OpenTURNS Users Day 2025

Bayesian estimation of the Gutenberg-Richter law
Application to the Elsinore fault

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Plan de la présentation

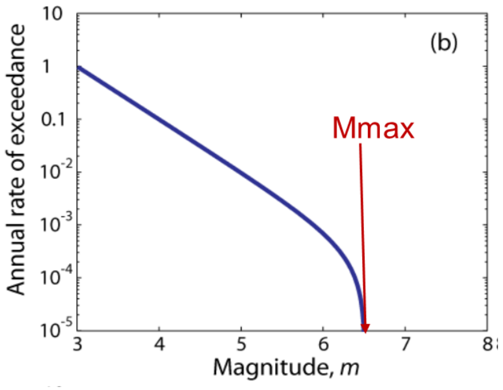
Bayesian estimation of the Gutenberg-Richter law
Application to the Elsinore fault

1. Geological data taken into account
2. Bayesian estimation
3. Normalisation constant
4. Elsinore fault, Californie

Bayesian estimation of the Gutenberg-Richter law

How to take into account geological data: Youngs & Coppersmith (1985) and Brune (1968)

PSHA (Probabilistic Seismic Hazard Assessment) analyses rely on a frequency-magnitude relationship for each source: .



❖ **Gutenberg-Richter law:**
Magnitude distribution: Exponential

❖ **Point Poisson process:**
Count distribution: Poisson

$\boxed{\text{Log } E[N(m)] = a - b\,m}$

Parameters to estimate

This relationship is adjusted purely statistically on the basis of catalog data. The Gutenberg-Richter law does not incorporate any physical information.

When a source is explicitly associated with a known active fault, it is possible to mobilize geological and paleoseismological observations to constrain the parameters of this relationship.

The work of Youngs and Coppersmith (1985), for example, makes it possible to use the mean slip rate along a fault to constrain its behavior over long periods (not covered by catalogs).

We also use the work of Brune (1968), who relates slip rate to the rate of release of seismic energy.

Bayesian estimation of the Gutenberg-Richter law

Modélisation probabiliste des séismes

Earthquakes are modeled by a Poisson point process :

- the annual distribution law is a Poisson distribution ,
- The magnitude distribution is a truncated exponential at m_{max}

❖ **Gutenberg-Richter law:**

$$\boxed{\text{Log } E[N(m)\} = a - b\, m}$$

Geological information

Catalogu data

Link between Slip rate and energy release rate (Brune 1968):

µ Shear modulus, A_f : fault plane surface, S: mean slip rate

$$\dot{M}_0^T = \mu A_f S$$

Link between Energy release rate and Poisson process intensity :

$$\dot{M}_0^T = \int_{m_0}^{m_{max}} \lambda_T(m,t) M_0(m)\, dm$$

$$\dot{M}_0^T = \frac{\mu_T b}{1 - e^{-b(m_{max}-m_0)\log 10}} \frac{M_0^{max}}{(b-c)} \left[e^{c(m_0 - m_{max})\log 10} - e^{-b(m_{max}-m_0)\log 10} \right]$$

Bayesian estimation of the Gutenberg-Richter law

Relationship Geological data – Short term data (catalogu)

Log E[N(m)} = a – b m

✓ Other parameter: **(a, b) => (μ_T , β) = (E[N(m_0)], b log(10))**

$$a = \log_{10}(\mu A_f S) - \log_{10}(K(b))$$

$$\mu_T = h(S)\ell(b)$$

$$\beta = b\log(10)$$

$$K(b) = \left(\frac{b}{b-c}\right)\left(\frac{10^{(c-b)m_0+d} - 10^{(c-b)m_{max}+d}}{1-10^{-bm_{max}}}\right)$$

$$\begin{aligned} h(S) &= S \\ \ell(b) &= \mu A_f\left(\frac{b-c}{b}\right)\left(\frac{10^{-bm_0} - 10^{-bm_{max}}}{10^{(c-b)m_0+d} - 10^{(c-b)m_{max}+d}}\right) \end{aligned}$$

Bayesian estimation of the Gutenberg-Richter law

Frequentist Likelihood of the Poisson point process: we consider the point Poisson process with intensity Λ of density λ such that $\Lambda(\mathcal{R}) = \int_{\mathcal{R}} \lambda(s)\, ds$, for any borelian R. We denote by $\mathcal{E}_N = (N, \{x_1, \ldots, x_N\})$ a realisation of the process inside R. Then the likelihood of process is: $\mathcal{L}(\Lambda|\mathcal{E}_N) = \exp(-\Lambda(\mathcal{R})) \prod_{i=1}^{N} \lambda(x_i)$

Log likelihood of the earthquakes Poisson point process:

$$\log \mathcal{L}(\beta, \mu_T|\mathcal{E}) = -D\mu_T + N \log \mu_T + N \log \beta - N \log \alpha(\beta) - \beta \sum_{k=1}^{N} m_k$$

$$\boxed{\beta = b \log(10)} \qquad \boxed{\mu_T = h(S)\ell(b)}$$

$$\log \mathcal{L}(b, S|\mathcal{E}) = -Dh(S)\ell(b) + N \log(h(S)\ell(b)) + N \log(b \log 10) - N\alpha(b \log(10)) - b \log 10 \sum_{k=1}^{N} m_k$$

The maximum likelihood estimator maximizes the likelihood with respect to the parameters.

Bayesian estimation of the Gutenberg-Richter law

Bayesian model likelihood:

$$\mathcal{L}_{bay}(b, S|\mathcal{E}) = \mathcal{L}(b, S|\mathcal{E})\pi_0(b, S)$$

Prior: expert information

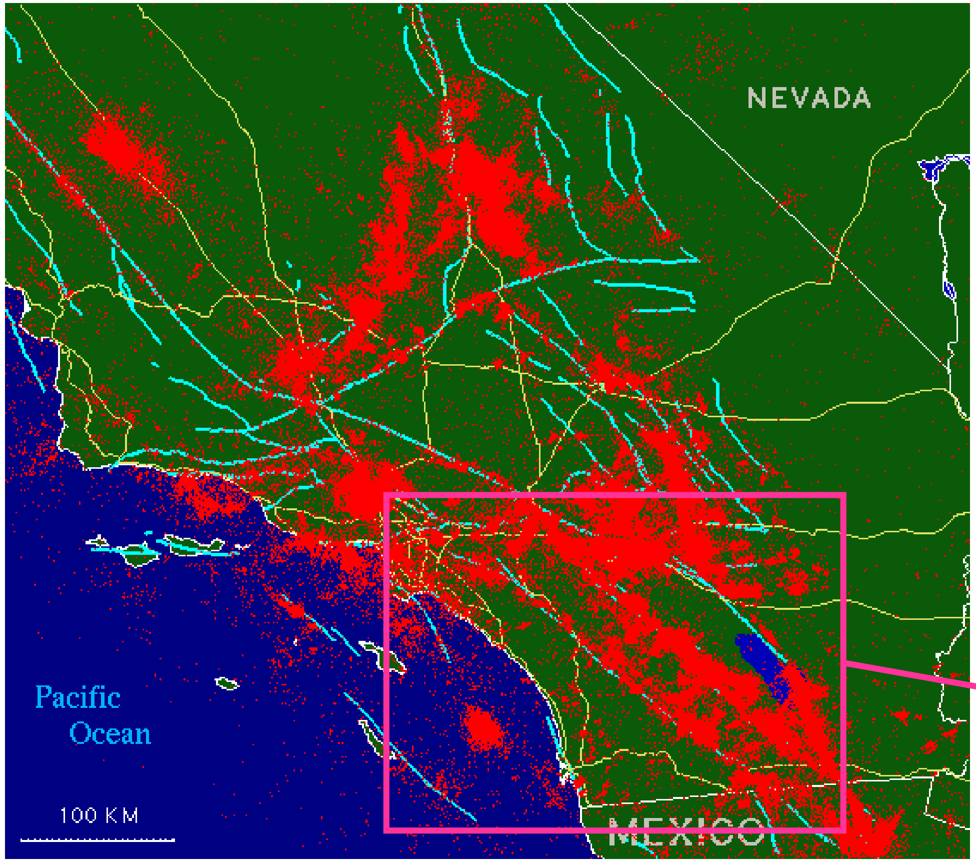
$$\boxed{\pi_f(b, S) = \frac{\mathcal{L}_{bay}(b, S|\mathcal{E})}{\int \mathcal{L}_{bay}(b, S|\mathcal{E})\, db\, dS}}$$

Normalisation constant to compute

- Essentially exact posterior laws
- No need for MCMC-type samplers (difficult to set up) ➔ Guaranty to get iid samples

Elsinore fault (California)

Elsinore fault (California)



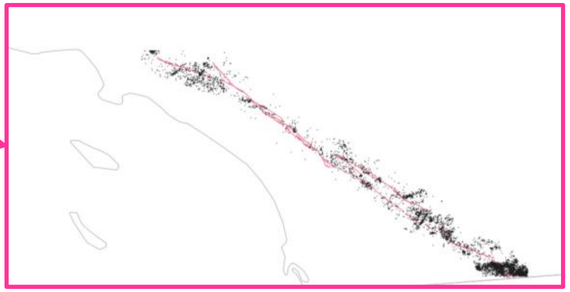
Sismicité du Sud de la Californie

Map of seismic activity recorded in Southern California between 1932 and 1996

Period: 01/01/1981 to 09/27/2024

Magnitudes >= 2, N = 3464 earthquakes

Only earthquakes located within a 0.1-degree buffer zone surrounding the surface fault line.



Sismicité de la faille Elsinore (Californie)

Bayesian estimation of the Gutenberg-Richter law

Normalisation constant to compute

$$I_f = \int \mathcal{L}_{bay}(b, S|\mathcal{E})\, db\, dS$$

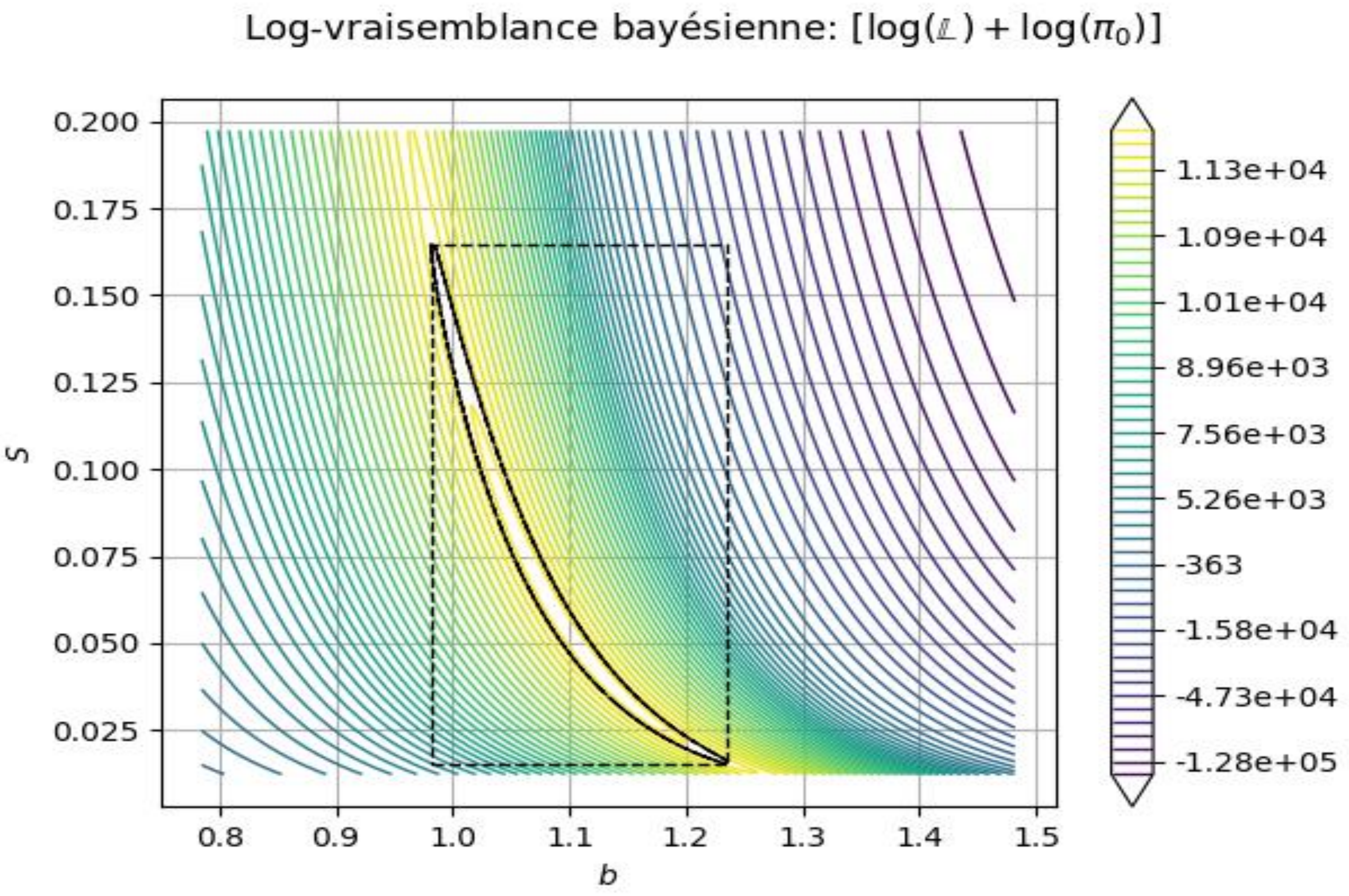
Difficult task?? ➔ YES!

$\mathcal{L}_{bay}(b, S|\mathcal{E})$

Max = e^{11400}
Min = $e^{-130000}$

➔ Machine representation problems 'the largest number that can be represented by a machine is $2.e^{308}$)

High gradients ➔ The mass of the function to be integrated is located in a small domain.



Bayesian estimation of the Gutenberg-Richter law

Calcul de la constante de normalisation

$$I_f = \int \mathcal{L}_{bay}(b, S|\mathcal{E})\, db\, dS$$

Three steps: Step 1: . Reduction of the likelihood integration domain ➔ (log) likelihood level line

- The integration domain is reduced to the smallest possible bounded domain S, outside of which the function to be integrated is negligible in relation to the values taken in S: this step facilitates the work of the integration algorithms.

$$\forall (b, S) \notin \mathcal{S},\ \mathcal{L}_{bay}(b, S|\mathcal{E}) \leq \varepsilon \max_{b,S} \mathcal{L}_{bay}(b, S|\mathcal{E}) \qquad \varepsilon = 10^{-10}$$

$$\Downarrow$$

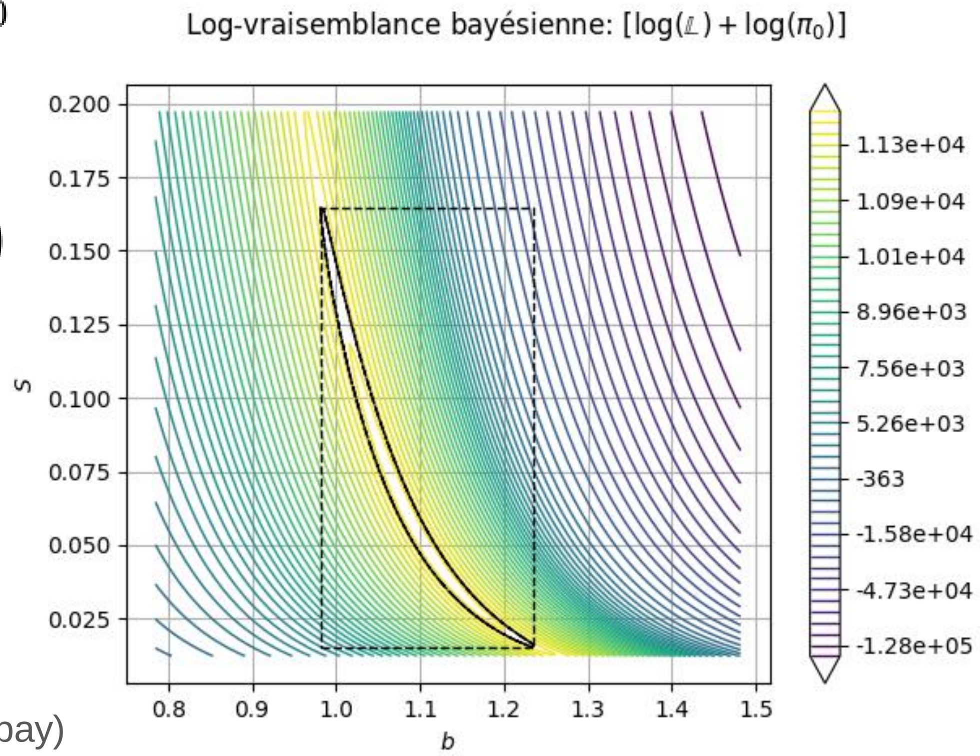
$$\forall (b, S) \notin \mathcal{S},\ \log \mathcal{L}_{bay}(b, S|\mathcal{E}) \leq \log \varepsilon + \max_{b,S} \log \mathcal{L}_{bay}(b, S|\mathcal{E})$$

$$\Downarrow$$

$$\mathcal{S} \subset [b_{inf}, b_{sup}]_{post} \times [S_{inf}, S_{sup}]_{post}$$

- We use the OptimizationPtroblem class to get the max
- We use the LevelSet class to get the iso-line
- We use the draw() method of an OpenTURNS function

```
>>> levelsetS = LevelSet(logLbay, Greater(), logEpsilon+maxlogLbay)
```



Bayesian estimation of the Gutenberg-Richter law

Calcul de la constante de normalisation

$$I_f = \int \mathcal{L}_{bay}(b, S|\mathcal{E})\, db\, dS \Longrightarrow I_f = \int_{\mathcal{S}} \mathcal{L}_{bay}(b, S|\mathcal{E})\, db\, dS$$

Step2: Renormalisation and Calibration ok K

- We reduce the variations of the function to be integrated to the domain S

$$I_f = \left(e^{KN}\right) J_f \quad \text{et} \quad J_f = \int_{\mathcal{S}} e^{N\left(\frac{\log \mathcal{L}_{bay}(b,S|\mathcal{E})}{N} - K\right)}\, db\, dS$$

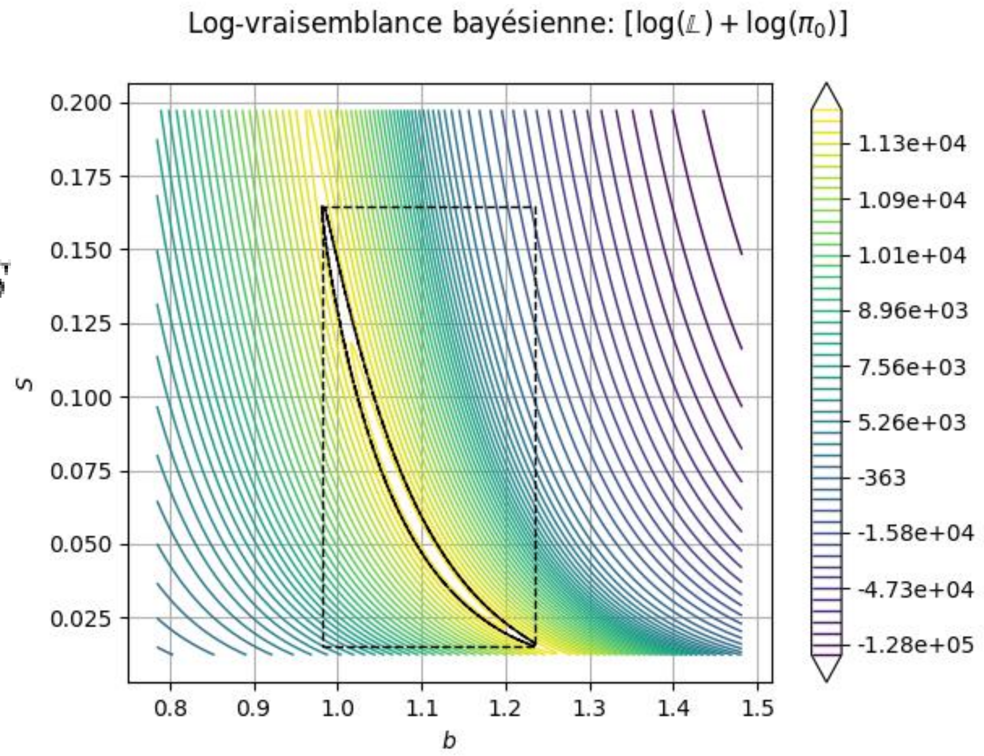
Plusieurs possibilités:

$$K_{moy} = \frac{1}{\mathcal{V}ol(\mathcal{S})} \int_{\mathcal{S}} \frac{1}{N} \mathcal{L}_{bay}(b, S|\mathcal{E})\, db\, dS$$

But fluctuations around the mean can be very large!

$$K_{max} = \max_{\mathcal{S}} \left(\frac{1}{N} \mathcal{L}_{bay}(b, S|\mathcal{E}) \right)$$

- We use OpenTURNS optimization algorithms and the OptimizationProblem class



For the Elsinore study, we considered the max:

K = 3,306, N = 3464

Bayesian estimation of the Gutenberg-Richter law

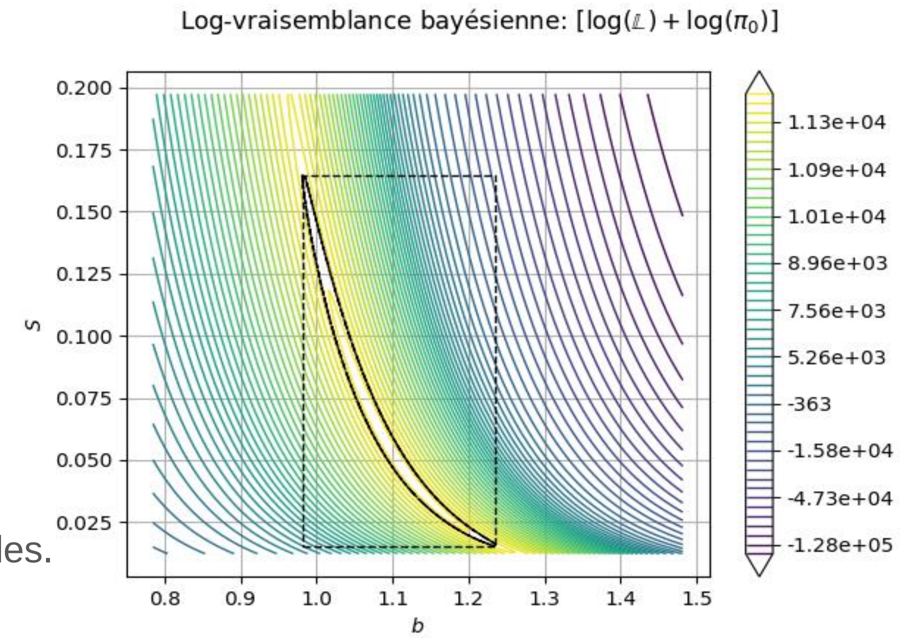
Calcul de la constante de normalisation

$$I_f = \int \mathcal{L}_{bay}(b, S|\mathcal{E})\, db\, dS \Longrightarrow I_f = \int_{\mathcal{S}} \mathcal{L}_{bay}(b, S|\mathcal{E})\, db\, dS$$

Step 3: Computation of J_f

❖ Calculation is straightforward

$$I_f = \left(e^{KN}\right) J_f \quad \text{et} \quad J_f = \int_{\mathcal{S}} \boxed{e^{N\left(\frac{\log \mathcal{L}_{bay}(b,S|\mathcal{E})}{N} - K\right)}}\, db\, dS \Longrightarrow F(b,S)$$



Plusieurs possibilités:

❖ Quadrature formulas on $[b_{inf}, b_{sup}]_{post} \times [S_{inf}, S_{sup}]_{post}$

And the Gauss-Legendre quadrature formula with the class GaussLegendre and its method integrate

❖ We consider $\mathcal{S}$ that we mesh with triangles and we use a quadrature formula to calculate the integral on each of the triangles.

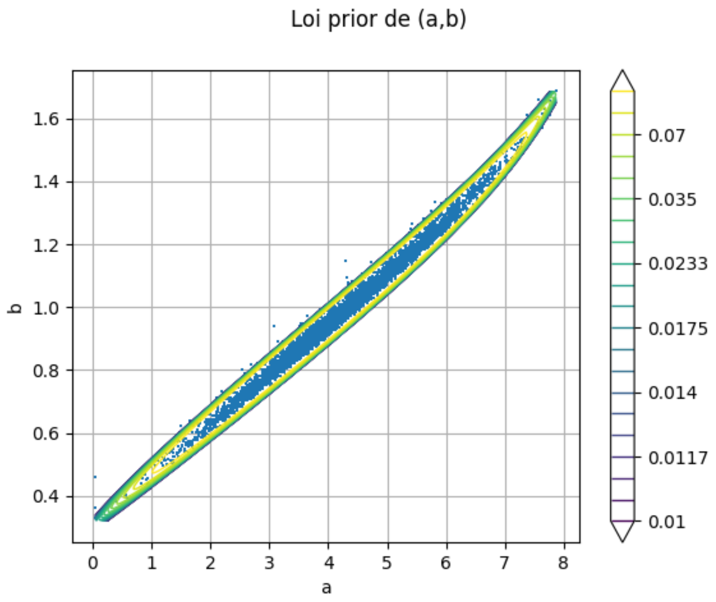
```
>>> levelsetS = LevelSet(logF, Greater(), logEpsilon)
>>> box = Interval([binf, Sinf], [bsup, Ssup])
>>> mesh = LevelSetMesher([Nmesh]*2).build(levelsetS, box)

>>> F = ComposedFunction(SymbolicFunction('x', 'exp(x)'), logF)
>>> J = SimplicialCubature().integrate(F, mesh)[0]
```

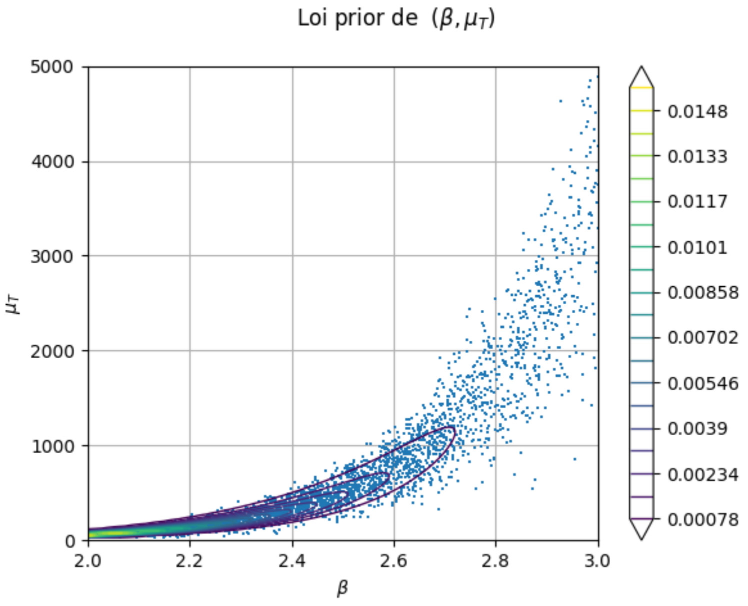
Log I = KN + log J_f

$$K = 3.306$$
$$I = e^{11443}$$
$$[b_{inf}, b_{sup}]_{post} = [0.98, 1.24]$$
$$[S_{inf}, S_{sup}]_{post} = [0.0149, 0.166]$$

Elsinore fault (California): Prior (b,S)

- $S \sim \mathcal{NT}(0.4, \sigma_S = 0.1, [0, +\infty[)$ en cm/an
- $b \sim \mathcal{NT}(1, \sigma_b = 0.2, [0, +\infty[)$
- (b, S) independent





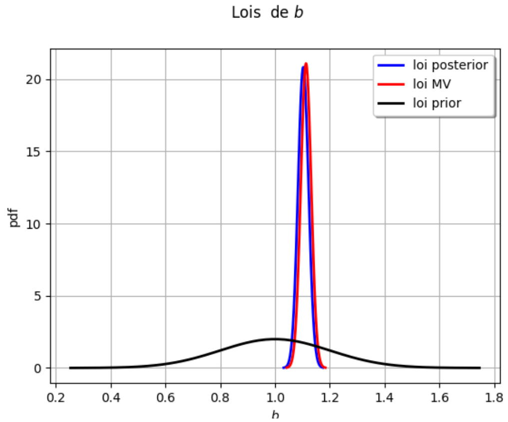
$$a = \log_{10}(\mu A_f S) - \log_{10}(K(b))$$

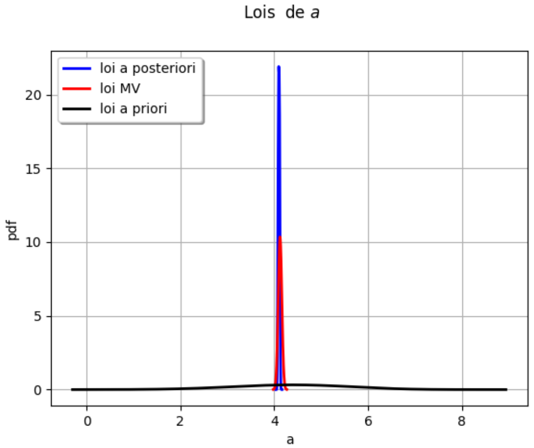
$$\mu_T = h(S)\ell(b)$$

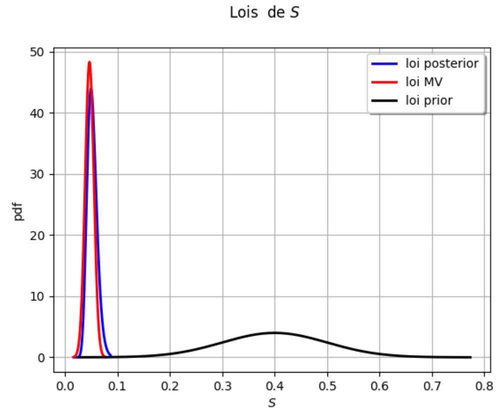
Prior distributions of a and μ_T are essentially exacte
(thanks to the distribution algebra of OpenTURNS)

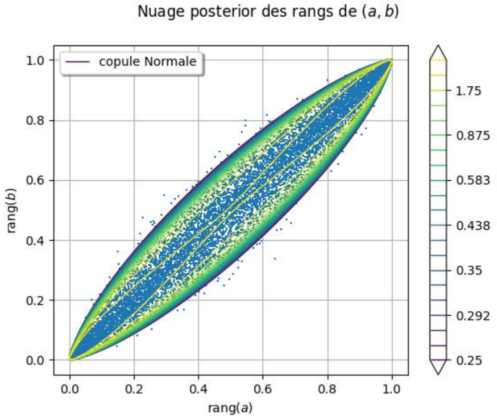
Elsinore fault (California)

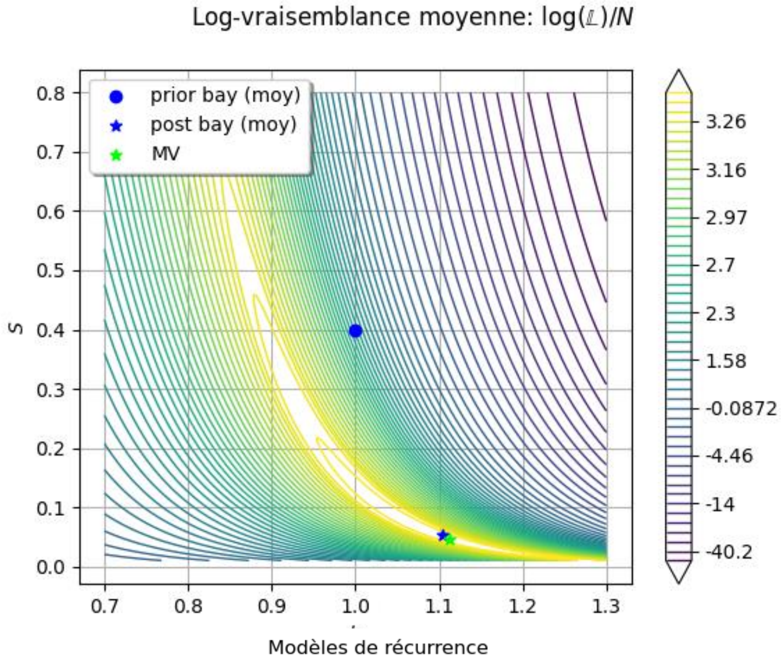
Results: From the whole catalogu (3464 seisms)

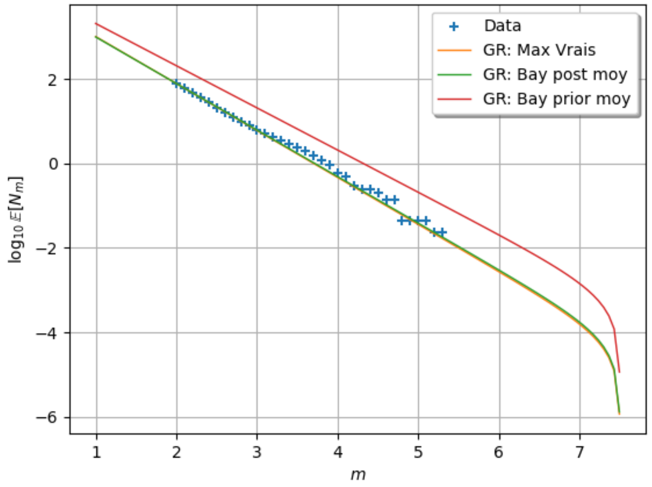








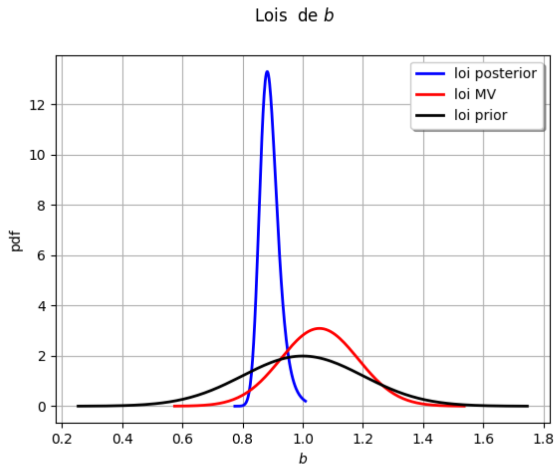


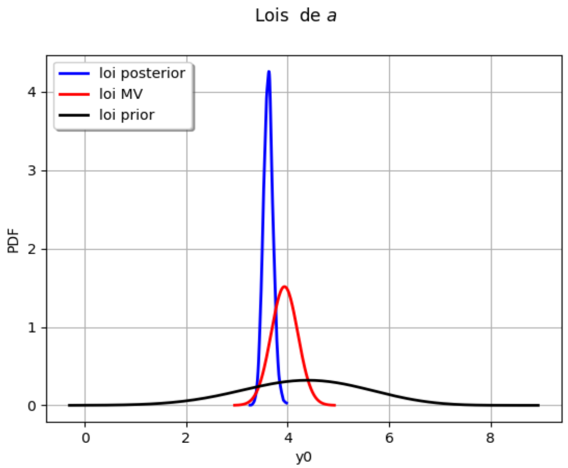


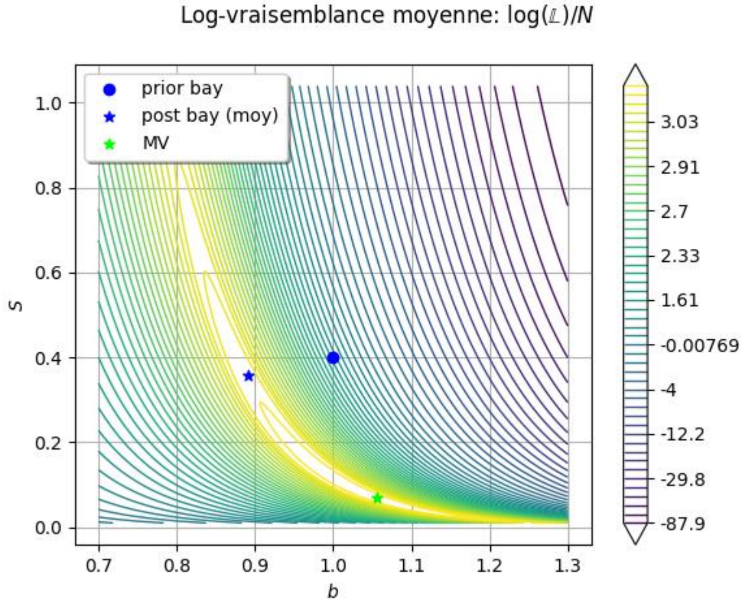
The prior tells a different story from the data .... but the posterior laws are essentially those of the data.

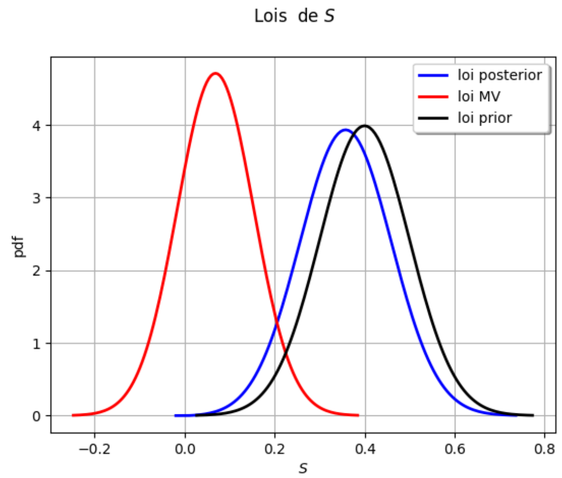
Elsinore fault (California)

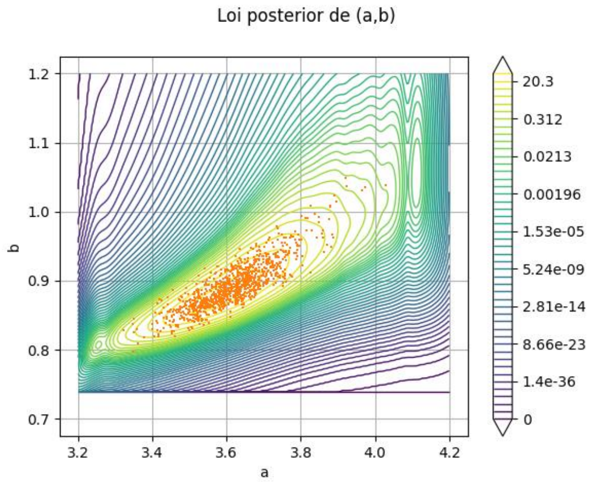
Results: Year 2017 only (67 seisms)

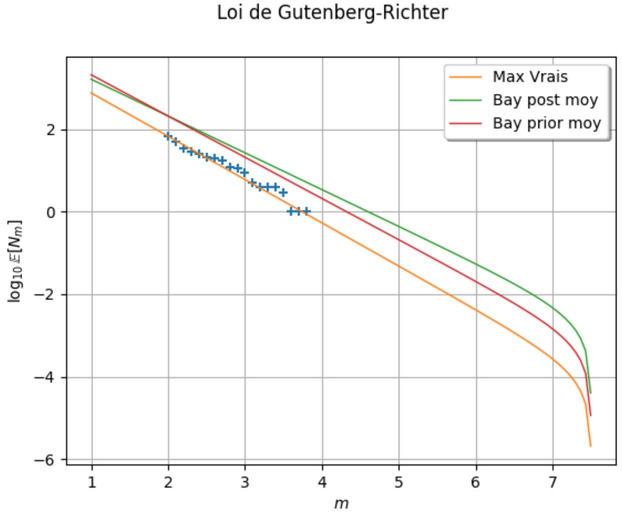












The prior tells a different story from the data ....
The posterior laws are influenced by expert knowledge,,,