Review of Probability Theory

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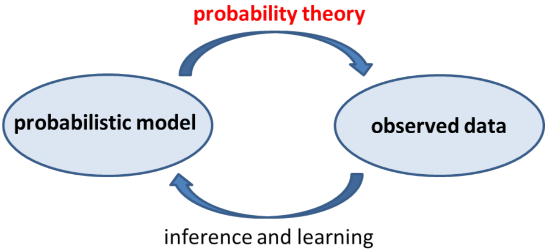
Lisboa, **Portugal**

LxMLS 2015: Lisbon Machine Learning School

July 16, 2015

Probability theory



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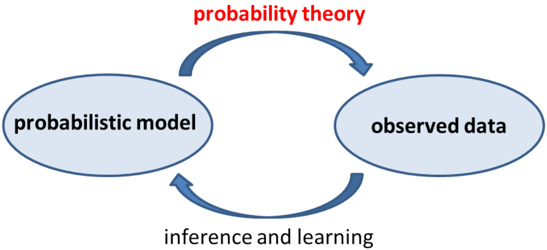

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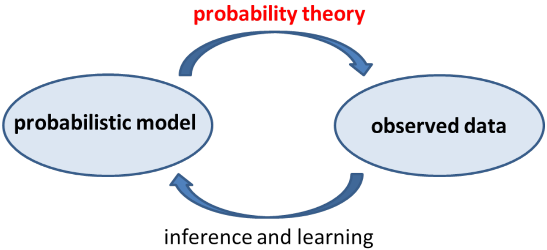

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- Great names of science: Cardano, Fermat, Pascal, Laplace, Kolmogorov, Bernoulli, Poisson, Cauchy, Boltzman, de Finetti, ...

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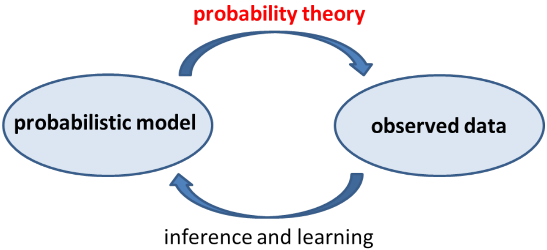

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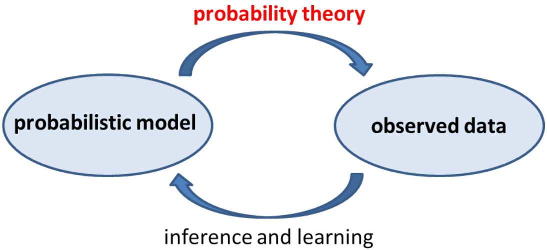

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- Great names of science: Cardano, Fermat, Pascal, Laplace, Kolmogorov, Bernoulli, Poisson, Cauchy, Boltzman, de Finetti, ...
- Natural tool to model uncertainty, information, knowledge, belief, ...
- ...thus also learning, decision making, inference, ...

What is probability?

- Classical definition: $\mathbb{P}(A) = \dfrac{N_A}{N}$

  ...with N mutually exclusive equally likely outcomes,
  N_A of which result in the occurrence of A. *Laplace, 1814*

  Example: $\mathbb{P}(\text{randomly drawn card is } \clubsuit) = 13/52$.

  Example: $\mathbb{P}(\text{getting 1 in throwing a fair die}) = 1/6$.

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- Subjective probability: $\mathbb{P}(A)$ is a degree of belief. *de Finetti, 1930s*

  ...gives meaning to $\mathbb{P}(\text{“it will rain tomorrow”})$.

Key concepts: Sample space and events

- Sample space $\mathcal{X}$ = set of possible outcomes of a random experiment.

  Examples:
  - Tossing two coins: $\mathcal{X} = \{HH, TH, HT, TT\}$
  - Roulette: $\mathcal{X} = \{1, 2, ..., 36\}$
  - Draw a card from a shuffled deck: $\mathcal{X} = \{A\clubsuit, 2\clubsuit, ..., Q\diamondsuit, K\diamondsuit\}$.

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- An event A is a subset of $\mathcal{X}$: $A \subseteq \mathcal{X}$ (also written $A \in 2^{\mathcal{X}}$).

  Examples:
  - "exactly one H in 2-coin toss": $A = \{TH, HT\} \subset \{HH, TH, HT, TT\}$.
  - "odd number in the roulette": $B = \{1, 3, ..., 35\} \subset \{1, 2, ..., 36\}$.
  - "drawn a $\heartsuit$ card": $C = \{A\heartsuit, 2\heartsuit, ..., K\heartsuit\} \subset \{A\clubsuit, ..., K\diamondsuit\}$

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- An event A is a measurable subset of $\mathcal{X}$, *i.e.*, $A \in \sigma(F)$, $F \in 2^{\mathcal{X}}$.
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  - $A \in \sigma(F) \Rightarrow A^c \in \sigma(F)$
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- For any $F \in 2^{\mathcal{X}}$, $\quad \emptyset \in \sigma(F)$, $\quad \mathcal{X} \in \sigma(F)$.

Kolmogorov's Axioms for Probability

- Probability is a function that maps events A into the interval $[0, 1]$.

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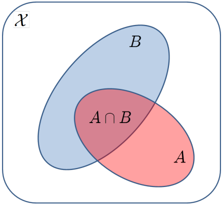
- From these axioms, many results can be derived. Examples:

  - $\mathbb{P}(\emptyset) = 0$
  - $C \subset D \;\Rightarrow\; \mathbb{P}(C) \leq \mathbb{P}(D)$
  - $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$
  - $\mathbb{P}(A \cup B) \leq \mathbb{P}(A) + \mathbb{P}(B)$ (union bound)



Conditional Probability and Independence

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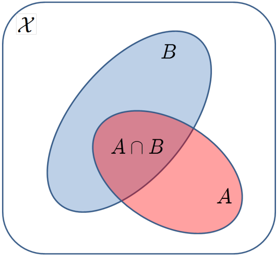

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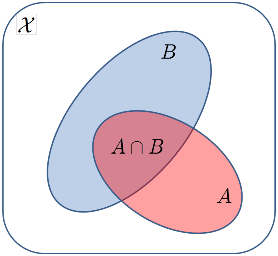

- Independence: A, B are independent (denoted $A \perp\!\!\!\perp B$) if and only if

$$\mathbb{P}(A \cap B) = \mathbb{P}(A)\,\mathbb{P}(B).$$

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- Example: $\mathcal{X} =$ "52 cards", $A = \{3\heartsuit, 3\clubsuit, 3\diamondsuit, 3\clubsuit\}$, and $B = \{A\heartsuit, 2\heartsuit, ..., K\heartsuit\}$; then, $\mathbb{P}(A) = 1/13$, $\mathbb{P}(B) = 1/4$

$$\mathbb{P}(A \cap B) \;=\; \mathbb{P}(\{3\heartsuit\}) = \frac{1}{52}$$

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Bayes Theorem

- Law of total probability: if $A_1, ..., A_n$ are a partition of $\mathcal{X}$

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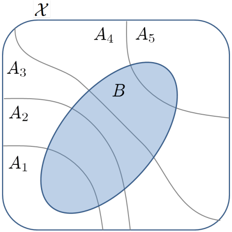

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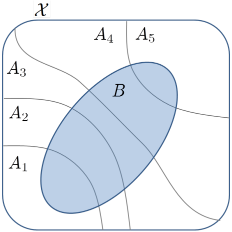
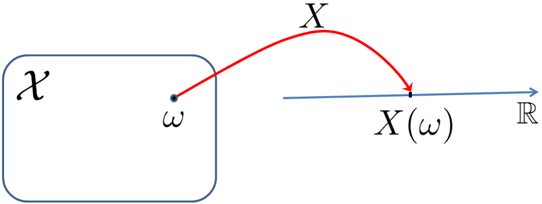


- Bayes' theorem: if $\{A_1, ..., A_n\}$ is a partition of $\mathcal{X}$

$$\mathbb{P}(A_i|B) = \frac{\mathbb{P}(B \cap A_i)}{\mathbb{P}(B)} = \frac{\mathbb{P}(B|A_i)\,\mathbb{P}(A_i)}{\sum_i \mathbb{P}(B|A_i)\mathbb{P}(A_i)}$$

Random Variables

- A (real) random variable (RV) is a function: $X : \mathcal{X} \to \mathbb{R}$



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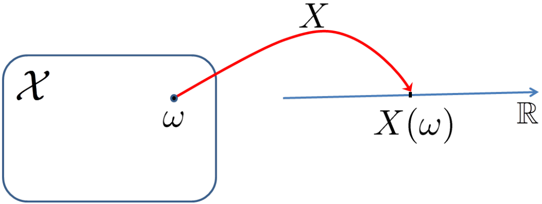

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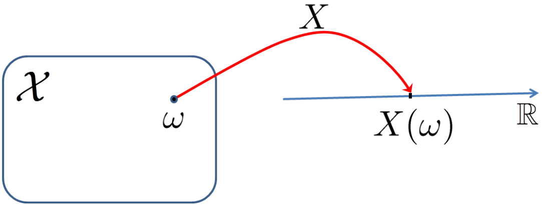

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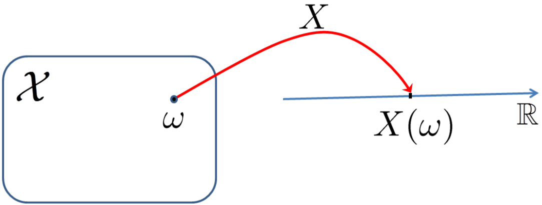

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  - Continuous RV: range of X is uncountable (*e.g.*, $\mathbb{R}$ or $[0,\ 1]$)
  - Example: number of head in tossing two coins,
    $\mathcal{X} = \{HH, HT, TH, TT\}$,
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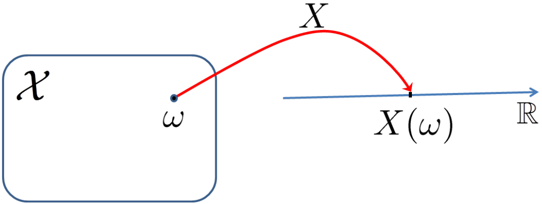

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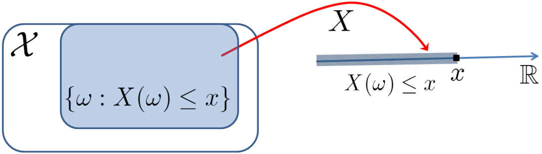
    Range of $X = \{0, 1, 2\}$.
  - Example: distance traveled by a tossed coin; range of $X = \mathbb{R}_+$.

Random Variables: Distribution Function

- Distribution function: $F_X(x) = \mathbb{P}(\{\omega \in \mathcal{X} : X(\omega) \leq x\})$



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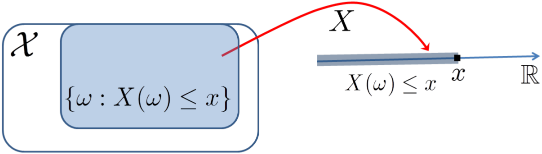
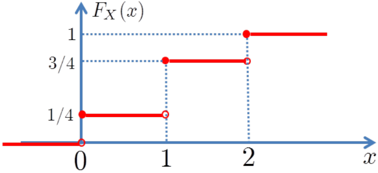
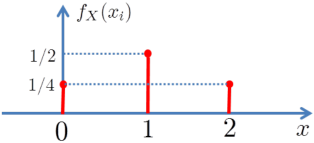


- Example: number of heads in tossing 2 coins; range$(X) = \{0, 1, 2\}$.





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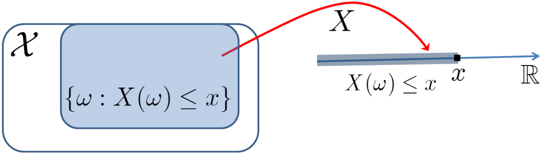

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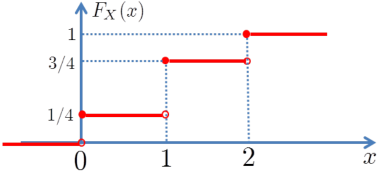

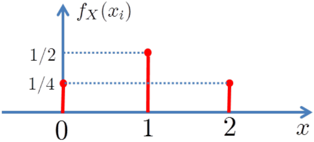

- Probability mass function (discrete RV): $f_X(x) = \mathbb{P}(X = x)$,

$$F_X(x) = \sum_{x_i \leq x} f_X(x_i).$$

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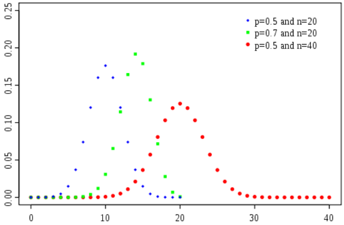
Binomial coefficients ("n choose x"):

$$\binom{n}{x} = \frac{n!}{(n-x)!\,x!}$$



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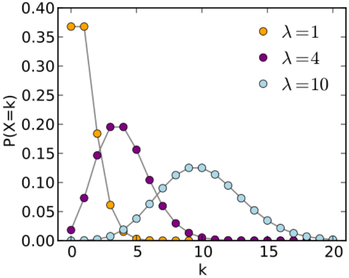
- Poisson(λ): $X \in \mathbb{N} \cup \{0\}$, pmf $f_X(x) = \dfrac{e^{-\lambda}\lambda^x}{x!}$

  Notice that $\sum_{x=0}^{\infty} \frac{\lambda^x}{x!} = e^{\lambda}$, thus $\sum_{x=0}^{\infty} f_X(x) = 1$.

  "...probability of the number of independent occurrences in a fixed (time/space) interval if these occurrences have known average rate"



Random Variables: Distribution Function

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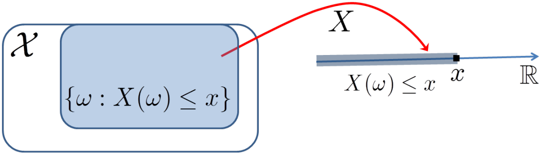

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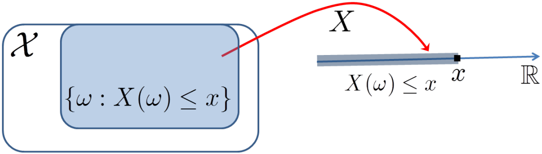

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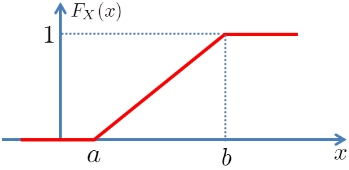

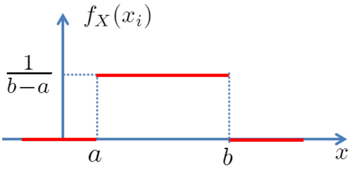

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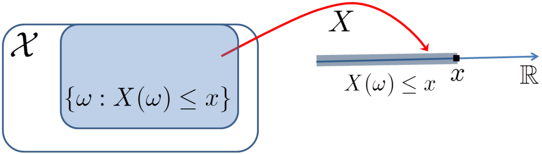

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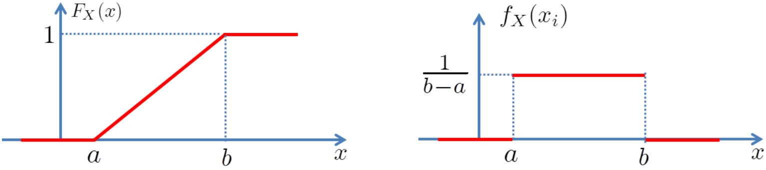

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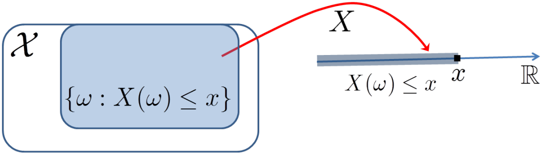

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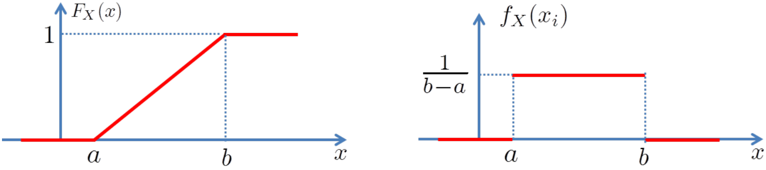

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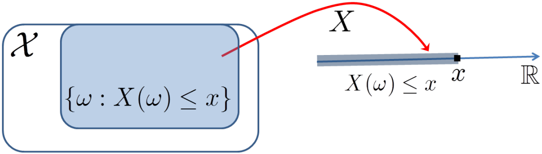

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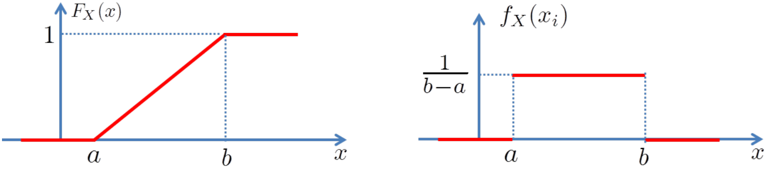

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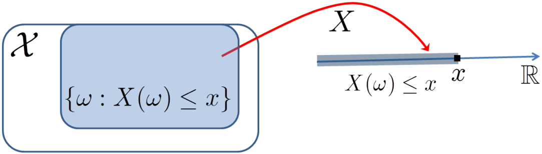

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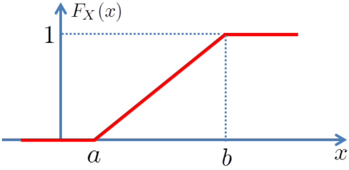

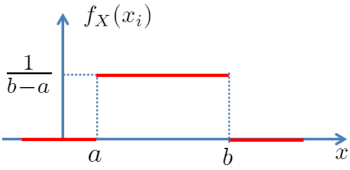

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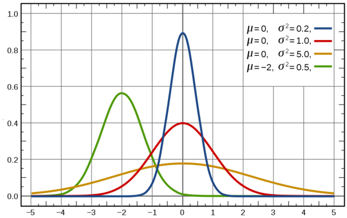

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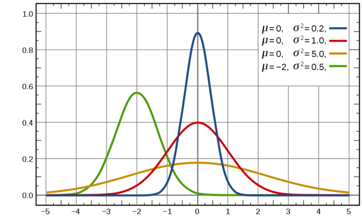

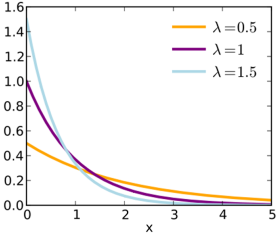

- Exponential: $f_X(x) = \text{Exp}(x; \lambda) = \begin{cases} \lambda e^{-\lambda x} & \Leftarrow \quad x \geq 0 \\ 0 & \Leftarrow \quad x < 0 \end{cases}$

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- Linearity of expectation:
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Moments of Random Variables

- Non-central moments of order k:

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- If k-th moment exists, then the j-th moment exists, for $j \leq k$.

$$\begin{aligned} \int_{-\infty}^{\infty} |x|^j\, f_X(x)\, dx &= \int_{|x|\leq 1} |x|^j\, f_X(x)\, dx + \int_{|x|>1} |x|^j\, f_X(x)\, dx \\ &\leq \int_{|x|\leq 1} f_X(x)\, dx + \int_{|x|>1} |x|^k\, f_X(x)\, dx \leq 1 + \mathbb{E}\big(|X|^k\big) < \infty \end{aligned}$$

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$$\mathbb{P}\big((X,Y) \in A\big) = \iint_A f_{X,Y}(x,y)\, dx\, dy, \qquad A \in \sigma(\mathbb{R}^2)$$

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- Also valid in the mixed case (*e.g.*, X continuous, Y discrete).

Joint, Marginal, and Conditional Probabilities: An Example

- A pair of binary variables $X, Y \in \{0, 1\}$, with joint pmf:

$f_{X,Y}(x, y)$	$Y = 0$	$Y = 1$
$X = 0$	1/5	2/5
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$f_{Y\|X}(y\|x)$	$Y = 0$	$Y = 1$
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An Important Multivariate RV: Multinomial

- Multinomial: $X = (X_1, ..., X_K)$, $X_i \in \{0, ..., n\}$, such that $\sum_i X_i = n$,

$$f_X(x_1, ..., x_K) = \begin{cases} \binom{n}{x_1\ x_2\ \cdots\ x_K} p_1^{x_1}\, p_2^{x_2} \cdots p_k^{x_K} & \Leftarrow \quad \sum_i x_i = n \\ 0 & \Leftarrow \quad \sum_i x_i \neq n \end{cases}$$

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- Generalizes the binomial from binary to K-classes.

- Example: tossing n independent fair dice, $p_1 = \cdots = p_6 = 1/6$.
  x_i = number of outcomes with i dots. Of course, $\sum_i x_i = n$.

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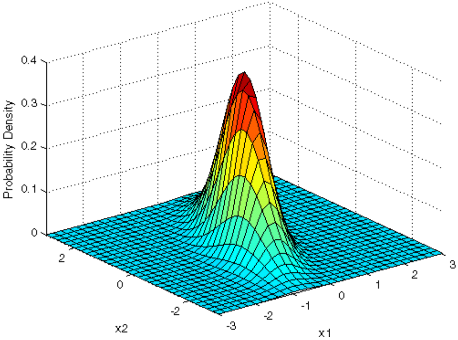

Covariance, Correlation, and all that...

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- Covariance of Gaussian RV, $f_X(x) = \mathcal{N}(x; \mu, C) \;\Rightarrow\; \text{cov}(X) = C$

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- $f_{X_b|X_a}(x_b|x_a) = \mathcal{N}\big(x_b; \mu_b + C_{ba}C_{aa}^{-1}(x_a - \mu_a), C_{bb} - C_{ba}C_{aa}^{-1}C_{ab}\big)$

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- Canonical parameter(s): $\eta(\theta)$
- Sufficient statistics: $\phi(x)$
- Partition function: $Z(\theta)$
- Curved exponential family: $\dim(\theta) < \dim(\eta(\theta))$

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  $f_X(x) = \exp\big(x \log p + (1-x)\log(1-p)\big) = \exp\big(x \log \frac{p}{1-p} + \log(1-p)\big)$,

  thus $\eta(p) = \log \frac{p}{1-p}$, $\phi(x) = x$, $Z(p) = \frac{1}{1-p}$, and $h(x) = 1$.

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  $f_X(x) = \exp\big(x \log p + (1-x)\log(1-p)\big) = \exp\big(x \log \frac{p}{1-p} + \log(1-p)\big)$,

  thus $\eta(p) = \log \frac{p}{1-p}$, $\phi(x) = x$, $Z(p) = \frac{1}{1-p}$, and $h(x) = 1$.

- Gaussian:

$$f_X(x) = \frac{\exp\big(-\frac{(x-\mu)^2}{2\sigma^2}\big)}{\sqrt{2\pi\sigma^2}} = \frac{\exp\big(-\frac{x^2}{2\sigma^2} + \frac{\mu x}{\sigma^2} - \frac{\mu^2}{2\sigma^2}\big)}{\sqrt{2\pi\sigma^2}},$$

  thus $\eta(\mu, \sigma^2) = [\mu/\sigma^2,\ -1/(2\sigma^2)]^T$, $\phi(x) = [x,\ x^2]^T$,
  $Z(\mu, \sigma^2) = \sqrt{2\pi\sigma^2}\exp\big(\frac{\mu^2}{2\sigma^2}\big)$, and $h(x) = 1$.

More on Exponential Families

- Independent identically distributed (i.i.d.) observations:

$$X_1, ..., X_m \sim f_X(x) = \frac{1}{Z(\eta)} h(x) \exp\left(\eta^T \phi(x)\right)$$

then

$$f_{X_1,...,X_m}(x_1, ..., x_m) = \frac{1}{Z(\eta)^m} \left(\prod_{j=1}^{m} h(x_i) \right) \exp\left(\eta^T \sum_{j=1}^{m} \phi(x_j) \right)$$

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- Expected sufficient statistics:

$$\frac{d \log Z(\eta)}{d\eta} = \frac{\frac{dZ(\eta)}{d\eta}}{Z(\eta)} = \frac{1}{Z(\eta)} \int \phi(x) h(x) \exp\left(\eta^T \phi(x)\right) dx = \mathbb{E}\left(\phi(X)\right)$$

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- In the Bayesian philosophy, all the knowledge about X is carried by

$$f_{X|Y}(x|y) = \frac{f_{Y|X}(y|x)\ f_X(x)}{f_Y(y)} = \frac{f_{Y,X}(y,x)}{f_Y(y)}$$

...the posterior (or a posteriori) pdf/pmf.

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where

$$\mathbb{E}[L(\widehat{x}, X)|Y = y] = \begin{cases} \displaystyle\int L(\widehat{x}, x)\, f_{X|Y}(x|y)\, dx, & \text{continuous (estimation)} \\ \displaystyle\sum_x L(\widehat{x}, x)\, f_{X|Y}(x|y), & \text{discrete (classification)} \end{cases}$$

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- Same criterion can be derived for continuous X, using $\lim_{\varepsilon \to 0} L_\varepsilon(\widehat{x}, x)$, where $L_\varepsilon(\widehat{x}, x) = 0$, if $|\widehat{x} - x| < \varepsilon$, and 1 otherwise.

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- Example: $n = 10$, observed $y = (1, 1, 1, 0, 1, 0, 0, 1, 1, 1)$, $\widehat{x}_{\mathrm{ML}} = 7/10$.

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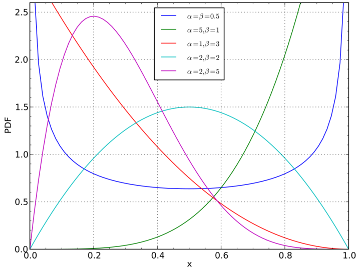

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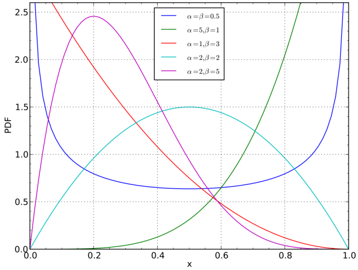

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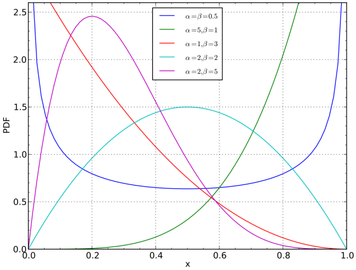

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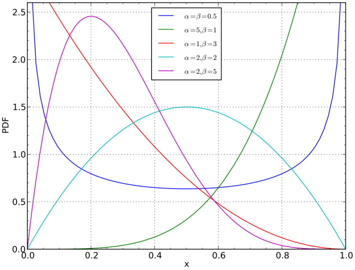

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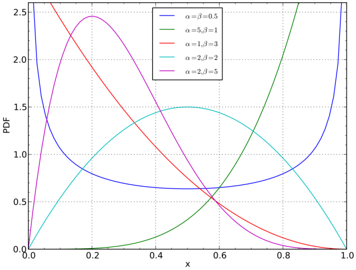

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- Does not apply to classification problems.

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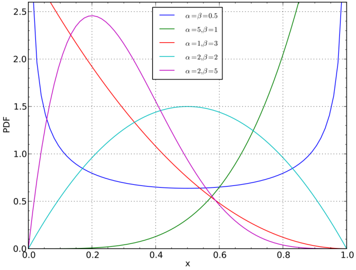

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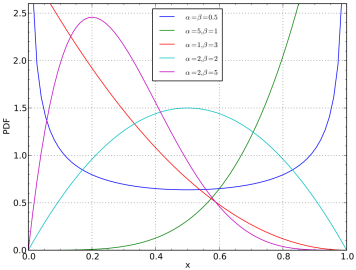

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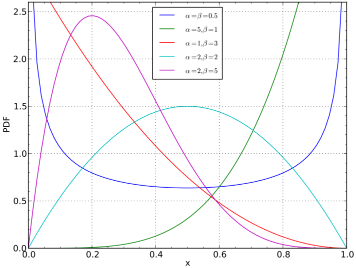

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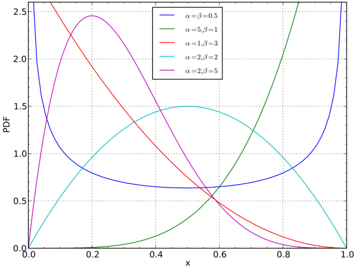

$\widehat{x}_{\text{MMSE}} \simeq 0.611 \;\; \left(\text{recall that } \widehat{x}_{\text{MAP}} = 0.625,\;\; \widehat{x}_{\text{ML}} = 0.7\right)$

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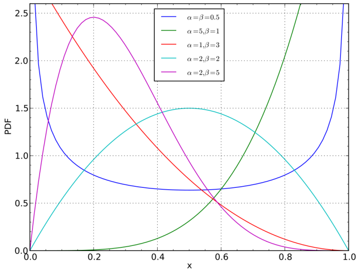
  - Posterior: $f_{X|Y}(x|y) = x^{\alpha-1+\sum_i y_i}(1-x)^{\beta-1+n-\sum_i y_i}$
  - MMSE: $\widehat{x}_{\text{MMSE}} = \frac{\alpha+\sum_i y_i}{\alpha+\beta+n}$
  - Example: $\alpha = 4$, $\beta = 4$, $n = 10$, $y = (1,1,1,0,1,0,0,1,1,1)$,



$$\widehat{x}_{\text{MMSE}} \simeq 0.611 \;\; \left(\text{recall that } \;\widehat{x}_{\text{MAP}} = 0.625, \;\; \widehat{x}_{\text{ML}} = 0.7\right)$$

- Conjugate prior equivalent to "virtual" counts; often called "smoothing" in NLP and ML.

The Bernstein-Von Mises Theorem

- In the previous example, we had
  $n = 10, \;\; y = (1, 1, 1, 0, 1, 0, 0, 1, 1, 1), \;\;$ thus $\sum_i y_i = 7$.
  With a Beta prior with $\alpha = 4$ and $\beta = 4$, we had

$$\widehat{x}_{\text{ML}} = 0.7, \quad \widehat{x}_{\text{MAP}} = \frac{3 + \sum_i y_i}{6 + n} = 0.625, \quad \widehat{x}_{\text{MMSE}} = \frac{4 + \sum_i y_i}{8 + n} \simeq 0.611$$

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- Consider $n = 100$, and $\sum_i y_i = 70$, with the same Beta(4,4) prior

$$\widehat{x}_{\text{ML}} = 0.7, \quad \widehat{x}_{\text{MAP}} = \frac{73}{106} \simeq 0.689, \quad \widehat{x}_{\text{MMSE}} = \frac{74}{108} \simeq 0.685$$

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  ... both Bayesian estimates are much closer to the ML.

- This illustrates an important result in Bayesian inference: the Bernstein-Von Mises theorem; under (mild) conditions,

$$\lim_{n\to\infty} \widehat{x}_{\text{MAP}} = \lim_{n\to\infty} \widehat{x}_{\text{MMSE}} = \widehat{x}_{\text{ML}}$$

  message: if you have a lot of data, priors don't matter much.

Important Inequalities

- Markov's ineqality: if $X \geq 0$ is an RV with expectation $\mathbb{E}(X)$, then

$$\mathbb{P}(X > t) \leq \frac{\mathbb{E}(X)}{t}$$

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- Chebyshev's inequality: $\mu = \mathbb{E}(Y)$ and $\sigma^2 = \mathrm{var}(Y)$, then

$$\mathbb{P}(|X - \mu| \geq s) \leq \frac{\sigma^2}{s^2}$$

...simple corollary of Markov's inequality, with $X = |Y - \mu|^2, \;\; t = s^2$

Other Important Inequalities

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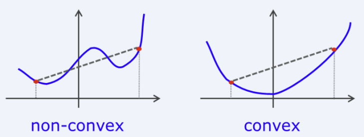
$$g(\alpha x + (1-\alpha)y) \leq \alpha g(x) + (1-\alpha)g(y)$$



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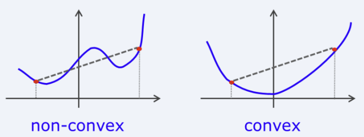

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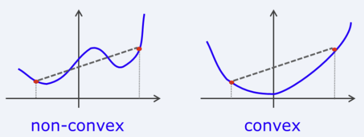

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Examples: $\mathbb{E}(X)^2 \leq \mathbb{E}(X^2) \;\Rightarrow\; \text{var}(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2 \geq 0.$
$\mathbb{E}(\log X) \leq \log \mathbb{E}(X)$, for X a positive RV.

Entropy and all that...

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- If $var(Y) = \sigma^2$, then $h(Y) \leq \frac{1}{2}\log(2\pi e \sigma^2)$

Kullback-Leibler divergence

Kullback-Leibler divergence (KLD) between two pmf:

$$D(f_X \| g_X) = \sum_{x=1}^{K} f_X(x) \log \frac{f_X(x)}{g_X(x)}$$

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$D(f_X \| g_X) = 0 \;\Leftrightarrow\; f_X(x) = g_X(x), \text{ almost everywhere}$

Mutual information

Mutual information (MI) between two random variables:

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MI is a measure of dependency between two random variables

Recommended Reading (Probability and Statistics)

- K. Murphy, "Machine Learning: A Probabilistic Perspective", MIT Press, 2012 (Chapter 2).
- L. Wasserman, "All of Statistics: A Concise Course in Statistical Inference", Springer, 2004.

Linear Algebra

- Linear algebra provides (among many other things) a compact way of representing, studying, and solving linear systems of equations

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- Example: the system

$$\begin{aligned} 4\,x_1 - 5\,x_2 &= -13 \\ -2\,x_1 + 3\,x_2 &= 9 \end{aligned}$$

can be written compactly as $\boxed{Ax = b}$, where

$$A = \begin{bmatrix} 4 & -5 \\ -2 & 3 \end{bmatrix}, \; b = \begin{bmatrix} -13 \\ 9 \end{bmatrix},$$

and can be solved as

$$x = A^{-1}b = \begin{bmatrix} 1.5 & 2.5 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} -13 \\ 9 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}.$$

Notation: Matrices and Vectors

- $A \in \mathbb{R}^{m\times n}$ is a matrix with m rows and n columns.

$$A = \left[\begin{array}{ccc} A_{1,1} & \cdots & A_{1,n} \\ \vdots & \ddots & \vdots \\ A_{m,1} & \cdots & A_{m,n} \end{array}\right].$$

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Matrix Transpose and Products

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- Transpose of sum: $(A+B)^T = A^T + B^T$.

Norms

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Norms

- The norm of a vector is (informally) its "length". Euclidean norm:

$$\|x\|_2 = \sqrt{\langle x, x \rangle} = \sqrt{x^T x} = \sqrt{\sum_{i=1}^{n} x_i^2}.$$

- More generally, the ℓ_p norm of a vector $x \in \mathbb{R}^n$, where $p \geq 1$,

$$\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p}.$$

- Notable case: the ℓ_1 norm, $\|x\|_1 = \sum_i |x_i|$.

- Notable case: the ℓ_∞ norm, $\|x\|_\infty = \max\{|x_1|, ..., |x_n|\}$.

- Notable case: the ℓ_0 "norm" (not): $\|x\|_0 = |\{i : x_i \neq 0\}|$.

Special Matrices

- The identity matrix $I \in \mathbb{R}^{n \times n}$ is a square matrix such that

$$I_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \qquad I = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

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- There are several algorithms to compute A^{-1}; general case, computational cost $O(n^3)$.

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