trajectory_utils: Mathematical background

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1. Cartpoles

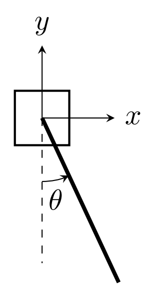

Figure 1: Cart–pole system. The cart is actuated by some means and is constrained to move along $\pm x$. Objectives are to swing the pole up to vertical - and/or stabilize it there. The physical cart is actuated by a belt drive, and the pole is a steel rod.

1.1 Equations of motion

By inspection, figure 1 offers:

$$x_r(t) = r sin\theta(t), r \in [0, L] \tag{1}$$
$$y_r(t) = -r cos\theta(t), r \in [0, L] \tag{2}$$
$$\dot{x_r}(t) = r cos\theta(t)\dot{\theta}(t) + \dot{x}(t), r \in [0, L] \tag{3}$$
$$\dot{y_r}(t) = r sin\theta(t)\dot{\theta}(t), r \in [0, L] \tag{4}$$

where r parameterizes the location of infinitesimal mass along the pole. It is straightforward to compute the Lagrangian $\mathcal{L} = T - V$ - the first step in deriving the equations of motion.

The kinetic energy T can be written as a sum of the cart kinetic energy and an integral over the pole:

$$T = \int_{r=0}^{r=L} (\dot{x}_r^2 + \dot{y}_r^2)\rho dr + \frac{1}{2} m_c \dot{x}^2 \tag{5}$$

where ρ is the pole mass per unit length. A bit of algebra yields:

$$T = m_p \frac{L^2}{6} \dot{\theta}^2 + \frac{1}{2} m_p L \cos\theta \dot{\theta} \dot{x} + \frac{1}{2} (m_p + m_c) \dot{x}^2 \tag{6}$$

where we have used the fact that $\rho L = m_p$.

The (gravitational) potential energy of the cart does not change, so we are left with an integral over the pole:

$$V = g \int_{r=0}^{r=L} y_r \rho dr \tag{7}$$

or, equivalently, the potential energy of a mass concentrated at the pole's center of mass:

$$V = -\frac{g m_p L}{2} \cos\theta \tag{8}$$

Euler-Lagrange now yields two equations:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) - \frac{\partial \mathcal{L}}{\partial x} = F_x \tag{9}$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) - \frac{\partial \mathcal{L}}{\partial \theta} = 0 \tag{10}$$

By substitution of expressions for T and V, and some algebra, this yields:

1.2 Cart force control

1.3 Cart velocity servo control

1.4 The `cvxpy` experience

2. Differential Drive Control trajectories

2.1 Signed Distance Function (SDF) for obstacle avoidance

2.2 Differentiable SDF in pyTorch

2.3 The `cvxpy` experience

References