Chapter 0 - Introduction (Skipped)

Chapter 1 - Regular Languages

Theorem 1.39 Every NFA has an equivalent DFA .... pg. 55
Theorem 1.45 - 1.49 Regular languages closure properties .... pg. 59
Theorem 1.54 A language is regular iff some regular expression describes it .... pg. 66
Example 1.73 - 1.77 Non-regular languages examples .... pg. 80-82

- $\{0^n 1^n \mid n \geq 0\}$
- $\{w \mid w \text{ has an equal number of 0s and 1s}\}$
- $\left\{ww \mid w \in \{0,1\}^*\right\}$
- $\left\{1^{n^2} \mid n \geq 0\right\}$
- $\left\{0^i 1^j \mid i > j\right\}$

NOTE More examples are given under EXERCISES *and* PROBLEMS*, pg. 83 - 93 in the book.*

Chapter 2 - Context-Free Languages

Definition 2.8 **Chomsky normal form** of context-free grammars .... pg. 109
Example 2.14 - 2.18 Context-free languages examples .... pg. 114-116

- $\{0^n 1^n \mid n \geq 0\}$
- $\left\{a^i b^j c^k \mid i, j, k \geq 0 \text{ and } i = j \text{ or } i = k\right\}$
- $\left\{ww^r \mid w \in \{0,1\}^*\right\}$

Theorem 2.20 A language is context free iff some PDA recognizes it .... pg. 117
Example 2.36 - 2.38 Non-context-free languages examples (diagrams) .... pg. 128-129

- $\{a^n b^n c^n \mid n \geq 0\}$
- $\left\{a^i b^j c^k \mid 0 \leq i \leq j \leq k\right\}$
- $\left\{ww \mid w \in \{0,1\}^*\right\}$

NOTE More examples are given under EXERCISES *and* PROBLEMS*, pg. 154 - 159.*

Chapter 3 - The Church-Turing Thesis

Definition 3.3 - 3.6 **Turing machine**, recognizable and decidable definitions .... pg. 168
Example 3.7 - 3.12 Turing machines examples (with diagrams) .... pg. 171-175

- $\left\{0^{2^n} \mid n \geq 0\right\}$
- $\left\{w\#w \mid w \in \{0,1\}^*\right\}$
- $\left\{a^i b^j c^k \mid i \times j = k \text{ and } i, j, k \geq 1\right\}$
- $\left\{\#x_1\#x_2\#\cdots\#x_l \mid \text{each } x_i \in \{0,1\}^* \text{ and } x_i \neq x_j \text{ for each } i \neq j\right\}$
- $\{w \mid w \text{ contains an equal number of 0s and 1s}\}$ *(proof on pg. 191)*

Theorem 3.13 - 3.19 Turing machines variants equivalence .... pg. 176-180

- Multitape ≡ single-tape
- Nondeterministic ≡ deterministic

Figure 3.20 Description of an **enumerator** .... pg. 180
Theorem 3.21 A language is Turing-recognizable iff it's enumerated .... pg. 181
Hilbert's problem $\{p \mid p \text{ is a polynomial with an integral root }\}$ is recognizable, but not decidable .... pg. 184

Example 3.23 $\{\langle G\rangle \mid G$ is a connected undirected graph $\}$ is decidable .... pg. 185

EXERCISES Useful theorems, unproved unless otherwise noted .... pg. 188-190

Note: items with an A symbol instead of a • symbol have a proof on page 191

- 2-PDAs are more powerful than 1-PDAs
- 3-PDAs are NOT more powerful than 2-PDAs
- A Variant: write-once Turing machine
- Variant: Turing machine with doubly infinite tape
- Variant: Turing machine with left reset
- Variant: Turing machine with stay put instead of left
- A language can be recognized by a deterministic queue automaton iff the language is Turing-recognizable
- A Decidable languages closure properties
- A Recognizable languages closure properties
- *NOTE Turing-recognizable languages are NOT closed under complement! Very useful.*

STUDY GUIDE's theorems. *Note: page numbers refer to the study guide, not to the book.*

Example Enumerator of $\left\{0^k 1^k \mid k \geq 0\right\}$ .... pg. 17

Exercise 1.9a Every finite language is decidable .... pg. 19

Chapter 4 - Decidability

Theorem 4.1 - 4.23 Common languages & decidabilities .... pg. 194-210

- A_{DFA} is decidable
- A_{NFA} is decidable
- A_{REX} is decidable
- EQ_{DFA} is decidable
- A_{CFG} is decidable
- E_{CFG} is decidable
- Every context-free language is decidable
- EQ_{CFG} is NOT decidable
- A_{TM} is NOT decidable
- $\overline{A_{\text{TM}}}$ is NOT Turing-recognizable

Figure 4.10 **regular** $\subset$ **context-free** $\subset$ **decidable** $\subset$ **Turing-recognizable** .... pg. 201

The diagonalization method .... pg. 202

Theorem 4.22 A language is decidable iff it is recognizable and co-recognizable ..... pg. 209

EXERCISES All of the languages below are **decidable** .... pg. 211-214

Note: items with an A symbol instead of a • symbol have a proof on page 214.

- ALL_{DFA}
- $\{\langle G\rangle \mid G$ is a CFG that generates $\varepsilon\}$
- $INFINITE_{\text{DFA}} = \{\langle A\rangle \mid A$ is a DFA and $L(A)$ is an infinite language$\}$
- $INFINITE_{\text{PDA}} = \{\langle M\rangle \mid M$ is a PDA and $L(M)$ is an infinite language$\}$
- A $\{\langle M\rangle \mid M$ is a DFA that doesn't accept any string containing an odd numbers of 1s $\}$
- $\{\langle R, S\rangle \mid R$ and S are regular expressions and $L(R) \subseteq L(S)\}$
- A $\{\langle G\rangle \mid G$ is a CFG over $\{0,1\}$ and $1^* \cap L(G) \neq \phi\}$

- $\{\langle G\rangle \mid G$ is a CFG over $\{0,1\}$ and $1^* \subseteq L(G)\}$
- $\{\langle M\rangle \mid M$ is a DFA that accepts w^R whenever it accepts $w\}$

A Say that an NFA is ***ambiguous*** if it accepts some string along two different computation branches. $AMBIG_{\text{NFA}} = \{\langle N\rangle \mid N$ is an ambiguous NFA$\}$ is decidable

A $BAL_{\text{DFA}} = \{\langle M\rangle \mid M$ is a DFA that accepts strings with an equal number of 0s and 1s$\}$

- $PAL_{\text{DFA}} = \{\langle M\rangle \mid M$ is a DFA that accepts palindromes$\}$
- $\{\langle M\rangle \mid M$ is a DFA that accepts strings with more 1s than 0s$\}$
- $\{\langle G,x\rangle \mid G$ is a CFG and x is a substring of some $y \in L(G)\}$
- $\{\langle G,k\rangle \mid G$ is a CFG and L(G) contains exactly k strings where $k \geq 0$ or $k = \infty\}$

STUDY GUIDE's theorems. *Note: page numbers refer to the study guide, not to the book.*

Chapter 5 - Reducibility

- $HALT_{\text{TM}}$ is undecidable
- E_{TM} is undecidable
- $REGULAR_{\text{TM}}$ is undecidable
- EQ_{TM} is undecidable
- $A_{\mathbf{LBA}}$ **is decidable**
- E_{LBA} is undecidable
- ALL_{CFG} is undecidable

NOTE Gray theorems are not in the syllabus

Note: items with an A symbol instead of a • symbol have a proof on page 242-244.

- EQ_{CFG} is undecidable
- EQ_{CFG} is co-Turing-recognizable

A A_{TM} is NOT mapping reducible to E_{TM}

A $\leq_m$ is a transitive relation

A If A is recognizable and $A \leq_m \overline{A}$, then A is decidable

- $\{\langle M \rangle \mid M$ is a TM that accepts w^R whenever it accepts $w\}$ is undecidable
- The following three languages are undecidable:
  - $\{\langle M, w \rangle \mid M$ is a two-tape TM that writes a nonblank symbol on its second tape when it is run on $w\}$
  - $\{\langle M \rangle \mid M$ is a two-tape TM that writes a nonblank symbol on its second tape when it is run on some input$\}$
  - $\{\langle M \rangle \mid M$ is a single-tape TM that writes a blank symbol over a nonblank symbol when it is run on some input$\}$
- $\{\langle M \rangle \mid M$ has a *useless state*$\}$ is undecidable
- $\{\langle M, w \rangle \mid M$ attempts to move its head left when it's on the left-most tape cell$\}$ is undecidable
- $\{\langle M, w \rangle \mid M$ attempts to move its head left at any point when run on $w\}$ is decidable
- *BB (busy beaver function)* is not a computable function
- PCP is decidable over the unary alphabet $\Sigma = \{1\}$
- PCP is undecidable over the binary alphabet $\Sigma = \{0, 1\}$
- SPCP *(silly PCP)* is decidable
- $AMBIG_{\text{CFG}}$ is undecidable (hint: reduction from PCP)
- There exists an undecidable subset of $\{1\}^*$
- A is Turing-recognizable iff $A \leq_m A_{\text{TM}}$
- A is decidable iff $A \leq_m 0^*1^*$
- Let $J = \{w \mid$ either $w = 0x$ for some $x \in A_{\text{TM}}$, or $w = 1y$ for some $y \in \overline{A_{\text{TM}}}\}$. Neither J nor $\overline{J}$ is Turing-recognizable
- There exists an undecidable language B where $B \leq_m \overline{B}$
- A_{2DFA} *(two-headed finite automaton)* is decidable
- E_{2DFA} is undecidable
- $A_{\text{2DIM-DFA}}$ *(two-dimensional finite automaton)* is undecidable
- The following three languages are undecidable *(prove using Rice's theorem)*:

  A $INFINITE_{\text{TM}} = \{\langle M \rangle \mid M$ is a TM and $L(M)$ is an infinite language$\}$
  - $\{\langle M \rangle \mid M$ is a TM and $1011 \in L(M)\}$
  - $ALL_{\text{TM}} = \{\langle M \rangle \mid M$ is a TM and $L(M) = \Sigma^*\}$
  - $OVERLAP_{\text{CFG}} = \{\langle G, H \rangle \mid G$ and H are CFGs where $L(G) \cap L(H) \neq \phi\}$
  - $PREFIX-FREE_{\text{CFG}} = \{\langle G \rangle \mid G$ is a CFG where $L(G)$ is a prefix-free$\}$
- $NECESSARY_{\text{CFG}} = \{\langle G, A \rangle \mid A$ is a necessary variable in $G\}$ is recognizable and undecidable
- $MIN_{\text{CFG}} = \{\langle G \rangle \mid G$ is a minimal CFG$\}$ is recognizable and undecidable

Chapter 6 - Advanced Topics in Computability Theory (Skipped)

Chapter 7 - Time Complexity

Note: items with an A symbol instead of a • symbol have a proof on page 329.

- P and NP closure properties *(NP star proof on pg. 329)*
- *CONNECTED* = $\{\langle G\rangle \mid G$ is a connected undirected graph$\}$ is in P
- *TRIANGLE* = $\{\langle G\rangle \mid G$ contains a 3-clique$\}$ is in P
- $ALL_{\text{DFA}} \in$ P, $EQ_{\text{DFA}} \in$ P
- *ISO* = $\{\langle G, H\rangle \mid G$ and H are isomorphic graphs$\}$ is in NP
- *MODEXP* = $\{\langle a, b, c, p\rangle \mid a, b, c, p \in \mathbb{N}^+$ such that $a^b \equiv c \pmod{p}\}$ is in P
- *PERM-POWER* = $\{\langle p, q, t\rangle \mid p = q^t$ where p and q are permutations on $\{1, \ldots, k\}\}$ is in P
- *UNARY-SSUM* $\in$ P *(definition on pg. 323 ex. 7.17)*
- **If P = NP**, then every language $A \in$ P, except $A = \phi$ and $A = \Sigma^*$, is NP-complete
- *PRIMES* = $\{m \mid m$ is a prime number in binary$\}$ is in NP
- Proving that *PATH* is not NP-complete would prove P $\neq$ NP
- *SPATH* $\in$ P, and *LPATH* is NP-complete *(definitions on pg. 324 ex. 7.21)*
- *DOUBLE-SAT* = $\{\langle\phi\rangle \mid \phi$ has at least two satisfying assignments$\}$ is NP-complete

A *HALF-CLIQUE* = $\{\langle G\rangle \mid G$ has a half clique$\}$ is NP-complete

- $CNF_2 \in$ P
- CNF_3 is NP-complete
- *MAX-CUT* = $\{\langle G, k\rangle \mid G$ has a cut of size k or more$\}$ is NP-complete
- *3COLOR* = $\{\langle G\rangle \mid G$ is colorable with 3 colors$\}$ is NP-complete

A *SOLITAIRE* = $\{\langle G\rangle \mid G$ is a winnable game configuration$\}$ is NP-complete

- $\{\langle p\rangle \mid p$ is a polynomial in several variables having an integral root$\}$ is ***NP-hard***
- If P = NP, then these problems can be solved in polynomial time:
  - Satisfying a boolean formula
  - Factoring integers

  A Finding a largest clique in an undirected graph *(proof on pg. 330)*
- Minimizing DFAs can be done in polynomial time *(Details on pg. 327 ex. 7.42)*
- P $\neq$ NP implies that NFAs cannot be minimized in polynomial time
- *2SAT* = $\{\langle\phi\rangle \mid \phi$ is a satisfiable 2cnf-formula$\} \in$ P
- TIME$(f(n))$ contains only the regular languages when $f(n) = o(n \log n)$
- P is closed under homomorphism iff P = NP

STUDY GUIDE's theorems. *Note: page numbers refer to the study guide, not to the book.*

- *INDEPENDENT-SET*
- *SET-COVER*
- *k-COLORING*
- *k-SET-PACKING*

Chapter 8 - Space Complexity

- *TQBF*
- *FORMULA-GAME*
- *GG* (ez wp)

Note: items with an A symbol instead of a • symbol have a proof on page 361.

- $\text{SPACE}(f(n))$ is the same whether we define this class using a single-tape TM or double-tape read-only TM when $f(n) \geq n$
- PSPACE closure properties

A $A_{\text{DFA}} \in \text{L}$

- Any PSPACE-hard language is also NP-hard

A NL closure properties

- $EQ_{\text{REX}} \in \text{PSPACE}$
- If every NP-hard language is also PSPACE-hard, then PSPACE = NP
- A_{LBA} is PSPACE-complete
- The following languages are in L:
  - The language of properly nested parentheses and brackets
  - $MULT = \{a\#b\#c \mid a, b, c \in \mathbb{N} \text{ and } a \times b = c\}$
  - $ADD = \{\langle x, y, z\rangle \mid x, y, z > 0 \text{ and } x + y = z\}$
  - $PAL\text{-}ADD = \{\langle x, y\rangle \mid x, y > 0 \text{ and } x + y \text{ is a palindrome}\}$
  - $\{\langle G\rangle \mid G \text{ is an undirected graph that contains a simple cycle}\}$
- $BIPARTITE = \{\langle G\rangle \mid G \text{ is a bipartite graph}\}$ is in NL
- The following languages are NL-complete:
  - $STRONGLY\text{-}CONNECTED = \{\langle G\rangle \mid G \text{ is a strongly connected graph}\}$
  - $BOTH_{\text{NFA}} = \{\langle M_1, M_2\rangle \mid M_1, M_2 \text{ are NFAs and } L(M_1) \cap L(M_2) \neq \phi\}$

- A_{NFA} E_{NFA} *2SAT*

A $\{\langle G\rangle \mid G$ is a directed graph that contains a directed cycle$\}$

- There exists an NL-complete context-free language

Section 9.1 - Hierarchy Theorems

Note: items with an A symbol instead of a • symbol have a proof on page 391.

A TIME(2^n) = TIME(2^{n+1})

A TIME(2^n) $\subsetneq$ TIME(2^{2n})

A NTIME(n) $\subsetneq$ PSPACE

- If NEXPTIME $\neq$ EXPTIME, then P $\neq$ NP *(proof in the study guide)*

A The *pad* function

- $E_{\text{REX}\uparrow}$ is in P

Chapter 10 - Advanced Topics in Complexity Theory

STUDY GUIDE's theorems. *Note: page numbers refer to the study guide, not to the book.*

- *COMPOSITES* $\in$ RP
- *PRIMES* $\in$ coRP
- RP $\subseteq$ BPP
- coRP $\subseteq$ BPP
- RP $\subseteq$ NP

Made by Yehonatan Simian

Appendix A - Closure Properties

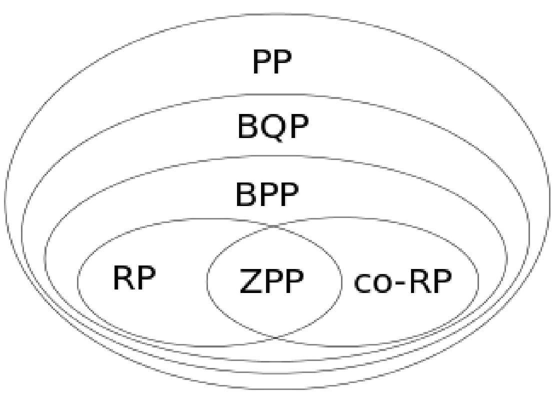
	$A \cup B$	$A \cap B$	$A \cdot B$	A^*	$\overline{A}$	homomorphism
Regular	✓	✓	✓	✓	✓	✓
Context-free	✓	✗	✓	✓	✗	✓
Decidable	✓	✓	✓	✓	✓	✗
Recognizable	✓	✓	✓	✓	✗	✓
P	✓	✓	✓	✓	✓	iff P=NP
NP	✓	✓	✓	✓	iff NP=coNP	
PSPACE	✓	✓	✓	✓	✓	
L	✓	✓	✓	iff L=NL		
NL	✓	✓	✓	✓	✓	
BPP	✓	✓			✓	

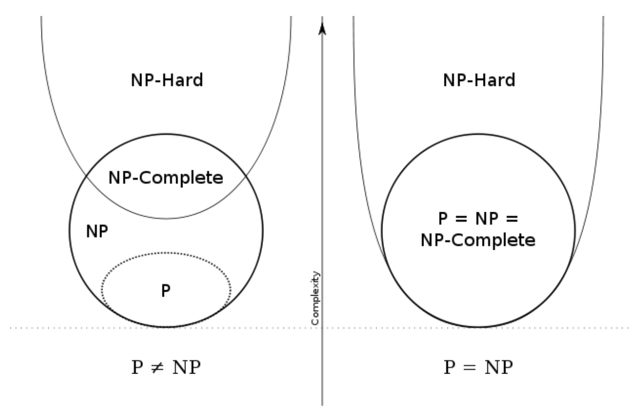
Appendix B - Classes Hierarchies

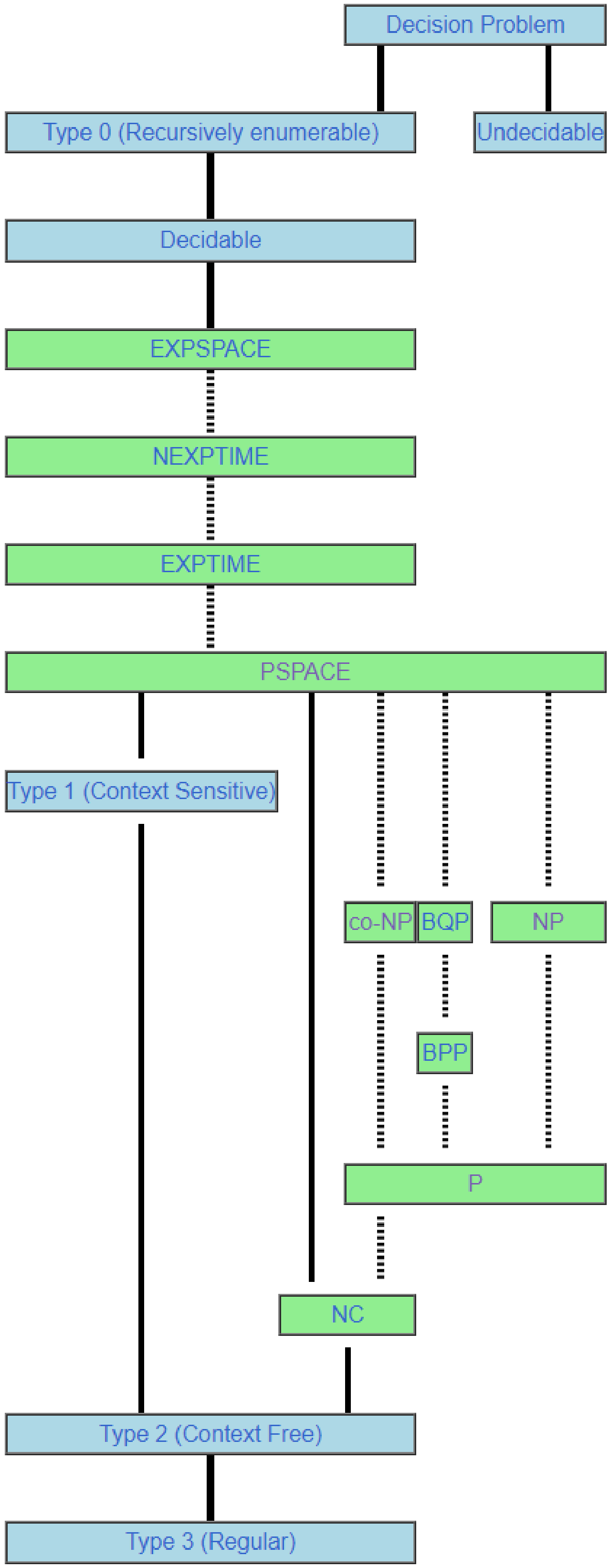
$$\text{Regular} \subset \text{Context-free} \subset \text{Decidable} \subset \text{Turing-recognizable}$$

$$\text{L} \subseteq \underset{\substack{\parallel \\ \text{coNL}}}{\text{NL}} \subseteq \text{P} \subseteq \text{NP} \subseteq \underset{\substack{\parallel \\ \text{NPSPACE}}}{\text{PSPACE}} \subseteq \text{EXPTIME} \subseteq \underset{\substack{\parallel \\ \text{NEXPSPACE}}}{\text{EXPSPACE}}$$

Note: NL $\subsetneq$ PSPACE, P $\subsetneq$ EXPTIME, PSPACE $\subsetneq$ EXPSPACE






Decision Problem
Type 0 (Recursively enumerable)
Undecidable
Decidable
EXPSPACE
NEXPTIME
EXPTIME
PSPACE
Type 1 (Context Sensitive)
co-NP
BQP
NP
BPP
P
NC
Type 2 (Context Free)
Type 3 (Regular)