HMMA 307 : Modèles linéaires avancés

REstricted Maximum Likelihood (REML)

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https://github.com/opheliecoiffier/REML

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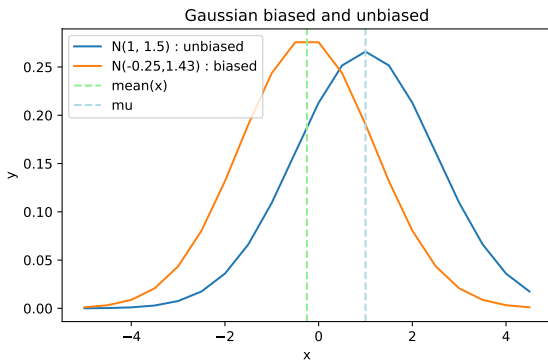
The main issue : biased variance

Figure: Difference between a Biased Normal distribution and an unbiased Normal distribution

Characterization of the biased variance

The expected value of the variance's biased estimator is not the variance estimator. That means :

Dimension : 1 the model
writes out :
$$y = X\beta + \varepsilon$$

where y $\sim \mathcal{N}(\mu, \sigma^2)$ and
$\varepsilon \sim \mathcal{N}(0, 1)$

$$\mathbb{E}[\sigma^2] = \frac{N-1}{N}\sigma^2 \neq \sigma^2$$

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where y $\sim \mathcal{N}(X\beta, \sigma^2 I_k)$ and $\varepsilon \sim \mathcal{N}(0, \sigma^2 I_k)$

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It is an underestimation of the true variance. We need to choose $\widehat{\sigma^2} = \frac{1}{N-k}(y - X\hat{\beta})^T(y - X\hat{\beta})$ to find an unbiased variance.

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1) Integrate the likelihood in relation to μ
2) Calculate the log of the previous result
3) Recognize the Maximum likelihood solution and the REML approach (the bias, the fee)
4) Determine the variance estimator

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An example : Data

Ind	Resp	Treat
1	10	0
1	25	1
2	3	0
2	6	1

In *Treat* column : 0 means that the individual gets the treatment and 1 means that it doesn't get the treatment.

An example : Codes

First, we use linear regression with Least Squared method (OLS)

```
linear_reg     =     sm.OLS(df.Resp,df.Treat)
linear_reg_fit    =     linear_reg.fit()
print(linear_reg_fit.summary())
```

Then, we use mixed effects model with REML

```
mixed_random    =     smf.mixedlm("Resp~Treat",df,
            groups=df['Ind'])
mixed_fit    =    mixed_random.fit()
print(mixed_fit.summary())
```

An example : Results

Method	Log-likelihood	σ^2	σ_s^2
OLS linear regression	-14.23	6.95	-
REML mixed effects model	-7.89	6.00	8.15

Both methods have the same value for the coefficients : $\beta_1 = 6.0$ and $\beta_2 = 15.5$

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Conclusion

The REML method allows to estimate the variance without bias.

The log-likelihood is different because we have a second term with the REML method : a fee. But we have the same coefficients β.

Bibliography

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