第十三章:

A作业:

13.5.

证明: $\because \frac{d}{dx}[x^{-\nu}J_\nu(x)] = -x^{-\nu}J_{\nu+1}(x)$

令 $\nu=1$.

$$J_2(x) = (-x)\left(\frac{J_1(x)}{x}\right)'$$
$$= -x\left[\frac{J_1'(x)x - J_1(x)}{x^2}\right]$$
$$= -J_1'(x) + \frac{J_1(x)}{x}$$

又 $\because J_1(x) = -J_0'(x)$ 代入

$$J_2(x) = J_0''(x) - \frac{1}{x}J_0'(x)$$

注: 本题中课本中 $J_0'(x)$ 应为 $J_0''(x)$

13.7. $\begin{cases} \nabla^2 u = 0 \\ u|_{\rho=a} = A\cos\phi \end{cases}$

解: $u = A_0 + B_0\ln\rho + \sum_{n=1}^{\infty}(A_n\rho^n + B_n\rho^{-n})e^{in\phi} + \sum_{n=-1}^{-\infty}(A_n\rho^n + B_n\rho^{-n})e^{in\phi}$

当 $\rho<a$ 时. $u|_{\rho=0}$ 有限.故:

$$u = A_0 + \sum_{n=1}^{\infty}A_n\rho^n e^{in\phi} + \sum_{n=-1}^{-\infty}B_n\rho^{-n}e^{in\phi}$$

$$u|_{\rho=a} = A_0 + \sum_{n=1}^{\infty}A_n a^n e^{in\phi} + \sum_{n=-1}^{-\infty}B_n a^{-n}e^{in\phi} = A\cos\phi$$
$$= \frac{A}{2}[e^{i\phi} + e^{-i\phi}]$$

比较两边系数得

$A_1 = \frac{A}{2a}$ $\quad B_{-1} = \frac{A}{2a}$.

故此 $u(\rho,\phi) = \frac{A}{2a}[\rho e^{i\phi} + \rho e^{-i\phi}]$
$$= \frac{A\rho}{a}\cos\phi.$$

当 $\rho>a$ 时 $u|_{\rho\to\infty}$ 有限:故:

$$u = A_0 + \sum_{n=1}^{\infty}B_n\rho^{-n}e^{in\phi} + \sum_{n=-1}^{-\infty}A_n\rho^n e^{in\phi}.$$

$$u|_{\rho=a} = A_0 + \sum_{n=1}^{\infty}B_n\frac{1}{a^n}e^{in\phi} + \sum_{n=-1}^{-\infty}A_n a^n e^{in\phi}.$$

$$= \frac{A}{2}e^{i\phi} + \frac{A}{2}e^{-i\phi}$$

$B_1 = \frac{aA}{2}$ $\qquad A_{-1} = \frac{aA}{2}$

故此 $u(\rho,\phi) = \frac{aA}{2}\left[\frac{1}{\rho}e^{i\phi} + \frac{1}{\rho}e^{-i\phi}\right]$
$$= \frac{Aa}{\rho}\cos\phi.$$

综合可知:

$$u(\rho,\phi) = \begin{cases} \frac{A\rho}{a}\cos\phi & (\rho<a) \\ \frac{Aa}{\rho}\cos\phi & (\rho>a) \end{cases}$$

13.8. $\begin{cases} \nabla^2 u = 0 \\ u(a,\phi) = \begin{cases} u_1 & 0<\phi<\pi \\ u_2 & \pi<\phi<2\pi \end{cases} \end{cases}$

当 $\rho<a$ 时 $u|_{\rho=0}$ 有限.

$$u = A_0 + \sum_{n=1}^{\infty}A_n\rho^n e^{in\phi} + \sum_{n=-1}^{-\infty}B_n\rho^{-n}e^{in\phi}.$$

$$u(a,\phi) = A_0 + \sum_{n=1}^{\infty}A_n a^n e^{in\phi} + \sum_{n=-1}^{-\infty}B_n a^{-n}e^{in\phi}.$$

① $n=0$

$$A_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi}f(\phi)d\phi = \frac{1}{2\pi}\int_0^{\pi}f(\phi)d\phi + \frac{1}{2\pi}\int_{\pi}^{2\pi}f(\phi)d\phi$$
$$= \frac{1}{2}[u_1+u_2]$$

② $n\geq1$ 时 利用正交性. 注意共轭正交.

$$A_n a^n\cdot 2\pi = \int_{-\pi}^{\pi}u(a,\phi)e^{-in\phi}d\phi.$$
$$= \frac{1}{-in}\left[(u_1-u_2)[(-1)^n-1]\right]$$

$$A_n = \frac{1}{-in2\pi a^n}\left[(u_1-u_2)[(-1)^n-1]\right]$$

③ $n\leq-1$ 时 利用共轭正交性:

$$B_n = \frac{a^n}{-in2\pi}\left[(u_1-u_2)[(-1)^n-1]\right]$$

综合可知 $\rho<a$.

$$u = \frac{u_1+u_2}{2} + \frac{2(u_1-u_2)}{\pi}\sum_{n=0}^{\infty}\left(\frac{\rho}{a}\right)^{2n+1}\frac{\sin(2n+1)\phi}{(2n+1)}$$

注: 圆域的 Dirichlet问题 对应 Bessel 方程中 $k^2=\lambda-\mu$. $\lambda=0$ (稳态问题) $\mu=0$ (与Z方向无关) 此时 $k^2\rho^2=0$ 方程为欧拉方程, 此时 ϕ 常微分方程周期性边界条件确定本征函数为 $e^{in\phi}$ 注意 n 为正负整数 $n=0$ 有为 $A_0+B_0\ln\rho$ ...

13.10.

解: $\begin{cases} \dfrac{\partial u(\rho,t)}{\partial t} - D\nabla^2 u(\rho,t) = 0 \\ u(R,t) = 0 \\ u(\rho,0) = u_0 \end{cases}$

由于该问题轴对称第一类齐次边界问题.

$$u(\rho,t) = \sum_{m=1}^{\infty} T_m(t) J_0(k_m^0 \rho)$$

本征值 $u = (k_m^0)^2 = \left(\dfrac{x_m^0}{R}\right)^2$

$$T_m'(t) + D\left(\frac{x_m^0}{R}\right)^2 = 0$$

$$T_m(t) = C_n \exp\left[-D\left(\frac{x_m^0}{R}\right)^2 t\right]$$

故则 $u(\rho,t) = \sum_{m=1}^{\infty} C_n \exp\left[-D\left(\frac{x_m^0}{R}\right)^2 t\right] J_0(k_m^0 \rho)$

$$u(\rho,0) = \sum_{m=1}^{\infty} C_n J_0(k_m^0 \rho) = u_0$$

利用正交性求得

$$C_n = \frac{2}{R^2[J_1(x_m^0)]^2}\int_0^R \rho u_0 J_0(k_m^0\rho)\,d\rho$$

$$= \frac{2u_0}{x_m^0 J_1(x_m^0)} \quad (\text{利用 } xJ_0(x) = [xJ_1(x)]' \text{ 化简})$$

故则

$$u(\rho,t) = 2u_0 \sum_{m=1}^{\infty} \frac{J_0(x_m^0 \rho/R)}{x_m^0 J_1(x_m^0)} e^{-D\left(\frac{x_m^0}{R}\right)^2 t}$$

B组习题.

13.2.

解: $u = A_0 + B_0 \ln\rho + \sum_{n=1}^{\infty}(A_n\rho^n + B_n\rho^{-n})e^{in\phi} + \sum_{n=-1}^{-\infty}(A_n\rho^n + B_n\rho^{-n})e^{in\phi}$

$\because u(\rho_2,\phi) = 0$

$\therefore A_0 + B_0\ln\rho_2 + \sum_{n=1}^{\infty}(A_n\rho_2^n + B_n\rho_2^{-n})e^{in\phi} + \sum_{n=-1}^{-\infty}(A_n\rho_2^n + B_n\rho_2^{-n})e^{in\phi} = 0$

$A_0 = -B_0\ln\rho_2$.

$n \geq 1$: $A_n = -B_n/\rho_2^{2n}$.

$u(\rho,\phi)$

$= B_0\ln\rho_2 + B_0\ln\rho$

$+ B_n\sum_{n=1}^{\infty}\left(-\dfrac{\rho^n}{\rho_2^{2n}} + \rho^{-n}\right)e^{in\phi}$

$+ B_n\sum_{n=-1}^{-\infty}\left(-\dfrac{\rho^n}{\rho_2^{2n}} + \rho^{-n}\right)e^{in\phi}$

$u(\rho_1,\phi) = B_0\ln\dfrac{\rho_2}{\rho_1} + B_n\sum_{n=1}^{\infty}\left(-\dfrac{\rho_1^n}{\rho_2^{2n}} + \dfrac{1}{\rho_1^n}\right)e^{in\phi}$

$+ B_n\sum_{n=-1}^{-\infty}\left(-\dfrac{\rho_1^n}{\rho_2^{2n}} + \dfrac{1}{\rho_1^n}\right)e^{in\phi}$

$= \dfrac{1}{2i}\left[e^{i\phi} - e^{-i\phi}\right]$

$B_1 = \dfrac{1}{2i}\dfrac{\rho_1\rho_2^2}{\rho_2^2-\rho_1^2} \qquad B_{-1} = \dfrac{1}{-2i}\dfrac{\rho_1}{\rho_1^2-\rho_2^2}$

故则 $u(\rho,\phi)$

$= \dfrac{\rho_1}{(\rho_1^2-\rho_2^2)}\cdot\dfrac{(\rho^2-\rho_2^2)\sin\phi}{\rho}$

注: 圆环问题中 ρ^n, ρ^{-n} 项一般要保留.

13.3.

解: 由 $u(a,\phi,z) = 0$

可知 $J_n(k\rho)$ 本征值为 $k_m^n = \dfrac{x_m^n}{a}$.

而有关Z方程:

$\begin{cases} \dfrac{d^2Z}{dz^2} = -uZ \\ Z(0) = Z(h) = 0 \end{cases}$

u 必取本征值为 $u = \left(\dfrac{k\pi}{h}\right)^2$.

故则 T的常微分方程 $\lambda - u = k^2$.

$\dfrac{d^2T}{dt^2} + \lambda c^2 T(t) = 0$. $\quad \lambda_{mk}^n = u_k + \left(\dfrac{x_m^n}{a}\right)^2$

解为 $T_n(t) = A_n\cos\sqrt{\lambda_n}ct + B_n\sin\sqrt{\lambda_n}ct$

$= A_n\cos w_n t + B_n\sin w_n t$.

其中 $\omega_{mk}^n = c\sqrt{\lambda} = c\sqrt{\left(\dfrac{x_m^n}{a}\right)^2 + \left(\dfrac{k\pi}{h}\right)^2}$

证毕: