习题

第一章

A组习题：

1.10.

解法1.

$$a+bi=\frac{x+iy}{x-yi}=\frac{(x+iy)^2}{x^2+y^2}=\frac{x^2-y^2}{x^2+y^2}+\frac{2xyi}{x^2+y^2}$$

$$a=\frac{x^2-y^2}{x^2+y^2}\qquad b=\frac{2xy}{x^2+y^2}$$

$$a^2+b^2=\frac{(x^2-y^2)^2+4x^2y^2}{(x^2+y^2)^2}=\frac{(x^2+y^2)^2}{(x^2+y^2)^2}=1$$

解法2：

$$a^2+b^2=(a+bi)(a-bi)=(a+bi)\overline{(a+bi)}$$

$$=\frac{x+iy}{x-yi}\cdot\overline{\left(\frac{x+iy}{x-yi}\right)}=\frac{x+iy}{x-yi}\cdot\frac{\overline{x+iy}}{\overline{x-yi}}=1$$

说明：方法一利用复数相等条件

方法二利用共轭复数性质 $|z|^2=z\cdot\bar z$

B组习题：

1.4

解：$\left|\frac{z-a}{1-\bar a z}\right|=r$.

$$|z-a|=r|a|\,|z-\tfrac{1}{\bar a}|$$

$$(z-a)(\bar z-\bar a)=r^2|a|^2(z-\tfrac{1}{\bar a})(\bar z-\tfrac{1}{a})$$

$$(1-r^2|a|^2)z\bar z-(a-r^2a)\bar z-[\bar a-r^2\bar a]z+|a|^2-r^2=0$$

令 $\lambda=\frac{a-r^2\bar a}{1-r^2|a|^2}$ 则 $\bar\lambda=\frac{\bar a-r^2a}{1-r^2|a|^2}$

$$R^2=\frac{r^2-|a|^2}{1-r^2|a|^2}$$

上式化为

$$z\bar z-\lambda\bar z-\bar\lambda z-R^2=0$$

$$|z-\lambda|^2=R^2+|\lambda|^2=\frac{r^2[1+|a|^4-a^2-\bar a^2]}{(1-r^2|a|^2)^2}$$

故圆心 $\lambda=\frac{\bar a-r^2a}{1-r^2|a|^2}$ $\because 1+|a|^4-a^2-\bar a^2>0$

故半径为 $\frac{r}{|1-r^2|a|^2|}\sqrt{1+|a|^4-(a^2+\bar a^2)}$

说明：结合圆方程表达形式

$z\bar z-\lambda\bar z-\bar\lambda z-R^2\Rightarrow|z-\lambda|^2=R^2+|\lambda|^2$

当 $R^2+|\lambda|^2>0$ 方程表示为圆．注意 $a\neq 1$

半径为 $\sqrt{R^2+|\lambda|^2}$ 圆心为 λ

1.6.

(1) 解法(1).

$$x^2+y^2=4\Rightarrow|z|=2$$

$$\because w=\frac{1}{z}\Rightarrow|w|=\frac{1}{2}$$

故映射到 w 平面为以坐标原点为圆心．半径为 $\frac{1}{2}$ 的圆．

解法(2)

$$x=\frac{z+\bar z}{2}=\frac{\frac{1}{w}+\frac{1}{\bar w}}{2}\quad(1)$$

$$y=\frac{z-\bar z}{2i}=\frac{\frac{1}{w}-\frac{1}{\bar w}}{2i}\quad(2)$$

$\because x^2+y^2=4$ 将 (1).(2) 代入

$$\therefore\frac{\frac{1}{w^2}+\frac{1}{\bar w^2}+\frac{2}{|w|^2}}{4}-\frac{\frac{1}{w^2}+\frac{1}{\bar w^2}-\frac{2}{|w|^2}}{4}=4$$

$$\therefore|w|^2=\frac{1}{4}\qquad|w|=\frac{1}{2}$$

故映射到 w 平面为以坐标原点为圆心．半径为 $\frac{1}{2}$ 的圆．

说明：解法(2)为一般解法，即对由一个平面曲线通过复映射到另一个复平面的曲线的求解．

1.12.

解．$x_n+iy_n=(1-i\sqrt3)^n=2^n\cdot(\frac{1}{2}-\frac{\sqrt3}{2}i)^n$

$$x_n-iy_n=2^n(\tfrac{1}{2}+\tfrac{\sqrt3}{2}i)^n$$

$x_ny_{n-1}-x_{n-1}y_n$ 为 $(x_n-iy_n)(x_{n-1}+iy_{n-1})$ 虚部

$$(x_n-iy_n)(x_{n-1}+iy_{n-1})=2^n\cdot(\tfrac{1}{2}+\tfrac{\sqrt3}{2}i)^n\cdot2^{n-1}\cdot(\tfrac{1}{2}+\tfrac{\sqrt3}{2}i)^{n-1}$$

$$=2^n\cdot2^{n-1}\cdot(\tfrac{1}{2}+\tfrac{\sqrt3}{2}i)$$

其虚部为 $2^{n-1}\cdot2^{n-1}\cdot\sqrt3=4^{(n-1)}\sqrt3$

故：$x_ny_{n-1}-x_{n-1}y_n=4^{(n-1)}\sqrt3$.

第二章：

A练习题：

2.2.(5)

解：$u=x^3-3xy^2$
$v=3x^2y-y^3$

$\frac{\partial u}{\partial x}=3x^2-3y^2$ $\frac{\partial u}{\partial y}=-6xy$

$\frac{\partial v}{\partial x}=6xy$ $\frac{\partial v}{\partial y}=3x^2-3y^2$

C-R条件：

$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}\Rightarrow 3x^2-3y^2=3x^2-3y^2$ 成立

$\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}\Rightarrow -6xy=-6xy$ 成立

故f(z)在复平面内可导.

2.5.

解 $\cos z+\sin z=0$

由余弦、正弦指数定义得：

$\frac{1}{2}(e^{iz}+e^{-iz})=-\frac{1}{2i}(e^{iz}-e^{-iz})$ 两边同乘 e^{iz}.

$e^{2iz}+1=i(e^{2iz}-1)$

$e^{2iz}=\frac{-1-i}{1-i}=-i=e^{i(2k\pi-\frac{\pi}{2})}$

故则 $z=k\pi-\frac{\pi}{4}$ k为整数.

注意在复数域求解 $\cos z$、$\sin z$ 化为 e^z 形式表达.

2.8 ①

解：$u=x^2-y^2+xy$.

$\frac{\partial u}{\partial x}=2x+y$ $\frac{\partial^2 u}{\partial x^2}=2$.

$\frac{\partial u}{\partial y}=-2y+x$ $\frac{\partial^2 u}{\partial y^2}=-2$

$\therefore \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$ 则 u为调和函数.

$f'(z)=\frac{\partial u}{\partial x}-i\frac{\partial u}{\partial y}$

$=2x+y-i(-2y+x)$

$=2x+2yi+y-xi$

$=2z-i(x+yi)$

$=(2-i)z$

2.9 (3)

解：(3) $u=\frac{y}{x^2+y^2}$ $f(1)=0$.

解：$\frac{\partial u}{\partial x}=\frac{-2xy}{(x^2+y^2)^2}$ $\frac{\partial u}{\partial y}=\frac{x^2-y^2}{(x^2+y^2)^2}$

方法(1) 偏积分法：

~~$\frac{\partial v}{\partial x}=-\frac{\partial u}{\partial y}=-\frac{x^2-y^2}{(x^2+y^2)^2}$~~

~~$v=\int\frac{-x^2+y^2}{(x^2+y^2)^2}dx+\varphi(y)$~~

~~$\therefore\frac{\partial v}{\partial y}=\frac{\partial\int\frac{-x^2+y^2}{(x^2+y^2)^2}dx}{\partial y}+\varphi'(y)=\frac{\partial u}{\partial x}=\frac{-2xy}{(x^2+y^2)^2}$~~

$\therefore\frac{\partial v}{\partial y}=+\frac{\partial u}{\partial x}=\frac{-2xy}{(x^2+y^2)^2}$.

$v=\int\frac{-2xy}{(x^2+y^2)^2}dy+\varphi(x)$

$\frac{\partial v}{\partial x}=\frac{\partial(\frac{x}{x^2+y^2})}{\partial x}+\varphi'(x)=-\frac{\partial u}{\partial y}=\frac{y^2-x^2}{(x^2+y^2)^2}$

$\varphi'(x)=0$ $\therefore\varphi(x)=C$.

$\therefore f(z)=u+iv=\frac{y}{x^2+y^2}+\frac{xi}{x^2+y^2}+C_1$.

$f(1)=\frac{i}{1}+C_1=0$ $C_1=-i$

$f(z)=\frac{y}{x^2+y^2}+(\frac{x}{x^2+y^2}-1)i$

$=\frac{i}{z}-i$

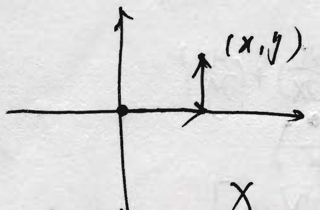
方法(2)

$dv=\frac{\partial v}{\partial x}dx+\frac{\partial v}{\partial y}dy$

$=-\frac{\partial u}{\partial y}dx+\frac{\partial u}{\partial x}dy$

$$\therefore = \frac{y^2-x^2}{(x^2+y^2)^2}\cdot dx - \frac{2xy}{(x^2+y^2)^2}dy$$

$$V(x,y)=\int_0^x \frac{y^2-x^2}{(x^2+y^2)^2}dx+\int_0^y \frac{-2xy}{(x^2+y^2)^2}dy$$



$$V(x,y) = \frac{x}{x^2+y^2} - 1 + C$$

$$\therefore f(z) = u + iV = \frac{y}{x^2+y^2} + \frac{xi}{x^2+y^2} + C_1$$

$$f(1) = i + C_1 = 0 \qquad C_1 = i$$

故 $f(z) = \frac{y}{x^2+y^2} + \left(\frac{x}{x^2+y^2} - 1\right)i$

$$= \frac{i}{z} - i$$

方法二.

$$f'(z) = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$$

$$= -\frac{2xy}{(x^2+y^2)^2} - i\frac{(x^2-y^2)}{(x^2+y^2)^2}$$

$$= \frac{(-i)\cdot(x-yi)^2}{(x^2+y^2)^2} = \frac{-i}{(x+yi)^2} = \frac{-i}{z^2}.$$

$$\therefore f(z) = \frac{i}{z} + C.$$

$$f(1) = i + C = 0 \qquad C = -i$$

$$\therefore f(z) = \frac{i}{z} - i.$$

2.10.

解 $w = \tan z = \frac{\sin z}{\cos z} = \frac{\frac{e^{iz}-e^{-iz}}{2i}}{\frac{e^{iz}+e^{-iz}}{2}} = \frac{e^{iz}-e^{-iz}}{i(e^{iz}+e^{-iz})}$

$$w = \frac{i(1-e^{2iz})}{e^{2iz}+1} \qquad e^{2iz} = \frac{i-w}{w+i}$$

$$2iz = Ln\left(\frac{i-w}{i+w}\right),\ z = -\frac{i}{2}Ln\left(\frac{i-w}{i+w}\right)$$

故则 $\operatorname{arctg} z = -\frac{i}{2}Ln\frac{i-z}{i+z}$.

注意 i 取 Ln 而非 ln

2.12.

(4) $(3-4i)^{1+i}$

解 利用 $z^s = e^{s\,Ln z}$

$$(3-4i)^{1+i} = e^{(1+i)Ln(3-4i)}$$

$$= e^{(1+i)[ln5 + i(\operatorname{arctg}\frac{4}{3}) + i2k\pi]}$$

$$= e^{[ln5 + \operatorname{arctg}\frac{4}{3} + 2k\pi) + i(ln5 - \operatorname{arctg}\frac{4}{3} + 2k\pi)]}$$

$$(k=0,\pm1,\pm2\ldots)$$

(7) $Ln(-3+4i)$

$$Ln(-3+4i) = ln5 + i\operatorname{arctg}\frac{4}{3} - \pi i + 2k\pi i$$

$$Ln(-3+4i) = ln5 + i\operatorname{arctg}\frac{4}{3} + (2k-1)\pi i$$

$$(k=0,\pm1,\pm2\cdots)$$

说明 利用 $z^s = e^{s\,Ln z}$ 化简

以 $Ln z = ln|z| + i\arg z + i2k\pi$.

B 陈燃.

2.1.

证明 由直角坐标系下柯西黎曼方程导出

$$\begin{cases}\frac{\partial u}{\partial \rho} = \frac{1}{\rho}\frac{\partial v}{\partial \varphi}. & \frac{\partial v}{\partial \rho} = -\frac{1}{r}\frac{\partial u}{\partial \varphi}\end{cases}$$

$$\begin{cases} x = \rho\cos\varphi \\ y = \rho\sin\varphi. \end{cases}$$

$$\frac{\partial u}{\partial \rho} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial \rho} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial \rho}$$

$$= \frac{\partial u}{\partial x}\cdot\cos\varphi + \frac{\partial u}{\partial y}\sin\varphi. \quad (1)$$

$$\frac{\partial u}{\partial \varphi} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \varphi} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \varphi} = -\frac{\partial u}{\partial x}\rho\sin\varphi + \frac{\partial u}{\partial y}\rho\cos\varphi \quad (2)$$

$$\frac{\partial v}{\partial \rho} = \frac{\partial v}{\partial x}\cos\varphi + \frac{\partial v}{\partial y}\sin\varphi \quad (3)$$

$$\frac{\partial v}{\partial \varphi} = -\frac{\partial v}{\partial x}\rho\sin\varphi + \frac{\partial v}{\partial y}\rho\cos\varphi. \quad (4).$$

由 C-R 方程:

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y} \quad \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}.$$

又①④有:

$$\frac{\partial u}{\partial \rho}=\frac{1}{\rho}\frac{\partial v}{\partial \phi}.$$

由②③ $\frac{\partial v}{\partial \rho}=-\frac{1}{\rho}\frac{\partial u}{\partial \phi}.$

即: $\frac{\partial u}{\partial \rho}=\frac{1}{\rho}\frac{\partial v}{\partial \phi} \quad \frac{\partial v}{\partial \rho}=-\frac{1}{\rho}\frac{\partial u}{\partial \phi}.$ (5)

证毕.

由(1)及(3)式.

$$\frac{\rho}{z}\left(\frac{\partial u}{\partial \rho}+i\frac{\partial v}{\partial \rho}\right)=\frac{\rho}{z}\left[\frac{\partial u}{\partial x}\cos\phi+\frac{\partial u}{\partial y}\sin\phi+i\frac{\partial v}{\partial x}\cos\phi+i\frac{\partial v}{\partial y}\sin\phi\right]$$

$$=\frac{\rho}{z}\left[\frac{\partial u}{\partial x}e^{i\phi}+i\frac{\partial v}{\partial x}e^{i\phi}\right]$$

$$=\frac{z}{z}\left[\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}\right]=f'(z) \quad (6)$$

由(5),(6).

$$f'(z)=\frac{\rho}{z}\left(\frac{\partial u}{\partial \rho}+i\frac{\partial v}{\partial \rho}\right)=\frac{1}{z}\left(\frac{\partial v}{\partial \phi}-i\frac{\partial u}{\partial \phi}\right)$$

$$=\frac{1}{iz}\left(\frac{\partial u}{\partial \phi}+i\frac{\partial v}{\partial \phi}\right)$$

2.5.

解 $z=0 \quad w(z)=\sqrt{z^2-1}=i$

$w(0)=\sqrt{-1}=\sqrt{1}\cdot e^{i\left[\frac{\pi+2k\pi}{2}\right]}=i$

$(k=0,1)$

则 $k=0$

取 $z_1=z^2-1 \quad w=w(z_1)=\sqrt{z_1}$

则 $w=\sqrt{|z_1|}\cdot\exp i\left(\frac{\arg(z_1)+2k\pi}{2}\right) \quad (k=0,1)$

当 $z=0 \quad z_1=-1. \quad w(z_1)=i$

故则 $k=0$

$w=\sqrt{|z_1|}\exp\frac{i\arg(z_1)}{2}$

故当 $z=i \quad z^2-1=z_1=-2.$

$w(i)=\sqrt{2}\,i$

2.7

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^2$$

$$=\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)(u^2+v^2)$$

$$=2\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2+u\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)+v\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)\right]$$

由于 $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=0$

故上式 $=2\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2\right]$

由 C-R 条件.

上式 $=2\left[2\left(\frac{\partial u}{\partial x}\right)^2+2\left(\frac{\partial v}{\partial x}\right)^2\right]$

$=4\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2\right]$

由于 $f'(z)=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}$

$|f'(z)|^2=\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2$

综合上式得:

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^2=4|f'(z)|^2$$