ} \cap \mathbb{Z}^n| - 0$$

= $L_{\mathbb{R}}((\triangle_n - \pi_n)^{\circ}; \lambda + 2).$

One can also show by solving inequalities that for n=2 the result holds, but we omit the technicalities here. Therefore, for all $n \in \mathbb{Z}_{\geq 2}$,

$$L_{\mathbb{R}}(\triangle_n - \pi_n; \lambda) = L_{\mathbb{R}}((\triangle_n - \pi_n)^{\circ}; \lambda + 2).$$

In particular, for all $n \in \mathbb{Z}_{\geq 2}$ and $t \in \mathbb{Z}_{\geq 0}$, \triangle_n is Gorenstein of index 2, i.e.,

$$L_{\mathbb{Z}}(\triangle_n; t) = L_{\mathbb{Z}}(\triangle_n^{\circ}; t+2).$$

Remark 5.13. If we extend the previous definition and define a polytope to be *real Gorenstein of index k* for an integer k, if the relations from Equation 2 hold for all real dilates t > k, we believe this matches the rational/real Gorenstein definitions given in [1] and we can say that $\triangle_n - \pi_n$ is "real Gorenstein" of index 2.

Dedication & Acknowledgments

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6. Appendix

Due to the length and technicalities of the proofs of the following claims, we have decided to include them in an appendix. Before beginning, it is necessary to outline a few more terms:

Given a collection of points $C = \{\mathbf{x}_1, \dots, \mathbf{x}_k\}$ in \mathbb{R}^n , we define a *pointed cone* with apex $\mathbf{v} \in \mathbb{R}^n$ as a set of the form

(4)
$$\operatorname{cone}_{\mathbf{v}}(C) := \left\{ \mathbf{v} + \sum_{i=1}^{k} \lambda_i (\mathbf{x}_i - \mathbf{v}) : \lambda_i \in \mathbb{R}_{\geq 0} \right\}.$$

This creates a cone that starts from the point \mathbf{v} and consists of the infinite region bounded by the k rays formed from \mathbf{v} through a given point \mathbf{x}_i .

Similarly, we define the pyramid over P from \mathbf{v} as

$$\mathsf{pyr}_v(P) := \left\{ \mathbf{v} + \sum_{i=1}^k \lambda_i (\mathbf{x}_i - \mathbf{v}) : \lambda_i \in \mathbb{R}_{\geq 0} \text{ and } \sum_{i=1}^k \lambda_i = 1 \right\}.$$

Note that the choice of point \mathbf{v} will affect the information that is encoded in the cone/pyramid. For a polytope P that is not full-dimensional, it is a careful selection of $\mathbf{v} \notin \mathsf{aff}(P)$ that allows dilates of P to perfectly encode themselves in 'slices' of the cone or pyramid over \mathbf{v} .

Definition 6.1. Let $S \subseteq \mathbb{R}^n$ be an (n-1)-dimensional affine subspace and take $\mathbf{x} \in \mathbb{R}^n - S$. Then the S-halfspace with \mathbf{x} denoted as $\mathcal{H}_{\mathbf{x}}(S)$ is defined as the unique halfspace in \mathbb{R}^n formed from the hyperplane S that includes the point \mathbf{x} .

Remark 6.2. Note that $\mathsf{pyr}_{\mathbf{v}}(P) = \mathsf{cone}_{\mathbf{v}}(P) \cap \mathcal{H}_{\mathbf{v}}(P)$ in our case where P is (n-1)-dimensional in n-space and $\mathbf{v} \not\in \mathsf{aff}(P)$.

The first claim shows that the lattice points in the real dilates of $\triangle'_{n+1} - \pi'_{n+1}$ correspond to the disjoint union of rational dilates of $\hat{\triangle}_n - \pi'_{n+1}$.

Claim 6.3. The following equality holds for all $\lambda \in \mathbb{R}_{>0}$:

$$\lambda(riangle_{n+1}'-\pi_{n+1}')\cap \mathbb{Z}^{n+1}=iguplus_{k=0}^{\lfloor \lambda n
floor}\left(rac{k}{n}(\hat{ riangle}_n-\pi_{n+1}')\cap \mathbb{Z}^{n+1}
ight).$$

Proof. Let $\mathbf{0}_{\mathbf{n}}$ be the origin in \mathbb{R}^n , d be the distance between $\mathbf{0}_{\mathbf{n}+1}$ and $\mathsf{aff}(\hat{\triangle}_n - \pi'_{n+1})$, and \mathbf{v} be any fixed vector on the plane $x_{n+1} = 0$ that is orthogonal to $\mathsf{aff}(\hat{\triangle}_n - \pi'_{n+1})$ such that $\mathbf{v} \cdot \pi'_{n+1} \geq 0$ (see Figure 5, for a concrete visual example).

We will show that the lattice points in λ -th dilate of $\triangle'_{n+1} - \pi'_{n+1}$ corresponds to the lattice points in the union of slices of $\lambda(\triangle'_{n+1} - \pi'_{n+1})$ obtained by intersecting the polytope with a infinite set of hyperplanes $\mathsf{aff}(\lambda(\hat{\triangle}_n - \pi'_{n+1})) + k\mathbf{v}$ for all real numbers $k \in [0, \frac{\lambda d}{|\mathbf{v}|}]$.

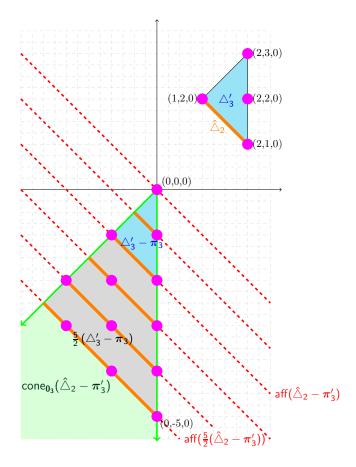


FIGURE 5. This figure corresponds to the set-up of the proof of Claim 6.3, where $\lambda = \frac{5}{2}, \ n = 2$, and distance $d = \sqrt{2}$.

Take $\mathcal{H}_{\mathbf{0}_{n+1}}(\lambda(\hat{\triangle}_n - \pi'_{n+1}))$ as defined using Definition 6.1. Then

$$\lambda(\triangle'_{n+1} - \pi'_{n+1}) \cap \mathbb{Z}^{n+1} = \lambda(\triangle'_{n+1} - \pi'_{n+1}) \cap \mathcal{H}_{\mathbf{0}_{n+1}}(\lambda(\hat{\triangle}_n - \pi'_{n+1})) \cap \mathbb{Z}^{n+1}$$

$$= \lambda(\triangle'_{n+1} - \pi'_{n+1}) \cap \left(\bigcup_{0 \le k \le \frac{\lambda d}{|\mathbf{v}|}} (\operatorname{aff}(\lambda(\hat{\triangle}_n - \pi'_{n+1})) + k\mathbf{v}) \right) \cap \mathbb{Z}^{n+1}$$

$$= \bigcup_{0 \le k \le \frac{\lambda d}{|\mathbf{v}|}} \left(\lambda(\triangle'_{n+1} - \pi'_{n+1}) \cap (\operatorname{aff}(\lambda(\hat{\triangle}_n - \pi'_{n+1})) + k\mathbf{v}) \cap \mathbb{Z}^{n+1}\right).$$
(5)

Since,

$$\mathsf{aff}(\hat{\triangle}_n - \pi'_{n+1}) = \{ \mathbf{x} \in \mathbb{R}^{n+1} | x_1 + \dots + x_n = -n, x_{n+1} = 0 \},$$

note that $\mathbf{v} = c(1, \dots, 1, 0) \in \mathbb{R}^{n+1}$ for some $c \in \mathbb{R}$.

Without loss of generality, take c=1 and consider $\mathbf{v}=(1,\ldots,1,0)\in\mathbb{R}^{n+1}$. Note that $d=|\mathbf{v}|$ and thus, we can proceed with (5) as follows:

$$\lambda(\triangle'_{n+1}-\pi'_{n+1})\cap\mathbb{Z}^{n+1}=\mathsf{cone}_{\mathbf{0}_{n+1}}(\lambda(\hat{\triangle}_n-\pi'_{n+1}))\cap\bigcup_{0\leq k\leq\lambda}\left((\mathsf{aff}(\lambda(\hat{\triangle}_n-\pi'_{n+1}))+k\mathbf{v})\cap\mathbb{Z}^{n+1}\right),$$

with $\mathbf{v} = (1, ..., 1, 0) \in \mathbb{R}^{n+1}$.

Now, we show that lattice points can only exist on the slices of $\lambda(\triangle'_{n+1} - \pi'_{n+1})$ obtained from the hyperplanes $\operatorname{aff}(\lambda(\hat{\triangle}_n - \pi'_{n+1})) + k\mathbf{v}$ for $k \in [0, \lambda]$ such that kn is a integer. Equivalently, we show that if $kn \notin [0, \lambda] \cap \mathbb{Z}$, then $\lambda(\triangle'_{n+1} - \pi'_{n+1}) \cap (\operatorname{aff}(\lambda(\hat{\triangle}_n - \pi'_{n+1})) + k\mathbf{v}) = \emptyset$. So, first we consider $\lambda(\triangle_{n+1} - \pi)$, which is $\lambda(\triangle'_{n+1} - \pi'_{n+1})$ before the projection to $x_{n+1} = 0$. Notice that the slice of $\lambda(\triangle_{n+1} - \pi)$ obtained from intersecting it with the hyperplane $H_h := \{\mathbf{x} \in \mathbb{R}^{n+1} | x_{n+1} = h\}$ only contains lattice points if h is an integer. Thus,

$$h \notin \mathbb{Z} \cap [-\lambda n, 0] \Longrightarrow (\lambda(\triangle_{n+1} - \pi) \cap \mathbb{Z}^{n+1}) \cap H_h = \emptyset.$$

Next, we show that the slice of $\lambda(\triangle_{n+1} - \pi)$ obtained from intersecting it with H_h is equivalent to the slice of $\lambda(\triangle_{n+1} - \pi)$ obtained from intersecting it with $H^{\text{vert}} + (1 - \frac{h}{n})\mathbf{v}$, where

$$H^{\mathrm{vert}} := \{ \boldsymbol{\mathsf{x}} \in \mathbb{R}^{n+1} | x_1 + \dots + x_n = -n \}.$$

Note that

$$\mathsf{aff}(\lambda(\triangle_{n+1} - \pi_{n+1})) = \{ \mathbf{x} \in \mathbb{R}^{n+1} | x_1 + \dots + x_{n+1} = 0 \}.$$

Now, we can derive the following:

$$\begin{split} H_h \cap \operatorname{aff}(\lambda(\triangle_{n+1} - \pi_{n+1})) &= \{\mathbf{x} \in \mathbb{R}^{n+1} | x_{n+1} = h\} \cap \{\mathbf{x} \in \mathbb{R}^{n+1} | x_1 + \dots + x_{n+1} = 0\} \\ &= \{\mathbf{x} \in \mathbb{R}^{n+1} | x_1 + \dots + x_n = -h, x_{n+1} = h\} \\ &= \{\mathbf{x} \in \mathbb{R}^{n+1} | x_1 + \dots + x_n = -h\} \\ &\qquad \qquad \cap \{\mathbf{x} \in \mathbb{R}^{n+1} | x_1 + \dots + x_{n+1} = 0\} \\ &= \left\{ \left(x_1 + \left(1 - \frac{h}{n} \right), \dots, x_n + \left(1 - \frac{h}{n} \right), x_{n+1} \right) \middle| x_1 + \dots + x_n = -n \right\} \\ &\qquad \qquad \cap \{\mathbf{x} \in \mathbb{R}^{n+1} | x_1 + \dots + x_{n+1} = 0\} \\ &= \left(\{\mathbf{x} \in \mathbb{R}^{n+1} | x_1 + \dots + x_n = -n\} + \left(1 - \frac{h}{n} \right) \mathbf{v} \right) \cap \operatorname{aff}(\lambda(\triangle_{n+1} - \pi)) \\ &= \left(H^{\operatorname{vert}} + \left(1 - \frac{h}{n} \right) \mathbf{v} \right) \cap \operatorname{aff}(\lambda(\triangle_{n+1} - \pi)). \end{split}$$

Hence,

$$\left(\lambda(\triangle_{n+1}-\pi_{n+1})\cap\mathbb{Z}^{n+1}\right)\cap H_k=\left(\lambda(\triangle_{n+1}-\pi_{n+1})\cap\mathbb{Z}^{n+1}\right)\cap\left(H^{\mathrm{vert}}+\left(1-\frac{h}{n}\right)\mathbf{v}\right).$$

Further, note that the slice of $\lambda(\triangle_{n+1} - \pi_{n+1})$ obtained from intersecting it with $H^{\text{vert}} + (1 - \frac{h}{n})\mathbf{v}$ has no lattice point on it if h is not an integer, i.e.,

$$h
ot \in \mathbb{Z} \cap [-\lambda n, 0] \Longrightarrow (\lambda(\triangle_{n+1} - \pi_{n+1}) \cap \mathbb{Z}^{n+1}) \cap \left(H^{\operatorname{vert}} + \left(1 - \frac{h}{n}\right)\mathbf{v}\right) = \emptyset.$$

Lastly, we show that the projection onto $x_{n+1} = 0$ of the slice of $\lambda(\triangle_{n+1} - \pi_{n+1})$ obtained from intersecting it with $H^{\text{vert}} + (1 - \frac{h}{n})\mathbf{v}$ is equivalent to the slice of $\lambda(\hat{\triangle}_n - \pi'_{n+1})$ obtained from intersecting it with $\text{aff}(\lambda(\hat{\triangle}_n - \pi'_{n+1})) + (1 - \frac{h}{n})\mathbf{v}$. So, consider

$$\mathsf{proj}_{\mathsf{x}_{n+1}=0}\left(\left(\lambda(riangle_{n+1}-oldsymbol{\pi}_{n+1})\cap\mathbb{Z}^{n+1}
ight)\cap\left(H^{\mathrm{vert}}+\left(1-rac{h}{n}
ight)oldsymbol{\mathsf{v}}
ight)
ight).$$

Then

$$\begin{aligned} \operatorname{proj}_{\mathsf{x}_{n+1}=0} \left(\left(\lambda(\triangle_{n+1} - \pi_{n+1}) \cap \mathbb{Z}^{n+1} \right) \cap \left(H^{\operatorname{vert}} + \left(1 - \frac{h}{n} \right) \mathbf{v} \right) \right) \\ &= \operatorname{proj}_{\mathsf{x}_{n+1}=0} \left(\lambda(\triangle_{n+1} - \pi_{n+1}) \cap \mathbb{Z}^{n+1} \right) \cap \operatorname{proj}_{\mathsf{x}_{n+1}=0} \left(H^{\operatorname{vert}} + \left(1 - \frac{h}{n} \right) \mathbf{v} \right) \\ &= \left(\lambda(\triangle'_{n+1} - \pi'_{n+1}) \cap \mathbb{Z}^{n+1} \right) \cap \left(\left\{ \mathbf{x} \in \mathbb{R}^{n+1} | \mathbf{x}_1 + \dots + \mathbf{x}_n = -n, \mathbf{x}_{n+1} = 0 \right\} + \left(1 - \frac{h}{n} \right) \mathbf{v} \right) \\ &= \left(\lambda(\triangle'_{n+1} - \pi'_{n+1}) \cap \mathbb{Z}^{n+1} \right) \cap \left(\operatorname{aff} (\lambda(\hat{\triangle}_n - \pi'_{n+1})) + \left(1 - \frac{h}{n} \right) \mathbf{v} \right). \end{aligned}$$

Thus,

$$h \not\in \mathbb{Z} \cap [-\lambda n, 0] \Longrightarrow (\lambda(\triangle'_{n+1} - \pi'_{n+1}) \cap \mathbb{Z}^{n+1}) \cap \left(\operatorname{aff}(\lambda(\hat{\triangle}_n - \pi'_{n+1})) + \left(1 - \frac{h}{n}\right)\mathbf{v}\right) = \emptyset.$$

Therefore,

$$kn \not\in \mathbb{Z} \cap [0, \lambda n] \Longrightarrow (\lambda(\triangle'_{n+1} - \pi'_{n+1}) \cap \mathbb{Z}^{n+1}) \cap \left(\mathsf{aff}(\lambda(\hat{\triangle}_n - \pi'_{n+1})) + k\mathbf{v} \right) = \emptyset.$$

To conclude, by "flossing" out all the slices of $\lambda(\triangle'_{n+1} - \pi'_{n+1})$ that don't contain lattice points, we can derive the following:

$$\bigcup_{0 \leq k \leq \lambda} \left(\lambda(\triangle'_{n+1} - \pi'_{n+1}) \cap \left(\operatorname{aff}(\lambda(\hat{\triangle}_n - \pi'_{n+1})) + k\mathbf{v} \right) \cap \mathbb{Z}^{n+1} \right) \\
= \biguplus_{\substack{0 \leq k \leq \lambda: \\ kn \in \mathbb{Z}}} \left(\lambda(\triangle'_{n+1} - \pi'_{n+1}) \cap \left(\operatorname{aff}(\lambda(\hat{\triangle}_n - \pi'_{n+1})) + k\mathbf{v} \right) \cap \mathbb{Z}^{n+1} \right) \\
= \biguplus_{\substack{k=0}} \left(\lambda(\triangle'_{n+1} - \pi'_{n+1}) \cap \left(\operatorname{aff}(\lambda(\hat{\triangle}_n - \pi'_{n+1})) + \frac{k}{n}\mathbf{v} \right) \cap \mathbb{Z}^{n+1} \right) \\
= \biguplus_{\substack{k=0}} \left(\operatorname{pyr}_{\mathbf{0}_{n+1}} (\lambda(\hat{\triangle}_n - \pi'_{n+1})) \cap \left(\operatorname{aff}(\lambda(\hat{\triangle}_n - \pi'_{n+1})) + \frac{k}{n}\mathbf{v} \right) \cap \mathbb{Z}^{n+1} \right) \\
= \biguplus_{\substack{k=0}} \left(\operatorname{cone}_{\mathbf{0}_{n+1}} (\hat{\triangle}_n - \pi'_{n+1}) \cap \left(\operatorname{aff}(\lambda(\hat{\triangle}_n - \pi'_{n+1})) + \frac{k}{n}\mathbf{v} \right) \cap \mathbb{Z}^{n+1} \right) \\
= \biguplus_{\substack{k=0}} \left(\frac{k}{n} (\hat{\triangle}_n - \pi'_{n+1}) \cap \mathbb{Z}^{n+1} \right) .$$

Hence, the claim holds.

Claim 6.4. For $\pi_n := L'n1$, we have

$$\sum_{k=0}^{\lfloor n \cdot \frac{t}{n+1} \rfloor} \left| \left(\frac{k}{n} + \frac{2n-1}{n} (\triangle_n - \pi_n) \right)^{\circ} \cap \mathbb{Z}^n \right| = \sum_{k=2n-1}^{\lceil n \cdot \frac{t}{n+1} + \frac{2n+1}{n+1} \rceil - 1} \left| \left(\frac{k}{n} (\triangle_n - \pi_n) \right)^{\circ} \cap \mathbb{Z}^n \right|.$$

Proof. By dividing the cases for when n+1|t and $n+1\nmid t$, we can derive the following:

$$\begin{split} \sum_{k=0}^{\lfloor n \cdot \frac{t}{n+1} \rfloor} \left| \left(\frac{k}{n} + \frac{2n-1}{n} (\triangle_n - \pi_n) \right)^{\circ} \cap \mathbb{Z}^n \right| \\ &= \begin{cases} \sum_{k=2n-1}^{\lceil n \cdot \frac{t}{n+1} + 2n \rceil - 1} \left| \left(\frac{k}{n} (\triangle_n - \pi_n) \right)^{\circ} \cap \mathbb{Z}^n \right| & \text{if } n+1 | t \\ \sum_{k=2n-1}^{\lceil n \cdot \frac{t}{n+1} + 2n - 1 \rceil - 1} \left| \left(\frac{k}{n} (\triangle_n - \pi_n) \right)^{\circ} \cap \mathbb{Z}^n \right| & \text{if } n+1 \nmid t. \end{cases} \end{split}$$

Note that if $n+1 \mid t$, then $n \cdot \frac{t}{n+1} + 2n \in \mathbb{Z}$, thus

$$\left\lceil n \cdot \frac{t}{n+1} + 2n \right\rceil = \left\lceil n \cdot \frac{t}{n+1} + 2n - \frac{n}{n+1} \right\rceil.$$

Alternatively, if
$$n+1 \nmid t$$
, then $0 < n \cdot \frac{t}{n+1} + 2n - 1 - \left \lceil n \cdot \frac{t}{n+1} + 2n - 1 \right \rceil \leq \frac{n}{n+1}$, thus

$$\left[n \cdot \frac{t}{n+1} + 2n - 1\right] = \left[n \cdot \frac{t}{n+1} + 2n - 1 + \frac{1}{n+1}\right].$$

Therefore,

$$\sum_{k=0}^{\lfloor n \cdot \frac{t}{n+1} \rfloor} \left| \left(\frac{k}{n} + \frac{2n-1}{n} (\triangle_n - \pi_n) \right)^{\circ} \cap \mathbb{Z}^n \right| = \sum_{k=2n-1}^{\lceil n \cdot \frac{t}{n+1} + \frac{2n+1}{n+1} \rceil - 1} \left| \left(\frac{k}{n} (\triangle_n - \pi_n) \right)^{\circ} \cap \mathbb{Z}^n \right|.$$

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