A basic example of continuous adjoint method

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December, 2010

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Index

Original (forward) problem

Continuous method

Adjoint (backward) problem

Comparison with discrete approach

Flowchart

Very simple examples

Introducing a transient term

Barcelona Supercomputing Center
Centro Nacional de Supercomputación

Index

Original (forward) problem

Continuous method

Adjoint (backward) problem

Comparison with discrete approach

Flowchart

Very simple examples

Introducing a transient term

Barcelona Supercomputing Center
Centro Nacional de Supercomputación

Original (forward) problem

The basic example (2D) to be studied has the following state equations:

$$
\begin{aligned}
\Delta u(x,y) &= \alpha^2, \qquad (x,y) \in \Omega && \text{(1a)}\\
\frac{\partial u}{\partial n}(x,y) &= \beta^2, \qquad (x,y) \in \Gamma_1 && \text{(1b)}\\
u(x,y) &= \gamma^2, \qquad (x,y) \in \Gamma_2 && \text{(1c)}
\end{aligned}
$$

with $\alpha, \beta, \gamma \in \mathbb{R}$ and $\partial\Omega = \Gamma_1 \cup \Gamma_2$.
The design variables are α, β and γ, and the cost functional is:

$$
J(u) := \frac{1}{2}\int_{\Gamma_1} (u(x,y) - \delta)^2 dS \tag{2}
$$

with $\delta \in \mathbb{R}$.

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Original (forward) problem

The objetive of this tutorial is to minimize the cost functional subject to the state equations and using one design variable as a minimization parameter. In summary:

$$
\begin{array}{ll}
\underset{\beta\in\mathbb{R}}{\text{minimize}} & \dfrac{1}{2}\displaystyle\int_{\Gamma_1}(u(x,y)-\delta)^2 dS \\
\text{subject to} & \begin{cases} \Delta u(x,y) & =\alpha^2, \quad (x,y)\in\Omega \\ \dfrac{\partial u}{\partial n}(x,y) & =\beta^2, \quad (x,y)\in\Gamma_1 \\ u(x,y) & =\gamma^2, \quad (x,y)\in\Gamma_2 \end{cases}
\end{array}
\tag{3}
$$

Barcelona Supercomputing Center
Centro Nacional de Supercomputación

Index

Original (forward) problem

Continuous method

Adjoint (backward) problem

Comparison with discrete approach

Flowchart

Very simple examples

Introducing a transient term

Barcelona Supercomputing Center
Centro Nacional de Supercomputación

Step 1: Differentiation of state equations

Differentiate the state equations (1) w.r.t. the design variable β:

$$\begin{aligned} D_\beta \Delta u(x,y) &= D_\beta \alpha^2, & (x,y) &\in \Omega \\ D_\beta \frac{\partial u}{\partial n}(x,y) &= D_\beta \beta^2, & (x,y) &\in \Gamma_1 \\ D_\beta u(x,y) &= D_\beta \gamma^2, & (x,y) &\in \Gamma_2 \end{aligned}$$

The resulting equations are (using the notation $\tilde{u} = D_\beta u$):

$$\Delta \tilde{u}(x,y) = 0, \qquad (x,y) \in \Omega \tag{5a}$$

$$\frac{\partial \tilde{u}}{\partial n}(x,y) = 2\beta, \qquad (x,y) \in \Gamma_1 \tag{5b}$$

$$\tilde{u}(x,y) = 0, \qquad (x,y) \in \Gamma_2 \tag{5c}$$

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Step 2: Build $< L\tilde{u}, \psi >= 0$

Multitply by a test function ψ and integrate by parts (twice) the equation (5a):

$$
\begin{aligned}
\int_{\Omega} \psi \Delta \tilde{u} dV &= 0 \\
\int_{\partial\Omega} \psi \frac{\partial \tilde{u}}{\partial n} - \tilde{u} \frac{\partial \psi}{\partial n} dS + \int_{\Omega} \tilde{u} \Delta \psi dV &= 0 \\
\int_{\Gamma_1} \psi \frac{\partial \tilde{u}}{\partial n} - \tilde{u} \frac{\partial \psi}{\partial n} dS + \int_{\Gamma_2} \psi \frac{\partial \tilde{u}}{\partial n} - \tilde{u} \frac{\partial \psi}{\partial n} dS + \int_{\Omega} \tilde{u} \Delta \psi dV &= 0 \\
\int_{\Gamma_1} \psi 2\beta - \tilde{u} \frac{\partial \psi}{\partial n} dS + \int_{\Gamma_2} \psi \frac{\partial \tilde{u}}{\partial n} - 0 \frac{\partial \psi}{\partial n} dS + \int_{\Omega} \tilde{u} \Delta \psi dV &= 0 \\
\int_{\Gamma_1} \psi 2\beta - \tilde{u} \frac{\partial \psi}{\partial n} dS + \int_{\Gamma_2} \psi \frac{\partial \tilde{u}}{\partial n} dS + \int_{\Omega} \tilde{u} \Delta \psi dV &= 0 \quad (6)
\end{aligned}
$$

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Step 3: Differentiate the cost functional

Differentiate the cost functional w.r.t. β:

$$\begin{aligned} D_\beta J(u) &= \frac{1}{2} D_\beta \int_{\Gamma_1} (u-\delta)^2 dS \\ &= \frac{1}{2} \int_{\Gamma_1} 2(u-\delta) D_\beta u dS \\ &= \int_{\Gamma_1} (u-\delta)\tilde{u} dS \end{aligned} \tag{7}$$

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Step 4: Couple equations

Couple the principal equations, adding (7)+(6):

$$\begin{aligned}
D_\beta J(u) &= D_\beta J(u) + (6) \\
&= \int_{\Gamma_1} (u-\delta)\tilde{u} dS \\
&\quad + \int_{\Gamma_1} \psi 2\beta - \tilde{u}\frac{\partial \psi}{\partial n} dS + \int_{\Gamma_2} \psi \frac{\partial \tilde{u}}{\partial n} dS + \int_\Omega \tilde{u}\Delta\psi dV \\
&= \int_{\Gamma_1} \tilde{u}\left[(u-\delta) - \frac{\partial \psi}{\partial n}\right] dS + \int_{\Gamma_2} \psi \frac{\partial \tilde{u}}{\partial n} dS + \int_\Omega \tilde{u}\Delta\psi dV \\
&\quad + \int_{\Gamma_1} \psi 2\beta dS
\end{aligned} \tag{8}$$

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Step 5: Impose constraints and build adjoint problem

Build the adjoint problem, setting the following constraints on equation (8):

$$\Delta\psi(x,y) = 0, \qquad (x,y)\in\Omega \tag{9a}$$

$$\frac{\partial\psi}{\partial n}(x,y) = u(x,y)-\delta, (x,y)\in\Gamma_1 \tag{9b}$$

$$\psi(x,y) = 0, \qquad (x,y)\in\Gamma_2 \tag{9c}$$

With this constraints, the differential of the cost function is as:

$$D_\beta J(u) = \int_{\Gamma_1} \psi 2\beta dS \tag{10}$$

with ψ solution of (9). This differential is used by a descent direction method to minimize the cost functional.

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Centro Nacional de Supercomputación

Index

Original (forward) problem

Continuous method

Adjoint (backward) problem

Comparison with discrete approach

Flowchart

Very simple examples

Introducing a transient term

Barcelona Supercomputing Center
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Adjoint (backward) problem

$$
\begin{aligned}
\Delta\psi(x,y) &= 0, \qquad\qquad (x,y) \in \Omega && \text{(11a)}\\
\frac{\partial\psi}{\partial n}(x,y) &= u(x,y) - \delta, (x,y) \in \Gamma_1 && \text{(11b)}\\
\psi(x,y) &= 0, \qquad\qquad (x,y) \in \Gamma_2 && \text{(11c)}
\end{aligned}
$$

BSC Barcelona Supercomputing Center
Centro Nacional de Supercomputación

Index

Original (forward) problem

Continuous method

Adjoint (backward) problem

Comparison with discrete approach

Flowchart

Very simple examples

Introducing a transient term

Barcelona Supercomputing Center
Centro Nacional de Supercomputación

Comparison with discrete approach

The derivatives obtained by both methods are:

- Continuous:

$$D_\beta J(u) = \int_{\Gamma_1} \psi 2\beta dS \qquad (12)$$

- Discrete:

$$D_\beta J_h(u_h) = \int_{\Gamma_1} \sum_{i\in\Gamma_1^h} \psi(x_i)\phi^{(i)}(x,y) 2\beta dS \qquad (13)$$

The same result is obtained for both methods if the discretization used in the continuous one is the same as in the discrete one.

Barcelona Supercomputing Center
Centro Nacional de Supercomputación

Index

Original (forward) problem

Continuous method

Adjoint (backward) problem

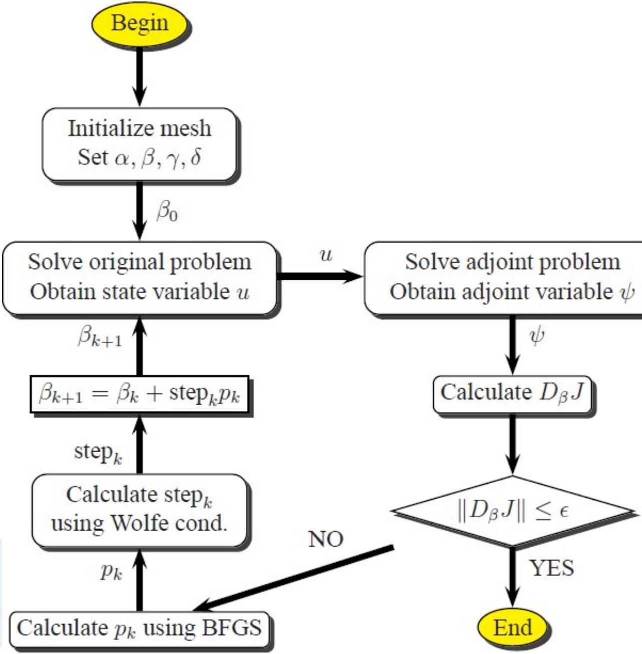
Comparison with discrete approach

Flowchart

Very simple examples

Introducing a transient term

Barcelona Supercomputing Center
Centro Nacional de Supercomputación



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Index

Original (forward) problem

Continuous method

Adjoint (backward) problem

Comparison with discrete approach

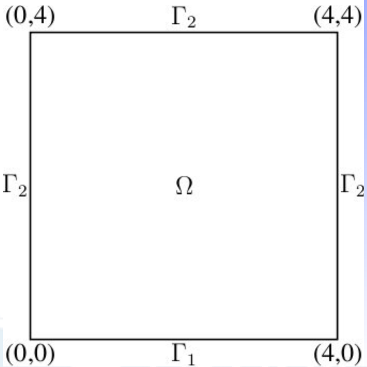
Flowchart

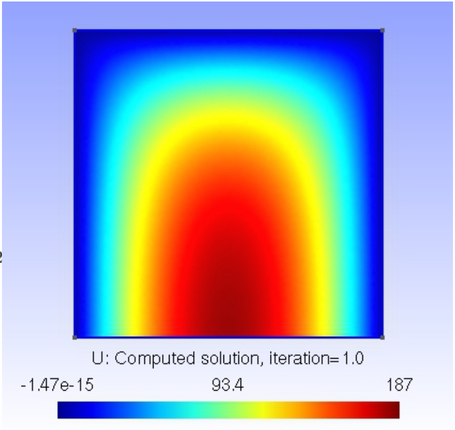
Very simple examples

Introducing a transient term

Barcelona Supercomputing Center
Centro Nacional de Supercomputación

Example 1





The initial design variables are:

$$\alpha = 10 \qquad \beta = 32.75$$

$$\gamma = 0 \qquad \delta = 10$$

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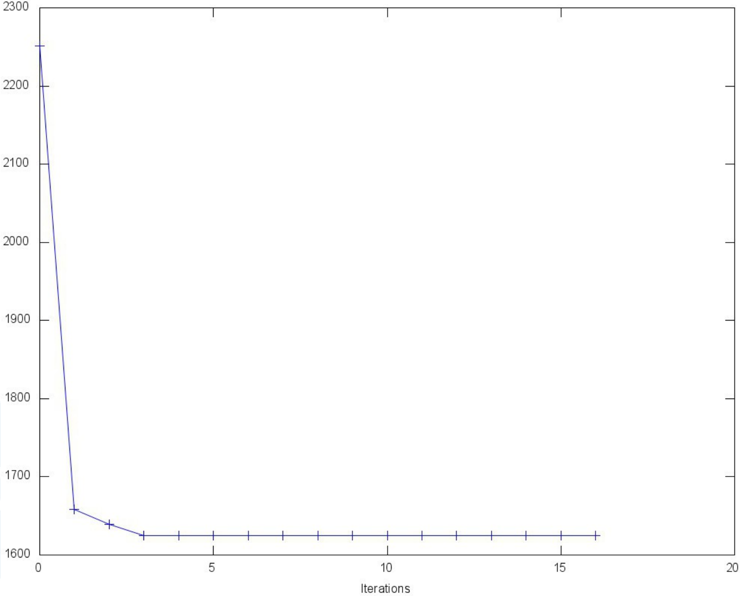
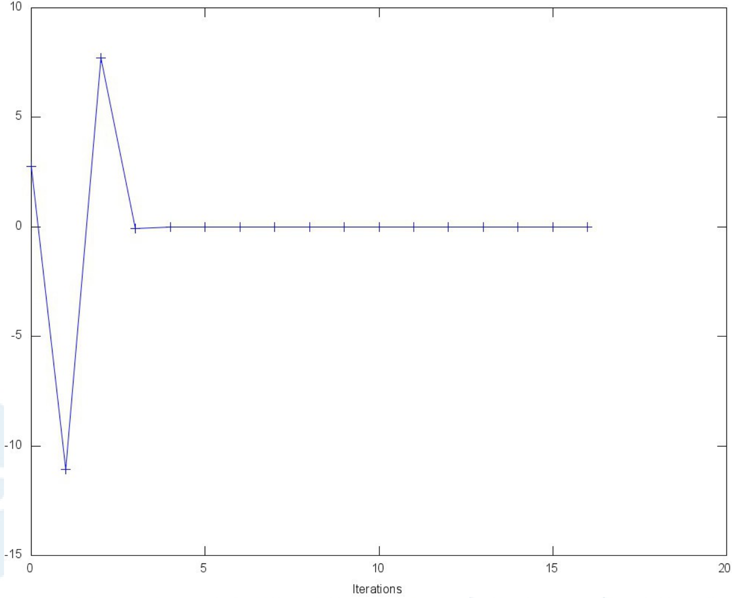
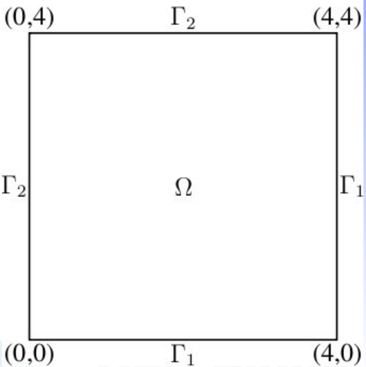
Results: Evolution of J

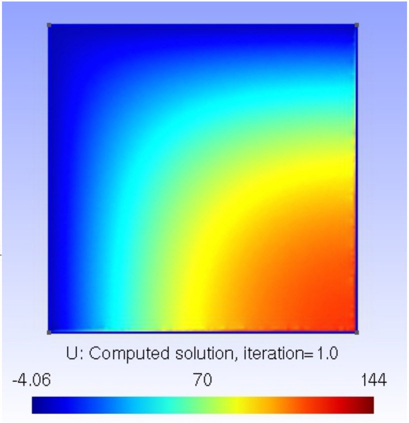


Results: Evolution of ∇J



Example 2





The initial design variables are:

$$\alpha = 5 \qquad \beta = 35$$
$$\gamma = 0 \qquad \delta = 10$$

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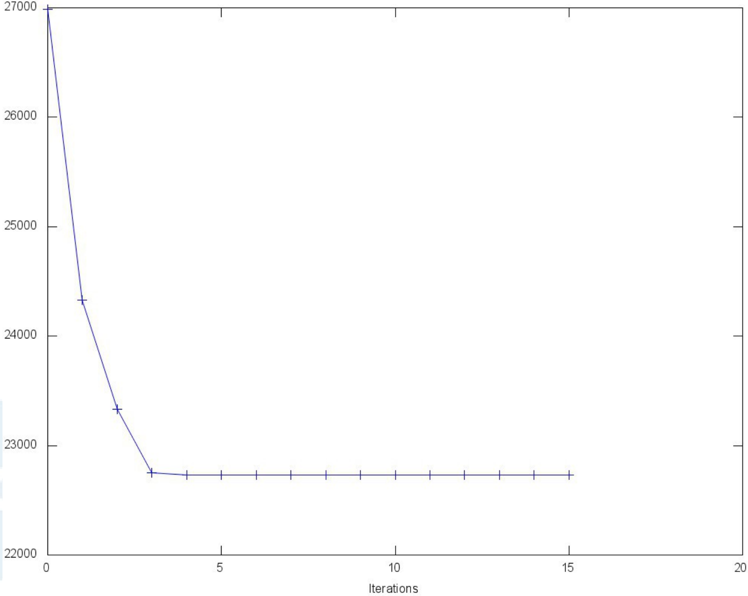
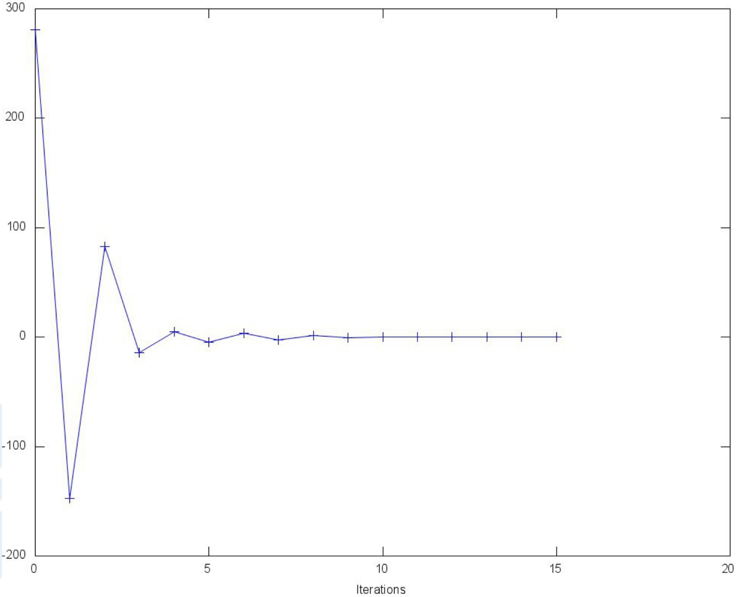
Results: Evolution of J



Results: Evolution of ∇J



Index

Original (forward) problem

Continuous method

Adjoint (backward) problem

Comparison with discrete approach

Flowchart

Very simple examples

Introducing a transient term

Barcelona Supercomputing Center
Centro Nacional de Supercomputación

Transient term

If the state equations have a transient term, the problem can be formulated as follows:

$$\underset{\beta\in\mathbb{R}}{\text{minimize}} \quad \frac{1}{2}\int_0^T \int_{\Gamma_1} (u(x,y;t)-\delta)^2 dSdt$$

$$\text{subject to} \quad \begin{cases} \partial_t u(x,y;t) + \Delta u(x,y;t) = \alpha^2, & (x,y)\in\Omega, \\ & t\in[0,T] \\ \dfrac{\partial u}{\partial n}(x,y;t) = \beta^2, & (x,y)\in\Gamma_1, \\ & t\in[0,T] \\ u(x,y;t) = \gamma^2, & (x,y)\in\Gamma_2, \\ & t\in[0,T] \\ u(x,y;0) = 0, & (x,y)\in\Omega \end{cases}$$

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Transient term

The adjoint problem is:

$$
\begin{aligned}
-\partial_t \psi(x, y; t) + \Delta\psi(x, y; t) &= 0, && (x, y) \in \Omega, t \in [0, T] \\
\frac{\partial \psi}{\partial n}(x, y; t) &= u(x, y; t) - \delta, && (x, y) \in \Gamma_1, t \in [0, T] \\
\psi(x, y; t) &= 0, && (x, y) \in \Gamma_2, t \in [0, T] \\
\psi(x, y; T) &= 0, && (x, y) \in \Omega
\end{aligned}
$$

In this case, we need to solve the adjoint equations with a **final** condition in $t = T$.

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Thanks for your attention!

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