Evaluating Statistical Methods for Nuclear Forensics Analysis

Preliminary Examination

Arrielle Opotowsky

University of Wisconsin-Madison

29 January 2018

WISCONSIN
UNIVERSITY OF WISCONSIN–MADISON

Outline

Research Overview

How does the ability to determine forensic-relevant spent nuclear fuel attributes using machine learning techniques degrade as less information is available?

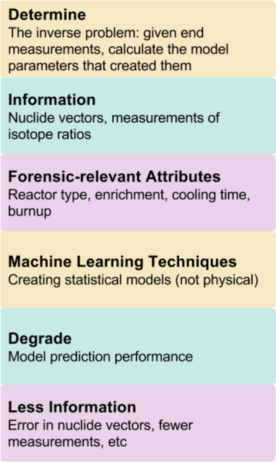


Figure 1: Definitions of terms within the main research question

Nuclear Security and Forensics

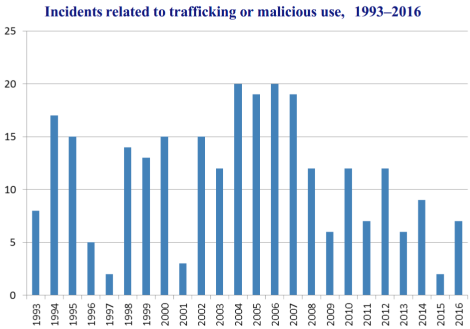


Figure 2: 24 years of incidents: HEU (12), Pu (2), Pu-Be neutron sources (4) [Obtained from: https://www.iaea.org/sites/default/files/17/12/itdb-factsheet-2017.pdf]

- FY2016 DHS DNDO budget : 0.3 bill
- FY2016 DOE NNSA nonpro budget : 1.6 bill

Needs in Nuclear Forensics

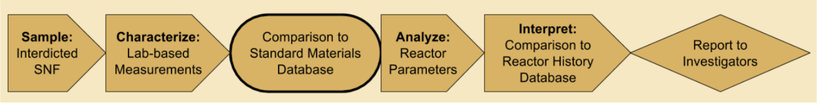


Figure 3: Typical techincal nuclear forensics workflow

Material-specific:

- Measurement needs
- Measurement techniques
- Forensic signatures

Challenges:

- Rapid characterization
- Forensics databases
    - Multidimensional
    - Inconsistent uncertainties
    - International cooperation

Computational Methods

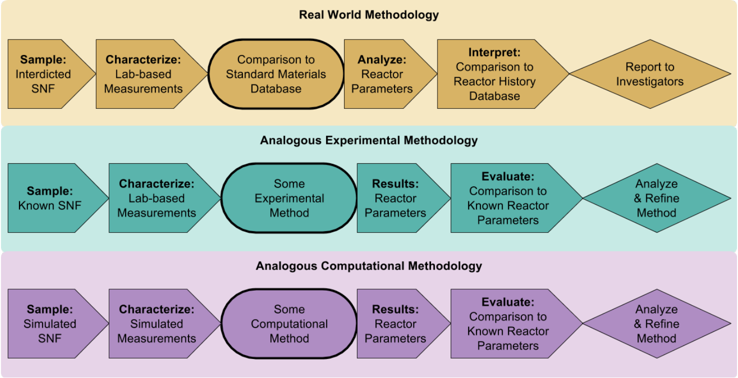


Figure 4: Nuclear forensics research: physical, experimental, and computational

Computational Methods

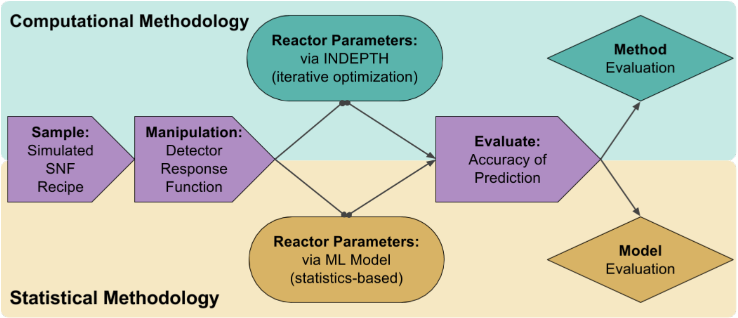


Figure 5: Comparison of two different computational approaches

Statistical Methods

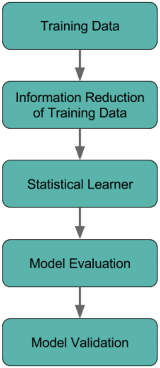


Figure 6: Workflow of a methodology using statistical models

- Training data: large set of SNF measurements
  - Labels (e.g., burnup)
  - Features (e.g., nuclide concs)
  - Instances (individual SNF recipe)
- Statistical learner
  - Machine learning algorithms
  - Algorithm parameters
  - Predict label of new instance
- Model evaluation
  - Diagnostic curves
    - Learning curves
    - Validation curves
  - Prediction error
    - Bias versus variance
    - Generalizability

Statistical Methods

		TRAINING DATA	TESTING DATA
Physical Motivation			
	Ideal World	*Lab*-Measured Mass Spectra	*Lab*-Measured Mass Spectra
	Real World	*Lab*-Measured Gamma Spectra	*Field*-Measured Gamma Spectra
Computational Representation			
	Ideal World	*Simulation*-Created Isotopics	*Simulation*-Created Isotopics
	Real World	DRF-*Derived* Gamma Spectra	DRF-*Derived* Gamma Spectra

Figure 7: Illustration of data set modularity

Outline

Nuclear Forensics Investigations

Post-detonation

- Collection: debris, swipe samples
- Characterization: rapid analysis of isotope ratios
- Goals
  - Inverse problem: reconstruct weapon design/yield
  - Safety: informing disaster response
- Data evaluation

Nuclear Forensics Investigations

Post-detonation

- Collection: debris, swipe samples
- Characterization: rapid analysis of isotope ratios
- Goals
  - Inverse problem: reconstruct weapon design/yield
  - Safety: informing disaster response
- Data evaluation

Pre-detonation

- Collection: depends on intercepted material
- Characterization: non-destructive and destructive
- Goals:
  - Inverse problem: material chain of custody
  - Safety: material handling and security
- Data evaluation

Nuclear Forensics as an Inverse Problem

Use Bayes' Framework:
$P(M|D) = \frac{P(D|M)P(M)}{P(D)}$

M : **M**odel parameters
D : Measured **D**ata

Physical System	Bayes Representation	Calculated from:	
Model Parameterization	Prior Probability : P(M)	Model Space	Simulation *Input* : Rxtr Parameters
Forward Problem	Marginal Likelihood : P(D)	Data Space	Simulation *Output* : SNF Recipes
	Likelihood : P(D|M)	Both	*Output* + *Input* = (Statistical) Model
Inverse Problem	Posterior Probability : P(M|D)	Both	(Statistical) Model : *Output->Input*

Table 1: Mapping the study of a physical system its Bayesian representation

Machine Learning

ML Tasks / Variable Types	Supervised	Unsupervised
Discrete	Classification	Clustering
Continuous	Regression	Dimensionality Reduction

Figure 8: Table depicting the different machine learning aglorithms depending on the task

Supervised Regression

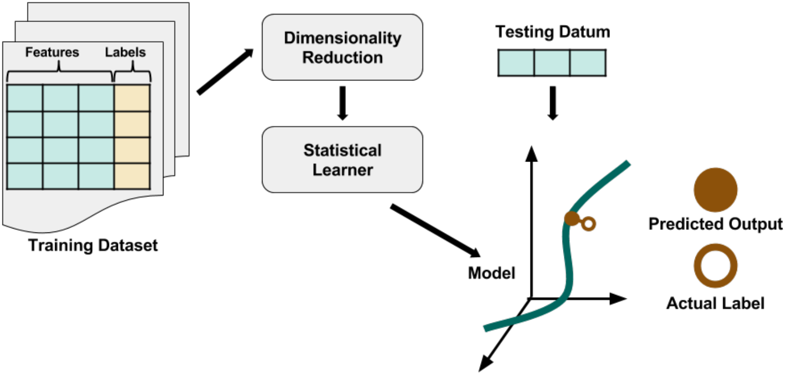


Figure 9: Schematic of a representative training and predicting workflow

Linear Models

Objective: minimize error over all training data wrt their labels
$F(\boldsymbol{X}) = \beta_0 + \sum_{j=1}^{p} x_j \beta_j$

Smoothing model using regularization by varying λ
$F(\boldsymbol{X}) = \beta_0 + \sum_{j=1}^{p} x_j \beta_j + \lambda \sum_{j=1}^{p} \beta_j^2$

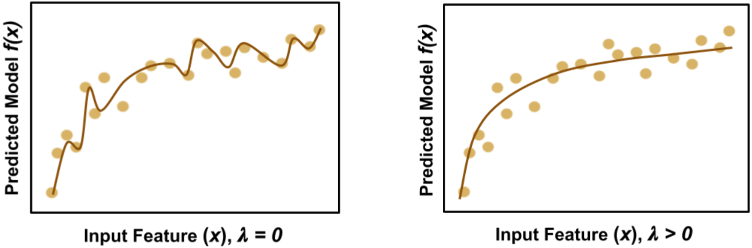


Figure 10: How regularization might affect the generalizability of an ML model

Nearest Neighbor Methods

Objective: minimum distance between test sample and training instance(s)

$$Y(\boldsymbol{X}) = \frac{1}{k} \sum_{x_i \in N_k(\boldsymbol{X})} y_i$$

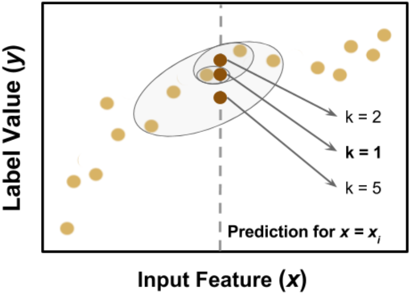


Figure 11: Illustration of the regularization effects by choosing k

Support Vector Machines and Regression

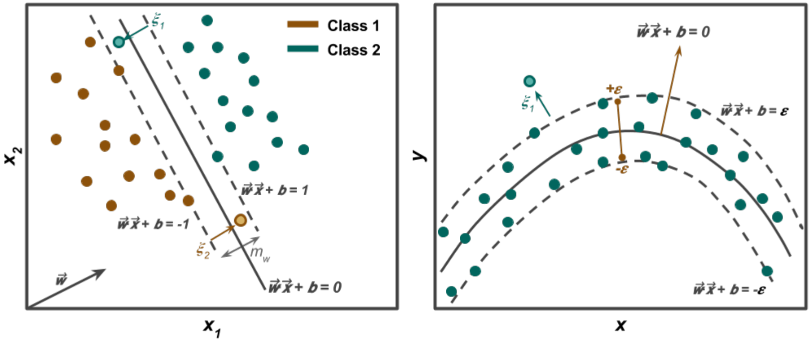


Figure 12: Classification with SVM and regression with SVR

Support Vector Regression with Many Dimensions

Objective: minimize margin width and outliers

$$min \ \frac{1}{2}\|w\|^2 + C\sum_i \xi_i$$

$$subject\ to: \ |y_i - (w\phi(x_i) + b)| \leq \varepsilon + \xi_i$$

$$where: \ w = \sum_i \alpha_i y_i \phi(x_i)$$

$$and: \ K(x_i, x_j) = \phi(x_i)\phi(x_j) = e^{\gamma \|x_i - x_j\|^2}$$

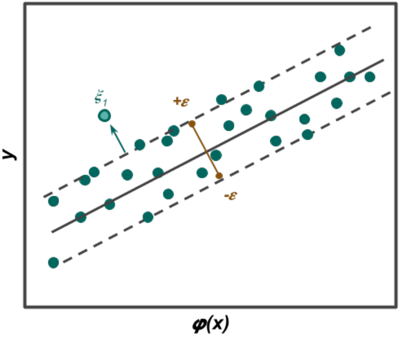


Figure 13: Diagram showing the use of the kernel trick with SVR

Model Smoothing

1. Regularization (reduce weights of features)
2. Dimensionality Reduction (delete features)
   - Manual : some metric
   - Manual : domain knowledge
   - Manual & Statistical : factor analysis
   - Statistical : principal components analysis (PCA)
   - Statistical : independent components analysis (ICA)

Types of Error

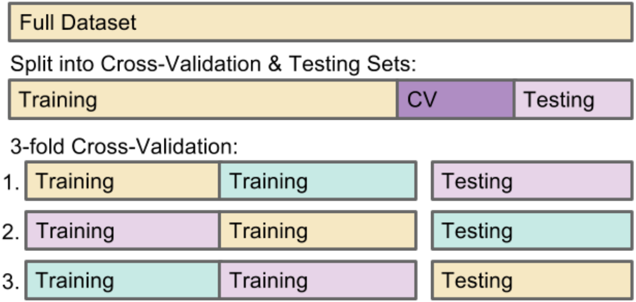


Figure 14: Diagram explaining the concept of k-fold cross-validation

Error Metrics

$$\text{Mean Squared Error (MSE)} : \frac{\sum_{i=1}^{n}(y_i - f(x_i))^2}{n}$$

$$\text{Mean Absolute Error (MAE)} : \frac{\sum_{i=1}^{n}|y_i - f(x_i)|}{n}$$

$$\text{Mean Absolute Percentage Error (MAPE)} : \frac{\sum_{i=1}^{n}\frac{|y_i - f(x_i)|}{y_i}}{n}$$

$$\text{Coefficient of Determination, } R^2 : \frac{\sum_{i=1}^{n}(f(x_i) - \overline{y})^2}{\sum_{i=1}^{n}(y_i - \overline{y})^2}$$

Sources of Error

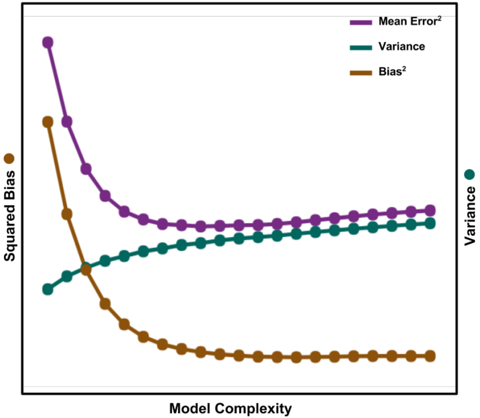
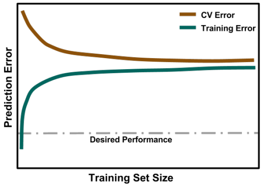
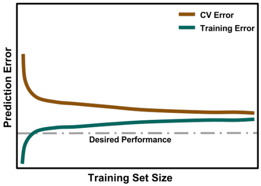


Figure 15: Bias and variance comprise the prediction error

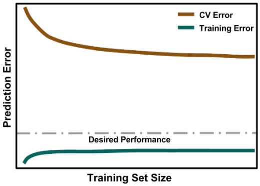
Training Set Size: Learning Curves



(a) High bias and low variance



(b) Balanced bias and variance



(c) Low bias and high variance

Figure 16: Learning curves for three training scenarios with respect to training set size

Model Complexity: Validation Curves

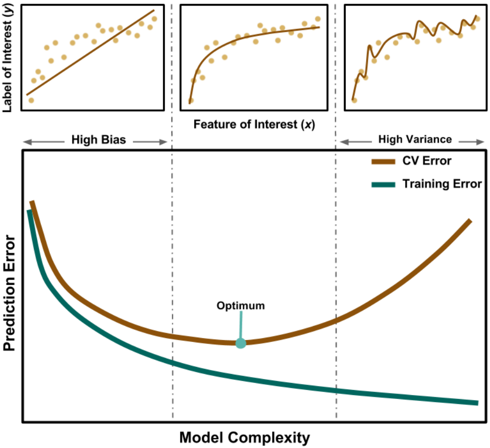


Figure 17: Validation curve showing different fitness of models with respect to model complexity

Model Comparison

$$Posterior = \frac{Likelihood * Prior}{Marginal\ Likelihood}$$

Probabilities	Calculation Method	Example
P(D|M) Likelihood	MLE or ML model prediction w/ CV	Given [M] : BWR, burnup = ***x*** GWd/MTU Then [D] : Pu-239 concentration = ***y***%
P(M) Prior	Histogram of simulation **inputs**	Given [D] : No direct information Then [M] : BWR, burnup = ***x*** GWd/MTU
P(D) Marginal L.	Histogram of simulation **outputs**	Given [M] : No direct information Then [D] : Pu-239 concentration = ***y***%
P(M|D) Posterior	Indirectly, from 3 probabilities above	Given [D] : Pu-239 concentration = ***y***% Then [M] : BWR, burnup = ***x*** GWd/MTU

Table 2: Table showing how each component of the model comparison framework will be computed

Computational Tools

- Training Data : SNF recipes from SCALE/ORIGEN-ARP [11, 13]
- Information Reduction
  - Gamma energies: ORIGEN
  - Computational gamma spectra: GADRAS [2]
- Statistics Toolkit : scikit-learn (python) [12]

Pre-detonation Materials of Interest

- UOC
- UOX powder
- SNF
- Reprocessed SNF
- Separated Pu
- Anything radioactive from the fuel cycle

Statistical Methods Employed

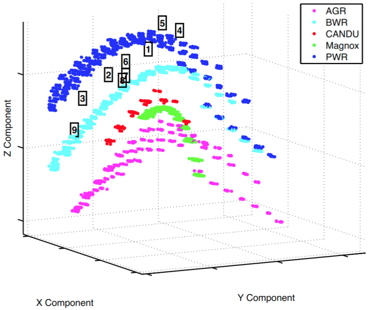


Figure 18: Unsupervised clustering for visualization separating reactor types [4]

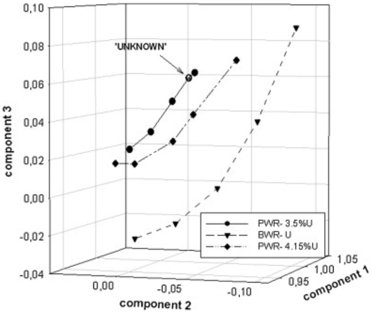


Figure 19: Factor analysis employed to determine provenance of unknown plutonium [8]

Statistical Methods Employed

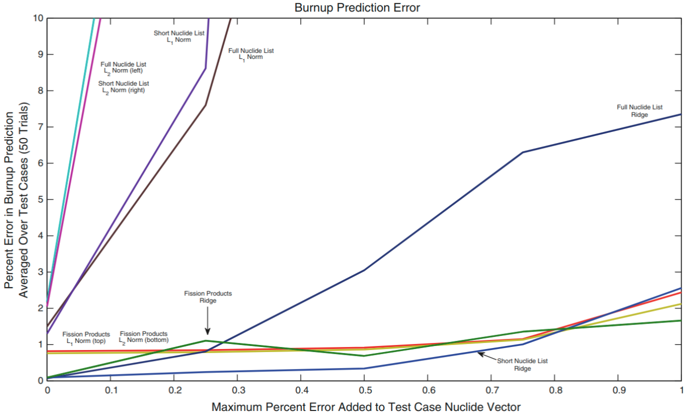


Figure 20: Burnup prediction error with respect to random nuclide error, using nearest neighbor & ridge regression methods [1]

Outline

Proposed Experiment Methodology

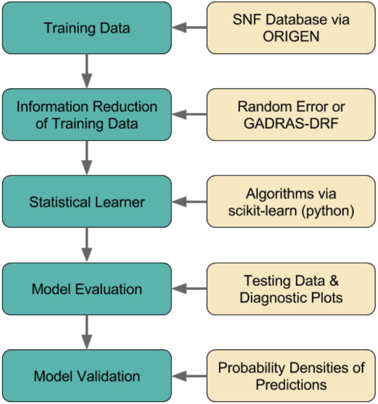


Figure 21: Workflow of the experiments with tools used for each step

Training Set

ORIGEN Rxtr	Rxtr Type	Enrichment
CE14x14	PWR	2.8
CE16x16	PWR	2.8
W14x14	PWR	2.8
W15x15	PWR	2.8
W17x17	PWR	2.8
S14x14	PWR	2.8
VVER440	PWR	3.60
VVER440_3.82	PWR	3.82
VVER440_4.25	PWR	4.25
VVER440_4.38	PWR	4.38
VVER1000	PWR	2.8
GE7x7-0	BWR	2.9
GE8x8-1	BWR	2.9
GE9x9-2	BWR	2.9
GE10x10-8	BWR	2.9
Abb8x8-1	BWR	2.9
Atrium9x9-9	BWR	2.9
SVEA64-1	BWR	2.9
SVEA100	BWR	2.9
CANDU28	PHWR	0.711
CANDU37	PHWR	0.711

Table 3: ORIGEN simulations [1]

	PWR	BWR	PHWR
Power Density [MW/MTU]	32	23	22
Burnup [MWd/MTU]	600–17700	600–12300	600–12300
Cooling Time		{1m, 7d, 30d, 1y}	

Table 4: Range of burnups and cooling times simulated for the training set [1]

Independent Testing Set

Reactor	Type	Enrichment	Cooling Time	Burnup
CANDU28	PHWR	0.711	{1m, 7d, 30d, 1y}	{1400, 5000, 11000}
CANDU28	PHWR	0.711	{3m, 9d, 2y}	{5000, 6120}
CE16x16	PWR	2.8	{1m, 7d, 30d, 1y}	{1700, 8700, 17000}
CE16x16	PWR	2.8	{3m, 9d, 2y}	{8700, 9150}
CE16x16	PWR	3.1	{7d, 9d}	{8700, 9150}
GE7x7-0	BWR	2.9	{1m, 7d, 30d, 1y}	{2000, 7200, 10800}
GE7x7-0	BWR	2.9	{3m, 9d, 2y}	{7200, 8800}
GE7x7-0	BWR	3.2	{7d, 9d}	{7200, 8800}

Table 5: Separate testing set used in previous work [1]

Initial Results

Algorithm	Error Origin	MAPE	RMSE [MWd/MTU]
Nearest Neighbor Regression	Testing Set	9.82	812.43
	5-fold Cross-Validation	2.24	421.41
Ridge Regression	Testing Set	15.68	1049.66
	5-fold Cross-Validation	0.08	13.08
Support Vector Regression	Testing Set	12.28	769.97
	5-fold Cross-Validation	2.08	188.07

Table 6: MAPE and RMSE for both CV and testing sets

Information Reduction

Demonstrated : Random error

Introduced $0\% < E_{max} < 10\%$

Each nuclide receives $[1 - E_{max}, 1 + E_{max}]$ error

Not Demonstrated : Systematic error

Gamma energies (ORIGEN), radionuclides only

Gamma spectra (GADRAS), reduced radionuclide observation

ML Model Prediction with Reduced Information

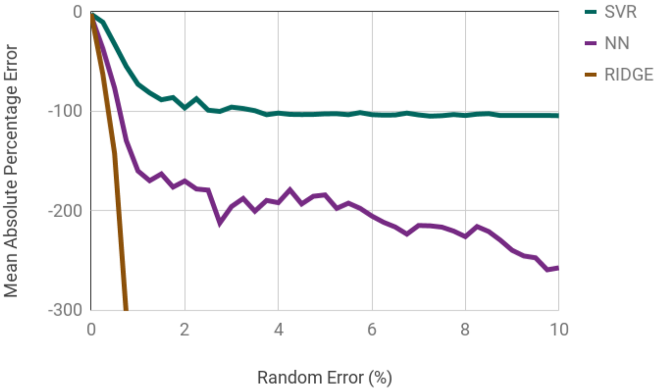


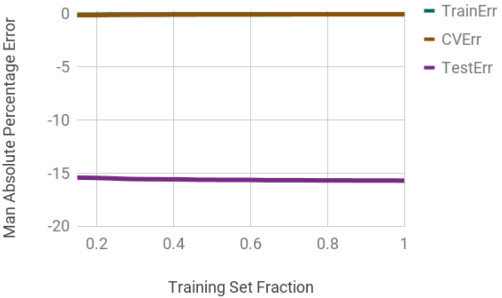
Figure 22: Negative MAPE for three algorithms given increasing random nuclide error

Algorithm Parameters

Algorithm	Parameter	Value
Nearest Neighbor Regression	*n*-neighbors	1
	Weights	uniform
	Distance Metric	L2: Euclidian Distance
Ridge Regression	Regularization, α	1.0
	Normalization	False
	Stopping Tolerance	0.001
Support Vector Regression	Kernel	Radial Basis Function
	Gamma, γ	0.001
	C	1000
	Epsilon, ε	0.1
	Stopping Tolerance	0.001

Table 7: Parameters chosen for demonstration; C and γ are not the default values

Learning Curves : Linear Model



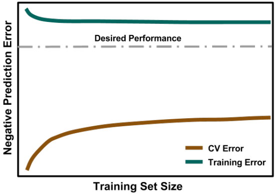
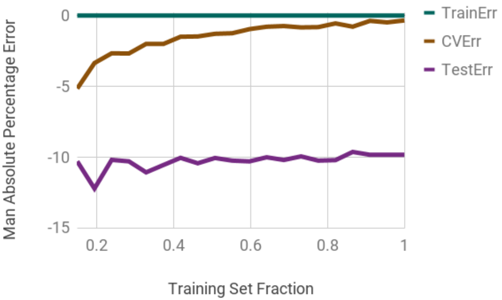


Figure 23: Learning curve and comparison schematic for ridge regression; $\alpha = 1.0$

Learning Curves : k-Nearest Neighbors



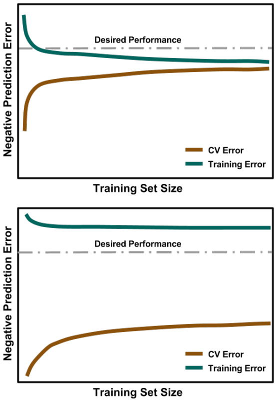


Figure 24: Learning curve and comparison schematic for k-NN regression; $k = 1$

Learning Curves: Support Vectors

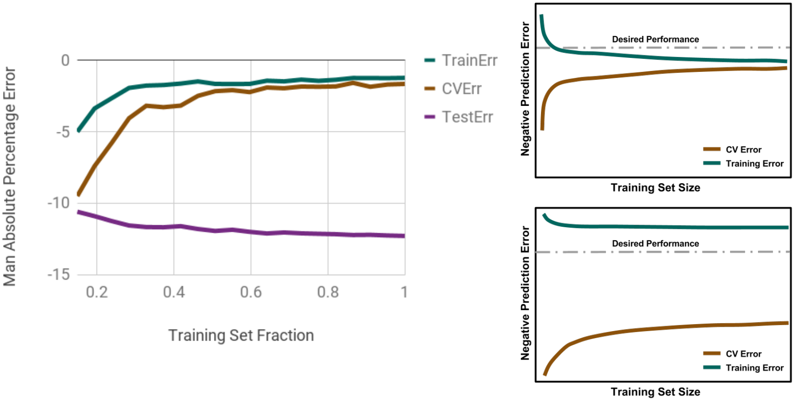
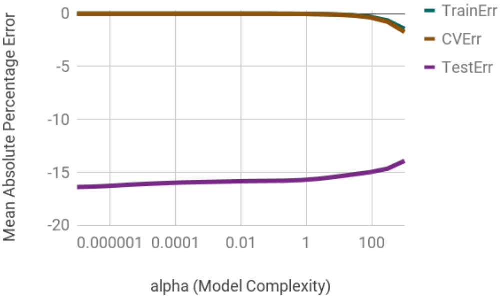


Figure 25: Learning curve and comparison schematic for SVR; $\gamma = 10^{-3}$ and $C = 10^3$

Validation Curves : Linear Model



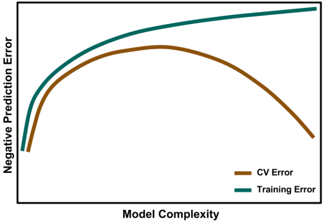


Figure 26: Validation curve and comparison schematic for ridge regression; $TrainingSetSize = 2313$

Validation Curves : *k*-Nearest Neighbors

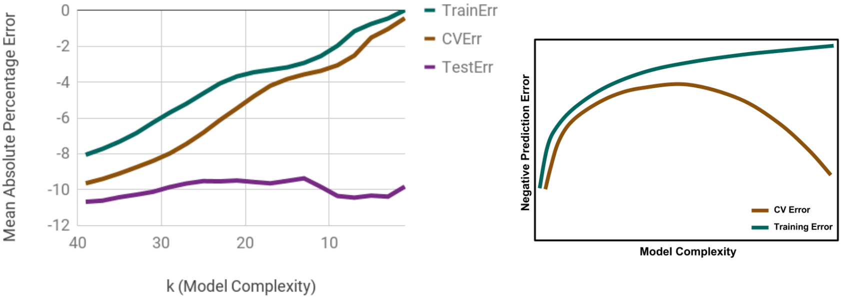


Figure 27: Validation curve and comparison schematic for k-NN regression; $TrainingSetSize = 2313$

Validation Curves : Support Vectors

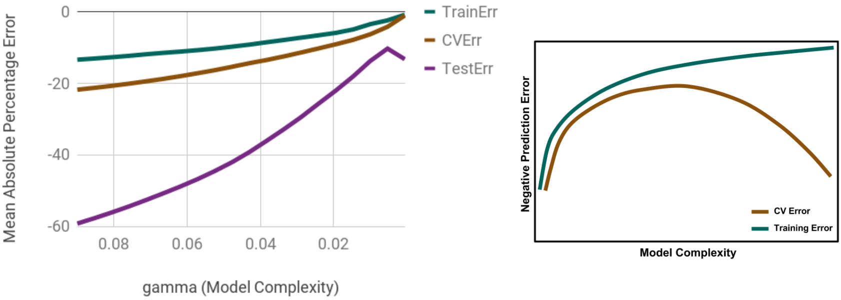


Figure 28: Validation curve and comparison schematic for SVR; $TrainingSetSize = 2313$ and $C = 10^3$

Note: Default γ would be $\frac{1}{2000} = 0.0005$

Outline

Research Proposal Preparations

1. Training set considerations
2. Finalizing set of algorithms
3. Computational resources

Reactor Type	U-235 Enrichment (%U-235)	Cooling Times	Burnup (MWd/te)
Advanced Gas Reactor (AGR)	1, 1.5, 2, 2.5, 3, 3.5, 4	1, 5, 10, 15, 20, 30, 40 years	1000, 5000, 10000, 15000, 20000, 25000, 30000, 35000, 40000, 45000
Boiling Water Reactor (BWR)	2, 2.5, 3, 3.5, 4	1, 5, 10, 15, 20, 30, 40 years	5000, 10000, 15000, 20000, 25000, 30000, 35000, 40000, 45000, 50000, 55000
Canada Deuterium Uranium (CANDU)	0.711, 1.2	0, 1, 2, 10, 15, 20, 30, 40 years	5000, 10000, 15000, 20000, 25000, 30000
Magnesium Non-Oxidizing (Magnox)	0.711	0, 16, 90, 115, 280, 365 days 2, 3, 4, 5, 10, 15, 20, 30, 40, 50, 60 years	500, 1,000, 2,000, 3,000, 4,000, 5,000, 6,000, 7,000, 8,000, 9,000, 10,000, 1,000, 12,000, 13,000, 14,000, 15,000
Pressurized Water Reactor (PWR)	2, 2.5, 3, 3.5, 4, 4.5	1, 5, 10, 15, 20, 30, 40 years	5000, 10000, 15000, 20000, 25000, 30000, 35000, 40000, 45000, 50000, 55000

Table 8: Example of a training data set based on comparison to the SFCOMPO database [4, 6]

Statistical Learning with Direct Isotopics

Goals : Understand limits of simplest scenario

1. Usefulness of statistical methods for reactor parameter prediction
2. Best performing methods

Variables

1. the complexity of the ML algorithm used,
2. feature reduction, and
3. different subsets of the decision space.

Statistical Learning with Direct Isotopics

Qualitative Hypotheses

- Complex algorithm will provide best behavior
- Manual preprocessing (feature reduction): speed, accuracy
- Reduction of decision space should help: PWR vs. BWR?

Risk Mitigation

- New algorithms: tree-based, neural nets, Bayesian MLE
- Statistical preprocessing: PCA, ICA
- New materials: Pu, UOC, Post-detonation (urban canyon [3])

Statistical Learning with Gamma Spectra

Goals : Understand limits of real-world scenario

1. Level of reduction in reactor parameter prediction
2. Best performing methods

Variables

1. the complexity of the ML algorithm used,
2. feature reduction (implicit), and
3. quality of training and/or testing data set.

Statistical Learning with Gamma Spectra

Qualitative Hypotheses

- Complex algorithm will provide best behavior
- Indirect isotopics = implicit feature reduction: less accurate
- Higher quality gamma spectra will yield better results

Risk Mitigation

- New algorithms: tree-based, neural nets, Bayesian MLE
- Further manual or statistical preprocessing
- Add isotope identification step

Statistical Learning with Reprocessed Fuel

Goals : Probe prediction performace in reprocessing scenario

1. Experiment with both direct and indirect isotopics
2. Fresh evaluation of preprocessing
3. Best performing methods for materials with multiple sources

Variables

1. the complexity of the ML algorithm used,
2. quality of training data set, and
3. type of preprocessing for feature reduction.

Statistical Learning with Reprocessed Fuel

Qualitative Hypotheses

- Complex algorithm will provide best behavior
- Reduced information will provide less accurate results
- ICA may outperform PCA, but factor analysis may outperform components analysis [7, 9, 10, 15, 14, 4, 5]

Risk Mitigation

- New algorithms: tree-based, neural nets, Bayesian MLE
- Manual preprocessing
- Results may be interesting even if prediction fails
- Ensemble methods or other creative solutions [15, 14]

Probability Distribution Functions

$$P(\boldsymbol{m}|\boldsymbol{d}) = C * P(\boldsymbol{d}|\boldsymbol{m}) * P(\boldsymbol{m})$$

$\boldsymbol{d}$: training data set

$\boldsymbol{m}$: model parameters

C: constant from marginal likelihood

$P(\boldsymbol{m})$: prior probability distribution

$P(\boldsymbol{d}|\boldsymbol{m})$: likelihood distribution function

$P(\boldsymbol{m}|\boldsymbol{d})$: posterior probability distribution

Typically, (cumulative) probability distributions are integrated probability density functions:

$$P(\boldsymbol{d}|\boldsymbol{m}) = \int_{\boldsymbol{d},\boldsymbol{m}} \rho(\boldsymbol{d}|\boldsymbol{m}) d\boldsymbol{m}$$

Estimating Density Functions

MLE is the CV error because the results are reported as: $\mu \pm \sigma$ [12]

MLE is not this simple for other methods that do not employ CV [16, 17]

Probabilities	Calculation Method
P(D|M) Likelihood	MLE or ML model prediction w/ CV
P(M) Prior	Histogram of simulation **inputs**
P(D) Marginal L.	Histogram of simulation **outputs**
P(M|D) Posterior	Indirectly, from 3 probabilities above

Table 9: The density functions that represent each probability are estimated from histograms or the ML model

Note: This implies the posterior is now only dependent on the likelihood.

Posterior Odds

Posterior odds = probability of ML model i being correct

1. Calculate $MLE_i = P_i(m|d)$ for statistical model i
2. Calculate $MLE_j = P_j(m|d)$ for statistical model j
3. Ratio of MLE_i to MLE_j to get Bayes' factor: $B_{ij} = \frac{P_i(d|m)}{P_j(d|m)}$
4. Take ratio of both Bayes' relations to get posterior odds: $\frac{P_i(m|d)}{P_j(m|d)} = B_{ij}\frac{P_i(m)}{P_j(m)}$

$\|\ln B_{ij}\|$	Probability	Likelihood Strength
< 1.0	< 0.750	Inconclusive
1.0	0.750	Weak
2.5	0.923	Moderate
5.0	0.993	Strong

Table 10: Model comparison using the Bayes' factor to describe the likelihood strength [16, 17]

Outline

Summary

Pre-detonation nuclear forensics analysis on SNF

- Demonstrated:
    - Simulation of training data set
    - Information reduction of training set
    - Burnup prediction using three statisitcal models: ridge, *k*-nearest neighbor, and support vector regression
    - ML model optimization via diagnostic curves
    - Plan for ML model comparison
- Simplifications:
    - Only predicted burnup (missing rxtr type, enrichment, cooling time)
    - Borrowed training set
    - No statistical feature reduction
    - No physically based information reduction
- Three experiments:
    - Nuclide concentrations
    - Gamma energies/spectra
    - Reprocessed SNF

References I

[1] Kenneth Dayman and Steven Biegalski.
Feasibility of fuel cycle characterization using multiple nuclide signatures.
Journal of Radioanalytical and Nuclear Chemistry, 296:195–201, 2013.

[2] Steven M. Horne, Gregory G Thoreson, Lisa A. Theisen, Dean J. Mitchell, Lee Harding, and Wendy A. Amai.
Gamma Detector Response and Analysis Software - Detector Response Function (GADRAS-DRF).
User's Manual, Sandia National Laboratories, Albuquerque, New Mexico, USA, Dec 2014.
Version 18.5; SAND2014-19465.

[3] Kenneth G.W. Inn, Jacqueline Mann, Jeffrey Leggitt, JoAnne Buscaglia, Simon Jerome, John Molloy, and William Pramenko.
Nuclear forensic reference materials for attribution of urban nuclear terrorism, 2015.
Presentation for NIST.

References II

[4] Andrew Jones, Phillip Turner, Colin Zimmerman, and J.Y. Goulermas.
Machine learning for classification and visualisation of radioactive substances for nuclear forensics.
In *Techniques and Methods for Safeguards, Nonproliferation and Arms Control Verification Workshop*, Portland, Oregon, May 2014.

[5] Andrew E. Jones, Phillip Turner, Colin Zimmerman, and John Y. Goulermas.
Classification of spent reactor fuel for nuclear forensics.
Analytical Chemistry, 86:5399–5405, 2014.

[6] Franco Michel-Sendis, Jesus Martinez-Gonzlez, and Ian Gauld.
Sfcompo 2.0 – a relational database of spent fuel isotopic measurements, reactor operational histories, and design data.
EPJ Web of Conferences, 146:06015, Jan 2017.

[7] G. Nicolaou.
Determination of the origin of unknown irradiated nuclear fuel.
Journal of Environmental Radioactivity, 86:313–318, 2006.

References III

[8] G. Nicolaou.
Provenance of unknown plutonium material.
Journal of Environmental Radioactivity, 99(10):1708–1710, 2008.

[9] G. Nicolaou.
Identification of unknown irradiated nuclear fuel through its fission product content.
Journal of Radioanalytical and Nuclear Chemistry, 279(2):503–508, 2009.

[10] G. Nicolaou.
Discrimination of spent nuclear fuels in nuclear forensics through isotopic fingerprinting.
Annals of Nuclear Energy, 72:130–133, Oct 2014.
Technical Note.

References IV

[11] Oak Ridge National Laboratory.
SCALE: A Comprehensive Modeling and Simulation Suite for Nuclear Safety Analysis and Design.
Code suite, Oak Ridge National Laboratory, Oak Ridge, Tennessee, USA, Aug 2016.
Version 6.2.1, ORNL/TM-2005/39, Available from Radiation Safety Information Computational Center as CCC-834.

[12] F. Pedregosa, G. Varoquaux, A. Gramfort, V. Michel, B. Thirion, O. Grisel, M. Blondel, P. Prettenhofer, R. Weiss, V. Dubourg, J. Vanderplas, A. Passos, D. Cournapeau, M. Brucher, M. Perrot, and E. Duchesnay.
Scikit-learn: Machine learning in Python.
Journal of Machine Learning Research, 12:2825–2830, 2011.

References V

[13] B.T. Rearden and M.A. Jessee.
Ch. 5 Depletion, Activation, and Spent Fuel Source Terms.
In *SCALE Code System: User Documentation*, pages 5–1–5–263, Oak Ridge, Tennessee, USA, Apr 2016. Oak Ridge National Laboratory.
Version 6.2.1; ORNL/TM-2005/39.

[14] Martin Robel and Michael J. Kristo.
Discrimination of source reactor type by multivariate statistical analysis of uranium and plutonium isotopic concentrations in unknown irradiated nuclear fuel material.
Journal of Environmental Radioactivity, 99(11):1789–1797, November 2008.

[15] Martin Robel, Michael J. Kristo, and Martin A. Heller.
Nuclear forensic inferences using iterative multidimensional statistics.
In *Proceedings of the Institute of Nuclear Materials Management 50th Annual Meeting*, Tuscon, AZ, USA, Jul 2009. Institute of Nuclear Materials Management.
LLNL-CONF-414001.

References VI

[16] Tan Bui-Thanh.
A Gentle Tutorial on Statistical Inversion using the Bayesian Paradigm.
Note. ICES REPORT 12-18, The University of Texas at Austin, The Institute for Computational Engineering and Sciences, May 2012.

[17] Roberto Trotta.
Bayes in the Sky: Bayesian Inference and Model Selection in Cosmology.
Contemporary Physics, 49(2):71–104, 2008.
Invited review.