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MATH 131 NOTES

Recall 1. If Q is a refinement of P, then $L(P, f, \alpha) \leq L(Q, f, \alpha)$ and $U(Q, f, \alpha) \leq U(P, f, \alpha)$.

Theorem 1. $\underline{\int_a^b} f d\alpha \leq \overline{\int_a^b} f d\alpha.$

Proof. Let P_1, P_2 be partitions of $[a, b]$. Consider $P = P_1 \cup P_2$, their common refinement. Then:

$$L(P_1, f, \alpha) \leq L(P, f, \alpha) \leq U(P, f, \alpha) \leq U(P_2, f, \alpha).$$

So $L(P_1, f, \alpha) \leq U(P_2, f, \alpha)$. Thus it follows that:

$$\underline{\int_a^b} f d\alpha = \sup\{L(P_1, f, \alpha | P_1\} \leq U(P_2, f, \alpha).$$

Then taking the infimum with respect to P_2, we have:

$$\underline{\int_a^b} f d\alpha \leq \overline{\int_a^b} f d\alpha.$$

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Theorem 2. *$f \in \mathcal{R}(\alpha)$ on $[a, b]$ if and only if for all $\epsilon > 0$ there exists a partition P such that:*

$$U(P, f, \alpha) - L(P, f, \alpha) < \epsilon.$$

Proof. ($\Leftarrow$) Let $\epsilon > 0$. We know:

$$L(P, f, \alpha) \leq \underline{\int} f d\alpha \leq \overline{\int} f d\alpha \leq U(P, f, \alpha).$$

Thus it follows:

$$\overline{\int} f d\alpha - \underline{\int} f d\alpha < \epsilon.$$

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Since this holds for all $\epsilon > 0$, we have equality.

($\Rightarrow$) Let $\epsilon > 0$. Then there exist partitions P_1, P_2 such that:

$$U(P_2, f, \alpha) - \int f d\alpha < \frac{\epsilon}{2},$$
$$\int f d\alpha - L(P_1, f, \alpha) < \frac{\epsilon}{2}.$$

Let $P = P_1 \cup P_2$. Then:

$$\begin{aligned} U(P, f, \alpha) &\leq U(P_2, f, \alpha) \\ &< \int f d\alpha + \frac{\epsilon}{2} \\ &< L(P_1, f, \alpha) + \epsilon \\ &\leq L(P, f, \alpha) + \epsilon. \end{aligned}$$

Therefore there exists some P such that $U(P, f, \alpha) - L(P, f, \alpha) < \epsilon$.

Theorem 3. *f is continuous o $[a, b]$ if and only if $f \in \mathcal{R}(\alpha)$ on $[a, b]$.*

Proof. at Let $\epsilon > 0$. We wish to show that there exist some partition P such that $U(P, f, \alpha) - L(P, f, \alpha) < \epsilon$. Notice that for any partition P,

$$U(P, f, \alpha) - L(P, f, \alpha) = \sum_{i=1}^{n} (M_i - m_i) \Delta \alpha_i$$

Note that there exists $\eta > 0$ such that $[\alpha(b) - \alpha(a)]\eta < \epsilon$. Since f is continuous on this interval, then f is uniformly continuous on $[a, b]$ (since the domain of the continuous function is compact). Thus there exists $\delta > 0$ such that:

$$|x - t| < \delta \implies |f(x) - f(t)| < \eta.$$

Choose P such that $\Delta x_i < \delta$. Then $M_i - m_i \leq \eta$. Thus

$$\begin{aligned} U(P, f, \alpha) - L(P, f, \alpha) &= \sum_{i=1}^{n} (M_i - m_i) \Delta \alpha_i \\ &\leq \eta \sum_{i=1}^{n} \Delta \alpha_i \\ &= \eta((\alpha(b) - \alpha(a)) < \epsilon. \end{aligned}$$

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Theorem 4. *If f is monotonic on $[a, b]$, and α is continuous, monotonically increasing, then $f \in \mathcal{R}(\alpha)$.*

Proof. Without loss of generality assume f is monotonically increasing. Let $\epsilon > 0$. We show there exists a partition P such that $U(P, f, \alpha) - L(P, f, \alpha) < \epsilon$.

Fix $n \in \mathbb{N}$. Then there exists a partition P such that $\Delta \alpha_i = \frac{\alpha(b) - \alpha(a)}{n}$. Then it follows:

$$\begin{aligned} U(P, f, \alpha) - L(P, f, \alpha) &= \sum_{i=1}^{n} (M_i - m_i) \Delta \alpha_i \\ &= \frac{\alpha(b) - \alpha(a)}{n} \sum_{i=1}^{n} (f(x_i) - f(x_{i-1})) \\ &= \frac{\alpha(b) - \alpha(a)}{n} (f(b) - f(a)). \end{aligned}$$

Then choose $n \in \mathbb{N}$ such that the above expression is less than ϵ. ∎

Theorem 5. *Let f be bounded on $[a, b]$ with only finitely many discontinuities. If α is continuous at each discontinuity of f, then $f \in \mathcal{R}(\alpha)$.*

Definition 1. Let $I(x) = \begin{cases} 0 & x \leq 0 \\ 1 & x > 0 \end{cases}$.

Theorem 6. *Suppose $\sum c_n$ converges, $c_n \geq 0$ and let $\{s_n\}$ be a sequence in $[a,b]$ such that $s_n \neq s_m$ if $n \neq m$. Let $\alpha(x) = \sum_{i=1}^{\infty} c_n I(x - s_n)$. If f is continuous on $[a,b]$, then $\int f d\alpha = \sum c_n f(s_n)$.*

Theorem 7. *Let α be monotonically increasing and $\alpha' \in \mathcal{R}$ on $[a,b]$. If f is bounded, then $f \in \mathcal{R}(a)$ if and only if $f\alpha' \in \mathcal{R}$. Moreover*

$$\int_a^b f d\alpha = \int_a^b f(x)\alpha'(x)dx.$$

Example 1. When calculating the moment of inertia of a rod with mass, it's given by a Riemann-Stieltjies integral. If x represents the distance from the point of rotation, $I = \int x^2 dm$. If $m'(x) = \rho(x)$, then $I = \int x^2 \rho(x) dx$.