第四讲　静态场问题（2）

电磁场的解析方法

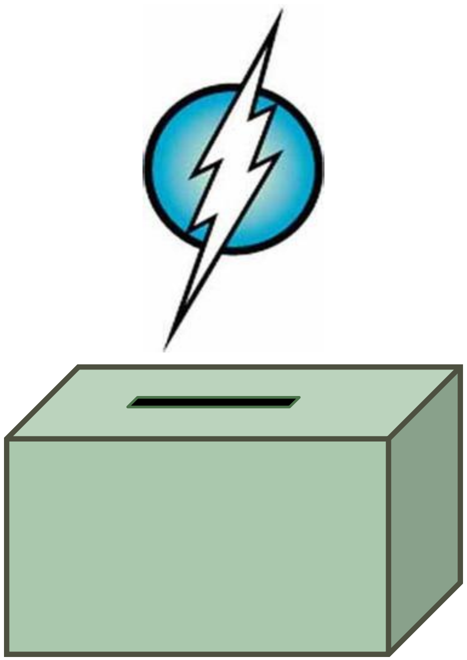
（第四章）

主要内容：

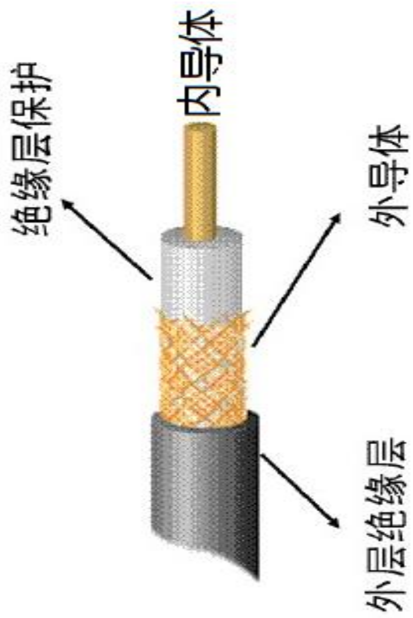
- 电磁场的定解问题
- 唯一性定理及应用
- 分离变量法及应用
- 格林函数法及应用
- 镜像方法及其应用

§1 唯一性定理

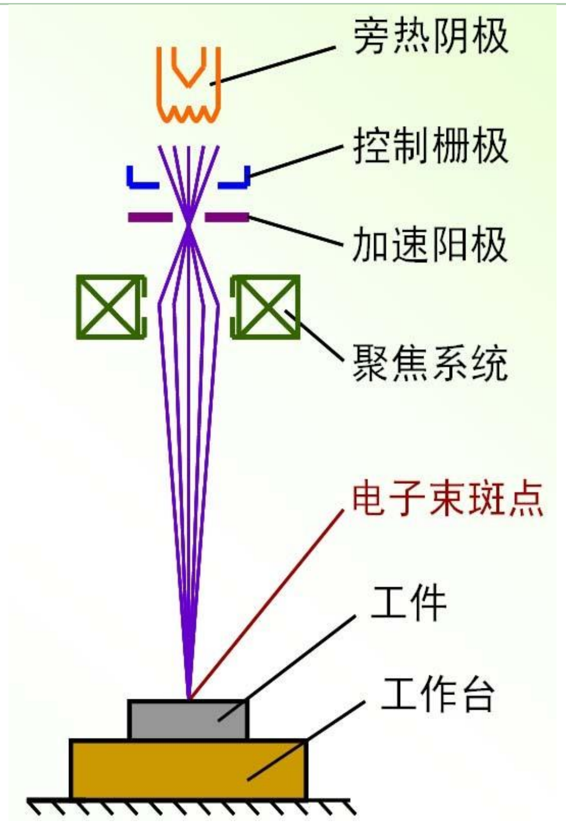
1. 电磁场的基本问题



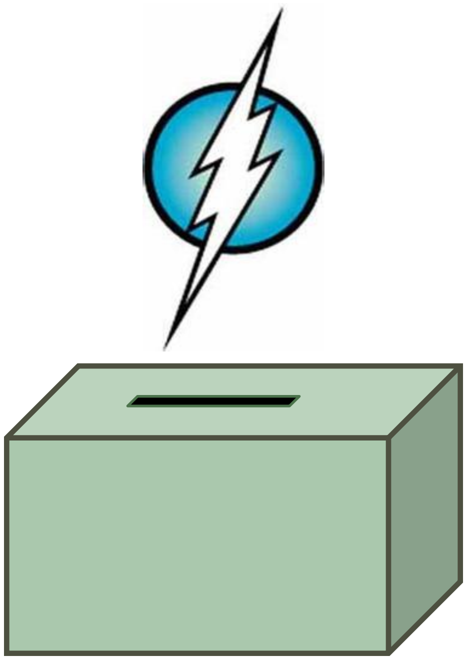
雷电冲击效应



同轴线信号传输



磁聚焦系统



雷电冲击效应：

闭合导体空间内耦合场的分布

缝隙形状、位置对耦合场影响

冲击波特性对耦合场的影响

非理想导体对耦合场的影响

闭合导体空间结构对耦合场影响

求一定空间中麦克斯韦方程组在一定边界和初始状态下的解

同轴线信号传输问题：

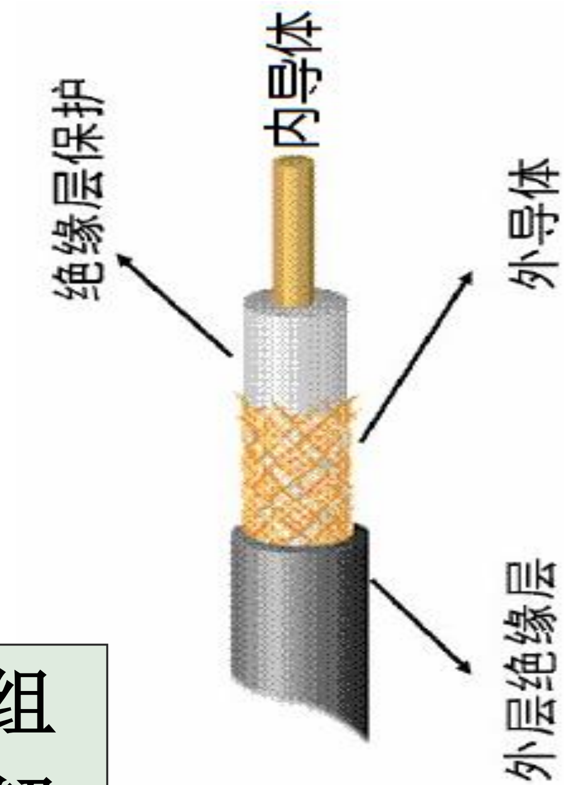
可传输信号的模式

可传输信号场的分布

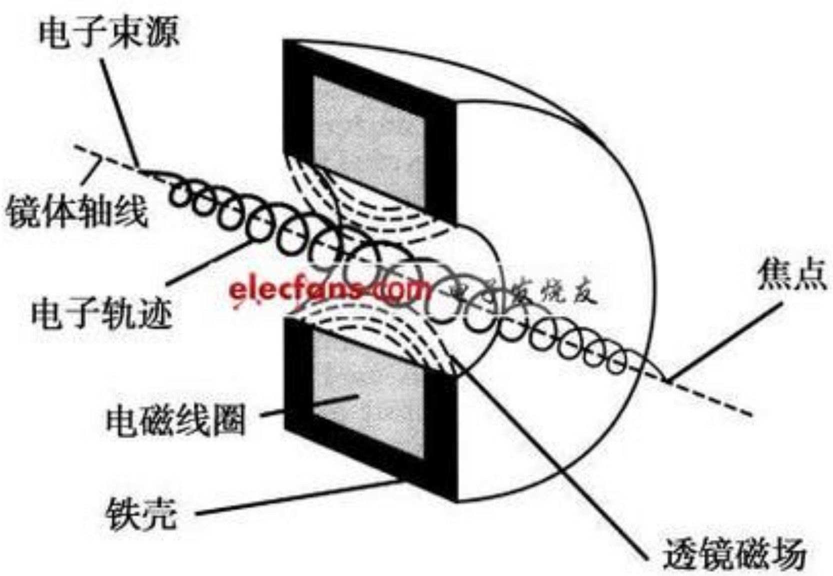
性能指标的设计

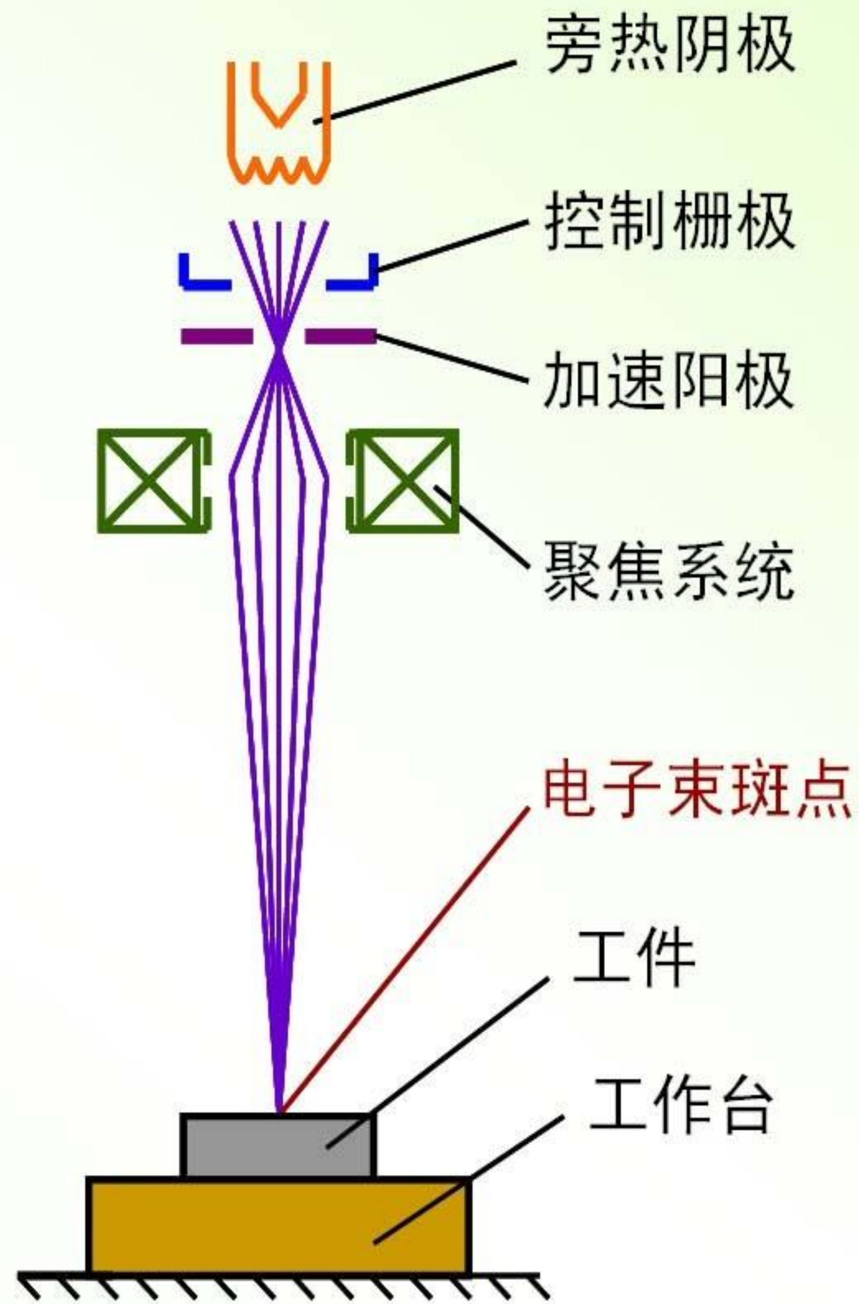
非理想导体的影响

求一定空间内麦克斯韦方程组在一定边界和初始状态下的解



磁聚焦系统的磁场设计

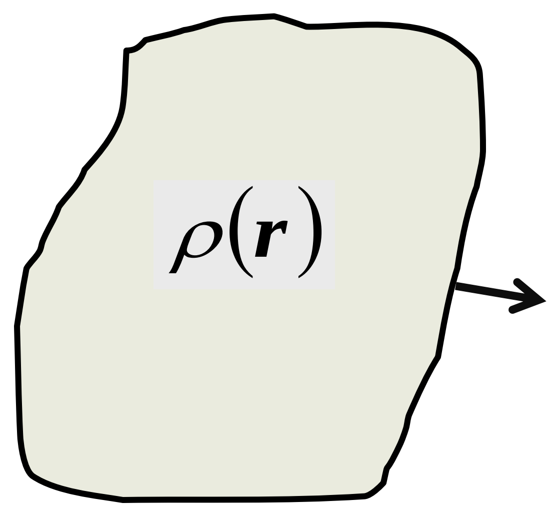

电子束源
镜体轴线
电子轨迹
elecfans.com 电子发烧友
焦点
电磁线圈
铁壳
透镜磁场


旁热阴极
控制栅极
加速阳极
聚焦系统
电子束斑点
工件
工作台

2. 静态电磁场的定解问题

① 静态电场 $\boldsymbol{E}(\boldsymbol{r}) = -\nabla\phi(\boldsymbol{r})$

$$\begin{cases} \nabla^2\phi(\boldsymbol{r}) = -\dfrac{1}{\varepsilon}\rho(\boldsymbol{r}) \\ \phi_1(\boldsymbol{r})\big|_S = \phi_2(\boldsymbol{r})\big|_S \text{或} \\ \varepsilon_1 \dfrac{\partial\phi}{\partial n}\Big|_S - \varepsilon_2 \dfrac{\partial\phi}{\partial n}\Big|_S = \rho_s \end{cases}$$

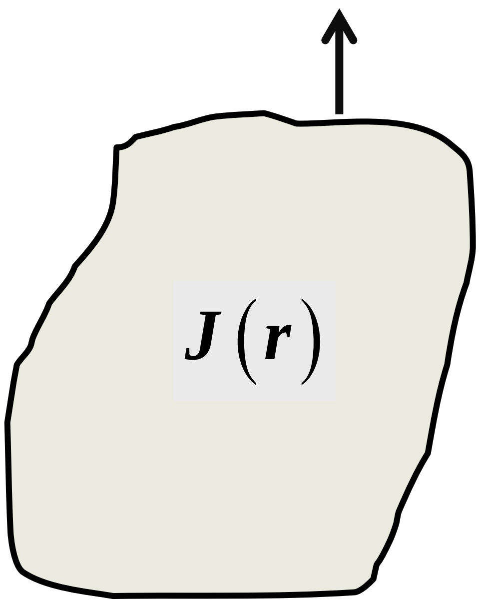


② 恒定电流磁场

引入磁矢势

$$\boldsymbol{B}(\boldsymbol{r})=\nabla\times\boldsymbol{A}(\boldsymbol{r})$$

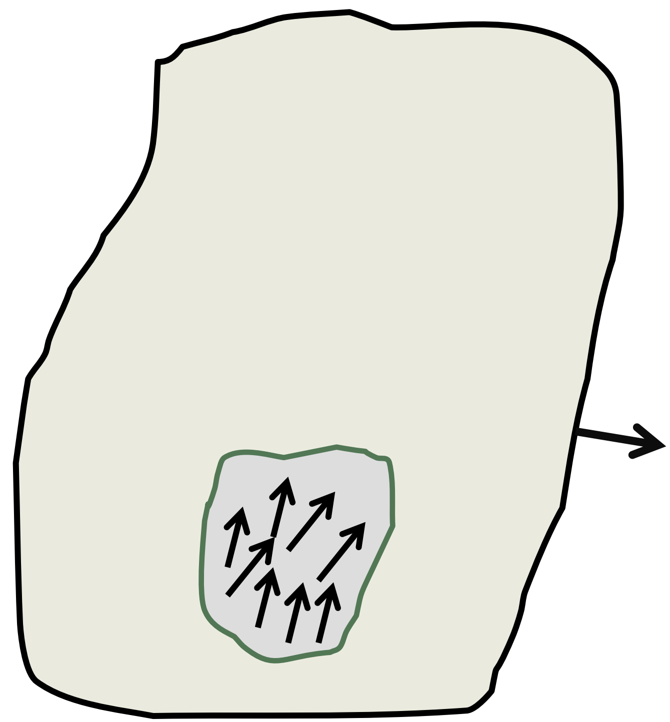
$$\begin{cases}\nabla^2\boldsymbol{A}(\boldsymbol{r})=-\mu\boldsymbol{J}(\boldsymbol{r})\\ \left(\boldsymbol{A}_2-\boldsymbol{A}_1\right)\Big|_{\text{边界面}}=0\end{cases}$$



无电流区域非时变磁场

$$\boldsymbol{H}(\boldsymbol{r}) = -\nabla \varphi_m(\boldsymbol{r})$$

$$\begin{cases} \nabla^2 \phi_m(\boldsymbol{r}) = -\dfrac{1}{\mu_0} \rho_m \\ \phi_{1m}(\boldsymbol{r})\big|_S = \phi_{2m}(\boldsymbol{r})\big|_S \\ \mu_1 \dfrac{\partial \phi_m}{\partial n}\Big|_S = \mu_2 \dfrac{\partial \phi_m}{\partial n}\Big|_S \\ \rho_m = -\mu_0 \nabla \cdot \boldsymbol{M}(\boldsymbol{r}) \end{cases}$$



静态（电或磁）场的定解问题

$$
\left\{
\begin{aligned}
&\nabla^2\phi(\mathbf{r}) = -\frac{1}{\kappa}\rho(\mathbf{r}) \\
&\phi_1(\mathbf{r})\big|_S = \phi_2(\mathbf{r})\big|_S \\
&\kappa_1\frac{\partial\phi}{\partial n}\Big|_S - \kappa_2\frac{\partial\phi}{\partial n}\Big|_S = \rho_s
\end{aligned}
\right\} \rightarrow
$$

对应三类求解的问题：

1）源、介质和边界求场

2）场、介质求边界形状

3）场、边界求介质特性

两个边界条件无需同时给出

$\phi(\mathbf{r})$ $\quad$ $\rho(\mathbf{r})$

解的三个基本问题：

存在性 唯一性 稳定性

2. 静态场的唯一性定理

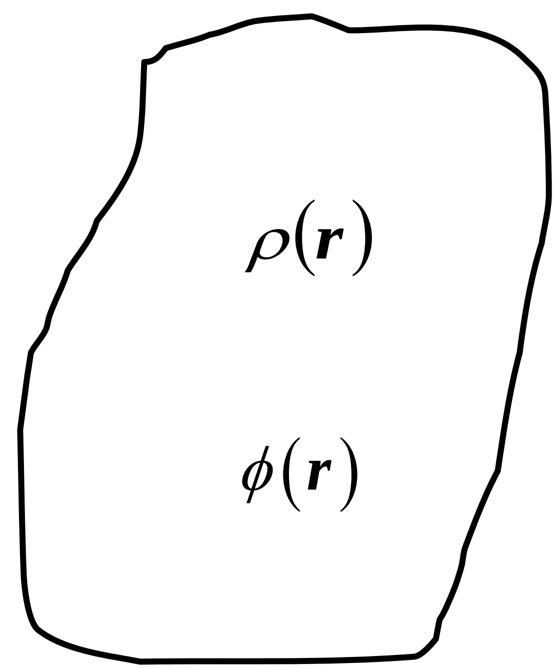
区域内源 $\rho(\boldsymbol{r})$ 已知，区域边界上

$$\phi(\boldsymbol{r})\Big|_{\text{界面}}=\psi(M)$$

$$\text{或}\ \frac{\partial\phi(\boldsymbol{r})}{\partial n}\Big|_{\text{界面}}=\xi(M)$$

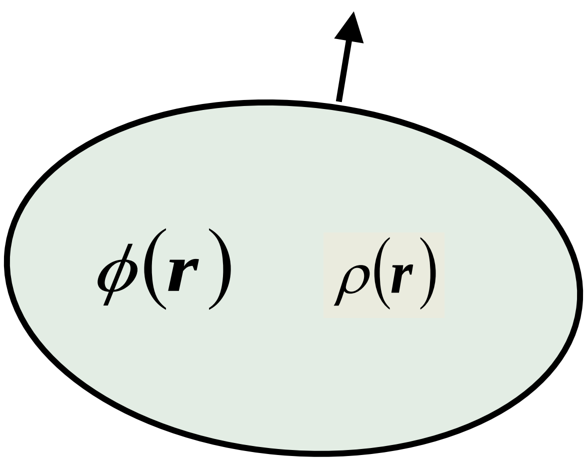
则区域内泊松方程存在唯一解；

边界上满足给定的边界条件。



证：设$\phi_1(\boldsymbol{r})$和$\phi_2(\boldsymbol{r})$满足方程和边界条件：

$$\begin{cases} \nabla^2\phi_{1,2}(\boldsymbol{r}) = -\dfrac{\rho(\boldsymbol{r})}{\kappa} \\ \phi_{1,2}(\boldsymbol{r})\big|_{\text{边界}} = \psi(M) \\ \text{或}\ \dfrac{\partial\phi_{1,2}(\boldsymbol{r})}{\partial n}\bigg|_{\text{边界}} = \varsigma(M) \end{cases}$$



如能够证明 $\phi_1(\boldsymbol{r})=\phi_2(\boldsymbol{r})$ 即证明解唯一

记：$\phi(\boldsymbol{r})=\phi_1(\boldsymbol{r})-\phi_2(\boldsymbol{r})$

根据线性叠加原理，$\phi(\boldsymbol{r})$满足方程：

$$\begin{cases} \nabla^2\phi(\boldsymbol{r})=0 \\ \phi(\boldsymbol{r})\big|_{边界}=0，或\dfrac{\partial}{\partial n}\phi(\boldsymbol{r})\big|_{边界}=0 \end{cases}$$

将$\phi(\boldsymbol{r})$代入格林公式并在区域求积分，得到：

$$\iiint_V \phi(\boldsymbol{r})\nabla^2\phi(\boldsymbol{r})\,\mathrm{d}V = \oiint_S \phi(\boldsymbol{r})\nabla\phi(\boldsymbol{r})\cdot \mathrm{d}\boldsymbol{s} - \iiint_V \nabla\phi(\boldsymbol{r})\cdot\nabla\phi(\boldsymbol{r})\,\mathrm{d}V$$

$$\rightarrow \oiint_S \phi(\boldsymbol{r})\nabla\phi(\boldsymbol{r})\cdot \mathrm{d}\boldsymbol{s} = \iiint_V \nabla\phi(\boldsymbol{r})\cdot\nabla\phi(\boldsymbol{r})\,\mathrm{d}V$$

$$\rightarrow \oiint_S \phi(\boldsymbol{r})\frac{\partial}{\partial n}\phi(\boldsymbol{r})\,\mathrm{d}s = \iiint_V \left|\nabla\phi(\boldsymbol{r})\right|^2 \mathrm{d}V$$

应用边界条件上式：

$$\iiint_V \left|\nabla\phi(\boldsymbol{r})\right|^2 \mathrm{d}V = 0 \Rightarrow \nabla\phi(\boldsymbol{r}) = 0 \Rightarrow \phi(\boldsymbol{r}) = A$$

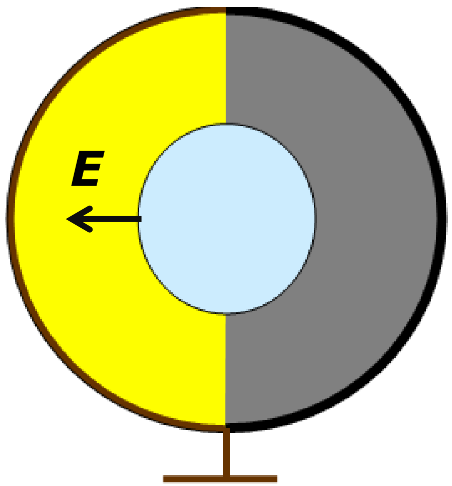
由于$\phi(\boldsymbol{r})$在区域边界上恒为零，可以得到

$$\phi(\boldsymbol{r}) = A = 0$$

即得到： $\phi_1(\boldsymbol{r}) = \phi_2(\boldsymbol{r})$

【例】同心导体球壳间充满两种介质。内导体带电荷 Q, 外导体接地，求两壳间电场的分布。

分析：内外导体等势，导体表面电场切向为零。介质中电场应只有径向分量。在介质交界面上：



$$E_{1t}=E_{2t}\quad D_{1n}=D_{2n}$$

因此设解为：$\begin{cases} \boldsymbol{E}_1(\boldsymbol{r}) = A_1 \dfrac{\boldsymbol{r}}{r^3}(\text{左半球}) \\ \boldsymbol{E}_2(\boldsymbol{r}) = A_2 \dfrac{\boldsymbol{r}}{r^3}(\text{右半球}) \end{cases}$

应用介质边界条件求得：$A_1 = A_2 = A$

$$\oint\!\!\!\oint_S \boldsymbol{D}\cdot \mathrm{d}\boldsymbol{S} = A\left[\iint_{\text{左半球面}} \varepsilon_1 \boldsymbol{E}_1 \cdot \mathrm{d}\boldsymbol{S} + \iint_{\text{右半球面}} \varepsilon_2 \boldsymbol{E}_2 \cdot \mathrm{d}\boldsymbol{S}\right] = Q \Rightarrow A = \frac{Q}{2\pi(\varepsilon_1 + \varepsilon_2)}$$

因此解为：$\boldsymbol{E}_1(\boldsymbol{r}) = \boldsymbol{E}_2(\boldsymbol{r}) = \dfrac{Q}{2\pi(\varepsilon_1+\varepsilon_2)}\dfrac{\boldsymbol{r}}{r^3}$

验证：

1) 所求的解是否满足方程！

2) 所求的解是否满足边界条件！

§2　分离变量方法

定解问题解的探讨

存在很多可能解？

$$\begin{cases} \nabla^2\phi(\boldsymbol{r}) = -\dfrac{1}{\kappa}\rho(\boldsymbol{r}) \\ \phi_1(\boldsymbol{r})\big|_S = \phi_2(\boldsymbol{r})\big|_S \\ \kappa_1 \dfrac{\partial\phi}{\partial n}\Big|_S - \kappa_2 \dfrac{\partial\phi}{\partial n}\Big|_S = \rho_s \end{cases}$$

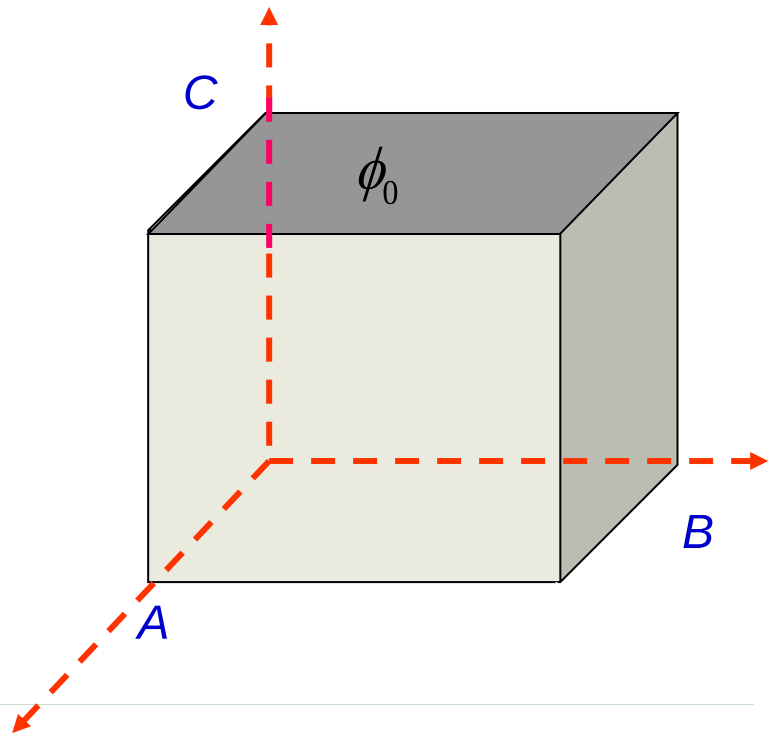
静态电场或磁场满足**普遍**规律的数学描述

静态电场或磁场的某个**特殊**边界状态数学描述

具有人为的任意性？

【例】长A、宽B、高C方形盒，上盖电位为ϕ_0，其余接地，求盒内电位。

$$\begin{cases} \nabla^2\phi(\boldsymbol{r}) = 0 \\ \phi(0,y,z) = \phi(x,0,z) = 0 \\ \phi(A,y,z) = \phi(x,B,z) = 0 \\ \phi(x,y,0) = 0 \\ \phi(x,y,C) = \phi_0 \end{cases}$$



设：$\phi(\boldsymbol{r}) = X(x)Y(y)Z(z)$

$$\begin{cases} \dfrac{1}{X}\dfrac{\mathrm{d}^2 X}{\mathrm{d}x^2} + \dfrac{1}{Y}\dfrac{\mathrm{d}^2 Y}{\mathrm{d}y^2} + \dfrac{1}{Z}\dfrac{\mathrm{d}^2 Z}{\mathrm{d}z^2} = 0 \\ \phi(A, y, z) = \phi(0, y, z) = 0 \Rightarrow X(A) = X(0) = 0 \\ \phi(x, B, z) = \phi(x, 0, z) = 0 \Rightarrow Y(B) = Y(0) = 0 \\ \phi(x, y, 0) = 0 \Rightarrow Z(0) = 0 \end{cases}$$

$$\begin{cases} \dfrac{\mathrm{d}^2 X}{\mathrm{d}x^2} = -k^2 X(x) \\ X(0) = X(A) = 0 \end{cases} \quad \begin{cases} \dfrac{\mathrm{d}^2 Y}{\mathrm{d}y^2} = -l^2 Y(x) \\ Y(0) = Y(B) = 0 \end{cases} \quad \begin{cases} \dfrac{\mathrm{d}^2 Z}{\mathrm{d}z^2} = -p^2 Z(z) \\ Z(0) = 0 \end{cases}$$

$$k^2 + l^2 + p^2 = 0$$

$$\begin{cases} X(x) = A_1 \sin \dfrac{n\pi}{A} x \quad , \quad k = \dfrac{n\pi}{A} \quad ,(n = 1,2,3,\cdots) \\ Y(y) = A_2 \sin \dfrac{m\pi}{B} y \quad , \quad l = \dfrac{m\pi}{A} \quad ,(m = 1,2,3,\cdots) \end{cases}$$

$$Z(z) = C_{kl}\, sinh \sqrt{k^2 + l^2}\, z \quad \leftarrow \; k^2 + l^2 + p^2 = 0$$

$$\phi(x,y,z) = \sum_{n,m=1}^{\infty} \phi_{nm}(x,y,z)$$

$$= \sum_{n,m=1}^{\infty} C_{nm} \sin \frac{n\pi}{A} x \sin \frac{m\pi}{B} y \sinh \left[\sqrt{\left(\frac{n}{A}\right)^2 + \left(\frac{m}{B}\right)^2}\, \pi z \right]$$

利用边界条件求出待定系数

$$\phi_0 = \sum_{n,m=1}^{\infty} C_{nm} \sinh\left[\sqrt{\left(\frac{n}{A}\right)^2 + \left(\frac{m}{B}\right)^2}\,\pi C\right] \sin\frac{n\pi x}{A} \sin\frac{m\pi y}{B}$$

$$C_{mm} = \frac{16\phi_0}{mn\pi^2 \sinh\left[\sqrt{\left(\frac{n}{A}\right)^2 + \left(\frac{m}{B}\right)^2}\,\pi C\right]}$$

1. 分离变量法的思想

分离变量法的思想：

化偏微方程为含待定参数的本征值方程；

求解本征值方程得本征值和本征函数；？

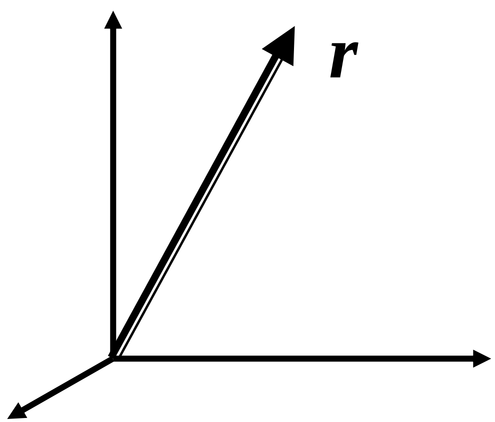
利用本征函数展开表示待求函数；？

待求函数转化为待求系数（展开）-代数方程；

通过边界条件确定展开系数，求出待求解。

分离变量方法的理论基础

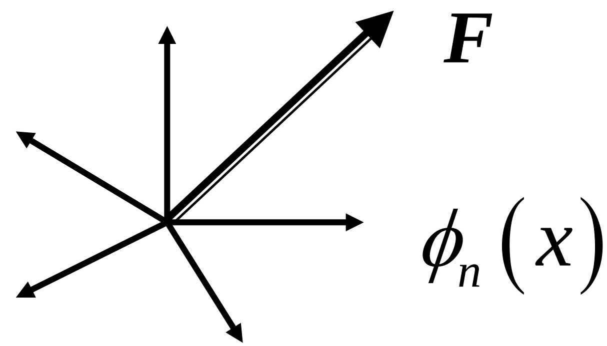
欧氏空间—3维



$$\boldsymbol{r}=\hat{e}_x x+\hat{e}_y y+\hat{e}_z z$$

正交完备基矢量3个

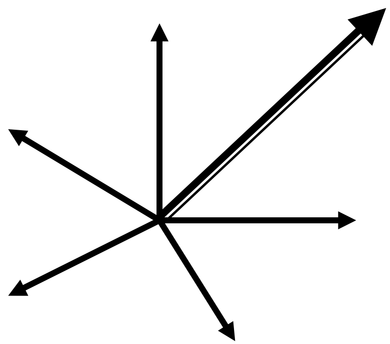
函数空间—无穷维



$$F(x)=\sum_{n,=1}^{\infty} C_n\phi_n(x)$$

正交完备基矢量（函数）无穷个

分离变量方法的理论基础



$F(x)$

$\phi_n(x)$

本征值问题：

本征值和本征函数

本征函数的完备性

无穷维正交完备基（矢量）函数

$$F(x)=\sum_{n,=1}^{\infty}C_n\phi_n(x)$$

方程在定义区域上的可能解构成解（无穷维函数）空间

$$\left(\phi(\boldsymbol{r}) = \sum_{n=1}^{\infty} C_n \phi_n(\boldsymbol{r})\right) \rightarrow \left\{\begin{array}{l} \nabla^2 \phi(\boldsymbol{r}) = -\dfrac{1}{\varepsilon_0}\rho(\boldsymbol{r}) \\ \phi_1(\boldsymbol{r})\big|_S = \phi_2(\boldsymbol{r})\big|_S \text{或} \\ \varepsilon_1 \dfrac{\partial \phi}{\partial n}\Big|_S - \varepsilon_2 \dfrac{\partial \phi}{\partial n}\Big|_S = \rho_s \end{array}\right\} \rightarrow \left[\begin{array}{l}\text{关于}\{C_n\} \\ \text{代数方程}\end{array}\right]$$

分离变量建立本征值方程

本征值问题
本征值和本征函数
本征函数的完备性

2. 分离变量方法的程序

① 提炼出定解问题的数学表达式

② 选取适合变量分离的正交坐标系

③ 方程和边界条件变量分离—本征值问题

④ 求解本征值方程，确定本征值和本征函数

⑤ 由本征函数构造定解问题的解

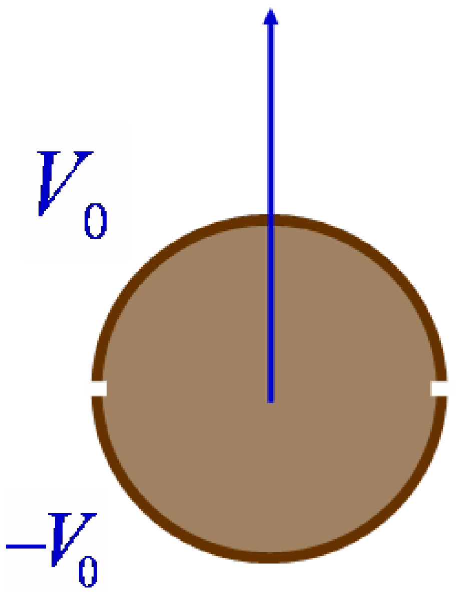
⑥ 利用边界条件确定展开系数，　验证解

正交曲线坐标系中$\nabla^2\phi(\boldsymbol{r})$的分离变量

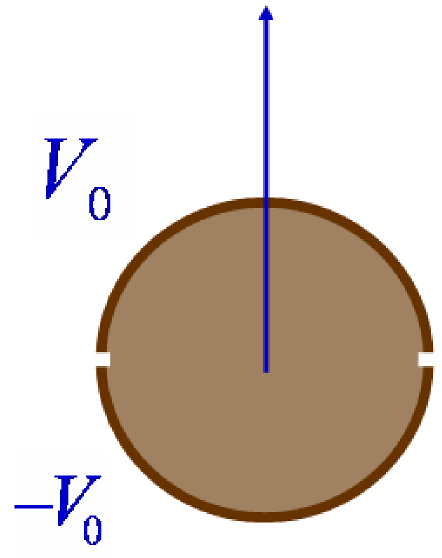
$$\nabla\phi(\boldsymbol{r})=\sum_{i=1}^{3}\hat{e}_{q_i}\frac{\partial\phi(\boldsymbol{r})}{h_i\partial q_i}$$

$$\nabla^2\phi(\boldsymbol{r})=\nabla\cdot\nabla\phi(\boldsymbol{r})=\frac{1}{h_1h_2h_3}\sum_{i=1}^{3}\frac{\partial}{\partial q_i}\left[\frac{h_1h_2h_3}{h_i}\frac{\partial\phi(\boldsymbol{r})}{h_i\partial q_i}\right]$$

【例3】 无穷长导体圆筒，半径为 a，厚度可以忽略不计。圆筒分成相等的两个半片，相互绝缘，其电位分别是V_0和$-V_0$，求筒内电位。



$$\begin{cases} \nabla^2\phi(\boldsymbol{r}) = \dfrac{1}{r}\dfrac{\partial}{\partial r}\left(r\dfrac{\partial\phi}{\partial r}\right) + \dfrac{1}{r^2}\dfrac{\partial^2\phi}{\partial\varphi^2} = 0 \\ \phi(a,\varphi) = V_0, (0 < \varphi < \pi) \\ \phi(a,\varphi) = -V_0, (\pi < \varphi < 2\pi) \end{cases}$$



圆柱坐标系： h_1=1， h_2=r， h_3=1

$$\nabla^2\phi(\boldsymbol{r}) = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi(\boldsymbol{r})}{\partial r}\right) + \frac{1}{r}\frac{\partial}{\partial\varphi}\left(\frac{\partial\phi(\boldsymbol{r})}{r\partial\varphi}\right) + \frac{\partial^2\phi(\boldsymbol{r})}{\partial z^2}$$

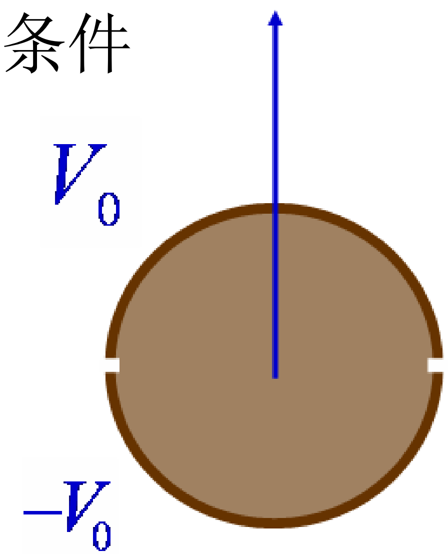
$$\phi(\boldsymbol{r}) = R(r)\Phi(\varphi) \Rightarrow \frac{\Phi}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}R}{\mathrm{d}r}\right) + \frac{R}{r^2}\frac{\mathrm{d}^2\Phi}{\mathrm{d}\varphi^2} = 0$$

$$\Phi''(\varphi) + n^2\Phi''(\varphi) = 0 \quad , \quad r\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}R}{\mathrm{d}r}\right) - n^2 R = 0$$

$$\begin{cases} \phi(a,\varphi) = V_0, (0 < \varphi < \pi) \\ \phi(a,\varphi) = -V_0, (\pi < \varphi < 2\pi) \end{cases} \Rightarrow R(a)\Phi(\varphi) = \begin{cases} V_0, (0 < \varphi < \pi) \\ -V_0, (\pi < \varphi < 2\pi) \end{cases}$$

直接对边界条件分离变量得不到本征值方程！

$$\begin{cases}\phi(r,\varphi+2n\pi)=\phi(r,\varphi)\\ \lim\limits_{r\to 0}\phi(r,\varphi)\to \text{有限值}\end{cases}\to \text{称为自然边界条件}$$



$$\begin{cases}\Phi''(\varphi)+n^2\Phi''(\varphi)=0\\ \Phi(\varphi+2n\pi)=\Phi(\varphi)\end{cases}$$

$$\to \Phi(\varphi)=\begin{pmatrix}\cos n\varphi\\ \sin n\varphi\end{pmatrix},(n=0,1,2,3,\cdots)$$

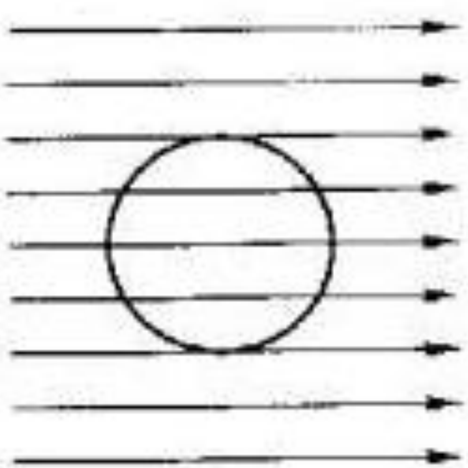
$$\begin{cases}r\dfrac{\mathrm{d}}{\mathrm{d}r}\left(r\dfrac{\mathrm{d}R}{\mathrm{d}r}\right)-n^2R=0\\ \lim\limits_{r\to 0}R(r)\to \text{有限值}\end{cases}\Rightarrow R(r)=\begin{pmatrix}a_0+b_0\ln r\\ a_nr^n+b_nr^{-n}\end{pmatrix},\begin{pmatrix}\text{当}n=0\\ \text{当}n\neq 0\end{pmatrix}$$

$$\phi(r,\varphi)=\sum_{n=0}^{\infty} r^{n}\left[A_{n}\cos n\varphi+B_{n}\sin n\varphi\right]$$

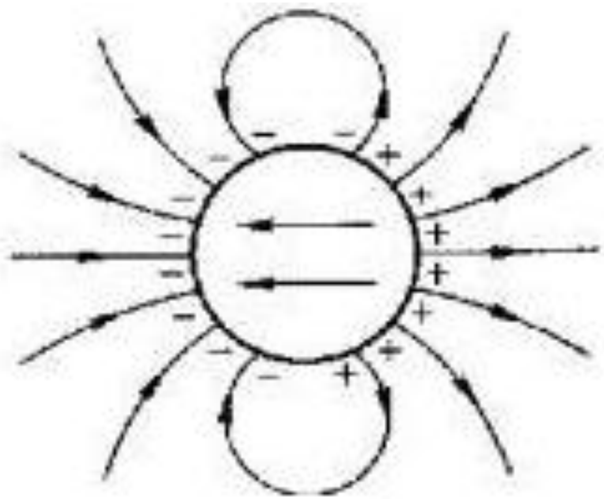
$$\phi(a,\varphi)=\sum_{n=0}^{\infty} a^{n}\left[A_{n}\cos n\varphi+B_{n}\sin n\varphi\right]=\begin{cases} V_{0} & , \quad (0<\varphi<\pi) \\ -V_{0} & , \quad (\pi<\varphi<2\pi) \end{cases}$$

$$\phi(r,\varphi)=\frac{4V_{0}}{\pi}\sum_{n=0}^{\infty}\frac{1}{2k+1}\left(\frac{r}{a}\right)^{2k+1}\sin(2k+1)\varphi$$

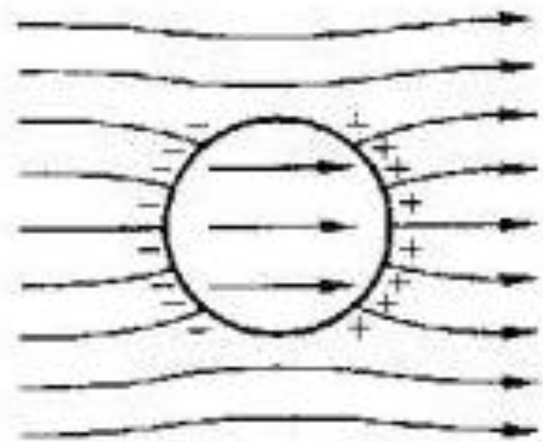
【例4】 将半径为 a 的介质球置于均匀电场中，求介质球内外电场分布。



均匀外电场

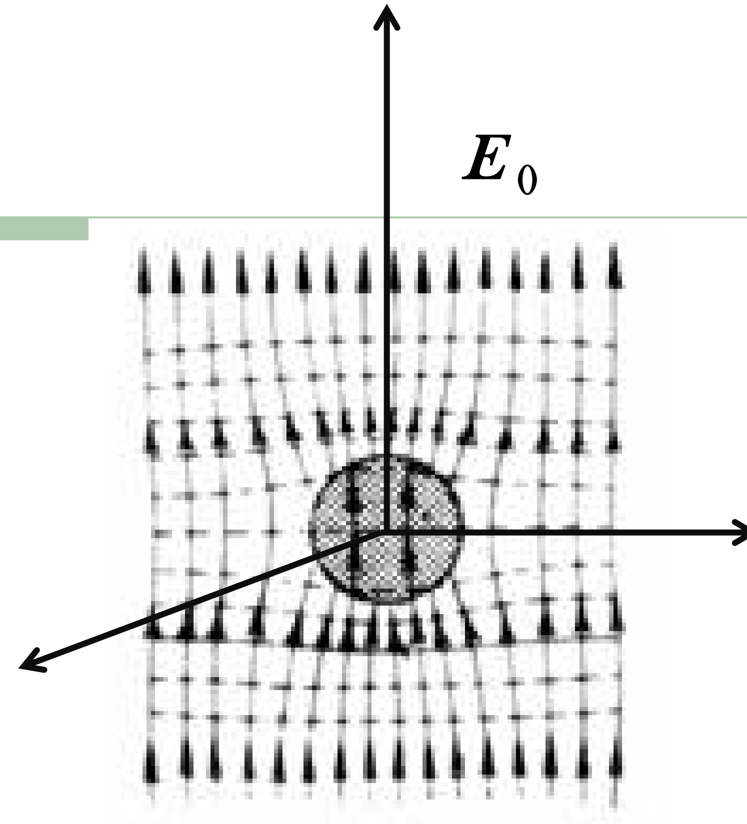


极化电荷附加电场



总的电场分布

$$\begin{cases}\nabla^2\phi_{0,i}\left(r,\theta,\varphi\right)=0\\ \phi_i\left(a,\theta,\varphi\right)=\phi_0\left(a,\theta,\varphi\right)\\ \varepsilon\dfrac{\partial\phi_i}{\partial r}\Big|_{r=a}=\varepsilon_0\dfrac{\partial\phi_0}{\partial r}\Big|_{r=a}\\ \lim\limits_{r\to\infty}-\nabla\phi_0=\hat{e}_z E_0\end{cases}$$



球坐标系：$h_1=1$，$h_2=r$，$h_3=r\sin\theta$

$$\nabla^2\phi(\boldsymbol{r})=\frac{1}{r^2}\left[\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right)+\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\phi}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2\phi(\boldsymbol{r})}{\partial\varphi^2}\right]$$

令：$\phi(r,\theta,\varphi)=R(r)\Theta(\theta)\Phi(\varphi)$

$$\begin{cases} r^2\dfrac{\mathrm{d}^2R(r)}{\mathrm{d}r^2}+2r\dfrac{\mathrm{d}R(r)}{\mathrm{d}r}-\mu R(r)=0 \\ \Phi''(\varphi)+m^2\Phi(\varphi)=0 \\ \dfrac{1}{\sin\theta}\dfrac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\dfrac{\mathrm{d}\Theta(\theta)}{\mathrm{d}\theta}\right)+\left[\mu-\dfrac{m^2}{\sin^2\theta}\right]\Theta(\theta)=0 \end{cases}$$

作变换：$x=\cos\theta,\quad y(x)=\Theta(\theta)$

$$\left(1-x^2\right)y''(x)-2xy'(x)+\left[\mu-\frac{m^2}{1-x^2}\right]y(x)=0$$

$$\begin{cases} \phi\left(r,\theta,\varphi+2n\pi\right)=\phi\left(r,\theta,\varphi\right) \\ \lim\limits_{\substack{r\to 0 \\ \theta\to 0,\pi}} \phi\left(r,\theta,\varphi\right)\to \text{有限值} \end{cases} \to \text{自然边界条件}$$

$$\begin{cases} \Phi''\left(\varphi\right)+m^2\Phi\left(\varphi\right)=0 \\ \Phi\left(\varphi\right)=\Phi\left(\varphi+2n\pi\right) \end{cases}$$

$$\begin{cases} \left(1-x^2\right)y''\left(x\right)-2xy'\left(x\right)+\left[\mu-\dfrac{m^2}{1-x^2}\right]y\left(x\right)=0 \\ y\left(x\right)\Big|_{x\to\pm 1}\to \text{有界} \end{cases}$$

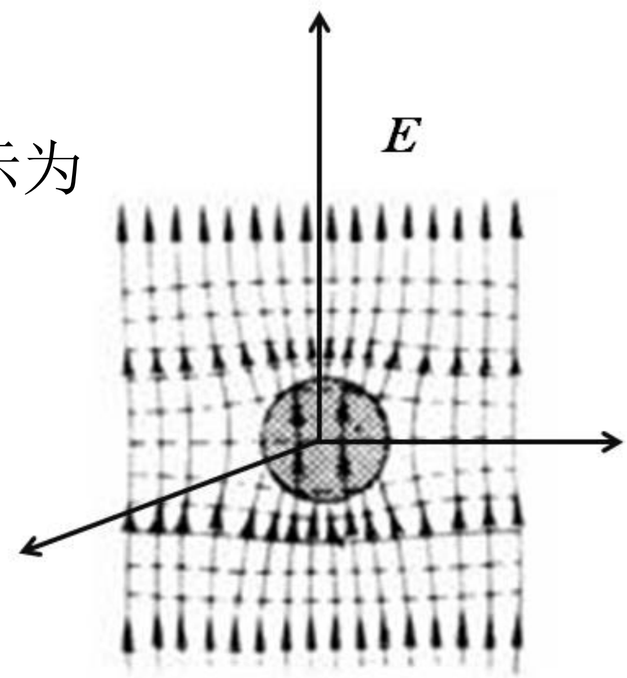
$$\begin{cases}\Phi''(\varphi)+m^2\Phi(\varphi)=0\\ \Phi(\varphi)=\Phi(\varphi+2n\pi)\end{cases}\rightarrow\Phi(\varphi)=\begin{Bmatrix}\cos n\varphi\\ \sin n\varphi\end{Bmatrix}\quad,\quad 0,1,2,3\ldots$$

任$[0,2\pi]$上分段连续有界函数可展开表示为

$$\Phi(\varphi)=\sum_n\left[A_n\cos n\varphi+B_n\sin n\varphi\right]$$

由于本物理问题的轴对称性

$$\Phi(\varphi)=C(\text{常数})\rightarrow m=0$$



$$\begin{cases}\left(1-x^2\right)y''(x)-2xy'(x)+\mu y(x)=0\\ y(x)\big|_{x\to\pm1}\to\text{有界}\end{cases}$$

本征值： $\mu=l(l+1),l=0,1,2,3.....$

本征函数： $y(x)=P_l(x)=P_l(\cos\theta)$

$$\begin{cases}P_0(x)=P_0(\cos\theta)=1\\ P_1(x)=P_1(\cos\theta)=\cos\theta\\ P_2(x)=P_2(\cos\theta)=\dfrac{1}{2}\left[3\cos^2\theta\text{-}1\right]\end{cases}$$

利用$\left\{P_l\left(\cos\theta\right)\right\}$的完备特性，区间$[-1,1]$或$[0，\pi]$上分段连续有界函数可展开表示为

$$\begin{cases} f\left(\cos\theta\right)=\sum_l A_l P_l\left(\cos\theta\right) \\ f\left(x\right)=\sum_l A_l P_l\left(x\right) \end{cases}$$

$$r^2\frac{\mathrm{d}^2R\left(r\right)}{\mathrm{d}r^2}+2r\frac{\mathrm{d}R\left(r\right)}{\mathrm{d}r}-l\left(l+1\right)R\left(r\right)=0 \quad \to \quad R\left(r\right)=\begin{pmatrix} r^l \\ r^{-(l+1)} \end{pmatrix}$$

球坐标系下拉普拉斯方程的特解可为

$$\phi_l(r,\theta)=R_l(r)\Theta_l(\cos\theta)$$

$$=\left(A_l r^l+B_l r^{-(l+1)}\right)P_l(\cos\theta)\quad,\quad l=0,1,2,3\ldots$$

$$\phi(r,\theta)=\sum_l \phi_l(r,\theta)=\sum_l R_l(r)\Theta_l(\cos\theta)$$

$$\begin{cases}\phi_i(r,\theta)=\sum_{l=0}^{\infty}\left(A_l r^l+B_l r^{-(l+1)}\right)P_l(\cos\theta) & ,r<a(\text{球内})\\ \phi_0(r,\theta)=\sum_{l=0}^{\infty}\left(C_l r^l+D_l r^{-(l+1)}\right)P_l(\cos\theta) & ,r>a(\text{球外})\end{cases}$$

$$\text{球外}: -\lim_{r\to\infty}\nabla\phi_0 = -\lim_{r\to\infty}\left[\hat{e}_r\frac{\partial\phi_0}{\partial r}+\hat{e}_\theta\frac{\partial\phi_0}{r\partial\theta}\right]=\hat{e}_z E_0$$

$$\to \lim_{r\to\infty}\phi_0 = -E_0 r\cos\theta$$

$$\to \phi_0\left(r,\theta\right)=\sum_{l=0}^{\infty}\left(C_l r^l + D_l r^{-(l+1)}\right)P_l\left(\cos\theta\right)$$

$$\to C_0 = 0, C_1 = -E_0, C_l = 0\left(l>1\right)$$

$$\phi_0\left(r,\theta\right) = -E_0 r\cos\theta + \sum_{l=0}^{\infty} D_l r^{-(l+1)} P_l\left(\cos\theta\right)$$

$$\text{球内}:\lim_{r\to 0}\phi(r,\theta,\varphi)\to\text{有限值}$$

$$\to\phi_i(r,\theta)=\sum_{l=0}^{\infty}\left(A_l r^l+B_l r^{-(l+1)}\right)P_l(\cos\theta)\to B_l=0$$

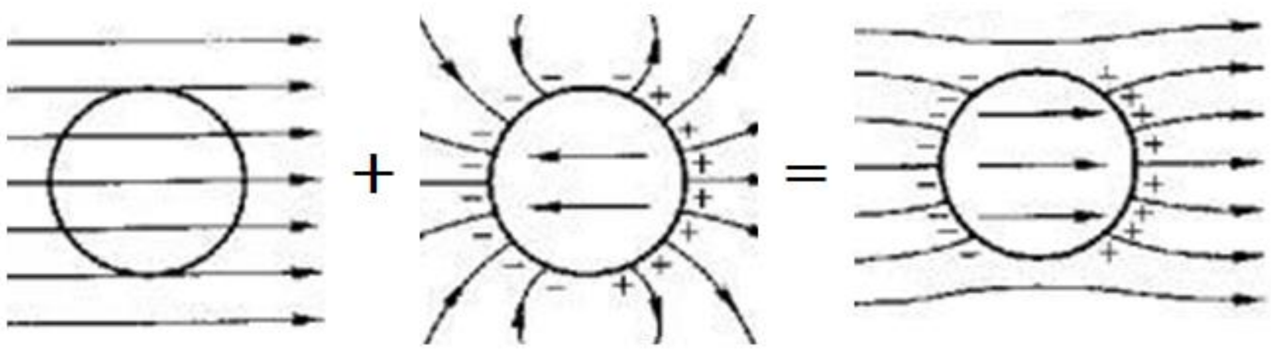
$$\begin{cases}\phi_i(r,\theta)=\sum\limits_{l=0}^{\infty}A_l r^l P_l(\cos\theta);\ \text{球内}\\ \phi_0(r,\theta)=-E_0 r\cos\theta+\sum\limits_{l=0}^{\infty}D_l r^{-(l+1)}P_l(\cos\theta)\ ;\ \text{球外}\end{cases}$$

$$
\begin{cases}
\phi_i(a,\theta,\varphi) = \phi_0(a,\theta,\varphi) \\
\varepsilon \dfrac{\partial \phi_i}{\partial r}\Big|_{r=a} = \varepsilon_0 \dfrac{\partial \phi_0}{\partial r}\Big|_{r=a}
\end{cases}
\rightarrow
\begin{cases}
A_1 = \dfrac{-3\varepsilon_0}{\varepsilon + 2\varepsilon_0} E_0 \quad , \; A_l = 0, \quad l \neq 1 \\
D_1 = \dfrac{(\varepsilon\text{-}\varepsilon_0) a^3}{\varepsilon + 2\varepsilon_0} E_0, \quad D_l = 0, \quad l \neq 1
\end{cases}
$$

$$
\begin{cases}
\phi_i(r,\theta) = \dfrac{-3\varepsilon_0}{\varepsilon + 2\varepsilon_0} E_0 r P_1(\cos\theta) \quad ; \; \text{球内} \\
\phi_0(r,\theta) = \left[\dfrac{\varepsilon - \varepsilon_0}{\varepsilon + 2\varepsilon_0} \dfrac{E_0 a^3}{r^2} - E_0 r \right] P_1(\cos\theta) \quad ; \; \text{球外}
\end{cases}
$$

球内电场:

$$\boldsymbol{E} = -\nabla \phi_i\left(r,\theta\right) = \frac{3\varepsilon_0 E_0}{\varepsilon + 2\varepsilon_0}\nabla\left[rP_1\left(\cos\theta\right)\right] = \frac{3\varepsilon_0 E_0}{\varepsilon + 2\varepsilon_0}\hat{e}_z$$



$$\phi_0(r,\theta) = -E_0 r\cos\theta + \frac{\varepsilon-\varepsilon_0}{\varepsilon+2\varepsilon_0}\frac{E_0 a^3}{r^2}\cos\theta$$

球内介质极化强度：

$$\boldsymbol{p} = \chi\varepsilon_0\boldsymbol{E} = (\varepsilon\text{-}\varepsilon_0)\boldsymbol{E} = \frac{\varepsilon\text{-}\varepsilon_0}{\varepsilon+2\varepsilon_0}3\varepsilon_0 E_0\hat{e}_z$$

介质球总电偶极矩：

$$\boldsymbol{P} = \frac{3}{4}\pi a^3\boldsymbol{p} = \chi\varepsilon_0\boldsymbol{E} = (\varepsilon\text{-}\varepsilon_0)\boldsymbol{E} = \frac{\varepsilon\text{-}\varepsilon_0}{\varepsilon+2\varepsilon_0}4\pi\varepsilon_0 a^3 E_0\hat{e}_z$$

电偶极矩产生的势：$\dfrac{1}{4\pi\varepsilon_0}\dfrac{\boldsymbol{P}\cdot\boldsymbol{r}}{r^3} = \dfrac{\varepsilon-\varepsilon_0}{\varepsilon+2\varepsilon_0}\dfrac{E_0 a^3}{r^2}\cos\theta$

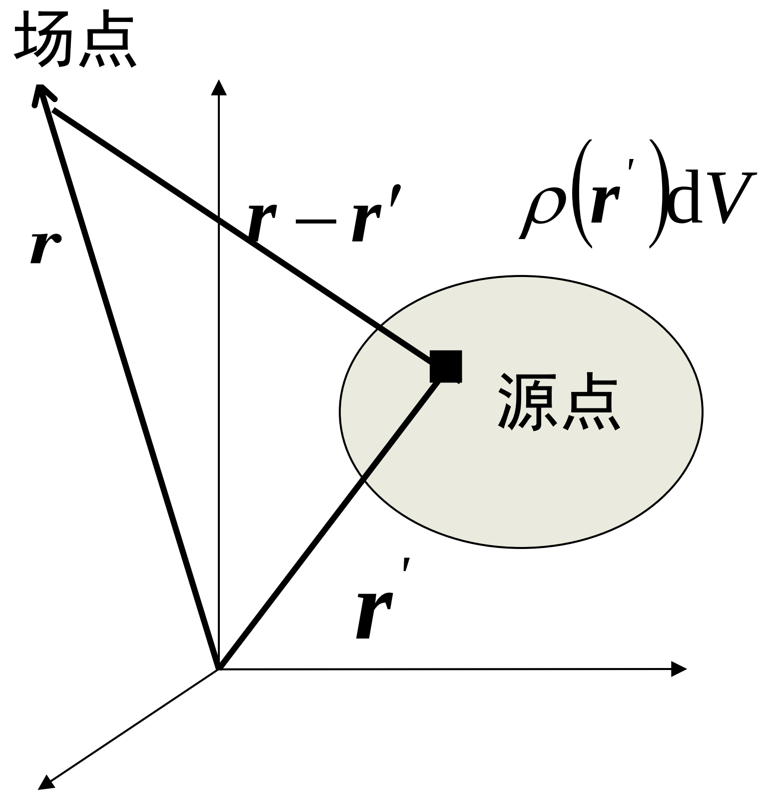
§ 3 格林函数法

1. 格林函数方法的思想

无界区域 体电荷在空间产生的电位：

$$\begin{cases}\nabla^2\phi(\boldsymbol{r}) = -\dfrac{\rho(\boldsymbol{r})}{\varepsilon} \\ \lim\limits_{r\to\infty}\phi(\boldsymbol{r})\to 0\end{cases}$$

$$\phi(\boldsymbol{r}) = \iiint_V \frac{\rho(\boldsymbol{r}')\mathrm{d}V'}{4\pi\varepsilon|\boldsymbol{r}-\boldsymbol{r}'|}$$



$$\left\{\begin{array}{l}\nabla^2 G(\boldsymbol{r},\boldsymbol{r}')=-\dfrac{\delta(\boldsymbol{r}-\boldsymbol{r}')}{\varepsilon}\\ \lim\limits_{r\to\infty} G(\boldsymbol{r},\boldsymbol{r}')\to 0\end{array}\right\}\to G(\boldsymbol{r},\ \boldsymbol{r}')=\frac{1}{4\pi\varepsilon\left|\boldsymbol{r}-\boldsymbol{r}'\right|}$$

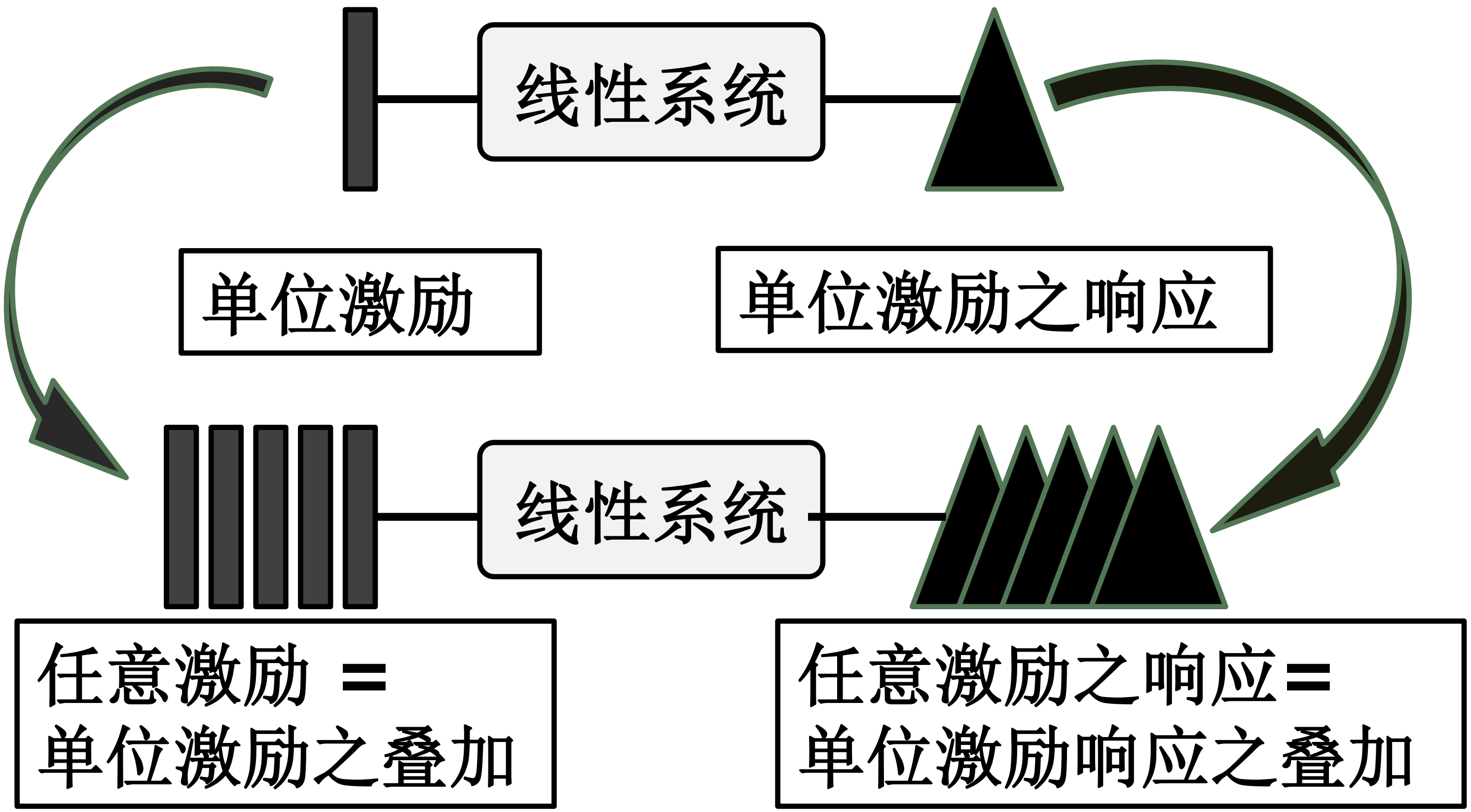
任意体电荷的电位为不同点处点电荷电位叠加，原问题转化求任意点单位电荷在空间电位问题。此即格林函数的基本思想。

$G(\boldsymbol{r},\boldsymbol{r}')$：$\boldsymbol{r}'$点单位电荷在$\boldsymbol{r}$产生的场

$G(\boldsymbol{r},\boldsymbol{r}')\rho(\boldsymbol{r}')\mathrm{d}V'$：$\boldsymbol{r}'$处点电荷$\rho(\boldsymbol{r}')\mathrm{d}V'$在$\boldsymbol{r}$产生的场

$$\phi(\boldsymbol{r})=\iiint_V G(\boldsymbol{r},\boldsymbol{r}')\rho(\boldsymbol{r}')\mathrm{d}V'$$

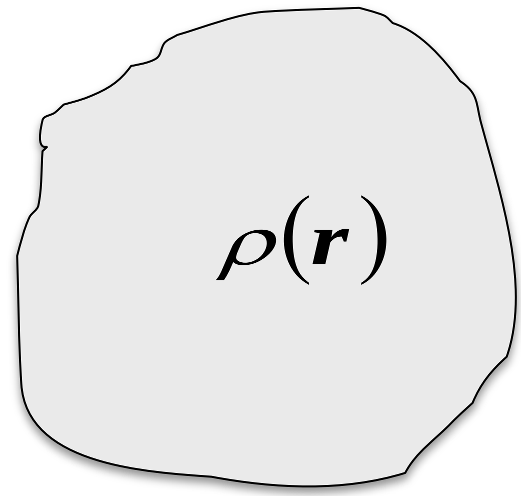
$\rightarrow V$内所有点处电荷产生电位的叠加


线性系统
单位激励
单位激励之响应
线性系统
任意激励 =
单位激励之叠加
任意激励之响应=
单位激励响应之叠加

2. 静态场的格林函数

任意静态场的定解问题：

$$\begin{cases} \nabla^2\phi(\boldsymbol{r}) = -\dfrac{\rho(\boldsymbol{r})}{\kappa} \\ \alpha\phi(M) + \beta\dfrac{\partial\phi(M)}{\partial n} = h(M) \end{cases}$$



格林函数的方程：
$$\begin{cases} \nabla^2 G(\boldsymbol{r},\boldsymbol{r}') = -\dfrac{1}{\kappa}\delta(\boldsymbol{r}-\boldsymbol{r}') \\ \alpha G(\boldsymbol{r},\boldsymbol{r}') + \beta\dfrac{\partial G(\boldsymbol{r},\boldsymbol{r}')}{\partial n} = 0 \end{cases}$$

应用格林定理：

$$\iiint_V (\nabla\varphi\cdot\nabla\phi+\varphi\nabla^2\phi)\mathrm{d}V=\oiint_S(\varphi\nabla\phi)\cdot\mathrm{d}S$$

以及互易性 $G(\boldsymbol{r}',\boldsymbol{r})-G(\boldsymbol{r},\boldsymbol{r}')=0$

$$\phi(\boldsymbol{r})=\iiint_V\rho(\boldsymbol{r}')G(\boldsymbol{r},\boldsymbol{r}')\mathrm{d}V'-\frac{\kappa}{\alpha}\oiint_s h(\boldsymbol{r}')\frac{\partial G(\boldsymbol{r},\boldsymbol{r}')}{\partial n'}\mathrm{d}s'$$

以静电场为例

还原 $h(M)$ 表达式，得到：

$$\phi(\boldsymbol{r}) = \iiint_V G(\boldsymbol{r},\boldsymbol{r}')\rho(\boldsymbol{r}')\mathrm{d}V'$$

$$+\varepsilon \oiint_s G(\boldsymbol{r},\boldsymbol{r}')\frac{\partial\phi(\boldsymbol{r}')}{\partial n'}\mathrm{d}s' - \varepsilon \oiint_s \phi(\boldsymbol{r}')\frac{\partial G(\boldsymbol{r},\boldsymbol{r}')}{\partial n'}\mathrm{d}s'$$

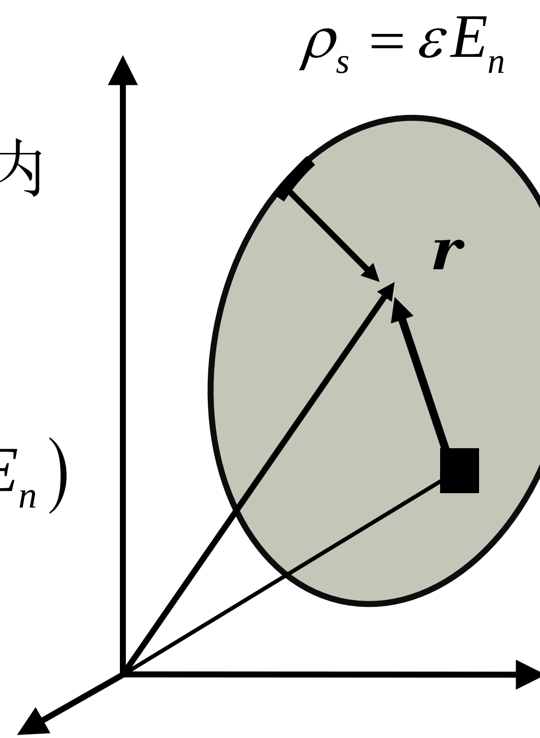
第一项：$\iiint\limits_V G\left(\boldsymbol{r},\boldsymbol{r}'\right)\rho\left(\boldsymbol{r}'\right)\mathrm{d}V'$表示区域内

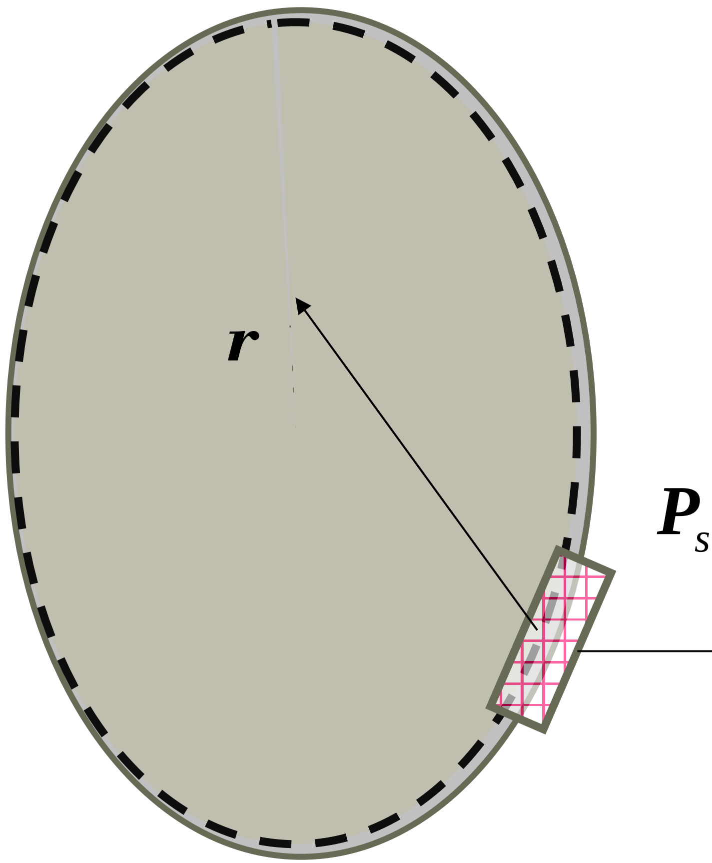
体电荷分布在 $\boldsymbol{r}$ 产生的电位

第二项：$\varepsilon\oiint\limits_s G(\boldsymbol{r},\boldsymbol{r}')\dfrac{\partial\phi(\boldsymbol{r}')}{\partial n'}\mathrm{d}s'$ $\left(\rho_s=\varepsilon E_n\right)$

$$=\oiint\limits_s G(\boldsymbol{r},\boldsymbol{r}')\rho_{\mathrm{s}}(\boldsymbol{r},\boldsymbol{r}')\mathrm{d}s'$$

边界面上面电荷在 $\boldsymbol{r}$ 产生的电位





第三项：$-\varepsilon\oint\!\!\!\oint_s \phi(\mathbf{r}')\dfrac{\partial G(\mathbf{r},\mathbf{r}')}{\partial n'}\mathrm{d}s'$

$$=\oint\!\!\!\oint_s \boldsymbol{P}(\mathbf{r}')\cdot\nabla' G(\mathbf{r},\mathbf{r}')\mathrm{d}s'$$

界面上偶极矩产生电位

$$-\varepsilon\phi\left(\mathbf{r}'\right)\hat{n}\cdot\nabla G\left(\mathbf{r},\mathbf{r}'\right)$$

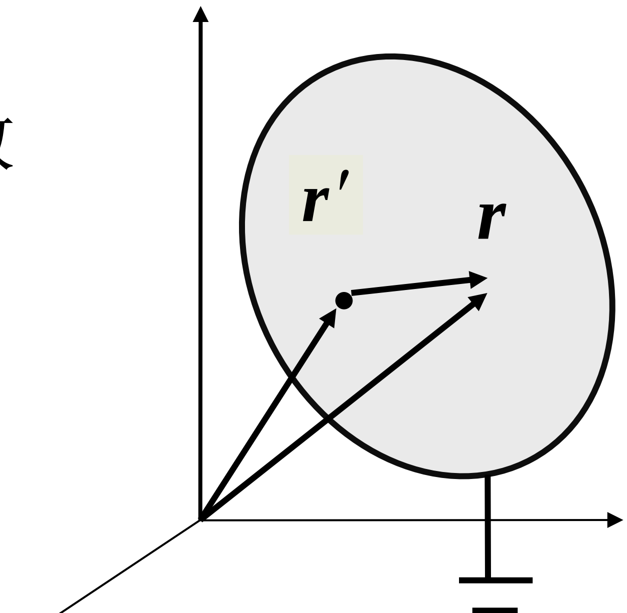
$$=-\boldsymbol{P}_s\cdot\nabla G\left(\mathbf{r},\mathbf{r}'\right)$$

两个典型特例

① 第一类边界条件的Green函数

$$\begin{cases} \nabla^2 \phi(\boldsymbol{r}) = -\dfrac{\rho(\boldsymbol{r})}{\kappa} \\ \phi(M) = \varphi(M) \end{cases}$$

$$\begin{cases} \nabla^2 G(\boldsymbol{r},\boldsymbol{r}') = -\dfrac{1}{\kappa}\delta(\boldsymbol{r}-\boldsymbol{r}') \\ G(\boldsymbol{r},\boldsymbol{r}')\big|_S = 0 \end{cases}$$

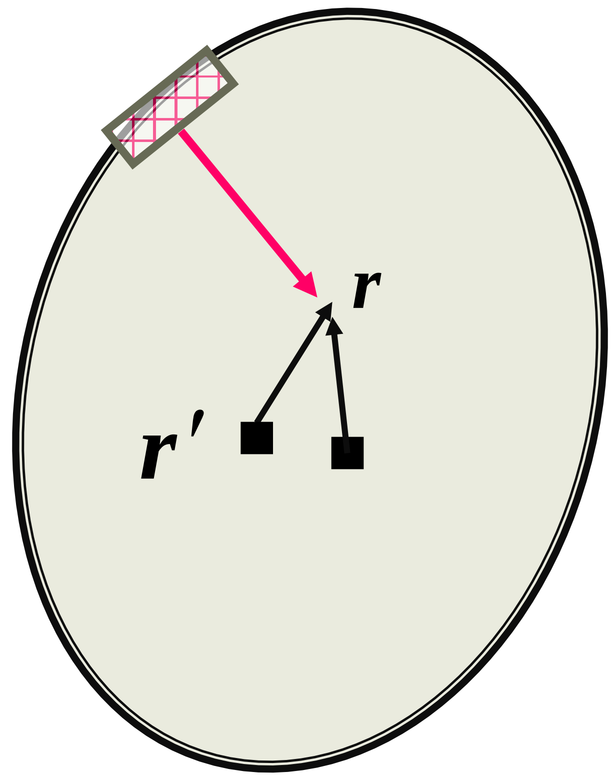


Green函数
的物理模型

$$\begin{cases} \nabla^2 \phi(\boldsymbol{r}) = -\dfrac{\rho(\boldsymbol{r})}{\kappa} \\ \phi(M) = \varphi(M) \end{cases}$$

$$\phi(\boldsymbol{r}) = \iiint_V \rho(\boldsymbol{r}')G(\boldsymbol{r},\boldsymbol{r}')\,\mathrm{d}V'$$

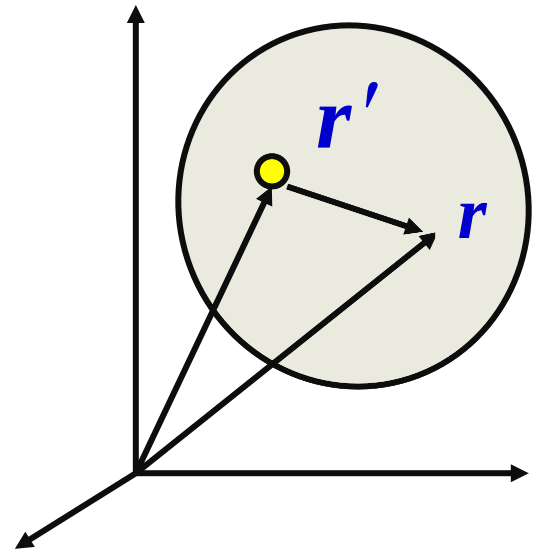
$$-\kappa \oint\!\!\!\oint_s \varphi(\boldsymbol{r}') \frac{\partial G(\boldsymbol{r},\boldsymbol{r}')}{\partial n'} \mathrm{d}s'$$



② 第二类边界条件的格林函数

$$\begin{cases} \nabla^2 \phi(\boldsymbol{r}) = -\dfrac{\rho(\boldsymbol{r})}{\kappa} \\ \dfrac{\partial \phi(r)}{\partial n} = \psi(M) \end{cases}$$

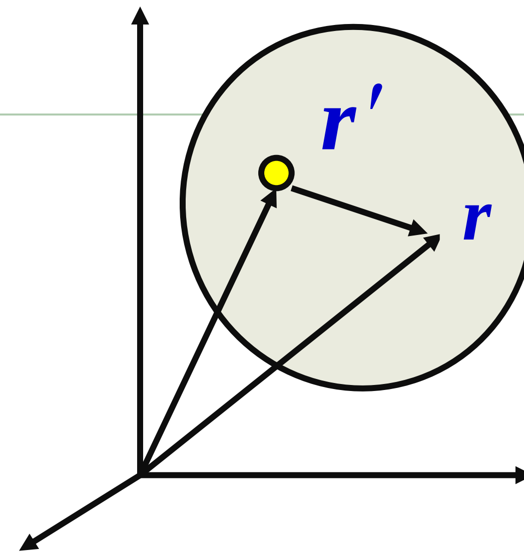
$$\begin{cases} \nabla^2 G(\boldsymbol{r},\boldsymbol{r}') = -\dfrac{1}{\kappa}\delta(\boldsymbol{r}-\boldsymbol{r}') \\ \left.\dfrac{\partial G(\boldsymbol{r},\boldsymbol{r}')}{\partial n}\right|_s = 0 \end{cases}$$



物理模型

$$\phi(r) = \iiint_V \rho(\boldsymbol{r}')G(\boldsymbol{r},\boldsymbol{r}')\mathrm{d}V' + \kappa \oiint_s G(\boldsymbol{r},\boldsymbol{r}')\psi(\boldsymbol{r}')\mathrm{d}s'$$

$$\begin{cases} \nabla^2 G(\boldsymbol{r},\boldsymbol{r}') = -\dfrac{1}{\kappa}\delta(\boldsymbol{r}-\boldsymbol{r}') \\ \dfrac{\partial G(\boldsymbol{r},\boldsymbol{r}')}{\partial n}\Big|_s = 0 \end{cases}$$

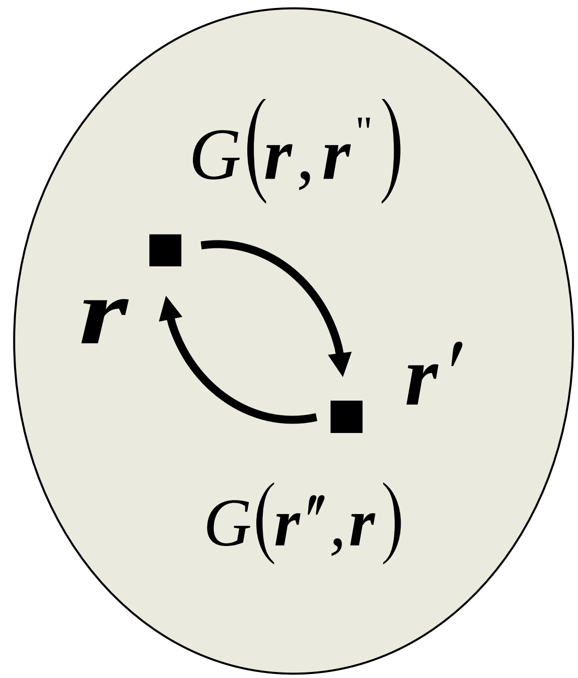


第二类边界条件下Green函数的物理意义：

表示绝热边界条件的封闭系统内单位热源产生的温度场分布。严格意义上的第二类边界条件下Green函数的解不存在？

3. 格林函数的互易性

$$G\left(\boldsymbol{r}',\boldsymbol{r}\right)-G\left(\boldsymbol{r},\boldsymbol{r}'\right)=0$$



物理意义：

$\boldsymbol{r}'$ 点的源在 $\boldsymbol{r}$ 点产生的场

$\boldsymbol{r}$ 点的源在 $\boldsymbol{r}'$ 点 产生的场

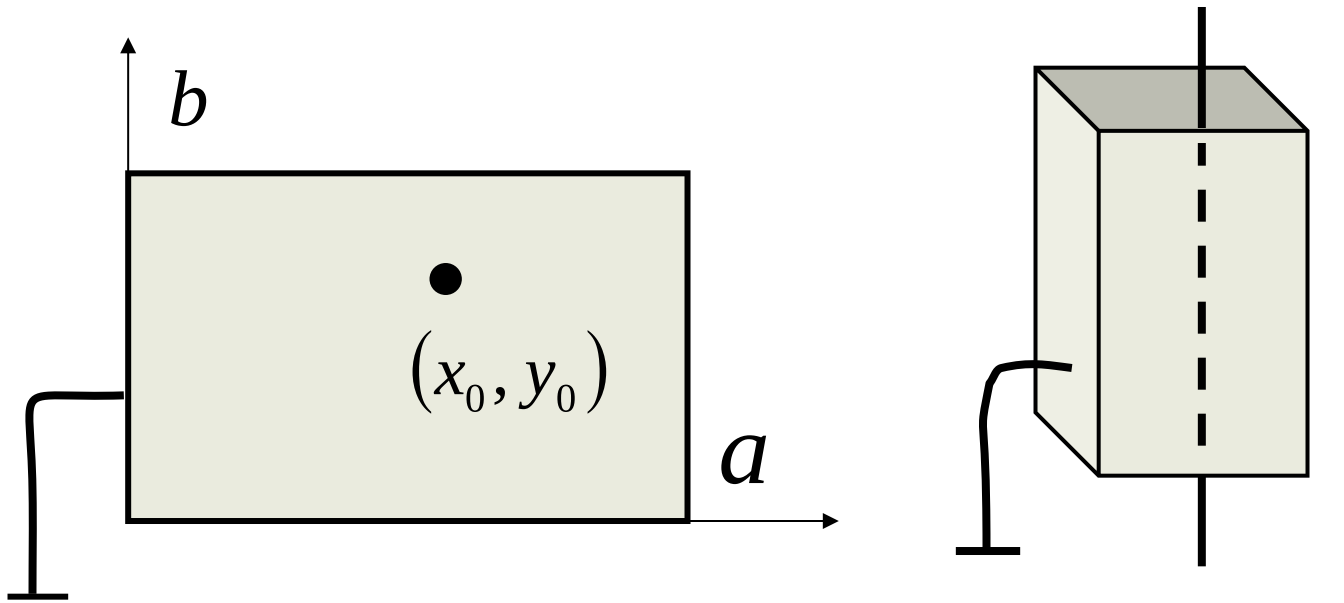
两者具有互易性 。

格林函数的求解：

格林函数本身是一个数学物理方程，
所有关于数学物理方程的求解方法也是
格林函数的求解方法，包括：

分离变量方法、积分变换方法
静电镜像方法、复变函数方法
积分公式方法、Fourier级数方法

【例3 】求无穷长矩形金属壳内单位线源的电位，矩形导体壳接地。



$$\begin{cases} \nabla^2 G(\boldsymbol{r}, \boldsymbol{r}_0) = -\dfrac{1}{\varepsilon}\delta(x - x_0, y - y_0) \\ G(\boldsymbol{r}, \boldsymbol{r}_0)\Big|_{\substack{x=0,a \\ y=0,b}} = 0 \end{cases}$$

设：$G(x, y \mid x_0, y_0) = \sum_{n,m=1}^{\infty} A_{nm} \sin\dfrac{n\pi}{a} x \sin\dfrac{m\pi}{b} y$ 理由

$$\nabla^2 G(x, y \mid x_0, y_0) = -\sum_{n,m=1}^{\infty} A_{nm}\pi^2 \left[\left(\frac{n}{a}\right)^2 + \left(\frac{m}{b}\right)^2\right] \sin\frac{n\pi x}{a} \sin\frac{m\pi y}{b}$$

$$\sum_{n,m=1}^{\infty} A_{nm}\left[\left(\frac{n\pi}{a}\right)^2+\left(\frac{m\pi}{b}\right)^2\right]\sin\frac{n\pi x}{a}\sin\frac{m\pi y}{b}=\frac{1}{\varepsilon}\delta(x-x_0)\delta(y-y_0)$$

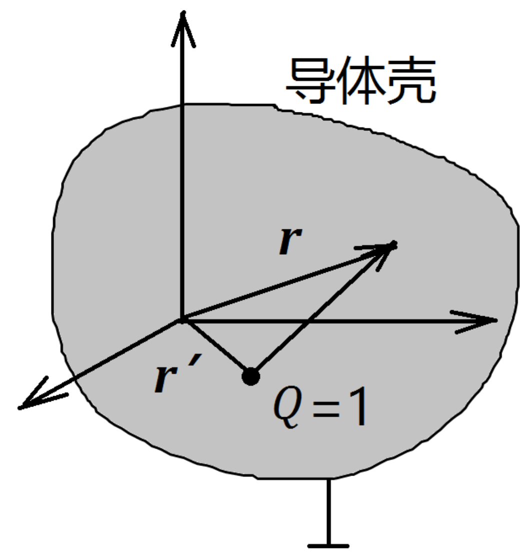
$$A_{nm}=\frac{4ab\sin\frac{n\pi}{a}x_0\sin\frac{m\pi}{b}y_0}{\varepsilon\pi^2\left[(nb)^2+(ma)^2\right]}$$

$$G(x,y\,|\,x_0,y_0)=\sum_{n,m=1}^{\infty}\frac{4ab\sin\frac{n\pi}{a}x_0\sin\frac{n\pi}{a}x\sin\frac{m\pi}{b}y_0\sin\frac{m\pi}{b}y}{\varepsilon\pi^2\left[(nb)^2+(ma)^2\right]}$$

§4 镜像方法

1. 镜像方法的基本思想

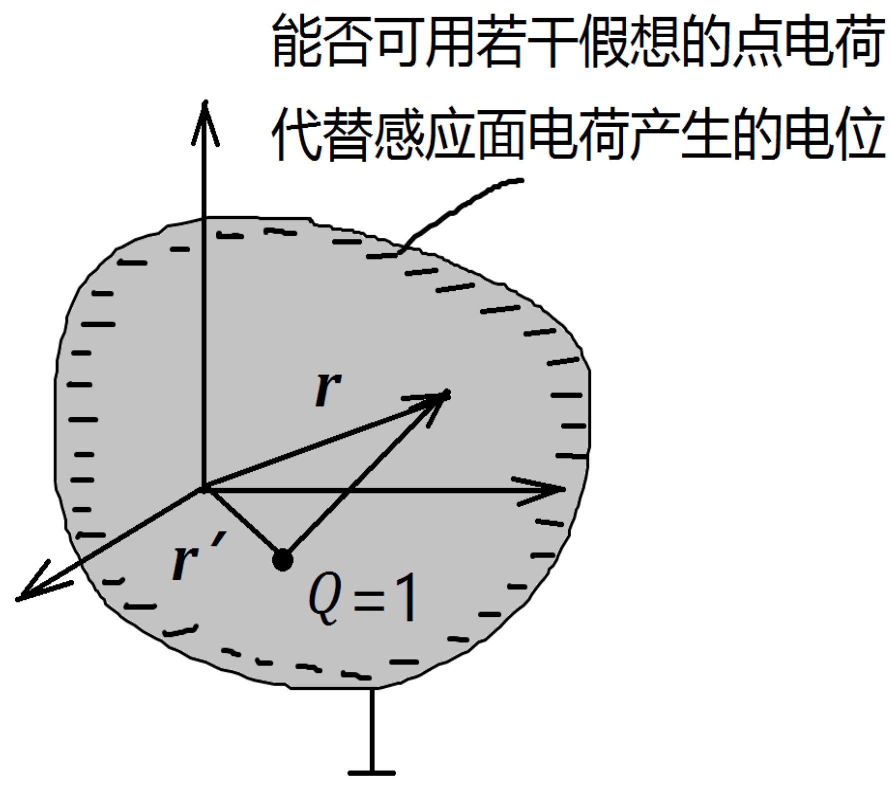
$$u(\boldsymbol{r},\boldsymbol{r}')=\left[\begin{array}{c}\text{点电荷直接}\\\text{产生的电位}\end{array}\right]+\left[\begin{array}{c}\text{边界感应面电}\\\text{荷产生的电位}\end{array}\right]$$



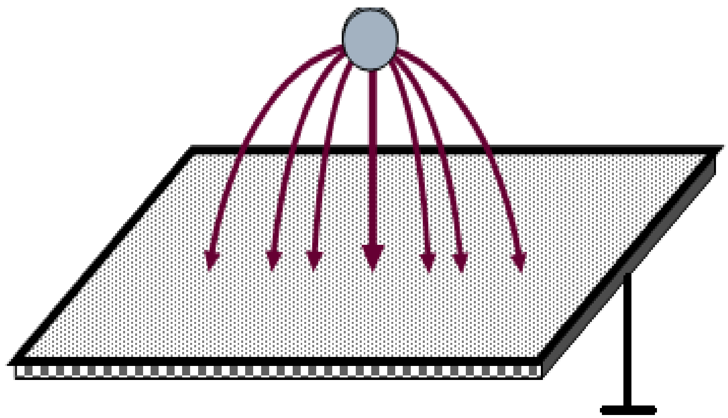
点电荷在空间产生的电位已知

求的是边界感应电荷产生的电位

寻找一个或者多个假想点电荷等效边界面上感应电荷的贡献，这种假想的一个或者多个点电荷称为像电荷—称为镜像法

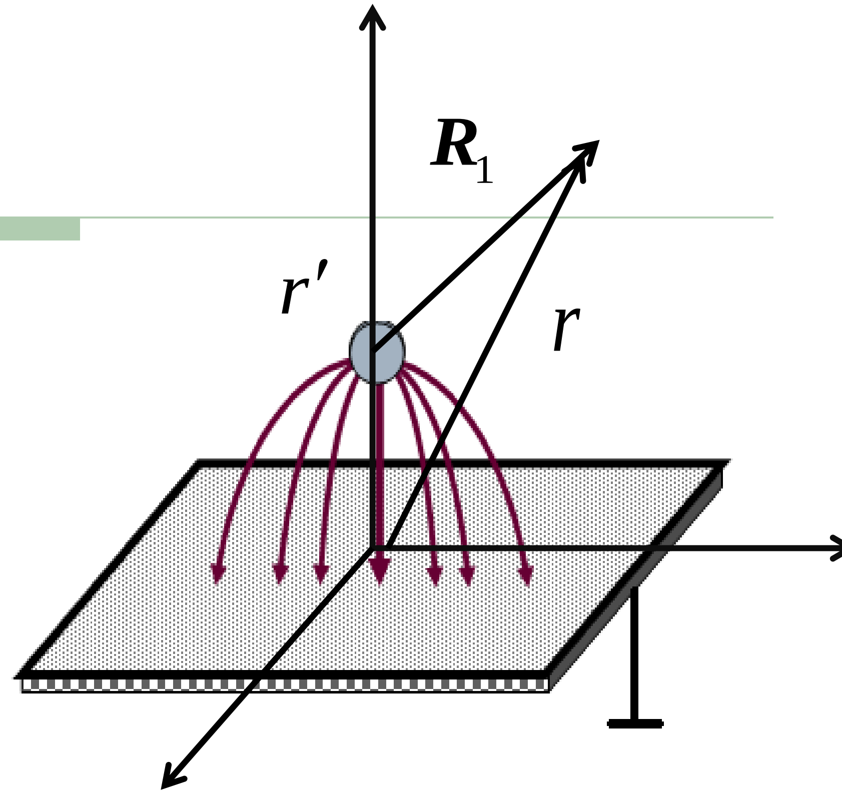


【例4-4】 无穷大接地导体板上单位点电荷在上半空间的电位。



$$\begin{cases} \nabla^2 G(\boldsymbol{r},\boldsymbol{r}') = -\dfrac{1}{\varepsilon_0}\delta(\boldsymbol{r}-\boldsymbol{r}') \quad , \ z>0 \\ G(\boldsymbol{r},\boldsymbol{r}')\big|_{z=0} = 0 \end{cases}$$

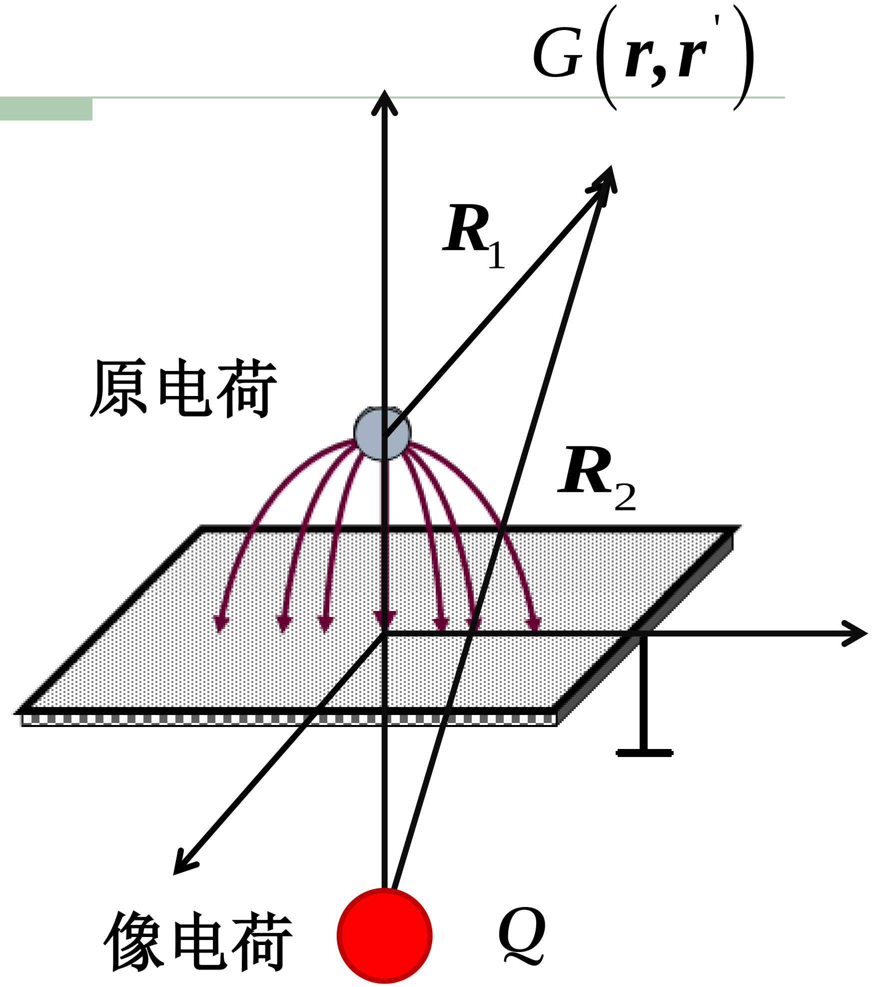
平板上方的电位为单位点电荷的产生的电位和导体平板面感应电荷产生电位的的叠加，即：



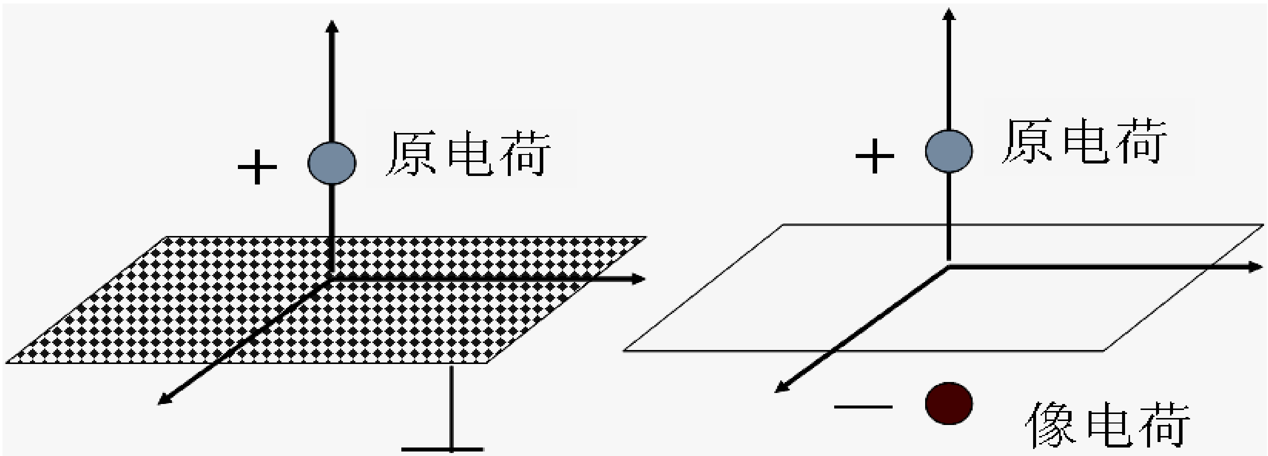
$$G(\boldsymbol{r},\boldsymbol{r}') = \frac{1}{4\pi\varepsilon_0 R_1} + \begin{bmatrix} \text{导体平面感应面电荷} \\ \text{在上半空间产生电位} \end{bmatrix}$$

如果能找到一个像电荷 Q，它在上半空间产生电位与导体平板感应面电荷在上半空间产生电位在等效，平板上方空间电位为

$$G(\boldsymbol{r},\boldsymbol{r}')=\frac{1}{4\pi\varepsilon_0 R_1}+\frac{Q}{4\pi\varepsilon_0 R_2}$$



像电荷的确定



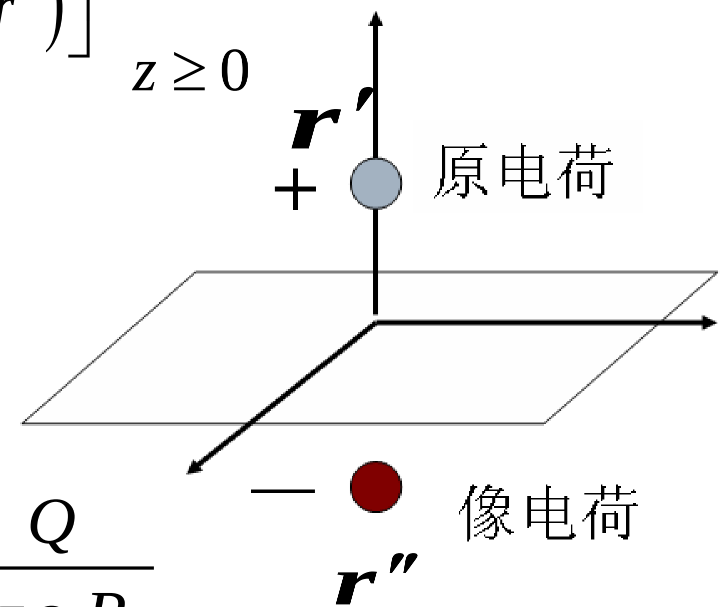
① 像电荷不能与原电荷在同一空间（满足方程）

② 原电荷-感应电荷中心-像电荷在一连线上（对称）

③ 像电荷与原电荷符号相反（感应原理）

④ 像电荷与原电荷在平面上的电位和为零（接地）

$$\begin{cases} \nabla^2 G(\boldsymbol{r},\boldsymbol{r}') = -\dfrac{1}{\varepsilon_0}\left[\delta(\boldsymbol{r}-\boldsymbol{r}') + Q\delta(\boldsymbol{r}-\boldsymbol{r}'')\right] \\ G(\boldsymbol{r},\boldsymbol{r}')\big|_{z=0} = 0 \end{cases} \quad z \geq 0$$



$$G(\boldsymbol{r},\boldsymbol{r}') = \frac{1}{4\pi\varepsilon_0 R_1} + \frac{Q}{4\pi\varepsilon_0 R_2}$$

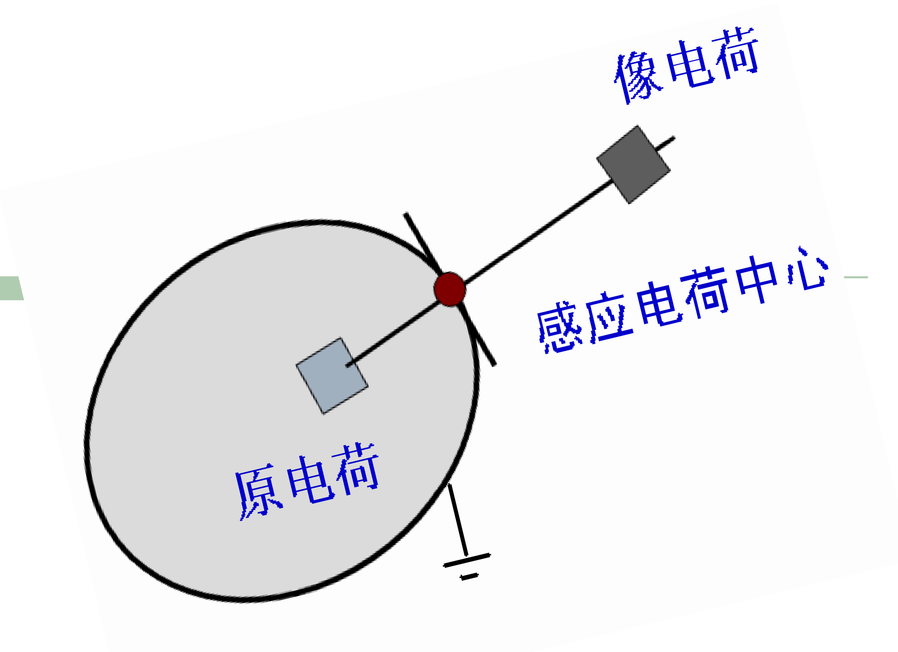
$$= \frac{1}{4\pi\varepsilon_0\sqrt{x^2+y^2+(z-h)^2}} + \frac{Q}{4\pi\varepsilon_0 R_2}$$

$$\boldsymbol{r}' = \hat{e}_z h \quad , \quad \boldsymbol{r}'' = -\hat{e}_z f \qquad , \qquad Q' < 0$$

$$G\left(\boldsymbol{r},\boldsymbol{r}'\right)\Big|_{z=0} = \left[\frac{1}{4\pi\varepsilon_0 R_1} + \frac{Q'}{4\pi\varepsilon_0 R_2}\right]_{z=0} = 0 \ , \ f = h \ , \ Q' = -1$$

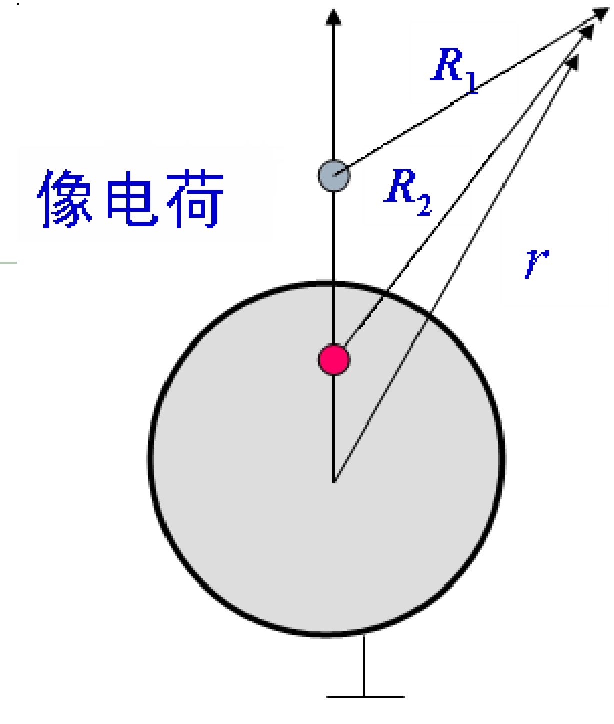
$$G\left(\boldsymbol{r},\boldsymbol{r}'\right) = \frac{1}{4\pi\varepsilon_0}\left[\frac{1}{\sqrt{x^2+y^2+\left(z-h\right)^2}} - \frac{1}{\sqrt{x^2+y^2+\left(z+h\right)^2}}\right]$$

确定像电荷的原则



- 找一个或几个假想电荷等效感应电荷的贡献
- 像电荷在区域的外部，与原电荷符号相反
- 像电荷位置与原电荷的位置互为共轭点对
- 利用边界条件确定像电荷大小和位置

【例4-5】接地导体球壳外部空间的格林函数



$$\begin{cases}\nabla^2 G(\boldsymbol{r},\boldsymbol{r}')=-\dfrac{1}{\varepsilon_0}\delta(\boldsymbol{r}-\boldsymbol{r}')\,, \quad r, \ r'>0 \\ G(\boldsymbol{r},\boldsymbol{r}')\big|_{r=a}=0\end{cases}$$

$$G(\boldsymbol{r},\boldsymbol{r}')=\frac{1}{4\pi\varepsilon_0 R_1}+\frac{Q'}{4\pi\varepsilon_0 R_2}\leftarrow\begin{cases}R_1=\sqrt{r^2+d_1^2-2ad_1\cos\theta} \\ R_2=\sqrt{r^2+d_2^2-2ad_2\cos\theta}\end{cases}$$

在导体球边界上：$$\left[\frac{1}{R_1}+\frac{Q'}{R_2}\right]_{r=a}=0$$

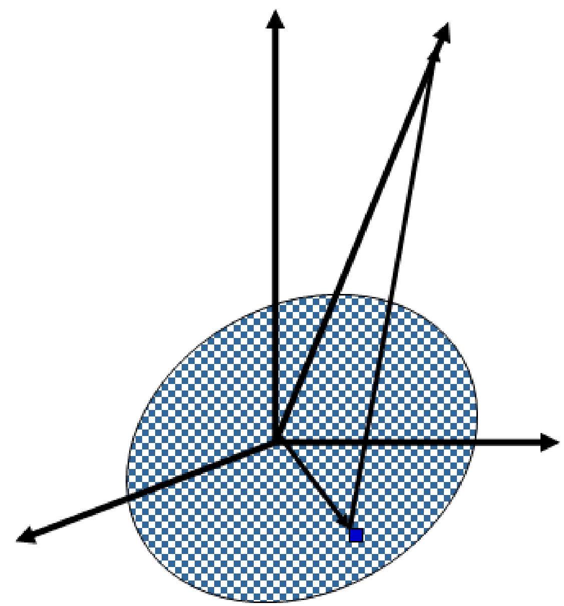
$$(a^2+d_2^2)-Q'(a^2+d_1^2)+2a\cos\theta(Q'^2d_1-d_2)=0$$

$$\begin{cases}(a^2+d_2^2)-Q'(a^2+d_1^2)=0\\ Q'^2d_1-d_2=0\end{cases}\Rightarrow\begin{cases}d_2=\dfrac{a^2}{d_1}\\ Q'=-\dfrac{a}{d_1}\end{cases}$$

§ 5 近似方法—多极矩展开

1. 无界区域势函数计算及意义

$$\phi(\boldsymbol{r})=\frac{1}{4\pi\varepsilon_0}\iiint_V \frac{\rho(\boldsymbol{r}')}{\left|\boldsymbol{r}-\boldsymbol{r}'\right|}\mathrm{d}V'$$



精确计算困难在于被积函数中包含了场点变量在内。即使借助计算机能够给出任意场点的数值，但数值结果的理解需要物理图像，以建立物理模型。

2. 电位函数多极矩展开

源区尺度小于源到场点的距离

将Taylor展开公式

$$f(\boldsymbol{r}+\mathrm{d}\boldsymbol{r})=\sum_n \frac{1}{n!}(\mathrm{d}\boldsymbol{r}\cdot\nabla)^n f(\boldsymbol{r})$$

$$\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}=\frac{1}{r}-\boldsymbol{r}'\cdot\nabla\left(\frac{1}{r}\right)+\frac{1}{2!}(\boldsymbol{r}'\cdot\nabla)^2\frac{1}{r}+\cdots+\frac{(-1)^n}{n!}(\boldsymbol{r}'\cdot\nabla)^n\frac{1}{r}+\cdots$$

$$\phi(\boldsymbol{r}) = \frac{1}{4\pi\varepsilon_0}\iiint\limits_V \rho(\boldsymbol{r}')\mathrm{d}V' \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}.(\boldsymbol{r}'\cdot\nabla)^n\frac{1}{r}$$

$$=\phi^{(0)}(\boldsymbol{r})+\phi^{(1)}(\boldsymbol{r})+\phi^{(2)}(\boldsymbol{r})+\cdots$$

$$=\frac{1}{4\pi\varepsilon_0}\iiint\limits_V \rho(\boldsymbol{r}')\mathrm{d}V'\left[1-\boldsymbol{r}'\cdot\nabla+\frac{1}{2}(\boldsymbol{r}'\cdot\nabla)^2+\cdots\right]\frac{1}{r}$$

$$\phi(\boldsymbol{r}) = \frac{1}{4\pi\varepsilon_0}\iiint_V \rho(\boldsymbol{r}')\mathrm{d}V' \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}.(\boldsymbol{r}'\cdot\nabla)^n\frac{1}{r}$$

$$= \phi^{(0)}(\boldsymbol{r}) + \phi^{(1)}(\boldsymbol{r}) + \phi^{(2)}(\boldsymbol{r}) + \cdots$$

$$\phi^{(0)}(\boldsymbol{r}) = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r} \quad \left(Q = \iiint_V \rho(\boldsymbol{r}')\mathrm{d}V'\right)$$

$$\phi^{(1)}(\boldsymbol{r}) = \frac{-1}{4\pi\varepsilon_0} - \boldsymbol{P}\cdot\nabla\left(\frac{1}{r}\right) \quad \left(\boldsymbol{P} = \iiint_V \boldsymbol{r}'\rho(\boldsymbol{r}')\mathrm{d}V'\right)$$

$$
\begin{aligned}
\phi^{(2)}(\boldsymbol{r}) &= \frac{1}{4\pi\varepsilon_0}\iiint_V \rho(\boldsymbol{r}')\mathrm{d}V'\frac{1}{2!}(\boldsymbol{r}'\cdot\nabla)^2\frac{1}{r} \\
&= \frac{1}{4\pi\varepsilon_0}\iiint_V \rho(\boldsymbol{r}')\mathrm{d}V'\frac{1}{2!}(\boldsymbol{r}'\cdot\nabla)(\boldsymbol{r}'\cdot\nabla)\frac{1}{r} \\
&= \frac{1}{4\pi\varepsilon_0}\iiint_V \rho(\boldsymbol{r}')\mathrm{d}V'\frac{1}{2!}(\boldsymbol{r}'\boldsymbol{r}':\nabla\nabla)\frac{1}{r} \\
&= \frac{1}{4\pi\varepsilon_0}\left[\frac{1}{6}\overleftrightarrow{\boldsymbol{D}}:\nabla\nabla\left(\frac{1}{r}\right)\right]
\end{aligned}
$$

$$\overleftrightarrow{\boldsymbol{D}} = \iiint_V 3\boldsymbol{r}'\boldsymbol{r}'\rho(\boldsymbol{r}')\mathrm{d}V' = \begin{bmatrix} \hat{e}_x\hat{e}_x D_{11} & \hat{e}_x\hat{e}_y D_{12} & \hat{e}_x\hat{e}_z D_{13} \\ \hat{e}_y\hat{e}_x D_{21} & \hat{e}_y\hat{e}_y D_{22} & \hat{e}_y\hat{e}_z D_{23} \\ \hat{e}_z\hat{e}_x D_{31} & \hat{e}_z\hat{e}_y D_{32} & \hat{e}_z\hat{e}_z D_{33} \end{bmatrix}$$

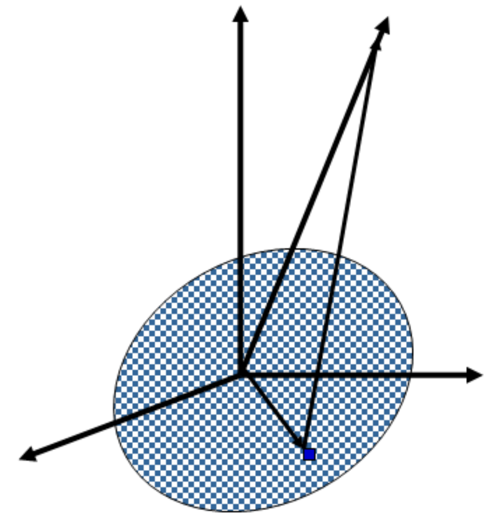
$$D_{ij} = \iiint_V 3x_i' x_j' \rho(x_1', x_2', x_3')\mathrm{d}x_1'\mathrm{d}x_2'\mathrm{d}x_3'$$

其中 $x_1' = x', x_2' = y', x_3' = z',\ \ i, j = 1 \to 3$

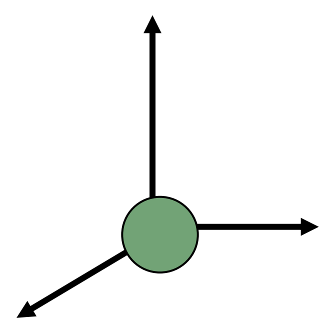
3. 各项意义—电多极矩概念

零级展开项：

$$\phi^{(0)}(\boldsymbol{r})=\frac{Q}{4\pi\varepsilon_0 r} \qquad Q=\iiint_V \rho(\boldsymbol{r}')\mathrm{d}V'$$



相当于将体积V中电荷集中于坐标原点的等效点电荷在远处产生的场

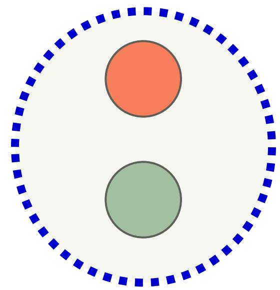


这是忽略体积V不同点处电荷元到场点距离差易所得到的近似结果。

一级展开项的物理意义

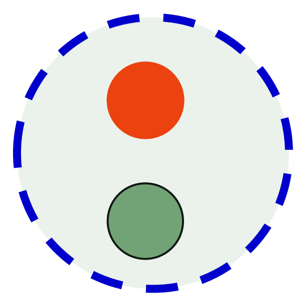
$$\phi^{(1)}(\boldsymbol{r})=\frac{-1}{4\pi\varepsilon_0}\iiint_V \boldsymbol{r}'\rho(\boldsymbol{r}')\mathrm{d}V'\cdot\nabla\left(\frac{1}{r}\right)=\frac{\boldsymbol{P}\cdot\boldsymbol{r}}{4\pi\varepsilon_0 r^3}$$

$$\boldsymbol{P}=\iiint_V \boldsymbol{r}'\rho(\boldsymbol{r}')\mathrm{d}V'=\hat{z}LQ$$



$$\phi^{(0)}(\boldsymbol{r})=\frac{Q}{4\pi\varepsilon_0 r}=0$$

是小电荷体中电荷分布的非均匀性所对应的电偶极矩的电位。



$$\phi^{(1)}(\boldsymbol{r})=\frac{\boldsymbol{P}\cdot\boldsymbol{r}}{4\pi\varepsilon_0 r^3}$$

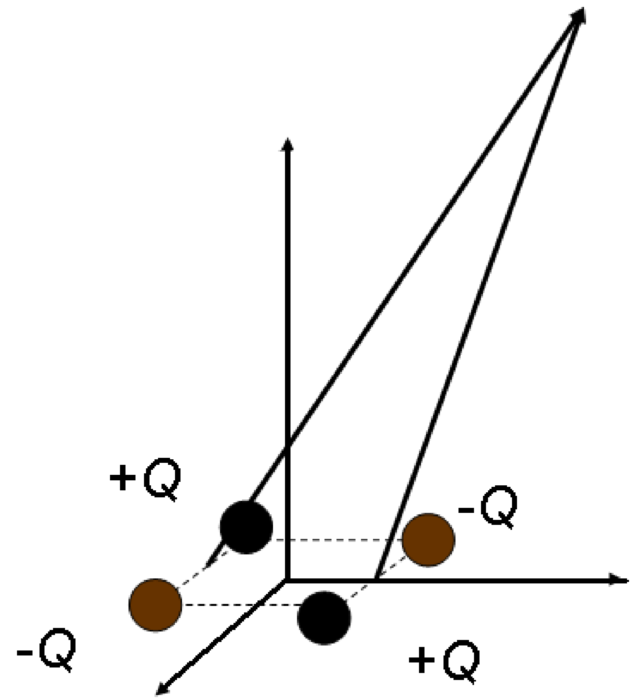
二级展开项意义

$$\phi^{(0)}(\boldsymbol{r}) = \frac{Q}{4\pi\varepsilon_0 r} = 0$$

$$\phi^{(1)}(\boldsymbol{r}) = \frac{-1}{4\pi\varepsilon_0}\iiint_V \boldsymbol{r}'\rho(\boldsymbol{r}')\mathrm{d}V' \cdot \nabla\left(\frac{1}{r}\right)$$

$$= \frac{\boldsymbol{P}\cdot\boldsymbol{r}}{4\pi\varepsilon_0 r^3} = 0$$

$$\phi^{(2)}(\boldsymbol{r}) = \frac{1}{4\pi\varepsilon_0}\frac{1}{3!}\overset{\leftrightarrow}{\boldsymbol{D}} : \nabla\nabla\left(\frac{1}{r}\right)$$



小电荷体系非均匀性对应的电四极矩所产生的电位

$$\phi(\boldsymbol{r})=\phi^{(2)}(\boldsymbol{r})=\frac{1}{4\pi\varepsilon_0}\frac{1}{3!}\overleftrightarrow{\boldsymbol{D}}:\nabla\nabla\left(\frac{1}{r}\right)$$

$$=\frac{3QL^2}{4\pi\varepsilon_0}\frac{1}{6}\begin{pmatrix} 0 & \hat{e}_x\hat{e}_y & 0 \\ \hat{e}_y\hat{e}_x & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} \hat{e}_x\hat{e}_x xx & \hat{e}_x\hat{e}_y xy & \hat{e}_x\hat{e}_z xz \\ \hat{e}_y\hat{e}_x yx & \hat{e}_y\hat{e}_y yy & \hat{e}_y\hat{e}_z yz \\ \hat{e}_z\hat{e}_x zx & \hat{e}_z\hat{e}_y zy & \hat{e}_z\hat{e}_z zz \end{pmatrix}\frac{6}{r^5}$$

$$=\frac{3QL^2}{4\pi\varepsilon_0}\frac{\cos\theta\sin\theta}{r^3}$$