LINEAR ALGEBRA REVIEW

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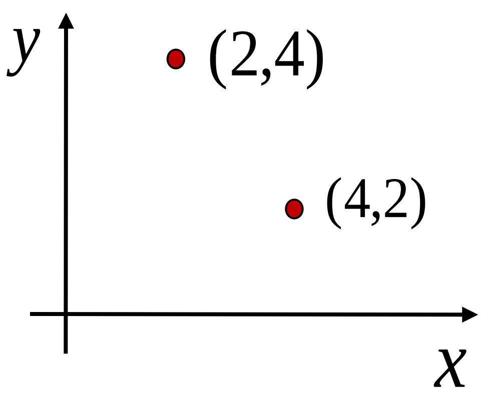
Overview

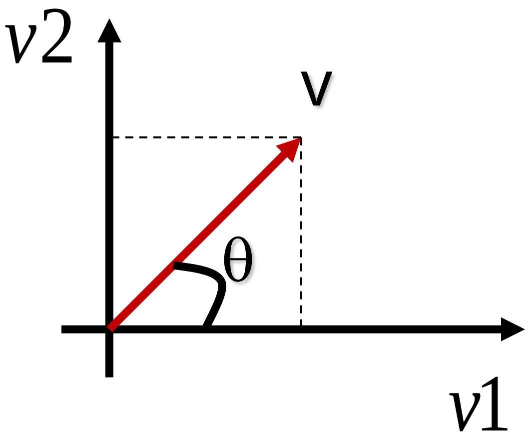
- Points and Lines
- Vectors
- Inner (Dot) Products
- Cross Products
- Orthonormal Basis
- Change of Orthonormal Basis
- Matrices and Operations

Points and Vectors

- Points specify locations in space (or in the plane)



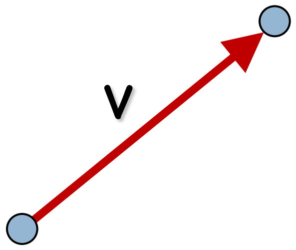
- Vectors have a magnitude and direction



Vectors

Vector

$$v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \in R^2$$



Magnitude or

Length

$$\|v\| = \sqrt{v_1^2 + v_2^2}$$

Unit Vector or

Normalized Vector

$$\|v\| = 1$$

Vectors

Vector

$$v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \in R^2$$

Orientation

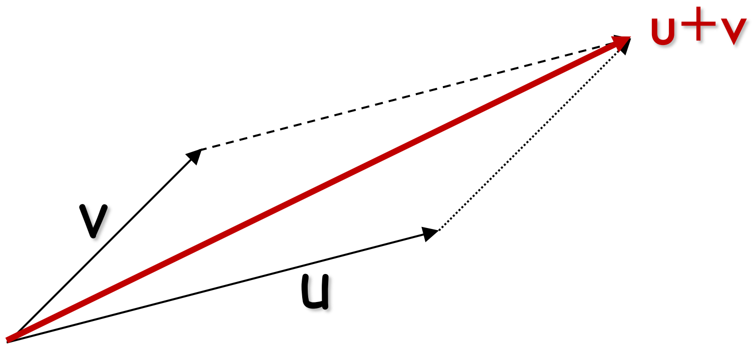
$$\theta = \tan^{-1}\left(\frac{v_2}{v_1}\right)$$

Zero Vector

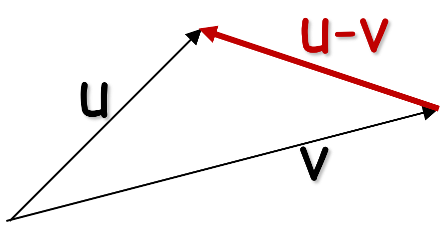
$$v = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Vector Addition and Subtraction

$$u + v = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \end{bmatrix}$$

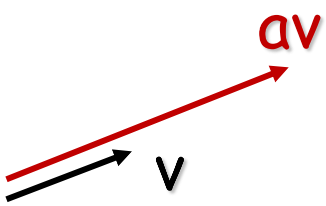


$$u - v = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} - \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} u_1 - v_1 \\ u_2 - v_2 \end{bmatrix}$$



Multiplication with a Scalar

$$av = a \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} av_1 \\ av_2 \end{bmatrix}$$

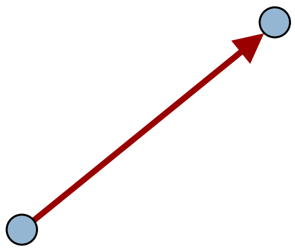


Point or Vector ??

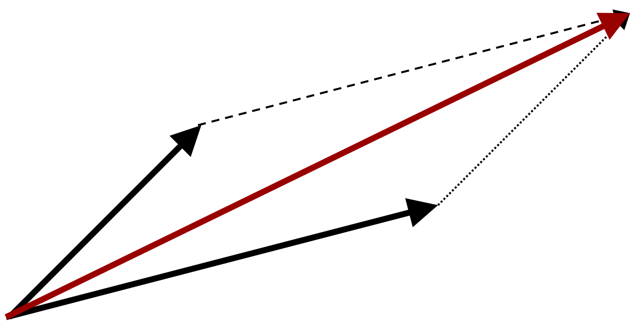
- Point + Vector = ?
- Vector + Vector = ?
- Point – Point = ?
- Point + Point = ?

Point or Vector

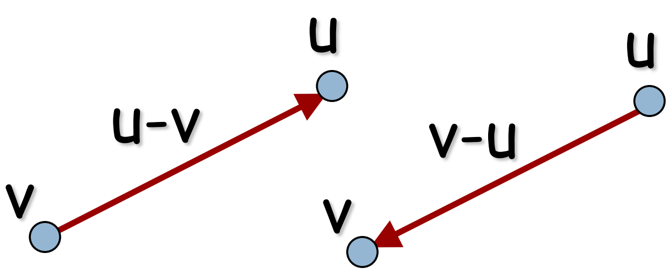
Point



Vector + Vector = Vector



Point - Point = Vector



Point + Point = Not Defined

Dot Product

- The dot product or inner product measures to what degree two vectors are aligned
- Notation:

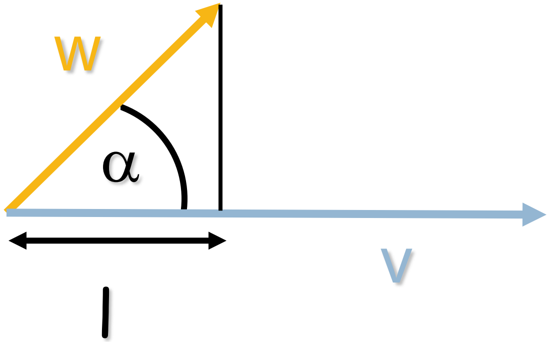
$$< v, w > = v^T w = v \cdot w$$

- Computation:

$$< v, w > = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = v_1 \cdot w_1 + v_2 \cdot w_2 + v_3 \cdot w_3$$

Dot Product

- Geometric Interpretation:



$$COS(\alpha) = \frac{l}{\|w\|} = \frac{\langle v, w \rangle}{\|v\| \cdot \|w\|}$$

Dot Product Calculation

- Two vectors v and w are perpendicular or normal iff

$$< v, w > = 0$$

Dot Product Calculation

- Note:

  Length = Magnitude = 2-Norm = Euclidian Norm

$$\|v\| = \sqrt{\langle v, v \rangle} = \sqrt{v^T v}$$

Dot Product Laws

- Commutative law:

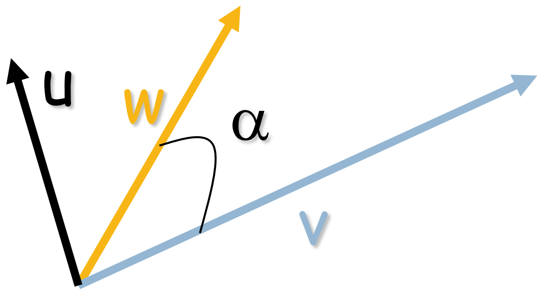
$$< v, w > = < w, v >$$

- Distributive law:

$$< v, w + u > = < v, w > + < v, u >$$

Cross Product

- Cross Product



- Notation: $u = v \times w$
- The cross product is a vector!
- Magnitude of u: proportional to the sine of the angle between v and w

  $$\| \mathrm{u} \| = \| v \times w \| = \| v \| \| w \| \sin \alpha$$

- *u* is perpendicular to *v* and *w*
- The direction of *u* follows the right hand rule

Cross Product Computation

- Computation:

$$u = v \times w = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \times \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{vmatrix} i & j & k \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = \begin{bmatrix} v_2 w_3 - w_2 v_3 \\ v_3 w_1 - w_3 v_1 \\ v_1 w_2 - w_1 v_2 \end{bmatrix}$$

Cross Product Laws

- Distributive law:

$$v \times (w + u) = v \times w + v \times u$$

- Commutative law:

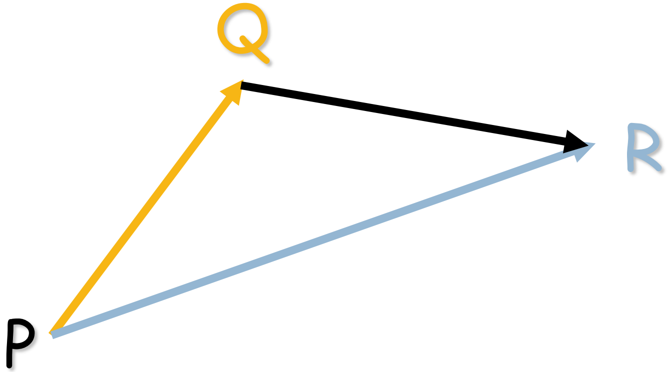
$$v \times w = -w \times v$$

- Associative law:

$$v \times (w \times u) \neq (v \times w) \times u$$

Cross Product and Triangle Area

- The cross product is related to the area of a triangle

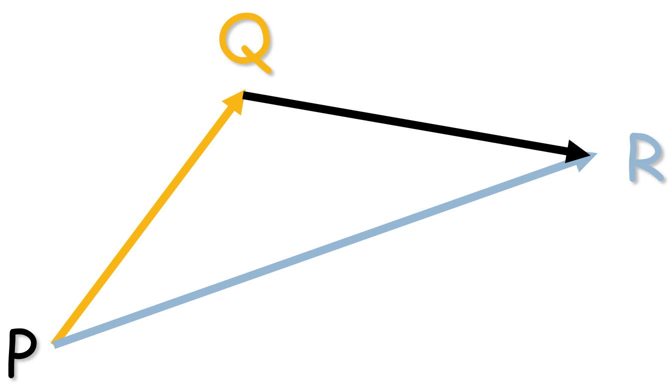


$$area = \frac{|PQ \times PR|}{2}$$

Triangle Area Example

Given a Triangle with
P(2,3,-1), Q(-1,2,3) and R(3,1,-2)

What is the area?



2D LINES

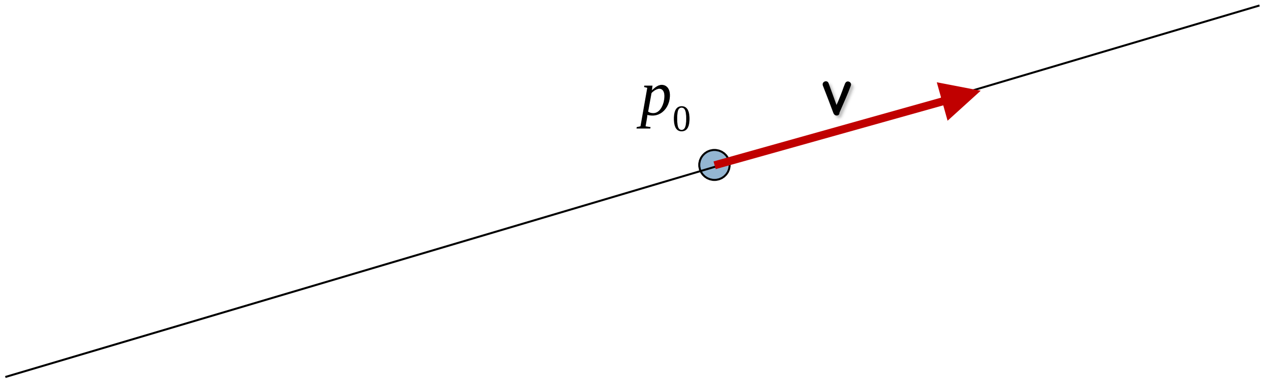
Defining a Line

- Two elements of 2D geometry define a line
  - two points
  - a point and a vector parallel to the line
  - a point and a vector perpendicular to the line
- Three different mathematical representations
  - parametric equation
  - implicit equation
  - explicit equation

Parametric Equation of a Line

- Parametric Line Equation:

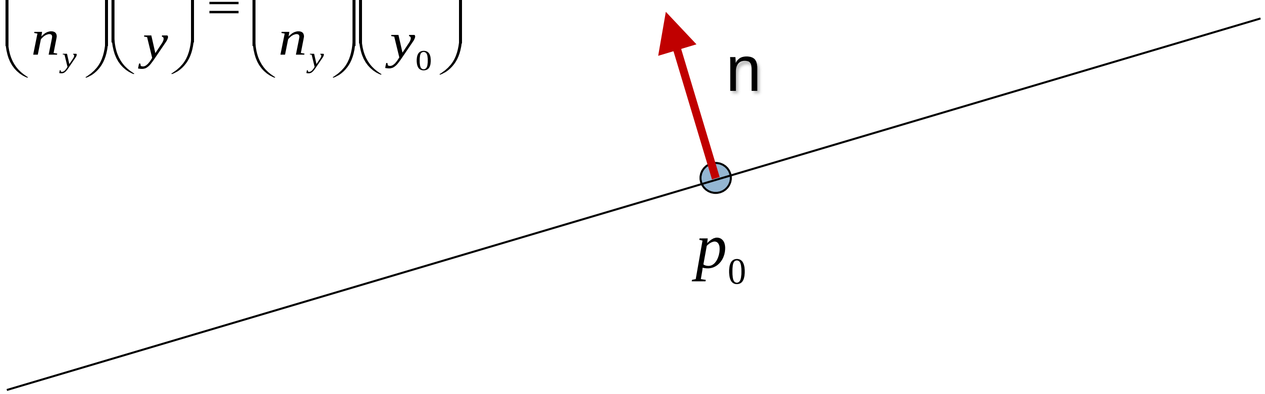
  $line(t) = p_0 + tv$ (Note t is a scalar)



Implicit Equation of a Line

- Line equation:

$$\begin{pmatrix} n_x \\ n_y \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} n_x \\ n_y \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$



- Alternative notation

$$n_x \cdot x + n_y \cdot y = c$$

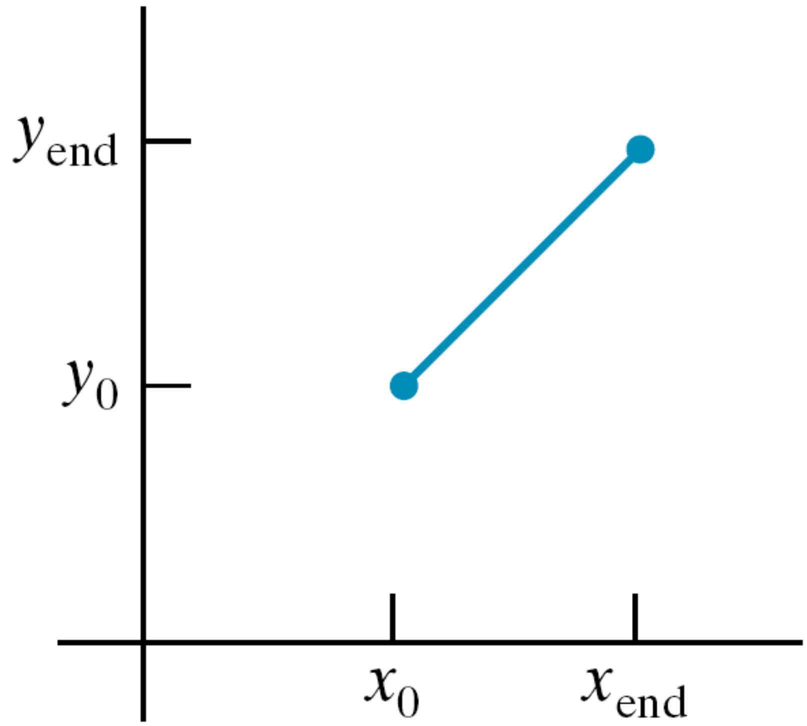
Slope Intercept Line Equation

- line equation: y = m*x + b

line path between two points:

$$m = \frac{y_{end} - y_0}{x_{end} - x_0}$$

$$b = y_0 - m \cdot x_0$$



Distance between a point and a line

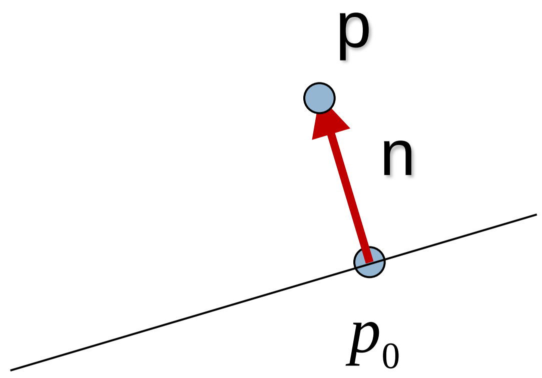
- Use the line equation:

$$n_x \cdot x + n_y \cdot y = c$$

- Divide the line equation by the length of n

- Compute the absolute value of:

$$n_x \cdot px + n_y \cdot py - c$$



Derivation

Find a point **q'** such that $(\mathbf{q} - \mathbf{q}')\perp \mathbf{v}$

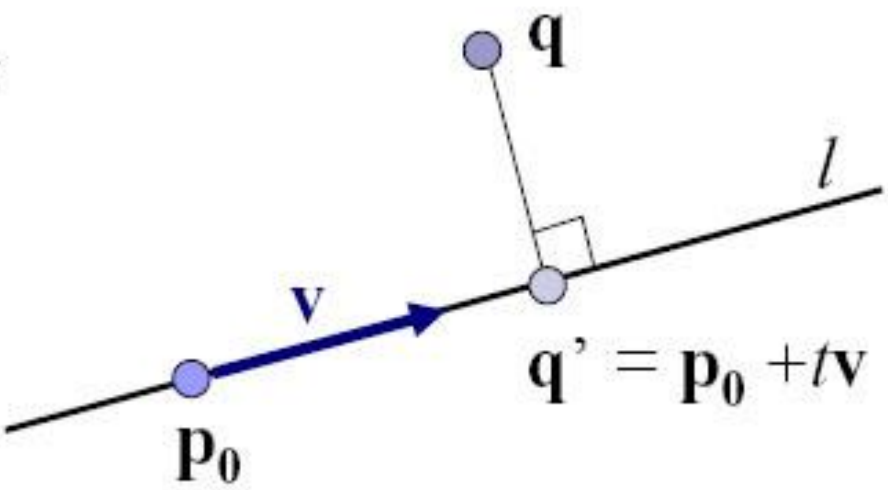
$\mathrm{dist}(\mathbf{q}, l) = \| \mathbf{q} - \mathbf{q}' \|$

$$< \mathbf{q} - \mathbf{q}',\ \mathbf{v} > = 0$$

$$< \mathbf{q} - (\mathbf{p}_0 + t\mathbf{v}),\ \mathbf{v} > = 0$$

$$< \mathbf{q} - \mathbf{p}_0,\ \mathbf{v} > -\ t < \mathbf{v}, \mathbf{v} > = 0$$

$$t = \frac{< \mathbf{q} - \mathbf{p}_0,\ \mathbf{v} >}{< \mathbf{v}, \mathbf{v} >} = \frac{< \mathbf{q} - \mathbf{p}_0,\ \mathbf{v} >}{\| \mathbf{v} \|^2}$$



$$\boxed{\mathrm{dist}^2(\mathbf{q},\ l) = \| \mathbf{q} - \mathbf{q}' \|^2 = \| \mathbf{q} - \mathbf{p}_0 \|^2 - \frac{< \mathbf{q} - \mathbf{p}_0,\ \mathbf{v} >^2}{\| \mathbf{v} \|^2}}$$

MATRICES

Matrix

- A matrix is a set of elements, organized into rows and columns

$$\begin{array}{cc} & \text{rows} \\ \text{columns} & \begin{bmatrix} a & b \\ c & d \end{bmatrix} \end{array}$$

Matrix Addition

- Addition

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a+e & b+f \\ c+g & d+h \end{bmatrix}$$

- General Form

$$C_{n\times m} = A_{n\times m} + B_{n\times m} \qquad c_{ij} = a_{ij} + b_{ij}$$

- Example

$$\begin{bmatrix} 2 & 5 \\ 3 & 1 \end{bmatrix} + \begin{bmatrix} 6 & 2 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} 8 & 7 \\ 4 & 6 \end{bmatrix}$$

Matrix Subtraction

- Subtraction

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} - \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a-e & b-f \\ c-g & d-h \end{bmatrix}$$

Matrix Multiplication

- Multiplication

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{bmatrix}$$

Multiply each row by each column

Matrix Multiplication

Multiplication:

$$C_{n\times p} = A_{n\times m} B_{m\times p}$$

A and B must have compatible dimensions

$$c_{ij} = \sum_{k=1}^{m} a_{ik} b_{kj}$$

$$A_{n\times n} B_{n\times n} \neq B_{n\times n} A_{n\times n}$$

Examples:

$$\begin{bmatrix} 2 & 5 \\ 3 & 1 \end{bmatrix} \cdot \begin{bmatrix} 6 & 2 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} 17 & 29 \\ \boxed{} & 11 \end{bmatrix}$$

$$\begin{bmatrix} 6 & 2 \\ 1 & 5 \end{bmatrix} \cdot \begin{bmatrix} 2 & 5 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 18 & 32 \\ 17 & 10 \end{bmatrix}$$

Transpose

- Notation: A^T
- Definition: $A^T_{(ij)} = A_{(ji)}$
- Examples:

$$\begin{bmatrix} 6 & 2 \\ 1 & 5 \end{bmatrix}^T = \begin{bmatrix} 6 & 1 \\ 2 & 5 \end{bmatrix} \qquad \begin{bmatrix} 6 & 2 \\ 1 & 5 \\ 3 & 8 \end{bmatrix}^T =$$

Rules

$$(AB)^T = B^T A^T$$

$$(A+B)^T = A^T + B^T$$

Symmetric Matrices

- A is symmetric iff $A^T = A$

Determinants

- Determinant (Note A must be square)

$$\det\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}$$

$$\det\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

- Example $\quad \det\begin{bmatrix} 2 & 5 \\ 3 & 1 \end{bmatrix} = 2 - 15 = -13$

Inverse Matrices

- Notation $AA^{-1} = A^{-1}A = I$

- I is the Identity (1s on the diagonal, 0s elsewhere)

- Inverse Computation:

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^{-1} = \frac{1}{a_{11}a_{22} - a_{21}a_{12}} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

Matrices

- Example

$$\begin{bmatrix} 6 & 2 \\ 1 & 5 \end{bmatrix}^{-1} = \frac{1}{28}\begin{bmatrix} 5 & -2 \\ -1 & 6 \end{bmatrix}$$

$$\begin{bmatrix} 6 & 2 \\ 1 & 5 \end{bmatrix}^{-1} \cdot \begin{bmatrix} 6 & 2 \\ 1 & 5 \end{bmatrix} = \frac{1}{28}\begin{bmatrix} 5 & -2 \\ -1 & 6 \end{bmatrix} \cdot \begin{bmatrix} 6 & 2 \\ 1 & 5 \end{bmatrix} = \frac{1}{28}\begin{bmatrix} 28 & 0 \\ 0 & 28 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

ORTHOGONAL BASIS

Orthonormal Basis

- Basis: a space is totally defined by a set of vectors – any point is a *linear combination* of the basis
- Ortho-normal: orthogonal + normal
- Orthogonal: dot product is zero
- Normal: magnitude is one
- Example: $x = \begin{bmatrix}1 & 0 & 0\end{bmatrix}^T$ $\quad x \cdot y = 0$

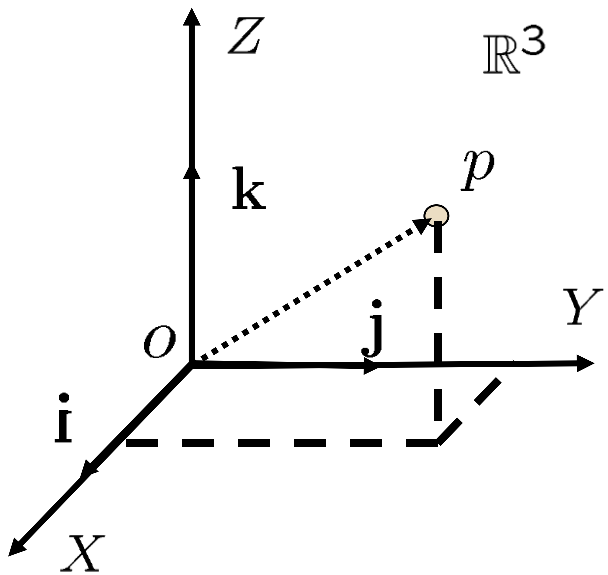
$$y = \begin{bmatrix}0 & 1 & 0\end{bmatrix}^T \qquad x \cdot z = 0$$

$$z = \begin{bmatrix}0 & 0 & 1\end{bmatrix}^T \qquad y \cdot z = 0$$

Orthonormal Coordinate in 3D

Standard basis vectors:

$$\mathbf{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad \mathbf{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad \mathbf{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$



Coordinates of a point p in space:

$$\boldsymbol{X} = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \in \mathbb{R}^3 \qquad \boldsymbol{X} = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = X.\mathbf{i} + Y.\mathbf{j} + Z.\mathbf{k}$$

Reading

- Reading
  - Read Appendix B & C in Interactive Computer Graphics

Questions?

- Ask now or e-mail later
- Acknowledgements
  - Previous instructors at Purdue
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  - Textbook (Ed Angel)
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