# 习题解答:

$$\boldsymbol{P}(t)=\iiint_V \boldsymbol{p}(\boldsymbol{r},t)\mathrm{d}V=\iiint_V \rho(\boldsymbol{r},t)\boldsymbol{r}\mathrm{d}V$$

$$\frac{\mathrm{d}\boldsymbol{P}(t)}{\mathrm{d}t}=\iiint_V \frac{\partial \boldsymbol{p}(\boldsymbol{r},t)}{\partial t}\mathrm{d}V=\iiint_V\left[\frac{\partial \rho(\boldsymbol{r},t)}{\partial t}\boldsymbol{r}+\rho(\boldsymbol{r},t)\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\right]\mathrm{d}V$$

介质空间: $\rho(\boldsymbol{r},t)\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}=0$, $\boldsymbol{J}(\boldsymbol{r},t)=\frac{\partial \rho(\boldsymbol{r},t)}{\partial t}\boldsymbol{r}$

导体空间: $\frac{\partial \rho(\boldsymbol{r},t)}{\partial t}\boldsymbol{r}=0$, $\boldsymbol{J}(\boldsymbol{r},t)=\rho(\boldsymbol{r},t)\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}$

$$\therefore \frac{\mathrm{d}\boldsymbol{P}(t)}{\mathrm{d}t}=\iiint_V \boldsymbol{J}(\boldsymbol{r},t)\mathrm{d}V$$

$\boldsymbol{p}(\boldsymbol{r},t)$ 单位体积电偶极矩 = 极化强度

问题 ？

1. 电荷只能分布在导体的表面 ？

2. $\nabla\left(\boldsymbol{k}\cdot\boldsymbol{r}\right)=\boldsymbol{k}\nabla\cdot\boldsymbol{r}$ ？

问题：

1. 场能密度两表达式是否正确？
2. 场能为什么可用两式计算？
3. 积分区域可否有差别？

# 第四讲 静态电磁场问题（1）

# 静态电磁场

（第三章）

静态电磁场：不随时间变化的电磁场

1）时间无关的电磁场—静态场

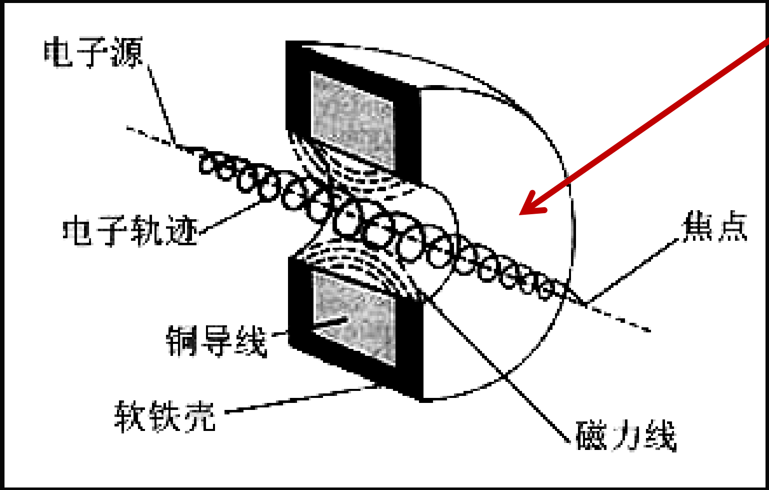
2）恒定电流的电场与磁场—恒定电磁场

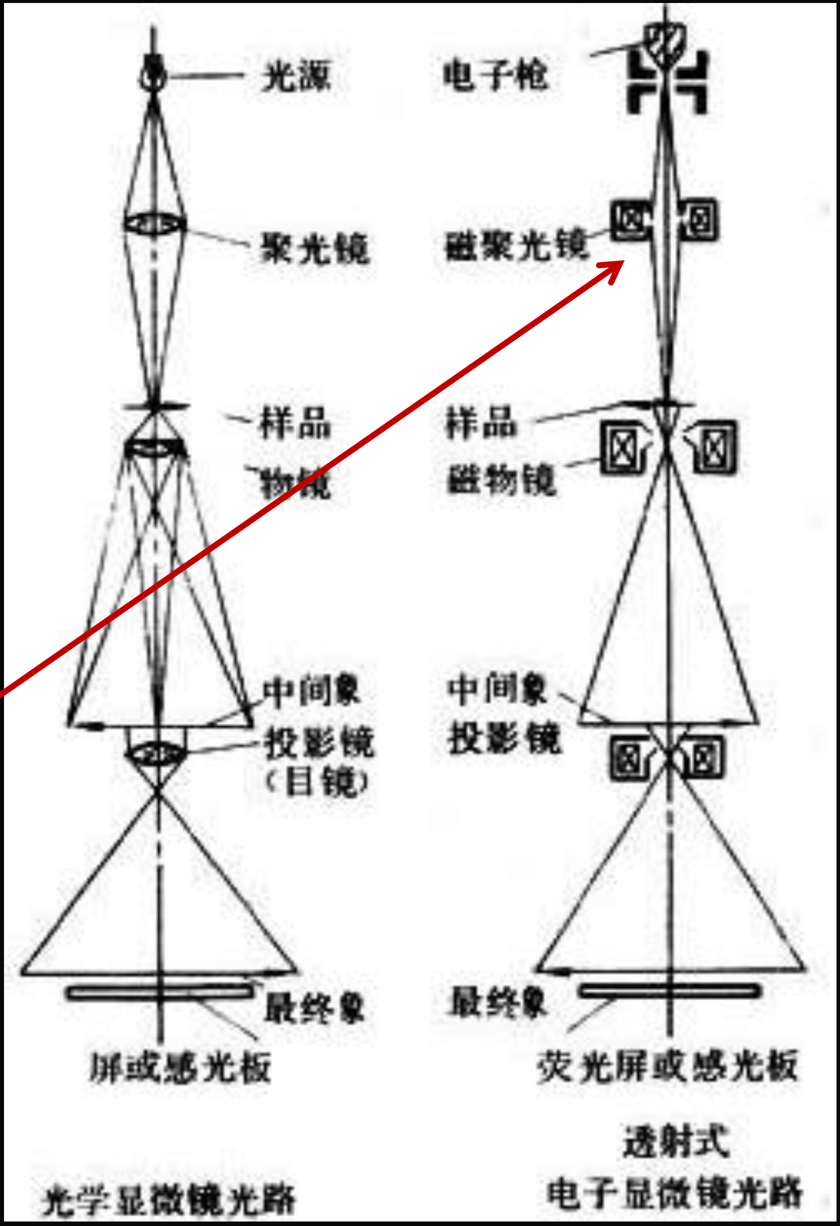
3）时变可忽略的电磁场—似稳场

尺度为米级 ⟵

**50Hz，波长6000000米**

- 能源与动力工程
- 新材料与新器件
- 存储与显示技术
- 功能仪器及设备
- 特种用途场科学设计



主要内容：

- 静态场的基本问题（电　磁）
- 静态（电　磁）场的能量
- 静态电磁系统的作用力问题
- 若干经典模型的计算问题

# §1 静态电场及方程

## 1 电位及其方程

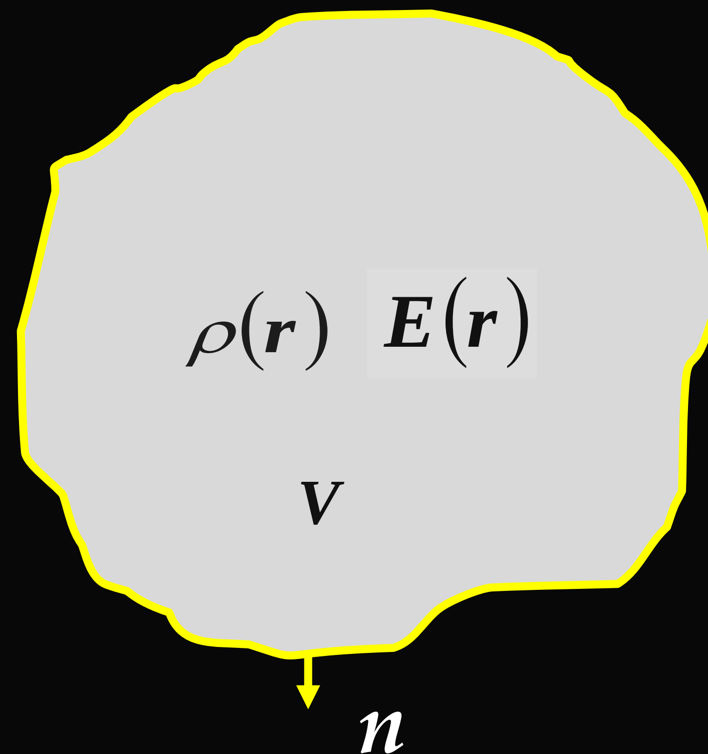
静电场，Maxwell方程组为

$$\nabla \cdot \boldsymbol{D}(\boldsymbol{r}) = \rho(\boldsymbol{r})$$

$$\nabla \times \boldsymbol{E}(\boldsymbol{r}) = 0$$

静电场为有散无旋矢量场

能表示为标量场的梯度



?

引入标量函数 $\phi(\boldsymbol{r})$，令 $\boldsymbol{E}(\boldsymbol{r}) = -\nabla\phi(\boldsymbol{r})$

得到 $\phi(\boldsymbol{r})$ 满足的方程

$$\nabla^2\phi(\boldsymbol{r}) = -\frac{\rho(\boldsymbol{r})}{\varepsilon}$$ （Poisson方程）

如果 $\rho(\boldsymbol{r}) = 0$ ，则为Laplace方程 $\nabla^2\phi(\boldsymbol{r}) = 0$

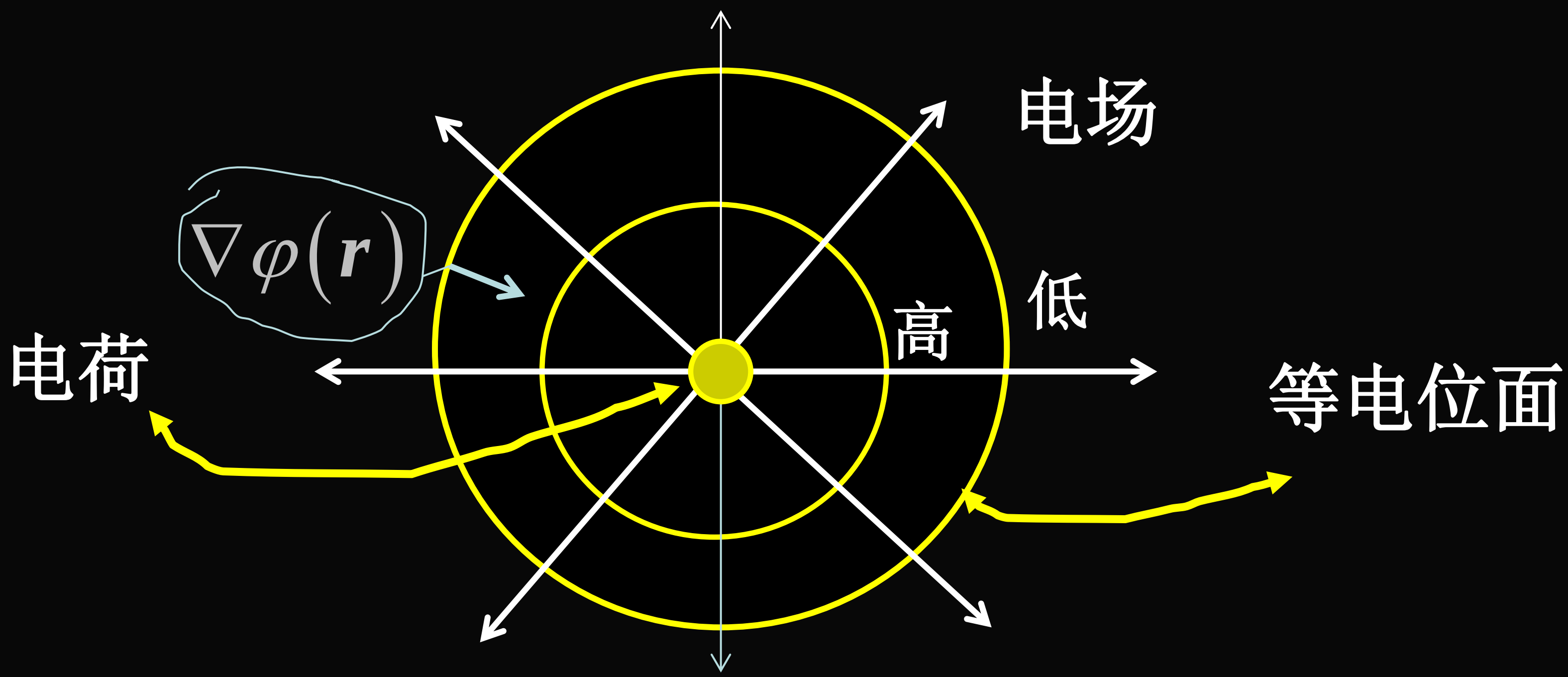
$\phi(\boldsymbol{r})$:称为电位函数

问题：静电场与电位是否一一对应？

电位函数是否能唯一确定静电场？

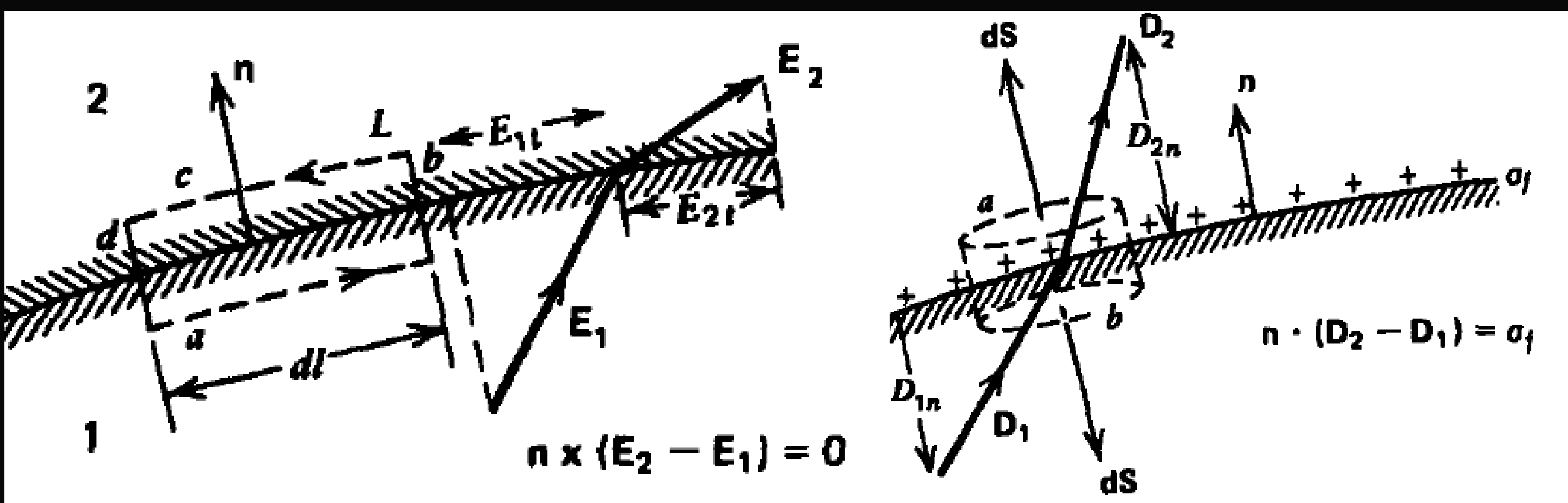
引入电位与电场关系 $\boldsymbol{E}(\boldsymbol{r}) = -\nabla\phi(\boldsymbol{r})$

负号源于电场力线与电位降低方向一致的习惯表达



## 2　电位的边界条件

引入电位函数，电场通过泊松方程求解，需将电场边界条件表达为电位边界条件

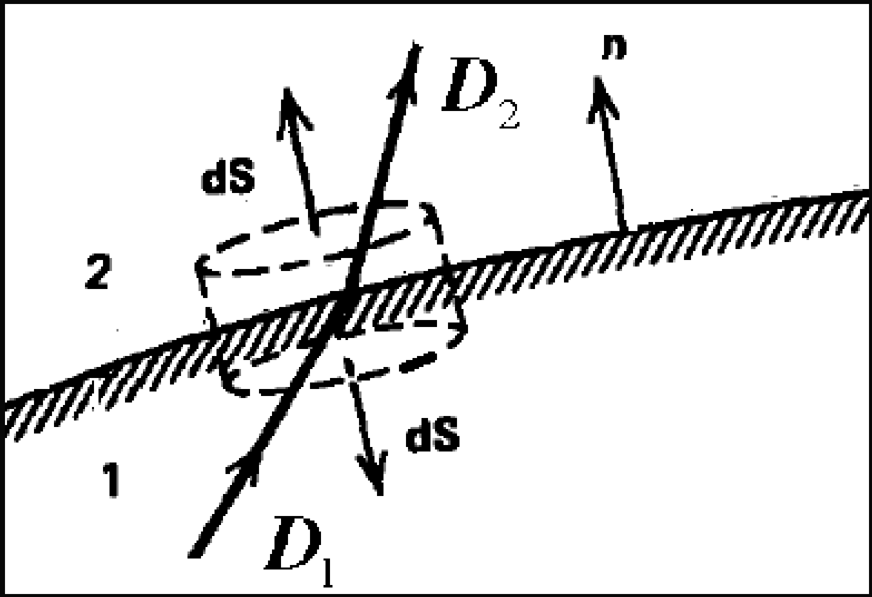


$$\nabla \cdot \boldsymbol{D}(\boldsymbol{r}) = \rho(\boldsymbol{r}) \leftrightarrow \oiint_{s} \boldsymbol{D} \cdot \mathrm{d}\boldsymbol{s} = \iiint_{V} \rho \mathrm{d}V$$

$$(\boldsymbol{D}_2 - \boldsymbol{D}_1) \cdot \hat{n} = \rho_s$$

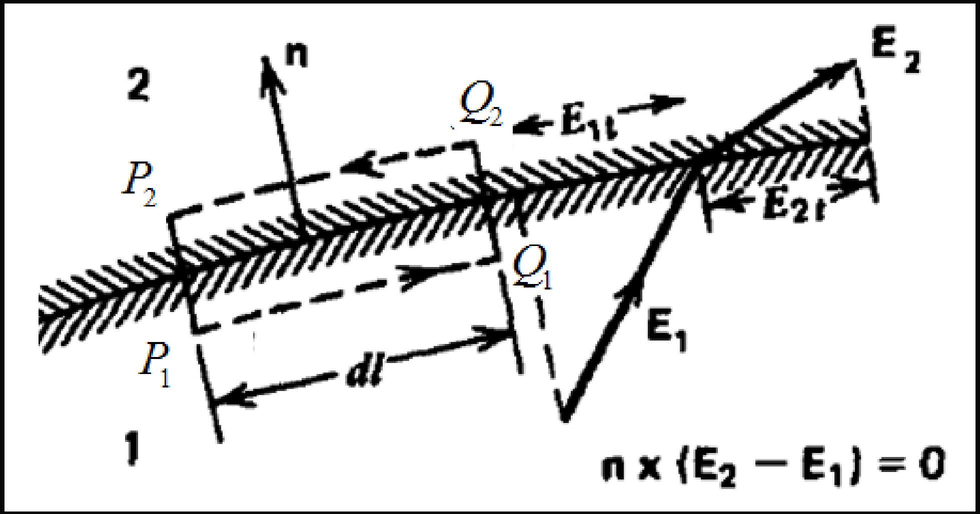
$$\left(\varepsilon_2 \nabla \phi_2 - \varepsilon_1 \nabla \phi_1\right) \cdot \hat{n} = \rho_s$$

$$\Rightarrow \varepsilon_2 \frac{\partial \phi_2}{\partial n} - \varepsilon_1 \frac{\partial \phi_1}{\partial n} = \rho_s$$



$$\phi(P_2)-\phi(P_1)=$$

$$-\lim_{P_1\to P_2}\left\{\int_{P_1}^{P_2}\boldsymbol{E}(\boldsymbol{r})\cdot\mathrm{d}\boldsymbol{l}\right\}=0$$




$$\phi(P_2)-\phi(P_1)=\phi(Q_2)-\phi(Q_1)$$

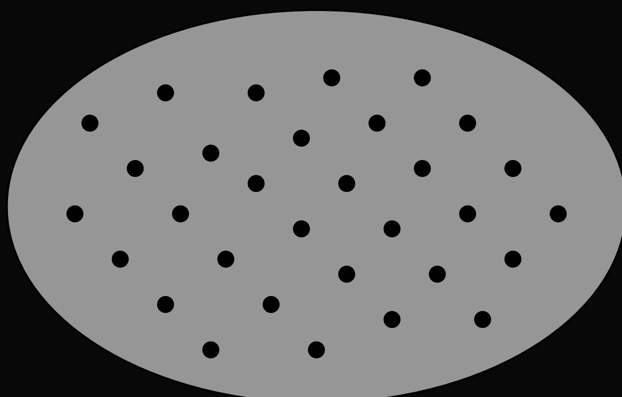
$$\to\phi(P_2)-\phi(Q_2)=\phi(P_1)-\phi(Q_1)$$

$$\to\oint_L\boldsymbol{E}\cdot\mathrm{d}\boldsymbol{L}=0\to\hat{n}\times(\boldsymbol{E}_2-\boldsymbol{E}_1)=0$$

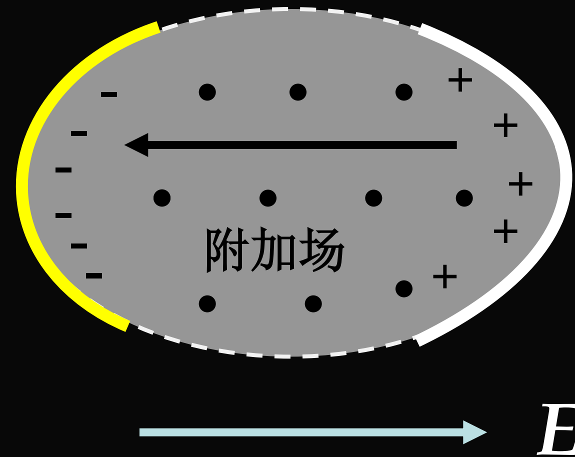
# 3　导体与导体边界条件

善于传导电流的物质称导体，反之称绝缘体

**条件：物质内存在大量可自由移动带电粒子**



没有外加电场




导体内存在大量可自由移动带电粒子，呈现电中性

达到静电平衡状态
导体内部电场为零

电场中的理想导体：

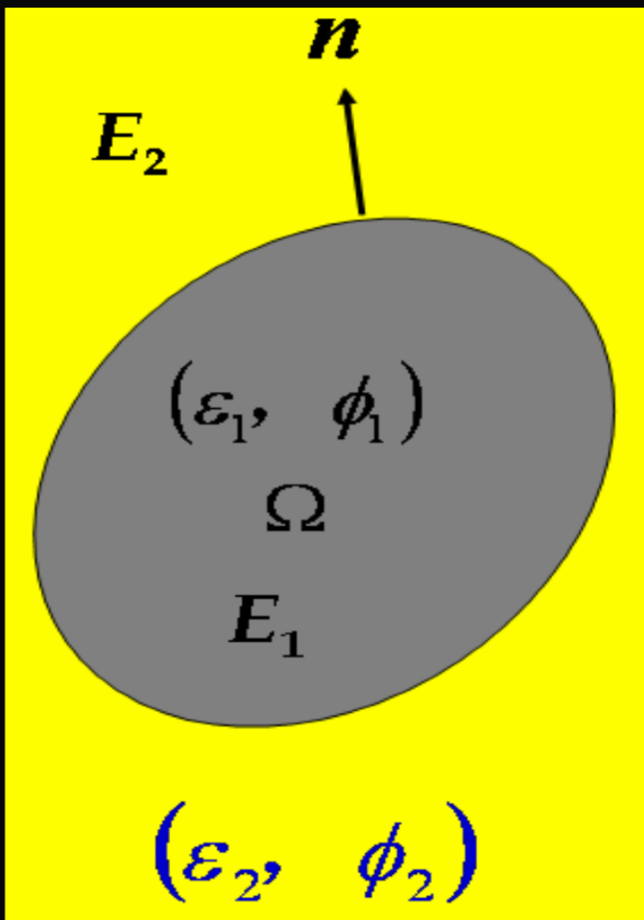
导体内部电场为零，为等势体；

导体边界面电场切向分量为零；

导体带电荷只分布在导体的表面　？

$$\begin{cases} \phi = \phi_0（\text{常数}） \\ -\varepsilon \dfrac{\partial \phi}{\partial n} = \rho_s \end{cases} \qquad \oiint_S \rho_s \mathrm{d}s = \begin{cases} Q \ (\text{导体所带电荷量}) \\ 0 \ (\text{导体不带电}) \end{cases}$$

# 4 静电场的定解问题




均匀介质空间静电场为

$$\begin{cases} \nabla^2\phi_1(\boldsymbol{r}) = -\dfrac{\rho(\boldsymbol{r})}{\varepsilon_1} \\ \left[\varepsilon_1 \dfrac{\partial \phi_1}{\partial n}\right]_S - \left[\varepsilon_2 \dfrac{\partial \phi_2}{\partial n}\right]_S = -\rho_s \\ \text{或} \quad \phi_1(\boldsymbol{r})\big|_S = \phi_2(\boldsymbol{r})\big|_S \end{cases}$$

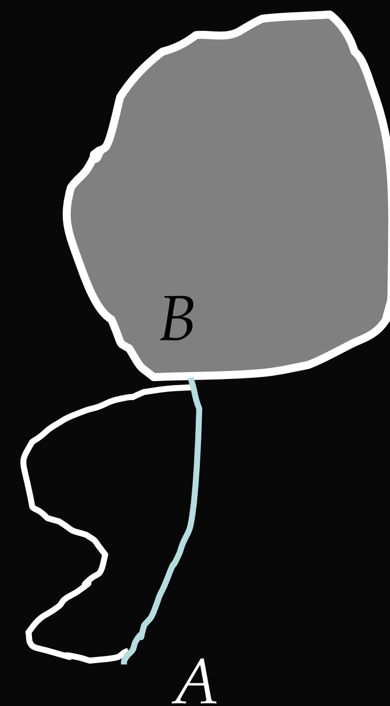
泊松方程定解问题之解

$$\left\{\begin{aligned}
&\nabla^2\phi_1(\boldsymbol{r}) = -\frac{\rho(\boldsymbol{r})}{\varepsilon_1} \\
&\left[\varepsilon_1\frac{\partial\phi_1}{\partial n}\right]_S - \left[\varepsilon_2\frac{\partial\phi_2}{\partial n}\right]_S = -\rho_s \\
&\text{或}\quad \phi_1(\boldsymbol{r})\big|_S = \phi_2(\boldsymbol{r})\big|_S
\end{aligned}\right\} \rightarrow \begin{cases}\text{物理原理方法}\\ \text{复变函数方法}\\ \text{积分变换方法}\\ \text{分离变量方法}\\ \text{格林函数方法}\\ \text{镜像原理方法}\\ \text{数值分析方法}\end{cases}$$

# 【例3-1】 导体电位与所带电量之比为常数。

导体带电荷$Q_1$ ，$\phi_1 = \int_B^A \boldsymbol{E}_1(\boldsymbol{r}) \cdot \mathrm{d}\boldsymbol{L}$

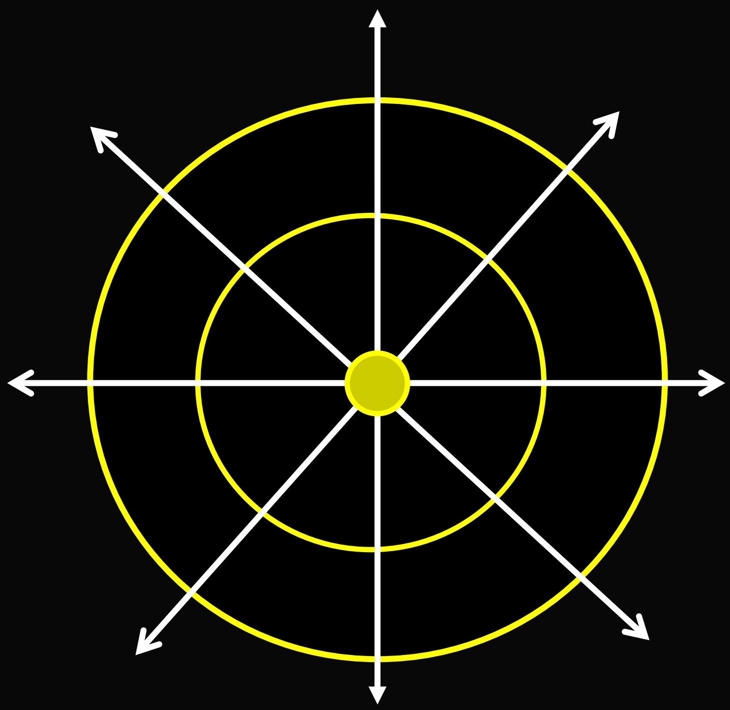
导体带电荷$Q_2$ ，$\phi_2 = \int_B^A \boldsymbol{E}_2(\boldsymbol{r}) \cdot \mathrm{d}\boldsymbol{L}$




$$\int_B^A \left[ \frac{\boldsymbol{E}_1(\boldsymbol{r})}{\phi_1} - \frac{\boldsymbol{E}_2(\boldsymbol{r})}{\phi_2} \right] \cdot \mathrm{d}\boldsymbol{L} = 0 \rightarrow \frac{\boldsymbol{E}_1(\boldsymbol{r})}{\phi_1} = \frac{\boldsymbol{E}_2(\boldsymbol{r})}{\phi_2}$$

$$\frac{1}{\phi_1} \oiint_S \boldsymbol{E}_1(\boldsymbol{r}) \cdot \mathrm{d}\boldsymbol{S} = \frac{1}{\phi_2} \oiint_S \boldsymbol{E}_2(\boldsymbol{r}) \cdot \mathrm{d}\boldsymbol{S} \rightarrow \frac{1}{\phi_1} \oiint_S \rho_{S1} \mathrm{d}S = \frac{1}{\phi_2} \oiint_S \rho_{S2} \mathrm{d}S \rightarrow \frac{Q_1}{\phi_1} = \frac{Q_2}{\phi_2}$$

【例3-2】 1）无界均匀介质空间点电荷电位；



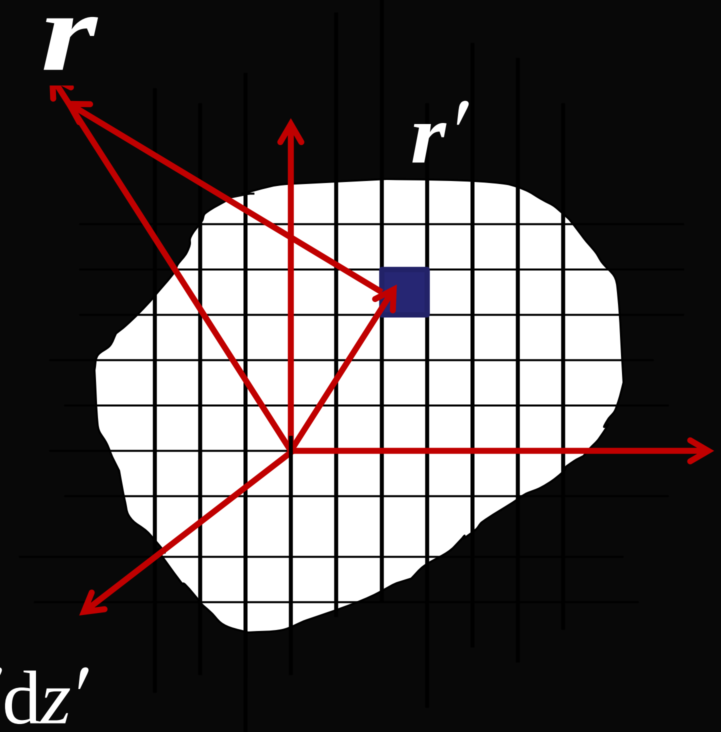
电场

$$\begin{cases} \nabla^2\phi(\boldsymbol{r}) = -\dfrac{q}{\varepsilon}\delta(\boldsymbol{r}) \\ \lim\limits_{r\to\infty}\phi(\boldsymbol{r}) = 0 \end{cases}$$

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{q\boldsymbol{r}}{4\pi\varepsilon r^3}$$

$$\phi(\boldsymbol{r}) = -\frac{q}{4\pi\varepsilon}\int_{\infty}^{r}\frac{\mathrm{d}r}{r^2} = \frac{q}{4\pi\varepsilon r}$$

$$\mathrm{d}\phi(\boldsymbol{r},\boldsymbol{r}') = \frac{\rho(\boldsymbol{r}')\delta V}{4\pi\varepsilon|\boldsymbol{r}-\boldsymbol{r}'|}$$

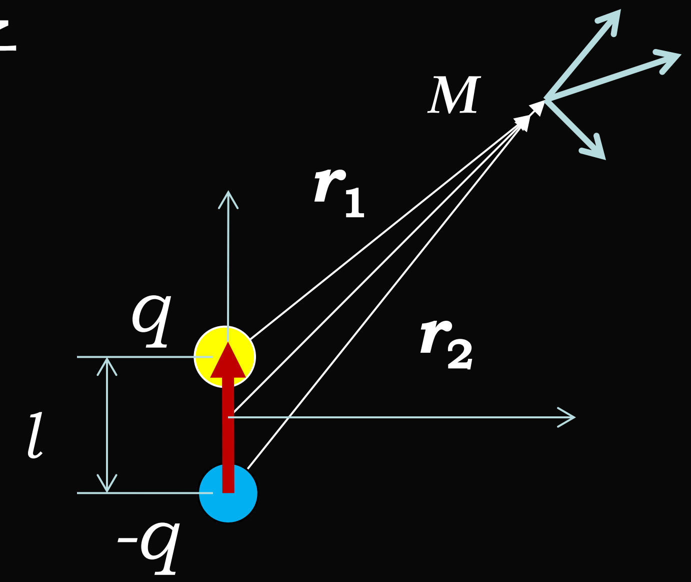



$$\phi(\boldsymbol{r}) = \iiint_V \frac{\rho(\boldsymbol{r}')\mathrm{d}\boldsymbol{r}'}{4\pi\varepsilon|\boldsymbol{r}-\boldsymbol{r}'|}$$

$$= \frac{1}{4\pi\varepsilon}\iiint_V \frac{\rho(x',y',z')\mathrm{d}x'\mathrm{d}y'\mathrm{d}z'}{\sqrt{(x-x')^2+(y-y')^2(z-z')^2}}$$

空间$\boldsymbol{r}$ 点的电位是全体电荷元产生电位的叠加

# 2）无界空间电偶极子远区的电场



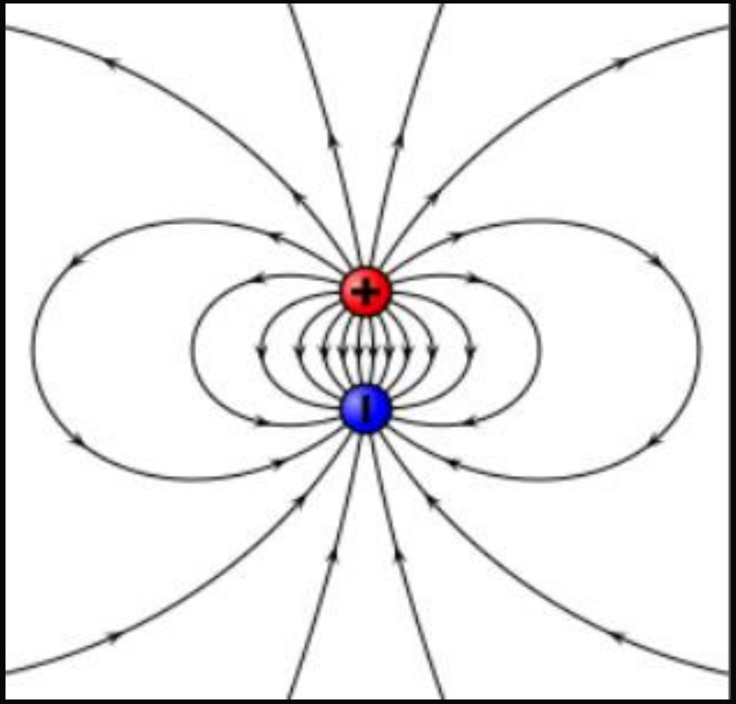

电偶极子（矩）： $\boldsymbol{P} = ql\hat{z}$

$$\phi(\boldsymbol{r}) = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$$

$$\phi(\boldsymbol{r}) \approx \frac{qL\cos\theta}{4\pi\varepsilon_0 r^2} = \frac{\boldsymbol{P}_e \cdot \boldsymbol{r}}{4\pi\varepsilon_0 r^3}$$

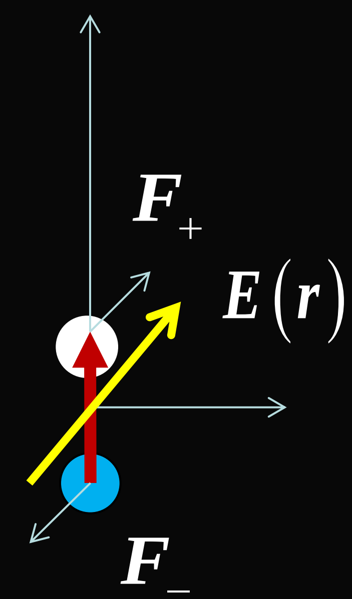
$$\boldsymbol{E}(\boldsymbol{r}) = -\nabla\varphi(\boldsymbol{r})$$

$$= \frac{1}{4\pi\varepsilon_0 r^3}\left(\hat{e}_r 2P_e\cos\theta + \hat{e}_\theta P_e\sin\theta\right)$$



## 3）外电场中电偶极子受力特性。

$$\boldsymbol{F}_+(\boldsymbol{r}) = q\boldsymbol{E}\left(0,0,\frac{l}{2}\right), \boldsymbol{F}_-(\boldsymbol{r}) = -q\boldsymbol{E}\left(0,0,-\frac{l}{2}\right)$$




$$\boldsymbol{F} = \boldsymbol{F}_+(\boldsymbol{r}) = q\left[\boldsymbol{E}\left(0,0,\frac{l}{2}\right) - \boldsymbol{E}\left(0,0,-\frac{l}{2}\right)\right]$$

$$= ql\left[\hat{e}_x \frac{\partial E_x}{\partial z} + \hat{e}_y \frac{\partial E_y}{\partial z} + \hat{e}_z \frac{\partial E_z}{\partial z}\right]_{(0,0,0)}$$

$$= ql\frac{\partial}{\partial z}\boldsymbol{E}(\boldsymbol{r})_{(0,0,0)} = (P\cdot\nabla)\boldsymbol{E}(\boldsymbol{r})_{(0,0,0)}$$

$\boxed{\text{均匀外恒为零}}$

均匀外场中电偶极子所受到的作用力恒为零

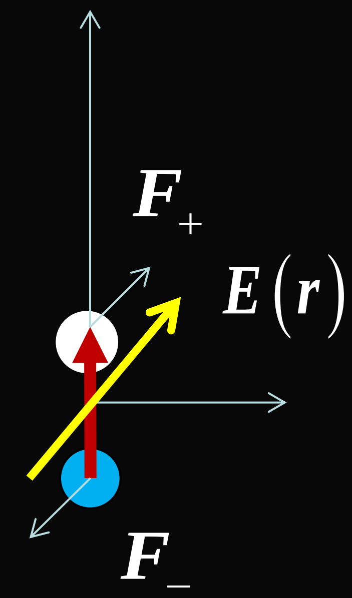
## 3）外电场中电偶极子受力特性。

$$\boldsymbol{L}(\boldsymbol{r}) = q\hat{e}_z \frac{l}{2} \times \left[ \boldsymbol{E}\left(0,0,\frac{l}{2}\right) + \boldsymbol{E}\left(0,0,-\frac{l}{2}\right) \right]$$

$$\because \boldsymbol{F}(\boldsymbol{r}+\delta\boldsymbol{r}) = \boldsymbol{F}(\boldsymbol{r}) + (\delta\boldsymbol{r}\cdot\nabla)\boldsymbol{F}(\boldsymbol{r})$$

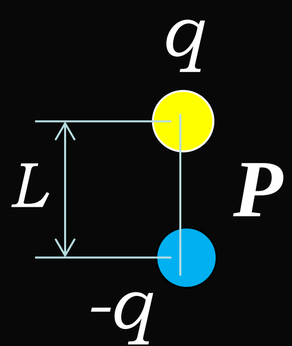
$$\therefore \boldsymbol{E}\left(0,0,\pm\frac{l}{2}\right) = \boldsymbol{E}(0,0,0) \pm \left( \frac{l}{2} \frac{\partial \boldsymbol{E}(\boldsymbol{r})}{\partial z} \right)_{(0,0,0)}$$

$$\boldsymbol{L}(\boldsymbol{r}) = \boldsymbol{P} \times \boldsymbol{E}(\boldsymbol{r})_{(0,0,0)}$$

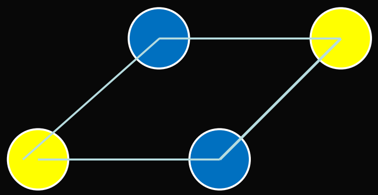



均匀外场中电偶极子所受到的作用力矩不为零

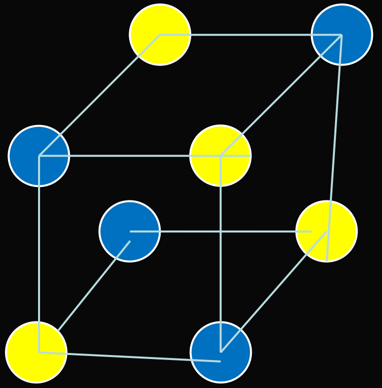
# 电多极矩概念




电偶极矩



电四极矩



电八极矩

电偶极矩：

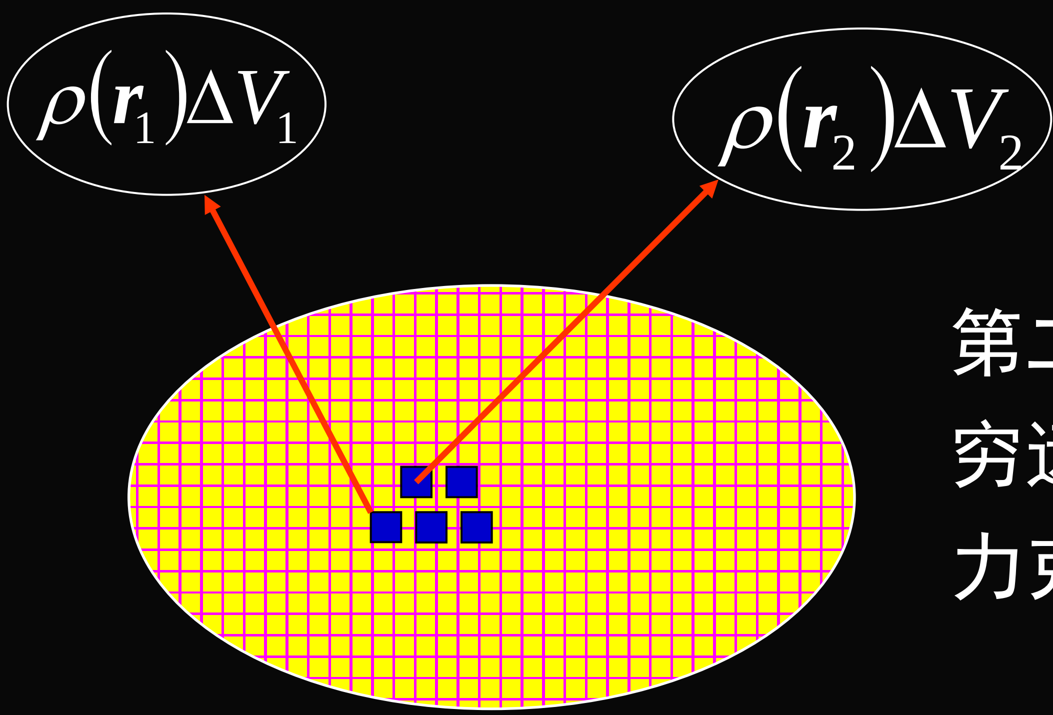
$$\boldsymbol{P} = \iiint_V \boldsymbol{r}'\,\rho(\boldsymbol{r}')\,\mathrm{d}V' = \hat{z}LQ$$

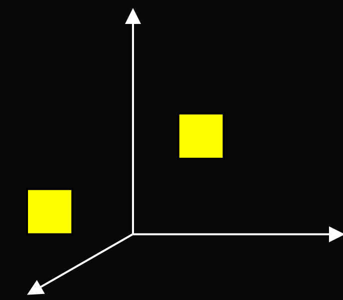
## 5 静电场的能量和能量密度

静电场对置于其中的电荷有力的作用，并对电荷作功。这说明静电场有能量。

根据能量守恒原理，静电场的能量等于电荷体建立过程中，外力克服静电力做功的总和

第一个小电荷元自从无穷远处移到$\boldsymbol{r}_1$，外力克服电场力做功为零



第二个小电荷元自从无穷远处移到$\boldsymbol{r}_2$点时，外力克服电场力作功为：

$$\mathrm{d}W_2 = -\int_{\infty}^{r_2} \rho(\boldsymbol{r}_2)\mathrm{dV}_2 \boldsymbol{E}_1 \cdot \mathrm{d}\boldsymbol{L} = \rho(\boldsymbol{r}_2)\mathrm{dV}_2 \phi_{12}$$

第三个小电荷元自从无穷远处移到$\boldsymbol{r}_3$点
外力克服电场力作功为：

$$\mathrm{d}W_3 = \rho\left(\boldsymbol{r}_3\right)\mathrm{dV}_3\phi_{13} + \rho\left(\boldsymbol{r}_3\right)\mathrm{dV}_3\phi_{23}$$

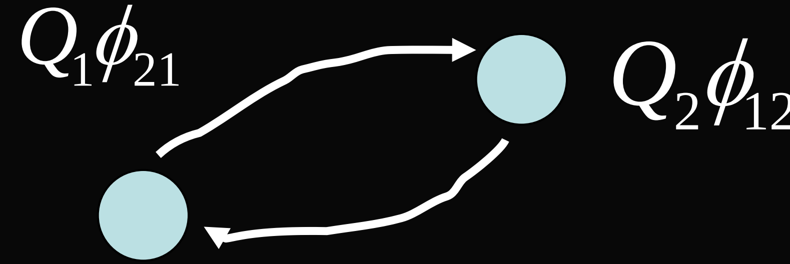
第 $\boldsymbol{n}$ 个小电荷元自从无穷远处移到 $\boldsymbol{r}_{\mathrm{n}}$ 点时，外力克服电场力作功为：

$$\mathrm{d}W_n = \rho\left(\boldsymbol{r}_n\right)\mathrm{dV}_n\phi_{1n} + \rho\left(\boldsymbol{r}_n\right)\mathrm{dV}_n\phi_{2n} + \rho\left(\boldsymbol{r}_n\right)\mathrm{dV}_n\phi_{3n}\cdots + \rho\left(\boldsymbol{r}_n\right)\mathrm{dV}_n\phi_{n-1,n}$$

$$W_e = \rho\left(\boldsymbol{r}_2\right)\mathrm{dV}_2\phi_{12} + \rho\left(\boldsymbol{r}_3\right)\mathrm{dV}_3\phi_{13} + \rho\left(\boldsymbol{r}_3\right)\mathrm{dV}_3\phi_{23} + \cdots$$

$$= \lim_{n\to\infty}\sum_{i=1}^{n}\sum_{j=1}^{i-1}\rho\left(\boldsymbol{r}_i\right)\mathrm{dV}_i\phi_{ji}$$

# 静电场的互易性




另一方面：

$$\mathrm{d}W_2=\rho(\boldsymbol{r}_2)\mathrm{d}V_2\phi_{12}=\frac{1}{2}\left[\rho(\boldsymbol{r}_2)\mathrm{d}V_2\phi_{12}+\rho(\boldsymbol{r}_1)\mathrm{d}V_1\phi_{21}\right]$$

$$\mathrm{d}W_3=\frac{1}{2}\left[\rho(\boldsymbol{r}_3)\mathrm{d}V_3\phi_{13}+\rho(\boldsymbol{r}_3)\mathrm{d}V_3\phi_{23}+\rho(\boldsymbol{r}_1)\mathrm{d}V_1\phi_{31}+\rho(\boldsymbol{r}_2)\mathrm{d}V_2\phi_{32}\right]$$

$$\mathrm{d}W_n=\frac{1}{2}\Big\{\left[\rho(\boldsymbol{r}_n)\mathrm{d}V_n\phi_{1n}+\rho(\boldsymbol{r}_n)\mathrm{d}V_n\phi_{2n}+\cdots+\rho(\boldsymbol{r}_n)\mathrm{d}V_n\phi_{n-1,n}\right]$$
$$+\left[\rho(\boldsymbol{r}_1)\mathrm{d}V_1\phi_{n1}+\rho(\boldsymbol{r}_2)\mathrm{d}V_2\phi_{n2}+\cdots+\rho(\boldsymbol{r}_{n-1})\mathrm{d}V_{n-1}\phi_{n,n-1}\right]\Big\}$$

$$W_e=\mathrm{d}W_2+\mathrm{d}W_3+\mathrm{d}W_4+\cdots+\mathrm{d}W_{\mathrm{n}}=\lim_{n\to\infty}\frac{1}{2}\sum_{i=1}^{n}\rho(\boldsymbol{r}_i)\mathrm{d}V_i\sum_{j=1(j\neq i)}^{n}\phi_{ji}$$

$$W_e = \mathrm{d}W_2 + \mathrm{d}W_3 + \mathrm{d}W_4 + \cdots + \mathrm{d}W_\mathrm{n} = \lim_{n\to\infty}\sum_{i=1}^{n}\sum_{j=1}^{i-1}\rho(\boldsymbol{r}_i)\mathrm{d}V_i\phi_{ji}$$

$$= \lim_{n\to\infty}\frac{1}{2}\sum_{i=1}^{n}\rho(\boldsymbol{r}_i)\mathrm{d}V_i\sum_{j=1(j\neq i)}^{n}\phi_{ji} = \lim_{n\to\infty}\frac{1}{2}\sum_{i=1}^{n}\rho(\boldsymbol{r}_i)\phi_i\mathrm{d}V_i$$

$$= \iiint_V \frac{1}{2}\rho(\boldsymbol{r})\phi(\boldsymbol{r})\mathrm{d}V$$

体电荷在 $(\boldsymbol{r}_i)$ 产生的电位

利用$\nabla\cdot\boldsymbol{D}=\rho$ 和 $\boldsymbol{E}(\boldsymbol{r})=-\nabla\phi(\boldsymbol{r})$

$$W_e=\frac{1}{2}\iiint_V\rho(\boldsymbol{r})\phi(\boldsymbol{r})\mathrm{d}V=\frac{1}{2}\iiint_V\nabla\cdot\boldsymbol{D}(\boldsymbol{r})\phi(\boldsymbol{r})\mathrm{d}V$$

$$=\frac{1}{2}\iiint_V\boldsymbol{D}(\boldsymbol{r})\cdot\boldsymbol{E}(\boldsymbol{r})\mathrm{d}V+\oiint_{S_\infty}\phi(\boldsymbol{r})\boldsymbol{D}(\boldsymbol{r})\cdot\mathrm{d}\boldsymbol{S}$$

$$=\iiint_V\frac{1}{2}\boldsymbol{D}(\boldsymbol{r})\cdot\boldsymbol{E}(\boldsymbol{r})\mathrm{d}V$$

静电场能量密度函数：$w_e=\dfrac{1}{2}\boldsymbol{D}(\boldsymbol{r})\cdot\boldsymbol{E}(\boldsymbol{r})$

$$W_e = \frac{1}{2}\iiint_V \rho(\boldsymbol{r})\phi(\boldsymbol{r})\mathrm{d}V \to ?\ w_e = \frac{1}{2}\rho(\boldsymbol{r})\phi(\boldsymbol{r})$$

$$W_e = \iiint_V \frac{1}{2}\boldsymbol{D}(\boldsymbol{r})\cdot\boldsymbol{E}(\boldsymbol{r})\mathrm{d}V \to w_e = \frac{1}{2}\boldsymbol{D}(\boldsymbol{r})\cdot\boldsymbol{E}(\boldsymbol{r})$$

静电场能量:

可通过电荷分布计算，也可通过电场计算

但场能量密度函数只能表示为电场的函数。

$$W_e = \frac{1}{2}\iiint_V \rho(\boldsymbol{r})\phi(\boldsymbol{r})\mathrm{d}V \to ?\ w_e = \frac{1}{2}\rho(\boldsymbol{r})\phi(\boldsymbol{r})$$

$$W_e = \iiint_V \frac{1}{2}\boldsymbol{D}(\boldsymbol{r})\cdot\boldsymbol{E}(\boldsymbol{r})\mathrm{d}V \to w_e = \frac{1}{2}\boldsymbol{D}(\boldsymbol{r})\cdot\boldsymbol{E}(\boldsymbol{r})$$

**问题：**

1. 场能密度两表达式是否正确？
2. 场能为什么可用两式计算？
3. 两式积分区域可否有差别？

将静电场能量公式应用到导体系，由于导体的电位为常数，从而得到导体系的能量为

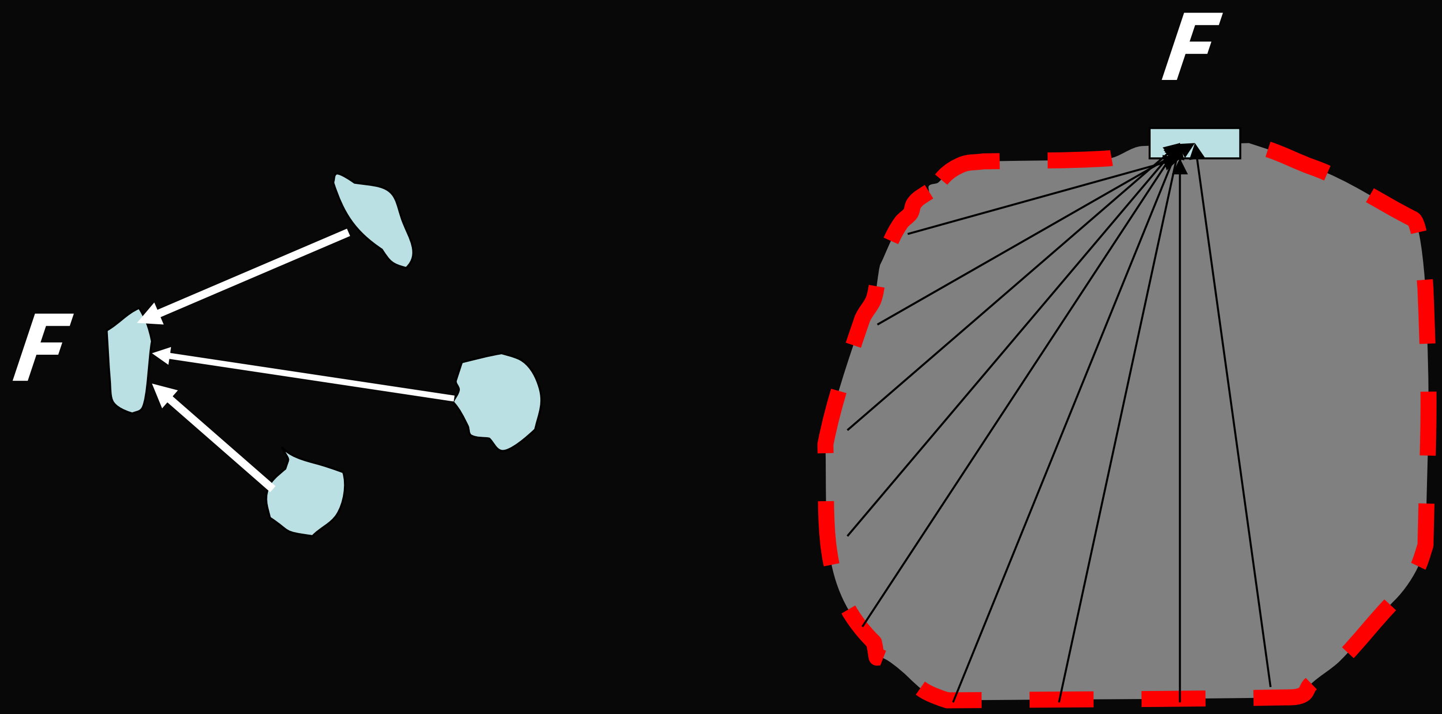
$$W_e = \frac{1}{2}\iiint_V \phi(\boldsymbol{r})\rho(\boldsymbol{r})\mathrm{d}V = \frac{1}{2}\sum \oiint_{s_i} \phi_i \rho_s \mathrm{d}s = \frac{1}{2}\sum \phi_i q_i$$

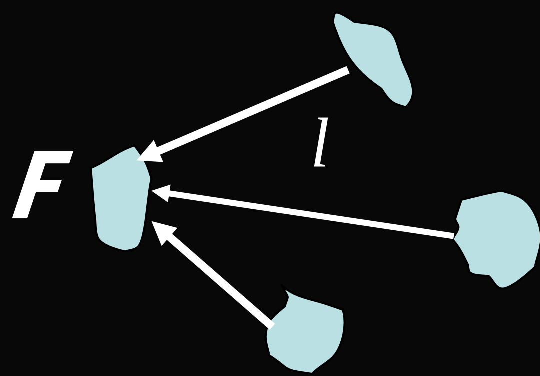
导体系相对于同一参考点的电位

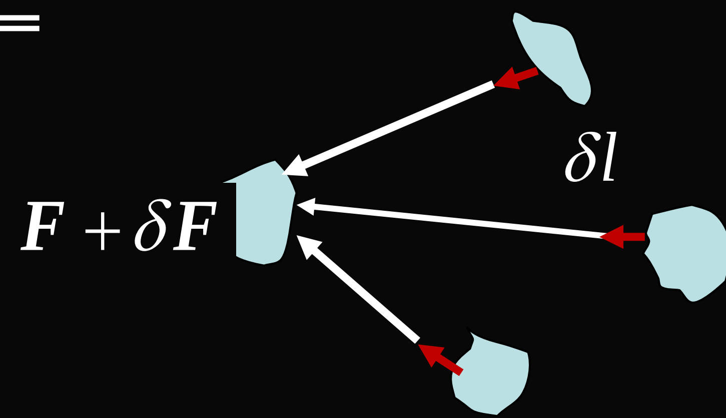
导体系的电荷量

# 6　荷电体系的静电作用力

根据库仑定律计算带点体受到作用力






电荷体空间结构 $l$

虚拟电荷有小位移

具有能量$W_e$

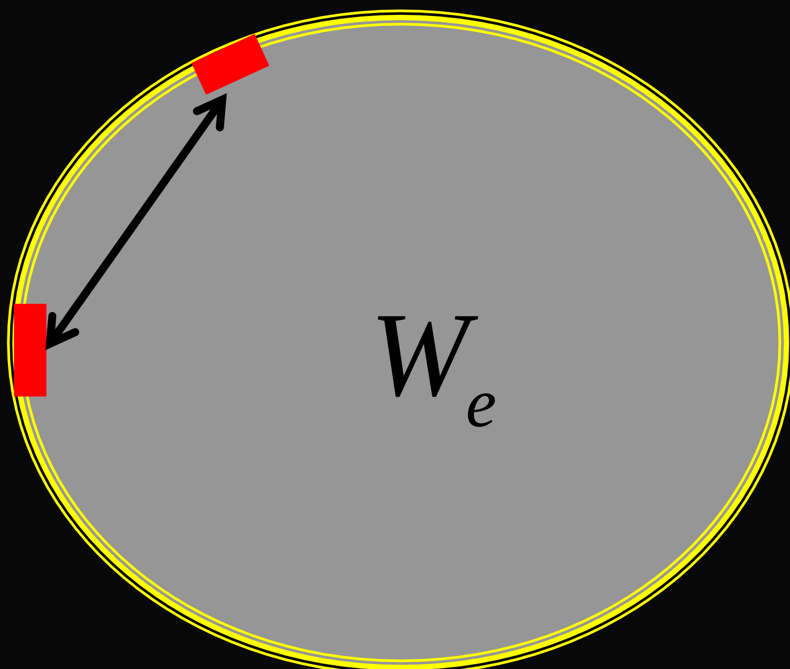
具有能量$W_e+\delta W_e$

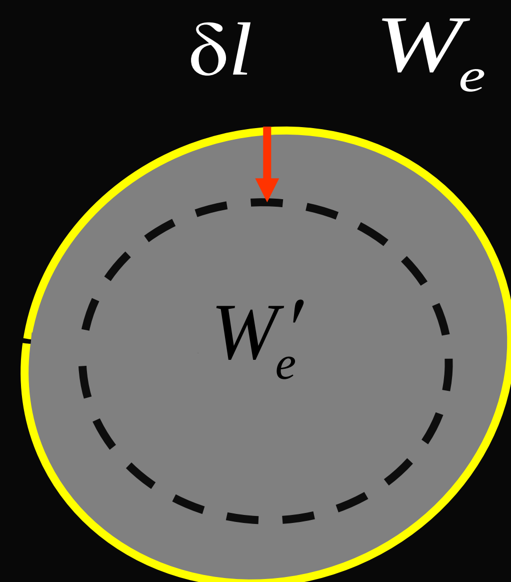
$$\boldsymbol{W_e} \leftrightarrow \boldsymbol{F}$$

$$W_e+\delta W_e \leftrightarrow \boldsymbol{F}+\delta \boldsymbol{F}$$

# 虚拟（功）仿真计算原理

$$\Delta W_e = W_e - W_e'$$




虚拟仿真 —— 现代科学研究的重要手段

虚功原理：

空间有一定结构的带电体系，静电能为 $W_e$ 。

假想该电荷体系的空间位形结构在静电力作用下发生位移 $\delta l$ ，变化后体系的静电能 $W_e'$，

静电力所作的假想功，称虚功为：

$$\delta A = \boldsymbol{F} \cdot \delta \boldsymbol{l}$$

该虚功等于电荷体系能量的减少

$$\boldsymbol{F}\cdot\delta\boldsymbol{L}=\delta A\rightarrow F_i=\frac{\delta A}{\delta x_i}=-\frac{\partial W}{\partial x_i}(i=1,2,3)$$

$$\boldsymbol{F}=-\left[\hat{e}_x\frac{\partial W_e}{\partial x}+\hat{e}_y\frac{\partial W_e}{\partial y}+\hat{e}_z\frac{\partial W_e}{\partial z}\right]=-\nabla W_e$$

① 将上式应用于电荷保持不变的导体系：结合导体系能量表达式，静电力为

$$\boldsymbol{F} = -\nabla W_e \Big|_{q=\text{常量}}$$

$$= -\frac{1}{2}\sum\left[\oiint_{s_i} \nabla\phi_i \rho_{si} \mathrm{ds}\right] \Leftarrow \boldsymbol{E} = -\nabla\phi_i$$

$$= \sum \oiint_{s_i} \frac{1}{2}\rho_{si}\boldsymbol{E}\mathrm{ds} = \sum \oiint_{s_i} \boldsymbol{f}\mathrm{ds}$$

导体面上单位面积受到静电力密度 $\boldsymbol{f}=\frac{1}{2}\rho_s\boldsymbol{E}\big|_{导体表面}$

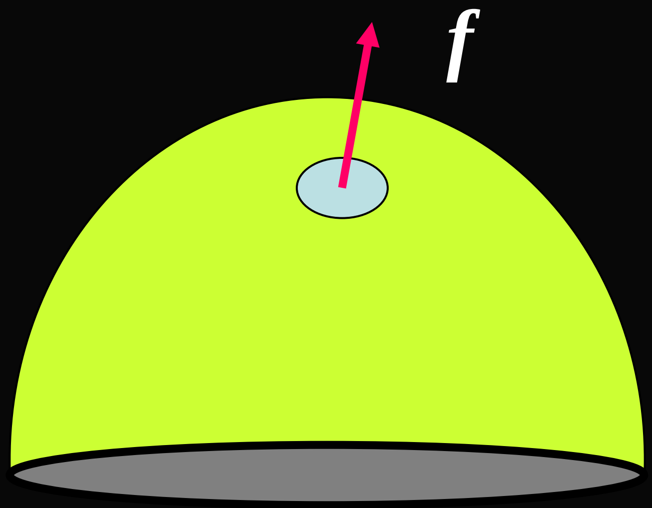
$\boldsymbol{E}\big|_{导体表面}$ 为系统**总电荷**在导体表面处产生的电场（含受力面元本身的电荷在内）

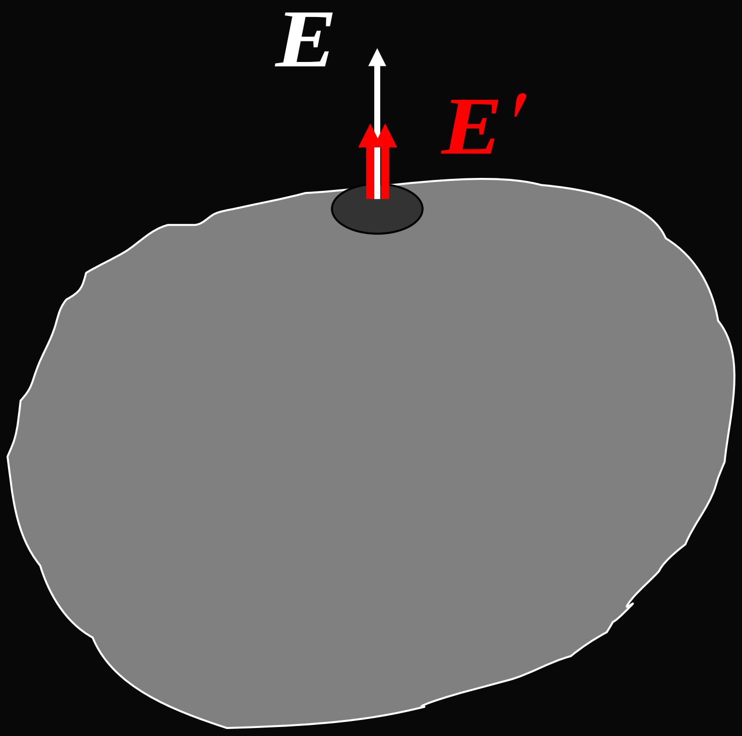
**问题：根据库仑定律**

$$\boldsymbol{f}=\rho_s\boldsymbol{E}'\big|_{导体表面}$$

**按照虚功原理得到：**

$$\boldsymbol{f}=\frac{1}{2}\rho_s\boldsymbol{E}\big|_{导体表面}$$






重要推论结果：

$$2E'\Big|_{体表面} = E\Big|_{体表面}$$

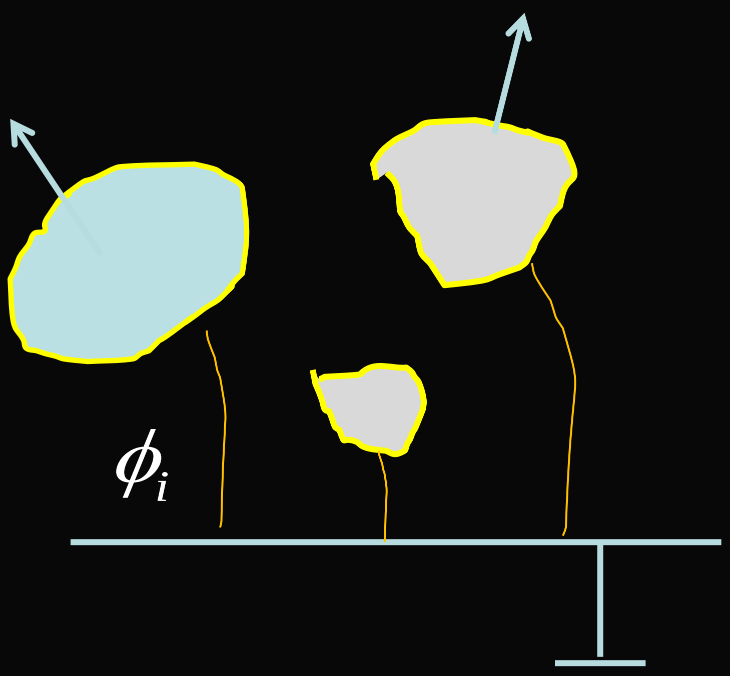
$\boldsymbol{E}$：导体总（面）电荷激发的总的电场

$\boldsymbol{E}'$：导体单位面电荷在其表面处激发的电场

② 应用于电位保持不变导体系：

如导体系与恒定电源相连，在导体系几何位置改变过程中，电位保持不变，而导体系电荷量发生变化。电源对导体系作功

$$\delta W = \sum \oiint \phi_i \delta \rho_{si} = \sum \phi_i \delta q_i$$

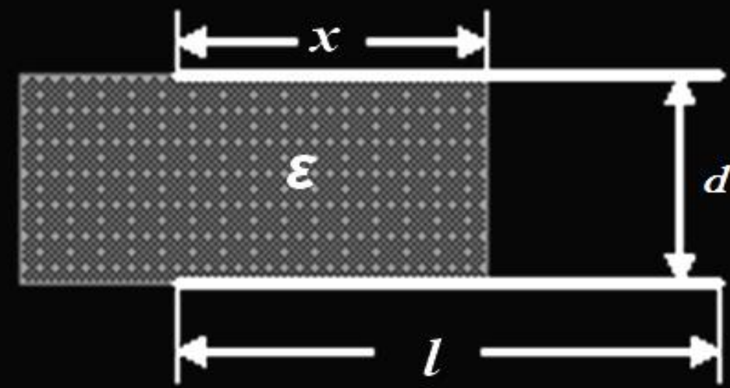


电源对导体系作功一部分电源克服导体系的静电场力输运电荷所作的功，转变为静电场能。另外一部分为导体系空间几何结构变化，电源克服静电场力所作的功。

$$\boldsymbol{F}\cdot\delta\boldsymbol{l}=\sum\varphi_i\delta q_i-\frac{1}{2}\sum\varphi_i\delta q_i$$

电源对系统所做的总功　　静电场能量改变量

$$\therefore\quad \boldsymbol{F}=\nabla W_e\Big|_{\varphi=C}$$




【例3】 平行板电容器宽长度为 $L$,宽度为$b$,间距为$d$ 。电容器两板极之间的部分区域充满了电介质。如果将平行板电容器接入电压为$V_0$ 的直流电源，求电容器的储能；求介质板在拉出时受到的作用力。

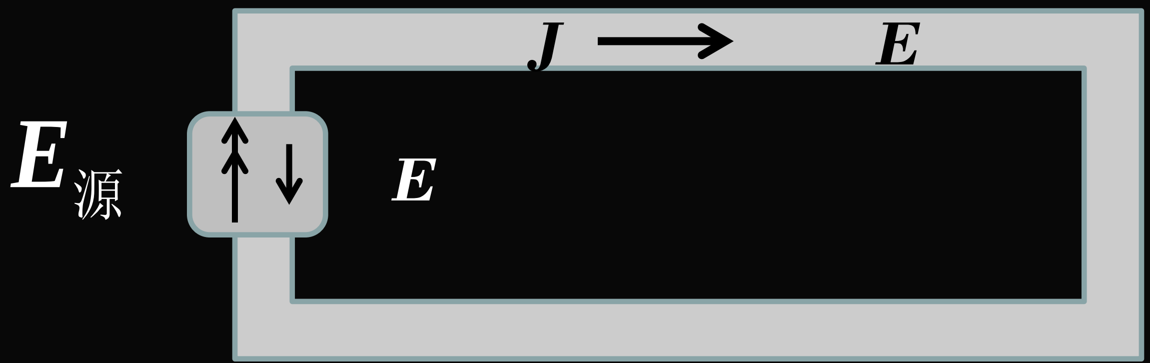
忽略边缘效应，两板极之间的电场为：

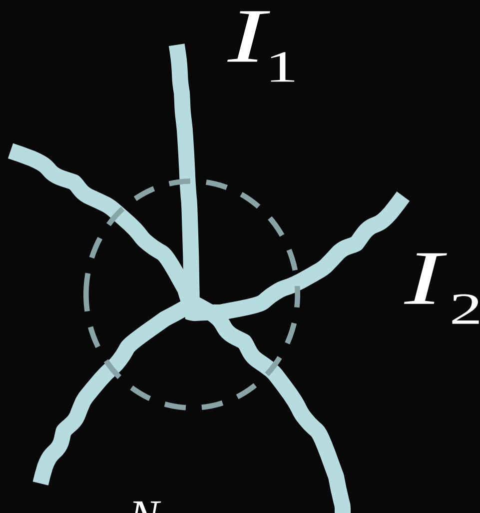
$$E=\frac{V_0}{d} \quad , \quad D=\begin{cases}\dfrac{V_0}{d}\varepsilon, & \text{介质中}\\ \dfrac{V_0}{d}\varepsilon_0, & \text{介质外}\end{cases}$$

$$W_e=\frac{1}{2}\iiint_V w_e(\boldsymbol{r})\mathrm{d}V=\frac{1}{2}\left(\frac{V_0}{d}\right)^2 db\left[\varepsilon_0(l-x)+\varepsilon x\right]$$

$$\boldsymbol{F}=\nabla W_e=\hat{e}_x\frac{1}{2}\left(\frac{V_0}{d}\right)^2 db(\varepsilon-\varepsilon_0)$$

# * 恒定电流的电场




$$\left.\begin{array}{l}\boldsymbol{J}=\sigma\boldsymbol{E}\\ \nabla\cdot\boldsymbol{E}=0\\ \nabla\times\boldsymbol{E}=0\end{array}\right\}\rightarrow\nabla\cdot\boldsymbol{J}=0\rightarrow\oint\!\!\oint_S\boldsymbol{J}\cdot\mathrm{d}\boldsymbol{S}=0\rightarrow\sum_{i=1}^{N}I_i=0$$

$$\oint_L\boldsymbol{E}\cdot\mathrm{d}\boldsymbol{l}=\oint_L\left(\boldsymbol{E}+\boldsymbol{E}_{源}\right)\cdot\mathrm{d}\boldsymbol{l}=\oint_L\left(\frac{\boldsymbol{J}}{\sigma}+\boldsymbol{E}_{源}\right)\cdot\mathrm{d}\boldsymbol{l}\rightarrow\sum_{i=1}^{N}\mathcal{E}_i=0$$

# § 3.4 恒定电流的磁场

## 1 恒定电流磁场满足的方程

恒定电流产生的磁场满足：

$$\begin{cases}\nabla\times\boldsymbol{H}(\boldsymbol{r})=\boldsymbol{J}(\boldsymbol{r})\\ \nabla\cdot\boldsymbol{B}(\boldsymbol{r})=0\end{cases}\leftrightarrow\begin{cases}\oint_L\boldsymbol{H}(\boldsymbol{r})\cdot\mathrm{d}\boldsymbol{l}=\iint_S\boldsymbol{J}(\boldsymbol{r})\cdot\mathrm{d}\boldsymbol{s}\\ \oiint_S\boldsymbol{B}(\boldsymbol{r})\cdot\mathrm{d}\boldsymbol{s}=0\end{cases}$$

引入矢量函数 $\boldsymbol{A}(\boldsymbol{r})$ ，磁感应强度可表示为

$$\boldsymbol{B}(\boldsymbol{r})=\nabla\times\boldsymbol{A}(\boldsymbol{r})$$

称矢量函数$\boldsymbol{A}(\boldsymbol{r})$为磁矢势。存在的问题是：

引入：$\boldsymbol{A}'(\boldsymbol{r})=\boldsymbol{A}(\boldsymbol{r})\pm\nabla\varphi(\boldsymbol{r})$

$$\boldsymbol{B}(\boldsymbol{r})=\nabla\times\boldsymbol{A}(\boldsymbol{r})=\nabla\times\boldsymbol{A}'(\boldsymbol{r})$$

$$\begin{pmatrix}\boldsymbol{A}(\boldsymbol{r})\\ \boldsymbol{A}'(\boldsymbol{r})\end{pmatrix}\Rightarrow \text{描述同一个}\boldsymbol{B}(\boldsymbol{r})$$

产生这一问题的原因？

造成磁矢势不唯一的原因是：

$\boldsymbol{A}(\boldsymbol{r})$旋度由 $\boldsymbol{B}(\boldsymbol{r})=\nabla\times\boldsymbol{A}(\boldsymbol{r})$确定

而 $\boldsymbol{A}(\boldsymbol{r})$的散度 $\nabla\cdot\boldsymbol{A}(\boldsymbol{r})$ 没有唯一确定。

为使 $A$ 与 $B$ 之间是唯一对应关系，对磁矢势须附加确定散度条件，才能唯一确定。

令磁矢势满足 $\nabla \cdot \boldsymbol{A}(\boldsymbol{r}) = 0$

$$\nabla \times \boldsymbol{B}(\boldsymbol{r}) = \nabla \times \nabla \times \boldsymbol{A}(\boldsymbol{r})$$

$$= \nabla\left[\nabla \cdot \boldsymbol{A}(\boldsymbol{r})\right] - \nabla^2 \boldsymbol{A}(\boldsymbol{r}) = \mu \boldsymbol{J}(\boldsymbol{r})$$

$$\Rightarrow \nabla^2 \boldsymbol{A}(\boldsymbol{r}) = -\mu \boldsymbol{J}(\boldsymbol{r})$$

这是一个矢量Poisson方程，包含三个标量Poisson方程，是恒定电流磁场的基本方程

# 2 边界条件

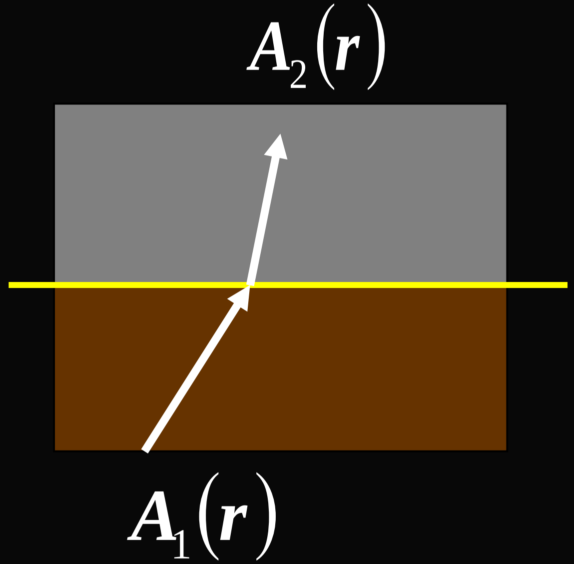
利用磁场在两介质边界上满足的条件

$$\begin{cases} \hat{n}\cdot\left(\boldsymbol{B}_2-\boldsymbol{B}_1\right)=0 \\ \hat{n}\times\left(\boldsymbol{H}_2-\boldsymbol{H}_1\right)=\boldsymbol{J}_s \end{cases} \rightarrow \begin{cases} \hat{\boldsymbol{n}}\cdot\left[\nabla\times\boldsymbol{A}_2-\nabla\times\boldsymbol{A}_1\right]=0 \\ \hat{\boldsymbol{n}}\times\left[\dfrac{\nabla\times\boldsymbol{A}_2}{\mu_2}-\dfrac{\nabla\times\boldsymbol{A}_1}{\mu_1}\right]=\boldsymbol{J}_s \end{cases}$$

此外还可利用磁矢势与磁场关系

$$\nabla\cdot\boldsymbol{A}(\boldsymbol{r})=0\rightarrow\oiint_S\boldsymbol{A}(\boldsymbol{r})\cdot\mathrm{d}\boldsymbol{S}=0$$

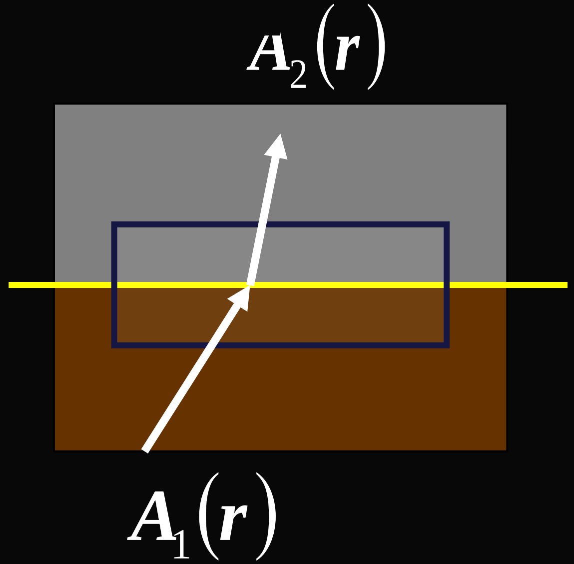
$$\boldsymbol{n}\cdot\left[\boldsymbol{A}_2-\boldsymbol{A}_1\right]=0\rightarrow A_{1n}=A_{2n}$$



$$\boldsymbol{B}(\boldsymbol{r}) = \nabla \times \boldsymbol{A}(\boldsymbol{r})$$

$$\rightarrow \iint_S \boldsymbol{B}(\boldsymbol{r}) \cdot \mathrm{d}\boldsymbol{s} = \iint_S \nabla \times \boldsymbol{A}(\boldsymbol{r}) \cdot \mathrm{d}\boldsymbol{s} = \oint_L \boldsymbol{A}(\boldsymbol{r}) \cdot \mathrm{d}\boldsymbol{L}$$

$$\rightarrow \boldsymbol{n} \times \left[\boldsymbol{A}_2 - \boldsymbol{A}_1\right] = 0 \rightarrow A_{2t} = A_{1t}$$




导出磁矢势的边界条件：

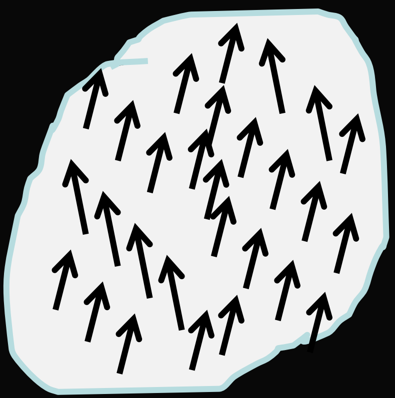
$$\left[\boldsymbol{A}_2 - \boldsymbol{A}_1\right]_{界面} = 0$$

# 3 恒定电流磁场的定解问题

$$\begin{cases}\nabla^2 \boldsymbol{A}(\boldsymbol{r}) = -\mu \boldsymbol{J}(\boldsymbol{r}) \\ \left(\boldsymbol{A}_2 - \boldsymbol{A}_1\right)\Big|_{\text{边界面}} = 0\end{cases} \rightarrow \begin{cases}\nabla^2 A_i(\boldsymbol{r}) = -\mu J_i(\boldsymbol{r}) \\ \left(\boldsymbol{A}_i - \boldsymbol{A}_j\right)\Big|_{\text{边界面}} = 0\end{cases} \quad i = 1,2,3$$

无界空间的基本解为：

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu}{4\pi}\iiint_V \frac{\boldsymbol{J}(\boldsymbol{r}')\mathrm{d}\boldsymbol{r}'}{|\boldsymbol{r}-\boldsymbol{r}'|}$$

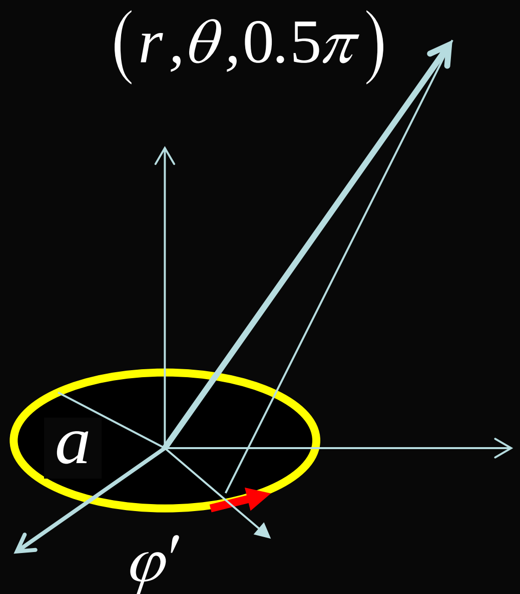


$$= \frac{\mu}{4\pi}\iiint_V \frac{\boldsymbol{J}(x',y',z')\mathrm{d}x'\mathrm{d}y'\mathrm{d}z'}{\sqrt{(x-x')^2+(y-y')^2(z-z')^2}}$$

# 4 小电流环—磁偶极矩

## 1）磁偶极矩产生的磁场

电流以 $z$ 为对称轴，磁矢势与 $\varphi$ 无关。



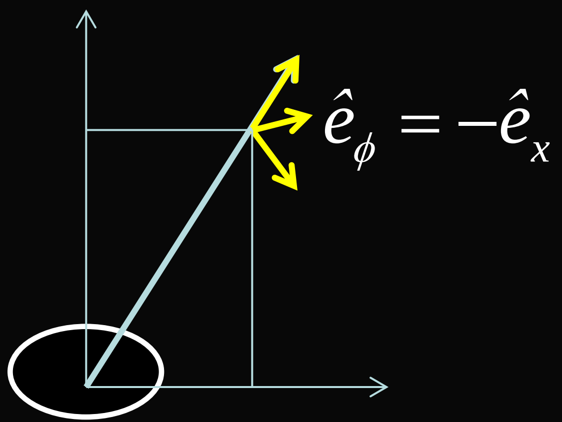

$$\boldsymbol{P}_m = \pi a^2 I\hat{\boldsymbol{n}}$$

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0}{4\pi}\oint_l \frac{I\mathrm{d}\boldsymbol{l}}{|\boldsymbol{r}-\boldsymbol{r}'|}$$

$$I\mathrm{d}\boldsymbol{l} = \left(-\hat{e}_x\sin\varphi' + \hat{e}_y\cos\varphi'\right)a\mathrm{d}\varphi'$$

$$R^{-1} = \left[r^2 + a^2 - 2ar\sin\theta\sin\varphi'\right]^{-1/2}$$

$$\approx \frac{1}{r}\left(1 + \frac{a}{r}\sin\theta\sin\varphi'\right)$$



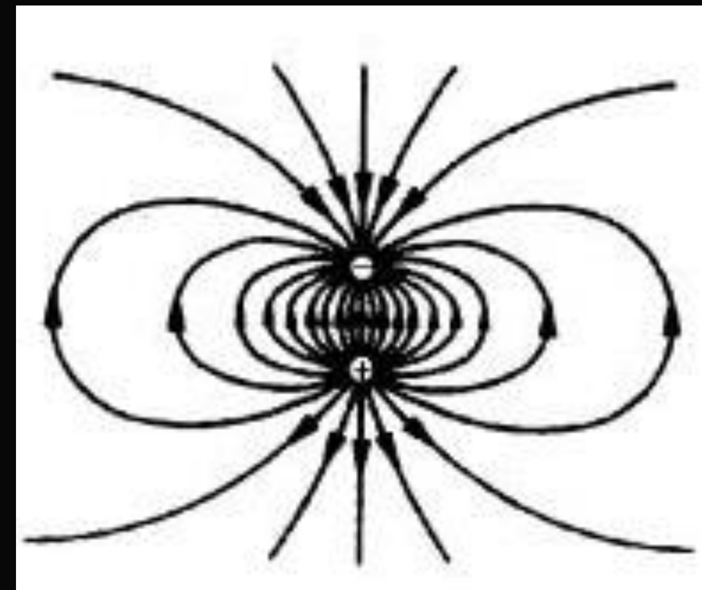

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0}{4\pi}\oint_l \frac{I\mathrm{d}\boldsymbol{l}}{\left|\boldsymbol{r}-\boldsymbol{r}'\right|}$$

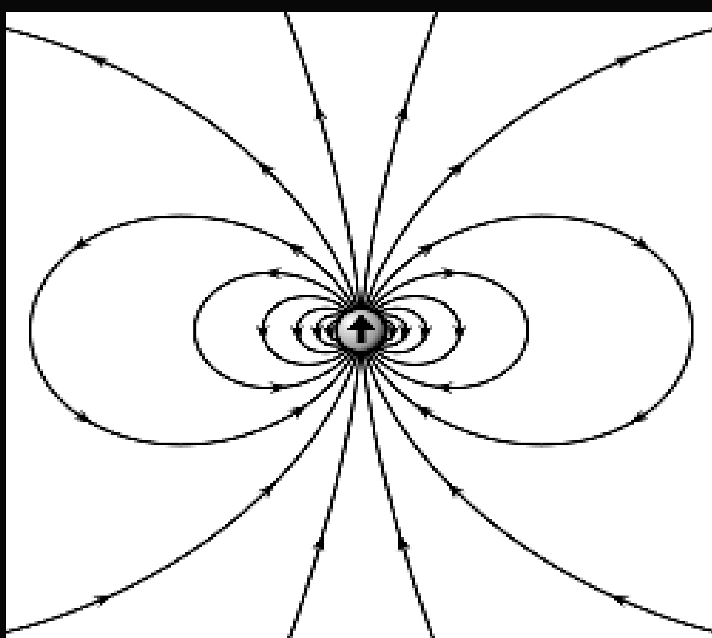
$$= \frac{\mu_0 I}{4\pi}\int_0^{2\pi}\frac{1}{r}\left(1+\frac{a}{r}\sin\theta\sin\phi'\right)\left(-\hat{e}_x\sin\phi' + \hat{e}_y\cos\phi'\right)a\,\mathrm{d}\phi'$$

$$= \frac{\mu_0 I\pi a^2}{4\pi r^2}\sin\theta(-\hat{e}_x) = \hat{e}_{\phi}\frac{\mu_0 I\pi a^2}{4\pi r^2}\sin\theta = \frac{\mu_0 \boldsymbol{P}_m \times \boldsymbol{r}}{4\pi r^3}$$

$$\boldsymbol{B}(\boldsymbol{r}) = \nabla\times\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0 p_m}{4\pi r^3}\left(2\hat{e}_r\cos\theta + \hat{e}_{\theta}\sin\theta\right)$$

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{P_e}{4\pi\varepsilon_0 r^3}\left(\hat{e}_r 2\cos\theta + \hat{e}_\theta \sin\theta\right)$$

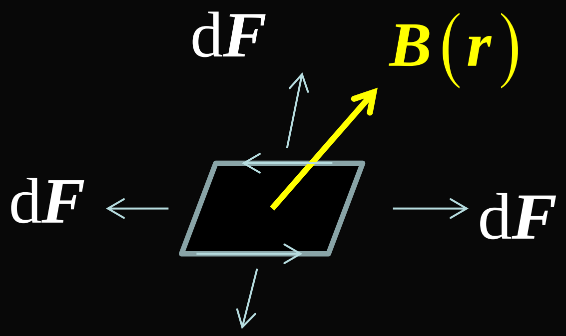




$$\boldsymbol{B}(\boldsymbol{r}) = \frac{\mu_0 p_m}{4\pi r^3}\left(2\hat{e}_r \cos\theta + \hat{e}_\theta \sin\theta\right)$$

## 2）外场中的磁偶极矩

$$\boldsymbol{P}_{\mathrm{m}}=\frac{1}{2}\iiint\limits_{V}\boldsymbol{r}'\times\boldsymbol{J}\left(\boldsymbol{r}'\right)\mathrm{d}\boldsymbol{r}'$$




$$\mathrm{d}\boldsymbol{F}=I\mathrm{d}\boldsymbol{l}\times\boldsymbol{B}\left(\boldsymbol{r}\right)$$

磁偶极矩在外磁场中受的力

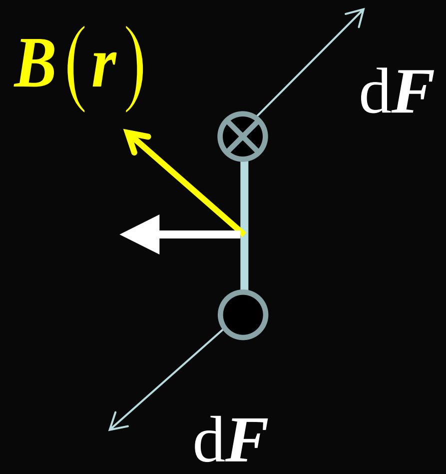
$$\boldsymbol{F}=\oint_{C}I\mathrm{d}\boldsymbol{l}\times\boldsymbol{B}\left(\boldsymbol{r}\right)=-\boldsymbol{B}\left(\boldsymbol{r}\right)\times\oint_{C}I\mathrm{d}\boldsymbol{l}$$

$$=-\boldsymbol{B}\left(\boldsymbol{r}\right)\times\iint\mathrm{d}\boldsymbol{S}\times\nabla I=0$$

# 磁偶极矩在外磁场中受的力矩

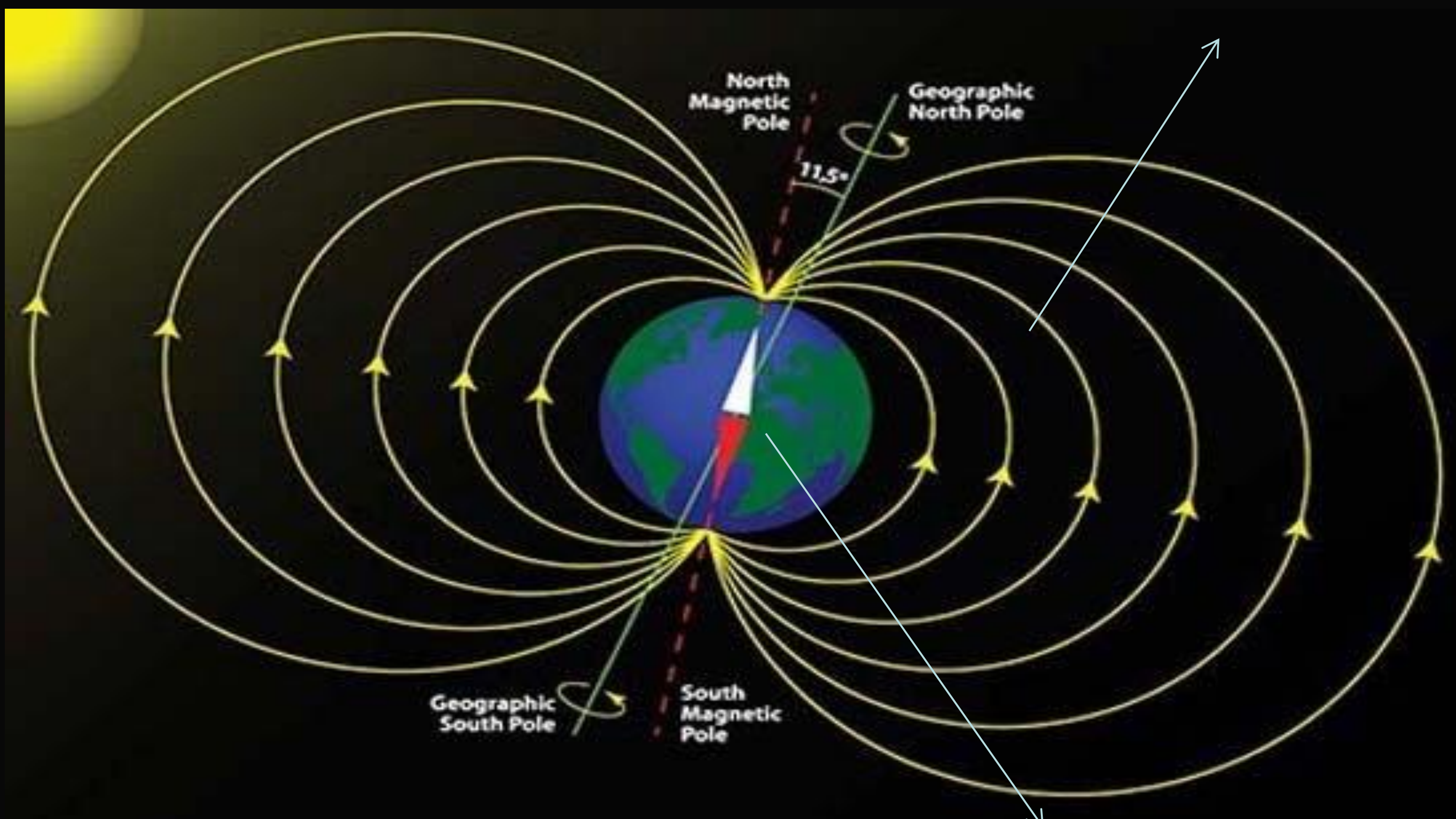
$$\boldsymbol{L}=\oint_C \boldsymbol{r}\times I\mathrm{d}\boldsymbol{l}\times\boldsymbol{B}(\boldsymbol{r})$$

$$=I\left[\oint_C \boldsymbol{r}\times\mathrm{d}\boldsymbol{l}\right]\times\boldsymbol{B}(\boldsymbol{r})=\boldsymbol{P}_m\times\boldsymbol{B}(\boldsymbol{r})\neq 0$$



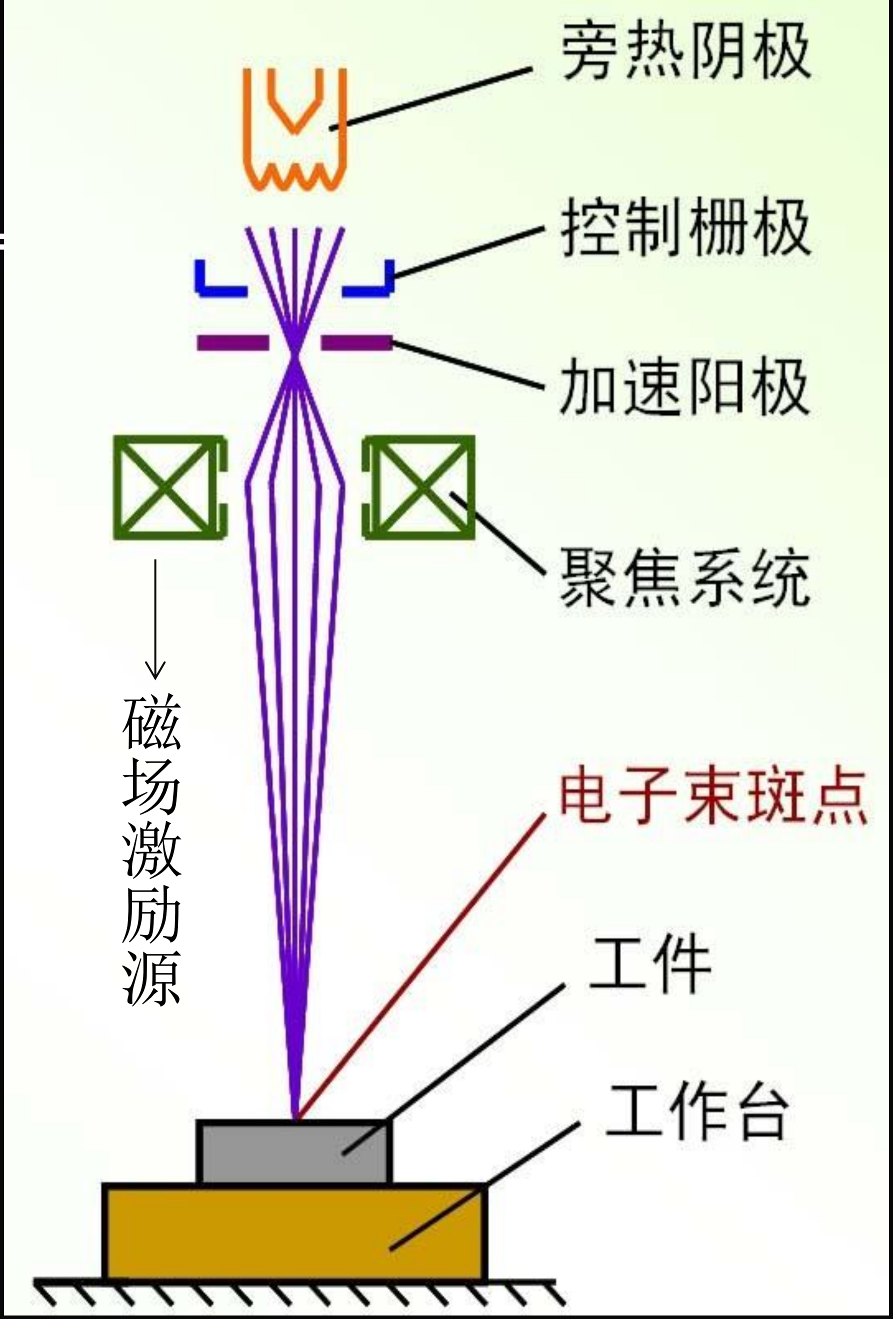
# 5 磁场的标位函数

无激励源磁场分布区 $\nabla\times\boldsymbol{H}(\boldsymbol{r})=0$




$\nabla\times\boldsymbol{H}(\boldsymbol{r})=\boldsymbol{J}$ 磁场激励源分布区

在无电流分布的区域，磁场为无旋矢量场，具有静电场特点。因此在不包含电流源的区域内的磁场可表示为某标量场的梯度。其分析方法与静态电场分析相同。



引入标量函数 $\phi_m(\boldsymbol{r})$，在无电流区域上磁场可以表示为：

$$\begin{cases} \nabla\times\boldsymbol{H}(\boldsymbol{r})=0 \\ \boldsymbol{H}(\boldsymbol{r})=-\nabla\phi_m(\boldsymbol{r}) \end{cases} \qquad \begin{cases} \nabla\times\boldsymbol{E}(\boldsymbol{r})=0 \\ \boldsymbol{E}(\boldsymbol{r})=-\nabla\phi(\boldsymbol{r}) \end{cases}$$

$\phi_m(\boldsymbol{r})$称为磁标位。必须注意的是，磁标位只能在没有传导电流的空间区域引入。这一方法对于讨论介质中磁场的求解方程方便。

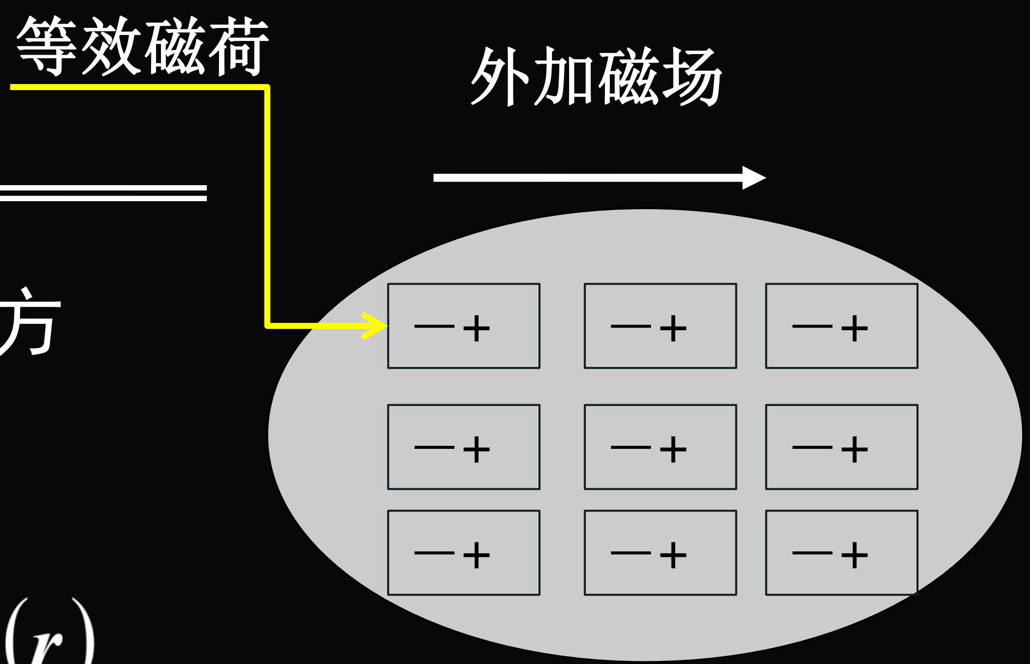
利用磁感应强度的无散特性和磁场定义，得到：

$$\nabla\cdot\boldsymbol{B}(\boldsymbol{r})=\nabla\cdot\left[\mu_0\boldsymbol{H}(\boldsymbol{r})+\boldsymbol{M}(\boldsymbol{r})\right]=0$$

定义假想磁荷密度为： $\rho_m=-\mu_0\nabla\cdot\boldsymbol{M}(\boldsymbol{r})$

得到磁场满足的方程： $\nabla\cdot\boldsymbol{H}(\boldsymbol{r})=-\dfrac{\rho_m}{\mu_0}$

介质中磁标位满足的方程及其边界条件是：




$$
\begin{cases}
\nabla^2 \phi_m(\boldsymbol{r}) = -\dfrac{1}{\mu_0}\rho_m(\boldsymbol{r}) \\
\phi_{m1}(\boldsymbol{r})\big|_S = \phi_{m2}(\boldsymbol{r})\big|_S \\
\mu_1 \dfrac{\partial \phi_{m1}}{\partial n}\Big|_S = \mu_2 \dfrac{\partial \phi_{m2}}{\partial n}\Big|_S
\end{cases}
\Leftarrow
\begin{cases}
\hat{n}\cdot(\boldsymbol{B}_2 - \boldsymbol{B}_1) = 0 \\
\hat{n}\times(\boldsymbol{H}_2 - \boldsymbol{H}_1) = 0
\end{cases}
$$

# 介质中电位和磁标位的比较

$$\left\{\begin{array}{l}\nabla\times\boldsymbol{E}(\boldsymbol{r})=0\\ \nabla\cdot\boldsymbol{E}(\boldsymbol{r})=\varepsilon_0^{-1}(\rho+\rho_P)\\ \boldsymbol{D}(\boldsymbol{r})=\varepsilon_0\boldsymbol{E}(\boldsymbol{r})+\boldsymbol{P}(\boldsymbol{r})\\ \boldsymbol{E}(\boldsymbol{r})=-\nabla\varphi(\boldsymbol{r})\\ \nabla^2\varphi(\boldsymbol{r})=-\varepsilon_0^{-1}(\rho+\rho_P)\\ \rho_p=-\nabla\cdot\boldsymbol{P}(\boldsymbol{r})\end{array}\right\}\leftrightarrow\left\{\begin{array}{l}\nabla\times\boldsymbol{H}(\boldsymbol{r})=0\\ \nabla\cdot\boldsymbol{H}(\boldsymbol{r})=\mu_0^{-1}\rho_m\\ \boldsymbol{B}(\boldsymbol{r})=\mu_0\boldsymbol{H}(\boldsymbol{r})+\boldsymbol{M}(\boldsymbol{r})\\ \boldsymbol{H}(\boldsymbol{r})=-\nabla\varphi_m(\boldsymbol{r})\\ \nabla^2\varphi_m(\boldsymbol{r})=-\mu_0^{-1}\rho_m\\ \rho_m=-\mu_0\nabla\cdot\boldsymbol{M}(\boldsymbol{r})\end{array}\right\}$$

$$\left.\begin{cases} \varphi_1(\boldsymbol{r})\big|_S = \varphi_2(\boldsymbol{r})\big|_S \\ \varepsilon_1 \dfrac{\partial \varphi}{\partial n}\Big|_S - \varepsilon_2 \dfrac{\partial \varphi}{\partial n}\Big|_S = \rho_s \end{cases}\right\} \leftrightarrow \left.\begin{cases} \varphi_{m1}(\boldsymbol{r})\big|_S = \varphi_{m2}(\boldsymbol{r})\big|_S \\ \mu_1 \dfrac{\partial \varphi_{m1}}{\partial n}\Big|_S = \mu_2 \dfrac{\partial \varphi_{m2}}{\partial n}\Big|_S \end{cases}\right\}$$

【例】证明 $\mu\to\infty$ 的磁性介质为等磁位体

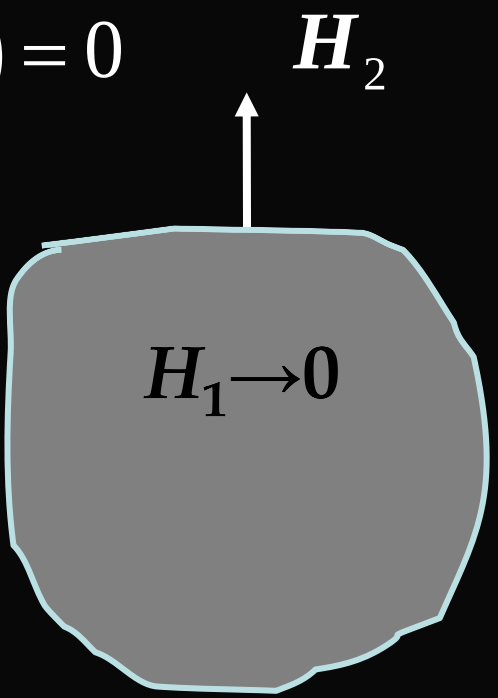
证：下标1代表磁性介质，2代表真空

$$\boldsymbol{n}\cdot\left(\boldsymbol{B}_2-\boldsymbol{B}_1\right)=0 \quad , \quad \boldsymbol{n}\times\left(\boldsymbol{H}_2-\boldsymbol{H}_1\right)=0$$

$$\boldsymbol{B}_2=\mu_0\boldsymbol{H}_2,\quad \boldsymbol{B}_1=\mu\boldsymbol{H}_1$$

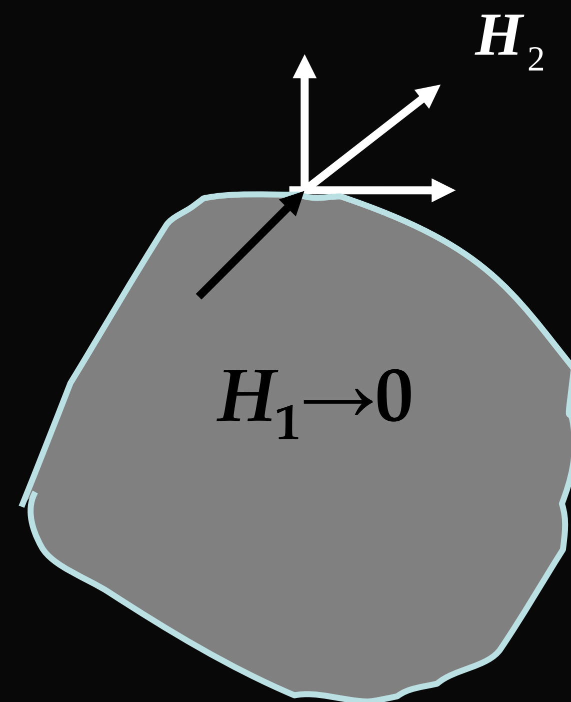
$$\mu_0 H_{2n}=\mu H_{1n}, H_{2t}=H_{1t}$$

$$\frac{H_{2t}}{H_{2n}}=\frac{\mu_0}{\mu}\frac{H_{1t}}{H_{1n}}\to 0\Rightarrow\left(\begin{matrix}H_{2t}\\ H_{1t}\end{matrix}\right)\to 0$$



$$\frac{\mu_0}{\mu} H_{2n} = H_{1n} \to 0$$

$$H_{1n} \to 0 \ , \ H_{1t} \to 0 \ , \ \boldsymbol{H}_1 \to 0$$




磁性介质中磁场为零等磁位体（与理想导体对应），称为理想导磁体

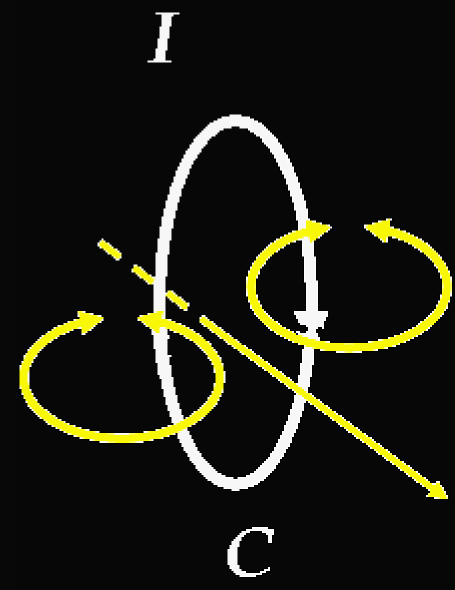
# §5 磁场的能量与作用力

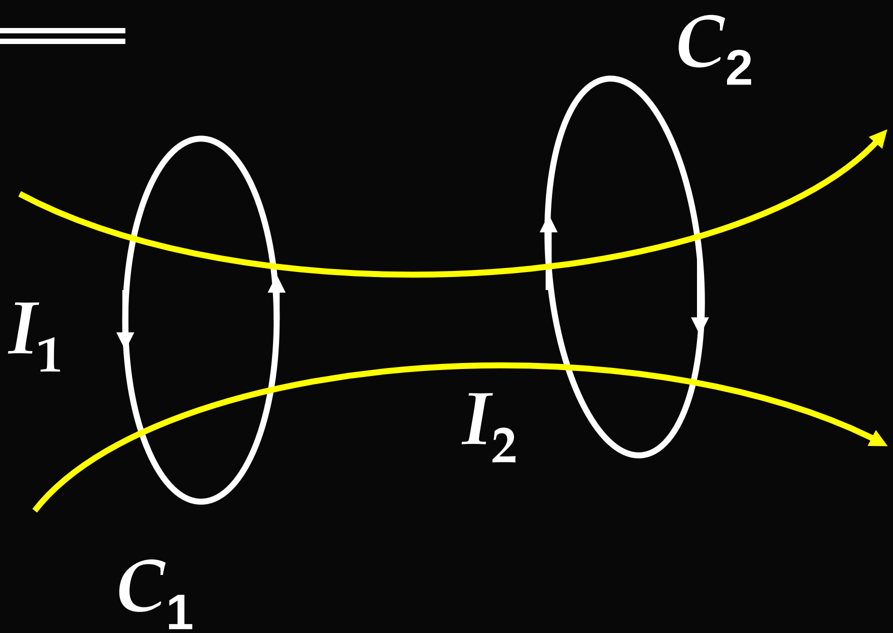
## 1. 自感与互感现象

### 自感现象：

闭合线圈 $C$ 自身电流变化

激发电动势的现象称为自感现象






互感现象：

线圈$C_1$中的电流的变化在线圈$C_2$激发电动势的现象称为互感现象

## 2、自感与互感系数

电流环$\boldsymbol{C_1}$在空间产生磁场，该磁场对以回路$\boldsymbol{C_2}$为边界的曲面的磁通量（称为磁通匝链数）为：

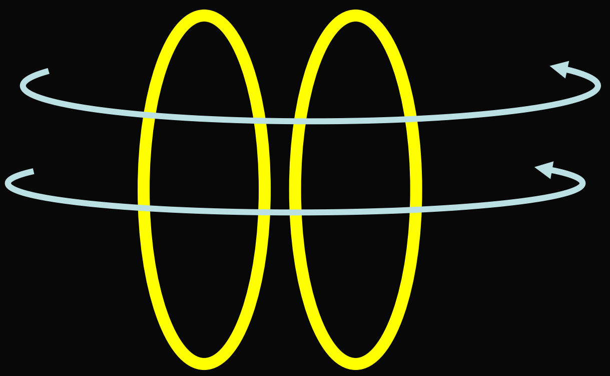
$$\psi_{12} = \iint_S \boldsymbol{B}(\boldsymbol{r}) \cdot \mathrm{d}\boldsymbol{s} = I_0 \iint_S \left( \frac{\mu}{4\pi} \oint_{C_1} \frac{\mathrm{d}\boldsymbol{l}_1 \times \boldsymbol{R}}{R^3} \right) \cdot \mathrm{d}\boldsymbol{s}$$

$$= \iint_S \nabla \times \boldsymbol{A}(\boldsymbol{r}) \cdot \mathrm{d}\boldsymbol{s} = I_0 \oint_{C_2} \frac{\mu}{4\pi} \oint_{C_1} \frac{\mathrm{d}\boldsymbol{l}_1 \cdot \mathrm{d}\boldsymbol{l}_2}{R}$$

比值： $$\frac{\psi_{12}}{I_0} = \frac{1}{I_0}\oint_{C_2} \boldsymbol{A}(\boldsymbol{r})\cdot \mathrm{d}\boldsymbol{l}_2 = \oint_{C_2}\left(\frac{\mu}{4\pi}\oint_{C_1}\frac{\mathrm{d}\boldsymbol{l}_1}{R}\right)\cdot \mathrm{d}\boldsymbol{l}_2$$

为与回路$C_1$上的电流强度无关，与空间介质磁导率、以及$C_1$和$C_2$的几何位形结构有关的常量。

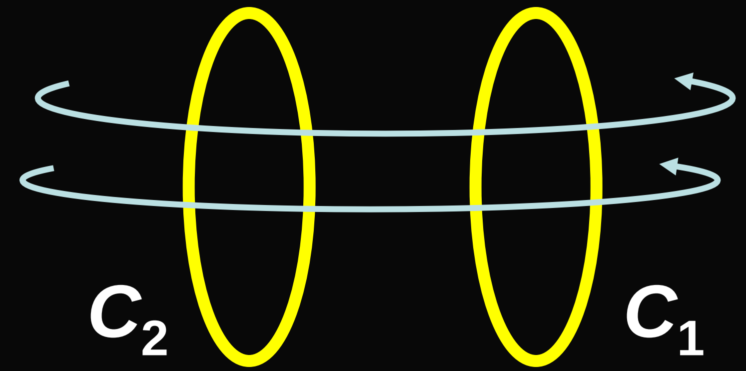
描述了载流线圈单位电流强度在另一回路为边界曲面上产生磁通量的能力，称之为电感系数。
它与电容、电阻一起构成了电路的基本参量。



$C_2 \rightarrow C_1$

$C_1$中的电流在其自身边界的曲面上产生磁通量与$C_1$上的电流强度之比为自感系数，记为$L$，使$C_2 \rightarrow C_1 = C$，得到：

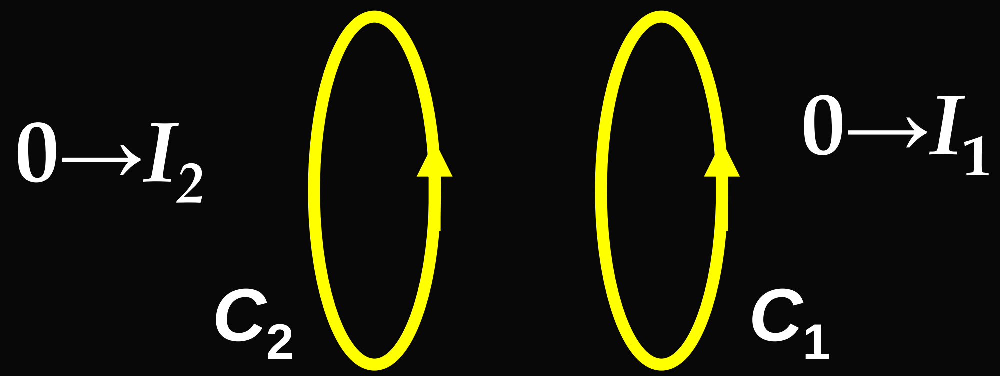
$$L = \frac{\psi_{11}}{I_0} = \frac{\mu}{4\pi}\oint_C \left( \oint_C \frac{\mathrm{d}\boldsymbol{l}}{R} \right) \cdot \mathrm{d}\boldsymbol{l}'$$




$C_1$中电流在$C_2$为边界的曲面上产生的磁通量与$C_1$中的电流强度之为互感系数，记为$M_{12}$

$$M_{12}=\frac{\psi_{12}}{I_0}=\oint_{C_2}\left(\frac{\mu}{4\pi}\oint_{C_1}\frac{\mathrm{d}\boldsymbol{l}_1}{R}\right)\cdot\mathrm{d}\boldsymbol{l}_2$$

# 3. 磁场的能量




磁场能来源：

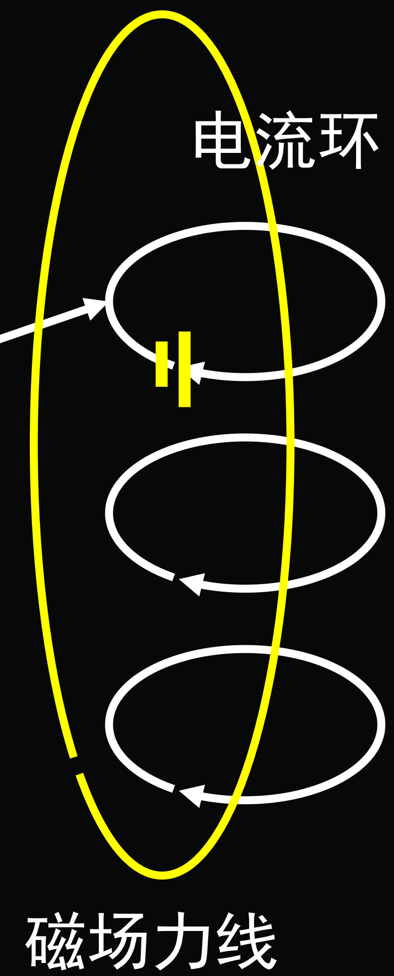
电源克服回路上感应电动势作功转变为线圈回路系统的磁场能（电流建立过程中没有其它形式的能量损耗）

dt 时间内电源对回路 $j$ 所作的功

$$\mathrm{d}W_j = -\varepsilon_j \mathrm{d}q_j = \frac{\mathrm{d}\psi_j}{\mathrm{d}t} i_j \mathrm{d}t = i_j \mathrm{d}\psi_j$$

$$\mathrm{d}W = \sum_{j=1}^{N} \mathrm{d}W_j = \sum_{j=1}^{N} i_j \mathrm{d}\psi_j$$

$$\psi_j = \sum_{k=1}^{N} M_{kj} i_k$$

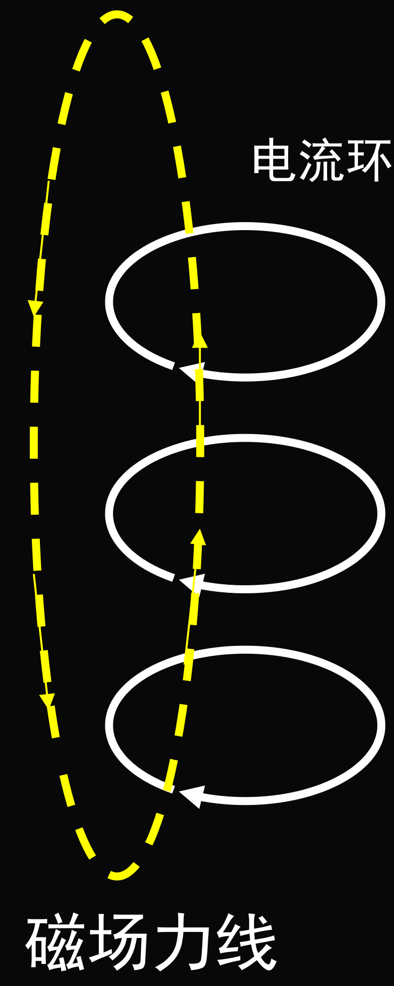


$$\mathrm{d}W=\sum_{j=1}^{N}\mathrm{d}W_j=\sum_{j=1}^{N}i_j\mathrm{d}\psi_j \qquad \psi_j=\sum_{k=1}^{N}M_{kj}i_k$$

设各线圈电流按同比例线性由 $0\rightarrow I_k$

$$W_m=W=\sum_{j=1}^{N}\sum_{k=1}^{N}I_jM_{kj}I_k\int_0^1\alpha\mathrm{d}\alpha$$

$$=\frac{1}{2}\sum_{j=1}^{N}\sum_{k=1}^{N}M_{kj}I_jI_k=\frac{1}{2}\sum_{j=1}^{N}I_j\psi_j$$



$$W_m=\frac{1}{2}\sum_{j=1}^{N}I_j\psi_j=\frac{1}{2}\sum_{j=1}^{N}I_j\oint_{C_j}\boldsymbol{A}\cdot\mathrm{d}\boldsymbol{l}_j$$

$$=\frac{1}{2}\sum_{j=1}^{N}\oint_{C_j}\boldsymbol{A}\cdot I_j\mathrm{d}\boldsymbol{l}_j=\iiint_V\frac{1}{2}\boldsymbol{A}\cdot\boldsymbol{J}\mathrm{d}V$$

$\frac{1}{2}\boldsymbol{A}(\boldsymbol{r})\cdot\boldsymbol{J}(\boldsymbol{r})$能否代表能量密度?

体积分区域的选择和其具体含意!

$$\boldsymbol{A}\cdot\nabla\times\boldsymbol{H}=\nabla\cdot\left(\boldsymbol{H}\times\boldsymbol{A}\right)+\boldsymbol{H}\cdot\nabla\times\boldsymbol{A}$$

$$W_m=\frac{1}{2}\iiint_V \boldsymbol{A}\left(\boldsymbol{r}\right)\cdot\nabla\times\boldsymbol{H}\left(\boldsymbol{r}\right)\mathrm{d}V$$

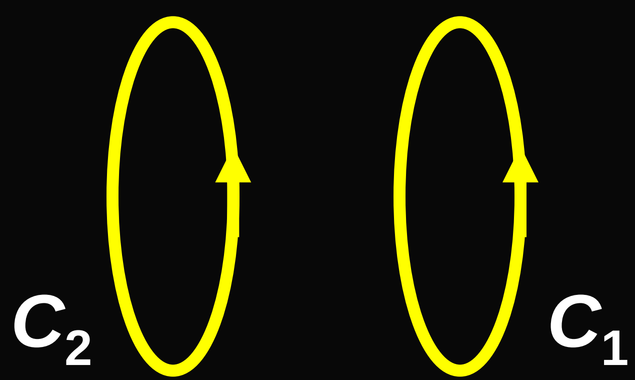
$$=\frac{1}{2}\iiint_V \boldsymbol{H}\left(\boldsymbol{r}\right)\cdot\nabla\times\boldsymbol{A}\left(\boldsymbol{r}\right)\mathrm{d}V-\frac{1}{2}\oiint_S\left(\boldsymbol{A}\times\boldsymbol{H}\right)\cdot\mathrm{d}\boldsymbol{s}=\iiint_V\frac{1}{2}\boldsymbol{H}\left(\boldsymbol{r}\right)\cdot\boldsymbol{B}\left(\boldsymbol{r}\right)\mathrm{d}V$$

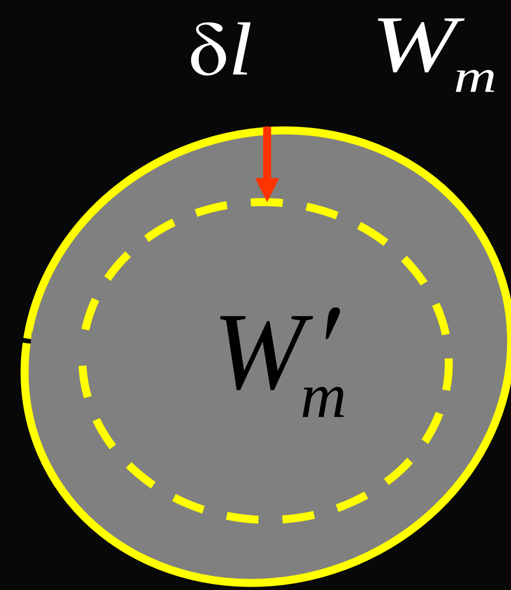
$$w_m\left(\boldsymbol{r}\right)=\frac{1}{2}\boldsymbol{B}\left(\boldsymbol{r}\right)\cdot\boldsymbol{H}\left(\boldsymbol{r}\right)$$

体积分区域的选择和其具体含意！

# 4 载流体系的磁场作用力

$$\Delta W_m = W_m - W_m'$$



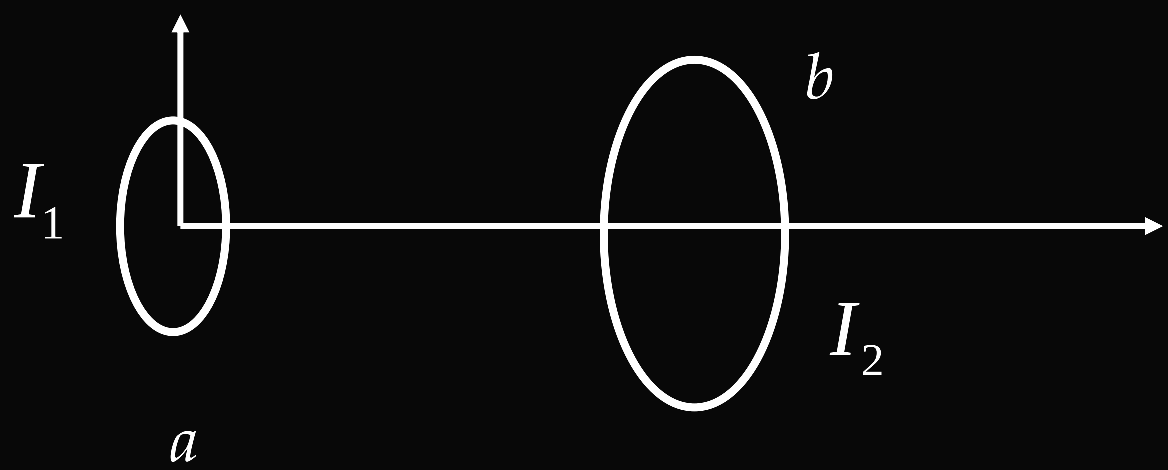
在磁场力的作用下，载流体系发生了小的位移 $\delta l$ ，磁场力所作的虚拟功为

$$\delta A = \boldsymbol{F} \cdot \delta \boldsymbol{l}$$

$$\boldsymbol{F}_m = -\left[ \hat{e}_x \frac{\partial W_m}{\partial x} + \hat{e}_y \frac{\partial W_m}{\partial y} + \hat{e}_z \frac{\partial W_m}{\partial z} \right] = -\nabla W_m$$

对于两个载流线圈，其磁场能为

$$W_m = \frac{1}{2}\sum_{j=1}^{2}\sum_{k=1}^{2} M_{kj} I_j I_k = \frac{1}{2} L_1 I_1^2 + \frac{1}{2} L_2 I_2^2 + M_{12} I_1 I_2$$



如果两线圈的几何形状和电流保持不变，

线圈 1 相对于线圈 2 有微小的位移 $\delta l$ ，

电源克服线圈感应电动势所做的功为

$$\delta W = \sum_{j=1}^{2} -\varepsilon_j \delta q_j = I_1 \delta \psi_1 + I_2 \delta \psi_2$$

$$= I_1 I_2 \delta M_{21} + I_1 I_2 \delta M_{12} = 2 I_1 I_2 \delta M$$

$$W_m = \frac{1}{2}\sum_{j=1}^{2}\sum_{k=1}^{2} M_{kj} I_j I_k = \frac{1}{2}L_1 I_1^2 + \frac{1}{2}L_2 I_2^2 + M_{12} I_1 I_2$$

一部分为磁场能的增量，数值为：$I_1 I_2 \delta M$

另一部分为线圈位移所消耗的能量

$$F_m\Big|_{\delta \boldsymbol{l}方向} \cdot \delta \boldsymbol{l} = \delta W - \delta W_m\Big|_{I恒定} = I_1 I_2 \delta M$$

两线圈在位移方向所受到的作用力为

$$F_m\Big|_{\delta \boldsymbol{l}方向} = \frac{\delta W - \delta W_m}{\delta l}\Big|_{I恒定} = I_1 I_2 \frac{\delta M}{\delta l}$$