**Chapter 11 Voting and Apportionment**

*Photo 11.1: Voting image. Example given here.*

Suppose a friend asked you, “When did you last vote?” What would your answer be? Maybe you would tell your friend that the last time you voted was during the last presidential election, or perhaps you would tell your friend that you prefer not to vote. When thinking about voting, presidential campaigns or advertisements for reelections may come to mind, but you can cast your vote in many ways. Have you liked a post, followed a creator, friended a stranger, or clicked a heart online today? In the digital age, it’s possible to vote several times a day. Voting systems are not only the machines that drive every democracy on Earth, but they are also the engines driving social media and many other aspects of life. A deeper understanding of these voting systems will enhance your ability to successfully engage with the world in which we live.

In this chapter, you will become one of the founders of the new democratic country of Imaginaria. You have a great responsibility to the people of this fledgling democracy. You have been tasked with writing the portion of the constitution that lays out voting procedures. In preparation for this important task, you will explore the various types of voting systems, from school board elections to Twitter wars. You will see how these types are alike, how they differ, and how they might be applied in Imaginaria. Most importantly, you will learn about the mathematically inherent advantages and disadvantages of various voting systems so that you can make informed choices to better the lives of the Imaginarians.

# [H1] Module 1 – Voting Methods

*Photo 11.2: Image of President Obama casting his vote.*

**

**After completing this module, you should be able to:**

1. Apply plurality voting to determine a winner LO 11.1.1.
2. Apply runoff voting to determine a winner LO 11.1.2.
3. Apply ranked-choice voting to determine a winner LO 11.1.3.
4. Apply Borda count voting to determine a winner LO 11.1.4.
5. Apply pairwise comparison and Condorcet voting to determine a winner LO 11.1.5.
6. Apply approval voting to determine a winner LO 11.1.6.
7. Compare and contrast voting methods to identify flaws LO 11.1.7.

Today is the day that you begin your quest to collaborate on the constitution of Imaginaria! Let’s begin by thinking about the selection of a leader who can serve as President. It seems straightforward; if the majority of citizens prefer a particular candidate, that candidate should win. But not all votes are decided by a simple majority. Why not? What are the options?

## [H2] Majority versus Plurality Voting

When an election involves only two options, a simple **majority** is a reasonable way to determine a winner.

**[DEFINITION]**

A **majority** is a number equaling more than half or greater than 50 percent of the total.<END>

Let’s take a look at the outcomes of U.S. presidential elections to understand more. Figure 11.1 displays the results of the 2000 U.S. Presidential Election. Like most presidential elections, this election involved more than two options. If that is the case, is it possible that no single candidate will receive more than half of the votes cast?

**Figure 11.1 The 2000 U.S. Presidential Election Popular Vote**

|  |  |
| --- | --- |
| **CANDIDATE (PARTY LABEL)** | **POPULAR VOTE TOTAL** |
| Al Gore (Democrat) | 50,999,897 |
| George W. Bush (Republican) | 50,456,002 |
| Ralph Nader (Green) | 2,882,955 |
| Patrick J. Buchanan (Reform/Independent) | 448,895 |
| Harry Browne (Libertarian) | 384,431 |
| Howard Phillips (Constitution) | 98,020 |
| Other | 134,900 |
| **Total:** | **105,405,100** |

Figure 11.1 shows the vote summary for all candidates on the ballot in at least one state.   
**For more details visit:** FEC (Federal Election Commission) <https://www.fec.gov/introduction-campaign-finance/election-and-voting-information/federal-elections-2000/president2000/>

### <example> EXAMPLE 1 - <title>Majority of Popular Vote in the 2000 U.S. Presidential Election</title>

Figure 11.1 shows the 2000 presidential popular vote summary for all candidates listed on at least one state ballot. Did any single candidate secure the majority of popular votes?

**<SOLUTION>**

Step 1–Calculate 50 percent of 105,405,100 by multiplying the decimal form of 50 percent, which is 0.50, by 158,394,605: 

Step 2–Determine the minimum number of votes needed to have a majority. The minimum number of votes required is the lowest counting number that is larger than 50 percent of the votes. To have a majority, an individual candidate must have more than 52,702,550; so, a majority candidate must have 52,702,551 votes or more.

Step 3–Compare the number of votes each candidate received to 52,702,551. According to the data in Figure 11.1, none of the candidates secured a majority. <END>

### [Your Turn] 1

According to the Cook Political Report, a total of 158,394,605 votes were cast in the 2020 U.S. presidential election. Of those, 81,281,502 were cast for Joe Biden, 74,222,593 for Donald Trump, and 2,890,510 for other candidates. Although U.S. presidential elections are not determined by popular vote, did either Joe Biden or Donald Trump secure a majority of the votes? <END>

**Answer**

Yes, Joe Biden won the majority. <END>

Unlike in the 2000 U.S. presidential election, a candidate won the majority of votes in the 2020 election (see Figure 11.1). It is a common occurrence for no single candidate to receive a majority of the votes in an election with more than two candidates. When this occurs, the candidate with the largest portion of the votes is said to have a **plurality**.

**[DEFINITION]**

A **plurality** occurs when a candidate receives more votes than any other candidate, but they may or may not have received a majority of the votes. <END>

### <example> EXAMPLE 2 - <title>Plurality of Popular Vote in the 2000 U.S. Presidential Election</title>

In the 2000 U.S. presidential election, which candidate had a plurality of popular votes? (Refer to Figure 11.1).

**<SOLUTION>**

Al Gore secured 50,999,897 votes which was more than any other single candidate. Therefore, he had a plurality of the popular votes. <END>

Your plans for Imaginarian elections will likely include primary elections, or preliminary elections to select candidates for a principal or general election. Review the results of the 2018 U.S. Senate primary for Maryland (Figure 11.2).

**Figure 11.2 Vote Counts in the 2018 U.S. Senate Republican Primary for Maryland**

|  |  |  |
| --- | --- | --- |
| **TOP FOUR REPUBLICAN**  **CANDIDATES** | **VOTES** | **PERCENTAGE OF PARTY VOTES** |
| Cambell, Tony | 51,426 | 29.22% |
| Chaffee, Chris | 42,328 | 24.05% |
| Grigorian, Christina J. | 30,756 | 17.48% |
| Graziani | 15,435 | 8.77% |
| **Total Votes** | 175,981 | 100% |

### [Your Turn] 2

Consider the results of the 2018 U.S. Senate primary for Maryland in Figure 11.2. Determine which candidate won the primary for the Republicans (R). Did the candidate win a majority or a plurality of Republican votes? <END>

**Answer**

Tony Cambell won a plurality of Republic votes. <END>

Consider how election by plurality, not majority, is the most common method of selecting candidates for public office.

### [WHO KNEW?] The U.S. Electorial College – Winning the Presidency without a Plurality

In the 2000 U.S. presidential election, Al Gore had a plurality of the popular votes, but he did not win the election. Why? This occurred because the U.S. President and Vice President are elected by electors rather than a direct vote by the citizens. The electors are part of the Electoral College, a body of people representing the states. Why was the Electoral College created? The Electoral College was created as a compromise between those authors of the U.S. Constitution who believed Congress should elect the President, and those who believed the citizens should vote directly. The popular vote was not recorded until the presidential election of 1824. Since then, only five presidents have been elected without winning a plurality of the popular vote: John Quincy Adams in 1824, Rutherford B. Hayes in 1876, Grover Cleveland in 1888, George Bush in 2000, and Donald Trump in 2016.

## [H2] Runoff Voting

Has your family ever debated what to have for dinner? Suppose your family is deciding on a restaurant and exactly half of you want to have pizza but the other half want hamburgers. How do you decide when the result is a tie? You need a tiebreaker!

Will the new democracy of Imaginaria need tiebreakers? You could use **runoff elections** in a situation in which no candidate satisfies the requirements to win the election.

**[DEFINITION]**

A **runoff election** is a second election held to determine a winner when no candidate has met the requirement to win the first election.

A **runoff voting system** is any voting system that includes a runoff election when the first round does not result in a winner. <END>

How would run-off voting work in Imaginaria? There are many types of **runoff voting systems**. The method for implementing a runoff election can vary widely, particularly in the criteria used to determine if a candidate will be on the ballot in the second election. For example, a **two-round system** is a runoff voting system in which only the top candidates advance to the runoff election. In some two-round systems, only the top two candidates are on the second ballot, or it may be any candidate who secures a certain percentage of the vote will advance. The **Hare Method** is another runoff voting system in which only the candidate(s) with the very least votes are eliminated. This can potentially result in several rounds of runoff elections.

**[DEFINITION]**

A **two-round system** is a voting method in which a majority is required, and, if there is no majority in the first round, the top two candidates (or candidates meeting another predetermined threshold) advance a runoff election.

The **Hare Method** of voting refers to a system with potentially multiple runoffs in which all candidates except the candidate(s) in last place advance to the next round until a single candidate has a majority. <END>

### <example> EXAMPLE 3 - <title>Runoff Election for Condominium Association President</title>

A condominium association elects a new president every two years by a two-round system of voting. If none of the candidates receive a majority, the association charter states that the top two candidates will be eligible to participate in a runoff election. In a particular year, five residents were nominated. The results of the first round are given in Figure 11.3.

**Figure 11.3 Condominium Association President Election First Round Votes**

|  |  |
| --- | --- |
| **Candidate** | **Votes in First Round** |
| Abou | 18 |
| Baiocchi | 10 |
| Campana | 5 |
| Dali | 11 |
| Eugene | 4 |

Is there a winner based on the first round? Why or why not?

<**SOLUTION>**

A majority of 48 total votes is required to win. Begin by finding 50 percent of 48, which is calculated as follows: . A majority is 25 or more. No candidate has a majority, so there is no winner based on the first round.

<END>

### [Your Turn] 3

The student government bylaws of a particular college require that a new president is elected annually by plurality voting. In the event of a tie, the bylaws require the candidate(s) with the fewest votes to be removed from the ballot and a runoff election to be held with the remaining candidates. This process is repeated until a single candidate receives a plurality and wins the election. The results of two voting rounds are given in Figure 11.4.

**Figure 11.4 Student Government President Election First and Second Round Votes**

|  |  |  |
| --- | --- | --- |
| **Candidate** | **Votes in First Round** | **Votes in Second Round** |
| Ferguson | 158 | 168 |
| Garcia | 103 | 104 |
| Hearn | 157 | 180 |
| Isaac | 123 | 123 |
| Jackman | 58 | Eliminated |
| Kelly | 72 | 74 |
| Lim | 158 | 180 |

Which two candidates tied in the second round? Does this mean there will be a third election?

<END>

**<SOLUTION>**

Hearn and Lim tied. Yes, there must be a third election.

<END>

### [H2] Steps to Determine Winner by Plurality or Majority Election with Runoff

Step 1 — If a majority is required to win the election, determine the number of votes needed to achieve a majority. This is the least whole number greater than 50 percent of the total votes. If a majority is not required, move to Step 2.

Step 2 — Count the number of votes for each candidate in the current round of voting. If a single candidate has enough votes to win a plurality, or a majority as appropriate, then you are done!Otherwise, eliminate a predetermined number of candidates based on the rules of the election. Elimination conditions may vary. For example, the rules may state that the candidate(s) with the fewest votes will be eliminated (as in the Hare method), or that only the candidates meeting a certain threshold will move on (as in a two-round system). Once the appropriate candidates are eliminated, move on to Step 3.

Step 3 — Hold a runoff election. If the runoff is simulated using a list of voter’s preferences, renumber the preferences to reflect the remaining number of options in such a way that the original order of preference is retained. Then repeat Step 2.

*Note: Steps 2 and 3 may be repeated as many times as necessary for voting procedures that allow multiple runoffs*.

### <example> EXAMPLE 4 - <title> Family Dinner Night</title>

The five members of the Chionilis family—Annette, Rene, Seema, Titus, and Galen—have decided to get takeout for dinner. They are trying to decide on a restaurant. The options are Rainbow China, Dough Boys Pizza, Taco City, or Caribbean Flavor. They will use majority election with runoffs where the restaurant with the fewest votes is eliminated in each round. The preferences of each family member are listed by first initial in Figure 11.5. An entry of 1 represents the person’s first choice; 2, their second; and so on. For example, Annette’s second choice is Dough Boys Pizza.

**Figure 11.5 Chionilis Family Restaurant Preferences**

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| **Options** | **A** | **R** | **S** | **T** | **G** |
| Rainbow China | 1 | 3 | 3 | 1 | 3 |
| Dough Boys Pizza | 2 | 2 | 1 | 2 | 1 |
| Taco City | 3 | 4 | 2 | 4 | 2 |
| Caribbean Flavor | 4 | 1 | 4 | 3 | 4 |

Use the information in Figure 11.5 to answer the following questions.

1. Which common type of runoff voting method is this?
2. List the results of each round of voting based on this information and determine which restaurant was ultimately chosen.

**<SOLUTION>**

1. The Hare Method.
2. Step 1 —- Determine the number of votes necessary to have a majority. There are five family members, so 50 percent of 5 is . A majority is three or more votes.

Step 2 — Count the number of votes for each restaurant in the first election. In a list of voter preferences, the 1s represent the top choice of each voter, which corresponds to their vote in the first round.

Results of Round 1:

* + Rainbow China — 2 votes
  + Dough Boys — 2 votes
  + Taco City — 0 votes
  + Caribbean Flavor — 1 vote

No restaurant received a majority. Eliminate Taco City which has the fewest first place votes.

**Figure 11.6 Chionilis Family Restaurant Preferences after Round 1 Elimination**

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| **Options** | **A** | **R** | **S** | **T** | **G** |
| Rainbow China | 1 | 3 | 3 | 1 | 3 |
| Dough Boys Pizza | 2 | 2 | 1 | 2 | 1 |
| Caribbean Flavor | 4 | 1 | 4 | 3 | 4 |

Step 3 — Hold a runoff election. In other words, hold a second round. Since we have a list of the voters’ preferences with the eliminated option removed, we will renumber the preferences as first, second, and third so that we keep the original order of preference. The result is that we will count the second-place vote of any voter whose first choice was eliminated.

**Figure 11.7 Chionilis Family Restaurant Preferences Round 2 Election**

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| **Options** | **A** | **R** | **S** | **T** | **G** |
| Rainbow China | 1 | 3 | 2 | 1 | 2 |
| Dough Boys Pizza | 2 | 2 | 1 | 2 | 1 |
| Caribbean Flavor | 3 | 1 | 3 | 3 | 3 |

Repeat Step 2 — Count the number votes for each restaurant in the first-round election. Since we are using a list of preferences, we need to count the number of 1s received by each restaurant.

Results of Round 2:

* + Rainbow China — 2 votes
  + Dough Boys — 2 votes
  + Caribbean Flavor — 1 vote

No single restaurant has three votes. Eliminate Caribbean Flavor, which has the fewest votes.

**Figure 11.8 Chionilis Family Restaurant Preferences Round 2 Elimination**

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| **Options** | **A** | **R** | **S** | **T** | **G** |
| Rainbow China | 1 | 3 | 2 | 1 | 2 |
| Dough Boys Pizza | 2 | 2 | 1 | 2 | 1 |

Repeat Step 3 — Hold another runoff election. This will be Round 3. Renumber the voters' preferences as first and second this time.

**Figure 11.9 Chionilis Family Restaurant Preferences Round 3 Election**

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| **Options** | **A** | **R** | **S** | **T** | **G** |
| Rainbow China | 1 | 2 | 2 | 1 | 2 |
| Dough Boys Pizza | 2 | 1 | 1 | 2 | 1 |

Repeat Step 2 — Count the number of first place votes for each remaining restaurant.

Results of Round 3:

* + Rainbow China — 2 votes
  + Dough Boys — 3 votes

Determine if any one choice has a majority. Yes! Dough Boys has three votes, so it is the winner! <END>

### [Your Turn] 4

There are six members on the board of a Parent Teacher Association (PTA) at a local elementary school: the president (P), the vice president (V), the recording secretary (R), the liaison to the administration (L), the treasurer (T). and the chief fundraiser (C). The board must decide which equipment to purchase for the classrooms with moneys from their annual fundraisers. The preferences of the board members are shown in Figure 11.10.

**Figure 11.10 Preferences of PTA Board Members**

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
|  | **P** | **V** | **R** | **L** | **T** | **C** |
| Option A | 3 | 4 | 1 | 3 | 4 | 1 |
| Option B | 1 | 3 | 2 | 1 | 3 | 2 |
| Option C | 2 | 2 | 3 | 2 | 1 | 4 |
| Option D | 4 | 1 | 4 | 4 | 2 | 3 |

The board uses plurality method and a runoff in the event of a tie, such that the option(s) with the least votes will be eliminated in each round. Which option will be chosen?

**Answer**

Option B. <END>

## [H2] Ranked-Choice Voting

In Example 4 and Your Turn 4, we were given a list containing each voter’s preferences. This is called a **preference ranking**, and Imaginaria could use a voting system that requires voters to complete a **ranked ballot**.

**[DEFINITION]**

A **preference ranking** is an ordering of a voter's preferences in a sequence.

A **ranked ballot** is a ballot in which a voter is required to give an ordering of their preferences. <END>

The vote for the Academy Awards uses a ranked ballot. Figure 11.11 provides an example of a ranked ballot for the 2020 Academy Award nominees for Best Director.

**Figure 11.11 Academy Award Ballot for Best Director**

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| **2020 Academy Award Ballot for Best Director** | | | | | |
| Candidate for Best Director | Rank top choice as 1, next choice as 2, and so on. | | | | |
| Martin Scorsese, *The Irishman* | * 1 | * 2 | * 3 | * 4 | * 5 |
| Todd Phillips, *Joker* | * 1 | * 2 | * 3 | * 4 | * 5 |
| Sam Mendes, *1917* | * 1 | * 2 | * 3 | * 4 | * 5 |
| Quentin Tarantino,  *Once Upon a Time in Hollywood* | * 1 | * 2 | * 3 | * 4 | * 5 |
| Bong Joon-ho, *Parasite* | * 1 | * 2 | * 3 | * 4 | * 5 |

As you decide on the voting methods that will be used in your new democracy, budget must be a consideration. You might consider a particular type of **ranked voting,** called **ranked-choice voting,** which simulates a series of runoff elections without the usual time and expense involved when voters must repeatedly return to the polls, like we did in Example 4.

**[DEFINITION]**

**Ranked voting** refers to any voting system in which a voter uses a ranked ballot, regardless of the method by which the winner is determined. <END>

The method of **ranked-choice** **voting (RCV),** also called **instant runoff voting (IRV)**, is a version of the Hare Method, using *preference ranking* so that, if no single candidate receives a majority, the least popular selections can be eliminated and the results can be recounted, without the need for more elections.

🚦 *Ranked voting can be confused with ranked-choice voting, but ranked voting is a more general category which includes ranked-choice voting and several other voting methods.*

As we explore examples of ranked voting, we will summarize the voters’ preference rankings using a table like Figure 11.12, which provides rankings for four options. The top row shows the number of ballots that ranked the options in the same order. Let’s practice interpreting the information in the table.

**Figure 11.12 Sample Preference Summary**

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| **Number of Ballots** | **100** | **200** | **150** | **75** |
| Option A | 1 | 4 | 3 | 4 |
| Option B | 2 | 3 | 4 | 2 |
| Option C | 4 | 2 | 1 | 1 |
| Option D | 3 | 1 | 2 | 3 |

### <example> EXAMPLE 5 - <title>Interpreting the Sample Preference Summary</title>

Refer to Figure 11.12 to answer the following questions.

1. How many voters ranked the options in the following order: Option A in fourth place, Option B in second place, Option C in first place, and Option D in third place?
2. How many ballots in total were collected?
3. How many voters indicated that Option C was their first choice?

**<SOLUTION>**

1. The column farthest to the right displays this ordering. The top entry in this column is 75; so, there were 75 voters who ranked the options in this way.
2. The sum of the top row gives the total number of ballots collected: . So, there were 525 ballots collected.
3. In the Option C row, there are two entries of 1 which indicate a first choice for that option. These occur in the last two columns. The sum of the top entries in these columns is . So, 225 voters indicated Option C as their first choice. <END>

*Photo 11.3: A picture of a kindergarten age child painting or playing with colors*



### [Your Turn] 5

A kindergarten class votes on their favorite colors using a ranked ballot. Use the results in Figure 11.13 to answer parts a, b, and c.

**Figure 11.13 Kindergarten Class Color Preference**

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| **Number of Ballots** | **4** | **6** | **4** | **7** |
| Red | 2 | 6 | 2 | 5 |
| Blue | 1 | 2 | 1 | 4 |
| Green | 6 | 5 | 5 | 2 |
| Yellow | 5 | 4 | 6 | 3 |
| Purple | 4 | 3 | 4 | 1 |
| Pink | 3 | 1 | 3 | 6 |

1. How many students voted in total?
2. How many students voted for yellow as their favorite color?
3. How many students in the class indicated that blue was their favorite color while green was their least favorite color? <END>

**<SOLUTION>**

1. 
2. None
3. 4

<END>

Now that we’ve covered how to read a summary of preference rankings, let’s practice using the ranked-choice method to determine the winner of an election. Recall that ranked-choice voting is still the Hare Method where the candidate with the very least number of votes is eliminated each round until a majority is attained. The difference here is that the voters have completed a ranked ballot, so they don’t have to visit the polls multiple times. Here are the steps for ranked-choice voting.

### [H2] Steps to Determine Winner by Ranked-Choice Voting

Step 1— Determine the number of votes needed to achieve a majority. This is the least whole number greater than 50 percent of the total votes.

Step 2 — Count the number of first place votes for each candidate. If a candidate has a majority, that candidate wins the election and we are done! Otherwise, eliminate the candidate(s) with the fewest votes and complete Step 3.

Step 3 — Reallocate the votes to the remaining candidates, and repeat Step 2.

🚦 *Be methodical to avoid arithmetic errors. Make sure that each time you count the number of first place votes they sum to the number of ballots.*

**VIDEO:** **How Does Ranked-Choice Voting Work?** <[**https://youtu.be/oHRPMJmzBBw**](https://youtu.be/oHRPMJmzBBw)**>**

### <example> EXAMPLE 6 - <title> Most Popular Color in Kindergarten</title>

Let’s review Figure 11.13 again. Now let’s determine which color would be selected based on these results using the ranked-choice method.

**<SOLUTION>**

Step 1 — Determine if any candidate received a majority. There were 21 ballots. Fifty percent of 21 is . A majority is 11.

Step 2 — Count the number of first place votes for each candidate. If a candidate has a majority, that candidate wins the election. Otherwise, eliminate the candidate(s) with the fewest votes.

* Red: 0
* Blue: 
* Green: 0
* Yellow: 0
* Purple: 7
* Pink: 6

Notice that , which is the total number of ballots. Confirming this helps to catch any arithmetic or counting errors. No candidate has a majority with 11 or more votes. We must eliminate red, green, and yellow which had the fewest votes with 0 each. The remaining votes that must be counted for Round 2 are in Figure 11.14.

**Figure 11.14 Kindergarten Class Color Preference Round 1 Elimination**

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| **Number of Ballots** | **4** | **6** | **4** | **7** |
| Blue | 1 | 2 | 1 | 4 |
| Purple | 4 | 3 | 4 | 1 |
| Pink | 3 | 1 | 3 | 6 |

Step 3 — Reallocate the votes to the remaining candidates. We can do this by numbering the choices as 1, 2, and 3 in such a way that the order of preference is retained as seen in Figure 11.15.

**Figure 11.15 Kindergarten Class Color Preference Round 2 Votes**

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| **Number of Ballots** | **4** | **6** | **4** | **7** |
| Blue | 1 | 2 | 1 | 2 |
| Purple | 3 | 3 | 3 | 1 |
| Pink | 2 | 1 | 2 | 3 |

Repeat Step 2—Count the number of first place votes for each candidate. If a candidate has a majority, that candidate wins the election. Otherwise, eliminate the candidate(s) with the fewest votes.

* Blue: 
* Purple: 7
* Pink: 6

Confirm that . Great! No candidate has 11 or more votes. We must eliminate pink which had the fewest votes with 6. The remaining votes that must be counted for Round 2 are in Figure 11.16.

**Figure 11.16 Kindergarten Class Color Preference Round 2 Elimination**

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| **Number of Ballots** | **4** | **6** | **4** | **7** |
| Blue | 1 | 2 | 1 | 2 |
| Purple | 3 | 3 | 3 | 1 |

Repeat Step 3 — Reallocate the votes to the remaining candidates. We can do this by numbering the choices as 1 and 2 as seen in Figure 11.17.

**Figure 11.17 Kindergarten Class Color Preference Round 3**

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| **Number of Ballots** | **4** | **6** | **4** | **7** |
| Blue | 1 | 1 | 1 | 2 |
| Purple | 2 | 2 | 2 | 1 |

Repeat Step 2 — Count the number of first place votes for each candidate. If a candidate has a majority, that candidate wins the election. Otherwise, eliminate the candidate(s) with the fewest votes.

* Blue: 



* Purple: 7

Blue has a majority and wins the election! <END>

**VIDEO:** **Determine Winner of Election by Ranked-choice Method (aka Instant Runoff) <**[**https://www.youtube.com/watch?v=RojxTmloAak**](https://www.youtube.com/watch?v=RojxTmloAak)**>**

### [Your Turn] 6

Suppose that 70 Star Wars fans were asked to vote for their favorite Star Wars character. They were given a ranked ballot, and the results are shown in Figure 11.18. Use ranked-choice voting to determine the winner.

**Figure 11.18 Favorite Original Star Wars Character Ballot Preferences**

|  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| **Number of Ballots** | **7** | **6** | **10** | **8** | **4** | **5** | **6** | **7** | **2** | **3** |
| Han Solo | 3 | 5 | 1 | 3 | 2 | 4 | 2 | 2 | 1 | 4 |
| Princess Leia | 2 | 1 | 3 | 6 | 4 | 3 | 3 | 1 | 3 | 2 |
| Luke Skywalker | 6 | 6 | 6 | 1 | 5 | 5 | 5 | 4 | 6 | 1 |
| Chewbacca | 4 | 4 | 5 | 4 | 6 | 1 | 6 | 5 | 4 | 5 |
| Yoda | 1 | 2 | 4 | 2 | 3 | 2 | 1 | 3 | 2 | 3 |
| R2-D2 | 5 | 3 | 2 | 5 | 1 | 6 | 4 | 6 | 5 | 6 |

<END>

**Answer**

Using ranked-choice voting, Yoda is determined to be the winner.<END>

## [H2] Borda Count Voting

Ranked-choice voting is one type of ranked voting which simulates multiple run-offs based on ranked ballots. Another type of ranked voting is the **Borda count method**.

**[DEFINITION]**

The **Borda count method** is a system of voting using ranked ballots in which each candidate is awarded points corresponding to the number of candidates ranked lower on each ballot.<END>

To understand how this works, let’s review the favorite colors of the Kindergarten class from Figure 11.19. Let’s focus on the votes represented by the first column of the preference summary.

**Figure 11.19 Kindergarten Class Color Preference**

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| **Number of Ballots** | **4** | **6** | **4** | **7** |
| Red | 2 | 6 | 2 | 5 |
| Blue | 1 | 2 | 1 | 4 |
| Green | 6 | 5 | 5 | 2 |
| Yellow | 5 | 4 | 6 | 3 |
| Purple | 4 | 3 | 4 | 1 |
| Pink | 3 | 1 | 3 | 6 |

Each student had six options. This first column tells us that four students ranked blue as their first choice, red as their second choice, pink as their third choice, purple as their fourth choice, yellow as their fifth choice, and green as their sixth choice. Blue was ranked higher than  other colors. For each of the four students who completed their ballot in this way, blue would receive five points. Since there were four ballots with this ordering, blue would receive points from the first column. To determine the total points for each candidate, we have to find the sum of the points they received in each column.



To determine the winner of a contest using the Borda count method, we must compare total number of points earned by each candidate. The candidate with the most points is the winner. Each row of the preference summary corresponds to a single candidate. To find the number of points received by a particular candidate in the preference summary, or their **Borda score**, we will need to focus on the row in which that candidate appears.

**DEFINITION**

**Borda score** is the number of points awarded to a particular candidate using the Borda count method.<END>

Before we practice determining the winner of a Borda count election, let’s examine how to find the Bora score for a single candidate.

### <example> EXAMPLE 7 - <title>Most Popular Color in Kindergarten Revisited</title>

Let’s review the ballots from the kindergarten class again. This time let’s determine the Borda score received by the color purple.

**Figure 11.20 Kindergarten Class Color Preference**

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| **Number of Ballots** | **4** | **6** | **4** | **7** |
| Red | 2 | 6 | 2 | 5 |
| Blue | 1 | 2 | 1 | 4 |
| Green | 6 | 5 | 5 | 2 |
| Yellow | 5 | 4 | 6 | 3 |
| Purple | 4 | 3 | 4 | 1 |
| Pink | 3 | 1 | 3 | 6 |

**<SOLUTION>**

Step 1 — Find the number of points received by the candidate in each column.

Column 1: 

Column 2: 

Column 3: 



Column 4: 

Step 2 — Find the sum of the points received in each column. This is the total number of points received by this candidate: 

This process can also be combined into one step as shown here.



Purple received 69 points in this election. <END>

🚦 *When calculating a Borda score in one step, be careful to use the correct order of operations. Perform the subtraction inside each pair of parentheses first, then perform each multiplication, and then perform each addition.*

### [Your Turn] 7

Consider the color preferences of the Kindergarten class once more. Refer to Figure 11.20. Using the Borda count method, determine the total number of points the color blue received. <END>

**Answer**

Blue received 78 points. <END>

Now let’s determine the winner of an election by comparing the Borda scores for each of the candidates.

### <example> EXAMPLE 8 - <title>Determine the Winner by Two Ranked Voting Methods</title>

Figure 11.21 displays a sample preference summary. Use it to answer parts a and b.

**Figure 11.21 Sample Preference Summary**

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| **Number of Ballots** | **95** | **90** | **110** | **115** |
| **Option A** | 4 | 4 | 1 | 1 |
| **Option B** | 2 | 2 | 2 | 2 |
| **Option C** | 3 | 1 | 3 | 4 |
| **Option D** | 1 | 3 | 4 | 3 |

1. Use the ranked-choice voting method to determine the winner of the election.
2. Use the Borda count method to determine the winner of the election.

**SOLUTION**

1. Step 1 — Determine the number of votes needed to achieve a majority. The number of ballots is . 50% of 400 is . So, 206 votes or more is a majority.  
   Step 2 — Count the number of first place votes for each candidate.
   * Option A has 
   * Option B has 0
   * Option C has 90
   * Option D has 95

Since Option A has a majority, Option A is the winner by the ranked-choice method.

1. The Borda scores would be:
   * Option A: 
   * Option B: 
   * Option C: 
   * Option D: 

Since Option B has a Borda score of 820 points, Option B is the winner by the Borda count method. <END>

The Borda count method may seem too complicated to even consider using for Imaginaria, but each voting method has its own pros and cons. The Borda count method, for example, favors **compromise candidates** over **divisive candidates**.

**[DEFINITION]**

A **divisive candidate** is simultaneously the first choice of a large portion of the voters, and the last choice of a large portion of the voters.

A **compromise candidate** is not the first choice of most of the voters but is more acceptable to the population as a whole than the other candidates.<END>

In Example 8, Candidate A was ranked first by 225 voters, but it was ranked last by 185 voters. No voters ranked Candidate A as second or third. It appears that, although Candidate A had the majority of first place votes, there was a significant minority who strongly disliked them. Candidate A was a divisive candidate. Candidate B, on the other hand, was the second choice of every voter, making Candidate B a good compromise. The Borda count method chose Candidate B, a compromise candidate, that was more acceptable to the population as a whole. This scenario is cited by both opponents and proponents of the Borda count method. If this had been an Imaginarian election, which outcome would you support?

**VIDEO:** **Determine Winner of Election by Borda Count Method < https://youtu.be/cKiTSyasia0>**

### [Your Turn] 8

The Imaginarian voter preferences are summarized in Figure 11.22. Use it to answer parts a, b, and c.

**Figure 11.22 Imaginarian Preference Ballot Summary**

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
| **Number of Ballots** | **12** | **19** | **27** | **29** | **31** | **21** |
| Candidate A | 1 | 1 | 2 | 2 | 3 | 3 |
| Candidate B | 2 | 3 | 1 | 3 | 1 | 2 |
| Candidate C | 3 | 2 | 3 | 1 | 2 | 1 |

1. Use the ranked-choice voting method to determine the winner of the election.
2. Use the Borda count method to determine the winner of the election.
3. Compare the results of the two methods. Did the same candidate win? What observations can you make about the results? <END>

**Answer**

1. Candidate B would be considered the winner using the ranked-choice voting method.
2. Candidate C would be considered the winner using the Borda count method.
3. Different candidates won. It appears that the vote counts were so close that a small shift in either direction could change the results of either method. <END>

## [H2] Pairwise Comparison and Condorcet Voting

We have discussed two kinds of ranked voting methods so far: ranked-choice and Borda count. A third type of ranked voting is the **pairwise comparison** **method**. This method is one of several **Condorcet voting methods** named for the Marquis de Condorcet, a French philosopher and mathematician who preferred the pairwise comparison method to the Hare method and made public arguments in its favor.

**[DEFINITION]**

The **pairwise comparison method** is a Condorcet voting method where each candidate receives a point for each candidate they would beat in a one-on-one election and half a point for each candidate they would tie. If one candidate earns more points than the others, then that candidate wins.

**Condorcet voting methods** are methods in which the candidates are ranked, and then the candidates are compared pairwise to each other. If a candidate beats all others in these pairwise competitions, the candidate wins. These methods vary in the way candidates are scored. There is not always a clear winner.

A **Condorcet candidate** is a candidate who wins each possible pairing in an election. <END>

### PEOPLE IN MATHEMATICS

Condorcet voting methods are named for the **Marquis de Condorcet**, a French philosopher and mathematician known for, among other accomplishments, writing “*Sur l’admission des femmes au droit de Cité*” (On the Admission of Women to the Rights of Citizenship), in *1789,* the first published essay on the political rights of women.

(**For more details visit: <**<https://oll.libertyfund.org/title/condorcet-on-the-admission-of-women-to-the-rights-of-citizenship>>.

If you include a Condorcet voting method in the constitution of Imaginaria, the election supervisors may want to use a pairwise comparison matrix like the one in Figure 11.23. It’s a tool used to list the number of wins associated with each pairing of two candidates. Each candidate will receive a point for each win and a half a point for each tie. Each pairing is listed twice, once for the number of wins of a candidate over a particular challenger and once for the number of wins of the challenger over that candidate.

**Figure 11.23 Pairwise Comparison Matrix for Three Candidates**

|  |  |  |  |
| --- | --- | --- | --- |
| **Opponent**  **Runner** | **A** | **B** | **C** |
| **A wins** | -- | A over B | A over C |
| **B wins** | B over A | -- | B over C |
| **C wins** | C over A | C over B | -- |

### [H2] Steps to Determine A Winner by Pairwise Comparison Method using Matrix

Step 1 — On the matrix, indicate a losing matchup by crossing out a box, , and tie match ups by drawing a slash through the box, .

Step 2 — Award each candidate 1 point for a win, half a point for a tie, and 0 points for a loss.

Step 3 — Identify the winner, which is the candidate with the most points.

**VIDEO:** **Determine Winner of Election by Pairwise Comparison Method <https://www.youtube.com/watch?v=w1NNK7Dn3E8>**

Before you decide on the pairwise comparison method for Imaginaria, review what’s involved in constructing a pairwise comparison matrix from a summary of ranked ballots. Then we can use the matrix to determine the winner of the election. Does the winner using the Borda method still win?

### <example>EXAMPLE 9 - <title>Construct and Use a Pairwise Comparison Matrix</title>

Consider the summary of ranked ballots shown in Figure 11.24. Determine the winner of an election using the pairwise comparison method.

1. Construct a pairwise comparison matrix for the sample summary of ranked ballots in Figure 11.24.
2. Use the pairwise comparison method to determine a winner.
3. Recall that in Example 8, Candidate A won by the ranked-ballot method, and Candidate B won by the Hare method. Did the same candidate win using the pairwise comparison method?
4. Is the winner a Condorcet candidate?

**Figure 11.24 Sample Summary of Ranked Ballots**

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| **Number of Ballots** | **95** | **90** | **110** | **115** |
| Option A | 4 | 4 | 1 | 1 |
| Option B | 2 | 2 | 2 | 2 |
| Option C | 3 | 1 | 3 | 4 |
| Option D | 1 | 3 | 4 | 3 |

**SOLUTION**

There are four candidates on the ballots. We will need a row and a column for each candidate in addition to the headings, so we will draw a five by five matrix.

**Figure 11.25 Pairwise Comparison Matrix for Four Candidates**

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| **Opponent**  **Runner** | **Option A** | **Option B** | **Option C** | **Option D** |
| **Option A wins** | -- | A over B | A over C | A over D |
| **Option B wins** | B over A | -- | B over C | B over D |
| **Option C wins** | C over A | C over B | -- | C over D |
| **Option D wins** | D over A | D over B | D over C | -- |

1. Refer to Figure 11.25 to determine the values that belong in each cell.

* A over B: A is preferred to B in columns 3 and 4. So, A scores  points.
* A over C: A is preferred to C in columns 3 and 4. So, A scores 225 points again.
* A over D: Similarly, A scores 225 points.
* B over A: B is preferred to A in columns 1 and 2. So, B scores  points.

Continuing in this way, we complete the pairwise comparison matrix (Figure 11.26).

**Figure 11.26 Pairwise Comparison Matrix for Four Candidates with Vote Counts**

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| **Opponent**  **Runner** | **Option A** | **Option B** | **Option C** | **Option D** |
| **Option A wins** | -- | AB  225 | AC  225 | AD  225 |
| **Option B wins** | BA  185 | -- | BC  320 | BD  315 |
| **Option C wins** | CA  185 | CB  90 | -- | CD  200 |
| **Option D wins** | DA  185 | DB  95 | DC  210 | -- |

1. Step 1 — Losing pairings are crossed off with an . In the event of a tie, we will draw a slash, . (See Figure 11.27)

Step 2 — Determine the number of points for each candidate by analyzing their row of wins. Each win is 1 point, each loss, , is 0 points, and each tie, , is half a point. Construct an additional column for each candidate’s points (See Figure 11.27).

**Figure 11.27 Pairwise Comparison Matrix for Four Candidates with Pairwise Winners and Points Column Added**

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| **Opponent**  **Runner** | **Option A** | **Option B** | **Option C** | **Option D** | **Points** |
| **Option A wins** | -- | AB  225 | AC  225 | AD  225 | 3 |
| **Option B wins** | BA  185 | -- | BC  320 | BD  315 | 2 |
| **Option C wins** | CA  185 | CB  90 | -- | CD  200 | 0 |
| **Option D wins** | DA  185 | DB  95 | DC  210 | -- | 1 |

Step 3 – The winner by the pairwise comparison method is option A with 3 points.

1. Option A was not the winner by the Hare method.
2. The winner, option A, is a **Condorcet candidate,** because option A won each pairwise comparison. <END>

🚦 *Notice that the pairwise vote totals are not used to determine the points. Vote totals are only used to determine a win or a loss. Avoid the common error of adding the values in each row to get the points.*

### [Your Turn] 9

According to Variety magazine, there were 8,649 eligible Oscar voters in 2020. Suppose that the voter preferences for the ballot for Best Director had been as shown in Figure 11.28.

**Figure 11.28 Voter Preferences for Best Director**

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| **NUMBER OF BALLOTS** | **2,400** | **2,100** | **1,900** | **1,200** | **1,100** |
| Martin Scorsese, *The Irishman* | 3 | 1 | 4 | 5 | 2 |
| Todd Phillips, *Joker* | 1 | 4 | 5 | 2 | 4 |
| Sam Mendes, *1917* | 5 | 5 | 3 | 4 | 5 |
| Quentin Tarantino,  *Once Upon a Time in Hollywood* | 4 | 2 | 1 | 3 | 3 |
| Bong Joon-ho, *Parasite* | 2 | 3 | 2 | 1 | 1 |

Refer to Figure 11.28 to answer each question.

1. Construct a pairwise comparison matrix for the Best Director ballots.
2. Who is the “Best Director” according to the pairwise comparison method?
3. Is the winner a Condorcet candidate? <END>

**Answer**

1. See Figure 11.29

**Figure 11.29 Best Director Pairwise Comparison Matrix**

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
| **Opponent**  **Runner** | **Scorsese (S)** | **Phillips (P)** | **Mendes (M)** | **Tarantino (T)** | **Bong (B)** | **Points** |
| **Scorsese wins** | -- | SP  5100 | SM  5600 | ST  5600 | SB  2100 | 3 |
| **Phillips wins** | PS  3600 | -- | PM  6800 | PT  3600 | PB  2400 | 1 |
| **Mendes wins** | MS  3100 | MP  1900 | -- | MT  0 | MB  0 | 0 |
| **Tarantino wins** | TS  3100 | TP  5100 | TM  8700 | -- | TB  4000 | 2 |
| **Bong wins** | BS  6600 | BP  6300 | BM  8700 | BT  4700 | -- | 4 |

1. Bong Joon-ho won according to the pairwise comparison method.
2. Yes the winner is a Condorcet candidate.<END>

### [H2] Three Key Questions

Before you decide if you want to use the pairwise comparison method for Imaginarian elections, let’s consider three questions that might affect your decision.

1. Is there always a winner?
2. If there is a winner, is the winner always a Condorcet candidate?
3. If there is a Condorcet candidate, does that candidate always win?

Let’s think about why these questions might be important to you if you chose the pairwise comparison method. First, if no candidate meets the criteria to win an election, you will need a backup plan such as a runoff election. Second, if the winner is not a Condorcet candidate, then there is at least one candidate who beat the winner in a pairwise matchup and the supporters of that candidate might question the validity of the election. Finally, if there is a Condorcet candidate who beat every other candidate in a pairwise matchup, it is reasonable to conclude that it would be unfair for anyone else to win. The rest of the examples in this section should illustrate these key concepts.

### <example> EXAMPLE 10 - <title>Rock, Paper, Scissors by Pairwise Comparison</title>

Suppose that three people are playing the game Rock, Paper, Scissors. The group keeps having a tie because Person A always picks rock, Person B always picks paper which beats rock, and Person C always picks scissors which beats paper, and is beaten by rock! This leads to a disagreement about which choice is best. They decide to use the pairwise comparison method determine the winner. Their preference rankings are given in Figure 11.30.

**Figure 11.30 Rock, Paper, Scissors Preference Ranking**

|  |  |  |  |
| --- | --- | --- | --- |
| **Voters** | **A** | **B** | **C** |
| **Rock (R)** | 1 | 3 | 2 |
| **Paper (P)** | 2 | 1 | 3 |
| **Scissors (S)** | 3 | 2 | 1 |

**<SOLUTION>**

Construct the comparison matrix (Figure 11.31).

**Figure 11.31 Rock, Paper, Scissors Pairwise Comparison Matrix**

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| **Opponent**  **Runner** | **Rock (R)** | **Paper (P)** | **Scissors (S)** | **Points** |
| **Rock wins** | -- | RP  2 | RS  1 | 1 |
| **Paper wins** | PR  1 | -- | PS  2 | 1 |
| **Scissors wins** | SR  2 | SP  1 | -- | 1 |

There is a tie! There is no winner. <END>

Example 10 illustrates the answer to the first key question. The pairwise comparison method does not always result in a winner. For example, much like the game of Rock, Paper, Scissors, it is possible for a cyclic pattern to emerge in which each candidate beats the next until the last candidate who beats the first (Figure 11.32).

**Figure 11.32 Rock, Paper, Scissors Cyclic Outcome**

### [Your Turn] 10

A pairwise comparison matrix is given in 11Figure 11.33. Determine the winner by the pairwise comparison method. If there is not a winner, explain why. If there is a winner, tell whether the winner is a Condorcet candidate.

**Figure 11.33 A Pairwise Comparison Matrix**

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
| **Opponent**  **Runner** | **P** | **Q** | **R** | **S** | **T** | **Z** |
| **P wins** | -- | PQ 1 | PR 4 | PS 6 | PT 4 | PZ5 |
| **Q wins** | QP 5 | -- | QR 6 | QS 4 | QT 1 | QZ 2 |
| **R wins** | RP 2 | RQ 0 | -- | RS 5 | RT 4 | RZ 1 |
| **S wins** | SP 0 | SQ 2 | SR 1 | -- | ST 6 | SZ 5 |
| **T wins** | TP 2 | TQ 5 | TR 2 | TS 0 | -- | TZ 4 |
| **Z wins** | ZP 1 | ZQ 4 | ZR 5 | ZS 1 | ZT 2 | -- |

<END>

**Answer**

Candidate P wins. Candidate P is not a Condorcet candidate because P lost to Candidate Q. <END>

Now, you have the answer to the second key question. The pairwise comparison matrix in 11Figure 11.33 is an example of a scenario where a winner is not a Condorcet candidate.

The answer to the third question is not as clear. If there is a Condorcet candidate, does that candidate win? So far, we have not come across a contradictory example where the Condorcet candidate didn’t win, but we cannot know with certainty that it is not possible by looking at examples. Instead, we will need to use some reasoning. Let’s review some particular cases of elections with a certain number of candidates, and then we will try to generalize the scenario to an election with  candidates.

### <example>EXAMPLE 11 - <title> Does the Condorcet Candidate Win?</title>

1. Suppose there is an election with five candidates—A, B, C, D, and E—and that Candidate C is a Condorcet candidate. How many points did Candidate C win?
2. What is the greatest number of points that any one of the other candidates could win?
3. Is it possible for Candidate C to lose or tie?

**<SOLUTION>**

1. In any pairwise election with five candidates, each candidate must compete against four other candidates. It follows that the most points a single candidate can win is four points, which would occur if the candidate won every matchup. As a Condorcet candidate, Candidate C won all the pairwise matchups against Candidates A, B, D, and E, earning four points.
2. The rest of the candidates lost to Candidate C. The most points a particular candidate could win if they won matchups with each of the other three candidates is three points.
3. Since Candidate C has four points and the rest of the candidates have three points or less, Candidate C is the winner. Therefore, it is not possible for Candidate C to tie or lose. <END>

### [Your Turn] 11

Suppose there is an election with 26 candidates, A through Z, and that Candidate C is a Condorcet candidate.

1. How many points did Candidate C win?
2. What is the greatest number of points that any one of the other candidates could win?
3. Is it possible for Candidate C to lose or tie? <END>

**Answer**

1. Candidate C won 25 points.
2. The greatest number of points another candidate could win is 24 points.
3. No, Candidate C cannot lose or tie. <END>

Let’s consider a general case where there are  candidates. One of the candidates is a Condorcet candidate. Since the Condorcet candidate wins all matchups, the Condorcet candidate wins  points. Since each of the other candidates lost to the Condorcet candidate, the most a single candidate could win is . Since the Condorcet candidate won  points and each other candidate won  points or fewer, the Condorcet candidate is the winner. You have your answer to the third key question! If there is a Condorcet candidate, that candidate is always the winner.

## [H2] Approval Voting

The last type of voting system you will consider for your budding democracy, is known as an **approval voting system**. This voting system has aspects in common with plurality voting and Condorcet voting methods, but it has characteristics that distinguish it from both.

**[DEFINITION]**

In an **approval voting** **system**, each voter may approve any number of candidates without rank or preference for one candidate over another among the approved candidates, and the candidate approved by the most voters wins.

An **approval voting ballot** shows a list of candidates with an option to approve or not approve each candidate. <END>

The term “approval voting” was not used until the 1970’s ([Brams, Steven J.](https://en.wikipedia.org/wiki/Steven_Brams" \o "Steven Brams); [Fishburn, Peter C.](https://en.wikipedia.org/wiki/Peter_C._Fishburn" \o "Peter C. Fishburn) (2007), [*Approval Voting*](https://books.google.com/books?id=e7h7evxSclIC&pg=PR5), Springer-Verlag, p. xv, [ISBN](https://en.wikipedia.org/wiki/ISBN_(identifier)) [978-0-387-49895-9](https://en.wikipedia.org/wiki/Special:BookSources/978-0-387-49895-9)) although its use has been documented as early as the 13th century (Brams, Steven J. (April 1, 2006). [*The Normative Turn in Public Choice*](https://web.archive.org/web/20100531093534/http:/www.nyu.edu/gsas/dept/politics/faculty/brams/normative_turn.pdf) (PDF)(Speech). Presidential Address to Public Choice Society. New Orleans, Louisiana.) Approval voting has the appeal of being simpler than ranked voting methods. It also allows an individual voter to support more than one candidate equally. This has appeal for those who do not want a split vote among a few mainstream candidates to lead to the election of a fringe candidate. It also has appeal for those who want an underdog to have a chance of success because voters will not worry about wasting their vote on a candidate who is not believed likely to win.

### <example> EXAMPLE 12 - <title>Rock, Paper, Scissors, Lizard, Spock</title>

Suppose that Person A/B/C were just about to give up on their game of Rock, Paper, Scissors when they were joined by Person D who reminded them that their updated version, Rock, Paper, Scissors, Lizard, Spock was a far superior game with the added rules that Lizard eats Paper, Paper disproves Spock, Spock vaporizes Rock, Rock crushes Lizard, Lizard poisons Spock, Spock smashes Scissors, and Scissors decapitates Lizard (Figure 11.34).

**Figure 11.34 Rock, Paper, Scissors, Lizard Spock Dominance**

Person D encourages their friends to hold a new election. This time, for the sake of simplicity, the group decides to use approval voting to determine the best move in the game. The summary of approval ballots is given in Figure 11.35.

**Figure 11.35 Approval Ballots for Rock, Paper, Scissors, Lizard, Spock**

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| **VOTERS** | **A** | **B** | **C** | **D** |
| Rock | Yes | No | No | No |
| Paper | No | Yes | No | No |
| Scissors | No | No | Yes | No |
| Lizard | No | No | No | Yes |
| Spock | Yes | Yes | Yes | Yes |

**SOLUTION**

Count the number of approval votes for each candidate by counting the number of “Yes” votes in each row of Figure 11.35.

* Rock: 1
* Paper: 1
* Scissors: 1
* Lizard: 1
* Spock: 4

Spock is the winning candidate, approved by four voters! <END>

### [Your Turn] 12

Recall the Chionilis family from Example 4 who were trying to decide on a restaurant for dinner. They are trying to decide on a restaurant again, but now they don’t want to deal with multiple run-offs or even ranking. They will use the approval voting method shown in Figure 11.36. Each family member will approve their top two choices. Rainbow China won when multiple run-offs were used. Find the winner for tonight’s dinner.

**Figure 11.36 Chionilis Family Restaurant Approval Ballots with Two Approvals Each**

|  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- |
| **Options** | **A** | **R** | **S** | **T** | **G** | **E** | **M** | **D** |
| Rainbow China | Yes | No | No | Yes | No | No | Yes | Yes |
| Dough Boys Pizza | Yes | Yes | Yes | Yes | No | Yes | No | Yes |
| Taco City | No | No | Yes | No | Yes | Yes | No | No |
| Caribbean Flavor | No | Yes | No | No | Yes | No | Yes | No |

<END>

**Answer**

Dough Boys Pizza is the winner for dinner.<END>

### <example>EXAMPLE 13 - <title>The Chionilis Family is Hungry Again!</title>

The eight members of the Chionilis family—Annette, Rene, Seema, Titus, Galen, Elena, Max and Demitri—have another decision to make. Approval voting worked out nicely the last time. They are going to use it again, but this time, Annette, Rene, Seema, and Galen are feeling a little indecisive. They can’t narrow their choice down to two. They will approve their three top choices, but the other family members will only approve two. These choices are reflected in Figure 11.37. Determine the restaurant that will be chosen.

**Figure 11.37 Chionilis Family Restaurant Approval Ballots with Varied Number of Approvals**

|  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- |
| **Options** | **A** | **R** | **S** | **T** | **G** | **E** | **M** | **D** |
| Rainbow China | Yes | Yes | Yes | Yes | Yes | No | Yes | Yes |
| Dough Boys Pizza | Yes | Yes | Yes | Yes | No | Yes | No | Yes |
| Taco City | Yes | No | Yes | No | Yes | Yes | No | No |
| Caribbean Flavor | No | Yes | No | No | Yes | No | Yes | No |

**<SOLUTION>**

Count the number of approval votes for each restaurant by counting the number of “Yes” votes in each row.

* Rainbow China: 7
* Dough Boys Pizza: 6
* Taco City: 4
* Caribbean Flavor: 3

This time, Rainbow China won! <END>

### [Your Turn] 13

What would be the outcome of the election if every member of the Chionilis family approved their top three choices from Example 4 Figure 11.5? <END>

**Answer**

There is a tie between Rainbow China and Dough Boys Pizza. <END>

## [H2] Compare and Contrast Voting Methods to Identify Flaws

Wow! We have covered a lot of options for the voting methods. Now, you need to decide which one is best for Imaginaria. Imaginarians might consider characteristics of certain voting systems desirable and others undesirable. In some cases, voters may consider these undesirable traits to be flaws in a voting system that are significant enough to motivate them to reject that system. If you are feeling a bit overwhelmed by this decision, maybe it would help to read about the experiences of others who have faced similar questions.

Consider the 2000 U.S. Presidential Election in which Green Party candidate Ralph Nadar and Reform Party candidate Pat Buchanan were on the ballet running against the mainstream candidates, Democrat Al Gore and Republican George W. Bush. The results are given in Figure 11.38.

**Figure 11.38 Florida Results in the 2000 U.S. Presidential Election**

|  |  |  |  |
| --- | --- | --- | --- |
| **Candidate** | **Party** | **Votes** | **Percentage** |
| **(G) George W. Bush** | Republican | 2,912,790 | 48.85% |
| **(A) Al Gore** | Democrat | 2,912,253 | 48.84% |
| **(R) Ralph Nadar** | Green | 97,488 | 1.63% |
| **(P) Pat Buchanan** | Reform | 17,484 | 0.29% |
| **(H) Harry Brown** | Libertarian | 16,415 | 0.28% |
| **7 Other Candidates** | Other | 6,680 | 0.11% |
| **Total** | | 5,963,110 |  |

In more than one state, Buchanan was able to split the Republican vote enough to allow Gore to win that state. Nadar split the Democrat vote in Florida and New Hampshire by enough votes to prevent Gore from winning those states. Had Gore won either state, he would have had enough electoral votes to win the election. Instead, Bush won. This is an example of a flaw in the plurality system of voting: the spoiler.

A spoiler is a less popular candidate who takes votes from a more popular candidate with similar positions, swinging the race to another candidate with vastly different views that they would not support. This encourages voters not to vote for the candidate that they perceive to be the best, but instead for the candidate they can live with who they perceive to have a better chance of winning. Some voters may prefer a method such as approval voting, which does not have this trait in common with plurality voting.

### <example>EXAMPLE 14 - <title>The Spoiler Controversy</title>

The results of the Florida 2000 U.S. Presidential Election are given in Figure 11.38. Because the vote counts for George W. Bush and Al Gore differed by only 537 votes, many Democrats blamed Ralph Nadar and the Green Party for their loss. Let’s consider how the election results might have differed if the approval voting method had been used.

Use Figure 11.38 and the following assumptions to extrapolate the results of an approval method election:

* 100 percent of Pat Buchanan supporters would approve George W. Bush.
* 100 percent of Ralph Nadar supporters would approve Al Gore.
* 72 percent of Libertarians would approve George W. Bush.
* 28 percent of Libertarians would approve Al Gore (as was roughly the known percentage at the time according to the Cato Institute).
* 50 percent of the supporters of other candidates would approve George Bush while   
  50 percent would approve Al Gore.

**SOLUTION**

Create a summary of approval ballots based on the given assumptions (Figure 11.38). For the Libertarian candidate, 72 percent of 16,415 of the votes is  and 28 percent is . For the other candidates, 50 percent of the votes is .

**Figure 11.39 Summary of Approval Ballots**

|  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- |
| **Number of Votes** | **2,912,790 (G)** | **2,912,253 (A)** | **97,488**  **(R)** | **17,484**  **(B)** | **11,819**  **(72% H)** | **4,596**  **(28% H)** | **3,340**  **(50% O)** | **3,340**  **(50% O)** |
| **(G) George W. Bush** | Yes | No | No | Yes | Yes | No | No | Yes |
| **(A) Al Gore** | No | Yes | Yes | No | No | Yes | Yes | No |
| **(R) Ralph Nadar** | No | No | Yes | No | No | No | No | No |
| **(P) Pat Buchanan** | No | No | No | Yes | No | No | No | No |
| **(H) Harry Brown** | No | No | No | No | Yes | Yes | No | No |
| **(O)7 Other Candidates** | No | No | No | No | No | No | Yes | Yes |

Count the number of approval votes for each candidate.

* George W. Bush: 
* Al Gore:
* Ralph Nadar: 
* Pat Buchanan: 
* Harry Brown: 
* Other Candidates: 

In this scenario, Al Gore is the winner. <END>

### [Your Turn] 14

Use the results from the state of Florida during the 2000 U.S. Presidential Election to extrapolate the results of an approval method election using the assumptions from Example 14 but with the additional assumptions that the supporters of Al Gore would all approve Ralph Nadar and that half of the supporters of George Bush would approve Pat Buchanan while half would approve Harry Brown. <END>

**Answer**

Al Gore is the winner. <END>

The results in Example 14 and Your Turn 14 highlight one of the characteristics of approval voting. Ralph Nadar moved up from a distant third place finish to a close second place finish when Al Gore’s supporters approved him on their ballots. In this way, fringe candidates have a better chance of winning, which some voters consider a flaw but others consider a benefit.

Another aspect of approval voting systems that is a concern to many voters is that candidates in approval elections might encourage their loyal supporters to approve them and only them to avoid giving support to any other candidate. If this occurred, the election in effect becomes a traditional plurality election. This is a flaw that cannot occur in an instant runoff system since all candidates are ranked.

### <example>EXAMPLE 15 - <title>Three Habitable Planets</title>

In the future, humans have explored distant solar systems and found three habitable planets which could be colonized. Since it will take all available resources to colonize one planet, humans must agree on the planet. Planet A has the most comfortable climate and most plentiful resources, but it is the farthest from Earth making travel to the planet a challenge. Planet B is half the distance but will require more resources to make comfortable. Planet C is the least suitable of the three and terraforming will be required, but it is close enough to make travel between Earth and Planet C possible on a more regular basis. Figure 11.40 provides the voter preferences for each planet.

**Figure 11.40 Preference Summary Colonization of Planets**

|  |  |  |  |
| --- | --- | --- | --- |
| **Percentage of Voters** | **45%** | **15%** | **40%** |
| **Planet A** | 1 | 3 | 3 |
| **Planet B** | 2 | 1 | 2 |
| **Planet C** | 3 | 2 | 1 |

If the entire population were able to vote, determine the winning planet using each of the methods listed below.

1. Plurality
2. Ranked-choice method
3. Borda count

**SOLUTION**

1. The plurality method only considers the top choice of each voter. By this system, Planet A has 45 percent of the vote, Planet B has 15 percent of the vote, and Planet C has 40 percent of the vote. Planet A wins.
2. Using either instant runoff or a two-round system, Planet B with only 15 percent of the vote will be eliminated in the first round. In Round 2, the 15 percent that voted for Planet B would vote for their second choice, Planet C. This leaves Planet A with 45 percent and Planet C with 55 percent. Planet C has a majority and wins the election.
3. To find the Borda score for each candidate, imagine there are exactly 100 voters. Then the summary of ranked ballots looks like Figure 11.41.

**Figure 11.41 Preference of 100 Voters**

|  |  |  |  |
| --- | --- | --- | --- |
| **Out of 100 Voters** | **45** | **15** | **40** |
| Planet A | 1 | 3 | 3 |
| Planet B | 2 | 1 | 2 |
| Planet C | 3 | 2 | 1 |

The Borda score for each candidate is as follows:

Planet A: 

Planet B: 

Planet C: 

Planet B wins. <END>

The election in Example 15 involves a scenario in which there are two extreme candidates, Planet A and Planet B, and a moderate candidate, Planet C. The supporters of the extreme candidates prefer the moderate candidate to the other extremist ones. This makes Planet C a compromise candidate. In this case, both the plurality method and ranked-choice voting resulted in the election of one of the extreme candidates, but the Borda count method elected the compromise candidate in this scenario. Depending on a person’s perspective, this may be perceived as a flaw in either ranked-choice and plurality systems, or the Borda count method.

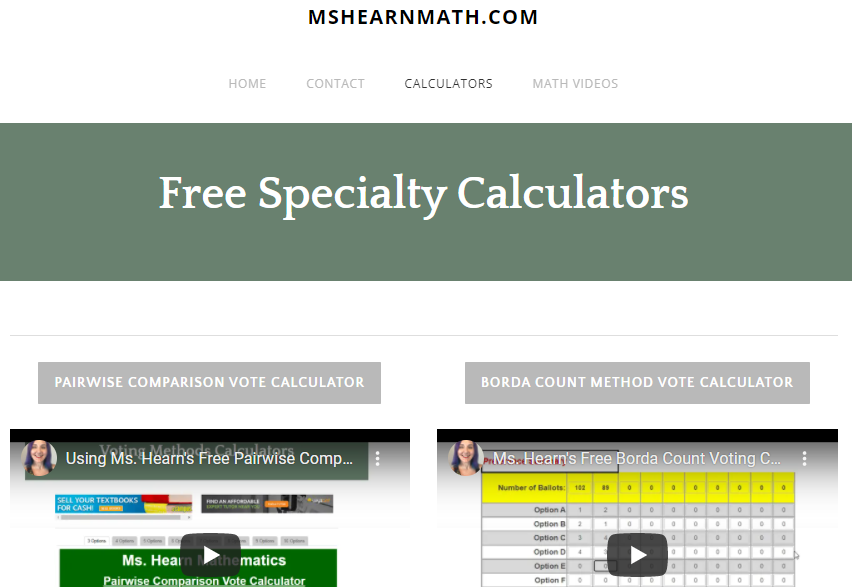
In Module 2, we will analyze the fairness of each voting system in greater detail using objective measures of fairness.

[TECH CHECK]

**Voting Calculators**

It is possible to create Excel spreadsheets that complete the calculations necessary to determine the winner of an election by various voting methods. In some cases, this work has already been done and posted online. As you practice applying the various voting methods that could be used in Imaginaria, quick internet search will lead to sites such as [www.mshearnmath.com/calculators](http://www.mshearnmath.com/calculators) with free specialty calculators.

[Screen 1111.1: Mshearnmath.com screenshot of free specialty calculators.]



These sites can be a great way to check your results! <END>

[CHECK YOUR UNDERSTANDING]

1. Name three voting methods that use a ranked ballot.

Answer: Answers may vary. Example: Ranked-choice, Borda count, and pairwise comparison.

1. Determine whether the following statement is true or false: The same ranked ballots may result in a different winner depending on which voting method is used.

Answer: True.

1. Determine whether the following statement is true or false.. A majority candidate is always a Condorcet candidate.

Answer: True

1. Fill in the blank. The \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ method is a system of voting using ranked ballots in which each candidate is awarded points corresponding to the number of candidates ranked lower on each ballot.

Answer: Borda count

1. Fill in the blank. The \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ method is a system of voting using ranked ballots (or multiple elections) in which each candidate receives a point for each candidate they would beat in a one-on-one election and half a point for each candidate they would tie.

Answer: pairwise comparison

1. Fill in the blank. The \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ method is a runoff voting system in which only the candidate(s) with the very least votes are eliminated.

Answer: Hare

1. Explain the differences between two-round voting and ranked-choice voting.

Answer: In two-round voting, only the top two candidates from Round 1 move on to Round 2, and there are only two rounds. In ranked-choice voting, all candidates except those in last place move on to the next round, and there can be many rounds of voting.

## EXERCISES

For the following questions, identify the winning candidate based on the described voter profile, if possible. If it is not possible, state so. Explain your reasoning.

1. In a plurality election, the candidates have the following vote counts: A 125, B 132, C 149, D 112.

Answer: Candidate C has a plurality and wins the election.

1. In the first round of a ranked-choice election with three candidates—A, B, and C. Candidate A received 55 first place rankings; Candidate B received 25; and Candidate C received 30.

Answer: Candidate A has a majority of the votes and wins the election in the first round.

1. The pairwise matchup points for each candidate were: A 1, B , D .

Answer: Candidate B won the most matchups and wins the election.

1. In a Borda count election, the candidates have the following Borda scores: A 15, B 11, C 12, D 16.

Answer: Candidate D has the highest Borda score and wins the election.

1. There is a pairwise comparison election with candidates A, B, and C. Candidate A had the most first choice rankings, Candidate B has the highest Borda score, and Candidate C is a Condorcet candidate.

Answer: In a pairwise comparison election, the Condorcet candidate always wins, so Candidate C wins.

1. In the first round of a ranked-choice election with three candidates—A, B, and C—Candidate A received 20 first place rankings, Candidate B received 25, and Candidate C received 30.

Answer: More information is required because no candidate secured a majority of the votes in the first round.

**Figure 11.42 Popular Vote in the 2016 U.S. Democratic Presidential Primary**

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| **O'Malley** | **De La Fuente** | **Clinton** | **Sanders** | **Other** |
| 110,227 | 67,331 | 17,174,432 | 13,245,671 | 322,276 |

*Source: Federal Election Commission*—*Federal Elections 2016 Report*

Use Figure 11.42 to answer questions 7 and 8.

1. Calculate the number of votes required to have a majority of the popular vote in the 2016 U.S. Democratic Presidential Primary.

Answer: 15,461,385

1. Which candidate had a plurality? Did this candidate have a majority?

Answer: Clinton. Yes.

**Figure 11.43 Popular Vote in the 2016 U.S. Republican Presidential Primary**

|  |  |
| --- | --- |
| **Candidate** | **Votes** |
| Bush | 281,189 |
| Trump | 13,783,037 |
| Cruz | 7,455,780 |
| Rubio | 3,354,067 |
| Carson | 822,242 |
| Kasich | 4,198,498 |
| Other | 337,714 |

*Source: Federal Election Commission*—*Federal Elections 2016 Report*

Use Figure 11.43 to answer the following questions.

1. Calculate the number of votes required to have a majority of the popular vote in the 2016 U.S. Republican Presidential Primary.

Answer: 15,116,263.50

1. Which candidate had a plurality? Did this candidate have a majority?

Answer: Trump. No.

Use Figure 11.42 and Figure 11.43 to answer the following questions.

1. Suppose the Republican Primary in 2016 was a two-round system. Would there be a second round? Why or why not? If so, which candidates would advance to the second round?

Answer: Yes, because no candidate had a majority. Trump and Cruz would have advanced.

1. Suppose the Democratic Primary in 2016 was a two-round system. Would there be a second round? Why or why not? If so, which candidates would advance to the second round?

Answer: No, because a candidate had a majority.

1. Suppose the Democratic Primary in 2016 used the Hare method. Would there be a second round? Why or why not?

Answer: No, because a candidate had a majority.

1. Suppose the Republican Primary in 2016 used the Hare method. Would there be a second round? Why or why not?

Answer: Yes, because no candidate had a majority.

**Figure 11.44 Sample Preference Summary for Voters A, B, C, D, and E**

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| **Options** | **A** | **B** | **C** | **D** | **E** |
| Candidate 1 | 1 | 3 | 3 | 1 | 3 |
| Candidate 2 | 2 | 1 | 1 | 2 | 4 |
| Candidate 3 | 3 | 4 | 2 | 4 | 1 |
| Candidate 4 | 4 | 2 | 4 | 3 | 2 |

Use Figure 11.44 and the Hare method to answer the following questions.

1. How many votes are needed to win by the Hare method?

Answer: 3

1. How many votes does each candidate receive in Round 1?

Answer: 2, 2, 1, and 0, respectively

1. Which candidates advance to Round 2?

Answer: Candidate 1, Candidate 2, and Candidate 3

1. How many votes does each remaining candidate receive in Round 2?

Answer: 2, 2, and 1, respectively

1. Will there be a third round? Why or why not?

Answer: Yes, no candidate has a majority.

1. Which candidate wins the election?

Answer: Candidate 1

**Figure 11.45 Sample Summary of Ranked Ballots**

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| **Number of Ballots** | **10** | **20** | **15** | **5** |
| Option A | 1 | 4 | 3 | 4 |
| Option B | 2 | 3 | 4 | 2 |
| Option C | 4 | 2 | 1 | 3 |
| Option D | 3 | 1 | 2 | 1 |

Use Figure 11.45 to answer the following questions.

1. How many votes were recorded, and how many are required to have a majority?

Answer: 50 votes were recorded. 26 is a majority.

1. How many voters indicated that Option A was their first choice?

Answer: 10

1. How many voters indicated that Option B was their first choice?

Answer: 0

1. How many voters indicated that Option A was their last choice?

Answer: 25

1. How many voters indicated that Option B was their last choice?

Answer: 15

1. Use ranked-choice voting to determine the two candidates in the final round and the number of votes they each receive in that round.

Answer: C receives 15 votes, D receives 35 votes

1. Is there a winning candidate? If so, which candidate? Justify your answer.

Answer: D is the winning candidate with 35 votes because 26 or more is a majority.

**Figure 11.46 Favorite New Heroes in Star Wars Sequels Ballot Preferences**

|  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| **Number of Ballots** | **7** | **6** | **10** | **8** | **4** | **5** | **6** | **7** | **2** |
| **Finn** | 3 | 5 | 1 | 3 | 2 | 4 | 2 | 2 | 1 |
| **Rey** | 2 | 1 | 3 | 2 | 4 | 3 | 3 | 1 | 3 |
| **Poe** | 1 | 6 | 4 | 1 | 5 | 5 | 5 | 4 | 2 |
| **BB8** | 4 | 2 | 2 | 4 | 6 | 1 | 6 | 5 | 4 |
| **Rose** | 6 | 4 | 5 | 6 | 3 | 2 | 1 | 3 | 6 |
| **Kylo** | 5 | 3 | 6 | 5 | 1 | 6 | 4 | 6 | 5 |

Suppose that 55 Star Wars fans were asked to vote for their favorite new Star Wars character. They were given a ranked ballot, and the results are shown in Figure 11.46. Use Figure 11.46 and ranked-choice voting to answer the following questions.

1. How many votes does each candidate get on the first round of voting?

Answer: Finn 12, Rey 13, Poe 15, BB8 5, Rose 6, Kylo 4

1. How many votes are required to get a majority?

Answer: 28

1. Which candidates remain in the final round, and how many votes do they have?

Answer: Rey 33, Finn 22

1. Who is the winner of the election?

Answer: Rey

Refer to Figure 11.45 to answer questions 32–33.

1. Find the Borda score for each candidate.

Answer: A 45, B 50, C 90, D 115

1. Compare your results from question 32 to those from question 26. Compare the winner and the second-place candidate using the Borda count method to those using the ranked-choice method. Are they the same?

Answer: Candidate D was the winner using either method. Candidate C was the runner up by ranked-choice voting

Refer back to Figure 11.Figure 11.4546 to answer the following questions.

1. Find the Borda score for each candidate.

Answer: Finn 189, Rey 202, Poe 132, BB8 130, Rose 105, Kylo 67

1. Compare your results from question 34 to those from question 30. Compare the winner and the second-place candidate using the Borda count method to those using the ranked-choice method. Are they the same?

Answer: Rey was the winner and Finn was in second place by either method.

**Figure 11.47 Sample Summary of Ranked Ballots**

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| **Number of Ballots** | **100** | **80** | **110** | **105** | **55** |
| Option A | 1 | 1 | 4 | 4 | 2 |
| Option B | 2 | 2 | 2 | 3 | 1 |
| Option C | 4 | 4 | 1 | 1 | 4 |
| Option D | 3 | 3 | 3 | 2 | 3 |

Use Figure 11.Figure 11.4647 to answer the following questions.

1. Do any candidates appear to be divisive candidates? Justify your answer.

Answer: Option C and Option A appear to be divisive because each is simultaneously the first choice of a large portion of the voters and the last choice of a large portion of the voters.

1. Do any candidates appear to be compromise candidates? Justify your answer.

Answer: Option B appears to be a compromise candidate because Option B is not the first choice of most voters, but Option B is the second choice of a significant portion of the population and is more acceptable to the population as a whole than the other candidates.

1. How many votes are required for a majority?

Answer: 226

1. Which candidate is eliminated first by the ranked-choice method?

Answer: Option D

1. Which candidate is eliminated second by the ranked-choice method?

Answer: Option B

1. Which candidate is the winner by the ranked-choice method?

Answer: Option A

1. What are the Borda scores for each candidate?

Answer: A 650, B 850, C 645, D 555

1. Which candidate is the winner by the Borda count method?

Answer: Option B.

1. Which method resulted in a win for the compromise candidate: ranked-choice voting or the Borda count method or both?

Answer: The Borda count method.

**Figure 11.48 Pairwise Comparison Matrix for Candidates Q, R, S and T**

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| **Opponent**  **Runner** | **Q** | **R** | **S** | **T** |
| **Q wins** | -- | QR 3 | QS 2 | QT 1 |
| **R wins** | RQ 1 | -- | RS 3 | RT 2 |
| **S wins** | SQ 2 | SR 1 | -- | ST 3 |
| **T wins** | TQ 3 | TR 2 | TS 1 | -- |

Use the pairwise comparison matrix in Figure 11.48 to answer the following questions.

1. Analyze the pairwise comparison matrix. Display the pairings in a table and indicate the winner of each matchup by marking an X through the losing matchups and a single slash through the ties.

Answer:

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| **Opponent**  **Runner** | **Q** | **R** | **S** | **T** |
| **Q wins** | -- | QR 3 | QS 2 | QT 1 |
| **R wins** | RQ 1 | -- | RS 3 | RT 2 |
| **S wins** | SQ 2 | SR 1 | -- | ST 3 |
| **T wins** | TQ 3 | TR 2 | TS 1 | -- |

1. Calculate the points received by each candidate in the pairwise comparison matrix.

Answer: Each candidate earns  points.

1. Determine whether there is a winner of the pairwise comparison election represented by the matrix. If there is a winner, determine if the winner is a Condorcet candidate.

Answer: There is no winner.

**Figure 11.49 Pairwise Comparison Matrix for Candidates U, V, W, X, and Y**

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| **Opponent**  **Runner** | **U** | **V** | **W** | **X** | **Y** |
| **U wins** | -- | UV 1 | UW 3 | UX 3 | UY 4 |
| **V wins** | VU 5 | -- | VW 6 | VX 4 | VY 1 |
| **W wins** | WU 3 | WV 0 | -- | WX 5 | WY 4 |
| **X wins** | XU 3 | XV 2 | XW 1 | -- | XY 6 |
| **Y wins** | YU 2 | YV 5 | YW 2 | YX 0 | -- |

Use the pairwise comparison matrix in Figure 11.49 to answer the following questions.

1. Analyze the pairwise comparison matrix. Display the pairings in a table and indicate the winner of each matchup.

Answer:

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| **Opponent**  **Runner** | **U** | **V** | **W** | **X** | **Y** |
| **U wins** | -- | UV 1 | UW 3 | UX 3 | UY 4 |
| **V wins** | VU 5 | -- | VW 6 | VX 4 | VY 1 |
| **W wins** | WU 3 | WV 0 | -- | WX 5 | WY 4 |
| **X wins** | XU 3 | XV 2 | XW 1 | -- | XY 6 |
| **Y wins** | YU 2 | YV 5 | YW 2 | YX 0 | -- |

1. Calculate the points received by each candidate in the pairwise comparison matrix.

Answer: U 2, V 3, W , X , Y 1

1. Determine whether there is a winner of the pairwise comparison election represented by the matrix. If there is a winner, determine if the winner is a Condorcet candidate and explain your reasoning.

Answer: V is the winner. V is not a Condorcet candidate because V lost the pairwise matchup to Y.

**Figure 11.50 Preference Rankings for Hogwarts Headmaster Election**

|  |  |  |  |
| --- | --- | --- | --- |
| **Percentage of vote** | **25%** | **40%** | **35%** |
| (S) Snape | 1 | 3 | 3 |
| (M) McGonagall | 3 | 1 | 2 |
| (F) Flitwick | 2 | 2 | 1 |

1. In J.K. Rowling’s Harry Potter series, the headmaster of Hogwarts, Albus Dumbledore, died. Imagine that an election is to be held to find his successor. Severus Snape, the head of Slytherin House will be running against the heads of Gryffindor and Ravenclaw, Minerva McGonagall and Filius Flitwick. Use the preference rankings for each candidate in Figure 11.50 to construct a pairwise comparison matrix.

Answer:

|  |  |  |  |
| --- | --- | --- | --- |
| **Opponent**  **Runner** | **S** | **M** | **F** |
| **S wins** | -- | VM 25% | VF 25% |
| **M wins** | MV 75% | -- | MF 40% |
| **F wins** | FV 75% | FM 60% | -- |

1. Analyze the pairwise comparison matrix you constructed for question 51. Display the pairings in a table and indicate the winner of each matchup.

Answer:

|  |  |  |  |
| --- | --- | --- | --- |
| **Opponent**  **Runner** | **S** | **M** | **F** |
| **S wins** | -- | VM 25% | VF 25% |
| **M wins** | MV 75% | -- | MF 40% |
| **F wins** | FV 75% | FM 60% | -- |

1. Use the pairwise comparison matrix from questions 51 and 52 to find the number of points earned by each candidate. Who is the winner by the pairwise comparison method?

Answer: S 0, M 1, F 2. Flitwick is the winner.

1. Is the winner of the Hogwarts headmaster election a Condorcet candidate? Explain how you know.

Answer: Yes, Flitwick is a Condorcet candidate because he won every pairwise matchup.

**Figure 11.51 The Women of Big Bang Theory Vote on Rock, Paper, Scissors, Lizard, Spock!**

|  |  |  |  |
| --- | --- | --- | --- |
| **VOTERS** | **Penny** | **Bernadette** | **Amy** |
| Rock | Yes | No | No |
| Paper | Yes | Yes | No |
| Scissors | Yes | Yes | Yes |
| Lizard | No | No | No |
| Spock | Yes | No | Yes |

1. The women of the Big Bang Theory decide to hold their own approval voting election to determine the best option in Rock, Paper, Scissors, Lizard, Spock. Use the summary of their approval ballots in Figure 11.51 to determine the number of votes for each candidate. Determine the winner, or state that there is none.

Answer: Rock 1, Paper 2, Scissors 3, Lizard 0, Spock 2. The winner is scissors.

**Figure 11.52 Summary of Sample Approval Voting Ballots for Candidates A, B, and C**

|  |  |  |  |
| --- | --- | --- | --- |
| **Percentage of vote** | **40%** | **35%** | **25%** |
| Candidate A | 1 | 3 | 2 |
| Candidate B | 2 | 1 | 3 |
| Candidate C | 3 | 2 | 1 |

Use Figure 11.52 for the following questions.

1. Which candidate is the winner by the ranked-choice method?

Answer: Candidate A.

1. Suppose thatthey used the approval method and each voter approved their top two choices. Which candidate is the winner by the approval method?

Answer: Candidate B.

1. Which candidate is the winner by the Borda count method?

Answer: Candidate B.

## MODULE 1 SUMMARY

### Key Terms

Majority

Plurality

Tie

Tiebreaker

Runoff election

Runoff voting system

Two-round system

Hare method

Preference ranking

Ranked ballot

Ranked-choice voting

Instant runoff voting

Borda count method

Borda score

Divisive candidate

Compromise candidate

Condorcet voting methods

Pairwise comparison method

Condorcet candidate

Approval voting system

Approval voting ballot

### Key Concepts

* In plurality voting, the candidate with the most votes wins.
* When a voting method does not result in a winner, runoff voting can be used to do so.
* Ranked-choice voting, also known as instant runoff voting, is one type of ranked voting system.
* Borda count is a type of ranked voting system in which each candidate is given a Borda score based on the number of candidates ranked lower than them on each ballot.
* When pairwise comparison is used, the winner will be the Condorcet candidate if one exists.
* Approval voting allows voters to give equally weighted votes to multiple candidates.
* When a voter finds a characteristic of a particular voting method unappealing, they may consider that characteristic a flaw in the voting method and look for an alternative method that does not have that characteristic.

### Videos

**How Does Ranked-Choice Voting Work? <**[**https://youtu.be/oHRPMJmzBBw**](https://youtu.be/oHRPMJmzBBw)>

**Determine Winner of Election by Ranked-choice Method (aka Instant Runoff) <**[**https://www.youtube.com/watch?v=RojxTmloAak**](https://www.youtube.com/watch?v=RojxTmloAak)

**Determine Winner of Election by Borda Count Method < https://youtu.be/cKiTSyasia0>**

**Determine Winner of Election by Pairwise Comparison Method < https://youtu.be/w1NNK7Dn3E8>**

**Websites**

**Specialty Calculators <http://**[**www.mshearnmath.com/calculators**](http://www.mshearnmath.com/calculators)**>**

**Who Votes on Oscars <**[**https://variety.com/feature/who-votes-on-oscars-academy-awards-how-voting-works-1203490944/**](https://variety.com/feature/who-votes-on-oscars-academy-awards-how-voting-works-1203490944/)**>**

**Condorcet on Women’s Voting Righte <**<https://oll.libertyfund.org/title/condorcet-on-the-admission-of-women-to-the-rights-of-citizenship>

**Federal Election Commission <https://www.fec.gov/resources/cms-content/documents/federalelections2018.pdf>**

**Cook Political Report <**<https://cookpolitical.com/2020-national-popular-vote-tracker>>

[H2] Module 2 — Fairness in Voting Methods

*Photo 11.3: Citizens involved in gatherings with signs about fair voting.*

**

**After completing this module, you should be able to:**

1. Compare and contrast fairness of voting using majority criterion LO 11.2.1.
2. Compare and contrast fairness of voting using head-to-head criterion LO 11.2.2.
3. Compare and contrast fairness of voting using monotonicity criterion LO 11.2.3.
4. Compare and contrast fairness of voting using irrelevant alternatives criterion LO 11.2.4.
5. Apply Arrow's impossibility theorem when evaluating voting fairness LO 11.2.5.

Now that we’ve covered a variety of voting methods and discussed their differences and similarities, you might be leaning toward one method over another. youYou will need to convince the other founders of Imaginaria that your preference will be the best for the country. Before your collaborators approve the inclusion of a voting method in the constitution, they will want to know that the voting method is a fair method. In this module, we will formally define the characteristics of a fair system. We will analyze each voting previously discussed to determine which characteristics of fairness they have, and which they do not. oftenIn order to guarantee one ideal, we must often sacrifice others.

**[H2] The Majority Criterion**

One of the most fundamental concepts in voting is the idea that most voters should be in favor of a candidate for a candidate to win and that a candidate should not win without majority support. This concept is known as the majority criterion.

**[DEFINITION]**

An election method satisfies the **majority criterion** if any candidate that is favored by more than half of voters is guaranteed to win. <END>

With respect to the four main ranked voting methods we have discussed—plurality, ranked-choice, pairwise comparison, and the Borda count method—we will explore two important questions:

* + 1. Which of these voting systems satisfy the majority criterion and which do not?
    2. Is it always “fair” for a voting system to satisfy the majority criterion?

Keep in mind that this criterion only applies when one of the candidates has a majority. So, the examples we will analyze will be based on scenarios in which a single candidate has more than 50 percent of the vote.

**EXAMPLE 16—Roommates Choose Fast Food**

It’s final exams week and seven college students are starving!! They must get food, but from which drive through? Their preferences are listed in 52.Figure 11.53. The majority have listed McDonalds as their top choice. Let’s calculate what the results of the election will be using various voting methods.

**Figure 11.53 Fast Food Restaurant Choices**

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
| **VOTERS** | **A** | **B** | **C** | **D** | **E** | **F** | **G** |
| (M) McDonalds | 1 | 1 | 1 | 1 | 5 | 5 | 5 |
| (B) Burger King | 2 | 2 | 2 | 2 | 2 | 2 | 2 |
| (T) Taco Bell | 4 | 4 | 3 | 4 | 3 | 3 | 1 |
| (O) Pollo Tropical | 3 | 5 | 4 | 3 | 4 | 1 | 3 |
| (I) Pizza Hut | 5 | 3 | 5 | 5 | 1 | 4 | 4 |

Use this information to find the winning restaurant using each of the voting methods in parts a and b, then answer the question in part c.

1. Plurality
2. Ranked-choice voting
3. Does the majority criterion apply? If so, for which of voting method(s), if any, did the majority criterion fail?

**SOLUTION**

1. For plurality voting, we only need to count the first-place votes for each candidate. In this case, McDonalds has four first place votes, which is a majority and wins the election automatically.
2. For ranked-choice voting, McDonalds also wins because it has a majority at the end of Round 1.
3. The majority criterion does apply because one candidate had a majority of the first-place votes. The majority criterion did not fail by either method, because in each case the majority candidate won. <END>

From Example 16 it appears that the plurality and ranked-choice voting methods satisfy the majority criterion, at least in this example.

**[Your Turn]16**

**Figure 11.54 Sample Summary of Ranked Ballots**

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| **VOTERS** | **L** | **M** | **N** | **O** | **P** |
| Option A | 2 | 2 | 3 | 2 | 1 |
| Option B | 1 | 4 | 1 | 4 | 2 |
| Option C | 3 | 1 | 4 | 1 | 3 |
| Option D | 4 | 3 | 2 | 3 | 4 |

Use the information in Figure 11.54 to find the winner using each of the voting methods in parts a and b, then answer the question in part c.

1. Plurality
2. Ranked-choice voting
3. Does the majority criterion apply? If so, for which voting method(s), if any, did the majority criterion fail?<END>

**Answer**

1. Using the plurality voting method, there would be a tie between Option B and Option C.
2. Option B
3. Using the ranked-choice voting method, Option B would be the winner.
4. No, there was not a majority candidate in Round 1. <END>

In general, the majority candidate always wins in a plurality election because the candidate that has more than half of the votes has more votes than any other candidate. The same is true for ranked-choice voting; and there will never be a need for a second round when there is a majority candidate. Let’s examine how some of the other voting methods stand up to the majority criterion.

EXAMPLE **17**—**Roommates Choose Fast Food**

Those seven college students are starving again! Their preferences haven’t changed (Figure 11.53). Let’s calculate if the results change when we use different voting methods.

**Figure 11.55 Fast Food Restaurant Choices**

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
| **VOTERS** | **A** | **B** | **C** | **D** | **E** | **F** | **G** |
| (M) McDonalds | 1 | 1 | 1 | 1 | 5 | 5 | 5 |
| (B) Burger King | 2 | 2 | 2 | 2 | 2 | 2 | 2 |
| (T) Taco Bell | 4 | 4 | 3 | 4 | 3 | 3 | 1 |
| (O) Pollo Tropical | 3 | 5 | 4 | 3 | 4 | 1 | 3 |
| (I) Pizza Hut | 5 | 3 | 5 | 5 | 1 | 4 | 4 |

Use this information to find the winning restaurant using each of the voting methods in parts a and b, then answer the question in part c.

1. Pairwise comparison
2. Borda count
3. Does the majority criterion apply? If so, for which voting method(s), if any, did the majority criterion fail?

**SOLUTION**

1. For pairwise comparison, notice that McDonalds is a Condorcet candidate because it wins every pairwise comparison. So, McDonalds is the winner.
2. For the Borda count, we must calculate the Borda score for each candidate: McDonalds is 16, Burger King is 21, Taco Bell is 13, Pollo Tropical is 12, Pizza Hut is 8. The winner is Burger King!
3. Yes, the majority criterion applies because McDonalds has the majority of first place votes. The majority criterion only fails using the Borda method. <END>

Example 17 demonstrates a concept that we also saw in Module 1—the Borda method frequently favors the compromise candidate over the divisive candidate. This can happen even when the divisive candidate has a majority, as it did in this example. Although a majority of the voters were in favor of McDonalds, a significant minority was strongly opposed to McDonalds, ranking it last. Since the Borda score includes all rankings, this strong opposition has an impact on the outcome of the election.

### [Your Turn] 17

**Figure 11.56 Sample Summary of Ranked Ballots**

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| **VOTERS** | **L** | **M** | **N** | **O** | **P** |
| Option A | 2 | 2 | 3 | 2 | 2 |
| Option B | 1 | 4 | 1 | 4 | 1 |
| Option C | 3 | 1 | 4 | 1 | 3 |
| Option D | 4 | 3 | 2 | 3 | 4 |

Use the information in Figure 11.56 to find the winner using each of the voting methods in parts a and b, then answer the question in part c.

1. Pairwise comparison
2. Borda count
3. Does the majority criterion apply? If so, for which voting method(s) did the majority criterion fail? <END>

**Answer**

1. B wins.
2. Using the pairwise comparison voting method, Option B wins.
3. Using the Borda count method, Option A and Option B tie.
4. Yes. The majority criterion fails for Borda count. <END>

Pairwise comparison will always satisfy the majority criterion because the candidate with the majority of first-place votes wins each pairwise matchup. While it is possible for the majority candidate to win by the Borda count method, it is not guaranteed. So, the Borda method fails the majority criterion. A summary of each voting method as it relates to the majority criterion is found in Figure 11.57.

**Figure 11.57 Summary of Voting Methods and Majority Criterion**

|  |  |
| --- | --- |
| **Voting Method** | **Majority Criterion** |
| Plurality | Satisfies |
| Ranked-choice | Satisfies |
| Pairwise comparison | Satisfies |
| Borda count | Violates |

If you prefer the Borda method, you might argue that its failure to satisfy the majority criterion is actually one of its strengths. As we saw in Examples 16 and 17, the majority have the power to vote for their own benefit at the expense of the minority. While four students were very enthusiastic about McDonalds, three students were strongly opposed to McDonalds. It is reasonable to say that the better option would be Burger King, the compromise candidate, which everyone ranked highly and no one strongly opposed. The Borda method is designed to favor a candidate that is acceptable to the population as a whole. In this way, the Borda method avoids a downfall of strict majority rule known as **the tyranny of the majority**.

**[DEFINITION]**

The **tyranny of the majority** is a situation in which a minority of a population is treated unfairly because their situation is different from the situation of the majority. <END>

The people of Imaginaria should know that the power of the majority to vote their will has serious implications for other groups. For example, according to the UCLA School of Law Williams Institute, the LGBTQ+ community in the United States makes up approximately 4.5 percent of the population. When elections occur that include issues that affect the LGBTQ+ community, members of the LGBTQ+ community depend on the 95.5 percent of the population who do not identify as LGBTQ+ to consider their perspectives when voting on issues such as same-sex marriage, the use of public restrooms by transgender people, and adoption by same-sex couples.

[PEOPLE IN MATHEMATICS]

In his book, *Democracy and Its Critics*, Robert Dahl wrote, “If a majority is not entitled to do so, then it is thereby deprived of its rights; but if a majority is entitled to do so, then it can deprive the minority of its rights.” Dahl was a renowned political theorist, but he is also considered to be a mathematician since his work utilizes ideas from an area of mathematics known as Game Theory (Mathematics Genealogy Project, NDSU Department of Mathematics with the American Mathematical Society).

[WHO KNEW?] THREE BRANCHES OF GOVERNMENT

Concerns about the consequences of majority rule are not new. In 1788, John Adams warned of the consequences of majority rule and he argued for three branches of government as a way to temper them. In the early 1800s, a young French aristocrat named Alexis de Tocqueville toured the United States and wrote *Democracy in America*, which focused on the impact of democracy on political and civil societies. He observed, even then, the dominance of the white majority over the Indigenous people and enslaved people, which was perpetuated by majority rule.

**VIDEO: Separation of powers and checks and balances <https://youtu.be/APcKEHAPYEg>**

<END>

[H2] **Head-to-Head Criterion**

Another fairness criterion you must consider as you select a voting method for Imaginaria is the **Condorcet criterion**, also known as the head-to-head criterion.

**DEFINITION**

An election method satisfies the **Condorcet criterion** provided that the Condorcet candidate wins the election whenever a Condorcet candidate exists.

A **Condorcet method** is any voting method that satisfies the Condorcet criterion. <END>

Recall from Module 1 that not every election has a Condorcet candidate; the Condorcet criterion will not apply to every election. Also recall that a Condorcet candidate cannot lose an election by pairwise comparison. So, the pairwise comparison voting method is said to satisfy the Condorcet criterion.

**<example>Example 18 - <title>Spending Tax Refund</title>**

A survey asked a random sample of 100 people in the United States to rank their priorities for spending their tax refund. The options were (V) go on vacation, (S) put into savings, (D) pay off debt, or (T) other. The pairwise comparison matrix for the results is in 57.Figure 11.58. Determine if the Condorcet criterion applies.

**Figure 11.58 Pairwise Comparison Matrix for Tax Refund Spending**

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| **Opponent**  **Runner** | **V** | **S** | **D** | **T** |
| **V wins** | -- | VS 33 | VD 37 | VT 100 |
| **S wins** | SV 67 | -- | SD 36 | ST 100 |
| **D wins** | DV 63 | DS 64 | -- | DT 96 |
| **T wins** | TV 0 | TS 0 | TD 4 | -- |

**<SOLUTION>**

The Condorcet criterion only applies when there is a Condorcet candidate. “Pay off debt” is a Condorcet candidate because D wins every matchup. Yes, the Condorcet criterion applies to this election. <END>

### [Your Turn] 18

Determine if the Condorcet criterion applies based on the summary of ranked ballots given in Figure 11.59.

**Figure 11.59 Sample Summary of Ranked Ballots**

|  |  |  |
| --- | --- | --- |
| **VOTEs** | **3** | **2** |
| Option A | 1 | 3 |
| Option B | 2 | 1 |
| Option C | 3 | 2 |

<END>

**Answer**

Yes, because Option A is a Condorcet candidate. <END>

**<example>Example 19 - <title>Spending Tax Refund</title>**

Let’s return to the survey about tax refund spending from Example 18. We know that the Condorcet criterion applies because Option D, “Pay off debt,” is a Condorcet candidate, which wins every pairwise match up. The pairwise comparison matrix in Example 18 was constructed from the summary of voter preferences in Figure 11.59.

**Figure 11.60 Tax Return Spending Ballot Summary**

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| **Votes** | **33** | **32** | **31** | **4** |
| On a vacation (V) | 1 | 3 | 3 | 2 |
| Put into savings (S) | 3 | 1 | 2 | 1 |
| Pay off debt (D) | 2 | 2 | 1 | 4 |
| Other (T) | 4 | 4 | 4 | 3 |

Use the information in the ballot summary from Figure 11.60 to find the winner and determine if the Condorcet criterion is satisfied in this election when each voting method used: (a) plurality, (b) ranked-choice voting, and (c) Borda count.

**SOLUTION**

1. V wins 33 first place votes; S, 36; D, 31; and T, 0. So candidate S, “Put into savings,” has a plurality and wins. Since the Condorcet candidate D didn’t win, the Condorcet criterion is violated.
2. Use the steps outlined in Module 1 for determining the winner of an election by ranked-choice voting, the application of the Hare method in which instant runoffs are used.

STEP 1—The number of votes needed to achieve a majority is 51.

STEP 2—As illustrated in part a, no candidate has a majority of first-place votes; so the candidate with the fewest votes, T, must be eliminated.

STEP 3 — Reallocate votes to the remaining candidates for the second round (60Figure 11.61).

**Figure 11.61 Summary of Ranked Ballots for Tax Refund Spending Round 2**

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| **Votes** | **33** | **32** | **31** | **4** |
| On a vacation (V) | 1 | 3 | 3 | 2 |
| Put into savings (S) | 3 | 1 | 2 | 1 |
| Pay off debt (D) | 2 | 2 | 1 | 3 |

Repeat Step 2 — Count the first-place votes for each candidate: V has 33 votes, S has 36 votes, D has 31. votes. Eliminate candidate D, “Pay off debt,” for the third round.

Repeat Step 3 — Reallocate votes to the remaining candidates for the third round (61Figure 11.62).

**Figure 11.62 Summary of Ranked Ballots for Tax Refund Spending Round 3**

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| **Votes** | **33** | **32** | **31** | **4** |
| On a vacation (V) | 1 | 2 | 2 | 2 |
| Put into savings (S) | 2 | 1 | 1 | 1 |

Repeat Step 2 — Count the first-place votes for each candidate: V has 33, S has 67. Candidate S, “Put into savings,” has a majority and wins. Since the Condorcet candidate D candidate D, “Pay off debt,” didn’t win, the Condorcet criterion is violated.

1. Calculate the Borda score for each candidate.

V:



S: 

D: 

T: 

Candidate D, “Pay off debt,” has the highest Borda score and wins. Since D was the Condorcet candidate, this election satisfies the Condorcet criterion. <END>

### [Your Turn] 19

Use the summary of ranked ballots ( Figure 11.59) from Your Turn 18 to find the winner and determine if the Condorcet criterion is satisfied when the voting method used is (a) plurality, (b) ranked-choice voting, and (c) Borda count. Recall that option A was a Condorcet candidate, so the Condorcet criterion applies. <END>

**Answer**

1. Option A. Yes, the Condorcet criterion is satisfied.
2. Option A. Yes, the Condorcet criterion is satisfied.
3. Option B. No, the Condorcet criterion is not satisfied. <END>

As we have seen in Example 19 and Your Turn 19, the plurality method, ranked-choice voting, and the Borda count method each fail the Condorcet criterion in some circumstances. Of the four main ranked voting methods we have discussed, only the pairwise comparison method satisfies the Condorcet criterion every time. A summary of each voting method as it relates to the Condorcet criterion is found in Figure 11.62.

**Figure 11.63 Voting Methods Satisfying the Condorcet Criterion**

|  |  |
| --- | --- |
| **Voting Method** | **Condorcet Criterion** |
| Plurality | Violates |
| Ranked-choice | Violates |
| Pairwise comparison | Satisfies |
| Borda count | Violates |

**[H2] Monotonicity Criterion**

The citizens of Imaginaria might be surprised to learn that it is possible for a voter to cause a candidate to lose by ranking that candidate higher on their ballot. Is that fair? Most voters would say, “Absolutely not!!” This is an example of a violation of the fairness criterion called the **monotonicity criterion**..

Consider a scenario in which voters are permitted a first round that is not binding, and then they may change their vote before the second round. Such a first round can be called a straw poll. Now, let’s suppose that a particular candidate won the straw poll. After that, several voters are convinced to increase their support, or **up-rank**, that winning candidate and no voters decrease that support. It is reasonable to expect that the winner of the first round will also win the second. Similarly, if some of the voters decide to decrease their support, or **down-rank**, a losing candidate, it is reasonable to expect that candidate will still lose in the second round.

**[DEFINITION]**

An election method satisfies the **monotonicity criterion** when no candidate is harmed by up-ranking nor helped by down-ranking, provided all other votes remain the same.

To **up-rank** a candidate means to increase support by ranking higher on a ranked ballot.

To **down-rank** a candidate means to decrease support by ranking lower on a ranked ballot.

<END>

You might be wondering why it’s called monotonicity criterion. In mathematics, the term monotonicity refers to the quality of always increasing or always decreasing. For example, a person’s age is monotonic because it always increases, whereas a person’s weight is not monotonic because it can increase or decrease. If the only changes to the votes for a particular candidate after a straw poll are in one direction, this change is considered monotonic.

If you are going to make an informed decision about which voting method to use in Imaginaria, you need to know which of the four main ranked voting methods we have discussed—plurality, ranked-choice, pairwise comparison, and the Borda count method—satisfy the monotonicity criterion.

*Photo 11.5: Image of dogs, preferably Standard Poodle, Labrador Retriever, Golden Retriever and/or Bulldog*

**<example> Example 20 - <title>Favorite Dog Breed by Plurality</title>**

The local animal shelter is having a vote-by-donation charity event. For a $10 donation, an individual can complete a ranked ballot indicating their favorite large dog breed: standard poodle, golden retriever, Labrador retriever, or bulldog. Use the summary of ballots is in Figure 11.64 to answer each question.

**Figure 11.64 Favorite Large Dog Breed Ballot Summary**

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| **Votes** | **42** | **53** | **61** | **24** |
| (S) Standard Poodle | 1 | 3 | 2 | 1 |
| (G) Golden Retriever | 3 | 1 | 4 | 4 |
| (L) Labrador Retriever | 4 | 2 | 1 | 2 |
| (B) Bulldog | 2 | 4 | 3 | 3 |

1. Determine the winner of the election by plurality.
2. Suppose that the 53 voters in the second column increased their ranking of the winner by 1. Determine the winner by plurality with the new rankings.
3. Does this election violate the monotonicity criterion?
4. Do you think the result of part c is also true for plurality voting and the monotonicity criterion in general? Why or why not?

**<SOLUTION>**

1. The number of votes for each candidate are: S 66, G 53, L 61, and B 0. The winner is the standard poodle.
2. If the 53 voters in the second column rank S as 2 and L as 3, then the number of votes for each candidate are: S with 66, G with 53, L with 61, and B with 0. The winner is still the standard poodle.
3. This election does not violate the monotonicity criterion because the winner was not hurt by up-ranking.
4. In general, increasing the ranking for a winner of a plurality election will either leave them with the same or more first place votes while leaving the other candidates with the same or fewer first place votes. So a plurality election will never violate the monotonicity criterion. <END>

### [Your Turn] 20

Use the Favorite Large Dog Breed Ballot Summary (Figure 11.64) from Example 20 to answer each question.

1. Determine the winner of the election by Borda count.
2. Suppose that the 61 voters in the third column increased their ranking of the winner by 1. Determine the winner by Borda count with the new rankings.
3. Does this election violate the monotonicity criterion?
4. Do you think the result of part c is true for Borda count and the monotonicity criterion in general? Why or why not? <END>

**Answer**

1. Standard Poodle.
2. Standard Poodle.
3. This election does not violate the monotonicity criterion.
4. Increasing the ranking for a winner of a Borda count election on a ballot will increase that candidate’s Borda score while decreasing another candidate’s Borda score, but leaving the remaining candidates’ Borda score unchanged. So, a Borda count election will never violate the monotonicity criterion. <END>

**<example> Example 21 - <title>Favorite Dog Breed by Ranked-Choice</title>**

In a previous example, we saw that the Summary of Ranked Ballots shown in Figure 11.65 results in the pairwise comparison matrix in Figure 11.66. Use this information to answer the questions.

**Figure 11.65 Sample Summary of Ranked Ballots**

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| **Number of Ballots** | **95** | **90** | **110** | **115** |
| Option A | 4 | 4 | 1 | 1 |
| Option B | 2 | 2 | 2 | 2 |
| Option C | 3 | 1 | 3 | 4 |
| Option D | 1 | 3 | 4 | 3 |

**Figure 11.66 Analyzed Pairwise Comparison Matrix for Sample Summary of Ranked Ballots**

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| **Opponent**  **Runner** | **Option A** | **Option B** | **Option C** | **Option D** | **Points** |
| **Option A wins** | -- | AB  225 | AC  225 | AD  225 | 3 |
| **Option B wins** | BA  185 | -- | BC  320 | BD  315 | 2 |
| **Option C wins** | CA  185 | CB  90 | -- | CD  200 | 0 |
| **Option D wins** | DA  185 | DB  95 | DC  210 | -- | 1 |

1. Determine the winner of the election by the pairwise comparison method.
2. Suppose that the 95 voters in the first column increased their ranking of the winner by 1. Determine the winner by the pairwise comparison method with the new rankings.
3. Does this election violate the monotonicity criterion?
4. Do you think the result of part c is true for the pairwise comparison method and the monotonicity criterion in general? Why or why not? <END>

**SOLUTION**

1. By the pairwise comparison method, Option A wins with three points.
2. If the 95 voters in the first column of Figure 11.65 increased their ranking of the winner by 1, then C would fall into fourth place and A would move up to third place on those ballots. This would only affect the matchup between A and C, and the result would be that A would gain 95 votes while C would lose 95 votes. This means A would have 320 votes and C would have 90. Since A already was ahead of C, this just puts A further ahead and causes no change to the election results.
3. Since the winner A is not hurt by an up-rank, and the loser C is not helped by a down-rank, this election is fair by the monotonicity criterion.
4. Yes, the monotonicity criterion would be satisfied by the pairwise comparison method, because an up-rank of the winner can never decrease the number of pairwise wins. Similarly, a down-rank can never increase the number of pairwise wins. <END>

### [Your Turn] 21

Use the Favorite Large Dog Breed Ballot Summary (63Figure 11.64) from Example 20 to answer each question.

1. Determine the winner of the election by the ranked-choice method.
2. Suppose that the 24 voters in the last column of Figure 11.64 increased their ranking of the winner by 1. Determine the winner by the ranked-choice method with the new rankings.
3. Does this election violate the monotonicity criterion?
4. Do you think the result in part c is true for the ranked-choice method and the monotonicity criterion in general? Why or why not?<END>

**Answer**

1. Labrador Retriever wins the election.
2. Golden Retriever wins the election.
3. This election violates the monotonicity criterion.
4. It is possible that the monotonicity criterion would be met in other ranked-choice election scenarios, but overall, the ranked-choice voting method is said to fail the monotonicity criterion even if it failed in only one scenario. <END>

The last few examples illustrate that the plurality method, pairwise comparison voting, and the Borda count method each satisfy the Monotonicity criterion. Of the four main ranked voting methods we have discussed, only the ranked-choice method violates the Monotonicity criterion. A summary of each voting method as it relates to the Condorcet criterion is found in Figure 11.63

**Figure 11.67 Voting Methods Satisfying the Monotonicity Criterion**

|  |  |
| --- | --- |
| **Voting Method** | **Monotonicity Criterion** |
| Plurality | Satisfies |
| Ranked-choice | Violates |
| Pairwise comparison | Satisfies |
| Borda count | Satisfies |

**[H2] Irrelevant Alternatives Criterion**

*Photo 11.6: Apple, cherry, blueberry pie or waiter bringing pie to a table.*



We have covered a lot about voting fairness, but there is one more fairness criterion that you and the other Imaginarians should know. Consider this well-known anecdote that is sometimes attributed to the American philosopher Sidney Morgenbesser:

A man is told by his waiter that the dessert options this evening are blueberry pie or apple pie. The man orders the apple pie. The waiter returns and tells him that there is also a third option, cherry pie. The man says, “In that case, I would like the blueberry pie.” (Gaming the vote: why elections aren't fair (and what we can do about it), William Poundstone, p. 50, [ISBN](https://www.wikiwand.com/en/ISBN_(identifier)) [0-8090-4893-0](https://www.wikiwand.com/en/Special:BookSources/0-8090-4893-0))

This story illustrates the concept of the Irrelevant Alternatives Criterion, also known as the **Independence of Irrelevant Alternatives Criterion (IIA)**,**),** which means that the introduction or removal of a third candidate should not change or reverse the rankings of the original two candidates relative to one another. In particular, if a losing candidate is removed from the race or if a new candidate is added, the winner of the race should not change.

**DEFINITION**

An election satisfies **Independence of Irrelevant Alternatives Criterion (IIA)** provided that for any two candidates A and B, with A preferred to B, the addition or removal of a candidate will not make B preferable to A. <END>

**<example>Example 22 - <title>Apple, Blueberry or Cherry?</title>**

Suppose that 30 students in a class are going to vote on whether to have apple, blueberry, or cherry pie. Use the summary of ranked ballots in Figure 11.68 to answer each question.

**Figure 11.68 Summary of Ranked Ballots for Pie Preferences**

|  |  |  |  |
| --- | --- | --- | --- |
| **Number of Ballots** | **14** | **12** | **4** |
| (A) Apple Pie | 1 | 3 | 3 |
| (B) Blueberry Pie | 2 | 1 | 2 |
| (C) Cherry Pie | 3 | 2 | 1 |

1. Determine the winner of the election by plurality.
2. Which candidate would win a plurality election if cherry pie were removed from the ballot?
3. Does this election violate the IIA?

**SOLUTION**

1. The number of first place votes for each candidate is: A with 14, B with 12, and C with 4. Apple pie has the most first-place votes and wins the election.
2. If cherry pie is removed from the ballot, then the four voters in the third column now rank blueberry pie as their first choice. So the four votes for C now belong to B. This means that blueberry pie has 16 votes compared to the 14 votes for apple pie. Blueberry pie now wins the plurality election.
3. Yes, the election violates the IIA because the removal of a losing candidate from the ballot changed the winner of the election. <END>

### [Your Turn] 22

The local animal shelter is having another vote-by-donation charity event! This time, for a $10 donation, an individual can complete a ranked ballot indicating their favorite small dog breed: Miniature from the ones provided: miniature poodle, Yorkshire terrier, or Chihuahua. Use the summary of ballots in Figure 11.69 to answer each question.

**Figure 11.69 Favorite Small Dog Breed Ballot Summary**

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| **Votes** | **53** | **42** | **24** | **11** |
| (P) Miniature Poodle | 1 | 3 | 3 | 2 |
| (Y) Yorkshire Terrier | 2 | 2 | 1 | 1 |
| (C) Chihuahua | 3 | 1 | 2 | 3 |

1. Determine the winner of the election by the ranked-choice method.
2. Determine the winner of the election if poodles are removed from the ballot.
3. Does this election violate IIA? <END>

**Answer**

1. Chihuahua.
2. Yorkshire Terrier.
3. Yes, this election violates IIA.<END>

**<example>Example 23 - <title>Best Fourth Wall Breaking Stare on The Office</title>**

The NBC sitcom *The Office* ran for nine years and has been one of the most popular streamed television shows of all time. One of the trademarks of the show was that characters would often break the fourth wall to communicate with the audience just by staring directly into the camera. In fact, there is a website dedicated to "*Office* stares" where you can watch over 700 of these stares! **For more details visit: <**  [**http://theofficestaremachine.com/**](https://www.data.gov/open-gov/)**>**

Suppose that 36 fans were asked which character had the best "*Office* stare." Use the ballot summary in Figure 11.70 to answer each question.

**Figure 11.70 Summary of Ranked Ballots for Best "*Office* Stare"**

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| **Number of Ballots** | **9** | **11** | **7** | **6** | **3** |
| (J) Jim Halpbert (John Krasinski) | 1 | 2 | 4 | 2 | 4 |
| (P) Pam Beesly-Halpert (Jenna Fischer) | 4 | 1 | 2 | 4 | 3 |
| (D) Dwight Schrute (Rainn Wilson) | 2 | 3 | 3 | 1 | 2 |
| (M) Michael Scott (Steve Carell) | 3 | 4 | 1 | 3 | 1 |

1. Determine the winner of the election by the pairwise comparison method.
2. Determine the winner of the election by the pairwise comparison method if Michael Scott is removed from the ballot.
3. Does this election violate IIA?

**SOLUTION**

1. Construct and analyze a pairwise comparison matrix (Figure 11.71).

**Figure 11.71 Pairwise Comparison Matrix for Jim, Pam, Dwight, and Michael**

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| **Opponent**  **Runner** | **J** | **P** | **D** | **M** | **Points** |
| **J wins** | -- | JP 15 | JD 20 | JM 26 | 2 |
| **P wins** | PJ 21 | -- | PD 18 | PM 11 |  |
| **D wins** | DJ 16 | DP 18 | -- | DM 26 |  |
| **M wins** | MJ 10 | MP 25 | MD 10 | -- | 1 |

Jim Halpbert wins with 2 points.

1. If Michael Scott is removed, the summary of ranked ballots becomes Figure 11.2.

**Figure 11.2 Summary of Ranked Ballots for Best "*Office* Stare" with Jim, Pam, and Dwight**

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| **Number of Ballots** | **9** | **11** | **7** | **6** | **3** |
| (J) Jim Halpbert (John Krasinski) | 1 | 2 | 3 | 2 | 3 |
| (P) Pam Beesly-Halpert (Jenna Fischer) | 3 | 1 | 1 | 3 | 2 |
| (D) Dwight Schrute (Rainn Wilson) | 2 | 3 | 2 | 1 | 1 |

Construct and analyze a pairwise comparison matrix (Figure 11.3).

**Figure 11.3 Pairwise Comparison Matrix for Jim, Pam, and Dwight**

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| **Opponent**  **Runner** | **J** | **P** | **D** | **Points** |
| **J wins** | -- | JP 15 | JD 20 | 1 |
| **P wins** | PJ 21 | -- | PD 18 |  |
| **D wins** | DJ 16 | DP 18 | -- | 0.5 |

Pam wins with  points.

1. Yes, this violates the IIA, because the winning candidate was hurt by the elimination of a losing candidate. <END>

### [Your Turn] 23

As in Example 23, use Figure 11.2 to answer the questions.

1. Determine the winner of the election by the Borda count method.
2. Determine the winner of the election by the Borda count method if Michael Scott is removed from the ballot.
3. Does this election violate IIA? <END>

**Answer**

1. Jim wins.
2. Pam wins.
3. Yes, this is a violation of IIA. <END>

We have seen that all four of the main voting systems we are working with fail the Irrelevant Alternatives Criterion (IIA). A summary of each voting method as it relates to the IIA criterion is found in Figure 11.4.

**Figure 11.4 Summary of Voting Methods and Irrelevant Alternatives Criterion**

|  |  |
| --- | --- |
| **Voting Method** | **Irrelevant Alternatives Criterion** |
| Plurality | Violates |
| Ranked-choice | Violates |
| Pairwise comparison | Violates |
| Borda count | Violates |

[WHO KNEW?] **Electronic Voting - Does Your Vote Count?**

In order for an election to be fair, voting must be accessible to everyone and every vote must be counted. When hundreds of thousands to millions of votes must be collected and counted in a short period of time, decidingit can be challenging to be on counting procedures that are accurate and secure. Electronic voting machines or even internet voting can speed up the process, but how reliable are these methods? This has been a subject for debate for years.

In a press release on August 3, 2007, California Secretary of State DebraBowen explained the results of an extensive review of electronic voting systems in her state. She said that transparency and auditability were key. She went on to say, “I think voters and counties are the victims of a federal certification process that hasn’t done an adequate job of ensuring that the systems made available to them are secure, accurate, reliable and accessible. Congress enacted the Help America Vote Act, which pushed many counties into buying electronic systems that—as we’ve seen for some time and we saw again in the independent UC review—were not properly reviewed or tested to ensure that they protected the integrity of the vote.” Secretary Bowden subsequently ordered that voting machines must have tighter security to be used in California. (DB07:042, Secretary of State Debra Bowen Moves to Strengthen Voter Confidence in Election Security Following Top-to-Bottom Review of Voting Systems, <https://sos.ca.gov/elections>)

In some instances, the use of electronic voting in parliamentary elections has been discontinued completely for security reasons. For example, according the National Democratic Institute, the Netherlands returned to all paper ballots and hand counting in 2006. Will you use voting machines or internet voting in Imaginaria?

Arrow’s Impossibility Theorem

So far, every one of the voting methods we have analyzed has failed one or more of the fairness criteria in one election or another.

**Figure 11.75 Summary of Voting Methods and Fairness Criteria**

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| **Voting Method** | **Majority Criterion** | **Condorcet Criterion**  **(Head-to-Head Criterion)** | **Monotonicity Criterion** | **(Independence of) Irrelevant Alternatives Criterion** |
| **Plurality** | Satisfies | Violates | Satisfies | Violates |
| **Ranked-choice** | Satisfies | Violates | Satisfies | Violates |
| **Pairwise comparison** | Satisfies | Satisfies | Violates | Violates |
| **Borda count** | Violates | Violates | Satisfies | Violates |

You might be wondering if there is a voting system you could recommend for Imaginaria that satisfies all the fairness criteria. If there is one, it remains to be discovered and it is not a voting system that is based solely on preference rankings. In 1972, Harvard Professor of Economics Kenneth J. Arrow received the Nobel Prize in Economics for proving **Arrow’s Impossibility Theorem**. (Kenneth J. Arrow – Facts. NobelPrize.org. Nobel Media AB 2021. Sun. 14 Mar 2021)

**[DEFINITION]**

**Arrow’s Impossibility Theorem** states that any voting system, either existing or yet to be created, in which the only information available is the preference rankings of the candidates, will fail to satisfy at least one of the following fairness criteria: the majority criterion, the Condorcet criterion, the monotonicity criterion, and the independence of irrelevant alternatives criterion. <END>

This theorem only applies to a specific category of voting systems—those for which the preference ranking is the only information collected. There are other types of voting systems to which the Impossibility Theorem does not apply. For example, there is a class of voting systems called **Cardinal voting systems** that allow for rating the candidates in some way.

**[DEFINITION]**

A **Cardinal voting system** is a system in which each voter gives a rating or grade to each candidate..<END>

“Rating” is different from “ranking” because a voter can give different candidates the same rating. Consider the five-star rating systems used by various industries, or the thumbs up/thumbs down rating system used on YouTube. Could there be a Cardinal voting system that does not violate any of the fairness criteria we have discussed? It’s possible, but more research must be done in order to prove it.

PEOPLE IN MATHEMATICS

In 1972, Kenneth J. Arrow, a Harvard Professor of Economics, received the Nobel Prize in Economics jointly with Sir John Hicks, another world renowned economist, for their contributions to economic theory. In particular, Professor Arrow proved mathematically that no ranked voting system meets all four of the fairness criteria discussed in this Module. The statement of this fact is known as Arrow’s Impossibility Theorem. (**For more details visit: <**[**https://www.nobelprize.org/prizes/economic-sciences/1972/arrow/facts/**](https://www.nobelprize.org/prizes/economic-sciences/1972/arrow/facts/)**>)**

CHECK YOUR UNDERSTANDING

1. Which fairness criterion is violated by all four of the main ranked voting methods presented in this chapter?

Answer: (Independence of) Irrelevant Alternatives Criterion

1. Which of the four main ranked voting methods presented in this chapter satisfies the Condorcet criterion?

Answer: Pairwise comparison

1. Which of the four main ranked voting methods presented in this module violates the majority criterion?

Answer: Borda count method

1. Which of the four main ranked voting methods presented in this module violates the monotonicity criterion?

Answer: Pairwise comparison

1. According to Arrow’s Impossibility Theorem, which of the four main ranked voting methods presented in this chapter violate at least one of the fairness criteria?

Answer: All of them: Plurality, ranked-choice, pairwise comparison, and Borda count

1. Determine whether the following statement is true or false and explain your reasoning: Any ranked election that violates the majority criterion also violates the Condorcet criterion.

Answer: True, because a majority candidate is always the Condorcet candidate.

1. Determine whether the following statement is true or false and explain your reasoning: Any ranked election that violates the Condorcet criterion also violates the majority criterion.

Answer: False, because the ranked-choice method violates the Condorcet criterion, but it doesn’t violate the majority criterion.

1. Does Arrow’s Impossibility Theorem apply to approval voting? Why or why not?

Answer: No. Arrow’s Impossibility Theorem does not apply to approval voting because it is not a ranked voting system

Determine whether each statement is true or false. Explain your reasoning.

1. Arrow’s Impossibility Theorem guarantees that ranked voting systems always lead to unfair elections.

Answer: False. Arrow’s Impossibility Theorem says that no ranked voting system is perfect and that voter profiles may arise that will lead to a violation of one or more fairness criteria, but it does not guarantee that those voter profiles will occur or are even likely to occur.

1. Approval voting is in the class of voting systems called Cardinal Voting systems.

Answer: True. Candidates are rated as approved or not approved, and voters can give multiple candidates the same rating.

EXERCISES

For the following questions, identify which fairness criteria, if any, are violated by characteristics of the described voter profile. Explain your reasoning.

1. In a plurality election, the candidates have the following vote counts: A 125, B 132, C 149, D 112. The pairwise matchup points for each candidate would have been: A 1, B 3, C 1, D 1.

Answer: C won the election by plurality, but B was the Condorcet candidate. This violates the Condorcet criterion.

1. In a Borda count election, the candidates have the following Borda scores: A 1245, B 1360, C 787. Candidate A received 55 percent of the first-place rankings.

Answer: Candidate A is the majority candidate, but Candidate B won the election. This violates the majority criterion.

1. In a pairwise comparison election, the four candidates initially received the following points for winning matchups: A 2, B ,, C 1, D . When candidate C dropped out of the election, the remaining candidates received: A 1, B , D .



Answer: Candidate A was preferred to B until C dropped out of the race. Then Candidate B was preferred to A. This is a violation of the Irrelevant Alternatives criterion.

1. In a Borda count election, the candidates have the following Borda scores: A 15, B 11, C 12, D 16. The pairwise matchup points for the same voter profiles would have been: A 2, B 0, C 1, D 3.

Answer: Candidate A won the Borda count election, but Candidate D was a Condorcet candidate. This violates the Condorcet criterion.

1. In a Borda count election, the candidates have the following Borda scores: A 15, B 11, C 12, D 16. When Candidate E was added to the ballot, the Borda scores became: A 25, B 21, C 15, D 24, E 18.

Answer: Candidate D was preferred to A, but that was reversed when Candidate E was added to the ballot. This violates the Irrelevant Alternatives criterion.

1. In a pairwise comparison election, Candidate C was a Condorcet candidate in a straw poll. When the actual election took place, several voters up-ranked Candidate C on their ballots, but no other changes were made to the voter preferences, and Candidate B won the election.

Answer: Up-ranking hurt a candidate. This violates the monotonicity criterion.

1. In a pairwise comparison election, Candidate A was in first place, Candidate B was in second place, and Candidate C was in third place. When the actual election tool place, the only changes were that several voters down-ranked Candidate B on their ballots, but the outcome remained the same.

Answer: This does not violate any fairness criteria.

**Figure 11.76 Sample Ballot Summary 1**

|  |  |  |
| --- | --- | --- |
| **Votes** | 49 | 51 |
| Candidate A | 3 | 1 |
| Candidate B | 1 | 2 |
| Candidate C | 2 | 3 |

Use Sample Ballot Summary 1 in Figure 11.76 to answer questions 7–14.

1. Determine the Borda score for each candidate and the winner of the election using the Borda count method.

Answer: The Borda score for each candidate is as follows: A 102, B 149, C 49; Winner. The winner is Candidate B.

1. Is there a majority candidate? If so, which candidate?

Answer: Yes, the majority candidate is Candidate A.

1. Does this Borda method election violate the majority criterion? Justify your answer.

Answer: Yes, the majority candidate is Candidate A, but Candidate B wins by the Borda count method.

1. Is there a Condorcet candidate? If so, which candidate?

Answer: Yes, the Condorcet candidate is A.

1. Does this Borda method election violate the Condorcet criterion? Justify your answer.

Answer: Yes, the Condorcet candidate is A, but B wins by the Borda count method.

1. If Candidate C is removed from the ballot, which candidate wins by the Borda count method?

Answer: ACandidate A wins using the Borda count method.

1. Does this Borda count method election violate IIA? Justify your answer.

Answer: Yes, Candidate A wins instead of B when C is removed from the ballot.

**Figure 11.77 Sample Ballot Summary 2**

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| **Votes** | **7** | **9** | **12** | **15** | **5** |
| Candidate A | 4 | 4 | 1 | 1 | 4 |
| Candidate B | 1 | 1 | 2 | 2 | 3 |
| Candidate C | 3 | 2 | 3 | 4 | 1 |
| Candidate D | 2 | 3 | 4 | 3 | 2 |

Use Sample Ballot Summary 2 in Figure 11.77 to answer the following questions.

1. Determine Borda score for each candidate and the winner of the election using the Borda count method.

Answer: The Borda score for each candidate is as follows: A 81, B 107, C 52, D 48; Winner. The winner is Candidate B.

1. Is there a majority candidate? If so, which candidate?

Answer: Yes, the majority candidate is A.

1. Does this Borda method election violate the majority criterion? Justify your answer.

Answer: Yes, the majority candidate is A, but B wins by the Borda method.

1. Is there a Condorcet candidate? If so, which candidate?

Answer: Yes, the Condorcet candidate is A.

1. Does the Borda method election violate the Condorcet criterion? Justify your answer.

Answer: Yes, the Condorcet candidate is A, but B wins by the Borda method.

1. Can an election that fails the majority criterion satisfy the Condorcet criterion? Why or why not?

Answer: No, because a majority candidate is automatically a Condorcet candidate.

**Figure 11.78 Sample Ballot Summary 3**

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| **Number of Ballots** | **10** | **7** | **5** | **5** | **4** |
| Candidate A | 1 | 3 | 3 | 3 | 4 |
| Candidate B | 3 | 2 | 1 | 4 | 1 |
| Candidate C | 2 | 4 | 2 | 1 | 2 |
| Candidate D | 4 | 1 | 4 | 2 | 3 |

Use Sample Ballot Summary 3 in Figure 11.78 to answer the following questions.

1. Determine the Borda score for each candidate and the winner of the election using the Borda count method.

Answer: The Borda score for each candidate is as follows: A 47 B 51, C 53, D 35; Winner. The winner is Candidate C.

1. Is there a majority candidate?

Answer: No.

1. Does the election violate the majority criterion? Justify your answer.

Answer: No. It can’t violate the majority criterion because no one has a majority.

1. Determine the winner by pairwise comparison.

Answer: Candidate B wins using the pairwise comparison method.

1. Is there a Condorcet candidate?

Answer: Candidate B is the Condorcet candidate.

1. Does the Borda election violate the Condorcet criterion? Justify your answer.

Answer: Yes, Candidate C won by the Borda method but B was a Condorcet candidate.

1. Determine the winner by the ranked-choice method.

Answer: Candidate D wins by the ranked-choice method.

1. Does the ranked-choice election violate the majority criterion? Justify your answer.

Answer: No. It can’t violate the majority criterion because no one has a majority.

1. Does the ranked-choice election violate the Condorcet criterion? Justify your answer.

Answer: Yes, the Condorcet candidate is B, but the winner by ranked-choice is D.

1. Can an election that fails the Condorcet criterion satisfy the majority criterion? Why or why not?

Answer: Yes, the ranked-choice method always satisfies the majority criterion but sometimes fails the Condorcet criterion.

**Figure 11.79 Sample Ballot Summary 4**

|  |  |  |  |
| --- | --- | --- | --- |
| **Number of Ballots** | **49** | **48** | **3** |
| Candidate A | 1 | 3 | 3 |
| Candidate B | 2 | 1 | 2 |
| Candidate C | 3 | 2 | 1 |

Use Sample Ballot Summary 4 in Figure 11.79 to answer the following questions.

1. Determine the winner of the election using the plurality method.

Answer: Candidate A wins using the plurality method.

1. Determine the winner by pairwise comparison.

Answer: Candidate B wins using pairwise comparison.

1. Is there a Condorcet candidate?

Answer: Candidate B is the Condorcet candidate.

1. Does this plurality election violate the Condorcet criterion? Justify your answer.

Answer: Yes, the Condorcet candidate is B, but A won by plurality.

1. If Candidate C is removed from the ballot, which candidate wins by plurality?

Answer: B

Answer: Candidate B wins by plurality if C is removed from the ballot.

1. Does this plurality violate the IIA? Explain your reasoning.

Answer: Yes, Candidate A was preferred to B, but that was reversed when C was removed from the ballot.

**Figure 11.80 Sample Ballot Summary 5**

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| **Number of Ballots** | **16** | **20** | **24** | **8** |
| Candidate A | 1 | 3 | 2 | 1 |
| Candidate B | 3 | 1 | 4 | 4 |
| Candidate C | 4 | 2 | 1 | 2 |
| Candidate D | 2 | 4 | 3 | 3 |

Use Sample Ballot Summary 5 in Figure 11.80 to answer the following questions.

1. Determine the winner of the election using the ranked-choice method.

Answer: Candidate C is the winner using the ranked-choice method.

1. Determine the winner by pairwise comparison.

Answer: TieUsing pairwise comparisons, there is a tie between Candidate A and C.

1. Is there a Condorcet candidate?

Answer: No

1. Does this ranked-choice election violate the Condorcet criterion? Justify your answer.

Answer: It does not violate the Condorcet criterion because there is no Condorcet Candidatecandidate.

1. If the four voters in the last column rank Candidate C ahead of A, which candidate wins by the ranked-choice method?

Answer: Candidate B wins.

1. Does this ranked-choice election violate the monotonicity criterion? Explain your reasoning.

Answer: Yes, Candidate C was the winner, but an increase in rank by some voters caused C to lose the election.

**Figure 11.81 Sample Ballot Summary 6**

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| **Number of Ballots** | **15** | **12** | **9** | **3** |
| Candidate A | 1 | 3 | 3 | 2 |
| Candidate B | 2 | 2 | 1 | 1 |
| Candidate C | 3 | 1 | 2 | 3 |

Use Sample Ballot Summary 6 in Figure 11.81 to answer the following questions.

1. Determine the winner of the election using the ranked-choice method.

Answer: Candidate A wins using the ranked-choice method.

1. How could it be demonstrated that this ranked-choice election violates IIA?

Answer: Removing either Candidate B or C from the election would result in Candidate A losing. This demonstrates that the election violates IIA.

1. Determine the winner of the election by the Borda method.

Answer: Candidate B wins using the Borda method.

1. Does this Borda method election violate the IIA? Why or why not?

Answer: No, if Candidate A is eliminated, then B wins; if Candidate C is eliminated, B wins.

1. Does this Borda method election violate the monotonicity criterion? Why or why not?

Answer: No because the Borda method always satisfies the monotonicity criterion.

**Figure 11.82 Sample Pairwise Comparison Matrix 1**

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| **Opponent**  **Runner** | **L** | **M** | **N** | **Points** |
| **L wins** | -- | LM 3 | LN 2 |  |
| **M wins** | ML 1 | -- | MN 3 |  |
| **N wins** | NL 2 | NM 1 | -- |  |

Use Figure 11.2 Sample Pairwise Comparison Matrix 1 to answer the following questions.

1. Which candidate wins the pairwise election?

Answer: L wins the pairwise election.

1. Determine the winner by pairwise comparison if N were removed from the ballot.

Answer: L wins by pairwise comparison if N were removed from the ballot.

1. Determine the winner by pairwise comparison if M were removed from the ballot.

Answer: There is no winner. L ties with N. if M is removed

1. Does this pairwise election satisfy the IIA?

Answer: No. L was favored over N in the original election, but removing a candidate hurts L’s status.

**[MODULE 2 SUMMARY]**

**Key Terms**

**majority criterion**

**tyranny of the majority**

**up-rank**

**down-rank**

**monotonicity criterion**

**Condorcet criterion**

**independence of irrelevant alternatives criterion (IIA)**

**cardinal voting system**

**Arrow’s Impossibility Theorem**

**Key Concepts**

* There are several common measures of voting fairness, including the majority criterion, the head-to head criterion, the monotonicity criterion, and the irrelevant alternatives criterion.
* According to Arrow’s Impossibility Theorem, each voting method in which the only information is the order of preference of the voters will violate one of these fairness criteria.

**Videos**

**Separation of powers and checks and balances <https://youtu.be/APcKEHAPYEg>**

[H1] Module 3 - Standard Divisors, Standard Quotas, and the Apportionment Problem

*PHOTO 11.7: People lining up for Covid19- vaccines*

******

**After completing this module, you should be able to:**

* + 1. Analyze the apportionment problem and applications to representation LO 11.3.1.
    2. Evaluate applications of standard divisors LO 11.3.2.
    3. Evaluate applications of standard quotas LO 11.3.3.

**[H2]The Apportionment Problem**

*PHOTO 11.8: A photo of several children eating/sharing cake or candy\*

******

In the new democracy of Imaginaria, there are four states: Fictionville, Pretendstead, Illusionham, and Mythbury. Each state will have representatives in the Imaginarian Legislature.l You might now have an agreement on which voting method your citizens will use to elect representatives. However, before that process can even begin, you must decided on how many representatives each state will receive. This decision will present its own challenges.

When sharing your birthday cake, it’s only fair that everyone gets the same portion size, right? You were apportioning the cake by dividing it up equally and giving everyone a slice. A great thing about cake is that you can slice it any way you want, but how do you **apportion** items that can’t be sliced? Suppose that you have a box of 16 ring pops, gem-shaped lollipops on a plastic ring. You are going to share the box with four other kids. Dividing the 16 ring pops among the group of five leads to a problem; after each person in the group gets three Ring Pops, there is still one left! Who gets the last one? This is the **apportionment problem**.

**[DEFINITION]**

To **apportion** a resource means to divide and distribute it among recipients.

The **apportionment problem** is how to fairly divide, or apportion, available resources that must be distributed to the recipients in whole, not fractional, parts. <END>

The apportionment problem applies to many aspects of life, including the representatives in the Imaginarian legislature! Figure 11.83 provides a short list of examples of resources that must be apportioned in whole parts, and the recipients of those resources.

**Figure 11.83 Applications of Apportionment**

|  |  |
| --- | --- |
| **Resource** | **Recipients** |
| Covid-19 Vaccines | Nations around the world |
| Airport Terminals | Airlines |
| Faculty Positions at a University | Departments |
| Public Schools | Communities |
| U.S. House of Representatives Seats | States |
| Parliamentary Seats | Political Parties |

Fair division of a resource is not necessarily equal division of the resource like when distributing cake slices. When distributing airport terminals amongst airlines, there are many factors to consider such as the size of the airline, the number and types of aircraft they have, and the demand for the service. In most cases, fairness is defined as being **proportional** to need in some way. In the case of the Covid-19 vaccine, the expectation would be that countries with larger populations get more doses of the vaccine. In the Imaginarian legislature, the expectation may be that the states with larger populations will receive the larger number of representatives. This concept is referred to as a **part-to-part ratio**.

**[DEFINITION]**

Two quantities are **proportional** if they have the same relative size, which can also be described as a constant part to part ratio.

A **part-to-part ratio** is a measure of the relationship between the relative quantities in two distinct groups. <END>

Suppose that a supermarket has a special on pies, two for $5. The first customer purchases four pies for $10, and the second customer purchases eight pies for $20. The dollar to pie ratio for the first customer is  and the dollar to pie ratio for the second customer is.. So, the dollar to pie ratio is constant. Although the customers do not spend the same amount of money, the amount each spent was proportional to the number of pies purchased.

Now suppose that the supermarket changed the special to $5 for the first pie, and $2 for each additional pie. In that case, four pies would cost $5+3($2)=$11, while 8 pies would cost $5+ 7($2)=$19. The dollar to pie ratios would be  and  respectively. This special does not result in a constant part to part ratio. The dollars spent are not proportional to the number of pies purchased.

Watch these videos to gain a better understanding of the concepts of constant ratio and part to part ratio:

**VIDEO: What is a ratio?** [**https://youtu.be/B4\_T6-rc35Y**](https://youtu.be/B4_T6-rc35Y)

**VIDEO: What are the Different Types of Ratios?** [**https://youtu.be/xwuFHj5O-kA**](https://youtu.be/xwuFHj5O-kA)

**<example> EXAMPLE 24 - <title>Ratio of Faculty to Students at a College</title>**

The data in Figure 11.84 is a comparison of the number of faculty members in each department at a particular college to the student head count in that department and the number of class sections in that department in the Spring semester. Use the information to answer the questions.

**Figure 11.84 Number of Faculty, Students, and Class Sections at a College**

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| **Department** | **Mathematics** | **English** | **History** | **Science** |
| **(S) Student Head Count** | 4800 | 2376 | 1536 | 2880 |
| **(C) Class Sections** | 120 | 108 | 48 | 96 |
| **(T) Total Faculty** | 30 | 27 | 12 | 24 |
| **(F) Full-Time Faculty** | 10 | 9 | 4 | 8 |
| **(P) Part-Time Faculty** | 20 | 18 | 8 | 16 |

* 1. Determine the ratios for each department: S to C, C to T, S to T, F to P
  2. What are the units of the ratios that you found?
  3. Which of these pairs, if any, has a constant part to part ratio? State the ratio.
  4. Does it appear that the total number of faculty positions were allocated to each department based on student head count, the number of class sections, or neither? Justify your answer.

**<SOLUTION>**

* 1. Divide the first quantity by the second. Answers are provided in Figure 11.85.
  2. Answers are provided in last column of Figure 11.85.

**Figure 11.85 Ratios and Units**

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| **Department** | **Mathematics** | **English** | **History** | **Science** | **Units of Ratios Found** |
| **S to C** |  |  |  |  | Students per class section |
| **C to T** |  |  |  |  | Class sections per faculty member |
| **S to T** |  |  |  |  | Students per faculty member |
| **F to P** |  |  |  |  | Full-time faculty member per part-time faculty member |

* 1. The ratio of class sections to faculty members is a constant ratio of four. The ratio of full-time faculty to part-time faculty is a constant ratio of .



* 1. It appears that the faculty positions were allocated based on the number of class sections because there is a constant ratio of four class sections per faculty member. <END>

**[Your Turn] 26**

4The SAT is to be administered at a high school. In preparation, pencils have been distributed to each of the classrooms based on the room capacity. Use the information in Figure 11.6 to answer each question.

**Figure 11.86 Room Capacities and Pencil Distribution**

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| **Room Number** | **B** | **C** | **D** | **E** |
| **Room capacity (number of student desks)** | 24 | 18 | 32 | 22 |
| **Number of pencils** | 36 | 27 | 48 | 33 |

a. Find the part to part ratio of desks to pencils for each room. Represent it as both a reduced fraction and a decimal rounded to the nearest hundredth as needed.

Answer: B 2/3; 0.67, C 2/3; 0.67, D 2/3; 0.67, and E 2/3; 0.67.

b.Find the part to part ratio of pencils to desks for each room. Represent it as both a reduced fraction and a decimal rounded to the nearest hundredth as needed.

Answer: B 3/2; 1.5, C 3/2; 1.5, D 3/2; 1.5, and E 3/2; 1.5

c.Have the pencils been distributed proportionally? If so, what is the constant ratio of pencils to desks? Give the units. <END>

Answer: Yes, the constant ratio is 3/2 pencils per desk.

There are some useful relationships between quantities that are proportional to each other. When there is a constant ratio between two quantities, the one quantity can be found by multiplying the other by that ratio. Remember the supermarket special on pies, 2 pies for $5? The ratio of dollars to pies is  and the ratio of pies to dollars is. These two values are reciprocals of each other,  and . This means that multiplying by one has the same effect as dividing by the other. This also means that knowing either constant ratio allows us to calculate the price given the number of pies. To find the cost of 20 pies, multiply by the ratio of dollars to pies or divide by the ratio of pies to dollars.

* .
* .

These patterns are true in general.

**FORMULA**

Let  be a particular item and  another such that there is a constant ratio of s to s

*  and 
* 
* <END>

**<example>EXAMPLE 25 - <title>Ratio of Faculty to Students at a College</title>**

Refer to the information given in Example 24.

* 1. If there are 32 class sections each semester in the Fine Art department, and the same ratio is used to determine the number of faculty members, how many faculty members would you expect to see in the Fine Art department?
  2. If the Health Sciences department has 6 full-time faculty members, how many part-time faculty members are in the department?

**<SOLUTION>**

* 1. Multiply the number of class sections by the ratio of faculty members per class section to find the number of faculty. Since there are 4 faculty members per class, the number of faculty members in 32 classes should be  faculty members.
  2. Multiply the number of full-time faculty by the ratio of part-time to full-time to find the number of part-time. Since ratio of full-time faculty to part-time faculty at the college is  or 1 full-time per 2 part time, the ratio of part-time to full-time is  part-time to 1 full-time; so the number of part-time faculty in a department with 6 full-time faculty should be  part-time faculty. <END>

**[Your Turn] 27**

Refer to the information in Your Turn 26.

* 1. Determine the number of pencils that would be allocated to a classroom F with 28 desks by multiplying the number of desks by the ratio of pencils to desks.
  2. Determine the number of pencils that would be allocated to a classroom F with 28 desks by dividing the number of desks by the ratio of desks to pencils.
  3. Assuming pencils continued to be distributed in the same manner, if there were another classroom, G, that received 51 pencils, how many desks would we expect to find in the room?

**Answer**

* + - * 1. 42 pencils would be allocated.
        2. 42 pencils would be allocated.
        3. 34 desks

<END>

The apportionment application that will be important to the founders of Imaginaria occurs in representative democracies in which elected persons represent a group. The United Kingdom, France, and India each have a parliament, and the United States has a Congress, just as Imaginaria will have a legislature! The citizens of a country must decide what portion of the representatives each group, such as a state or province or even a political party, will have. A larger portion of representatives means greater influence over policy.

**<example>Example 26 - <title>Ratio of US Representatives to State Population</title>**

Figure 11.87 contains a list of the five U.S. states with the greatest number of representatives in the U.S. House of Representatives along with the population of that state in 2021. Use the information in the table to answer the questions.

**Figure 11.87 The First through Fifth Ranked States by Number of Representatives**

|  |  |  |
| --- | --- | --- |
| State | Representative Seats | State Population |
| (CA) California | 53 | 39,613,000 |
| (TX) Texas | 36 | 29,730,300 |
| (NY) New York | 27 | 19,300,000 |
| (FL) Florida | 27 | 21,944,600 |
| (PA) Pennsylvania | 18 | 12,804,100 |

* 1. What is the ratio of State Population to Representative Seats for each state to the nearest hundred thousand?
  2. What is the ratio of Representative Seats to State Population for each state rounded to seven decimal places?
  3. What is the ratio of Representative Seats to State Population for each state rounded to six decimal places?
  4. Does there appear to be a constant ratio? Justify your answer.

**<SOLUTION>**

* 1. CA 700,000; TX 800,000; NY 700,000; FL 800,000; PA 700,000
  2. CA 0.0000013; TX 0.0000012; NY 0.0000014; FL 0.0000012; PA 0.0000014
  3. CA 0.000001; TX 0.000001; NY 0.000001; FL 0.000001; PA 0.000001
  4. The ratio of State Population to Representative Seats seems to be either 700,000 or 800,000. There does appear to be a constant ratio of about 0.000001 of Representative Seats to State Population if we round off to the sixth decimal place. <END>

**[Your Turn] 28**

Figure 11.88 contains a list of the five US states ranked sixth through tenth in the number of representatives in the U.S. House of Representatives, along with the population of that state in 2021. Use the information in the table to answer the questions.

**Figure 11.88 The Sixth through Tenth Ranked U.S. States by Number of Representatives**

|  |  |  |
| --- | --- | --- |
| State | Representative Seats | State Population |
| (IL) Illinois | 18 | 12,804,100 |
| (OH) Ohio | 16 | 11,714,600 |
| (MI) Michigan | 14 | 9,992,430 |
| (GA) Georgia | 14 | 10,830,000 |
| (NC) North Carolina | 13 | 10,701,000 |

* 1. What is the ratio of State Population to Representative Seats for each state to the nearest hundred thousand?
  2. What is the ratio of Representative Seats to State Population for each state rounded to seven decimal places?
  3. What is the ratio of Representative Seats to State Population for each state rounded to six decimal places?
  4. Does there appear to be a constant ratio? Are the results similar to the top five states? <END>

**<SOLUTION>**

* 1. IL 700,000; OH 700,000; MI 700,000; GA 700,000; NC 800,000
  2. IL 0.0000014; OH 0.0000014; MI 0.0000014; GA 0.0000013; NC 0.0000012
  3. IL 0.000001; OH 0.000001; MI 0.000001; GA 0.000001; NC 0.000001
  4. The ratio of State Population to Representative Seats seems to be either 700,000 or 800,000 to 1. There does appear to be a constant ratio of about 0.000001 to 1 of Representative Seats to State Population when rounding to six decimal places. This is the same as the top five states. <END>

Apportionment problems often involve rounding off decimals. There is a short lesson on rounding in this video.

**VIDEO: Math Antics – Rounding <**[**https://youtu.be/fd-E18EqSVk**](https://youtu.be/fd-E18EqSVk)**>**

You might be wondering why the ratio doesn’t appear to be quite the same depending on the rounding of the values. We will see that the key to this variation lies in the fractions. Just like the five children sharing 16 ring pops, there are going to be leftovers and there are many methods for deciding what to do with those leftovers.

**[H2] The Standard Divisor**

There are two houses of congress in the United States: the Senate and the House of Representatives. Each state has two senators, but the number of representatives depends on the population of the state. The number of representative seats in the U.S. House of Representatives is currently set by law to be 435. In order to distribute the seats fairly to each state, the ratio of the population of the U.S. to the number of representative seats must be calculated. This ratio is called the **standard divisor**.

Although apportionment applies to many other scenarios, such as the pencil distribution during the SAT, the terminology of apportionment is based on the House of Representatives scenario. So, the terms **states**, **seats**, **house size**, **populations**, and standard divisor will all take on a more general meaning.

**[DEFINITION]**

The **standard divisor** is the ratio of the total population to the house size, which is the number of members of the total population represented by one seat.

The **states** are the recipients of the apportioned resource.

The **seats** are the units of the resource being apportioned.

The **house size** is the total number of seats to be apportioned.

The **population** is the measurement of the state’s size.

The **total population** is the sum of the state populations.

<END>

**FORMULA**

<END>

**<example> EXAMPLE 27 - <title>The Standard Divisor of the U.S. House of Representatives 2021</title>**

As of this writing, the Census.gov website U.S. Population clock showed a population of 332,693,997. There are 435 seats in the U.S. House of Representatives. Find the standard divisor rounded to the nearest tenth.

**SOLUTION**

Dividing  people by 435 seats, there are 758960.6 people per representative. <END>

**[Your Turn] 29**

By the end of the first U.S. Congress in 1791, there were 13 states, 65 representative seats, and approximately 3,929,214 citizens. Find the standard divisor rounded to the nearest tenth. <END>

**Answer**

**60449.4**

Whether the standard divisor is less than, equal to, or greater than 1 depends on the ratio of the population to the number of seats.

* **The standard divisor will be equal to 1** if the total population is equal to the number of seats. This would mean that each member of the population is allocated their own personal seat.
* **The standard divisor will be a number between 0 and 1** when the total population is less than the number of seats. This means that each member of the population is allocated more than one seat.
* **The standard divisor will be a number greater than 1** when the total population is greater than the number of seats. This means that a certain number of members of the population will share 1 seat.

If the population is five children and the house consists of five pieces of candy, the standard divisor is  meaning each child gets one candy. If the population is five children and ten pieces of candy, the standard divisor is  meaning that each child gets more than one candy. If the population is five children and four pieces of candy, the standard divisor is  meaning that each child gets less than one candy.

If the seats in the Imaginarian legislature are distributed to the states based on population, then the house size will be less than the population and we should expect the standard divisor to be a number greater than 1.

*Photo 11.8: Image of School Resource Office (police officer at a school).*

**

**<example>Example 28 - <title>School Resource Officers in Brevard County, Florida</title>**

The public schools in a certain county have been allotted 349 school resource officers to be distributed among 327 public schools attended by approximately 271,500 students.

* 1. Identify the states, seats, house size, state population, and total population in this apportionment scenario.
  2. Describe the ratio the standard divisor represents in this scenario and calculate the standard divisor to the nearest tenth.

**<SOLUTION>**

* 1. The states are the schools in that county. The seats are the school resource officers. The house size is the number of school resource officers, which is 349. The state population is the number of students in a particular school, which was not given. The total population consists of the sum of the school populations, which is 271,500.
  2. The standard divisor is the ratio of the total population to the house size, which is the number of students served by each resource officer. Divide  students per officer. <END>

### [Your Turn] 30

The Hernandez family and the Higgins family went trick-or-treating together for Halloween last year. They returned with 313 pieces of candy, which they will now apportion to the families. The Hernandez family has three children and the Higgins family has four children.

* 1. Identify the states, seats, house size, state population and total population in this apportionment scenario.
  2. Describe the ratio the standard divisor represents in this scenario and calculate the standard divisor to four decimal places. <END>

**Answer**

* 1. The states are the Hernandez family and the Higgins family. The seats are the pieces of candy. The house size is 313. The state populations are three in the Hernandez family and four in the Higgins family. The total population is 7.
  2. The standard divisor is the ratio of the number of children to the number of pieces of candy. children per piece of candy. <END>

**[H2] The Standard Quota**

Once the standard divisor for the Imaginarian legislature is calculated, the next task is to determine the number of seats that each state should receive, which is referred to as the state’s **standard quota**. Unless all the states have the same population, each state will receive a different number of seats because the quantities will be proportionate to the state populations. To determine those amounts, we will use an idea we learned earlier. Recall that, when the number of units of item  is proportionate to the number of units of item , we have:



In this case, we are trying to calculate the number of seats a state should be apportioned, the state’s standard quota.So  So  would refer to seats allocated to a particular state, while  would refer to the state population. This means that the ratio of  to  is the ratio of the total population to house size, which is the standard divisor. So in apportionment terms, we have the following formula.

**FORMULA**

seats<END>

**[DEFINITION]**

A state’s **standard quota** is the number of seats that ideally would be apportioned to the state, which is the ratio of the state population to the standard divisor. <END>

**<example>Example 29 - <title> The Standard Quota of the U.S. House of Representatives 2021</title>**

Example 27 outlined that the Census.gov website U.S. Population clock showed a population of 330,147,881, there are 435 seats in the U.S. House of Representatives, and the standard divisor was 758,960.6 people per representative. The state of California has a population of approximately 39,613,000. Use these values to determine the standard quota for California to two decimal places.

**<SOLUTION>**

representatives. <END>

### [Your Turn] 31

By the end of the first U.S. Congress in 1791, there were 13 states, 65 representative seats, and approximately 3,929,214 citizens. In that year, the state of Delaware had a population of approximately 59,000. people. Use this information and the standard divisor you found in Your Turn 29 to find the standard quota rounded to two decimal places. <END>

**Answer**

0.98

EXAMPLE 30—Apportionment of Laptops in a Science Department

The science department of a high school has received a grant for 34 laptops. They plan to apportion them among their six classrooms based on each classroom’s student capacity. Use the values in Figure 11.89 to find the standard quota for each classroom.

**Figure 11.89 Classroom Capacities**

|  |  |
| --- | --- |
| **Room** | **Students** |
| A | 30 |
| B | 25 |
| C | 28 |
| D | 32 |
| E | 24 |
| F | 27 |

**<SOLUTION>**

Step 1 - Identify the State Population, Total Population, and the House Size. The states are the classrooms, and the State Populations are listed in the table. The Total Population is the sum of the State Populations, which is 166. The House Sizehouse size is the number of seats, or laptops, to be allocated, which is 34.

Step 2 - Calculate the Standard Divisor by dividing the Total Population by the House Size.

students per laptop.



Step 3 - Calculate the standard quota by dividing the State Population by the Standard Divisors (Figure 11.90).

**Figure 11.90 Apportionment of Laptops**

|  |  |  |
| --- | --- | --- |
| **Room** | **Room Capacity** | **Room’s Standard Quota** |
| A | 30 | laptops |
| B | 25 | laptops |
| C | 28 | laptops |
| D | 32 | laptops |
| E | 24 | laptops |
| F | 27 | laptops |

Step 4—Find the sum of the standard quotas.. This is only slightly off from the number of laptops—34—which can be caused by rounding off in previous steps. This is a good indication that the calculations were correct. If you find that the value of the sum of the standard quotas is significantly different from the house size (number of seats), it is possible that the standard divisor was calculated using too few decimal places. Calculate the standard divisor and standard quotas again but round off to a greater number of decimal places. <END>

**[Your Turn] 32**

This year the Hernandez family and the Higgins family were joined by the Ho family for Halloween trick-or-treating. The Hernandez family has three children, the Higgins family has four children, and the Ho family has two children. This time, they collected 527 pieces of candy, which they are going to apportion based on the number of children. Find the standard quota for each family. Round all values to four decimal places. If traditional rounding methods are applied to determine the actual number whole number values of pieces of candy received by each family, do the values sum to 527? <END>

**Answer**

**Figure 11.1 Family’s Standard Quotas**

|  |  |
| --- | --- |
| **Family** | **Family’s Standard Quota** |
| Hernandez | 175.4390 candies |
| Higgins | 233.9181 candies |
| Ho | 116.9591 candies |
| Total | 526.3162 |

The sum is very close to 527. <END>

CHECK YOUR UNDERSTANDING

In the following questions, assume there is a constant ratio between units of A and units of B. Two students are having a discussion. Determine who is correct: Student 1, Student 2, both, or neither.

1. Student 1 says that the number of units of A is the product of the number of units of B times the ratio of A to B.   
   Student 2 says that the number of units of A is the quotient of the number of units of B divided by the ratio of A to B.

Answer: Student 1 is correct.

1. Student 1 says that the number of units of A is the quotient of the number of units of B divided by the ratio of B to A.   
   Student 2 says that the number of units of B is the quotient of the number of units of A divided by the ratio of A to B.

Answer: Both are correct.

1. Suppose there are 110 of item A and 55 of item B.   
   Student 1 says the ratio of A to B is . Student 2 says the ratio of B to A is .

Answer: Both are correct.

1. Suppose the constant ratio of A to B is 0.75. Student 1 says the ratio of A to B is . Student 2 says the ratio of B to A is .



Answer: Neither are correct.

1. Student 1 says that . Student 2 says that .



Answer: Student 2 is correct.

1. Student 1 says that . Student 2 says that .



Answer: Student 2.

1. Student 1 says that.   
   Student 2 says that



Answer: Both are correct.

**EXERCISES**

In the following questions, identify the states, the seats, and the state population (the basis for the apportionment) in the given scenarios.

* + 1. A parent has 25 pieces of candy to split among their fourchildren. They will earn the candy based on how many minutes of chores they children did this week.

Answer: States are the children, the candies are the seats, and the minutes of chores each child completed are the state population.

* + 1. The board of trustees of a college has recently approved the installation of 70 new emergency blue lights in five parking lots. The number of lights in each lot will be proportionate to the size of the parking lot, which is to be measured in acres.

Answer: The states are the parking lots, the emergency lights are the seats, and the state population is the size of the parking lot in acres.

1. The reading coach at an elementary school has 52 prizes to distribute to students as a reward for time spent reading.

Answer: States are the students, the prizes are the seats, and the time spent reading is the state population.

1. “Top officials from Operation Warp Speed, the [U.S.] government’s program to fast-track the development and delivery of COVID-19 vaccines, announced they’ve allocated 6.4 million doses of COVID-19 vaccines to states based on their total populations.” (*The Coronavirus Crisis*, by Pien Huang, Shots Health News From NPR, npr.org, November 24, 2020)

Answer: States are the U.S. States, the COVID-19 vaccines are the seats, and the state populations are the state populations.

1. Refer to question 4, except suppose that the COVID-19 vaccine allocations were based on the most vulnerable population, residents aged 65 and over.   
   Answer: States are the U.S. States, the COVID-19 vaccines are the seats, and the state populations are the portion of residents aged 65 and over.

In and the following questions, use the given information to find the standard divisor to the nearest hundredth. Include the units.

1. The total population is 2,235 automobiles, and the number of seats is 14 warehouses.

Answer: The standard divisor is 159.64 automobiles per warehouse.

1. The total population is 135 hospitals, and the number of seats is 200 respirators.

Answer: The standard divisor is 0.68 hospitals per respirator.

In the following questions, use the given information to find the standard quota. Include the units.

1. The state population is eight residents in a unit, and the standard divisor is 1.75 residents per parking space.

Answer: The standard quota is 4.57 parking spaces.

1. The state population is 52 ICU patients each week, and the number of seats is 6.5 patients per respirator.

Answer: The standard quota is eight respirators.

1. The total population is 145 basketball players, the number of seats is 62 trophies, and the state population is 14 basketball players on Team Tigers.

Answer: The standard quota is 5.99 trophies.

1. The total population is 12 giraffes, the number of seats is nine water troughs, and the state population is three giraffes in Enclosure C.

Answer: The standard quota is 2.25 water troughs.

In items the following questions, use Figure 11.92 which shows student head count, class section, and total faculty in each of four college departments.

**Figure 11.92 Number of Faculty, Students, and Class Sections at a College**

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| **Department** | **(M) Math** | **(E) English** | **(H) History** | **(S) Science** | **(C) College Overall** |
| **(S) Student Head Count** | 4800 | 2376 | 1536 | 2880 | 87118 |
| **(C) Class Sections** | 120 | 108 | 48 | 96 | 3712 |
| **(F) Faculty Members** | 30 | 27 | 12 | 24 | 928 |

* 1. Determine the F to S ratios for each department rounded to four decimal places as needed. What are the units?

Answer: M 0.0063, E 0.0114, H 0.0078, S 0.0083. The units are faculty members per student.

* 1. Determine the C to F ratios for each department rounded to four decimal places as needed. What are the units?

Answer: M 4, E 4, H 4, S 4. The units are class sections per faculty member.

* 1. What is the F to S ratio for the college overall (include? Include units. How does it compare to the F to S ratios for individual departments?

Answer: 0.0107 faculty members per student. It is greater than M, H, and S but less than E.

* 1. What is the overall C to F ratio (include? Include units)?. How does it compare to the C to F ratios for individual departments?

Answer: Four classes per faculty member. It is the same.

* 1. Does there appear to be a constant F to S ratio? If so, what is the ratio? If not, what implications does this have about the different departments?

Answer: No, there does not. They are all different. Some departments have more faculty members per student and others have fewer.

* 1. Does there appear to be a constant C to F ratio? If so, what is the ratio? If not, what implications does this have about the different departments?

Answer: Yes. It is four classes per faculty member.

* 1. If the departments are the states, the students are the population, and the faculty members are the seats, use the College Overall column to determine the standard divisor for the apportionment of the faculty rounded to two decimal places as needed. Include the units.

Answer: 93.88 students per faculty member

* 1. If the departments are the states, the classes are the population, and the faculty members are the seats, use the Overall College column to determine the standard divisor rounded to two decimal places as needed. Include the units.

Answer: Four classes per faculty member.

* 1. Use the standard divisor from question 18 to find the standard quota for each department rounded to two decimal places as needed. What are the units?

Answer: M 51.13, E 25.31, H 16.36, S 30.68. The units are faculty members.

* 1. Use the standard divisor from 19question 12 to find the standard quota for each department rounded to two decimal places as needed.

Answer: M 30, E 27, H 12, S 24. The units are faculty members.

In Wakanda, the domain of the Black Panther, King T’Challa has six fortress cities. In Wakandan, the word “birnin” means “fortress city.” King T’Challa has found 111 Vibranium artifacts that must be distributed among the fortress cities of Wakanda. He has decided to apportion the artifacts based on the number of residents of each birnin. Use Figure 11.93 to answer the following questions.

**Figure 11.93 Populations by Major Wakandan Cities**

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
| Fortress  Cities | Birnin  Djata (D) | Birnin T’Chaka (T) | Birnin Zana (Z) | Birnin S’Yan (S) | Birnin Bashenga (B) | Birnin Azzaria (A) | Total Population |
| Residents | 26,000 | 57,000 | 27,000 | 18,000 | 64,000 | 45,000 | 237,000 |

* 1. Identify the states, the seats, and the state population (the basis for the apportionment) in this scenario.

Answer: The seats are the Vibranium artifacts, the states are the fortress cities, and the state populations are the city populations.

* 1. Find the standard divisor for the apportionment of the Vibranium artifacts. Round to the nearest tenth as needed. Include the units.

Answer: The standard divisor is 2135.1 people per artifact.

* 1. Find each Birnin’sbirnin’s standard quota for the apportionment of the Vibranium artifacts. Round to the nearest hundredth as needed. What are the units?

Answer: D 12.18, T 26.70, Z 12.65, S 8.43, B 29.98, A 21.08. The units are artifacts.

* 1. Find the sum of the standard quotas. Is it reasonably close to the number of artifacts available for distribution?

Answer: The sum is 111.02. Yes, the total is extremely close to the number of artifacts.

Suppose that 6.4 million doses of COVID-19 vaccine are to be distributed among U.S. States. The vaccines will either be distributed based on the total state population or based on the number of people over 65 years old. Use 93Figure 11.94 to answer items 26–31.

**Figure 11.94 The First through Fifth Ranked States by Population and Total U.S. Population**

|  |  |  |  |
| --- | --- | --- | --- |
| **State** | **State Population** | **State Population Age 65+** | **Percentage of State Population 65+** |
| (CA) California | 39,613,000 | 5,669,000 | 14.3% |
| (TX) Texas | 29,730,300 | 3,602,000 | 12.6% |
| (NY) New York | 19,300,000 | 3,214,000 | 16.4% |
| (FL) Florida | 21,944,600 | 4,358,000 | 20.5% |
| (PA) Pennsylvania | 12,804,100 | 2,336,000 | 18.2% |
| (US) United States | 330,151,000 | 52,345,000 | 15.8% |

* 1. Find the standard divisor for the apportionment of the vaccine doses by population using the estimate for the total U.S. population. Round to the nearest tenth as needed. Include the units.

Answer: The standard divisor is 51.6 people per dose of vaccine.

* 1. Find each state’s standard quota for the apportionment of the vaccine doses. Round to the nearest tenth as needed. What are the units?

Answer: Each state’s standard quota is as follows: CA is 767,693.8 doses of the vaccine; TX is 576,168.6 doses of the vaccine; NY is 374,031.0 doses; FL is 425,282.9, doses; and PA is 248,141.5 doses.

* 1. Find the standard divisor for the apportionment of the vaccine doses by population age 65 and older using the estimate for the total U.S. population of people aged 65 and older. Round to the nearest tenth as needed. Include the units.

Answer: The standard divisor is 8.2 people over 65 per dose of vaccine.

* 1. Find each state’s standard quota for the apportionment of the vaccine doses by total state population. Round to the nearest tenth as needed. What are the units?

Answer: Each state’s standard quota is as follows: CA is 691,341.5 doses of vaccine; TX is 439,268.3 doses; NY is 391,951.2 doses; FL is 531,463.4 doses; and PA is 284,878.0 doses.

* 1. Compare the standard quota for each state based on the entire state population found in question 20 to the standard quota for each state based on the portion of the population age 65 and older found in item 29.question 22. Which states would receive more doses of vaccine if the apportionment were based on the population of people age 65 and older?

Answer: New York, Florida, and Pennsylvania would receive more doses.

* 1. Approximately 15.8 percent of the U.S. residents are age 65 and older.
     1. Which of the five states listed have a percentage of residents age 65 and older greater than 15.8%? percent?
     2. Which of the five states listed have a percentage of residents age 65 and older less than 15.8%? percent?
     3. Explain the correlation.

Answer:

a. New York, Florida, and Pennsylvania.

b. California and Texas.

c. The states with a higher percentage of residents aged 65 and older receive more vaccines if the apportionment is based on the portion of the population aged 65 and older.

Children from five families—the Chorro family, the Eswaran family, the Javernick family, the Lahde family, and the Stolly family—joined a town-wide Easter egg hunt.. When they returned with their baskets, they had 827 eggs! They decided to share their eggs amongst the families based on the number of children in each family. Use the information in 94Figure 11.95 to answer the questions.

**Figure 11.95 Families in the Easter Egg Hunt**

|  |  |
| --- | --- |
| **Family** | **Number of Children** |
| (C) Chorro | 3 |
| (E) Eswaran | 2 |
| (J) Javernick | 4 |
| (L) Lahde | 1 |
| (S) Stolly | 5 |

* 1. Identify the states, the seats, and the state population (the basis for the apportionment) in this scenario.

Answer: The seats are the Easter eggs, the states are the families, and the state populations are the children in each family.

* 1. Find the standard divisor for the apportionment of the Easter eggs. Round to five decimal places as needed. Include the units.

Answer: The standard divisor is 0.01814 children per Easter egg.

* 1. Find each family’s standard quota for the apportionment of the Easter eggs. Round to the nearest hundredth as needed. What are the units?

Answer: Each family’s standard quota is as follows: C has 165.38 eggs, E has 110.25 eggs, J has 220.51 eggs, L has 55.13, eggs, and S has 275.63 eggs.

* 1. Find the sum of the standard quotas from item 34. Is the sum reasonably close to the number of Easter eggs available for distribution?

Answer: The sum is 826.90. Yes, the total is 0.10 less than the number of Easter eggs.

**[MODULE 1 SUMMARY]**

**Key Terms**

**apportion**

**apportionment problem**

**part-to-part ratio**

**proportional**

**representative democracies**

**states**

**seats**

**house size**

**state population**

**total population**

**standard divisor**

**standard quota**

**Key Concepts**

* Apportionment involves distributing resources proportionally to subsets of a population.
* Calculations are based on state population, total population, and the house size, or number of seats to be apportioned.
* Calculations involve ratios called the standard divisor and the standard quota.

**Formulas**

Let  be a particular item and  another such that there is a constant ratio of  to .

*  and 
* 
* 





*  and 
* 
* 





Videos

**What is a ratio? <**[**https://youtu.be/B4\_T6-rc35Y**](https://youtu.be/B4_T6-rc35Y)**>**

**What are the Different Types of Ratios? <**[**https://youtu.be/xwuFHj5O-kA**](https://youtu.be/xwuFHj5O-kA)**>**

**Math Antics – Rounding <**[**https://youtu.be/fd-E18EqSVk**](https://youtu.be/fd-E18EqSVk)**>**

# [H1] Module 4 - Apportionment Methods

*Photo 11.9: A-10C Thunderbolt Aircraft or children sitting in a classroom with laptops*

****

**After completing this module, you should be able to:**

* + - 1. Describe and interpret the Apportionment Problem LO 11.4.1apportionment problem.
      2. Apply Hamilton’s Method LO 11.4.2
      3. Describe and interpret the quota rule LO 11.4.3.
      4. Apply Jefferson’s Method LO 11.4.4
      5. Apply Adams's Method LO 11.4.5
      6. Apply Webster’s Method LO 11.4.6
      7. Compare and contrast apportionment methods LO 11.4.7.
      8. Identify and contrast flaws in various apportionment methods LO 11.4.8.

## [H2] A Closer Look at the Apportionment Problem

In Module 3, we calculated the standard divisor and the standard quotas in various apportionment scenarios. The results of those calculations routinely led to fractions and decimals of units. However, the seats in the House of Representatives, laptops in a classroom, or a variety of other resources, are indivisible, meaning they cannot be divided up into fractional parts. This leaves a decision to be made. For example, if the standard quota for the number of laptops to be distributed to a classroom is 12.44 units, how do we deal with the fractional part of 0.44? It is unclear if the classroom should receive 12 units, 13 units, or some other value. Let’s try traditional rounding to the nearest whole number value. <END>

### <example>Example 30 - <title>Installing Emergency Lights</title>

The board of trustees of a college has recently approved the installation of 70 new emergency blue lights in three parking lots. The number of lights in each lot will be proportionate to the size of the parking lot, which is to be measured in acres. The total number of acres is 34; so the standard divisor is . The standard quota for each lot is listed in Figure 11.96. Use this information to answer each question.

**Figure 11.96 Apportionment of Emergency Blue Lights**

|  |  |  |
| --- | --- | --- |
| **Lot** | **Acres** | **Lot’s Standard Quota** |
| A | 15 | emergency blue lights |
| B | 9 | emergency blue lights |
| C | 10 | emergency blue lights |

* 1. Use traditional rounding to determine the number of lights assigned to each lot.
  2. Find the sum of the values from part a.
  3. Does the sum found in part b equal the number of lights available?

**<SOLUTION>**

* 1. If traditional rounding is used, there will be 31, 19, and 21 lights distributed to each lot respectively.
  2. The total of these values is 71.
  3. No, the total from part b is one more than the number of lights available. In other words, one of the parking lots must get 1 fewer light than apportioned. <END>

### [Your Turn] 33

Recall from Example 32, the science department of a high school has received a grant for 34 laptops. They plan to apportion them among their six classrooms based on each classroom’s student capacity. Use the standard quotas in Figure 11.97 to answer each question.

**Figure 11.97 Apportionment of Laptops**

|  |  |  |
| --- | --- | --- |
| **Room** | **Room Capacity** | **Room’s Standard Quota** |
| A | 30 | laptops |
| B | 25 | laptops |
| C | 28 | laptops |
| D | 32 | laptops |
| E | 24 | laptops |
| F | 27 | laptops |

* 1. Use traditional rounding to determine the number of laptops assigned to each classroom.
  2. Find the sum of the values from part a.
  3. Does the sum found in part b equal the number of laptops available? <END>

**Answer**

* 1. 6, 5, 6, 7, 5, 6
  2. 35
  3. No. <END>

Example 31 demonstrates that we cannot successfully apportion indivisible resources by rounding off each standard quota using traditional rounding. This leaves us with a problem. What is a fair way to distribute the fractional parts of the standard quotas? We will refer to this as the **apportionment problem**. Several methods for making this decision will be discussed.

**[DEFINITION]**

The **apportionment problem** is the search for a fair approach to apportionment of indivisible resources when the standard quotas result in fractional parts.

An **apportionment method** is a method for solving the apportionment problem that does allocate a greater or lesser number of seats than are available. <END>

### <example>EXAMPLE 31 - <title>A-10C Thunderbolt II Aircraft</title>

In 2015, the U.S. Air Force had a fleet of approximately 281 A-10C Thunderbolt II aircraft. Suppose that the Air Force administration wanted to distribute 27 aircrafts across six bases based on the number of qualified pilots stationed at those bases. Use the information in Figure 11.98 to answer each question.

**Figure 11.978 Air Force Pilots**

|  |  |
| --- | --- |
| **Base** | **Pilots** |
| (A) Alpha | 13 |
| (B) Bravo | 12 |
| (C) Charlie | 5 |
| (D) Delta | 16 |
| (E) Echo | 7 |
| (F) Foxtrot | 9 |

1. Identify the states, the seats, and the state population (the basis for the apportionment) in this scenario.
2. Find the standard divisor for the apportionment of the aircraft. Round to four decimal places as needed. Include the units.
3. Find each air force base’s standard quota for the apportionment of the aircraft. Round to the nearest hundredth as needed. What are the units?
4. How does this example demonstrate the apportionment problem? Will traditional rounding solve the problem?

**<SOLUTION>**

1. The states are the bases, the seats are the aircraft, and the state populations are the pilots at a given base.
2.  pilots per aircraft.
3. A , B , C , D , E , F. . The units are aircraft.
4. This example demonstrates the apportionment problem because it is not possible to send a fractional number of aircraft to an air force base. On the other hand, if we use traditional rounding methods to get whole numbers, the results are  aircraft will be apportioned, which is one less than the number of aircraft that were supposed to be apportioned.<END>

### [Your Turn] 34

The reading coach at an elementary school has 13 gift cards to distribute to their three students as a reward for time spent reading. When they calculated the standard quota for each student based on the number of minutes they student had read, the results were: 4.49 gift cards, 4.03 gift cards, and 4.48 gift cards. How does this demonstrate the apportionment problem? <END>

**Answer**

It is not possible to give a fractional part of a gift card. Also, traditional rounding to the nearest integer results in four gift cards for each student, which leaves one extra gift card. <END>

## [H2] Hamilton's Method of Apportionment

One of the problems encountered when standard quotas are transformed into whole numbers using traditional rounding is that it is possible for the sum of the values to be greater than the number of seats available. A reasonable way to avoid this is to always round down, even when the first decimal place is five or greater. For example, a standard quota of 12.33 and a standard quota of 12.99 would both round down to 12. This is called the **lower quota**.

**[DEFINITION]**

The **lower quota** is the standard quota rounded down to the greatest whole number less than or equal to itself. <END>

### <example> Example 32 - <title>Lower Quota for Apportionment of Aircraft</title>

In Example 31, Air Force administration wanted to distribute 27 aircrafts across six bases based on the number of qualified pilots stationed at those bases. The standard quotas for each base are listed in Figure 11.99 Use this information to answer the questions.

**Figure 11.99 Standard Quota for A-10C Thunderbolt II Aircraft**

|  |  |
| --- | --- |
| **Base** | **Standard Quota** |
| (A) Alpha | aircraft |
| (B) Bravo | aircraft |
| (C) Charlie | aircraft |
| (D) Delta | aircraft |
| (E) Echo | aircraft |
| (F) Foxtrot | aircraft |

1. Give the lower quota for each Air Force base.
2. Find the sum of the lower quotas. By how much does this sum fall short of the actual number of aircraft?

**<SOLUTION>**

1. Round down. The lower quota for each air force base is 5, 5, 2, 6, 3, 3, respectively.
2. The sum is 24. This is 3 fewer than the actual number of aircraft. <END>

### [Your Turn] 35

In Example 32, we used the standard quotas for the apportionment of 70 new emergency blue lights in three parking lots based on acreage. The standard quota for each lot is listed in Figure 11.100. Use this information to answer each question.

**Figure 11.100 Apportionment of Emergency Blue Lights**

|  |  |  |
| --- | --- | --- |
| **Lot** | **Acres** | **Lot’s Standard Quota** |
| A | 15 | emergency blue lights |
| B | 9 | emergency blue lights |
| C | 10 | emergency blue lights |

1. Give the lower quota for each parking lot.
2. Find the sum of the lower quotas.
3. By how much does this sum fall short of the actual number of emergency lights? <END>

**Answer**

* 1. 30, 18, 20
  2. 68
  3. 2

If the standard quotas are all rounded down, their sum will always be less than or equal to the house size. Then, it would only remain to find a fair way to distribute any remaining seats. Alexander Hamilton, who was a general in the American Revolution, author of the Federalist Papers, and the first U.S. Secretary of the Treasury, took this approach to apportionment.

**[H2] Steps for Hamilton’s Method of Apportionment**

Step 1 - Find the standard divisor.

Step 2 - Find each state’s standard quota.

Step 3 - Give each state the state’s lower quota (with each state receiving at least 1 seat).

Step 4 - Give each remaining seat one at a time to the states with the largest fractional parts of their standard quotas until no seats remain.

Step 5 - Check the solution by confirming that the sum of the modified quotas equals the house size.

**VIDEO Apportionment: Hamilton’s Method <https://www.youtube.com/watch?v=6qRJGw4oXo4 >**

### <example> Example 33 - <title>Hawaiian School Districts</title>

Suppose that the Hawaii State Department of Education has a budget for 616 schools and is doing a research study to determine the equitable number of schools to have in each of the five counties based on the residents under 19 years old, This data is provided in Figure 11.100. Using the Hamilton method, calculate how many schools would be funded in each state.

**Figure 11.101 Populations of Hawaiian Counties**

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
| **County** | **Hawaii** | **Honolulu** | **Kalawao** | **Kauai** | **Maui** | **Total** |
| **Residents under age 19** | 46,310 | 224,230 | 20 | 16,560 | 38,450 | 325,570 |

**<SOLUTION>**

Step 1 - Calculate the standard divisor. Divide the total population, 325570, by the house size, 616 seats. The standard divisor is 528.52.

Step 2 - Find each state’s standard quota (Figure 11.102).

**Figure 11.102 Hawaiian States’ Standard Quotas**

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
| **County** | **Hawaii** | **Honolulu** | **Kalawao** | **Kauai** | **Maui** | **Total** |
| **Standard Quota** |  |  |  |  |  | 616 |

Step 3 - Find each state’s lower quota and their sum. (Figure 11.103)

**Figure 11.103 Hawaiian States’ Lower Quotas**

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
| **County** | **Hawaii** | **Honolulu** | **Kalawao** | **Kauai** | **Maui** | **Total** |
| **Lower Quota** | 87 | 424 | 1 | 31 | 72 | 615 |

Step 4 - Compare the sum of the states’ lower quotas, 615, to the house size, 616. One seat remains to be apportioned and must be given to the state with the largest fractional part: Maui with 0.75. So, the final Hamilton quotas are as follows: Hawaii 87, Honolulu 424, Kalawao 1, Kauai 31, and Maui 73.

Step 5 - Find the total to confirm the sum of the quotas equals the house size, 616. . The apportionment is complete. <END>

### [Your Turn] 36

In the country of Imaginaria, there will be four states: Fictionville, Pretendstead, Illusionham, and Mythbury. Suppose there will be 35 seats in the legislature of Imaginaria. Use Hamilton’s method of apportionment to determine the number of seats in each state based on the populations in Figure 11.1034.

**Figure 11.103 Populations of Imaginarian States**

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| **State** | **Fictionville** | **Pretendstead** | **Illusionham** | **Mythbury** | **Total** |
| **Population** | 71,000 | 117,000 | 211,000 | 1,194,000 | 1,593,000 |

<END>

**Answer**

The final Hamilton apportionment is Fictionville as follows:1, Pretendstead 3, Illusionham5, and Mythbury 26.

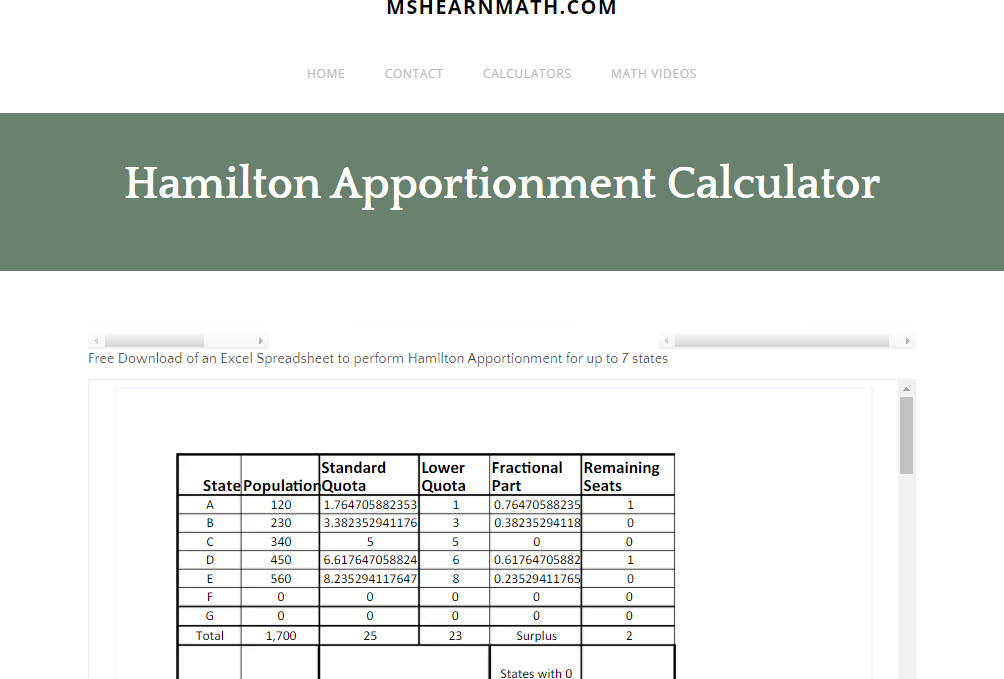
<END>

TECH CHECK

**Apportionment Calculators**

Check out websites such as <https://www.mshearnmath.com/hamilton-method-calculator.html> for a free Hamilton apportionment calculator.

[Screen 1.112: Mshearnmath.com screenshot of Hamilton Apportionment Calculator.]



This can be a useful tool to confirm your results! <END>

## [H2] The Quota Rule

A characteristic of an apportionment that is considered favorable is when the final quota values all either result from rounding down or rounding up from the standard quotas. The value that results from rounding down is called the lower quota, and the value that results from rounding up is called the **upper quota**.

**[DEFINITION]**

The **upper quota** is the lowest whole number greater than the standard quota. The upper quota is exactly one unit greater than the lower quota.

The **quota rule** says that the number of seats allocated to a given state should always be either the upper quota or the lower quota for that state. <END>

As we explore more methods of apportionment, we will consider whether they satisfy the quota rule. If a scenario exits in which a particular apportionment allocates a value greater than the upper quota or less than the lower quota, then that apportionment violates the quota rule and the apportionment method that was used violates the quota rule.

### <example> Example 34 - <title>Which Apportionment Method Satisfies the Quota Rule?</title>

Several apportionment methods have been used to allocate 125 seats to ten states and the results are shown in Figure 11.104. Determine which of these apportionments does not satisfy the quota rule and justify your answer.

**Figure 11.104 Do Apportionments Satisfy the Quota Rule?**

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
|  | **State A** | **State B** | **State C** | **State D** | **State E** | **State F** | **State G** |
| **Standard Quota** | 41.26 | 16.00 | 5.77 | 2.64 | 7.82 | 10.47 | 0.21 |
| **Lower Quota** | 41 | 16 | 5 | 2 | 7 | 10 | 0 |
| **Upper Quota** | 42 | 17 | 6 | 3 | 8 | 11 | 1 |
| **Method X** | 43 | 16 | 5 | 2 | 7 | 10 | 1 |
| **Method Y** | 41 | 16 | 6 | 2 | 8 | 10 | 1 |
| **Method Z** | 42 | 16 | 7 | 3 | 7 | 9 | 1 |

**<SOLUTION>**

Look for states such that the number of seats allocated differs from the lower or upper quota. Method X violates the quota rule because State A receives 43 seats instead of 41 or 42. Method Z violates the quota rule because State C receives 7 seats instead of 5 or 6 and State F receives 9 instead of 10 or 11. <END>

### [Your Turn] 37

Apportionment Method V has been used to allocate 125 seats to ten states as shown in Figure 11.105 Determine if the apportionment satisfies the quota rule and justify your answer.

**Figure 11.105 Does the Apportionments Satisfy the Quota Rule?**

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
|  | **State A** | **State B** | **State C** | **State D** | **State E** | **State F** | **State G** |
| **Standard Quota** | 41.26 | 10.70 | 16.00 | 13.11 | 17.00 | 5.77 | 2.64 |
| **Lower Quota** | 41 | 10 | 16 | 13 | 17 | 5 | 2 |
| **Upper Quota** | 42 | 11 | 17 | 14 | 18 | 6 | 3 |
| **Method V** | 42 | 11 | 17 | 13 | 18 | 4 | 2 |

<END>

**Answer**

Method V violates the quota rule because State F receives 4 seats instead of 5 or 6. <END>

It is possible for an apportionment method to satisfy the quota rule in some scenarios but violate it in others. However, because the Hamilton method always begins with the lower quota and either adds one to it or keeps it the same, the final Hamilton quota will always consist of values that are either lower quota values or upper quota values. When an apportionment method has this characteristic, it is said to satisfy the quota rule. So, we can say:

**The Hamilton method of apportionment satisfies the quota rule.**

Although the Hamilton method of apportionment satisfies the quota rule, it can result in some unexpected outcomes which has caused it to pass in and out of favor of the U.S. government over the years. There are several apportionment methods that have been popular alternatives, such as Jefferson’s method of apportionment that the founders of Imaginaria should consider.

## [H2] Jefferson’s Method of Apportionment

Another approach to dealing with the fractional parts of the standard quotas is to modify the standard divisor so that the total of the resulting modified lower quotas is the necessary number of seats. This is the approach used by Jefferson.

**[DEFINITION]**

A **modified divisor** is an adjustment to the standard divisor. <END>

In Jefferson’s method, the change to the standard divisor is made so that the total of the modified lower quotas equals the house size. The change in the standard divisor to get the modified divisor is relatively small. There is not a formula for this. The modified divisor is found by “guess and check”.. It is important to remember that *increasing* the divisor *decreases* the quotas, but *decreasing* the divisor *increases* the quotas. So, if you need a larger quota, try reducing the divisor, and if you need a smaller quota, try increasing the divisor.

### <example>Example 35 - <title>Modifying a Standard Divisor</title>

Suppose the population of a state is 50 and the standard divisor is 12.5.

1. Find the state’s standard quota.
2. Increase the standard divisor by 2 units and use the modified divisor to determine the modified quota for the state.
3. Decrease the modified divisor from part b by 1.5 units and use the new modified divisor to determine the modified quota for the state.
4. Choose any value of divisor between the value of the modified divisor from b and the value of the modified divisor from c and use it to determine the modified quota for the state.
5. Which modified quota was the largest, the modified quota from part b, from part c, or from part d? Explain why.

**<SOLUTION>**

1. The state’s standard quota is 
2. The modified divisor is 14.5. The modified quota is 
3. The modified divisor is 13. The modified quota is 
4. One value between 13 and 14.5 is 13.5. With a modified divisor of 13.5, the modified quota is 
5. The modified quota from part c was the largest because the divisor was the smallest of the three. Dividing the same number by a smaller value gives a larger result. <END>

### [Your Turn] 38

Suppose the population of a state is 12 and the standard divisor is 0.225.

1. Find the state’s standard quota.
2. Decrease the standard divisor by 0.200 units and use the modified divisor to determine the modified quota for the state.
3. Increase the modified divisor from part b by 0.100 units and use the new modified divisor to determine the modified quota for the state.
4. Choose any value of divisor between the value of the modified divisor from b and the value of the modified divisor from c and use it to determine the modified quota for the state.
5. Which modified quota was the smallest, the modified quota from part b, from part c, or from part d? Explain why. <END>

**Answer**

1. 53.33
2. 480
3. 96
4. Answers may vary. With a modified divisor of 0.100, the modified quota is 120.
5. The modified quota from part c was the smallest, because the divisor was the largest of the three. Dividing the same number by a larger value gives a smaller result. <END>

When you use Jefferson’s method, you might have to adjust the divisor several times find modified lower quotas that sum to the house size. First, guess what the divisor should be based on the sum of the lower quotas and then increase or decrease it from there based on whether the sum needs to be smaller or larger respectively. If the result still does not produce lower quotas that sum to the house size, adjust again. Keep a record of the values that didn’t work to help you narrow your search.

### [H2] Steps for Jefferson’s Method of Apportionment

Step 1 - Find the standard divisor.

Step 2 - Find each state’s quota. This will be the standard quota the first time Step 2 is completed and the standard divisor is used, but Step 2 may be repeated as needed using a modified divisor and resulting in modified quotas.

Step 3 - Find the states’ lower quotas (with each state receiving at least one seat), and their sum.

Step 4 - If the sum from Step 3 equals the number of seats, the apportionment is complete. If the sum of the lower quotas is less than the number of seats, reduce the standard divisor. If the sum of the lower quotas is greater than the number of seats, increase the standard divisor. Return to Step 2 using the modified divisor.

### <example> Example 36 - <title>Hawaiian State Representative Districts</title>

As in Example 34, suppose that the Hawaii State Department of Education has a budget for 616 schools and is doing a research study to determine the equitable number of schools to have in each of five counties based on the residents under the age of 19. Use the data from Figure 11.100 and Jefferson’s method to apportion the schools to the counties.

**<SOLUTION>**

Steps 1, 2, and 3 are the same in the Hamilton and Jefferson methods of apportionment. In Example 34, these steps resulted in lower quotas as shown in Figure 11.107.

**Figure 11.107 Hawaiian States’ Lower Quotas**

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
| **County** | **Hawaii** | **Honolulu** | **Kalawao** | **Kauai** | **Maui** | **Total** |
| **Standard Quota** |  |  |  |  |  | 616 |
| **Lower Quota** | 87 | 424 | 1 | 31 | 72 | 615 |

Step 4 - Compare the sum of the states’ lower quotas, 615, to the house size, 616. Since 615 is less than 616, use a modified divisor that is less than the standard divisor of 528.52. Try 526.00. Return to Step 2.

Repeat Steps 2 and 3 - Find each state’s modified quota, lower quota, and the sum of the lower quotas based on the modified divisor of 526 (Figure 11.108).

**Figure 11.108 Hawaiian States’ Modified Quota and Lower Quota**

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
| **County** | **Hawaii** | **Honolulu** | **Kalawao** | **Kauai** | **Maui** | **Total** |
| **Modified Quota** |  |  |  |  |  | 616 |
| **Lower Quota** | 88 | 426 | 1 | 31 | 72 | 618 |

Repeat Step 4 - The new sum of the lower quotas is 2 units greater than 616. We have overshot the goal. So, increase the divisor to a value between 526.00 and 528.52. Try 527.00.

Repeat Steps 2 and 3 - Find each state’s modified quota, lower quota, and the sum of the lower quotas based on the modified divisor of 526.00 (Figure 11.109).

**Figure 11.109 Hawaiian States’ Modified Quota and Lower Quota**

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
| **County** | **Hawaii** | **Honolulu** | **Kalawao** | **Kauai** | **Maui** | **Total** |
| **Modified Quota** |  |  |  |  |  | 616 |
| **Lower Quota** | 87 | 425 | 1 | 31 | 72 | 616 |

Repeat Step 4 - The new sum of the lower quotas equals the house size. The apportionment is complete.

The apportionment is: Hawaii County 87, Honolulu County 425, Kalawao County 1, Kauai 31, and Maui 72 schools. <END>

When using Jefferson’s method, the modified divisors you use may be different from what another person chooses, but final apportionment values will be the same.

### [Your Turn] 39

Let’s return to the Imaginarian states of Fictionville, Pretendstead, Illusionham, and Mythbury as in Your Turn 3.4 Suppose that there are going to be 35 seats in the legislature. This time use Jefferson’s method of apportionment to determine the number of seats in each state based on the populations in Figure 11.103. How many seats would each state receive?<END>

**<SOLUTION>**

Each state would receive the following seats: Fictionville 1, Pretendstead 2, Illusionham 4, and Mythbury 28. <END>

Notice that, in this apportionment, Mythbury received more than the upper quota. Since this apportionment of representatives to Imaginarian states by Jefferson’s method does not satisfy the quota rule, we say that:

**Jefferson’s method violates the quota rule.**

We have discussed two apportionment methods: one that satisfies the quota rule and one that does not. Before you decide which method to use in Imaginaria, there are a couple more options to consider.

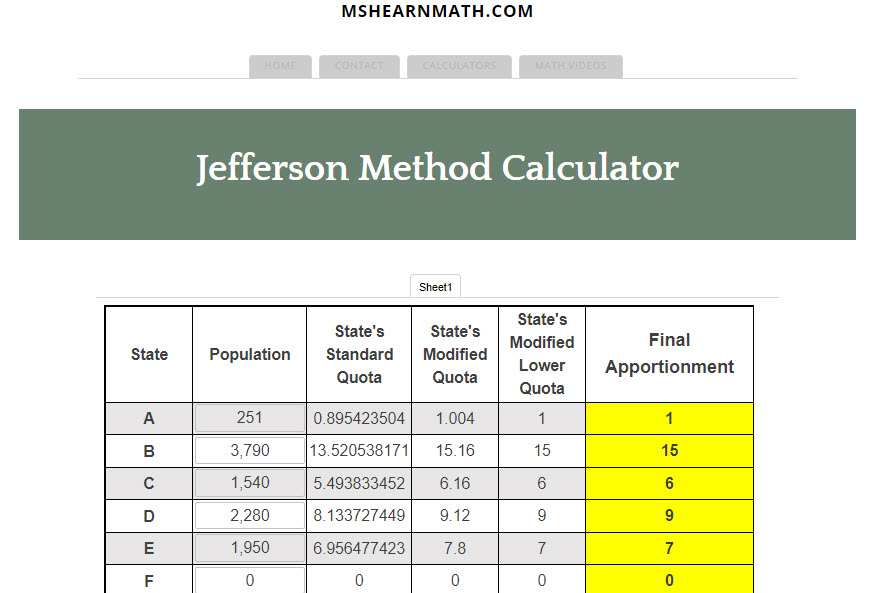
**Video: Jefferson Apportionment Method <https://www.youtube.com/watch?v=\_ZI2-yZl8Z8>**

**[TECH CHECK]**

**Apportionment Calculators**

It is possible to create Excel spreadsheets that complete the calculations necessary to complete a Jefferson Apportionment. In some cases, this work has already been done and posted online. Check out websites such <https://www.mshearnmath.com/jefferson-method-calculator.html> for a free Jefferson apportionment calculator.

[Screen 12.113: Mshearnmath.com screenshot of Jefferson Apportionment Calculator.]



This can be a useful tool to confirm your results! <END>

## [H2] Adams’s Method of Apportionment

Adams’s method of apportionment is another method of apportionment that is based on a modified divisor. However, instead of basing the changes on the sum of the lower quotas, as Jefferson did, Adams used the upper quotas.

### Steps for Adams’s Method of Apportionment

Step 1 - Find the standard divisor.

Step 2 - Find each state’s quota. This will be the standard quota the first time Step 2 is completed, and the standard divisor is used, but Step 2 may be repeated as needed using a modified divisor and resulting in modified quotas.

Step 3 - Find the states’ upper quotas and their sum.

Step 4 - If the sum from Step 3 equals the number of seats, the apportionment is complete. If the sum of the upper quotas is less than the number of seats, reduce the standard divisor. If the sum of the upper quotas is greater than the number of seats, increase the standard divisor. Return to Step 2 using the modified divisor.

### <example>Example 37 - <title>Hawaiian School Districts</title>

As in Example 36 and Example 37, suppose that the Hawaii State Department of Education has a budget for 616 schools and is doing a research study to determine the equitable number of schools to have in each of the five counties based on the residents under the age of19. This time, use the data from Figure 11.100 and the Adams method to apportion the schools to the counties.

**<SOLUTION>**

Steps 1 and 2 are the same in the Jefferson and Adams methods. As in Example 36, the standard divisor is 528.52.

Step 3 - Find each state’s upper quota and their sum (Figure 11.10910).

**Figure 11.110 Upper Quotas of Hawaiian Counties**

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
| **County** | **Hawaii** | **Honolulu** | **Kalawao** | **Kauai** | **Maui** | **Total** |
| **Standard Quota** |  |  |  |  |  | 616 |
| **Upper Quota** | 88 | 425 | 1 | 32 | 73 | 619 |

Step 4 - Compare the sum of the states’ upper quotas, 619, to the house size, 616. Since 619 is greater than 616, we need to reduce the size of the quotas. Use a modified divisor that is greater than the standard divisor of 528.52. Try 534.00. Return to Step 2.

Repeat Steps 2 and 3 - Find each state’s modified quota, upper quota, and the sum of the upper quotas based on the modified divisor of 534 (Figure 110.1).

**Figure 110.1 Modified Quotas of Hawaiian Counties**

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
| **County** | **Hawaii** | **Honolulu** | **Kalawao** | **Kauai** | **Maui** | **Total** |
| **Modified Quota** |  |  |  |  |  | 616 |
| **Upper Quota** | 88 | 420 | 1 | 32 | 72 | 613 |

Repeat Step 4 - The new sum of the upper quotas is 3 units less than 616. Larger quotas are needed. So, decrease the divisor to a value between 534.00 and 528.52. Try 532.00.

Repeat Steps 2 and 3 - Find each state’s modified quota, upper quota, and the sum of the upper quotas based on the modified divisor of 532.00 (Figure 111.2).

**Figure 111.2 Modified Quotas of Hawaiian Counties**

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
| **County** | **Hawaii** | **Honolulu** | **Kalawao** | **Kauai** | **Maui** | **Total** |
| **Modified Quota** |  |  |  |  |  | 616 |
| **Upper Quota** | 88 | 422 | 1 | 32 | 73 | 616 |

Repeat Step 4 - The new sum of the upper quotas equals the house size. The apportionment is complete.

The apportionment is Hawaii County 88, Honolulu County 422, Kalawao County 1, Kauai 32, and Maui 73 schools. <END>

When using Adams’s method, just as with Jefferson’s method, the modified divisors you use may be different from what another person chooses, but final apportionment values will be the same.

### [Your Turn] 40

Let’s return to Imaginaria again as we did in Your Turns 36 and 37. There are four states: Fictionville, Pretendstead, Illusionham, and Mythbury. Assume there will be 35 seats in the legislature of Imaginaria. Use Adams's method of apportionment to determine the number of seats in each state based on the populations in Figure 11.103. How many seats would each state receive?<END>

**Answer**

Fictionville 2, Pretendstead 3, Illusionham 5, and Mythbury 25. <END>

In this apportionment, Mythbury received less than the state’s lower quota. So, this apportionment is an example of a scenario in which the Adams method violates the quota rule.

**Adams’s method of apportionment violates the quota rule.**

So far, only Hamilton’s method satisfies the quota rule, but there is one more apportionment method you should consider for Imaginaria.

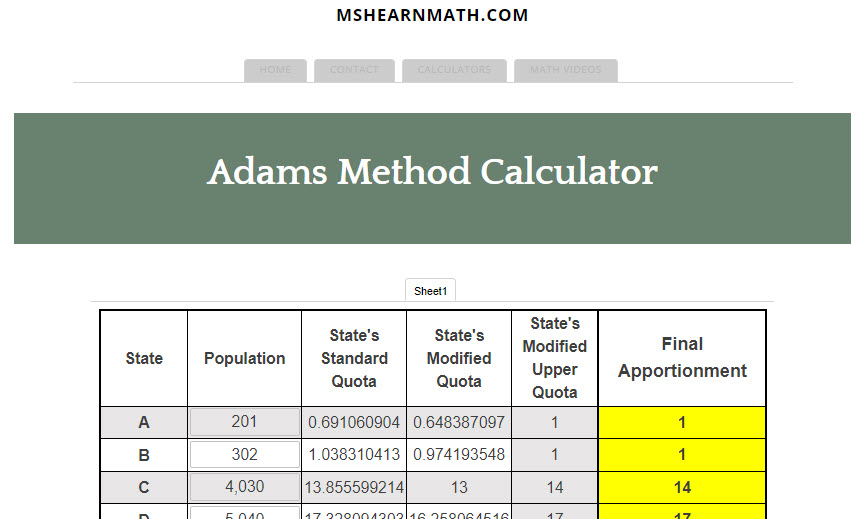
**Video: Adams Method Apportionment by Hand and Calculator <https://www.youtube.com/watch?v=IrPyTKD3wqI>**

**[TECH CHECK]**

**Apportionment Calculators**

Check out websites such as <https://www.mshearnmath.com/adams-method-calculator.html> for a free Adams Method apportionment calculator.

[Screen 12.114: Mshearnmath.com screenshot of Adams Apportionment Calculator.]



This can be a useful tool to confirm your results! <END>

## [H2] Webster’s Method of Apportionment

Webster’s method of apportionment is another method of apportionment that is based on a modified divisor. However, instead of basing the changes on the sum of the lower quotas, as Jefferson did or the sum of the upper quotas as Adams did, Webster used traditional rounding.

### Steps for Webster’s Method of Apportionment

Step 1 - Find the standard divisor.

Step 2 - Find each state’s quota. This will be the standard quota the first time Step 2 is completed, and the standard divisor is used, but Step 2 may be repeated as needed using a modified divisor and resulting in modified quotas.

Step 3 - Round each state’s quota to the nearest whole number and find the sum of these values.

Step 4 - If the sum of the rounded quotas equals the number of seats, the apportionment is complete. If the sum of the rounded quotas is less than the number of seats, reduce the divisor. If the sum of the rounded quotas is greater than the number of seats, increase the divisor. Return to Step 2 using the modified divisor.

When using Webster’s method, just as with Jefferson’s method, the modified divisors you use may be different from what another person chooses, but final apportionment values will be the same.

### <example>Example 38 - <title>Hawaiian School Districts</title>

As in Examples 36, 37, and 39 use the data from Figure 12.100 to apportion 616 schools to Hawaiian counties. This time, use Webster’s method.

**<SOLUTION>**

Steps 1 and 2 are the same in the Jefferson, Adams, and Websters methods. As in the previous examples, the standard divisor is 528.52.

Step 3 - Find each state’s rounded quota and their sum (Figure 112.3).

**Figure 112.3 Upper Quotas of Hawaiian Counties**

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
| **County** | **Hawaii** | **Honolulu** | **Kalawao** | **Kauai** | **Maui** | **Total** |
| **Standard Quota** |  |  |  |  |  | 616 |
| **Rounded Quota** | 88 | 424 | 1 | 31 | 73 | 617 |

Step 4 - Compare the sum of the states’ rounded quotas, 617, to the house size, 616. Since 617 is greater than 616, we need to reduce the size of the quotas. Use a modified divisor that is greater than the standard divisor of 528.52. Try 534.00. Return to Step 2.

Repeat Steps 2 and 3—Find each state’s modified quota, rounded quota, and the sum of the rounded quotas based on the modified divisor of 534 (Figure 113.4).

**Figure 113.4 Modified Quotas of Hawaiian Counties**

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
| **County** | **Hawaii** | **Honolulu** | **Kalawao** | **Kauai** | **Maui** | **Total** |
| **Modified Quota** |  |  |  |  |  | 616 |
| **Upper Quota** | 87 | 420 | 1 | 31 | 72 | 612 |

Repeat Step 4 - The new sum of the rounded quotas is 4 units less than 616. Larger quotas are needed. So, decrease the divisor to a value between 534.00 and 528.52. Try 530.00.

Repeat Steps 2 and 3 - Find each state’s modified quota, rounded quota, and the sum of the rounded quotas based on the modified divisor of 530.00 (Figure 114.5).

**Figure 114.5 Modified Quotas of Hawaiian Counties**

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
| **County** | **Hawaii** | **Honolulu** | **Kalawao** | **Kauai** | **Maui** | **Total** |
| **Modified Quota** |  |  |  |  |  | 616 |
| **Upper Quota** | 87 | 423 | 1 | 31 | 73 | 615 |

Repeat Step 4 - The new sum of the rounded quotas is 1 unit less than 616. Larger quotas are needed. So, decrease the divisor to a value between 528.52 and 530.00. Try 529.5.

Repeat Steps 2 and 3 - Find each state’s modified quota, rounded quota, and the sum of the rounded quotas based on the modified divisor of 529.5 (Figure 115.6).

**Figure 115.6 Modified Quotas of Hawaiian Counties**

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
| **County** | **Hawaii** | **Honolulu** | **Kalawao** | **Kauai** | **Maui** | **Total** |
| **Modified Quota** |  |  |  |  |  | 616 |
| **Upper Quota** | 87 | 423 | 1 | 31 | 73 | 615 |

Repeat Step 4 - The new sum is still only 1 unit less than 616. Larger quotas are needed, but not much larger. So, decrease the divisor to a value between 528.52 and 529.5. Try 529.30.

Repeat Steps 2 and 3 - Find each state’s modified quota, rounded quota, and the sum of the rounded quotas based on the modified divisor of 529.30 (Figure 116.7).

**Figure 116.7 Modified Quotas of Hawaiian Counties**

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
| **County** | **Hawaii** | **Honolulu** | **Kalawao** | **Kauai** | **Maui** | **Total** |
| **Modified Quota** |  |  |  |  |  | 616 |
| **Upper Quota** | 87 | 424 | 1 | 31 | 73 | 616 |

Repeat Step 4 - The new sum of the rounded quotas equals the house size. The apportionment is complete.

The apportionment is Hawaii County 87, Honolulu County 424, Kalawao County 1, Kauai 31, and Maui 73 schools. <END>

### [Your Turn] 41

If you use Webster’s method to apportion 35 legislative seats to the 4 states of Imaginaria, Fictionville, Pretendstead, Illusionham, and Mythbury, with the populations given in Figure 11.103 in Your Turn 34, what is the resulting apportionment? <END>

**Answer**

The apportionment is Fictionville 2, Pretendstead 2, Illusionham 5, and Mythbury 26. <END>

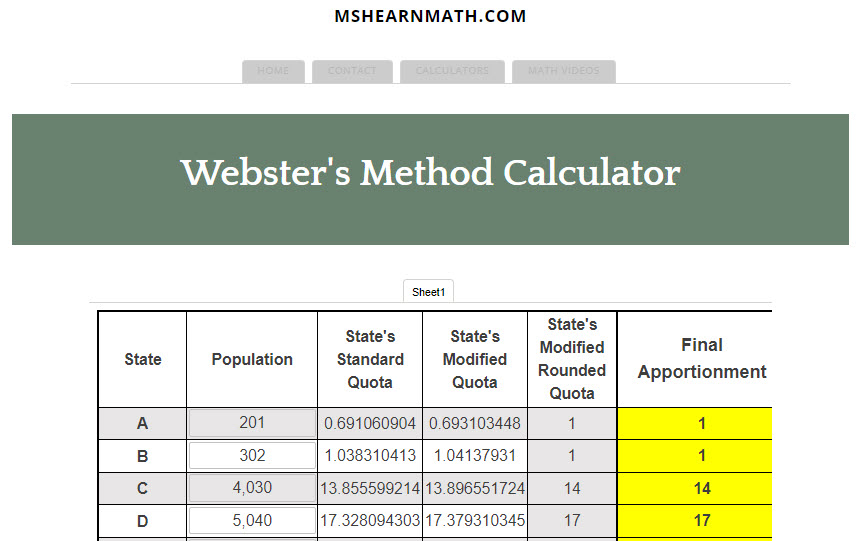
So far, we know that the Hamilton method satisfies the quota rule, while the Jefferson and Adams methods do not. The apportionments in Example 38 and Your Turn 38 are both scenarios in which the Webster method satisfies the quota rule. Does it always? We have a little more work to do to find out. However, one thing is clear. Not all apportionment methods have the same results. Before you make such an important decision for Imaginaria, it’s important to think about the differences in the apportionments that result from these four methods. How will the differences affect the citizens of Imaginaria?

**[TECH CHECK]**

**Apportionment Calculators**

Check out websites such as <https://www.mshearnmath.com/webster-method-calculator.html> for a free Webster’s Method apportionment calculator.

[Screen 12.115: Mshearnmath.com screenshot of Webster’s Apportionment Calculator.]



This can be a useful tool to confirm your results! <END>

## [H2] Comparing Apportionment Methods

Recall that the four apportionment methods discussed in this chapter differ in two main ways:

* Whether or not a modified divisor is used
* The type of rounding of the quotas that is used

How might these differences affect Imaginarians? In the next two examples, we will compare the results when different apportionment methods are applied to the same scenario.

### <example> Example 39 - <title>Hawaiian School Districts with Different Apportionment Methods<title>

Let’s use the results from Examples 34, 37, 38, and 39 to compare the four apportionment methods we have discussed. Figure 117.8 summarizes the results of the results of the Hamilton, Jefferson, Adams and Webster methods when applied to the apportionment of 616 schools to Hawaiian counties.

* 1. Do any of the apportionment methods result in the same apportionment? If so, which ones?
  2. Which apportionment method would the citizens of the largest county likely favor most and least? Justify your answer.
  3. As a group, which apportionment method would the citizens of the other four counties likely favor most and least? Justify your answer.

**Figure 117.8 Apportionment of Schools by Different Methods**

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| **County** | **Hawaii** | **Honolulu** | **Kalawao** | **Kauai** | **Maui** |
| **Under 19 years old** | 46,310 | 224,230 | 20 | 16,560 | 38,450 |
| **Hamilton** | 87 | 424 | 1 | 31 | 73 |
| **Jefferson** | 87 | 425 | 1 | 31 | 72 |
| **Adams** | 88 | 422 | 1 | 32 | 73 |
| **Webster** | 87 | 424 | 1 | 31 | 73 |

**<SOLUTION>**

* 1. Yes, the Hamilton and Webster methods result in the same apportionment.
  2. The largest county is Honolulu. The citizens would likely favor the Jefferson method of apportionment most since they received the most seats by that method. They would likely favor the Adams method of apportionment least because they received the least number of seats by that method.
  3. As a group, the other four counties received 192 seats by either the Hamilton or Webster method, 194 seats by the Adams method, and 191 seats by the Jefferson method. They would likely favor the Adams method the most and favor the Jefferson methods the least. <END>

### [Your Turn] 42

In Your Turn 34, 37, 38, and 39, you apportioned 35 legislative seats among the four states of Imaginaria using the Hamilton, Jefferson, Adams, and Webster methods of apportionment. To understand how the differences in the apportionments might affect Imaginarians, answer these questions.

* + 1. Which apportionment method would the citizens of the largest state likely favor most and least? Justify your answer.
    2. As a group, which apportionment method would the citizens of the other three states likely favor most and least? Justify your answer. <END>

**Answer**

1. The largest state is Mythbury. The citizens would likely favor the Jefferson method of apportionment most since they received the most seats by that method. They would likely favor the Adams and Webster methods of apportionment least because they received the least number of seats by those method.
2. As a group the other three states received nine seats by either the Hamilton method, seven seats by the Jefferson Method, and ten seats by either the Adams method or the Webster method. They would likely favor the Adams method and Webster method most and favor the Jefferson methods least. <END>

The Adams method favored the smaller states and the Jefferson method favored the larger states in the previous example, but is this the case in general?

Since the Jefferson method begins with the lower quotas, any adjustment to the quotas will be an increase. As you have seen, this is accomplished by using a modified divisor that is smaller than the standard divisor. The next example compares the impact of a decreasing divisor on the modified quotas of large states to the impact of the same size decrease on small states.

### <example> Example 40 - <title>Effect of Decreasing Divisors on Modified Quotas</title>

Figure 118.9 displays the effect of reducing the size of the divisor. Observe the effect this has on the modified quotas of smaller states versus larger states. Use Figure 118.9 to answer each question.

**Figure 118.9 Effect of Decreasing Divisors**

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
|  |  | **Modified Quotas** | | |
| **State** | **Population** | **Divisor:10,500** | **Divisor: 10,000** | **Divisor: 9,500** |
| A | 10,000 | 0.95 | 1 | 1.05 |
| B | 100,000 | 9.52 | 10 | 10.53 |
| C | 1,000,000 | 95.24 | 100 | 105.26 |

* 1. When the divisor decreases from 10,500 to 10,000, how many representatives are gained by each state based on the lower quota?
  2. When the divisor decreases from 10,000 to 9,500, how many representatives are gained by each state based on the lower quota?
  3. Which state gains the most representatives each time the divisor is decreased?

**<SOLUTION>**

* 1. Since a state must have at least one seat, State A begins with 1 seat and still has one seat. State B begins with 9 seats and increases to 10 seats. State C begins with 95 seats and increases to 100 seats. So, State A gains 0, B gains 1, and C gains 5 seats.
  2. State A begins with 1 and still has 1. State B begins with 10 and still has 10. State C begins with 100 and increases to 105. So, State A gains 0, State B gains 0, and State C gains 5.
  3. State C, the largest state, gains the most representatives each time the divisor is decreased. <END>

Example 41 demonstrates that the Jefferson method is biased toward states with larger populations because the modified divisor is smaller than the standard divisor. On the other hand, the Adams method, which begins with the upper quotas, must increase the standard divisor in order to reduce the quotas. Once again, the effect on the number of seats is greater for the larger states, but this time they are decreased. This means that the Adams method favors states with smaller populations.

### [Your Turn] 43

Figure 119.20 displays the effect of increasing the size of the divisor. Observe the effect this has on the modified quotas of smaller states versus larger states. Use Figure 119.20 to answer each question.

**Figure 119.20 Effect of Increasing Divisors**

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
|  |  | **Modified Quotas** | | |
| **State** | **Population** | **Divisor:11,500** | **Divisor: 12,000** | **Divisor: 12,500** |
| A | 10,000 | 0.87 | 0.83 | 0.8 |
| B | 100,000 | 8.7 | 8.33 | 8 |
| C | 1,000,000 | 86.96 | 83.33 | 80 |

* 1. When the divisor increases from 11,500 to 12,000, how many representatives are lost by each state based on the upper quota?
  2. When the divisor decreases from 12,000 to 12,500, how many representatives are lost by each state based on the upper quota?
  3. Which state loses the most representatives each time the divisor is increased? <END>

**Answer**

* 1. State A loses 0, seats State B loses 0, and State C loses 3 .
  2. State A loses 0 seats, State B loses 1, and State C loses 3.
  3. State C, the largest state, loses the most representatives each time the divisor is increased. <END>

**[WORK IT OUT]**

Let’s apply your knowledge of apportionment! In this class activity, you will use the actual census data to determine rates of population growth, explore the difference between popular vote and electoral vote, and make predictions based on historical data. Watch the video, download the student worksheet, and then let’s get started!

**Video: The U.S. Census and the Amazing Apportionment Machine** [**https://www.youtube.com/watch?v=RUCnb5\_HZc0**](https://www.youtube.com/watch?v=RUCnb5_HZc0)

**Student worksheet** [**https://www.census.gov/content/dam/Census/programs-surveys/sis/activities/2020/stateside-k-12/ss78-3-student.pdf**](https://www.census.gov/content/dam/Census/programs-surveys/sis/activities/2020/stateside-k-12/ss78-3-student.pdf)

**Teacher instructions** [**https://www.census.gov/content/dam/Census/programs-surveys/sis/activities/2020/stateside-k-12/ss78-3-teacher.pdf**](https://www.census.gov/content/dam/Census/programs-surveys/sis/activities/2020/stateside-k-12/ss78-3-teacher.pdf)

<END>

## [H2] Flaws in Apportionment Methods

As we have seen, different apportionment methods can have the same results in some scenarios but different results in others. Citizens of states which receive fewer seats with a particular apportionment method will view the apportionment method as flawed and argue in favor of a different method. This inevitably creates debates regarding the use of one method over another. Methods that favor larger states are likely to be challenged by smaller states, methods that favor smaller states are likely to be challenged by larger states, and methods that violate the Quota Rule are likely to be challenged by states of any size depending on the circumstances.

Suppose that the State of Hawaii House of Representatives has 51 representatives, each with their own district. There are also 5 counties in Hawaii. Suppose that redistricting was underway, and the representative districts were to be apportioned to each county based on population. The apportionment results from the use of the Jefferson, Adams, and Webster Methods are given in Figure 11.121. Which apportionment method, if any, are the citizens of each county likely to reject? Explain the argument that they are likely to make.

**Figure 11.121 Comparison of Jefferson, Adams, and Webster Methods**

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| **County** | **Hawaii** | **Honolulu** | **Kalawao** | **Kauai** | **Maui** |
| **Population** | 201,500 | 974,600 | 100 | 72,300 | 167,400 |
| **Lower Quota** | 7 | 35 | 0 | 2 | 6 |
| **Upper Quota** | 8 | 36 | 1 | 3 | 7 |
| **Jefferson** | 7 | 35 | 1 | 2 | 6 |
| **Adams** | 7 | 34 | 1 | 3 | 6 |
| **Webster** | 7 | 34 | 1 | 3 | 6 |

**<SOLUTION>**

Hawaii, Kalawao, and Maui receive the same number of seats regardless of the method used. Citizens of Honolulu would likely reject the Adams and Webster methods arguing that they violate the quota rule. Citizens of Kauai would probably reject the Jefferson method based on the argument that it unfairly favors the larger states. <END>

Example 42 demonstrates that the Webster method of apportionment, like the Jefferson and Adams methods, violates the quota rule in some circumstances.

**The Jefferson, Adams, and Webster methods of apportionment all violate the quota rule.**

The Hamilton method is the only method which satisfies the quota rule in all scenarios. It also consistently favors neither larger nor smaller states. Unfortunately, it can have some strange and results in certain circumstances, which you will see in the next module.

**[WHO KNEW?] GERRYMANDERING – A SUBTLE WAY TO IMPACE APPORTIONMENT**

In addition to your choice of voting method and your choice of apportionment method, there is another important decision to make which could potentially have a huge impact on the fairness of elections in Imaginaria—the creation of electoral districts. In Example 42, we imagined that there were 51 state legislators in Hawaii, each representing their own district. But how did the legislators decide on the boundaries for these districts? Typically, boundaries are drawn so that each district has approximately the same number of residents, but the percentage of residents in each district with a particular political affiliation can swing the power from one group to another. When the districts are drawn to impact the power of a political party or ethnic minority group, this is called gerrymandering. For example, districts can be drawn so that a particular group is spread thinly across districts, increasing the likelihood that they will not have strong representation.

Ap ne bola tha umesh sir ki esko delete mar de

**<a title="Staff of Boston Gazette, Public domain, via Wikimedia Commons" href="https://commons.wikimedia.org/wiki/File:TheGerry-mander.jpg"><img width="256" alt="TheGerry-mander" src="https://upload.wikimedia.org/wikipedia/commons/thumb/b/bb/TheGerry-mander.jpg/256px-TheGerry-mander.jpg"></a>**

"The Gerry-mander" first appeared in this cartoon-map in the Boston Gazette, March 26, 1812.

<END>

**[PEOPLE IN MATHEMATICS]**

Jonathon Mattingly is a mathematician who was featured in a Scientific American article titled, *The Mathematicians Who Want to Save Democracy*. Professor Mattingly is a mathematician at Duke University in North Carolina and he runs election simulations based on alternate versions of electoral districts in order to analyze the effects of gerrymandering. He has even been asked to testify as an expert witness in court. Mattingly and other mathematicians who are working on the problem will potentially have an impact on the redistricting that will occur as a result of the 2020 census. (Cindy Arnold, *The Mathematicians Who Want to Save Democracy*, Nature, June 7, 2017.) <END>

[CHECK YOUR UNDERSTANDING]

* 1. Which of the four apportionment methods discussed in this module does not use a modified divisor?

Answer: The Hamilton method.

* 1. Which of the four apportionment methods discussed in this module satisfies the quota rule?

Answer: The Hamilton method.

* 1. Which of the four apportionment methods discussed in this module is biased toward states with larger populations?

Answer: The Jefferson method.

* 1. Which of the four apportionment methods discussed in this module is biased toward states with smaller populations?

Answer: The Adams method.

* 1. Which of the four apportionment methods discussed in this module begin the apportionment with a state’s upper quota and adjust down?

Answer: The Adams method.

* 1. Which of the four apportionment methods discussed in this module begin the apportionment with a states’ lower quota and adjust up?

Answer: The Hamilton and Jefferson methods.

* 1. Which of the four apportionment methods discussed in this module use traditional rounding?

Answer: Webster’s method.

* 1. Does the change from a standard divisor to a modified divisor tend to change the number of seats for larger or smaller states more?

Answer: Larger

## EXERCISES

For the following questions, use the standard quotas given in Figure 11.122.

**Figure 11.122 Sample Standard Quotas for Scenarios X, Y, and Z**

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
|  | State A | State B | State C | State D | State E | State F | Total Seats |
| Scenario X | 17.63 | 26.62 | 10.81 | 16.01 | 13.69 | 15.24 | 100 |
| Scenario Y | 12.37 | 7.59 | 71.71 | 6.75 | 5.76 | 20.81 | 125 |
| Scenario Z | 3.53 | 31.56 | 2.95 | 5.12 | 9.84 | NA | 53 |

* + - 1. Round the standard quota for each state in Scenario X using traditional rounding. Find the sum of the modified quotas. What is the difference between the sum and the house size?

Answer: A 18, B 27, C 11, D 16, E 14, F 15. The sum of the modified quotas is 101. The sum is 1 greater than the house size.

* + - 1. Round the standard quota for each state in Scenario Y using traditional rounding. Find the sum of the modified quotas. What is the difference between the sum and the house size?

Answer: A 12, B 8, C 72, D 7, E 6, F 21. The sum of the modified quotas is 126. The sum is 1 greater than the house size.

* + - 1. Round the standard quota for each state in Scenario Z using traditional rounding. Find the sum of the modified quotas. What is the difference between the sum and the house size?

Answer: A 4, B 32, C 3, D 5, E 10. The sum of the modified quotas is 54. The sum is 1 greater than the house size.

* + - 1. Find the lower quota for each state in Scenario Y. If each state is allocated its lower quota, how many seats remain to be allocated?

Answer: A 12, B 7, C 71, D 6, E 5, F 20; 4 seats remain to be allocated.

* + - 1. Find the lower quota for each state in Scenario X. If each state is allocated its lower quota, how many seats remain to be allocated?

Answer: A 17, B 26, C 10, D 16, E 13, F 15, 3 seats remain to be allocated.

* + - 1. Find the lower quota for each state in Scenario Z. If each state is allocated its lower quota, how many seats remain to be allocated?

Answer: A 3, B 31, C 2, D 5, E 9; 3 seats remain to be allocated.

* + - 1. Find the upper quota for each state in Scenario X and determine how much the sum of the upper quotas exceeds the house size.

Answer: A 18, B 27, C 11, D 17, E 14, F 16. The sum of the upper quotas is 103, so it exceeds the house size by 3.

* + - 1. Find the upper quota for each state in Scenario Y and determine how much the sum of the upper quotas exceeds the house size.

Answer: A 13, B 8, C 72, D 7, E 6, F 21. The sum of the upper quotas is 127, so it exceeds the house size by 2.

* + - 1. Find the upper quota for each state in Scenario Z and determine how much the sum of the upper quotas exceeds the house size.

Answer: A 4, B 32, C 3, D 6, E 10. The sum of the upper quotas is 55, so it exceeds the house size by 2.

* + - 1. Determine the Hamilton apportionment for Scenario Y.

Answer: A 12, B 7, C 71+1=72, D 6+1=7, E 5+1=6, F 20+1=21

* + - 1. Determine the Hamilton apportionment for Scenario X.

Answer: A 17+1=18, B 26, C 10+1=11, D 16, E 13+1=14, F 15

* + - 1. Determine the Hamilton apportionment for Scenario Z.

Answer: A 3, B 31+1=32, C 2+1=3, D 5, E 9+1=10

For the following questions, use the information in Figure 11.123.

**Figure 11.123 Standard and Final Quotas for Methods X, Y, and Z**

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
|  | **State A** | **State B** | **State C** | **State D** | **State E** |
| **Standard Quota** | 1.67 | 3.33 | 5.00 | 6.67 | 8.33 |
| **Method X** | 2 | 2 | 5 | 7 | 9 |
| **Method Y** | 1 | 3 | 5 | 7 | 9 |
| **Method Z** | 1 | 3 | 5 | 6 | 10 |

* + - 1. Does the apportionment resulting from method X satisfy the quota rule? Why or why not?

Answer: The apportionment does not satisfy the quota rule because the upper and lower quotas for State B are 3 and 4, but State B received 2 seats.

* + - 1. Does the apportionment resulting from method Z satisfy the quota rule? Why or why not?

Answer: The apportionment does not satisfy the quota rule because the upper and lower quotas for State E are 8 and 9, but State E received 10 seats.

* + - 1. Does the apportionment resulting from method Y satisfy the quota rule? Why or why not?

Answer: The apportionment does satisfy the quota rule. The states each receive either the upper or lower quota.

For the following questions, use the following information and the information in Figure 11.124. In the movie Black *Panther*, the hero lives in the fictional country of Wakanda. Imagine that 111 Vibranium artifacts must be distributed among the fortress cities, or birnin, of Wakanda based on the population of each birnin.

**Figure 11.124 Populations and Standard Quotas of Major Wakandan Cities**

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
| Fortress  Cities | Birnin  Djata (D) | Birnin T’Chaka (T) | Birnin Zana (Z) | Birnin S’Yan (S) | Birnin Bashenga (B) | Birnin Azzaria (A) | Total |
| Residents | 26,000 | 57,000 | 27,000 | 18,000 | 64,000 | 45,000 | 237,000 |
| Standard Quota | 12.18 | 26.70 | 12.65 | 8.43 | 29.98 | 21.08 | 111 |

* + - 1. Modify the standard quota for each state using traditional rounding. Find the sum of the modified quotas. What is the difference between the sum and the house size?

Answer: D 12, T 27, Z 13, S 8, B 30, A 21. The sum of the modified quotas is 111. The difference between the sum and house size is 0.

* + - 1. Find the standard lower quota for each state. If each state is allocated its lower quota, how many seats remain to be allocated?

Answer: D 12, T 26, Z 12, S 8, B 29, A 21. 3 seats remain to be allocated.

* + - 1. Find the standard upper quota for each state and determine how much the sum of the upper quotas exceeds the house size.

Answer: D 13, T 27, Z 13, S 9, B 30, A 22. The sum of the upper quotas exceeds the house size by 3.

* + - 1. Use the Hamilton method to apportion the artifacts.

Answer: D 12, T 26+1=27, Z 12+1=13, S 8, B 29+1=30, A 21

* + - 1. Find the modified lower quota for each state using a modified divisor of 2000. Is the sum of the modified quotas too high, too low, or equal to the house size?

Answer: D 13, T 28, Z 13, S 9, B 32, A 22. The sum of the modified quotas is too high.

* + - 1. Find the modified lower quota for each state using a modified divisor of 2100. Is the sum of the modified quotas too high, too low, or equal to the house size?

Answer: D 12, T 27, Z 12, S 8, B 30, A 21. The sum of the modified quotas is too low.

* + - 1. Use the Jefferson method to apportion the artifacts. Determine if it is necessary to modify the divisor. If so, indicate the value of the modified divisor.

Answer: D 12, T 27, Z 13, S 8, B 30, A 21. It is necessary to modify the divisor. Modified divisors from 2065 to 2076 will result in this apportionment.

* + - 1. Does the Jefferson method result in an apportionment that satisfies or violates the quota rule in this scenario?

Answer: The apportionment satisfies the quota rule.

* + - 1. Find the modified upper quota for each state using a modified divisor of 2250. Is the sum of the modified quotas too high, too low, or equal to the house size?

Answer: D 12, T 26, Z 12, S 8, B 29, A 20. The sum of the modified quotas is too low.

* + - 1. Find the modified upper quota for each state using a modified divisor of 2150. Is the sum of the modified quotas too high, too low, or equal to the house size?

Answer: D 13, T 27, Z 13, S 9, B 30, A 21. The sum of the modified quotas is too high.

* + - 1. Use the Adams method to apportion the artifacts. Determine if it is necessary to modify the divisor. If so, indicate the value of the modified divisor.

Answer: D 12, T 26, Z 13, S 9, B 30, A 21. It is necessary to modify the divisor. Modified divisors from 2193 to 2206 will result in this apportionment.

* + - 1. Does the Adams method result in an apportionment that satisfies or violates the quota rule in this scenario?

Answer: The apportionment satisfies the quota rule.

* + - 1. Which method of apportionment, Jefferson or Adams, is a resident of Birnin T’Chaka likely to prefer? Justify your answer.

Answer: A resident of Birnin T’Chaka would likely prefer the Jefferson method of apportionment to the Adams method because they receive 27 artifacts instead of 26.

* + - 1. Use the Webster method to apportion the artifacts. Determine if it is necessary to modify the divisor. If so, indicate the value of the modified divisor.

Answer: D 12, T 27, Z 13, S 8, B 30, A 21. The standard divisor works, so it is not necessary to modify the divisor.

* + - 1. Does the Webster method result in an apportionment that satisfies or violates the quota rule in this scenario?

Answer: The quota rule is satisfied.

* + - 1. Which of the four methods of apportionment from this module (Hamilton, Jefferson, Adams, or Webster) are the residents of Birnin S’Yan likely to prefer? Justify your answer.

Answer: The residents of Birnin S’Yan are likely to prefer the Adams method of apportionment because they receive 9 artifacts instead of 8.

For the following questions, use the following information and the information in Figure 11.125.

Children from five families—the Chorro family, the Eswaran family, the Javernick family, the Lahde family, and the Stolly family—joined a town Easter egg hunt. When they returned with their baskets, they had 827 eggs! They decided to share their eggs amongst the families based on the number of children in each family.

**Figure 11.125 Populations and Standard Quotas for Easter Eggs**

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
| **Family** | **(C) Chorro** | **(E) Eswaran** | **(J) Javernick** | **(L) Lahde** | **(S) Stolly** | **Total** |
| **Children** | 3 | 2 | 4 | 1 | 5 | 15 |
| **Standard Quota** | 155.04 | 103.36 | 206.72 | 103.36 | 258.40 | 827 |

* + - 1. Modify the standard quota for each state using traditional rounding. Find the sum of the modified quotas. What is the difference between the sum and the house size?

Answer: C 165, E 110, J 221, L 55, S 276. The sum of the modified quotas is 827. The sum is the house size.

* + - 1. Find the standard lower quota for each state. If each state is allocated its lower quota, how many seats remain to be allocated?

Answer: C 165, E 110, J 220, L 55, S 275. 2 seats remain to be allocated.

* + - 1. Find the standard upper quota for each state, and determine how much the sum of the upper quotas exceeds the house size.

Answer: C 166, E 111, J 221, L 56, S 276; The sum of the upper quotas exceeds the house size by 3.

* + - 1. Use the Hamilton method to apportion the Easter eggs.

Answer: C 165, E 110, J 221, L 55, S 276

* + - 1. Find the modified lower quota for each state using a modified divisor of 0.01800. Is the sum of the modified quotas too high, too low, or equal to the house size?

Answer: C 166, E 111, J 222, L 55, S 277. The sum of the modified quotas is too high.

* + - 1. Find the modified lower quota for each state using a modified divisor of 0.01810. Is the sum of the modified quotas too high, too low, or equal to the house size?

Answer: C 165, E 110, J 220, L 55, S 276. The sum of the modified quotas is too low.

* + - 1. Use the Jefferson method to apportion the Easter eggs. Determine if it is necessary to modify the divisor. If so, indicate the value of the modified divisor.

Answer: C 165, E 110, J 221, L 55, S 276. It is necessary to modify the divisor. Modified divisors from 0.01808 to 0.01809 will result in this apportionment.

* + - 1. Does the Jefferson method result in an apportionment that satisfies or violates the quota rule in this scenario?

Answer: The apportionment satisfies the quota rule.

* + - 1. Find the modified upper quota for each state using a modified divisor of 0.1820. Is the sum of the modified quotas too high, too low, or equal to the house size?

Answer: C 165, E 110, J 220, L 55, S 275. The sum of the modified quotas is too low.

* + - 1. Find the modified upper quota for each state using a modified divisor of 0.1816. Is the sum of the modified quotas too high, too low, or equal to the house size?

Answer: C 166, E 111, J 221, L 56, S 276. The sum of the modified quotas is too high.

* + - 1. Use the Adams method to apportion the Easter eggs. Determine if it is necessary to modify the divisor. If so, indicate the value of the modified divisor.

Answer: C 165, E 110, J 221, L 55, S 276. It is necessary to modify the divisor. Modified divisor 0.0181818177 will result in this apportionment.

* + - 1. Does the Adams method result in an apportionment that satisfies or violates the quota rule in this scenario?

Answer: The apportionment satisfies the quota rule.

* + - 1. Use the Webster method to apportion the Easter eggs. Determine if it is necessary to modify the divisor. If so, indicate the value of the modified divisor.

Answer: C 165, E 110, J 221, L 55, S 276. The standard divisor works, so it is not necessary to modify the divisor.

* + - 1. Does the Webster method result in an apportionment that satisfies or violates the quota rule in this scenario?

Answer: The quota rule is satisfied.

For the following questions, suppose that the State of Delaware received 2000 packs of COVID-19 vaccines, with ten doses per pack. These (unopened) packs must be distributed to the three counties based on total population. Use the population information in Figure 11.126 to determine how many packs of vaccine will be distributed to each county based on the given apportionment method.

**Figure 11.126 Populations of Delaware Counties**

|  |  |  |  |
| --- | --- | --- | --- |
| County | (N) New Castle | (K) Kent | (S) Sussex |
| Residents | 558,753 | 180,786 | 234,225 |

* + - 1. Hamilton’s Method

Answer: N 1148, K 371, S 481

* + - 1. Jefferson’s Method

Answer: N 1148, K 371, S 481

* + - 1. Adams's Method

Answer: N 1147, K 372, S 481

* + - 1. Webster’s Method

Answer: N 1148, K 371, S 481

* + - 1. Notice that the apportionments found in questions 42, 43, 44, and 45 all satisfy the quota rule. Does this contradict the statement, “The Jefferson, Adams, and Webster methods of apportionment all violate the quota rule”? Why or why not?

Answer: This is not a contradiction. An apportionment method may satisfy the quota rule in some scenarios but not others. If even one apportionment created using a given apportionment method violates the quota rule, then that apportionment method is said to violate the quota rule.

## [MODULE 4 SUMMARY]

### Key Terms

**apportionment problem**

**apportionment method**

**lower quota**

**upper quota**

**quota rule**

### Key Concepts

* Hamilton’s method of apportionment utilizes the standard divisor and standard lower quotas, and it distributes any remaining seats based on the size of the fractional parts of the standard lower quota. Hamilton’s method satisfies the quota rule and favors neither larger nor smaller states.
* Jefferson’s method of apportionment utilizes a modified divisor which is adjusted so that the modified lower quotas sum to the house size. Jefferson’s method violates the quota rule and favors larger states.
* Adams’s method of apportionment utilizes a modified divisor which is adjusted so that the modified upper quotas, sum to the house size. Adams’s method violates the quota rule and favors smaller states.
* Webster’s method of apportionment utilizes a modified divisor which is adjusted so that the modified state quotas, rounded using traditional rounding, sum to the house size. Webster’s method violates the quota rule, but favors neither larger nor smaller states.

### Videos

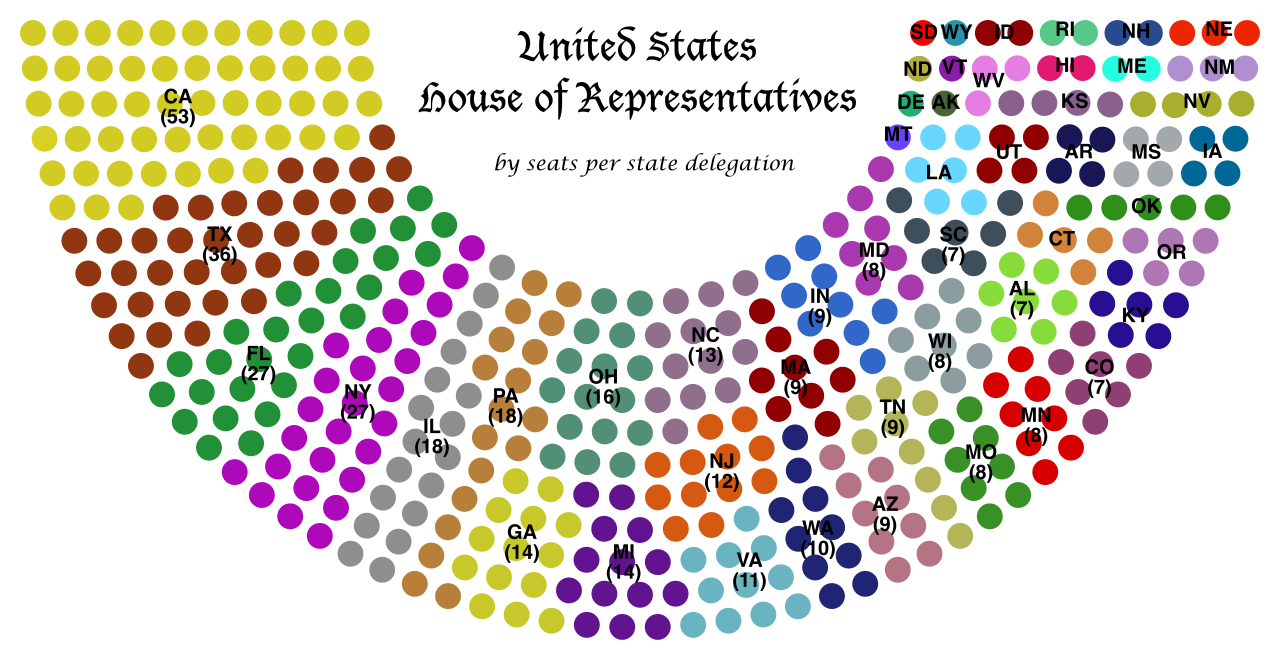
**VIDEO Apportionment: Hamilton’s Method** [**https://www.youtube.com/watch?v=6qRJGw4oXo4**](https://www.youtube.com/watch?v=6qRJGw4oXo4%20)

**Video: Adams Method Apportionment by Hand and Calculator <https://www.youtube.com/watch?v=IrPyTKD3wqI>**

**Video: Jefferson Apportionment Method <https://www.youtube.com/watch?v=\_ZI2-yZl8Z8>**

# [H1] Module 5 - Fairness in Apportionment Methods

*Photo 11.11: Diagram of the seats in the House of Representatives color coded by state represented.*



**After completing this module, you should be able to:**

* + 1. Describe and illustrate the Alabama paradox LO 11.5.1
    2. Describe and illustrate the population paradox LO 11.5.2
    3. Describe and illustrate the new-states paradox LO 11.5.3
    4. Identify ways to promote fairness in apportionment methods LO 11.5.4

## [H2] Apportionment Paradoxes

The citizens of Imaginaria will want the apportionment method to be as fair as possible. There are certain characteristics that they would reasonably expect from a fair apportionment.

* If the house size is increased, the state quotas should all increase or remain the same but never decrease.
* If one state’s population is growing more rapidly than another state’s population, the faster growing state should not lose a seat while a slower growing state maintains or gains a seat.
* If there is a fixed number of seats, adding a new state should not cause an existing state to gain seats while others lose them.

However, apportionment methods are known to contradict these expectations. Before you decide on the right apportionment for Imaginarians, let’s explore these **apportionment paradoxes**.

**[DEFINITION]**

An **apportionment paradox** is when an apportionment method produces results that seem to contradict reasonable expectations of fairness. <END>

There is a lot that the founders of Imaginaria can learn from U.S. history. The constitution of the United States requires that the seats in the House of Representatives be apportioned according to the results of the census that occurs every decade, but the number of seats and the apportionment method is not stipulated. Over the years, several different apportionment methods and house sizes have been used and scrutinized for fairness. This scrutiny has led to the discovery of several of these apportionment paradoxes.

## [H2] The Alabama Paradox

At the time of the 1880 U.S. Census, the Hamilton method of apportionment had replaced the Jefferson method. The number of seats in the House of Representatives was not fixed. To achieve the fairest apportionment possible, the house sizes were chosen so that the Hamilton and Webster methods would result in the same apportionment. The chief clerk of the Census Bureau calculated the apportionments for house sizes between 275 and 350. There was a surprising result that became known as the **Alabama paradox**. Alabama would receive eight seats with a house size of 299, but only receive seven seats if the house size increased to 300. (Michael J. Caulfield (Gannon University), "Apportioning Representatives in the United States Congress - Paradoxes of Apportionment," Convergence (November 2010), DOI:10.4169/loci003163)

**[DEFINITION]**

The **Alabama paradox** occurs when an increase in house size reduces a state’s quota. <END>

Let’s explore how the procedures of Hamilton’s method could cause the Alabama paradox to occur.

### <example>Example 41 - <title>The 1880 Alabama Quota</title>

The 1880 census recorded the population of Alabama as 1,513,401 and that of the U.S. as 62,979,766.

1. Calculate the standard divisor and standard quota for the State of Alabama based on a house size of 299.
2. Calculate the standard divisor and standard quota for the State of Alabama based on a house size of 300.
3. Did the standard quota increase or decrease when the house size increased?
4. Consider the Hamilton method of apportionment. Explain how Alabama’s final quota could be smaller with a larger standard quota.

**<SOLUTION>**

* 1. The standard divisor is  citizens per seat. The standard quota for Alabama is  seats.
  2. The standard divisor is  citizens per seat. The standard quota for Alabama is seats.
  3. The standard quota increased.
  4. In each case, the state would receive the lower quota of 7 and then be awarded one more seat if the fractional part of the standard quota were high enough relative to the fractional parts of the other states’ standard quotas. When the house size was 299, Alabama received one of the remaining seats after the lower quotas were distributed. When the house size was 300, Alabama did not receive one of the remaining seats after the lower quotas were distributed. It must have been the case that either the fractional part 0.2090 ranked lower amongst the other fractional parts of the state quotas than the fractional part 0.1850 did, or there were fewer remaining seats, or both. <END>

After the 1900 census, the Census Bureau again calculated the apportionment based on various house sizes. It was determined that Colorado would receive three seats with a house size of 356, but only two seats with a house size of 357.

### [Your Turn] 44

The 1900 census recorded the population of Colorado as 539,700 and that of the U.S. as 76,212,168.

1. Calculate the standard divisor and standard quota for the State of Colorado based on a house size of 356.
2. Calculate the standard divisor and standard quota for the State of Colorado based on a house size of 357.
3. Did the standard quota increase or decrease when the house size increased?
4. Consider the Hamilton method of apportionment. Explain how Colorado’s final quota could be smaller with a larger standard quota. <END>

**Answer**

* 1. The standard divisor is  citizens per seat. The standard quota for Colorado is  seats.
  2. The standard divisor is  citizens per seat. The standard quota for Colorado is seats.
  3. The standard quota increased.
  4. It must have been the case that either the fractional part 0.5281 ranked lower amongst the other fractional parts of the state quotas than the fractional part 0.5210 did, or there were fewer remaining seats to be distributed, or both. <END>

### <example>–Example 42 - <title>Hamilton’s Method and the Alabama Paradox</title>

Suppose that States A and B each have a population of 6, while State C has a population of 2.

* 1. Use the Hamilton method to apportion 10 seats.
  2. Use the Hamilton method to apportion 11 seats.
  3. Does this example demonstrate the Alabama paradox? If so, how?

**<SOLUTION>**

1. Step 1 - The total population is 14. The standard divisor is  individuals per seat.

Step 2 - The states’ standard quotas are as follows: A , B , and C 

Step 3 - The states’ lower quotas are as follows: A 4, B 4, and C 1.

Step 4 - The sum of the lower quotas is 9, which means there is one seat remaining to be apportioned. State C has the highest fractional part and receives the additional seat.

Step 5 - The final apportionment is as follows: A 4, B 4, and C 2, which sums to 10.

1. Step 1 - The total population is 14. The standard divisor is  individuals per seat.

Step 2 - The states’ standard quotas are: A , B , and C .

Step 3 - The states’ lower quotas are: A 4, B 4, and C 1.

Step 4 - The sum of the lower quotas is 9, which means there are two seats remaining to be apportioned. A and B have the highest fractional parts and receive the additional seats.

Step 5 - The final apportionment is: A 5, B 5, and C 1.

1. Yes, this demonstrates the Alabama paradox because State C receives two seats if the house size is 10, but only one seat if the house size is 11. <END>

### [Your Turn] 45

Suppose that the founders of Imaginaria decide to have a parliament that apportions seats to four political parties based on the portion of the vote each party has earned. Also, suppose that Party A has 56.7 thousand votes, Party B has 38.5 thousand votes, Party C has 4.2 thousand votes, and Party D has 0.6 thousand votes.

* 1. Use the Hamilton method to apportion 323 seats.
  2. Use the Hamilton method to apportion 324 seats.
  3. Does this example demonstrate the Alabama paradox? If so, how? <END>

**Answer**

1. The final apportionment is: A 183, B 124, C 14, and D 2, which sums to 323.
2. The final apportionment is: A 183, B 125, C 13, and D 2, which sums to 324.
3. Yes, this demonstrates the Alabama paradox because State C receives 14 seats if the house size is 323, but only 13 seats if the house size is 324. <END>

## [H2] The Population Paradox

It is important for the founders of Imaginaria to keep in mind that the populations of states change as time passes. Some populations grow and some shrink. Some populations increase by a large amount while others increase by a small amount. These changes may necessitate a **reapportionment** of seats. It would be reasonable for Imaginarians to expect that the state with a population that has grown more than others will gain a seat before the other states. Once again, this is not always the case with the Hamilton method of apportionment.

**[DEFINITION]**

A **reapportionment** is the recalculation of state quotas due to a change in population.

The **population growth rate** of a state is the ratio of the change in the population to the original size of the population, often expressed as a percentage. This value is positive if the population is increasing and negative if the population is decreasing.

The **population paradox** occurs when a state with an increasing population loses a seat while a state with a decreasing population either retains or gains seats. More generally, the population paradox occurs when a state with a higher population growth rate loses seats while a state with a lower population growth rate retains or gains seats. <END>

Notice that the population paradox definitions has two parts. If either part applies, then the population paradox has occurred. The first part of the definition only applies when one state has a decreasing population. The second part of the definition applies in all situations, whether there is a state with a decreasing population or not. It will be easier to identify situations that involve a decreasing population. The other situations requires the calculation of a growth rate. The reason that we don’t have to calculate a growth rate when one state has a decreasing population and the other has an increasing population is that increasing population has a positive growth rate which is always greater than the negative growth rate of a decreasing population*.*

🚦 *A state must lose a seat in order for the population paradox to apply. It is not enough for a state with a lower growth rate to gain a seat while a state with a higher growth rate retains the same number of seats.*

### <example> Example 43 - <title>Apportionment of Respirators to Hospitals</title>

Suppose that 18 respirators are to be apportioned to

three hospitals based on their capacities. The Hamilton method is used to allocate the respirators in 2020, then to reallocate based on new capacities in 2021. The results are in Figure 11.127. How does this demonstrate the population paradox?

**Figure 11.127 Hospital Capacities and Apportionment of Respirators**

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
| **Hospital** | **Capacity in 2020** | **Respirators in 2020** | **Change in Capacity** | **Growth Rate =** | **Capacity in 2021** | **Respirators in 2021** |
| A | 825 | 9 | 57 |  | 882 | 9 |
| B | 613 | 7 | 13 |  | 626 | 6 |
| C | 239 | 2 | 3 |  | 242 | 3 |

**<SOLUTION>**

Hospital B lost a respirator while hospital C gained one, even though hospital B had a higher growth rate than hospital C. <END>

### [Your Turn] 46

Suppose that 18 respirators are to be apportioned to three hospitals based on their capacities. The respirators are allocated based on the Hamilton method in 2020, then reallocated based on new capacities in 2021. The results are in Figure 11.128 How does this demonstrate the population paradox?

**Figure 11.128 Hospital Capacities and Apportionment of Respirators**

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
| **Hospital** | **Capacity in 2020** | **Respirators in 2020** | **Change in Capacity** | **Growth Rate =** | **Capacity in 2021** | **Respirators in 2021** |
| A | 237 | 5 | 6 |  | 243 | 6 |
| B | 889 | 21 | 69 |  | 958 | 21 |
| C | 674 | 16 | 18 |  | 692 | 15 |

<END>

**Answer**

Hospital C lost a respirator while hospital A gained a seat, but hospital C has a higher growth rate than hospital A. <END>

**[FORMULA]**

The growth rate of a population can be calculated by subtracting the previous population size from the current population size, and then dividing the difference by the previous population size.

<END>

🚦 *Make sure to calculate the subtraction before the division. If you are entering the values in a calculator, it is helpful to put parentheses around the subtracted terms.*

### <example>Example 44 - <title>The Congress of Costaguana</title>

The country of Costaguana has three states: Azuera with a population of 894,000; Punta Mala with a population of 696,000; and Esmeralda with a population of 215,000. There are 38 seats in the Congress of Costaguana. The apportionment of the seats is determined by Hamilton’s method to be: 19 for Azuera, 15 for Punta Mala, and 4 for Esmerelda. A census reveals that the population has grown and the seats must be reapportioned. If Azuera now has 953,000 residents, Punta Mala now has 706,000 residents, and Esmerelda now has 218,000 residents, how many seats will each state receive upon reapportionment? How is this an example of the population paradox?

**<SOLUTION>**

The Hamilton reapportionment is: 19 for Azuera, 14 for Punta Mala, and 5 for Esmerelda. This is an example of the population paradox because Punta Mala lost a seat to Esmerelda, even though Punta Mala’s population grew by 1.44 percent while Esmerelda’s only grew by 1.40 percent. <END>

### [Your Turn] 47

The country of Elbonia has three states: Mudston with a population of866,000; WallaWalla with apopulation of 626,000; and Dilberta with apopulation of 256,000. There are 38 seats in the Congress of Elbonia. The apportionment of the seats is determined by Hamilton’s method to be: 19 for Mudston, 14 for WallaWalla, and 5 for Dilberta. A census reveals that the population has grown and the seats must be reapportioned. If Mudston now has 921,000 residents, WallaWalla now has 640,000 residents, and Dilberta now has 260,000 residents, how many seats will each state receive upon reapportionment? How is this an example of the population paradox? <END>

**Answer**

The Hamilton reapportionment is: 19 for Mudston, 13 for WallaWalla, and 6 for Dilberta. This is an example of the population paradox because WallaWalla lost a seat to Dilberta, even though WallaWalla’s population grew by 2.24 percent while Dilberta’s only grew by 1.56 percent. <END>

## [H2] The New-States Paradox

As a founder of Imaginaria, you might also consider the possibility that Imaginaria could annex nearby lands and increase the number of states. This occurred several times in the United States such as when Oklahoma became a state in 1907. The Househ size was increased from 386 to 391 to accommodate Oklahoma’s quota of five seats. When the seats were reapportioned using Hamilton’s method, New York lost a seat to Maine despite the fact that their populations had not changed. This is an example of the **new-state paradox**.

**[DEFINITION]**

The **new-state paradox** occurs when the addition of a new state is accompanied by an increase in seats to maintain the standard ratio of population to seats, but one of the existing states loses a seat in the resulting reapportionment. <END>

### <example>Example 45 - <title>New State of Oscuridad</title>

The country of San Lorenzo has grown from two states to three. The house size of the congress has been increased by eight and the seats have been reapportioned to accommodate the new state of Oscuridad. See Figure 11.129. The constitution mandates the use of the Hamilton method of apportionment. Use this information to answer the questions.

**Figure 11.129 Apportionment of San Lorenzo Congressional Seats**

|  |  |  |  |
| --- | --- | --- | --- |
| **State** | **Population  (in hundreds)** | **Original Apportionment** | **Reapportionment** |
| Clara | 7100 | 39 | 40 |
| Velasco | 9080 | 51 | 50 |
| Oscuridad | 1500 | Not Applicable | 8 |

* 1. What was the original house size?
  2. What is the new house size?
  3. How is this reapportionment an example of the new-states paradox?

**<SOLUTION>**

* 1. 39+51=90
  2. 40+50+8=98
  3. The original state of Velasco lost a seat to the original state of Clara when the new state of Oscuridad was added. <END>

### [Your Turn] 48

The country of Narnia has grown from two states to three. The house size of the congress has been increased by five and the seats have been reapportioned to accommodate the new state of Chippingford. See Figure 11.130. The constitution mandates the use of the Hamilton method of apportionment. Use this information to answer the questions.

**Figure 11.130 Apportionment of Narnia Congressional Seats**

|  |  |  |  |
| --- | --- | --- | --- |
| **State** | **Population  (in hundreds)** | **Original Apportionment** | **Reapportionment** |
| Beruna | 7600 | 39 | 40 |
| Beaversdam | 9720 | 51 | 50 |
| Chippingford | 1000 | Not Applicable | 5 |

* 1. What was the original house size?
  2. What is the new house size?
  3. How is this reapportionment an example of the new-states paradox? <END>

**Answer**

* 1. 90
  2. 95
  3. The original state of Beaversdam lost a seat to the original state of Beruna when the new state of Chippingford was added. <END>

### <example>Example 45 - <title>The Growing Country of Gulliversia</title>

The country of Gulliversia has two states: Lilliput with a population of 700,000 and Brobdingnag with a population of 937,000. The constitution of Gulliversia requires that the 90 congressional seats be apportioned by Hamilton’s method. Lilliput has received 38 seats while Brobdingnag has received 52 seats. Recently, the island of Houyhnhnmsland with a population of 170,000 has joined the union, becoming a state of Gulliversia. When Houyhnhnmsland is included, nine additional seats must be apportioned to maintain the same ratio of seats to citizens. Use Hamilton’s method to reapportion the 99 seats to the three states. How is the resulting apportionment an example of the new-states paradox?<END>

**<SOLUTION>**

The reapportionment gives 39 seats to Lilliput, 51 seats to Brobdingnag, and 9 seats to Houyhnhnmsland. This is an example of the new-states paradox because the original state of Brobdingnag lost a seat to the original state of Lilliput when the new state was added to the union. <END>

### [Your Turn] 49

The country of Neverland has two states: Neverwood with a population of 760,000 and Mermaids Lagoon with a population of 943,000. The constitution of Neverland requires that the 84 congressional seats be apportioned by Hamilton’s method. Neverwood has received 37 seats while Mermaids Lagoon has received 47 seats. Recently, the island of Marooners Rock with a population of 190,000 has joined the union, becoming a state of Neverland. When Marooners Rock is included, nine additional seats must be apportioned to maintain the same ratio of seats to citizens. Use Hamilton’s method to reapportion the 93 seats to the three states. How is the resulting apportionment an example of the new-states paradox? <END>

**Answer**

The reapportionment gives 38 seats to Neverwood, 46 seats to Mermaids Lagoon, and 9 seats to Marooners Rock. This is an example of the new-states paradox because the original state Mermaids Lagoon lost a seat to the original state Neverwood when the new state was added to the union. <END>

When a new state is added, it is necessary to determine the amount that the house size must be increased to retain the original ratio of population to seats, in other words to keep the original standard divisor. To calculate the new house size, divide the new population by the original standard divisor, and round to the nearest whole number.

**[FORMULA]**

 rounded to the nearest whole number. <END>

### <example>Example 46 - <title>Oklahoma Joins the Union</title>

Oklahoma was admitted as the 46th state on November 16, 1907. Before Oklahoma joined the union, the U.S. population was approximately 75,030,000 and the House of Representatives had 386 seats. The new state had a population of approximately 970,000. Use this information to estimate the original standard divisor to the nearest hundred, the new population, the new house size, and the number of seats Oklahoma should receive.

**<SOLUTION>**



New Population



There are  new seats to be apportioned to Oklahoma. <END>

### [Your Turn] 50

New Mexico was admitted as the 47th state on January 6, 1912. Before New Mexico joined the union, the U.S. population was approximately 76,000,000 and the House of Representatives had 391 seats. The new state had a population of approximately 300,000. Use this information to estimate the original standard divisor to the nearest hundred, the new population, the new house size, and the number of seats New Mexico should receive. <END>

**Answer**



New Population



There are  new seats to be apportioned to New Mexico. <END>

## [H2] The Search for the Perfect Apportionment Method

The ideal apportionment method would simultaneously satisfy the following four fairness criteria.

* Satisfy the quota rule
* Avoid the Alabama paradox
* Avoid the population paradox
* Avoid the new-states paradox

We have seen that the Hamilton method allows the Alabama paradox, the population paradox, and the new-states paradox in some apportionment scenarios. Let’s explore the results of the other methods of apportionment we have discussed in some of the same scenarios.

### <example>Example 47 - <title>Orange Grove and the New-States Paradox</title>

The incorporated town of Orange Grove consists of two subdivisions: The Oaks with 1,254 residents and The Villages with 10,746 residents. A council with 100 members supervises the municipality’s operations. The council votes to annex an unincorporated subdivision called The Lakes with a population of 630. They plan to increase the size of the council to maintain the ratio of seats to residents such that the new council will have 100 seats plus the number of seats given to The Lakes. Use each of the following apportionment methods and indicated number of additional seats to find the original and new apportionment and determine whether the new-state paradox occurs.

1. Jefferson’s method with five additional seats.
2. Adams’s method with six additional seats.
3. Webster’s method with five additional seats

**<SOLUTION>**

1. Using a modified divisor of 119, the original apportionment would have been: The Oaks 10 and The Villages 90. Using a modified divisor of 119, the new apportionment would be: The Oaks 10, The Villages 90, and The Lakes 5. The new-state paradox does not occur.
2. Using a modified divisor of 121, the original apportionment would have been: The Oaks 11 and The Villages 89. Using a modified divisor of 121, the new apportionment would be: The Oaks 11, The Villages 89, and The Lakes 6. The new-state paradox does not occur.
3. Using the standard divisor of 120, the original apportionment would have been: The Oaks 10 and The Villages 90. Using a modified divisor of 119.5, the new apportionment would be: The Oaks 10, The Villages 90, and The Lakes 5. The new-state paradox does not occur. <END>

### [Your Turn] 51

Suppose that in 2016, States A, B, and C had populations of 13 million, 12 million, and 112 million, respectively. In 2020, State A has grown by 1 million residents, State B has lost 1 million residents, and State C has gained 2 million residents. Compare the apportionments in 2016 to 2020 using each method given below. Which of the four methods violate(s) the population paradox in this scenario? <END>

**Answer**

Only Hamilton’s method violates the population paradox. <END>

In Example 50 and Your Turn 50, we saw that neither the population paradox nor the new-states paradox occurred when using the Jefferson, Adams, and Webster methods. It turns out that, although all three of these divisor methods violate the quota rule, none of them ever causes the population paradox, new-states paradox, or even the Alabama paradox. On the other hand, the Hamilton method satisfies the quota rule, but will cause the population paradox, the new-states paradox, and the Alabama paradox in some scenarios.

In 1983, mathematicians Michel Balinski and Peyton Young proved that no method of apportionment can simultaneously satisfy all four fairness criteria.

**[DEFINITION]**

The **Balinkski-Young Impossibility Theorem** states that any apportionment method that does not violate the quota rule must produce paradoxes, and any apportionment method that does not produce paradoxes must violate the quota rule. <END>

There are other apportionment methods that satisfy different subsets of these fairness criteria. For example, the mathematicians, Balinski and Young who proved the Balinski-Young Impossibility Theorem created a method that both satisfies the quota rule and is free of the Alabama paradox. (Balinski, Michel L.; Young, H. Peyton (November 1974). "A New Method for Congressional Apportionment". *Proceedings of the National Academy of Sciences*. **71** (11): 4602–4606.) However, no method may always follow the quota rule and simultaneously be free of the population paradox.  (Balinski, Michel L.; Young, H. Peyton (September 1980). "The Theory of Apportionment" *(PDF)*. Working Papers. International Institute for Applied Systems Analysis. WP-80-131.)

So, as you and your fellow founders of Imaginaria make the important decision about the right apportionment method for Imaginaria, do not look for a perfect apportionment method. Instead, look for an apportionment method that best meets the needs and concerns of Imaginarians.

**[WORK IT OUT]**

Let’s apply your knowledge of reapportionment! In this class activity, you will use the actual census data to reallocate seats in the U.S. House of Representatives. Open the interactive website, download the student worksheet, and let’s get started!

**Interactive Website: My Congressional District** [**https://www.census.gov/mycd/**](https://www.census.gov/mycd/)

**Student worksheet:** [**https://www.census.gov/content/dam/Census/programs-surveys/sis/activities/2020/stateside-k-12/sshs-3-student.pdf**](https://www.census.gov/content/dam/Census/programs-surveys/sis/activities/2020/stateside-k-12/sshs-3-student.pdf)

**Teacher instructions:** [**https://www.census.gov/content/dam/Census/programs-surveys/sis/activities/2020/stateside-k-12/sshs-3-teacher.pdf**](https://www.census.gov/content/dam/Census/programs-surveys/sis/activities/2020/stateside-k-12/sshs-3-teacher.pdf)

[CHECK YOUR UNDERSTANDING]

In the following questions, a scenario is given. Determine if the scenario is an example of a quota rule violation, the Alabama paradox, the population paradox, the new-states paradox, or none of these.

* + 1. A city purchased five new firetrucks and apportioned them among the existing fire stations. Although your neighborhood fire station has the same proportion of the city’s firetrucks as before the new ones were purchased, it now has one fewer.

Answer: The Alabama paradox

* + 1. The school resources officers in a county were reapportioned based on the most recent census. The number of students at Chapel Run Elementary went up while the number of students at Panther Trail Elementary went down. However, Chapel Run now has one fewer resources officer while Panther Trail has one more than it did previously.

Answer: Population paradox

* + 1. The standard quota for States A, B, and C are 1.19, 2.73, and 5.71 respectively. State A received 1 seat, State B received 3 seats, and State C received 4 seats.

Answer: Quota rule violation

* + 1. A corporation that owns several hospitals purchased an additional hospital, causing the doctor to patient ratio to decrease.

Answer: None of these

* + 1. When the city of Cocoa annexed an adjacent unincorporated community, the number of seats on the city council was increased to maintain the standard ratio of citizens to seats, but one existing community of Cocoa still lost a seat on the city council to another existing community of Cocoa when the new community was added.

Answer: New-states paradox

In the following questions, a type of scenario is described. Indicate which paradox could arise in a scenario of this kind.

* + 1. A reapportionment occurs because the populations of the states change, and the house size remains the same.

Answer: The population paradox could occur.

* + 1. A reapportionment occurs because the house size increases, and the populations of the states remain the same.

Answer: The Alabama paradox could occur.

* + 1. A reapportionment occurs because an additional state is added to the union, the populations of the original states remain the same, and the house size is increased to correspond to the population of the new state.

Answer: The new-state paradox could occur.

## EXERCISES

In the following questions, determine if the scenario violates the Alabama paradox. Justify your answer.

* + 1. A company with an office in each of four cities must distribute 145 new Chromebooks to the four offices. It is determined that Office A will receive 42, Office B will receive 17, Office C will receive 35, and Office D will receive 51. At the last minute, it is discovered that there are 146 Chromebooks. When they are reapportioned, Office A receives 42, Office B receives 16, Office C receives 36, and Office D receives 52.

Answerer: Since Office B loses a Chromebook when the number available increases, this is an example of the Alabama paradox.

* + 1. A county with three towns has 30 garbage trucks to apportion. Attenborough receives 6 trucks, Breckenridge receives 8 trucks, and Cabbotsville receives 16 trucks, in proportion to their populations. When one additional truck is purchased, the reallocation results in 5 trucks for Attenborough, 9 for Breckenridge, and 8 for Cabbotsville though there has been no change in populations.

Answer: The Alabama paradox does occur because Attenborough loses a seat.

* + 1. A group of 60 dentists work for a company that runs five offices. The dentists have been apportioned to the offices by the number of patients. Office A receives 14 dentists, Office B receives 11, Office C receives 11, Office D receives 12, and Office E receives 12. When a new dentist joins the group, the new apportionment gives the following: A 14, B 11, C 11, D 12, and E 13.

Answer: No office lost a dentist, so this is not an example of the Alabama paradox.

In the following questions, determine if the scenario violates the population paradox. Justify your answer.

* + 1. A company with locations in three cities plans to give 200 achievement awards that shall be apportioned to the three cities by population. City A with 9,150 employees is allotted 61 awards, City B with 6,040 employees is allotted 40 awards, and City C with 14,810 employees is allotted 99 awards to distribute. Then it is discovered that the number of employees is out of date. The awards are reallocated based on the new populations: City A with 9,180; City B with 6,040; and City C with 14,930. It turns out that the apportionment remains the same.

Answer: No, this does not violate the population paradox because none of the cities lost an award.

* + 1. A soccer club must apportion soccer balls to the teams among four age brackets based on the number of teams in each bracket. Bracket U8 receives 32 balls, U12 receives 46 balls, U15 receives 29 balls, and U18 receives 25 balls. Mid-season, the balls are reapportioned. U8 has decreased by 10 percent, U12 has increased by 20 percent, U15 remains the same, and U18 has increased by 10 percent. The reapportionment gives 33 balls to U8, 46 balls to U12, 29 balls to U15, and 14 balls to U18.

Answer: This violates the population paradox because the population of U18 was growing but they lost a ball to U8, which had a decreasing number of teams.

* + 1. READ (Reading Education Assistance Dogs) trains and certifies therapy animals to serve in classrooms and help children develop a love of reading in a low-stress environment. Suppose that 20 therapy teams are apportioned to three schools based on their populations. When the population of School A decreases by 10 percent, the population of School B increases by 10 percent, and the population of School C remains constant, the apportionment remains the same.

Answer: Since no school lost a therapy team, the population paradox has not been violated.

In the following questions, determine if the scenario violates the new-states paradox. Justify your answer.

* + 1. A charity organization has 851 volunteers in Country A and 3449 volunteers in Country B. There are 43 lead organizers that must be apportioned to the two locations. Country A receives 9 while Country B receives 34. When the operations are expanded to Country C with 725 volunteers, 7 new lead organizers are added to the team. When the lead organizers are reapportioned, Country A receives 9, Country B receives 34, and Country C receives 7 lead organizers.

Answer: No, this does not violate the new-states paradox because the original countries retained the same number of lead organizers.

* + 1. A country has two states. There are 22 seats in the legislature. State A has 4 seats, and State B has 18 seats. When State C joins the union, the number of representatives is increased by five. Under the new apportionment, State A receives 3 seats, State B receives 19 seats, and State C receives 5 seats.

Answer: Yes, one of the original states, A, has lost a seat to the other original state, B.

* + 1. In the Garunga Solar System, there are four inhabited planets. Three of the planets are members of the United Association of Garungan Planets (UAGP). The UAGP has 67 seats. Planet Angluertka has 40 seats, planet Bangluertka has 27 seats, and planet Clangluertka has 13 seats. When planet Danggluertka decides to join the UAGP, 14 seats are added to accommodate them proportionately. The new allocation of seats gives 40 seats to Angluertka, 26 seats to Bangluertka, 14 seats to Clangluertka, and 14 seats to Danggluertka.

Answer: Yes, one of the original three planets, Bangluertka, has lost a seat to another original UAGP member, Clangluertka.

In the following questions, use the Hamilton method of apportionment to answer the questions.

* + 1. When the number of seats changed from 147 to 148, the standard quotas changed from A 44.24, B 17.35, C 37.12, and D 48.29 to A 45.54, B 17.47, C 37.37, and D 48.62.
       - 1. How did the increase in seats impact the apportionment?

Answer: It changed from A 44, B 18, C 37, and D 48 to A 45, B 17, C 37, and D 49, so State B lost a seat while States A and D each gained one.

* + - * 1. Is this apportionment an example of a paradox? Justify your answer.

Answer: Yes, this violates the Alabama paradox because an increase in the number of seats decreased state A’s quota.

* + 1. When the number of seats changed from 126 to 127, the standard quotas changed from A 9.57, B 29.49, C 33.89, D 28.43, and E 24.61 to A 9.65, B 29.72, C 34.16, D 38.66, and E 24.81.
       - 1. How did the increase in seats impact the apportionment?

Answer: It changed from A 10, B 29, C 34, D 28, and E 25 to A 9, B 30, C 34, D 29, and E 25. State A loses a seat, while State D gains a seat.

* + - * 1. Is this apportionment an example of a paradox? Justify your answer.

Answer: Yes, this violates the Alabama paradox because the increase in the number of seats decreased State A’s quota.

* + 1. When the number of seats changed from 25 to 26, the standard quotas changed from A 2.21, B 5.25, C 11.27, and D 6.27 to A 2.30, B 5.46, C 11.72, and D 6.52.
       - 1. How did the increase in seats impact the apportionment?

Answer: The only change was to increase State D from 6 to 7.

* + - * 1. Is this apportionment an example of a paradox? Justify your answer.

Answer: No. This type of scenario is susceptible to the Alabama paradox, but no state lost a seat, so it did not occur.

* + 1. When the number of seats changed from 25 to 26, the standard quotas changed from A 2.43, B 5.42, and C 8.15 to A 2.46, B 5.47, and C 8.07.
       - 1. How did the increase in seats impact the apportionment?

Answer: From A 3, B 5, and C 8 to A 2, B 6, and C 8.

* + - * 1. Is this apportionment an example of a paradox? Justify your answer.

Answer: Yes, this violates the Alabama paradox because the increase in the number of seats decreased State B’s quota.

* + 1. The house size is 18. When the population of State A increases by 11.76 percent, State B increases by 16.22 percent, and State C increases by 12.18 percent, the standard quotas change from A 2.46, B 6.00, and C 9.53 to A 2.41, B 6.25, and C 9.34.
       - 1. How did the change in populations impact the apportionment?

Answer: From A 2, B 6, and C 10 to A 3, B 6, and C 9.

* + - * 1. Is this apportionment an example of a paradox? Justify your answer.

Answer: Yes, this is an example of the population paradox because State C loses one seat to State A but the population growth of State C was greater than the population growth of State A.

* + 1. The house size is 100. When the population of State A increases by 20 percent, State B increases by 10 percent, State C increases by 30 percent, and the populations of States D, E, and F remain the same. The standard quotas change from A 12.50, B 25.00, C 9.38, D 18.75, E 12.50 and F 21.88 to A 13.91, B 25.51, C 11.30, D 17.39, E 11.59, and F 20.29.
       - 1. How did the change in populations impact the apportionment?

Answer: From A 12, B 25, C 9, D 19, E 13, and F 22 to A 14, B 26, C 11, D 17, E 12, and F 20.

* + - * 1. Is this apportionment an example of a paradox? Justify your answer.

Answer: No. This scenario is susceptible to the population paradox because the populations changed while the number of seats stayed the same. However, the states that lost seats had no population increase while all the other states had population increases and gained seats; so the population paradox did not occur.

* + 1. The house size is 17. When the population of State A increases by 12.73 percent, State B increases by 15.63 percent, and State C increases by 12.90 percent, the standard quotas change from A 2.53, B 5.90, and C 8.57 to A 2.51, B 5.99, and C 8.50.
       - 1. How did the change in populations impact the apportionment?

Answer: From A 2, B 6, and C 9, to A 3, B 6, and C 8.

* + - * 1. Is this apportionment an example of a paradox? Justify your answer.

Answer: Yes, this is an example of the population paradox because State C loses one seat to State A, but the population growth of State C was greater than the population growth of State A.

* + 1. The house size is 38 seats. When the population of State A increases by 6.60 percent, State B increases by 1.44 percent, and State C increases by 1.40 percent, the standard quotas change from A 18.82, B 14.65, and C 4.53 to A 19.59, B 14.29, and C 4.41.
       - 1. How did the change in populations impact the apportionment?

Answer: From A 19, B 15, and C 4 to A 19, B 14, and C 5.

* + - * 1. Is this apportionment an example of a paradox? Justify your answer.

Answer: Yes, this is an example of the population paradox because State B loses one seat to State C, but the population growth of State B was greater than the population growth of State C.

* + 1. The house size is 24 seats. When the population of State A increases by 28 percent, State B increases by 26 percent, and State C increases by 15 percent, the standard quotas change from A 3.38, B 6.32, and C 14.30 to A 3.63, B 6.67, and C 13.71.
       - 1. How did the change in populations impact the apportionment?

Answer: From A 4, B 6, and C 14 to A 3, B 7, and C 14.

* + - * 1. Is this apportionment an example of a paradox? Justify your answer.

Answer: Yes, this is an example of the population paradox because State B loses one seat to State C, but the population growth of State B was greater than the population growth of State C.

* + 1. The house size was 60. There were three states with standard quotas of A 4.18, B 15.38, and C 40.44. A fourth state was annexed, and the house size was increased to 65. The new standard quotas are A 4.17, B 15.33, C 40.32, and D 5.18.
       1. How did the additional state impact the apportionment?

Answer: From A 4, B 15, and C 41, to A 4, B 16, C 40, and D 5.

* + - 1. Is this apportionment an example of a paradox? Justify your answer.

Answer: Yes, this is an example of the new-states paradox because one of the original states, C, lost a seat because of the addition of a new state.

* + 1. The house size was 50. There were three states with standard quotas of A 9.41, B 24.42, and C 16.17. A fourth state was annexed, and the house size was increased to 66. The new standard quotas are A 9.36, B 24.30, C 16.09, and D 16.25.
       1. How did the additional state impact the apportionment?

Answer: From A 9, B 25, and C 16 to A 10, B 24, C 16, and D 16.

* + - 1. Is this apportionment an example of a paradox? Justify your answer.

Answer: Yes, this is an example of the new-states paradox because one of the original states, B, lost a seat because of the addition of a new state.

* + 1. The house size was 27. There were three states with standard quotas of A 6.39, B 11.40, and C 9.21. A fourth state was annexed, and the house size was increased to 35. The new standard quotas are A 6.38, B 11.37, C 9.19, and D 8.06.
       1. How did the additional state impact the apportionment?

Answer: From A 6, B 12, and C 9 to A 7, B 11, C 9, and D 8.

* + - 1. Is this apportionment an example of a paradox? Justify your answer.

Answer: Yes, this is an example of the new-states paradox because one of the original states, B, lost a seat because of the addition of a new state.

* + 1. The house size was 100. There were three states with standard quotas of A 26.09, B 30.43, and C 43.48. A fourth state was annexed, and the house size was increased to 122. The new standard quotas are A 26.14, B 30.50, C 43.57, and D 21.78.
       1. How did the additional state impact the apportionment?

Answer: From A 26 B 30, and C 44 to A 26, B 30, C 44, and D 22.

* + - 1. Is this apportionment an example of a paradox? Justify your answer.

Answer: No. This type of scenario is in susceptible to the new-states paradox, but none of the original states lost a seat, so the new-states paradox did not occur.

In the following questions, use the information in Figure 11.131 to answer the questions.

**Figure 11.131 Change in House Size**

|  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| **State** | **A** | **B** | **C** | **D** | **E** | **F** | **G** | **H** | **I** | **J** | **K** | **L** | **P** | | **Q** | **R** |
| **Population** | 624 | 1219 | 979 | 3462 | 7470 | 4264 | 5300 | 263 | 809 | 931 | 781 | 676 | 150 | | 250 | 350 |
| **Original House Size** | 38 | | | 204 | | | | 126 | | | | | | 50 | | |
| **Updated House Size** | 39 | | | 205 | | | | 127 | | | | | | 51 | | |

* + 1. Consider States A, B, and C.
       1. Determine the apportionment for States A, B, and C with the original house size using the Hamilton method.

Answer: A 8, B 17, C 13

* + - 1. Determine the apportionment for States A, B, and C with the updated house size using the Hamilton method.

Answer: A 9, B 17, C 13

* + - 1. Does the change in the house size and use of the Hamilton method cause the Alabama paradox? Explain your reasoning.

Answer: No. The increase in house size didn’t cause a state to lose a seat.

* + 1. Consider States D, E, F, and G.
       1. Determine the apportionment for States D, E, F, and G with the original house size using the Hamilton method.

Answer: D 35, E 74, F 42, G 53

* + - 1. Determine the apportionment for States D, E, F, and G with the updated house size using the Hamilton method.

Answer: D 34, E 75, F 43, G 53

* + - 1. Does the change in the house size and use of the Hamilton method cause the Alabama paradox? Explain your reasoning.

Answer: Yes, State D lost a seat when the house size was increased.

* + 1. Consider States H, I, J, K, and L.
       1. Determine the apportionment for States H, I, J, K, and L with the original house size using the Hamilton method.

Answer: H 10, I 29, J 34, K 28, L 25

* + - 1. Determine the apportionment for States H, I, J, K, and L with the updated house size using the Hamilton method.

Answer: H 9, I 30, J 34, K 29, L 25

* + - 1. Does the change in the house size and use of the Hamilton method cause the Alabama paradox? Explain your reasoning.

Answer: Yes, State H lost a seat when the house size was increased.

* + 1. Consider States P, Q, and R.
       1. Determine the apportionment for States P, Q, and R with the original house size using the Hamilton method.

Answer: P 10, Q 17, R 23

* + - 1. Determine the apportionment for States P, Q, and R with the updated house size using the Hamilton method.

Answer: P 10, Q 17, R 24

* + - 1. Does the change in the house size and use of the Hamilton method cause the Alabama paradox? Explain your reasoning.

Answer: No, an increase in house size didn’t cause a state to lose a seat.

In the following questions, use the information in Figure 11.132 to answer the questions.

**Figure 11.132 Change in State Populations**

|  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| **State** | **A** | **B** | **C** | **D** | **E** | **F** | **G** | **H** | **I** | **X** | **Y** | **Z** | **P** | **Q** | **R** | **S** |
| **Original Population** | 889 | 674 | 237 | 12,032 | 10,789 | 995 | 901 | 1683 | 3808 | 56 | 125 | 182 | 6,534 | 7,832 | 13,959 | 20,515 |
| **Updated Population** | 958 | 692 | 243 | 14,124 | 9,726 | 2,304 | 1156 | 2125 | 4369 | 63 | 141 | 213 | 6,534 | 7,810 | 13,992 | 21,164 |
| **Population Growth Rate** | 7.76% | 2.67% | 2.53% | % | % | % | 28.3% | 26.3% | 14.7% | % | % | % | % | % | % | % |
| **House Size** | 42 | | | 135 | | | 24 | | | 16 | | | 40 | | | |

* + 1. Calculate the population growth rates  for States D, E, and F. Give answer as a percentage rounded to one decimal place.

Answer: D 17.4 percent, E -9.9 percent, F 131.6 percent

* + 1. Calculate the population growth rates  for States P, Q, R, and S. Give answer as a percentage rounded to one decimal place.

Answer: P 0 percent, Q -0.2 percent, R 0.2 percent, S 3.2 percent

* + 1. Calculate the population growth rates  for States X, Y, and Z. Give answer as a percentage rounded to one decimal place.

Answer: X 12.5 percent, Y 12.8 percent, Z 17.0 percent

* + 1. Consider States A, B, and C.
       1. Determine the Hamilton apportionment for States A, B, and C with the original population.

Answer: A 21, B 16, C 5

* + - 1. Determine the Hamilton apportionment for States A, B, and C with the updated population.

Answer: A 21, B 15, C 6

* + - 1. Does the increase in population of States A, B, and C from the original population to the updated population and the use of the Hamilton method cause the population paradox? Explain your reasoning.

Answer: No, none of the states lost seats when the population grew.

* + 1. Consider States D, E, and F.
       1. Determine the Hamilton apportionment for States D, E, and F with the original population.

Answer: D 68, E 61, F 6

* + - 1. Determine the Hamilton apportionment for States D, E, and F with the updated population.

Answer: D 73, E 50, F 12

* + - 1. Does the increase in population and the use of the Hamilton method cause the population paradox? Explain your reasoning.

Answer: No, the only state to lose seats also had the lowest growth rate.

* + 1. Consider States G, H, and I.
       1. Determine the Hamilton apportionment for States G, H, and I with the original population.

Answer: G 4, H 6, I 14

* + - 1. Determine the Hamilton apportionment for States G, H, and I with the updated population.

Answer: G 3, H 7, I 14

* + - 1. Does the increase in population and the use of the Hamilton method cause the population paradox? Explain your reasoning.

Answer: Yes, State G had the highest growth rate, but they lost a seat while the other states gained seats or remained the same.

* + 1. Consider States X, Y, and Z.
       1. Determine the Hamilton apportionment for States X, Y, and Z with the original population.

Answer: X 2, Y 6, Z 8

* + - 1. Determine the Hamilton apportionment for States X, Y, and Z with the updated population.

Answer: X 3, Y 5, Z 8

* + - 1. Does the increase in population and the use of the Hamilton method cause the population paradox? Explain your reasoning.

Answer: Yes, X had a lower growth rate than Y, but Y lost a seat to X.

* + 1. Consider States P, Q, R, and S.
       1. Determine the Hamilton apportionment for States P, Q, R, and S with the original population.

Answer: P 5, Q 6, R 12, S 17

* + - 1. Determine the Hamilton apportionment for States P, Q, R, and S with the updated population.

Answer: P 5, Q 7, R 11, S 17

* + - 1. Does the increase in population and the use of the Hamilton method cause the population paradox? Explain your reasoning.

Answer: Yes, State R lost a seat while States P and Q with lower growth rates gained a seat or remained the same.

In the following questions, use the information in Figure 11.133 to answer the questions.

**Figure 11.133 New State Added**

|  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| **State** | **A** | **B** | **C** | **D** | **E** | **F** | **G** | **H** | **P** | **Q** | **R** | **K** | **L** | **M** | **T** | **U** | **V** | **W** |
| **Population** | 627 | 1,287 | 973 | 815 | 520 | 1,510 | 1,060 | 950 | 1,222 | 473 | 225 | 1,688 | 7,912 | 1,448 | 150 | 250 | 350 | 450 |
| **Original House Size** | 25 | | |  | 50 | | |  | 100 | |  | 48 | |  | 50 | | |  |
| **New House Size** | 32 | | | |  | | | |  | | |  | | |  | | | |

* + 1. Consider States S A, B, and C.
       1. Calculate the standard divisor based on the original house size.

Answer: 115.48

* + - 1. Use the Hamilton method to apportion the seats.

Answer: A 6, B 11, and C 8.

* + 1. Consider States S E, F, and G.
       1. Calculate the standard divisor based on the original house size.

Answer: 61.80

* + - 1. Use the Hamilton method to apportion the seats.

Answer: E 8, F 25, and G 17.

* + 1. Consider States P and Q.
       1. Calculate the standard divisor based on the original house size.

Answer: 16.95

* + - 1. Use the Hamilton method to apportion the seats.

Answer: P 72 and Q 28.

* + 1. Consider States K and L.
       1. Calculate the standard divisor based on the original house size.

Answer: 200

* + - 1. Use the Hamilton method to apportion the seats.

Answer: K 8 and L 40.

* + 1. Suppose that States A, B, and C annex State D and increase the house size proportionately.
       1. Calculate the standard divisor based on the new house size.

Answer: 115.69

* + - 1. Use the Hamilton method to reapportion the seats.

Answer: A 6, B 11, C 8, and D 7.

* + - 1. Does the new-states paradox occur? (Refer to question 35).

Answer: No because none of the original states lost a seat.

* + 1. Suppose that States E, F, and G annex State H and increase the house size proportionately.
       1. Determine the new house size, , that is necessary.

Answer: 

* + - 1. Calculate the standard divisor based on the new house size.

Answer: 62.15

* + - 1. Use the Hamilton method to reapportion the seats.

Answer: E 9, F 24, G 17, and H 15.

* + - 1. Does the new-states paradox occur? (Refer to question 36.)

Answer: Yes, State E lost a seat to State F as a result of the new state joining.

* + 1. Suppose that States P and Q annex State R and increase the house size proportionately.
       1. Determine the new house size, , that is necessary.

Answer: 

* + - 1. Calculate the standard divisor based on the new house size.

Answer: 16.99

* + - 1. Use the Hamilton method to reapportion the seats.

Answer: P 72, Q 28, and R 13.

* + - 1. Does the new-states paradox occur? (Refer to question 37.)

Answer: No because the original states did not lose seats.

* + 1. Suppose that States K and L annex State M and increase the house size proportionately.
       1. Determine the new house size, , that is necessary.

Answer: 

* + - 1. Calculate the standard divisor based on the new house size.

Answer: 200.87

* + - 1. Use the Hamilton method to reapportion the seats.

Answer: K 9, L 39, and M 7.

* + - 1. Does the new-states paradox occur? (Refer to question 38.)

Answer: Yes, original State L lost a seat.

* + 1. Suppose that States S T, U, and V annex State W and increase the house size proportionately.
       1. Calculate the standard divisor based on the original house size and only States T, U, and V.

Answer: 15

* + - 1. Use the Hamilton method to apportion the seats to T, U and V.

Answer: T 10, U 17, and V 23.

* + - 1. Determine the new house size when W is annexed.

Answer: 60

* + - 1. Calculate the standard divisor based on the new house size.

Answer: 15

* + - 1. Use the Hamilton method to reapportion the seats.

Answer: T 10, U 17, V 23, W 30

* + - 1. Does the new-states paradox occur? (Refer to part b.)

Answer: No because none of the states lose a seat.

* + 1. Suppose 24 seats are apportioned to States A, B, and C with populations of 16, 15, and 125 respectively. Then the populations of States A, B, and C change to 17, 15, and 126 respectively.
       1. Demonstrate that the population paradox occurs when the Hamilton method is used.

Answer: The original Hamilton apportionment is: A 3, B 2, and C 19. The reapportionment is: A 2, B 3, and C 19. The population paradox occurs when a state with a higher population growth rate loses seats while a state with a lower population growth rate retains or gains seats. In this case, A grew while B did not, yet A lost a seat to B.

* + - 1. Determine if the population paradox occurs when the Webster method is used. Justify your answer.   
         Answer: The original Hamilton apportionment is: A 2, B 2, and C 20. The reapportionment is: A 3, B 2, and C 19. The population growth rates were: A 6.25 percent, B 0 percent, and C 0.8%. percent The only state to gain a seat was A, which had the highest growth rate. The only state to lose a seat was B which had the lowest growth rate. So, the population paradox does not occur.
    1. Suppose that 10 seats are apportioned to States S A, B, and C with populations 6, 6, and 2 respectively. Then the number of seats is increased to 11. Demonstrate that the Alabama paradox occurs when the Hamilton method is used.

Answer: The initial apportionment is: A 4, B 4, C 2. The reapportionment is: A 5, B 5, C 1. State C loses a seat when the house size increases.

## [MODULE 5 SUMMARY]

### Key Terms

**apportionment paradox**

**Alabama Paradox**

**Reapportionment**

**population growth rate**

**population paradox**

**new-state paradox**

**Balinkski-Young Impossibility Theorem**

### Key Concepts

* Several surprising outcomes can occur when apportioning seats that voters may find unfair: Alabama paradox, population paradox, and new-state paradox.
* Apportionment methods are susceptible to apportionment paradoxes and may violate the quota rule.
* The Balinsky-Young Impossibility Theorem indicates that no apportionment can satisfy all fairness criteria.

### Formulas





Videos*<Note to author: if none, leave out>*

# CHAPTER SUMMARY

**Key Terms**

Majority

Plurality

Tie

Tiebreaker

Runoff election

Runoff voting system

Two round system

Hare method

Preference ranking

Ranked ballot

Ranked-choice voting

Instant runoff voting

Borda count method

Borda score

Divisive candidate

Compromise candidate

Condorcet voting methods

Pairwise comparison method

Condorcet candidate

Approval voting system

Approval voting ballot

majority criterion

tyranny of the majority

up-rank

down-rank

monotonicity criterion

Independence of Irrelevant Alternatives Criterion (IIA)

Cardinal voting system

Arrow’s Impossibility Theorem

apportion

apportionment problem

part to part ratio

proportional

representative democracies

states

seats

house size

state population

total population

standard divisor

state’s standard quota

apportionment paradox

Alabama Paradox

Reapportionment

population growth rate

population paradox

new-state paradox

Balinkski-Young Impossibility Theorem

**Key Concepts**

**Formulas**

**Videos**

[PROJECTS]

### The First Census - Challenges of Collecting Population Data

As you and your fellow founders write the Imaginarian constitution, you must also design systems to collect accurate information about the population of Imaginaria. Why is this important and what are the challenges you will face? Let’s find out! Research and answer each question. (Questions adapted from “Authorizing the First Census–The Significance of Population Data,” Census.gov, United States Census Bureau)

1. The First Congress laid out a plan for collecting this data in Chapter II, *An Act Providing for the Enumeration of the Inhabitants of the United States*, which was approved March 1, 1790. What group did Congress select to carry out the first enumeration? Why did they choose this group? What might be the advantages and disadvantages to this approach.
2. Choose at least two challenges faced by the U.S. government during the first enumeration and explain how the information gathered might have helped address them.
3. President James Madison was a U.S. Representative who participated in the census debate in the first congress. What were the main points of his remarks and how were they relevant to the overall debate about the first enumeration?
4. What issues do you think the U.S. Census Bureau encounters today as it continues to collect and process data about the U.S. population that might be significant to you and the other founders of Imaginaria?<END>

### [H2] The Party System - How Many Political Parties is Enough?

The electoral system of Imaginaria will likely involve multiple political parties. The way these parties interact with the system may be determined by the founders of the new democracy. Let’s explore the ways in which political parties interact with the governments of various democracies by researching and answering each of the following questions.

1. What concerns did the founders of the United States have about political parties? Have any of their concerns become a reality? Were political parties addressed in the U.S. Constitution? How did they become such a large part of politics in the U.S.? As a founder of Imaginaria, would you address political parties in your constitution?
2. What is frontloading? Does our current system of frontloading impact fair representation? Why do two small, racially homogeneous states hold their primaries first? Do you think this impacts the final results?
3. How does the interaction political parties and the electoral system in the United States differ from that of other countries? Give at least three specific examples. Of these three, which would you be most likely to use as a model for Imaginaria?<END>

### [H2] Technology and Voting—How has the Digital Age Impacted Elections?

Unlike most other countries, Imaginaria will be founded in the digital age. Let’s explore the impact this might have on how you choose to set up your electoral system. Research and answer each question.

1. Participation in an electoral system if very important. In what ways does the internet positively affect participation? In what ways does it negatively affect participation. What roll, if any, should government of Imaginaria play in tempering or promoting the impact of the internet?
2. Find at least three examples of governments that utilize internet voting around the world. What concerns have slowed the spread of this technology? What are the advantages and disadvantages of internet voting? Would you be in favor of internet voting for Imaginaria? Why or why not?

## Chapter Review

**Module 1 - Voting Methods**

1. In a plurality election, the candidates have the following vote counts: A 125, B 132, C 149, and D 112.

Answer: Candidate C has a plurality and wins the election.

**Figure 11.134 Sample Preference Summary**

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| **Options** | **A** | **B** | **C** | **D** | **E** |
| Candidate 1 | 1 | 3 | 3 | 1 | 3 |
| Candidate 2 | 2 | 1 | 1 | 2 | 4 |
| Candidate 3 | 3 | 4 | 2 | 4 | 1 |
| Candidate 4 | 4 | 2 | 4 | 3 | 2 |

Use Figure 11.4 to answer questions 2 and 3.

1. Which candidate has a plurality?

Answer: Candidate 2.

1. Does the plurality candidate have a majority?

Answer: No.

1. Determine the winner of the election by the Hare method based on the Sample Preference Summary in Figure 11.4.

Answer: Candidate 1 wins.

**Figure 11.135 Sample Preference Summary**

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| **Number of Ballots** | **10** | **20** | **15** | **5** |
| Option A | 1 | 4 | 3 | 4 |
| Option B | 2 | 3 | 4 | 2 |
| Option C | 4 | 2 | 1 | 3 |
| Option D | 3 | 1 | 2 | 1 |

Use Figure 11.135 to answer questions 5 and 6.

1. Use ranked-choice voting to determine the two options in the final round and the number of votes they each receive in that round.

Answer: Option C receives 15 votes and Option D receives 35 votes.

1. Is there a winning option? If so, which option? Justify your answer.

Answer: Option D is the winning option with 35 votes because 26 or more is a majority.

**Figure 11.136 Sample Summary of Ranked Ballots**

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| **Number of Ballots** | **100** | **80** | **110** | **105** | **55** |
| Candidate A | 1 | 1 | 4 | 4 | 2 |
| Candidate B | 2 | 2 | 2 | 3 | 1 |
| Candidate C | 4 | 4 | 1 | 1 | 4 |
| Candidate D | 3 | 3 | 3 | 2 | 3 |

Use Figure 11.136 to answer questions 7 and 8.

1. What are the Borda scores for each candidate?

Answer: A 650, B 850, C 645, and D 555.

1. Which candidate is the winner by the Borda count method?

Answer: Candidate B.

**Figure 11.137 Pairwise Comparison Matrix For Candidates U, V, W, X, and Y**

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| **Opponent**  **Runner** | **U** | **V** | **W** | **X** | **Y** |
| **U wins** | -- | UV 1 | UW 3 | UX 3 | UY 4 |
| **V wins** | VU 5 | -- | VW 6 | VX 4 | VY 1 |
| **W wins** | WU 3 | WV 0 | -- | WX 5 | WY 4 |
| **X wins** | XU 3 | XV 2 | XW 1 | -- | XY 6 |
| **Y wins** | YU 2 | YV 5 | YW 2 | YX 0 | -- |

Use Figure 11.137 to answer questions 9 and 10.

1. Calculate the points received by each candidate in pairwise comparison matrix.

Answer: U 2, V 3, W ,, X , and Y 1.

1. Determine the winner of the pairwise comparison election represented by matrix. If there is a winner, determine if the winner is a Condorcet candidate and explain your reasoning. If there is no winner, indicate this.

Answer: V is the winner. V is not a Condorcet candidate because V lost the pairwise matchup to Y.

**Figure 11.138 The Ladies of Big Bang Theory Vote on Rock, Paper, Scissors, Lizard, Spock!**

|  |  |  |  |
| --- | --- | --- | --- |
| **VOTERS** | **Penny** | **Bernadette** | **Amy** |
| Rock | Yes | No | No |
| Paper | Yes | Yes | No |
| Scissors | Yes | Yes | Yes |
| Lizard | No | No | No |
| Spock | Yes | No | Yes |

Use Figure 11.138 to answer question 11.

1. The ladies of the Big Bang Theory decide to hold their own approval voting election to determine the best option in Rock, Paper, Scissors, Lizard, Spock. Use the summary of their approval ballots to determine the number of votes for each candidate. Determine the winner, or state that there is none.

Answer: Rock 1, Paper 2, Scissors 3, Lizard 0, Spock 2. The winner is scissors.

**Figure 11.139 Summary of Sample Approval Voting Ballots for A, B, and C**

|  |  |  |  |
| --- | --- | --- | --- |
| **Percentage of vote** | **40%** | **35%** | **25%** |
| Candidate A | 1 | 3 | 2 |
| Candidate B | 2 | 1 | 3 |
| Candidate C | 3 | 2 | 1 |

Use Figure 11.139 to answer the following questions.

1. Which candidate is the winner by the ranked-choice method?

Answer: Candidate A wins.

1. Suppose that you used the approval method and each voter approved their top two choices. Which candidate is the winner by the approval method?

Answer: Candidate B wins.

1. Which candidate is the winner by the Borda count method?

Answer: Candidate B wins.

**Module 2 - Fairness in Voting Methods**

1. In a Borda count election, the candidates have the following Borda scores: A 1245, B 1360, and C 787. Candidate A received 55% of the first place rankings. Identify which fairness criteria, if any, are violated by characteristics of the described voter profile in this Borda count election. Explain your reasoning.

Answer: Candidate A is the majority candidate, but candidate B won the election. This violates the majority criterion.

**Figure 11.140 Sample Ballot Summary 5**

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| **Number of Ballots** | **8** | **10** | **12** | **4** |
| Option A | 1 | 3 | 2 | 1 |
| Option B | 3 | 1 | 4 | 4 |
| Option C | 4 | 2 | 1 | 2 |
| Option D | 2 | 4 | 3 | 3 |

Use Figure 11.140 to the following questions.

1. Determine Borda score for each candidate, and the winner of the election using the Borda count method.

Answer: A 70, B 38, C 64, and D 32. The winner is A.

1. Is there a majority candidate? If so, which candidate?

Answer: No, there is no majority candidate.

1. Does the Borda method election violate the majority criterion? Justify your answer.

Answer: No, it does not violate the majority criterion.

1. In a Borda count election, the candidates have the following Borda scores: A 15, B 11, C 12, and D 16. The pairwise matchup points for the same voter profiles would have been A 2, B 0, C 1, and D 3. Identify which fairness criteria, if any, are violated by characteristics of the described voter profile in this Borda election. Explain your reasoning.

Answer: Candidate D won Borda count election as well as Condorcet candidate.

**Figure 11.141 Sample Ballot Summary 5**

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| **Number of Ballots** | **8** | **10** | **12** | **4** |
| Option A | 1 | 3 | 2 | 1 |
| Option B | 3 | 1 | 4 | 4 |
| Option C | 4 | 2 | 1 | 2 |
| Option D | 2 | 4 | 3 | 3 |

Use Figure 11.141 to the following questions.

1. Determine the winner of the election using the ranked-choice method.

Answer: Option C wins.

1. If the four voters in the last column rank C ahead of A, which candidate wins by the ranked-choice method?

Answer: Option B wins.

1. Does this ranked-choice election violate the monotonicity criterion? Explain your reasoning.

Answer: Yes, Option C was the winner, but an increase in rank by some voters caused C to lose the election.

**Figure 11.142 Sample Ballot Summary 6**

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| **Number of Ballots** | **15** | **12** | **9** | **3** |
| Option A | 1 | 3 | 3 | 2 |
| Option B | 2 | 2 | 1 | 1 |
| Option C | 3 | 1 | 2 | 3 |

Use Figure 11.142 to answer the following questions.

1. Determine the winner of the election by the Borda method.

Answer: Option B wins.

1. Does this Borda method election violate the IIA? Why or why not?

Answer: No, if. If Option A is eliminated, then Option B wins, and if C is eliminated, B wins.

1. Which of the ranked voting methods in this chapter, if any, meets the majority criterion, the head-to-head criterion, the monotonicity criterion, and the irrelevant alternatives criterion?

Answer: None of them according to Arrow’s Impossibility Theorem.

**Module 3 - Standard Divisors, Standard Quotas & the Apportionment Problem**

1. Identify the states, the seats, and the state population (the basis for the apportionment) in the given scenario: The reading coach at an elementary school has 52 prizes to distribute to their students as a reward for time spent reading.

Answer: States are the students, the prizes are the seats, and the time spent reading is the state population.

1. Use the given information to find the standard divisor to the nearest hundredth. Include the units. The total population is 2,235 automobiles, and the number of seats is 14 warehouses.

Answer: The standard divisor is 159.64 automobiles per warehouse.

1. Use the given information to find the standard quota. Include the units. The state population is eight residents in a unit, and the standard divisor is 1.75 residents per parking space.

Answer: The standard quota is 4.57 parking spaces.

### Module 4—Apportionment Methods

1. Which of the four apportionment methods discussed in this module does not use a modified divisor?

Answer: Hamilton

**Figure 11.143 Sample Standard Quotas for Scenario X**

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
|  | **State A** | **State B** | **State C** | **State D** | **State E** | **State F** | **Total Seats** |
| **Scenario X** | 17.63 | 26.62 | 10.81 | 16.01 | 13.69 | 15.24 | 100 |

1. Determine the Hamilton apportionment for Scenario X in Figure 11.144.

Answer: A 17+1=18, B 26, C 10+1=11, D 16, E 13+1=14, and F 15

**Figure 11.144 Sample Standard Quotas for Apportionment Methods X**

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
|  | **State A** | **State B** | **State C** | **State D** | **State E** |
| **Standard Quota** | 1.67 | 3.33 | 5.00 | 6.67 | 8.33 |
| **Apportionment Method X** | 2 | 2 | 5 | 7 | 9 |

1. Does the apportionment resulting from method X in Figure 11.144 satisfy the quota rule? Why or why not?

Answer: The apportionment does not satisfy the quota rule because the upper and lower quotas for State B are 3 and 4, but State B received 2 seats.

For the following questions, use this information and Figure 11.145. In Wakanda, the domain of the Black Panther, King T’Challa, has six fortress cities. In Wakandan, the word “birnin” means “fortress city.” King T’Challa has found 111 Vibranium artifacts that must be distributed among the fortress cities of Wakanda. He has decided to apportion the artifacts based on the number of residents of each birnin.

**Figure 11.145 Populations and Standard Quotas by Major Wakandan Cities**

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
| Fortress  Cities | Birnin  Djata (D) | Birnin T’Chaka (T) | Birnin Zana (Z) | Birnin S’Yan (S) | Birnin Bashenga (B) | Birnin Azzaria (A) | Total |
| Residents | 26,000 | 57,000 | 27,000 | 18,000 | 64,000 | 45,000 | 237,000 |
| Standard Quota | 12.18 | 26.70 | 12.65 | 8.43 | 29.98 | 21.08 | 111 |

1. Does the Jefferson method result in an apportionment that satisfies or violates the quota rule in this scenario?

Answer: The apportionment satisfies the quota rule.

1. Find the modified upper quota for each state using a modified divisor of 2250. Is the sum of the modified quotas too high, too low, or equal to the house size?

Answer: D 12, T 26, Z 13, S 9, B 29, and A 21. The sum of the modified quotas is too low.

1. Use the Adams method to apportion the artifacts. Determine if it is necessary to modify the divisor. If so, indicate the value of the modified divisor.

Answer: D 12, T 26, Z 13, S 9, B 30, and A 21. It is necessary to modify the divisor. Modified divisors from 2193 to 2206 will result in this apportionment.

1. Does the Adams method result in an apportionment that satisfies or violates the quota rule in this scenario?

Answer: The apportionment satisfies the quota rule.

1. Use the Webster method to apportion the artifacts. Determine if it is necessary to modify the divisor. If so, indicate the value of the modified divisor.

Answer: D 12, T 27, Z 13, S 8, B 30, and A 21. The standard divisor works, so it is not necessary to modify the divisor.

1. Does the Webster method result in an apportionment that satisfies or violates the quota rule in this scenario?

Answer: The quota rule is satisfied.

1. Which of the four methods of apportionment from this module are the residents of Birnin S’Yan likely to prefer? Justify your answer.

Answer: The residents of Birnin S’Yan are likely to prefer the Adams method of apportionment because they receive nine artifacts instead of eight.

1. Does the change from a standard divisor to a modified divisor tend to change the number of seats for larger or smaller states more?

Answer: Larger

1. Which of the four apportionment methods—Jefferson, Adams, Hamilton, or Webster—satisfies the quota rule?

Answer: Hamilton

### Module 5—Fairness in Apportionment Methods

1. A city purchased five new firetrucks and apportioned them among the existing fire stations. Although your neighborhood fire station has the same proportion of the city’s firetrucks as before the new ones were purchased, it now has one fewer. Is this scenario an example of a quota rule violation, the Alabama paradox, the population paradox, the new-states paradox, or none of these?

Answer: Alabama paradox

1. When the number of seats changed from 25 to 26, the standard quotas changed from A 2.21, B 5.25, C 11.27, and D 6.27 to A 2.30, B 5.46, C 11.72, and D 6.52.
   1. How did the increase in seats impact the apportionment?

Answer: The only change was to increase D from 6 to 7.

* 1. Is this apportionment an example of a paradox? Justify your answer.

Answer: No. This type of scenario is susceptible to the Alabama paradox, but no state lost a seat so the paradox did not occur.

1. The school resources officers in a county were reapportioned based on the most recent census. The number of students at Chapel Run Elementary went up while the number of students at Panther Trail Elementary went down, but Chapel Run now has 1 fewer resources officers while Panther Trail has one more than it did previously. Is this scenario an example of a quota rule violation, the Alabama paradox, the population paradox, the new-states paradox, or none of these?

Answer: Population paradox

1. The house size is 24 seats. When the population of A increases by 28 percent, B increases by 26 percent, and C increases by 15 percent, the standard quotas change from A 3.38, B 6.32, and C 14.30 to A 3.63, B 6.67, and C 13.71.
   1. How did the change in populations impact the apportionment?

Answer: From A 4, B 6, and C 14 to A 3, B 7, and C 14.

* 1. Is this apportionment an example of a paradox? Justify your answer.

Answer: Yes, this is an example of the population paradox because State B loses one seat to State C, but the population growth of State B was greater than the population growth of State C.

1. When the city of Cocoa annexed an adjacent unincorporated community, the number of seats on the city council was increased to maintain the standard ratio of citizens to seats, but one existing community of Cocoa still lost a seat on the city council to another existing community of Cocoa when the new community was added. Is this scenario an example of a quota rule violation, the Alabama paradox, the population paradox, the new-states paradox, or none of these?

Answer: New-states paradox

1. The house size was 27. There were three states with standard quotas of A 6.39, B 11.40, and C 9.21. A fourth state was annexed, and the house size was increased to 35. The new standard quotas are A 6.38, B 11.37, C 9.19, and D 8.06.
   * + 1. How did the additional state impact the apportionment?

Answer: From A 6, B 12, and C 9 to A 7, B 11, C 9, and D 8.

* + - 1. Is this apportionment an example of a paradox? Justify your answer.

Answer: Yes, this is an example of the new-states paradox because one of the original states, B, lost a seat because of the addition of a new state.

In the following questions, suppose 11 seats are apportioned to States A, B, and C with populations of 50, 129, and 181 people respectively. Then the populations of States A, B, and C change to 57, 151, and 208 respectively.

1. Demonstrate that the population paradox occurs when the Hamilton method is used.

Answer: The population of State A increased by 14 percent. The population of State B increased by 17 percent. The original apportionment is: A 1, B 4, and C 6. The second apportionment is: A 2, B 4, and C 5. State A gained a seat while seat by State B did not.

1. Demonstrate that the population paradox does not occur when the Jefferson method is used. Justify your answer.

Answer: The original apportionment is: A 1, B 4, and C 6, and the second apportionment is the same; the population paradox didn’t occur.

1. Demonstrate that the population paradox does not occur when the Adams method is used. Justify your answer.

Answer: The original apportionment is A 2, B 4, and C 5. The second apportionment is the same so the population paradox didn’t occur.

1. Demonstrate that the population paradox does not occur when the Webster method is used. Justify your answer.

Answer: The original apportionment is A 2, B 4, and C 5. The second apportionment is the same so the population paradox didn’t occur.

# CHAPTER TEST

**Module 1 – Voting Methods**

**Figure 12.145 Sample Preference Summary A**

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
| **Number of Ballots** | **28** | **5** | **30** | **5** | **16** | **16** |
| **Option L** | 3 | 2 | 1 | 1 | 2 | 3 |
| **Option R** | 1 | 1 | 3 | 2 | 3 | 2 |
| **Option E** | 2 | 3 | 2 | 3 | 1 | 1 |

**Figure 12.146 Sample Preference Summary B**

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| **Number of Ballots** | **12** | **17** | **15** | **13** |
| Option A | 1 | 2 | 4 | 3 |
| Option B | 3 | 1 | 3 | 2 |
| Option C | 4 | 3 | 1 | 4 |
| Option D | 2 | 4 | 2 | 1 |

Use preference summary B from Figure 12.146 to answer the following exercises.

1. Determine the winner of the election by plurality.

Answer: Option B

1. Determine the Borda scores for each candidate to determine the winner by Borda count method.

Answer: A 83, B 104, C 62, D 93; Winner is Option B.

1. Create and analyze a pairwise comparison matrix based on the preference summary to determine the winner of the election by pairwise comparison.

Answer:

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
|  | **over A** | **over B** | **over C** | **over D** | **Points** |
| **A wins** | -- | 12 | 42 | 29 | 2 |
| **B wins** | 45 | -- | 42 | 17 | 2 |
| **C wins** | 15 | 15 | -- | 32 | 1 |
| **D wins** | 28 | 40 | 25 | -- | 1 |

There is no winner. There is a tie between Option A and Option B.

Use Sample Preference Summary A from Figure 1.45 to answer exercise 4.

1. Use ranked-choice voting to determine the winner of the election.

Answer: Option L

**Module 2 – Fairness in Voting Methods**

In exercises the following exercises, identify which fairness criteria, if any, are violated by characteristics of the described voter profile. Explain your reasoning

1. In a Borda count election, the candidates have the following Borda scores: A 1345, B 1260, C 685. Candidate B received 51% of the first-place rankings.

Answer: Candidate B is the majority candidate, but candidate A won the election. This violates the majority criterion.

1. In a plurality election, the candidates have the following percentages of first place votes: A 25, B 21, C 30, D 24. The pairwise matchup points for the same voter profiles would have been A 3, B 0, C 2, D 2.

Answer: Candidate C won the plurality election, but candidate A was a Condorcet candidate. This violates the Condorcet criterion.

**Figure 12.147 Sample Preference Summary A**

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| **Number of Ballots** | **13** | **14** | **11** | **12** |
| Option A | 2 | 1 | 3 | 3 |
| Option B | 3 | 2 | 4 | 1 |
| Option C | 4 | 4 | 1 | 2 |
| Option D | 1 | 3 | 3 | 4 |

Use the Sample Preference Summary A in Figure 12.145 to answer the following exercises.

1. Determine the winner by ranked-choice voting if two of the voters in the second column up-rank the original winner. Refer to Exercise 4. Which fairness criterion, if any, is violated?

Answer: Option E wins, violating the monotonicity criteria.

1. Determine the winner by ranked-choice voting if candidate R is removed from the election. Refer to Exercise 4. Which fairness criterion, if any, is violated?

Answer: Option E wins, violating IIA criterion.

**Module 3 – Standard Divisors, Standard Quotas & the Apportionment Problem**

The incorporated town of Orange Grove consists of two subdivisions: The Oaks with 1,254 residents, and The Villages with 10,746 residents. A council with 100 members supervises the municipality’s operations with representation proportionate to the number of residents. Use this information to answer the following exercises.

1. Identify the states, the seats, and the state population (the basis for apportionment) in the given scenario.

Answer: The states are the subdivisions, the seats are the council members, and the state populations are the number of residents in a given subdivision.

1. Determine the standard divisor for the apportionment

Answer: 120

1. Determine each state’s standard quota rounded to two decimal places.

Answer: The Oaks 10.45, The Villages 89.55

**Module 4 – Apportionment Methods**

Air Force administration wanted to distribute 27 aircraft across 6 bases based on the number of qualified pilots stationed at those bases. The standard quota is 2.2963. The standard quotas for each base are listed in Figure 11.148. Use this information to complete the following exercises.

**Figure 11.148 Standard Quota for A-10C Thunderbolt II Aircraft**

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
| **Air Force Base** | **(A) Alpha** | **(B) Bravo** | **(C) Charlie** | **(D) Delta** | **(E) Echo** | **(F) Foxtrot** |
| **Pilots** | 13 | 12 | 5 | 16 | 7 | 9 |
| **Standard Quota** | 5.66 | 5.23 | 2.18 | 6.97 | 3.05 | 3.92 |

1. Determine the states’ lower quotas and the states’ upper quotas.

Answer: Lower quotas are A 5, B 5, C 2, D 6, E 3, F 3. Upper quotas are A 6, B 6, C 3, D 7, E 4, F 4

1. Use Adams’s method to apportion the aircraft.

Answer: A 6, B 5, C 2, D 7, E 3, F 4

1. Use Jefferson’s method to apportion the aircraft.

Answer: A 6, B 5, C 2, D 7, E 3, F 4

**Figure 11.149 Apportionment of Hawaiian School Districts by Various Methods**

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| **County** | **Hawaii** | **Honolulu** | **Kalawao** | **Kauai** | **Maui** |
| **Lower Quota** | 87 | 424 | 0 | 31 | 72 |
| **Upper Quota** | 88 | 425 | 1 | 32 | 73 |
| **Jefferson** | 87 | 425 | 1 | 31 | 72 |
| **Adams** | 88 | 422 | 1 | 32 | 73 |
| **Webster** | 87 | 424 | 1 | 31 | 73 |

The apportionment of 616 schools to 5 Hawaiian counties by various methods is displayed in Figure 11.149. Use this information to answer the following exericises.

1. Apportionment by which methods, if any, fail to satisfy the quota rule? Explain your reasoning.

Answer: An apportionment violates the quota rule if the number of seats received by a state is neither the upper quota nor the lower quota. In this case, the Adams apportionment violated the quota rule because the county of Honolulu only received 422.

**Module 5 – Fairness in Apportionment Methods**

As in exercises 9 – 11, the incorporated town of Orange Grove consists of two subdivisions: The Oaks with 1,254 residents, and The Villages with 10,746 residents. A council with 100 members supervises the municipality’s operations. The Hamilton method was used to apportion the council seats. The Oaks has 10 seats on the council, while The Villages has 90 seats. The council votes to annex an unincorporated subdivision called The Lakes with a population of 630. They plan to increase the size of the council to maintain the ratio of seats to residents such that the new council will have 100 seats plus the number of seats given to The Lakes.

Use this information to answer the following exercises.

1. What is the standard divisor from the original apportionment?

Answer: 120

1. What is the new house size?

Answer: 105

1. Use the Hamilton method to reapportion the seats.

Answer: The Oaks 10, The Villages 89, and The Lakes 5.

1. Is the reapportionment an example of the new-states paradox? If so, how?

Answer: Yes, The Villages loses a seat to The Oaks.

In exercises 20 and 21, determine if the reapportionment violates the Alabama paradox, the population paradox, or neither. Justify your answer.

1. States A, B, C, and D received 21, 25, 26, and 28 seats respectively. When the population remains the same, but house size is increased, the reapportionment is A 20, B 26, C 27, and D 29.

Answer: The Alabama paradox occurred, because the number of seats increased, the population remained the same, but state A lost a seat.

1. States A, B, C, and D received 21, 25, 26, and 28 seats respectively. When the house size remains the same, the population of state A increased, the population of state B decreased, and the populations of states C and D remained the same, the reapportionment is A 20, B 26, C 26, and D 28.

Answer: The population paradox occurred because state A lost a seat to state B despite A having a higher population growth rate.