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MATH 131 NOTES

Recall 1. $\overline{E} = E \cup E'$.

Theorem 1. *$E \subseteq X$. $\overline{E}$ is closed.*

Proof. We will show $X \setminus \overline{E}$ is open. Let $x \in X \setminus \overline{E}$. We must show x is in an interior point of $X \setminus \overline{E}$. Since $x \in X \setminus \overline{E}$, we know x is not a limit point of E. Thus there exists an open neighborhood U of x such that U is completely contained in $X \setminus E$. Also note that U contains no limit points of E. Indeed, let $p \in E'$. Suppose $p \in U$. Thus there exists a point $q \neq p$, $q \in U$, $q \in E$. However, we know $U \subseteq X \setminus E$. Thus there does not exist a point $p \in E' \cap U$. Thus $U \subseteq X \setminus E \cap X \setminus E' = X \setminus (E \cap E') = X \setminus \overline{E}$. Thus x is an interior point of $X \setminus \overline{E}$ This holds for all $x \in X \setminus \overline{E}$. Thus $X \setminus \overline{E}$ is open. Therefore $\overline{E}$ is closed. □

Theorem 2. *$E = \overline{E} \iff E$ is closed.*

Proof. ($\Rightarrow$) Done, by previous theorem. ($\Leftarrow$). Suppose E is closed. Therefore it contains all of its limit points, i.e. $E' \subseteq E$. Thus $\overline{E} = E \cup \overline{E} = E$. □

Theorem 3. *Let $E \subseteq F \subseteq X$. If F is closed, then $\overline{E} \subseteq F$.*

Proof. Let $p \in E'$. Let U be an open neighborhood of p. There exists $q \neq p$ such that $q \in U$ and $q \in E$. Since $E \subseteq F$, we know $q \in F$. Therefore p is a limit point of F. Since F is closed, $p \in F$. Thus $E' \subseteq F$. Therefore $\overline{E} = E \cup E' \subseteq F$. □

Remark 1.

$$\overline{E} = \bigcap_{\substack{F=\overline{F},\\ E \subseteq F \subseteq X}} F.$$

Question 1. Is (a, b) open? Consider this interval in $\mathbb{R}$. Consider any $c \in (a, b)$. Indeed we can find a neighborhood of c completely contained in (a, b). Now consider the interval in $\mathbb{R}^2$. In this case it does not. Thus the question is ambiguous.

Definition 1. *Let $Y \subseteq X$ be metric spaces. A subset $E \subseteq Y$ is* open relative to Y *if and only if there is an open set $U \subseteq X$ such that:*

$$E = U \cap Y.$$

Example 1. Let $Y = (a, b]$ and $X = \mathbb{R}$. Then $(c, b]$ is open in Y for any $c \in (a, b)$.

Compact Sets

Definition 2. *An* open cover *of $E \subseteq X$ is a collection $\{U_\alpha\}$ of open sets of X (U_α is open in X) such that $E \subseteq \bigcup_\alpha U_\alpha$.*

Example 2. Consider the following examples:

- Let $E = X = \mathbb{R}$. Consider $U_n = (n-1, n+1)$.
- Let $E = \mathbb{Z}, X = \mathbb{R}$. Consider $U_n = (n - 1/2, n + 1/2)$.
- Let $E = [0,1], X = \mathbb{R}$. Consider $U_n = (-2/n, 2/n)$.
- Let $E = (0,1), X = \mathbb{R}$. Consider $U_n = (0, 1/n), V_n = (1/n, 1)$. Consider then $\{U_n\} \cup \{V_n\}$.
- Let $E = \{1,3,4\}, X = \mathbb{R}$. Let $U_n = (n - 1/2, n + 1/2)$.

Definition 3. *A set $K \subseteq X$ is* compact *if and only if every open cover $\{U_\alpha\}$ contains a finite sub cover.* $(K \subseteq U_{\alpha_1} \cup \cdots \cup U_{\alpha_k})$