100-ish integrations

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1. $$\int \frac{x}{x^2+3}\,dx$$

2. $$\int \frac{1}{x\ln x}\,dx$$

3. $$\int \frac{x}{\sqrt{x^2+4}}\,dx$$

4. $$\int \frac{3x-3}{x^2+3}\,dx$$

5. $$\int \sin x\,\cos^2 x\,dx$$

6. $$\int \sin x\,\sec^2 x\,dx$$

7. $$\int \cos^2\frac{x}{2}\,dx$$

8. $$\int x\sin x\,dx$$

9. $$\int x\sec^2 x\,dx$$

10. $$\int \tan^{-1} 2x\,dx$$

11. $$\int \frac{x^3}{x^2+1}\,dx$$

12. $$\int \frac{x}{(x+1)(x+2)}\,dx$$

13. $$\int \frac{(x-1)(x+1)}{(x-2)(x+3)}\,dx$$

14. $$\int \frac{2x-1}{x^2+2x+2}\,dx$$

15. $$\int \frac{x^3}{2x+1}\,dx$$

16. $$\int \frac{1-x}{\sqrt{1+x-x^2}}\,dx$$

17. $$\int \frac{1}{x^2\sqrt{1-x^2}}\,dx$$

18. $$\int \frac{1}{a^2-x^2}\,dx$$

19. $$\int \frac{1}{x(x^2-a^2)}\,dx$$

20. $$\int \frac{u}{\sqrt{u}+1}\,du$$

21. $$\int \frac{\sin^{-1} x}{\sqrt{1-x^2}}\,dx$$

22. $$\int \frac{1}{x(\ln x)^3}\,dx$$

23.
$$\int \sec^4 2x \, dx$$

24.
$$\int \frac{1}{x^2(x+1)} \, dx$$

25.
$$\int \frac{1}{x^2(x^2+1)} \, dx$$

26.
$$\int \frac{1}{(x^2+1)^2} \, dx$$

27.
$$\int \tan^2 x \, dx$$

28.
$$\int \frac{\sin x}{3+\cos x} \, dx$$

29.
$$\int \frac{1}{1+\cos^2 x} \, dx$$

30.
$$\int \frac{1}{10+8\cos x} \, dx$$

31.
$$\int \frac{1}{8+10\cos x} \, dx$$

32.
$$\int \frac{\sin x}{8+10\cos x} \, dx$$

33.
$$\int \cos^2 x - \sin^2 x \, dx$$

34.
$$\int x^2 \sin x \, dx$$

35.
$$\int \frac{x^2}{(x+1)(x+2)(x+3)} \, dx$$

36.
$$\int \frac{e^x}{e^x+1} \, dx$$

37.
$$\int \frac{1}{2\sin^2 x + 18\cos^2 x} \, dx$$

38.
$$\int x^2 \, e^{3x^3-5} \, dx$$

39.
$$\int x^3 \ln x \, dx$$

40.
$$\int \ln x^3 \, dx$$

41.
$$\int \log_2 x^3 \, dx$$

42.
$$\int \frac{1}{e^x + e^{-x}} \, dx$$

43.
$$\int \left(4x^2+5x-1\right)^{3/2} (8x+5) \, dx$$

44.
$$\int \frac{1}{(x^2+1)(x^2+4)} \, dx$$

45.
$$\int \frac{1}{x^2+x+1} \, dx$$

46.
$$\int \frac{1}{x^2+x-1} \, dx$$

47.
$$\int e^x \sin x \, dx$$

48.
$$\int \frac{1}{\sqrt{x^2-x}} \, dx$$

49.
$$\int \frac{1+2x}{2+x} \, dx$$

50.
$$\int \frac{x^2}{\sqrt{x^2+4}} \, dx$$

51.
$$\int \frac{\sin x}{3\cos^2 x + 2\sin^2 x}\, dx$$

52.
$$\int \frac{x^3}{1-x^4}\, dx$$

53.
$$\int \frac{1}{\sin x \cos x}\, dx$$

54.
$$\int \ln\sqrt{x+1}\, dx$$

55.
$$\int \frac{1}{e^x+1}\, dx$$

56.
$$\int \frac{\sec^2 x}{\tan^2 x - 4\tan x - 5}\, dx$$

57.
$$\int \sin 2x \cos x\, dx$$

58.
$$\int \frac{x}{(1+x)(x^2+x+1)}\, dx$$

59.
$$\int \frac{1}{2x^2+3x+1}\, dx$$

60.
$$\int \sqrt{4-x^2}\, dx$$

61.
$$\int \frac{1}{\sqrt{x+1}-\sqrt{x}}\, dx$$

62.
$$\int x\sqrt{9+x^2}\, dx$$

63.
$$\int \sec^2 x\, \tan^3 x\, dx$$

64.
$$\int x^2 e^{-x}\, dx$$

65.
$$\int x e^{-x^2}\, dx$$

66.
$$\int \sin x \tan x\, dx$$

67.
$$\int \sin^3 x \cos^2 x\, dx$$

68.
$$\int \frac{x^2+1}{x^2-x}\, dx$$

69.
$$\int \frac{1}{\sqrt{x-1}+(x-1)}\, dx$$

70.
$$\int \frac{3x^2}{1+x^6}\, dx$$

71.
$$\int \sec x\, dx$$

72.
$$\int_0^3 \frac{x}{\sqrt{x+3}}\, dx$$

73.
$$\int_1^3 \frac{1}{x(1+x^2)}\, dx$$

74.
$$\int_1^3 \frac{\ln x}{x}\, dx$$

75.
$$\int_0^1 \sin^{-1} x\, dx$$

76.
$$\int_1^3 \frac{x+2}{\sqrt{-3+4x-x^2}}\, dx$$

77.
$$\int_1^2 \frac{1}{x^2+4x+3}\, dx$$

78.
$$\int_0^1 x\sqrt{1-x^2}\, dx$$

79.
$$\int x \ln x \, dx$$

80.
$$\int x^2 e^{-x} \, dx$$

81.
$$\int \frac{x+1}{x^3+x^2+x+1} \, dx$$

82.
$$\int_0^1 \frac{e^{2x}}{e^x+1} \, dx$$

83.
$$\int_{-a}^{a} \sqrt{a^2-x^2} \, dx$$

84.
$$\int_{-a}^{a} x\sqrt{x^2-a^2} \, dx$$

85.
$$\int_{-\pi/2}^{\pi/2} \sin x \cos x \, dx$$

86.
$$\int_0^{\pi/4} \sec^2 x \tan x \, dx$$

87.
$$\int_1^2 \frac{e^{\sqrt{x}}}{\sqrt{x}} \, dx$$

88.
$$\frac{\ln(\sin^{-1} x)}{\sqrt{1-x^2}} \, dx$$

89.
$$\int_0^1 \frac{3+2x}{4+x^2} \, dx$$

90. Show that if $I_n = (\ln x)^n$ then

$$I_n = [x(\ln x)^n] - nI_{n-1}$$

91. Show that if $I_n = x^n \ln x$ then

$$I_n = \left[\frac{x^n \ln x}{n+1}\right] - \frac{1}{n+1} I_{n-1}$$

92. Show that if $I_n = \int \sin^n ax\, dx$ then

$$nI_n = -\frac{1}{a}\cos ax \sin^{n-1} ax + (n-1)I_{n-2}$$

93. Show that if $I_n = \int_0^{\pi/4} \tan^n x\, dx$ then

$$I_n = \frac{1}{n-1} I_{n-2}$$

94. Show that if $I_{n,m} = \int x^m (\ln x)^m\, dx$ then

$$I_{n,m} = \frac{x^{n+1}(\ln x)^m}{n+1} - \frac{m}{n+1} I_{n,m-1}$$

95. Show that if $I_n = \int \sec^n ax\, dx$ then

$$I_n = \left[\frac{1}{a}\frac{\sin ax}{\cos^{n-1} ax}\right] - (n-2)I_n + (n-2)I_{n-2}$$

96. Show that if $I_n = \int_0^1 (1-x^2)^n\, dx$ then

$$\left(1 + \frac{1}{2n}\right) I_n = I_{n-1}$$

97. Show that $I_n = \int_0^1 \frac{x^n}{\sqrt{ax+b}}\, dx$ then

$$(2n+1)\, I_n = \left[\frac{2x^n}{a}(ax+b)^{1/2}\right] - \frac{2bn}{a} I_n$$

98. Show that if $I_n = \int x^n \sqrt{ax+b}\, dx$ then

$$\left(1 + \frac{2n}{3}\right) I_n = \frac{2}{3a} x^n (ax+b)^{3/2} - \frac{2nb}{3a} I_{n-1}$$

99. Show that if $I_n = \frac{1}{(x^2+a^2)^n}\, dx$ then

$$I_n = \frac{x}{a^2\left(x^2+a^2\right)^{n-1}} + \frac{(2n-3)}{a^2} I_{n-1} - (2n-3)I_n$$

100. Show that if $I_n = \int x^n e^{ax}\, dx$ then

$$I_n = \frac{x^n e^{ax}}{a} - \frac{n}{a} I_{n-1}$$

Solutions

1. $$\int \frac{x}{x^2+3}\,dx$$

   Since derivative of x^2+3 is $2x$

$$\begin{aligned} &= \frac{1}{2}\int \frac{2x}{x^2+3}\,dx \\ &= \frac{1}{2}\ln|x^2+3| + C \end{aligned}$$

2. $$\int \frac{1}{x\ln x}\,dx$$

   Since derivative of $\ln x$ is $1/x$

$$\begin{aligned} &= \int \frac{1/x}{\ln x}\,dx \\ &= \ln\left|\,\ln|x|\,\right| + C \end{aligned}$$

3. $$\int \frac{x}{\sqrt{x^2+4}}\,dx$$

   Since derivative of x^2+4 is $2x$

$$\begin{aligned} &= \frac{1}{2}\int \frac{2x}{\sqrt{x^2+4}}\,dx \\ &= \frac{1}{2}\int 2x(x^2+4)^{-1/2}\,dx \\ &= \frac{1}{2}\left[\frac{(x^2+4)^{1/2}}{1/2}\right] \\ &= (x^2+4)^{1/2} + C \end{aligned}$$

4. $$\int \frac{3x-3}{x^2+3}\,dx$$

   Since derivative of x^2+3 is $2x$

$$\begin{aligned} &= \int \frac{3x}{x^2+3}\,dx - \int \frac{3}{x^2+3}\,dx \\ &= \frac{3}{2}\int \frac{2x}{x^2+3}\,dx - 3\int \frac{1}{x^2+3}\,dx \\ &= \frac{3}{2}\ln|x^2+3| - \frac{3}{\sqrt{3}}\tan^{-1}\frac{x}{\sqrt{3}} + C \end{aligned}$$

5.

$$\int \sin x \, \cos^2 x \; dx$$

Since derivative of $\cos x$ is $-\sin x$

$$= -1 \int -\sin x \, (\cos x)^2 \; dx$$
$$= -\frac{1}{3} \, (\cos x)^3 + C$$

6.

$$\int \sin x \, \sec^2 x \; dx$$

Since derivative of $\cos x$ is $-\sin x$

$$= -1 \int -\sin x \, (\cos x)^{-2} \; dx$$
$$= -\frac{(\cos x)^{-1}}{-1} + C$$
$$= \sec x + C$$

7.

$$\int \cos^2 \frac{x}{2} \; dx$$

Using $\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$

$$= \frac{1}{2} \int 1 + \cos(2 \times \frac{x}{2}) \; dx$$
$$= \frac{1}{2} \left[x + \sin x\right] + C$$

8.

$$\int x \sin x \; dx$$

Use integration by parts

$$\begin{Bmatrix} u = x & v' = \sin x \\ u' = 1 & v = -\cos x \end{Bmatrix}$$

$$\int uv' = [uv] - \int u'v \; dx$$
$$= -x \, \cos x - \int -\cos x \; dx$$
$$= -x \, \cos x + \sin x + C$$

9.

$$\int x \sec^2 x \; dx$$

Use integration by parts

$$\begin{Bmatrix} u = x & v' = \sec^2 x \\ u' = 1 & v \ = \tan x \end{Bmatrix}$$

$$\begin{aligned} \int uv' &= [uv] - \int u'v \; dx \\ &= x \; \tan x - \int \tan x \; dx \\ &= x \; \tan x - \frac{1}{-1} \int \frac{-\sin x}{\cos x} \; dx \\ &= x \; \tan x + \ln|\cos x| + C \end{aligned}$$

10.

$$\int \tan^{-1} 2x \; dx$$

Use integration by parts

$$\begin{Bmatrix} u = \tan^{-1} 2x & v' = 1 \\ u' = \dfrac{2}{4x^2+1} & v \ = x \end{Bmatrix}$$

$$\begin{aligned} \int uv' &= [uv] - \int u'v \; dx \\ &= x \; \tan^{-1} 2x - \int \frac{2x}{4x^2+1} \; dx \\ &= x \; \tan^{-1} 2x - \frac{1}{4} \int \frac{8x}{4x^2+1} \; dx \\ &= x \; \tan^{-1} 2x - \frac{1}{4} \ln|4x^2+1| + C \end{aligned}$$

11.

$$\int \frac{x^3}{x^2+1} \; dx$$

(§) Use polynomial division as numerator has higher degree than denominator

$$\begin{aligned} &= \int x + \frac{-x}{x^2+1} \; dx \\ &= \int x \; dx - \frac{1}{2} \int \frac{2x}{x^2+1} \; dx \\ &= \frac{x^2}{2} - \frac{1}{2} \ln|x^2+1| + C \end{aligned}$$

(§) Since

$$\begin{array}{r|l} & x \\ \hline x^2+1 & x^3 \\ & -x^3 - x \\ \hline & -x \end{array}$$

Meaning the fraction can be split

$$\frac{x^3}{x^2+1} \equiv x + \frac{-x}{x^2+1}$$

12.

$$\int \frac{x}{(x+1)(x+2)}\ dx$$

(§) Use partial fraction as numerator has lower degree than denominator

$$\begin{aligned} &= \int \frac{-1}{x+1} + \frac{2}{x+2}\ dx \\ &= -\ln|x+1| + 2\ln|x+2| + C \end{aligned}$$

(§) By letting

$$\begin{aligned} \frac{x}{(x+1)(x+2)} &\equiv \frac{A}{x+1} + \frac{B}{x+2} \\ &\equiv \frac{A(x+2)+B(x+1)}{(x+1)(x+2)} \\ &\equiv \frac{x(A+B)+(2A+B)}{(x+1)(x+2)} \end{aligned}$$

Compare coefficients

$$\{A+B=1 \qquad 2A+B=0\}$$

Solve simultaneously for each

$$\{A=-1 \qquad B=2\}$$

So the fraction can be split

$$\frac{x}{(x+1)(x+2)} \equiv \frac{-1}{x+1} + \frac{2}{x+2}$$

13.

$$\int \frac{(x-1)(x+1)}{(x-2)(x+3)}\ dx$$

Since numerator and denominator have same degree, re-express numerator in terms of denominator

$$\begin{aligned} &= \int \frac{x^2-1}{x^2+x-6}\ dx \\ &= \int \frac{(x^2+x-6)+(-x+6)-1}{x^2+x-6}\ dx \\ &= \int 1 + \frac{-x+5}{(x-2)(x+3)}\ dx \end{aligned}$$

(§) Use partial fraction since numerator has lower degree than denominator

$$= \int 1 + \frac{3/5}{x-2} + \frac{-8/5}{x+3}\ dx$$
$$= \int 1\ dx + \frac{3}{5}\int \frac{1}{x-2}\ dx$$
$$- \frac{8}{5}\int \frac{1}{x+3}\ dx$$
$$= x + \frac{3}{5}\ln|x-2| - \frac{8}{5}\ln|x+3| + C$$

(§) By letting

$$\frac{-x+5}{(x-2)(x+3)} \equiv \frac{A}{x-2} + \frac{B}{x+3}$$
$$\equiv \frac{A(x+3)+B(x-2)}{(x-2)(x+3)}$$
$$\equiv \frac{x(A+B)+(3A-2B)}{(x+1)(x+2)}$$

Compare coefficients

$$\{A+B=-1 \qquad 3A-2B=5\}$$

Solve simultaneously for each

$$\{A=3/5 \qquad B=-8/5\}$$

Meaning the fraction can be split

$$\frac{-x+5}{(x-2)(x+3)} \equiv \frac{3/5}{x-2} + \frac{-8/5}{x+3}$$

14.

$$\int \frac{2x-1}{x^2+2x+2}\ dx$$

Instead of partial fraction, since derivative of denominator is $2x+2$, it is similar to the numerator, re-express it in terms of $2x+2$

$$= \int \frac{(2x+2)-3}{x^2+2x+2}\ dx$$
$$= \int \frac{2x+2}{x^2+2x+2} + \frac{-3}{x^2+2x+2}\ dx$$
$$= \ln|x^2+2x+2| - 3\int \frac{1}{(x+1)^2+1}\ dx$$
$$= \ln|x^2+2x+2| - 3\tan^{-1}(x+1) + C$$

15.

$$\int \frac{x^3}{2x+1}\, dx$$

(§) Use polynomial division as numerator has higher degree than denominator

$$\begin{aligned}
&= \int \frac{1}{2}x^2 - \frac{1}{4}x + \frac{1}{8} + \frac{-1/8}{2x+1}\, dx \\
&= \frac{1}{6}x^3 - \frac{1}{8}x^2 + \frac{1}{8}x + \frac{-1/8}{2}\int \frac{2}{2x+1}\, dx \\
&= \frac{1}{6}x^3 - \frac{1}{8}x^2 + \frac{1}{8}x - \frac{1}{16}\ln|2x+1| + C
\end{aligned}$$

(§) Since

$$\begin{array}{r l}
 & \frac{1}{2}x^2 - \frac{1}{4}x + \frac{1}{8} \\ \cline{2-2}
2x+1 \big) & x^3 \\
 & -\, x^3 - \frac{1}{2}x^2 \\ \cline{2-2}
 & -\, \frac{1}{2}x^2 \\
 & \frac{1}{2}x^2 + \frac{1}{4}x \\ \cline{2-2}
 & \frac{1}{4}x \\
 & -\, \frac{1}{4}x - \frac{1}{8} \\ \cline{2-2}
 & -\, \frac{1}{8}
\end{array}$$

Meaning the fraction can be split

$$\frac{x^3}{2x+1} \equiv \frac{1}{2}x^2 - \frac{1}{4}x + \frac{1}{8} + \frac{-1/8}{2x+1}$$

16.

$$\int \frac{1-x}{\sqrt{1+x-x^2}}\, dx$$

Since the derivative of $1+x-x^2$ is $-2x+1$, re-express numerator in terms of $-2x+1$

$$\begin{aligned}
&= \frac{1}{2}\int \frac{-2x+2}{\sqrt{1+x-x^2}}\, dx \\
&= \frac{1}{2}\int \frac{(-2x+1)+1}{\sqrt{1+x-x^2}}\, dx \\
&= \frac{1}{2}\int \frac{-2x+1}{\sqrt{1+x-x^2}} + \frac{1}{\sqrt{1+x-x^2}}\, dx
\end{aligned}$$

Complete the square on quadratic expression inside square root, as the fraction often integrates to an inverse sine

$$\begin{aligned}
&= \frac{1}{2}\int (-2x+1)(1+x-x^2)^{-1/2}\,dx \\
&\quad + \frac{1}{2}\int \frac{1}{\sqrt{5/4-(x-1/2)^2}}\,dx \\
&= \frac{1}{2}\frac{(1+x-x^2)^{1/2}}{1/2} \\
&\quad + \frac{1}{2}\sin^{-1}\left(\frac{x-1/2}{\sqrt{5}/2}\right) + C \\
&= \sqrt{1+x-x^2} + \frac{1}{2}\sin^{-1}\left(\frac{2x-1}{\sqrt{5}}\right) + C
\end{aligned}$$

17.

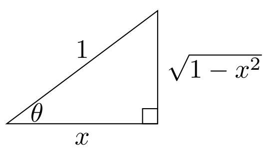
$$\int \frac{1}{x^2\sqrt{1-x^2}}\,dx$$

Use integration by substitution

$$\left\{ x = \cos\theta \qquad \frac{dx}{d\theta} = -\sin\theta \right\}$$

$$\begin{aligned}
&= \int \frac{1}{\cos^2\theta\sqrt{1-\cos^2\theta}} \times -\sin\theta\,d\theta \\
&= \int \frac{-1}{\cos^2\theta\sin\theta} \times -\sin\theta d\theta \\
&= \int -\sec^2\theta\,dx \\
&= -\tan\theta + C
\end{aligned}$$

Since $\cos\theta = \frac{x}{1}$, draw a right angle triangle with adjacent side x and hypotenuse 1, then use Pythagoras theorem to calculate its opposite side



So $\tan\theta = \frac{\sqrt{1-x^2}}{x}$, then substitute it

$$= -\frac{\sqrt{1-x^2}}{x} + C$$

18.

$$\int \frac{1}{a^2-x^2}\,dx \qquad a \text{ is constant}$$

(§) Factorise denominator then use partial fractions as numerator has lower degree than denominator

$$
\begin{aligned}
&= \int \frac{1}{(a-x)(a+x)}\, dx \\
&= \int \frac{1/2a}{a+x} + \frac{1/2a}{a-x}\, dx \\
&= \int \frac{1}{2a}\frac{1}{a+x}\, dx - \frac{1}{2a}\int \frac{-1}{a-x}\, dx \\
&= \frac{1}{2a}\ln|a+x| - \frac{1}{2a}\ln|a-x| + c
\end{aligned}
$$

(§) By letting

$$
\begin{aligned}
\frac{1}{(a-x)(a+x)} &\equiv \frac{A}{a-x} + \frac{B}{a+x} \\
&\equiv \frac{A(a+x)+B(a-x)}{(a-x)(a+x)} \\
&\equiv \frac{x(A+B)+a(A-B)}{(a-x)(a+x)}
\end{aligned}
$$

Compare coefficients

$$\left\{A+B=0 \qquad A-B=\frac{1}{a}\right\}$$

Solve simultaneously for each

$$\left\{A=\frac{1}{2a} \qquad B=\frac{1}{2a}\right\}$$

Meaning the fraction can be split

$$\frac{1}{(a-x)(a+x)} \equiv \frac{1/2a}{a-x} + \frac{1/2a}{a+x}$$

19.

$$\int \frac{1}{x(x^2-a^2)}\, dx$$

(§) Use partial fractions as numerator has lower degree than denominator

$$
\begin{aligned}
&= \int \frac{-1/a^2}{x} + \frac{(1/a^2)x+0}{x^2-a^2}\, dx \\
&= -\frac{1}{a^2}\int \frac{1}{x}\, dx + \frac{1}{2a^2}\int \frac{2x}{x^2-a^2}\, dx \\
&= -\frac{1}{a^2}\ln|x| + \frac{1}{2a^2}\ln|x^2-a^2| + C
\end{aligned}
$$

(§) By letting

$$\begin{aligned}\frac{1}{x(x^2-a^2)} &\equiv \frac{A}{x} + \frac{Bx+C}{x^2-a^2} \\ &\equiv \frac{A(x^2-a^2)+(Bx+C)x}{x(x^2-a^2)} \\ &\equiv \frac{x^2(A+B)+x(C)+a^2(-A)}{x(x-a^2)}\end{aligned}$$

Compare coefficients

$$\left\{A+B=0 \qquad -A=\frac{1}{a^2} \qquad C=0\right\}$$

Solve simultaneously for each

$$\left\{A=-\frac{1}{a^2} \qquad B=\frac{1}{a^2} \qquad C=0\right\}$$

Meaning the fraction can be split

$$\frac{1}{x(x^2-a^2)} \equiv \frac{-1/a^2}{x} + \frac{(1/a^2)x+0}{x^2-a^2}$$

20.

$$\int \frac{u}{\sqrt{u}+1}\,du$$

Use integration by substitution

$$\left\{u=x^2 \qquad \frac{du}{dx}=2x\right\}$$

$$\begin{aligned} &= \int \frac{x^2}{x+1} \times 2x\;dx \\ &= \int \frac{2x^3}{x+1}\;dx\end{aligned}$$

(§) Use polynomial division as numerator has higher degree than denominator

$$\begin{aligned} &= \int 2x^2-2x+2+\frac{-2}{x+1}\;dx \\ &= \frac{2x^3}{3}-x^2+2x-2\ln|x+1|+C\end{aligned}$$

Since $u=x^2$, substitute x terms back into u terms using $x=u^{1/2}$

$$= \frac{2}{3}u^{3/2}-u+2u^{1/2}-2\ln|u^{1/2}+1|+C$$

(§) Since

$$\begin{array}{r}
2x^2 - 2x + 2 \\
x+1 \overline{\big) \; 2x^3 \qquad\qquad\qquad} \\
\underline{-2x^3 - 2x^2 \qquad\quad} \\
-2x^2 \qquad\quad \\
\underline{2x^2 + 2x \quad} \\
2x \quad \\
\underline{-2x - 2} \\
-2
\end{array}$$

Meaning the fraction can be split

$$\frac{2x^3}{x+1} = (2x^2 - 2x + 2) + \frac{-2}{x+1}$$

21.

$$\int \frac{\sin^{-1} x}{\sqrt{1-x^2}}\, dx$$

Use integration by substitution

$$\left\{ u = \sin^{-1} x \qquad \frac{du}{dx} = \frac{1}{\sqrt{1-x^2}} \right\}$$

$$\begin{aligned}
&= \int \sin^{-1} x \frac{1}{\sqrt{1-x^2}}\, dx \\
&= \int u\, du \\
&= \frac{1}{2} u^2 + C \\
&= \frac{1}{2} \left(\sin^{-1} x\right)^2 + C
\end{aligned}$$

22.

$$\int \frac{1}{x(\ln x)^3}\, dx$$

Use integration by substitution

$$\left\{ u = \ln x \qquad \frac{du}{dx} = \frac{1}{x} \right\}$$

$$\begin{aligned}
&= \int \frac{1}{(\ln x)^3} \frac{1}{x}\, dx \\
&= \int \frac{1}{u^3}\, du \\
&= \int u^{-3}\, du \\
&= \frac{1}{-2} u^{-2} + C
\end{aligned}$$

Substitute u terms back into x terms

$$= -\frac{1}{2}(\ln x)^{-2} + C$$

23.

$$\int \sec^4 2x \; dx$$

$$\begin{aligned} &= \int \left(\sec^2 2x\right)\left(\sec^2 2x\right) \quad dx \\ &= \int \left(1 + \tan^2 2x\right)\left(\sec^2 2x\right) \; dx \end{aligned}$$

Use integration by substitution

$$\left\{ u = \tan 2x \qquad \frac{du}{dx} = 2\sec^2 2x \right\}$$

$$\begin{aligned} &= \int (1 + u^2) \times \frac{1}{2} \; du \\ &= \frac{1}{2}u + \frac{1}{6}u^3 + C \\ &= \frac{1}{2}\tan 2x + \frac{1}{6}\tan^3 2x + C \end{aligned}$$

24.

$$\int \frac{1}{x^2(x+1)} \; dx$$

(§) Use partial fractions as numerator has lower degree than denominator

$$\begin{aligned} &= \frac{1}{x^2} + \int \frac{-1}{x} + \frac{1}{x+1} \; dx \\ &= -\frac{1}{x} - \ln x + \ln|x+1| + C \end{aligned}$$

(§) By letting

$$\begin{aligned} \frac{1}{x^2(x+1)} &\equiv \frac{A}{x^2} + \frac{B}{x} + \frac{C}{x+1} \\ &\equiv \frac{A(x+1) + B(x+1)x + Cx^2}{x^2(x+1)} \\ &\equiv \frac{(B+C)x^2 + (A+B)x + A}{x^2(x+1)} \end{aligned}$$

Compare coefficients

$$\{C + B = 0 \qquad A + B = 0 \qquad A = 1\}$$

Solve simultaneously for each

$$\{A = 1 \qquad B = -1 \qquad C = 1\}$$

Meaning the fraction can be split

$$\frac{1}{x^2(x+1)} \equiv \frac{1}{x^2} + \frac{-1}{x} + \frac{1}{x+1}$$

25.

$$\int \frac{1}{x^2(x^2+1)}\ dx$$

(§) Use partial fractions as numerator has lower degree than denominator

$$\begin{aligned} &= \int \frac{1}{x^2} - \frac{1}{x^2+1}\ dx \\ &= -\frac{1}{x} - \tan^{-1} x + C \end{aligned}$$

(§) By letting

$$\frac{1}{x^2(x^2+1)} \equiv \frac{A}{x^2} + \frac{B}{x} + \frac{Cx+D}{x^2+1}$$

Multiply both sides by $x^2(x^2+1)$

$$\begin{aligned} &\equiv A(x^2+1) + B(x^3+x) + (Cx+D)x^2 \\ &\equiv (B+C)x^3 + (A+D)x^2 + Bx + A \end{aligned}$$

Compare coefficients

$$\begin{Bmatrix} B + C = 0 & \qquad B = 0 \\ A + D = 0 & \qquad C = 1 \end{Bmatrix}$$

Solving simultaneously for each

$$\{A = 1 \quad B = 0 \quad C = 0 \quad D = -1\}$$

Meaning the fraction can be split

$$\frac{1}{x^2(x+1)} \equiv \frac{1}{x^2} + \frac{-1}{x^2+1}$$

26.

$$\int \frac{1}{(x^2+1)^2}\ dx$$

Use integration by substitution

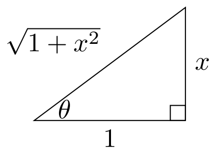
$$\left\{x = \tan\theta \qquad \frac{dx}{d\theta} = \sec^2\theta\right\}$$

$$= \int \frac{1}{(\tan^2\theta + 1)^2} \sec^2 x \; d\theta$$
$$= \int \frac{1}{(\sec^2\theta)^2} \sec^2\theta \; d\theta$$
$$= \int \cos^2\theta \; d\theta$$

Using trig identity $\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$ and $\sin 2\theta = 2\sin\theta\cos\theta$

$$= \int \frac{1}{2} + \frac{1}{2}\cos 2\theta \; d\theta$$
$$= \frac{1}{2}\theta + \frac{1}{4}\sin 2\theta + C$$
$$= \frac{1}{2}\theta + \frac{1}{4} 2\sin\theta\cos\theta + C$$

Since $\tan\theta = \frac{x}{1}$, draw a right angle triangle with opposite side x and adjacent side 1 and use Pythagoras theorem to calculate its hypotenuse



Therefore $\sin\theta = \frac{x}{\sqrt{1+x^2}}$ and $\cos\theta = \frac{1}{\sqrt{1+x^2}}$, then substitute them

$$= \frac{1}{2}\tan^{-1} x + \frac{1}{2}\frac{x}{\sqrt{1+x^2}}\frac{1}{\sqrt{1+x^2}} + C$$
$$= \frac{1}{2}\left(\tan^{-1} x + \frac{x}{1+x^2}\right) + C$$

27.

$$\int \tan^2 x \; dx$$

Since $1 + \tan^2\theta = \sec^2 x$

$$= \int \sec^2 x - 1 \; dx$$
$$= \tan x - x + C$$

28.

$$\int \frac{\sin x}{3 + \cos x} \; dx$$

Since derivative of $3 + \cos x$ is $-\sin x$

$$= \frac{1}{-1}\int \frac{-\sin x}{3 + \cos x} \; dx$$
$$= -\ln|3 + \cos x| + C$$

29.

$$\int \frac{1}{1+\cos^2 x}\, dx$$

Use $\cos^2 x = \frac{1}{2}(1+\cos 2x)$

$$= \int \frac{1}{1+\frac{1}{2}(1+\cos 2x)}\, dx$$
$$= \int \frac{2}{3+\cos 2x}\, dx$$

Use modified t-method where $t = \tan x$ and

$$\left\{\cos 2x = \frac{1-t^2}{1+t^2} \qquad \frac{dx}{dt} = \frac{1}{1+t^2}\right\}$$

$$= \int \frac{2}{3+\left(\frac{1-t^2}{1+t^2}\right)} \times \frac{1}{1+t^2}\, dt$$
$$= \int \frac{2}{3+3t^2+1-t^2}\, dt$$
$$= \int \frac{1}{2+t^2}\, dt$$
$$= \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{t}{\sqrt{2}}\right) + C$$

Substitute back into x terms using $t = \tan x$

$$= \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{1}{\sqrt{2}}\tan x\right) + C$$

30.

$$\int \frac{1}{10+8\cos x}\, dx$$

Use t-method where $t = \tan\frac{x}{2}$ and

$$\left\{\cos x = \frac{1-t^2}{1+t^2} \qquad \frac{dx}{dt} = \frac{2}{1+t^2}\right\}$$

$$= \int \frac{1}{10+8\frac{1-t^2}{1+t^2}} \times \frac{2}{1+t^2}\, dt$$
$$= \int \frac{2}{10(1+t^2)+8(1-t^2)}\, dt$$
$$= \int \frac{1}{9+t^2}\, dt$$
$$= \frac{1}{3}\tan^{-1}\left(\frac{t}{3}\right) + C$$

Substitute back into x terms using $t = \tan x$

$$= \frac{1}{3}\tan^{-1}\left(\frac{1}{3}\tan\frac{x}{2}\right) + C$$

31.

$$\int \frac{1}{8+10\cos x}\, dx$$

Use t-method where $t = \tan\frac{x}{2}$ and

$$\left\{ \cos x = \frac{1-t^2}{1+t^2} \qquad \frac{dx}{dt} = \frac{2}{1+t^2} \right\}$$

$$\begin{aligned}
&= \int \frac{1}{8+10\left(\frac{1-t^2}{1+t^2}\right)} \times \frac{2}{1+t^2}\, dt \\
&= \int \frac{2}{8(1+t^2)+10(1-t^2)}\, dt \\
&= \int \frac{1}{9-t^2}\, dt
\end{aligned}$$

(§) Use partial fractions as numerator has lower degree than denominator

$$\begin{aligned}
&= \int \frac{1/6}{3+t} + \frac{1/6}{3-t}\, dt \\
&= \frac{1}{6}\ln|3+t| - \frac{1}{6}\ln|3-t| + C
\end{aligned}$$

Substitute back into x terms using $t = \tan\frac{x}{2}$

$$= \frac{1}{6}\ln\left|3+\tan\frac{x}{2}\right| - \frac{1}{6}\ln\left|3-\tan\frac{x}{2}\right| + C$$

(§) By letting

$$\begin{aligned}
\frac{1}{9-t^2} &\equiv \frac{A}{3+t} + \frac{B}{3-t} \\
&\equiv \frac{A(3-t)+B(3+t)}{(3+t)(3-t)} \\
&\equiv \frac{(A+B)t+(3B-3A)}{x^2(x+1)}
\end{aligned}$$

Compare coefficients

$$\{B - A = 0 \qquad 3B + 3A = 1\}$$

Solve simultaneously for each

$$\{A = 1/6 \qquad B = 1/6\}$$

Meaning the fraction can be split

$$\frac{1}{9-t^2} \equiv \frac{1/6}{3+t} + \frac{1/6}{3-t}$$

32.

$$\int \frac{\sin x}{8 + 10\cos x}\ dx$$

Since derivative of $8 + 10\cos x$ is $-10\sin x$

$$= \frac{1}{-10}\int \frac{-10\sin x}{8 + 10\cos x}\ dx$$
$$= -\frac{1}{10}\ln|8 + 10\cos x| + C$$

33.

$$\int \cos^2 x - \sin^2 x\ dx$$

Using $\cos 2x = \cos^2 x - \sin^2 x$

$$= \int \cos 2x\ dx$$
$$= -\frac{1}{2}\sin 2x + C$$

34.

$$\int x^2 \sin x\ dx$$

Integration by parts can be applied twice in a question, especially if it has $\sin x$ or $\cos x$

$$\left\{\begin{matrix} u = x^2 & v' = \sin x \\ u' = 2x & v = -\cos x \end{matrix}\right\}$$

$$\int uv'\ dx = [uv] - \int u'v\ dx$$
$$= \left[x^2 \sin x\right] - \int -2x\cos x\ dx$$
$$= x^2 \sin x + \int 2x \cos x\ dx$$

Reapply integration by part on the integral

$$\left\{\begin{matrix} u = 2x & v' = \cos x \\ u' = 2 & v = \sin x \end{matrix}\right\}$$

$$= x^2 \sin x + [2x\sin x] - \int 2\sin x\ dx$$
$$= x^2 \sin x + 2x\sin x + 2\cos x + C$$

35.

$$\int \frac{x^2}{(x+1)(x+2)(x+3)}\ dx$$

(§) Use partial fractions as numerator has lower degree than denominator

$$= \int \frac{1/2}{x+1} + \frac{-4}{x+2} + \frac{9/2}{x+3}\, dx$$
$$= \frac{1}{2}\ln|x+1| - 4\ln|x+2|$$
$$+ \frac{9}{2}\ln|x+3| + C$$

(§) By letting

$$\frac{x^2}{(x+1)(x+2)(x+3)} \equiv \frac{A}{x+1} + \frac{B}{x+2} + \frac{C}{x+3}$$

Multiply both sides by $(x+1)(x+2)(x+3)$

$$x^2 \equiv A(x+2)(x+3) + B(x+1)(x+3) + C(x+1)(x+2)$$

Substitute $x = -1$ to get

$$1 \equiv 2A + 0B + 0C$$

Substitute $x = -2$ to get

$$4 \equiv 0A - B + 0C$$

Substitute $x = -3$ to get

$$9 \equiv 0A + 0B + 2C$$

Solving simultaneously for each

$$\{A = 1/2 \qquad B = -4 \qquad C = 9/2\}$$

Meaning fraction can be split

$$\frac{x^2}{(x+1)(x+2)(x+3)} \equiv \frac{1/2}{x+1} + \frac{-4}{x+2} + \frac{9/2}{x+3}$$

36.

$$\int \frac{e^x}{e^x+1}\, dx$$

Since derivative of $e^x + 1$ is e^x

$$= \ln|e^x + 1| + C$$

37.

$$\int \frac{1}{2\sin^2 x + 18\cos^2 x}\, dx$$

Divide all terms by $\cos^2 x$

$$\begin{aligned} &= \int \frac{\sec^2 x}{2\tan^2 x + 18}\, dx \\ &= \frac{1}{2}\int \frac{1}{\tan^2 x + 9} \times \sec^2 x\, dx \end{aligned}$$

Use integration by substitution

$$\left\{ u = \tan x \qquad \frac{du}{dx} = \sec^2 x \right\}$$

$$\begin{aligned} &= \frac{1}{2}\int \frac{1}{u^2 + 9}\, du \\ &= \frac{1}{2}\left[\frac{1}{3}\tan^{-1}\left(\frac{u}{3}\right)\right] + C \end{aligned}$$

Substitute $u = \tan x$

$$= \frac{1}{6}\tan^{-1}\left(\frac{1}{3}\tan x\right) + C$$

38.

$$\int x^2\, e^{3x^3-5}\, dx$$

Since derivative of $3x^3 - 5$ is $9x^2$

$$\begin{aligned} &= \frac{1}{9}\int 9x^2\, \left(e^{3x^3-5}\right)\, dx \\ &= \frac{1}{9}e^{3x^3-5} + C \end{aligned}$$

39.

$$\int x^3 \ln x\, dx$$

Use integration by parts

$$\left\{ \begin{array}{ll} u = \ln x & v' = x^3 \\ u' = \dfrac{1}{x} & v = \dfrac{1}{4}x^4 \end{array} \right\}$$

$$\begin{aligned} \int uv'\, dx &= [uv] - \int u'v\, dx \\ &= \left[\frac{1}{4}x^4 \ln x\right] - \frac{1}{4}\int x^3\, dx \\ &= \frac{1}{4}x^4 \ln x - \frac{1}{16}x^4 + C \end{aligned}$$

40.

$$\int \ln x^3 \ dx$$

Use log law $\ln x^n = n \ln x$

$$= \int 3 \ln x \ dx$$

Use integration by parts

$$\begin{Bmatrix} u = \ln x & v' = 3 \\ u' = \dfrac{1}{x} & v = 3x \end{Bmatrix}$$

$$\begin{aligned} \int uv' \ dx &= [uv] - \int u'v \ dx \\ &= [3x \ln x] - \int 3 \ dx \\ &= 3x \ln x - 3x + C \end{aligned}$$

41.

$$\int \log_2 x^3 \ dx$$

Use log law $\log_2 x^n = n \log_2 x$

$$= 3 \int \log_2 x \ dx$$

Use change of base formula $\log_a b = \frac{\ln a}{\ln b}$

$$\begin{aligned} &= 3 \int \frac{\ln x}{\ln 2} \ dx \\ &= \int \frac{3}{\ln 2} \ln x \ dx \end{aligned}$$

Let:

$$\begin{Bmatrix} u = \ln x & v' = \dfrac{3}{\ln 2} \\ u' = \dfrac{1}{x} & v = \dfrac{3}{\ln 2} x \end{Bmatrix}$$

$$\begin{aligned} \int uv' \ dx &= [uv] - \int u'v \ dx \\ &= \left[\frac{3}{\ln 2} x \ln x\right] - \int \frac{3}{\ln 2} \ dx \\ &= \left[\frac{3}{\ln 2} x \ln x\right] - \frac{3}{\ln 2} x + C \end{aligned}$$

42.

$$\int \frac{1}{e^x + e^{-x}}\ dx$$

With exponentials, making all having positive powers makes integration easier

$$= \int \frac{1}{e^x + e^{-x}} \times \frac{e^x}{e^x}\ dx$$

$$= \int \frac{1}{e^{2x} + 1} \times e^x\ dx$$

Use integration by substitution

$$\left\{ u = e^x \qquad \frac{du}{dx} = e^x \right\}$$

$$= \int \frac{1}{u^2 + 1}\ du$$

$$= \tan^{-1} u + C$$

Substitute back into x terms using $u = e^x$

$$= \tan^{-1}(e^x) + C$$

43.

$$\int \left(4x^2 + 5x - 1\right)^{3/2} (8x + 5)\ dx$$

Using integration by substitution

$$\left\{ u = 4x^2 + 5x - 1 \qquad \frac{du}{dx} = 8x + 5 \right\}$$

$$= \int u^{3/2}\ du$$

$$= \frac{u^{5/2}}{5/2} + C$$

$$= \frac{2}{5}\left(4x^2 + 5x - 1\right)^{5/2} + C$$

44.

$$\int \frac{1}{(x^2 + 1)(x^2 + 4)}\ dx$$

(§) Use partial fractions as numerator has lower degree than denominator

$$= \int \frac{-1/3}{x^2 + 1} + \frac{1/3}{x^2 + 4}\ dx$$

$$= -\frac{1}{3}\int \frac{1}{x^2 + 1}\ dx + \frac{1}{3}\int \frac{1}{x^2 + 4}\ dx$$

$$= -\frac{1}{3}\tan^{-1} x + \frac{1}{3} \times \frac{1}{2}\tan^{-1}\frac{x}{2} + C$$

$$= -\frac{1}{3}\tan^{-1} x + \frac{1}{6}\tan^{-1}\frac{x}{2} + C$$

(§) By letting

$$\frac{1}{(x^2+1)(x^2+4)} \equiv \frac{Ax+B}{x^2+1} + \frac{Cx+D}{x^2+4}$$

Multiply both sides by $(x^2+1)(x^2+4)$

$$\begin{aligned} 1 &\equiv (Ax+B)(x^2+1) \\ &\quad + (Cx+D)(x^2+4) \\ 1 &\equiv (A+C)x^3 + (B+D)x^2 \\ &\quad + (A+4C)x + (B+4D) \end{aligned}$$

Comparing coefficients

$$\left\{ \begin{aligned} A+C &= 0 & B+D &= 0 \\ A+4C &= 0 & B+4D &= 1 \end{aligned} \right\}$$

Solving simultaneously for each

$$\left\{ A = 0 \quad B = -\frac{1}{3} \quad C = 0 \quad D = \frac{1}{3} \right\}$$

Meaning the fraction can be split

$$\frac{1}{(x^2+1)(x^2+4)} \equiv \frac{-1/3}{x^2+1} + \frac{1/3}{x^2+4}$$

45.

$$\int \frac{1}{x^2+x+1}\, dx$$

Complete the square on the denominator, as it may integrate into an inverse tan or logs

$$= \int \frac{1}{(x+1/2)^2+3/4}\, dx$$

Use integration by substitution

$$\left\{ u = x + \frac{1}{2} \qquad \frac{du}{dx} = 1 \right\}$$

$$\begin{aligned} &= \int \frac{1}{u^2+(\sqrt{3}/2)^2}\, du \\ &= \frac{1}{\sqrt{3}/2} \tan^{-1}\left(\frac{u}{\sqrt{3}/2}\right) + C \end{aligned}$$

Substitute back into x terms using $u = x + \frac{1}{2}$

$$= \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + C$$

46.

$$\int \frac{1}{x^2+x-1}\,dx$$

Complete the square on the denominator, as it may integrate into an inverse tan or logs

$$= \int \frac{1}{(x+1/2)^2-5/4}\,dx$$

Use integration by substitution

$$\left\{u = x+\frac{1}{2} \qquad \frac{du}{dx} = 1\right\}$$

$$\begin{aligned}
&= \int \frac{1/\sqrt{5}}{u+\sqrt{5}/2} + \frac{1/\sqrt{5}}{u-\sqrt{5}/2}\,du \\
&= \frac{1}{\sqrt{5}}\int \frac{1}{u+\sqrt{5}/2}\,du \\
&\quad + \frac{1}{\sqrt{5}}\int \frac{1}{u-\sqrt{5}/2}\,du \\
&= \frac{1}{\sqrt{5}}\ln\left|u+\frac{\sqrt{5}}{2}\right| \\
&\quad - \frac{1}{\sqrt{5}}\ln\left|u-\frac{\sqrt{5}}{2}\right| + C
\end{aligned}$$

Substitute back into x terms using $u = x+\frac{1}{2}$

$$\begin{aligned}
&= \frac{1}{\sqrt{5}}\ln\left|x+\frac{1}{2}+\frac{\sqrt{5}}{2}\right| \\
&\quad - \frac{1}{\sqrt{5}}\ln\left|x+\frac{1}{2}-\frac{\sqrt{5}}{2}\right| + C
\end{aligned}$$

47.

$$\int e^x \sin x\,dx$$

Integration by parts can be applied twice in a question, especially if it has $\sin x$ or $\cos x$

$$\left\{\begin{matrix} u = e^x & v' = \sin x \\ u' = e^x & v = -\cos x \end{matrix}\right\}$$

$$\int uv'\,dx = [uv] - \int u'v\,dx$$

$$\int e^x \sin x\,dx = -e^x\cos x + \int e^x \cos x\,dx$$

(§) Apply integration by parts on integral

$$\int e^x \sin x \, dx =$$

$$-e^x \cos x + e^x \sin x - \int e^x \sin x \, dx$$

Combine the integrals

$$2 \int e^x \sin x \, dx = -e^x \cos x + e^x \sin x$$
$$\int e^x \sin x \, dx = \frac{1}{2} e^x (\sin x - \cos x) + C$$

(§) By letting

$$\begin{Bmatrix} u = e^x & v' = \cos x \\ u' = e^x & v = \sin x \end{Bmatrix}$$

The integral becomes

$$\int e^x \cos x \, dx = e^x \sin x - \int e^x \sin x \, dx$$

48.

$$\int \frac{1}{\sqrt{x^2 - x}} \, dx$$

Complete the square on the denominator

$$= \int \frac{1}{\sqrt{(x - \frac{1}{2})^2 - \frac{1}{4}}} \, dx$$

Use integration by substitution

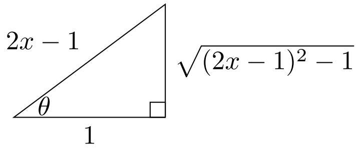
$$\begin{Bmatrix} x = \frac{1}{2} \sec\theta + \frac{1}{2} & \frac{dx}{d\theta} = \frac{1}{2} \frac{\tan\theta}{\cos\theta} \end{Bmatrix}$$

$$= \int \frac{1}{\sqrt{\frac{1}{4} \sec^2\theta - \frac{1}{4}}} \times \frac{1}{2} \frac{\tan\theta}{\cos\theta} \, d\theta$$
$$= \int \frac{1}{\frac{1}{2} \tan\theta} \times \frac{1}{2} \frac{\tan\theta}{\cos\theta} \, d\theta$$
$$= \int \sec\theta \, d\theta$$

(§) Since $\int \sec\theta \, d\theta = \ln|\sec\theta + \tan\theta| + C$

$$= \ln|\sec\theta + \tan\theta| + C$$

Since $\sec\theta = \frac{2x-1}{1}$, draw a right angle triangle with hypotenuse side $2x - 1$ and adjacent side 1 and use Pythagoras theorem to calculate its opposite side



Which gives

$$\tan\theta = \sqrt{(2x-1)^2-1}$$

Substitute back into equation

$$= \ln|2x-1+\sqrt{4x^2-4x+1-1}|+C$$
$$= \ln|2x-1+2\sqrt{x(x-1)}|+C$$

(§) Refer to question 71 showing

$$\int \sec\theta\, d\theta = \ln|\sec\theta+\tan\theta|+C$$

49.

$$\int \frac{1+2x}{2+x}\,dx$$

Since numerator and denominator have same degree, re-express numerator in terms of denominator

$$= \int \frac{2(2+x)-3}{2+x}\,dx$$
$$= \int 2-\frac{3}{2+x}\,dx$$
$$= 2x-3\ln|2+x|+C$$

50.

$$I = \int \frac{x^2}{\sqrt{x^2+4}}\,dx$$

Use Integral by parts

$$\left\{\begin{matrix} u = x & v' = \dfrac{x}{\sqrt{x^2+4}} \\ u' = 1 & v = (x^2+4)^{1/2} \end{matrix}\right\}$$

$$\int uv'\,dx = [uv] - \int u'v\,dx$$

$$I = x\sqrt{x^2+4} - \int \sqrt{x^2+4}\,dx$$
$$I = x\sqrt{x^2+4} - \int \frac{x^2+4}{\sqrt{x^2+4}}\,dx$$
$$I = x\sqrt{x^2+4} - \int \frac{x^2}{\sqrt{x^2+4}}\,dx$$
$$- \int \frac{4}{\sqrt{x^2+4}}\,dx$$

Since $I = \int \frac{x^2}{\sqrt{x^2+4}}\, dx$

$$I = x\sqrt{x^2+4} - I - \int \frac{4}{\sqrt{x^2+4}}\, dx$$
$$2I = x\sqrt{x^2+4} - \int \frac{4}{\sqrt{x^2+4}}\, dx$$

Use integration by substitution

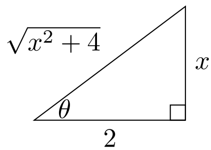
$$\left\{ x = 2\tan\theta \qquad \frac{dx}{d\theta} = 2\sec^2\theta \right\}$$

$$2I = x\sqrt{x^2+4} - \int \frac{4 \times 2\sec^2\theta}{\sqrt{4\tan^2\theta + 4}}\, d\theta$$
$$I = \frac{x}{2}\sqrt{x^2+4} - 2\int \sec\theta\; d\theta$$

(§) Since $\int \sec\theta\; d\theta = \ln|\sec\theta + \tan\theta| + C$

$$I = \frac{x}{2}\sqrt{x^2+4} - 2\ln|\sec\theta + \tan\theta| + C$$

Since $\tan\theta = \frac{x}{2}$, draw a right angle triangle with opposite side x and adjacent side 2 and use Pythagoras theorem to calculate its hypotenuse side



Which gives

$$\sec\theta = \frac{\sqrt{x^2+4}}{2}$$

Substitute back into equation

$$I = \frac{x}{2}\sqrt{x^2+4} - 2\ln\left|\frac{1}{2}\sqrt{x^2+4} + \frac{x}{2}\right|$$

(§) Refer to question 71 showing

$$\int \sec\theta\; d\theta = \ln|\sec\theta + \tan\theta| + C$$

51.

$$\int \frac{\sin x}{3\cos^2 x + 2\sin^2 x}\, dx$$

$$= \int \frac{\sin x}{\cos^2 x + 2\cos^2 x + 2\sin^2 x}\ dx$$
$$= \int \frac{\sin x}{\cos^2 x + 2}\ dx$$
$$= \frac{1}{-1}\int \frac{1}{\cos^2 x + 2} \times -\sin x\ dx$$

Use integration by substitution

$$\left\{ u = \cos x \qquad \frac{du}{dx} = -\sin x \right\}$$

$$= -\int \frac{1}{u^2+2}\ dx$$
$$= -\frac{1}{\sqrt{2}}\tan^{-1}\frac{u}{\sqrt{2}} + C$$

Substitute back into x terms using $u = \cos x$

$$= -\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{1}{\sqrt{2}}\cos x\right) + C$$

52.

$$\int \frac{x^3}{1-x^4}\ dx$$

Since derivative of $1-x^4$ is $-4x^3$

$$= \frac{1}{-4}\int \frac{-4x^3}{1-x^4}\ dx$$
$$= -\frac{1}{4}\ln|1-x^4| + C$$

53.

$$\int \frac{1}{\sin x \cos x}\ dx$$

Since: $\sin 2x = 2\sin x \cos x$

$$= \int 2 \times \frac{1}{\sin 2x}\ dx$$

Use modified t-method where $t = \tan x$ and

$$\left\{ \sin 2x = \frac{2t}{1+t^2} \qquad \frac{dx}{dt} = \frac{1}{1+t^2} \right\}$$

$$= \int 2\ \frac{1+t^2}{2t}\ \frac{1}{1+t^2}\ dt$$
$$= \int \frac{1}{t}\ dt$$
$$= \ln|t| + C$$

Substitute back into x terms using $t = \tan x$

$$= \ln|\tan x| + C$$

54.

$$\int \ln \sqrt{x+1}\ dx$$

Use log law where $\ln a^n = n \ln a$

$$= \int \ln(x+1)^{1/2}\ dx$$
$$= \int \frac{1}{2} \ln(x+1)\ dx$$

Use Integral by parts

$$\begin{Bmatrix} u = \ln(x+1) & v' = \dfrac{1}{2} \\ u' = \dfrac{1}{x+1} & v = \dfrac{1}{2}x \end{Bmatrix}$$

$$\begin{aligned}
&= [uv] - \int u'v\ dx \\
&= \frac{1}{2}x\ln(x+1) - \int \frac{x/2}{x+1}\ dx \\
&= \frac{1}{2}x\ln(x+1) - \frac{1}{2}\int \frac{(x+1)-1}{x+1}\ dx \\
&= \frac{1}{2}x\ln(x+1) - \frac{1}{2}\int 1 + \frac{-1}{x+1}\ dx \\
&= \frac{1}{2}x\ln(x+1) - \frac{1}{2}x + \frac{1}{2}\ln|x+1| + C
\end{aligned}$$

55.

$$\int \frac{1}{e^x+1}\ dx$$

Since derivative of an exponential function is also an exponential function

$$\begin{aligned}
&= \int \frac{1}{e^x+1} \times \frac{e^{-x}}{e^{-x}}\ dx \\
&= \frac{1}{-1}\int \frac{1}{1+e^{-x}} \times -e^{-x}\ dx
\end{aligned}$$

Use integration by substitution

$$\left\{ u = e^{-x} \qquad \frac{du}{dx} = -e^{-x} \right\}$$

$$\begin{aligned}
&= -\int \frac{1}{u+1}\ du \\
&= -\ln|u+1| + C
\end{aligned}$$

Substitute back into x terms using $u = e^{-x}$

$$= -\ln|e^{-x}+1| + C$$

56.

$$\int \frac{\sec^2 x}{\tan^2 x - 4\tan x - 5}\ dx$$

Use integration by substitution

$$\left\{ u = \tan x \qquad \frac{du}{dx} = \sec^2 x \right\}$$

$$\begin{aligned} &= \int \frac{1}{u^2 - 4u - 5}\ du \\ &= \int \frac{1}{(u-5)(u+1)}\ du \end{aligned}$$

(§) Use partial fraction to split the fraction

$$\begin{aligned} &= \int \frac{1/6}{u-5} + \frac{-1/6}{u+1}\ du \\ &= \frac{1}{6}\ln|u-5| - \frac{1}{6}\ln|u+1| + C \\ &= \frac{1}{6}\ln|\tan x - 5| - \frac{1}{6}\ln|\tan x + 1| + C \end{aligned}$$

(§) By letting

$$\begin{aligned} \frac{1}{(u-5)(u+1)} &\equiv \frac{A}{u-5} + \frac{B}{u+1} \\ \frac{1}{(u-5)(u+1)} &\equiv \frac{(A+B)u + (A-5B)}{(u-5)(u+1)} \end{aligned}$$

Comparing coefficients

$$\{A + B = 0 \qquad A - 5B = 1\}$$

Solving simultaneously for each

$$\{A = 1/6 \quad B = -1/6\}$$

Meaning the fraction can be split

$$\frac{1}{(u-5)(u+1)} \equiv \frac{1/6}{u-5} + \frac{-1/6}{u+1}$$

57.

$$\int \sin 2x \cos x\ dx$$

Since $\sin 2\theta = 2\sin\theta\cos\theta$

$$\begin{aligned} &= \int (2\sin x \cos x)\cos x\ dx \\ &= \frac{2}{-1}\int (-\sin x)\cos^2 x\ dx \end{aligned}$$

Since derivative of $\cos x$ is $-\sin x$

$$= -\frac{2}{3}\cos^3 x + C$$

58.

$$\int \frac{x}{(1+x)(x^2+x+1)}\,dx$$

(§) Use partial fraction

$$= \int \frac{-1}{1+x} + \frac{x+1}{x^2+x+1}\,dx$$

Within the integral, since derivative of denominator x^2+x+1 is $2x+1$, express the numerator in terms of $2x+1$

$$\begin{aligned}
&= -\ln\left|1+x\right| + \frac{1}{2}\int \frac{2x+2}{x^2+x+1}\,dx \\
&= -\ln\left|1+x\right| \\
&\quad + \frac{1}{2}\int \frac{2x+1}{x^2+x+1} + \frac{1}{x^2+x+1}\,dx
\end{aligned}$$

Complete the square on the denominator in right side integral

$$\begin{aligned}
&= -\ln\left|1+x\right| + \frac{1}{2}\int \frac{2x+1}{x^2+x+1}\,dx \\
&\quad + \frac{1}{2}\int \frac{1}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}\,dx \\
&= -\ln\left|1+x\right| + \frac{1}{2}\ln\left|x^2+x+1\right| \\
&\quad + \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + C
\end{aligned}$$

(§) By letting

$$\frac{x}{(1+x)(x^2+x+1)} \equiv \frac{A}{1+x} + \frac{Bx+C}{x^2+x+1}$$

Multiply both sides by $(1+x)(x^2+2+1)$

$$\begin{aligned}
x &\equiv A(x^2+x+1) + (Bx+C)(1+x) \\
&\equiv (A+B)x^2 + (A+B+C)x + (A+C)
\end{aligned}$$

Compare coefficients

$$\left\{\begin{matrix} A+B=0 & \qquad A+B+C=1 \\ A+C=0 & \end{matrix}\right\}$$

Solve simultaneously for each

$$\{A=-1 \qquad B=1 \qquad C=1\}$$

Meaning the fraction can be split

$$\frac{x}{(1+x)(x^2+x+1)} \equiv \frac{-1}{1+x} + \frac{x+1}{x^2+x+1}$$

59.

$$\int \frac{1}{2x^2+3x+1}\ dx$$

(§) Factorise the denominator and use partial fraction

$$\begin{aligned}&= \int \frac{1}{(2x+1)(x+1)}\ dx\\ &= \int \frac{2}{2x+1} + \frac{-1}{x+1}\ dx\\ &= \ln\left|2x+1\right| - \ln\left|x+1\right| + C\end{aligned}$$

Note (§) apply Partial Fraction, by letting

$$\frac{1}{(2x+1)(x+1)} \equiv \frac{A}{2x+1} + \frac{B}{x+1}$$

Multiply both sides by $(2x+1)(x+1)$

$$\begin{aligned}1 &\equiv A(x+1) + B(2x+1)\\ 1 &\equiv (A+2B)x + (A+B)\end{aligned}$$

Compare coefficients

$$\{A+2B=0 \qquad A+B=1\}$$

Solve simultaneously for each

$$\{A=2 \qquad B=-1\}$$

Meaning the fraction can be split

$$\frac{1}{(2x+1)(x+1)} \equiv \frac{2}{2x+1} + \frac{-1}{x+1}$$

60.

$$\int \sqrt{4-x^2}\ dx$$

Use integration by substitution

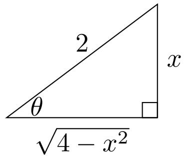
$$\left\{x = 2\sin\theta \qquad \frac{dx}{d\theta} = 2\cos\theta\right\}$$

$$\begin{aligned}&= \int \sqrt{4-4\sin^2 x} \times 2\cos\theta\ d\theta\\ &= \int 4\cos^2\theta\ d\theta\end{aligned}$$

Since $cos^2\theta = \frac{1}{2}(1 + 2\cos 2\theta)$

$$= \int 2 + 2\cos 2\theta \, d\theta$$
$$= 2\theta + \sin 2\theta + C$$

Since $\sin\theta = \frac{x}{2}$, draw a right angle triangle with opposite side x and hypotenuse side 2 and use Pythagoras theorem to calculate its opposite side



Gives

$$\cos\theta = \frac{\sqrt{4 - x^2}}{2}$$

Substitute back in equation

$$= 2\theta + 2\sin\theta\cos\theta + C$$
$$= 2\sin^{-1}\frac{x}{2} + 2 \times \frac{x}{2} \times \frac{\sqrt{4 - x^2}}{2} + C$$
$$= 2\sin^{-1}\left(\frac{x}{2}\right) + \frac{x}{2}\sqrt{4 - x^2} + C$$

61.

$$\int \frac{1}{\sqrt{x+1} - \sqrt{x}} \, dx$$

Rationalise the denominator

$$= \int \frac{1}{\sqrt{x+1} - \sqrt{x}} \times \frac{\sqrt{x+1} + \sqrt{x}}{\sqrt{x+1} + \sqrt{x}} \, dx$$
$$= \int (x+1)^{1/2} + x^{1/2} \, dx$$
$$= \frac{2}{3}(x+1)^{3/2} + \frac{2}{3}x^{3/2} + C$$

62.

$$\int x\sqrt{9 + x^2} \, dx$$

Since derivative of $9 + x^2$ is $2x$

$$= \frac{1}{2}\int 2x(9 + x^2)^{1/2} \, dx$$
$$= \frac{1}{2}\left[\frac{(9 + x^2)^{3/2}}{3/2}\right]$$
$$= \frac{1}{3}(9 + x^2)^{3/2} + C$$

63.

$$\int \sec^2 x \, \tan^3 x \, dx$$

Since derivative of $\tan x$ is $\sec^2 x$

$$= \frac{1}{4} \tan^4 x + C$$

64.

$$\int x^2 e^{-x} \, dx$$

Use integration by parts

$$\begin{Bmatrix} u = x^2 & v' = e^{-x} \\ u' = 2x & v = -e^{-x} \end{Bmatrix}$$

$$\begin{aligned} \int uv' &= [uv] - \int u'v \, dx \\ &= -x^2 e^{-x} - \int -2xe^{-x} \, dx \\ &= -x^2 e^{-x} + \int 2xe^{-x} \, dx \end{aligned}$$

Use integration by parts on the integral

$$\begin{Bmatrix} u = 2x & v' = e^{-x} \\ u' = 2 & v = -e^{-x} \end{Bmatrix}$$

$$\begin{aligned} &= -x^2 e^{-x} + \left\{ -2xe^{-x} - \int -2e^{-x} \, dx \right\} \\ &= -x^2 e^{-x} - 2xe^{-x} - 2e^{-x} + C \end{aligned}$$

65.

$$\int xe^{-x^2} \, dx$$

Since derivative of $-x^2$ is $-2x$

$$\begin{aligned} &= \frac{1}{-2} \int (-2x) e^{-x^2} \, dx \\ &= -\frac{1}{2} e^{-x^2} + C \end{aligned}$$

66.

$$\int \sin x \tan x \, dx$$

$$
\begin{aligned}
&= \int \frac{\sin^2 x}{\cos x}\, dx \\
&= \int \frac{1-\cos^2 x}{\cos x}\, dx \\
&= \int \frac{1}{\cos x} - \cos x\, dx \\
(\S) &= \ln|\sec x + \tan x| - \sin x + C
\end{aligned}
$$

($\S$) Refer to question 71 showing

$$\int \sec\theta\, d\theta = \ln\left|\sec\theta + \tan\theta\right| + C$$

67.

$$\int \sin^3 x \cos^2 x\, dx$$

$$
\begin{aligned}
&= \int \sin x \left(1-\cos^2 x\right)\cos^2 x\, dx \\
&= \int \sin x \cos^2 x - \sin x \cos^4 x\, dx \\
&= -\frac{\cos^3 x}{3} - -\frac{\cos^5 x}{5} + C \\
&= -\frac{1}{3}\cos^3 x + \frac{1}{5}\cos^5 x + C
\end{aligned}
$$

68.

$$\int \frac{x^2+1}{x^2-x}\, dx$$

Since numerator and denominator have same degree, re-express numerator in terms of denominator

$$
\begin{aligned}
&= \int \frac{x^2-x}{x^2-x} + \frac{x+1}{x^2-x}\, dx \\
&= \int 1 + \frac{x+1}{x(x-1)}\, dx
\end{aligned}
$$

($\S$) Since numerator has lower polynomial degree, use Partial Fraction

$$
\begin{aligned}
(\S) &= \int 1 + \frac{-1}{x} + \frac{2}{x-1}\, dx \\
&= x - \ln|x| + 2\ln|x-1| + C
\end{aligned}
$$

($\S$) By letting

$$\frac{x+1}{x(x-1)} \equiv \frac{A}{x} + \frac{B}{x-1}$$

Multiply both sides by $x(x-1)$

$$x+1 \equiv A(x-1)+Bx$$
$$x+1 \equiv (A+B)x - A$$

Compare coefficients

$$A+B=1 \qquad -A=1$$

Solve simultaneously for each

$$A=-1 \qquad B=2$$

Therefore

$$\frac{x+1}{x(x-1)} \equiv \frac{-1}{x} + \frac{2}{x-1}$$

69.

$$\int \frac{1}{\sqrt{x-1}+(x-1)}\,dx$$

When a part of denominator is a root, factorise it out

$$= \frac{2}{1}\int \frac{1}{1+\sqrt{x-1}} \times \frac{1}{2\sqrt{x-1}}\,dx$$

Use integration by substitution

$$\left\{ u=\sqrt{x-1} \qquad \frac{du}{dx} = \frac{1}{2\sqrt{x-1}} \right\}$$

$$= 2\int \frac{1}{1+u}\,du$$
$$= 2\ln\left|1+u\right| + C$$
$$= 2\ln\left|1+\sqrt{x-1}\right| + C$$

70.

$$\int \frac{3x^2}{1+x^6}\,dx$$

Use integration by substitution

$$\left\{ u=x^3 \qquad \frac{du}{dx} = 3x^2 \right\}$$

$$\int \frac{1}{1+u^2}\,du$$
$$= \tan^{-1} u + C$$
$$= \tan^{-1} x^3 + C$$

71.

$$\int \sec x \; dx$$

Using t-method where $t = \tan \frac{x}{2}$

$$\left\{ \cos x = \frac{1-t^2}{1+t^2} \qquad \frac{dx}{dt} = \frac{2}{1+t^2} \right\}$$

$$\begin{aligned} &= \int \frac{1+t^2}{1-t^2} \times \frac{2}{1+t^2} \; dt \\ &= \int \frac{2}{1-t^2} \; dt \end{aligned}$$

(§) Use partial fraction

$$\begin{aligned} &= \int \frac{1}{1+t} + \frac{1}{1-t} \; dx \\ &= \ln\left|1+t\right| - \ln\left|1-t\right| + C \\ &= \ln\left|\frac{1+t}{1-t}\right| + C \end{aligned}$$

Convert back into x terms using

$$\left\{ \cos x = \frac{1-t^2}{1+t^2} \qquad \tan x = \frac{2t}{1-t^2} \right\}$$

$$\begin{aligned} &= \ln\left|\frac{1+t}{1-t} \times \frac{1+t}{1+t}\right| + C \\ &= \ln\left|\frac{1+2t+t^2}{1-t^2}\right| + C \\ &= \ln\left|\frac{1+t^2}{1-t^2} + \frac{2t}{1-t^2}\right| + C \\ &= \ln\left|\sec x + \tan x\right| + C \end{aligned}$$

(§) By letting

$$\frac{2}{(1-t)(1+t)} \equiv \frac{A}{1-t} + \frac{B}{1+t}$$

Multiply both sides by $(1-t)(1+t)$

$$\begin{aligned} 2 &\equiv A(1+t) + B(1-t) \\ 2 &\equiv (A-B)t + (A+B) \end{aligned}$$

Compare coefficients

$$A - B = 0 \qquad A + B = 2$$

Solve simultaneously for each

$$A = 1 \qquad B = 1$$

Therefore

$$\frac{1}{(1-t)(1+t)} \equiv \frac{1}{1-t} + \frac{1}{1+t}$$

72.

$$\int_0^3 \frac{x}{\sqrt{x+3}}\, dx$$

Use integration by substitution

$$\left\{ u = x + 3 \qquad \frac{du}{dx} = 1 \right\}$$

Limits change to

$$\left\{ \begin{matrix} x = 3 \to u = 6 \\ x = 0 \to u = 3 \end{matrix} \right\}$$

$$\begin{aligned}
&= \int_3^6 \frac{u-3}{\sqrt{u}}\, du \\
&= \int_3^6 u^{1/2} - 3u^{-1/2}\, du \\
&= \left[\frac{2}{3} u^{3/2} - 6u^{1/2} \right]_3^6 \\
&= 4\sqrt{3} - 2\sqrt{6}
\end{aligned}$$

73.

$$\int_1^3 \frac{1}{x(1+x^2)}\, dx$$

(§) Use partial fraction

$$\begin{aligned}
&= \int_1^3 \frac{1}{x} + \frac{-x}{1+x^2}\, dx \\
&= \int_1^3 \frac{1}{x}\, dx + \frac{1}{-2} \int \frac{2x}{1+x^2}\, dx \\
&= \Big[\ln|x| \Big]_1^3 - \frac{1}{2} \Big[\ln|1+x^2| \Big]_1^3 \\
&= \ln 3 - \frac{1}{2} \ln 5
\end{aligned}$$

(§) By letting

$$\frac{1}{x(1+x^2)} \equiv \frac{A}{x} + \frac{Bx + C}{1+x^2}$$

Multiply both sides by $x(1+x^2)$

$$1 \equiv A(1+x^2) + (Bx+C)x$$
$$1 \equiv (A+B)x^2 + Cx + A$$

Compare coefficients

$$A+B=0 \qquad C=0 \qquad A=1$$

Solve simultaneously for each

$$A=1 \qquad B=-1 \qquad C=0$$

Meaning the fraction can be split

$$\frac{1}{x(1+x^2)} \equiv \frac{1}{x} + \frac{-1x+0}{1+x^2}$$

74.

$$\int_1^3 \frac{\ln x}{x}\, dx$$

Since derivative of $\ln x$ is $\frac{1}{x}$

$$\begin{aligned} &= \int_1^3 (\ln x) \times \frac{1}{x}\, dx \\ &= \left[\frac{1}{2}(\ln x)^2\right]_1^3 \\ &= \frac{1}{2}(\ln 3)^2 \end{aligned}$$

75.

$$\int_0^1 \sin^{-1} x\, dx$$

Use Integral by parts

$$\left\{ \begin{matrix} u = \sin^{-1} x & & v' = 1 \\ u' = \dfrac{1}{\sqrt{1-x^2}} & & v = x \end{matrix} \right\}$$

$$\begin{aligned} \int uv' &= [uv] - \int u'v\, dx \\ &= \left[x \sin^{-1} x\right]_0^1 - \int_0^1 \frac{x}{\sqrt{1-x^2}}\, dx \\ &= \frac{\pi}{2} - \frac{1}{-2}\int_0^1 -2x(1-x^2)^{-1/2}\, dx \\ &= \frac{\pi}{2} + \frac{1}{2}\left[\frac{(1-x^2)^{1/2}}{1/2}\right]_0^1 \\ &= \frac{\pi}{2} - 1 \end{aligned}$$

76.

$$\int_1^3 \frac{x+2}{\sqrt{-3+4x-x^2}}\,dx$$

Complete the square on the denominator

$$= \int_1^3 \frac{x+2}{\sqrt{1-(2-x)^2}}\,dx$$

Use integration by substitution by letting

$$\left\{u = 2-x \qquad \frac{du}{dx} = -1\right\}$$

Limits change to

$$\begin{Bmatrix} x = 3 \to u = -1 \\ x = 1 \to u = 1 \end{Bmatrix}$$

$$= \int_1^{-1} \frac{4-u}{\sqrt{1-u^2}} \times -1\,du$$

Use the integral property

$$-\int_a^b f(x)\,dx = \int_b^a f(x)\,dx$$

$$\begin{aligned}
&= \int_{-1}^1 \frac{4-u}{\sqrt{1-u^2}}\,du \\
&= \int_{-1}^1 \frac{4}{\sqrt{1-u^2}} - \frac{u}{\sqrt{1-u^2}}\,du \\
&= 4\int_{-1}^1 \frac{1}{\sqrt{1-u^2}}\,du \\
&\qquad - \frac{1}{2}\int_{-1}^1 2u(1-u^2)^{-1/2}du \\
&= 4\left[\sin^{-1} u\right]_{-1}^1 - \frac{1}{2}\left[\frac{(1-u^2)^{1/2}}{1/2}\right]_{-1}^1 \\
&= 4\pi
\end{aligned}$$

77.

$$\int_1^2 \frac{1}{x^2+4x+3}\,dx$$

(§) Using Partial fractions

$$= \int_1^2 \frac{1}{(x+3)(x+1)}\,dx$$

$$
\begin{aligned}
&= \int_1^2 \frac{-1/2}{x+3} + \frac{1/2}{x+1}\ dx \\
&= -\frac{1}{2}\Big[\ln|x+3|\Big]_1^2 + \frac{1}{2}\Big[\ln|x+1|\Big]_1^2 \\
&= -\frac{1}{2}\ln\left(\frac{5}{2}\right) + \frac{1}{2}\ln\left(\frac{3}{2}\right)
\end{aligned}
$$

(§) By letting

$$\frac{1}{(x+3)(x+1)} \equiv \frac{A}{x+3} + \frac{B}{x+1}$$

Multiply both sides by $(x+3)(x+1)$

$$
\begin{aligned}
1 &\equiv A(x+1) + B(x+3) \\
1 &\equiv (A+B)x + (A+3B)
\end{aligned}
$$

Comparing coefficients

$$A + B = 0 \qquad A + 3B = 1$$

Solving simultaneously for each

$$A = -1/2 \qquad B = 1/2$$

Therefore

$$\frac{1}{(x+3)(x+1)} \equiv \frac{-1/2}{x+3} + \frac{1/2}{x+1}$$

78.

$$\int_0^1 x\sqrt{1-x^2}\ dx$$

Since derivative of $1-x^2$ is $-2x$

$$
\begin{aligned}
&= \frac{1}{-2}\int_0^1 -2x(1-x^2)^{1/2}\ dx \\
&= -\frac{1}{2}\left[\frac{(1-x^2)^{3/2}}{3/2}\right]_0^1 \\
&= \frac{1}{3}
\end{aligned}
$$

79.

$$\int x\ln x\ dx$$

Use integration by parts

$$\begin{Bmatrix} u = \ln x & v' = x \\ u' = 1/x & v = 1 \end{Bmatrix}$$

$$\int uv' \, dx = [uv] - \int u'v \, dx$$
$$\int x \ln x \, dx = \left[\frac{x^2}{2} \ln x\right] - \int \frac{x}{2} \, dx$$
$$= \frac{x^2}{2} \ln x - \frac{x^2}{4} + C$$

80.

$$\int x^2 e^{-x} \, dx$$

Use integration by parts

$$\begin{Bmatrix} u = x^2 & v' = e^{-x} \\ u' = 2x & v = -e^{-x} \end{Bmatrix}$$

$$\int uv' \, dx = [uv] - \int u'v \, dx$$
$$\int x^2 e^{-x} \, dx = \left[-x^2 e^{-x}\right] - \int -2x e^{-x} \, dx$$
$$= -x^2 e^{-x} + \int 2x e^{-x} \, dx$$

Reapply integration by parts

$$\begin{Bmatrix} u = 2x & v' = e^{-x} \\ u' = 2 & v = -e^{-x} \end{Bmatrix}$$

$$= -x^2 e^{-x} + \left[-2x e^{-x}\right] - \int -2e^{-x} \, dx$$
$$= -x^2 e^{-x} + -2x e^{-x} - 2e^{-x} + C$$

81.

$$\int \frac{x+1}{x^3 + x^2 + x + 1} \, dx$$

If possible, factorise the denominator

$$\int \frac{x+1}{x^2(x+1) + (x+1)} \, dx$$
$$\int \frac{x+1}{(x+1)(x^2+1)} \, dx$$
$$\int \frac{1}{x^2+1} \, dx$$
$$= \tan^{-1} x + C$$

82.

$$\int_0^1 \frac{e^{2x}}{e^x + 1} \, dx$$

Since derivative of denominator $e^x + 1$ is e^x

$$\int_0^1 \frac{e^x}{e^x+1} \times e^x \; dx$$

Use integration by substitution

$$= \left\{ u = e^x + 1 \qquad \frac{du}{dx} = e^x \right\}$$

Limits change to

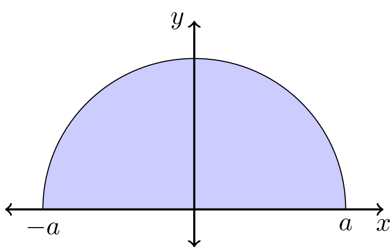
$$\begin{Bmatrix} x = 1 \to u = e + 1 \\ x = 0 \to u = 2 \end{Bmatrix}$$

$$\begin{aligned} &= \int_2^{e+1} \frac{u-1}{u} \; du \\ &= \int_2^{e+1} 1 - \frac{1}{u} \; du \\ &= \Big[u \Big]_2^{e+1} - \Big[\ln|u| \Big]_2^{e+1} \\ &= e - 1 - \ln\left|\frac{e+1}{2}\right| \end{aligned}$$

83.

$$\int_{-a}^{a} \sqrt{a^2 - x^2} \; dx \qquad \text{a is constant}$$

This is the same as finding area under a semi circle of radius a



Therefore

$$\int_{-a}^{a} \sqrt{a^2 - x^2} \; dx = \frac{1}{2}\pi a^2$$

84.

$$\int_{-a}^{a} x\sqrt{x^2 - a^2} \; dx$$

$$= \frac{1}{2}\int_{-a}^{a} 2x(x^2-a^2)^{1/2}\,dx$$
$$= \frac{1}{2}\left[\frac{(x^2-a^2)^{3/2}}{3/2}\right]_{-a}^{a}$$
$$= \frac{1}{3}\left[(x^2-a^2)^{3/2}\right]_{-a}^{a}$$
$$= 0$$

Shortcut: as it is an integral of an odd function with limits of same number but opposite signs, making it zero

85.

$$\int_{-\pi/2}^{\pi/2} \sin x \cos x\,dx$$

$$= \frac{1}{2}\int_{-\pi/2}^{\pi/2} 2\sin x\cos x\,dx$$
$$= \frac{1}{2}\int_{-\pi/2}^{\pi/2} \sin 2x\,dx$$
$$= \frac{1}{2}\left[\frac{-\cos 2x}{2}\right]_{-\pi/2}^{\pi/2}$$
$$= 0$$

Shortcut: as it is an integral of an odd function with limits of same number but opposite signs. Which will be zero

86.

$$\int_{0}^{\pi/4} \sec^2 x \tan x\,dx$$

Since derivative of $\tan x$ is $\sec^2 x$

$$= \left[\frac{(\tan x)^2}{2}\right]_{0}^{\pi/4}$$
$$= \frac{1}{2}$$

87.

$$\int_{1}^{2} \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$$

Since derivative of the power $\sqrt{x}$ is $\frac{1}{\sqrt{x}}$

$$= \int_{1}^{2} e^{\sqrt{x}}\frac{1}{\sqrt{x}}\,dx$$
$$= \left[e^{\sqrt{x}}\right]_{1}^{2}$$
$$= e^{\sqrt{2}} - e$$

88.

$$\int_{1/2}^{\sqrt{3}/2} \frac{\ln(\sin^{-1} x)}{\sqrt{1-x^2}}\, dx$$

Since derivative of $\sin^{-1}$ is $\frac{1}{\sqrt{1-x^2}}$

$$\int_{1/2}^{\sqrt{3}/2} \ln(\sin^{-1} x) \times \frac{1}{\sqrt{1-x^2}} \times\ dx$$

Use integration by substitution

$$\left\{ u = \sin^{-1} x \qquad \frac{du}{dx} = \frac{1}{\sqrt{1-x^2}} \right\}$$

Limits change to

$$\left\{ \begin{array}{c} x = \dfrac{\sqrt{3}}{2} \to u = \dfrac{\pi}{3} \\ x = \dfrac{1}{2} \to u = \dfrac{\pi}{6} \end{array} \right\}$$

$$= \int_{\pi/6}^{\pi/3} \ln u \, du$$

(§) Use integration by parts

$$\left\{ \begin{array}{cc} u = x & v' = \ln x \\ u' = 1 & v = 1/x \end{array} \right\}$$

$$= \Big[x \ln |x| \Big]_{\pi/6}^{\pi/3} - \int_{\pi/6}^{\pi/3} 1 \, du$$

$$= \frac{\pi}{3} \ln \frac{\pi}{3} - \frac{\pi}{6} \ln \frac{\pi}{6} - \frac{\pi}{6}$$

89.

$$\int_0^1 \frac{3+2x}{4+x^2}\, dx$$

Split the fraction as the denominator hints possible integration into inverse tan

$$= \int_0^1 \frac{3}{4+x^2}\, dx + \int_0^1 \frac{2x}{4+x^2}\, dx$$

$$= 3\left[\frac{1}{2} \tan^{-1} \frac{x}{2} \right]_0^1 + \Big[\ln |4+x^2| \Big]_0^1$$

$$= \frac{3}{2} \tan^{-1} \left(\frac{1}{2} \right) + \ln \frac{5}{4}$$

90. Show that if $I_n = x^n \ln x$ then

$$I_n = \left[\frac{x^n \ln x}{n+1}\right] - \frac{1}{n+1} I_{n-1}$$

Use Integral by parts

$$\begin{Bmatrix} u = \ln x & v' = x^n \\ u' = \dfrac{1}{x} & v = \dfrac{x^n}{n+1} \end{Bmatrix}$$

$$\begin{aligned} \int uv' &= [uv] - \int u'v \, dx \\ I_n &= \left[\frac{x^n \ln x}{n+1}\right] - \frac{1}{n+1} \int x^{n-1} \, dx \\ I_n &= \left[\frac{x^n \ln x}{n+1}\right] - \frac{1}{n+1} I_{n-1} \end{aligned}$$

91. Show that if $I_n = \int (\ln x)^n \, dx$ then

$$I_n = [x(\ln x)^n] - nI_{n-1}$$

Use Integral by parts

$$\begin{Bmatrix} u = (\ln x)^n & v' = 1 \\ u' = \dfrac{n}{x}(\ln x)^{n-1}) & v = x \end{Bmatrix}$$

$$\begin{aligned} \int uv' &= [uv] - \int u'v \, dx \\ I_n &= [x(\ln x)^n] - \int n(\ln x)^{n-1} \, dx \\ I_n &= [x(\ln x)^n] - nI_{n-1} \end{aligned}$$

92. Show that if $I_n = \int \sin^n ax \, dx$ then

$$nI_n = -\frac{1}{a} \cos ax \sin^{n-1} ax + (n-1)I_{n-2}$$

Common strategy is to separate one of the $\sin ax$

$$I_n = \int \sin^{n-1} ax \times \sin ax \, dx$$

Use integration by parts where

$$\begin{Bmatrix} u = \sin^{n-1} ax & v' = \sin ax \\ u' = (n-1)\sin^{n-2} ax \times a \cos ax & v = \dfrac{1}{a} \times -\cos ax \end{Bmatrix}$$

$$\int uv'\,dx = [uv] - \int u'v\,dx$$

$$\int \sin^n ax\,dx = \left[-\frac{1}{a}\cos ax \sin^{n-1} ax\right] - (n-1)\int -\sin^{n-2} ax \times \cos^2 ax\,dx$$

$$I_n = \left[-\frac{1}{a}\cos ax \sin^{n-1} ax\right] + (n-1)\int \sin^{n-2} ax \times (1-\sin^2 ax)\,dx$$

$$I_n = \left[-\frac{1}{a}\cos ax \sin^{n-1} ax\right] + (n-1)\int \sin^{n-2} ax - \sin^n ax\,dx$$

$$I_n = \left[-\frac{1}{a}\cos ax \sin^{n-1} ax\right] + (n-1)I_{n-2} - (n-1)I_n$$

Combine the I_n terms

$$nI_n = -\frac{1}{a}\cos ax \sin^{n-1} ax + (n-1)I_{n-2}$$

93. Show that if $I_n = \int_0^{\pi/4} \tan^n x\,dx$ then

$$I_{n,m} = \frac{x^{n+1}(\ln x)^m}{n+1} - \frac{m}{n+1} I_{n,m-1}$$

Common strategy is to separate $\tan^2 ax$ and use $\tan^2 ax + 1 = \sec^2 ax$

$$I_n = \int_0^{\pi/4} \tan^{n-2} x \times \tan^2 x\,dx$$

$$I_n = \int_0^{\pi/4} \tan^{n-2} x \times (\sec^2 x - 1)\,dx$$

$$I_n = \int_0^{\pi/4} \tan^{n-2} x \times \sec^2 x - \tan^{n-2} x\,dx$$

$$I_n = \int_0^{\pi/4} \tan^{n-2} x \times \sec^2 x\,dx - \int_0^{\pi/4} \tan^{n-2} x\,dx$$

Since derivative of $\tan x$ is $\sec^2 x$

$$I_n = \left[\frac{\tan^{n-1} x}{n-1}\right]_0^{\pi/4} - I_{n-2}$$

$$I_n = \frac{1}{n-1} - I_{n-2}$$

94. Show that if $I_{n,m} = \int x^m (\ln x)^m\,dx$ then

$$I_{n,m} = \frac{x^{n+1}(\ln x)^m}{n+1} - \frac{m}{n+1} I_{n,m-1}$$

Use integration by parts

$$\left\{\begin{array}{ll} u = (\ln x)^m & v' = x^n \\ u' = \dfrac{m(\ln x)^{m-1}}{x} & v = \dfrac{x^{n+1}}{n+1} \end{array}\right\}$$

$$I_{n,m} = \frac{x^{n+1}(\ln x)^m}{n+1} - \int \frac{mx^n(\ln x)^{n-1}}{n+1}$$
$$I_{n,m} = \frac{x^{n+1}(\ln x)^m}{n+1} - \frac{m}{n+1} I_{n,m-1}$$

95. Show that if $I_n = \int \sec^n ax \, dx$ then

$$I_n = \left[\frac{1}{a}\frac{\sin ax}{\cos^{n-1} ax}\right] - (n-2)I_n + (n-2)I_{n-2}$$

Common strategy is to separate $\sec^2 ax$ and use $\tan^2 ax + 1 = \sec^2 ax$

$$I_n = \int \sec^{n-2} ax \times \sec^2 ax \, dx$$
$$I_n = \int \cos^{-n+2} ax \times \sec^2 ax \, dx$$

Use integration by parts

$$\begin{Bmatrix} u = \cos^{-n+2} ax & v' = \sec^2 ax \\ u' = (-n+2)(\cos ax)^{-n+1} \times a(-\sin ax) & v = \dfrac{1}{a}\tan ax \end{Bmatrix}$$

$$\int uv' \, dx = [uv] - \int u'v \, dx$$
$$I_n = \left[\frac{1}{a}\frac{\sin ax}{\cos^{n-1} ax}\right] - \int (n-2)\sin^2 ax (\cos ax)^{-n} \, dx$$
$$I_n = \left[\frac{1}{a}\frac{\sin ax}{\cos^{n-1} ax}\right] - (n-2)\int (1-\cos^2 ax)(\cos ax)^{-n} \, dx$$
$$I_n = \left[\frac{1}{a}\frac{\sin ax}{\cos^{n-1} ax}\right] - (n-2)\int \cos^{-n} ax - \cos^{2-n} ax \, dx$$
$$I_n = \left[\frac{1}{a}\frac{\sin ax}{\cos^{n-1} ax}\right] - (n-2)\int \sec^n ax - \sec^{n-2} ax \, dx$$
$$I_n = \left[\frac{1}{a}\frac{\sin ax}{\cos^{n-1} ax}\right] - (n-2)I_n + (n-2)I_{n-2}$$

Combine the I_n terms together

$$(n-1)I_n = \frac{1}{a}\frac{\sin ax}{\cos^{n-1} ax} + (n-2)I_{n-2}$$

96. Show that if $I_n = \int_0^1 (1-x^2)^n \, dx$ then

$$\left(1 + \frac{1}{2n}\right) I_n = I_{n-1}$$

Common strategy to take one of $1-x^2$ out

$$\begin{aligned}
I_n &= \int_0^1 (1-x^2)^{n-1} \times (1-x^2)\ dx \\
&= \int_0^1 (1-x^2)^{n-1}\ dx - \int_0^1 x \times x(1-x^2)^{n-1}\ dx \\
&= I_{n-1} - \int_0^1 x \times x(1-x^2)^{n-1}\ dx
\end{aligned}$$

Use integration by parts on the integral $\int uv'\ dx = [uv] - \int u'v\ dx$

$$\left\{\begin{matrix} u = x & v' = x(1-x^2)^{n-1} \\ u' = 1 & v = \dfrac{1}{-2n}(1-x^2)^n \end{matrix}\right\}$$

$$\begin{aligned}
I_n &= I_{n-1} - \left\{ \left[\frac{x(1-x^2)^n}{-2n}\right]_0^1 - \int_0^1 \frac{1}{-2n}(1-x^2)^n\ dx \right\} \\
I_n &= I_{n-1} - 0 + \int_0^1 \frac{1}{-2n}(1-x^2)^n\ dx \\
I_n &= I_{n-1} - \frac{1}{2n} I_n
\end{aligned}$$

Combine the I_n terms together

$$I_n = \left(1 + \frac{1}{2n}\right) I_n = I_{n-1}$$

97. Show that $I_n = \int_0^1 \frac{x^n}{\sqrt{ax+b}}\ dx$ then

$$(2n+1)\, I_n = \left[\frac{2x^n}{a}(ax+b)^{1/2}\right] - \frac{2bn}{a} I_n$$

Use integration by parts

$$\left\{\begin{matrix} u = x^n & v' = (ax+b)^{-1/2} \\ u' = nx^{n-1} & v = \dfrac{2}{a}(ax+b)^{1/2} \end{matrix}\right\}$$

$$\begin{aligned}
I_n &= \left[\frac{2x^n}{a}(ax+b)^{1/2}\right] - \frac{2n}{a}\int x^{n-1}(ax+b)^{1/2}\ dx \\
I_n &= \left[\frac{2x^n}{a}(ax+b)^{1/2}\right] - \frac{2n}{a}\int (ax+b)x^{n-1}(ax+b)^{-1/2}\ dx \\
I_n &= \left[\frac{2x^n}{a}(ax+b)^{1/2}\right] - \frac{2n}{a}\int ax^n(ax+b)^{-1/2} + 2bx^{n-1}(ax+b)^{-1/2}\ dx \\
I_n &= \left[\frac{2x^n}{a}(ax+b)^{1/2}\right] - 2nI_n - \frac{2bn}{a} I_{n-1}
\end{aligned}$$

Combine the I_n terms

$$(2n+1)\, I_n = \left[\frac{2x^n}{a}(ax+b)^{1/2}\right] - \frac{2bn}{a} I_n$$

98. Show that if $I_n = \int x^n \sqrt{ax+b}\, dx$ then

$$\left(1 + \frac{2n}{3}\right) I_n = \frac{2}{3a} x^n (ax+b)^{3/2} - \frac{2nb}{3a} I_{n-1}$$

Apply integration by parts where

$$\begin{Bmatrix} u = x^n & v' = (ax+b)^{1/2} \\ u' = nx^{n-1} & v = \dfrac{(ax+b)^{3/2}}{3/2 \times a} \end{Bmatrix}$$

$$\begin{aligned}
I_n &= [uv] - \int u'v\, dx \\
I_n &= \frac{2}{3a} x^n (ax+b)^{3/2} - \int \frac{2n}{3a} x^{n-1} (ax+b)(ax+b)^{1/2}\, dx \\
I_n &= \frac{2}{3a} x^n (ax+b)^{3/2} - \frac{2n}{3a} \int ax^n (ax+b)^{1/2} + bx^{n-1}(ax+b)^{1/2}\, dx \\
I_n &= \frac{2}{3a} x^n (ax+b)^{3/2} - \frac{2n}{3} I_n - \frac{2n}{3a} b I_{n-1}
\end{aligned}$$

Combine the I_n terms

$$\left(1 + \frac{2n}{3}\right) I_n = \frac{2}{3a} x^n (ax+b)^{3/2} - \frac{2nb}{3a} I_{n-1}$$

99. Show that if $I_n = \frac{1}{(x^2+a^2)^n}\, dx$ then

$$I_n = \frac{x}{a^2\left(x^2+a^2\right)^{n-1}} + \frac{(2n-3)}{a^2} I_{n-1} - (2n-3) I_n$$

Use integration by substitution

$$\left\{ x = a\tan\theta \qquad \frac{dx}{d\theta} = a\sec^2\theta \right\}$$

$$\begin{aligned}
I_n &= \int \frac{1}{(a^2 + a^2\tan^2\theta)^n} \times a\sec^2\theta\, d\theta \\
I_n &= \int \frac{1}{(a^2\sec^2\theta)^n} \times a\sec^2\theta\, d\theta \\
I_n &= \frac{1}{a^{2n-1}} \int \cos^{2n-2}\theta\, d\theta
\end{aligned}$$

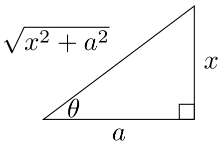
Use integration by parts on the integral

$$\begin{Bmatrix} u = \cos^{2n-3}\theta & v' = \cos\theta \\ u' = (2n-3)\cos^{2n-4}\theta(-\sin\theta) & v = \sin\theta \end{Bmatrix}$$

$$I_n = \frac{1}{a^{2n-1}}\left\{\left[\sin\theta\cos^{2n-3}\theta\right] + (2n-3)\int \sin^2\theta\cos^{2n-4}\theta\ d\theta\right\}$$

$$I_n = \frac{1}{a^{2n-1}}\left[\sin\theta\cos^{2n-3}\theta\right] + \frac{1}{a^{2n-1}}(2n-3)\int \sin^2\theta\cos^{2n-4}\theta\ d\theta$$

To convert back into x terms, since $\tan\theta = \frac{x}{a}$, draw a right angle triangle with opposite side t and adjacent side a and use Pythagoras theorem to calculate its hypotenuse



Gives

$$\sin\theta = \frac{x}{\sqrt{x^2+a^2}} \qquad \cos\theta = \frac{a}{\sqrt{x^2+a^2}} \qquad \frac{d\theta}{dx} = \frac{1}{a}\ \frac{a^2}{x^2+a^2}$$

Substitute back in equation

$$\begin{aligned} I_n &= \frac{1}{a^{2n-1}}\left[\frac{x}{\sqrt{x^2+a^2}}\ \frac{a^{2n-3}}{\left(\sqrt{x^2+a^2}\right)^{2n-3}}\right] \\ &\quad + \frac{1}{a^{2n-1}}(2n-3)\int \frac{x^2}{x^2+a^2}\ \frac{a^{2n-4}}{(x^2+a^2)^{n-2}}\ \frac{1}{a}\ \frac{a^2}{x^2+a^2}\ dx \\ I_n &= \frac{x}{a^2\left(x^2+a^2\right)^{n-1}} + \frac{(2n-3)}{a^2}\int \frac{x^2}{\left(x^2+a^2\right)^n}\ dx \\ I_n &= \frac{x}{a^2\left(x^2+a^2\right)^{n-1}} + \frac{(2n-3)}{a^2}\int \frac{\left(x^2+a^2\right)-a^2}{\left(x^2+a^2\right)^n}\ dx \\ I_n &= \frac{x}{a^2\left(x^2+a^2\right)^{n-1}} + \frac{(2n-3)}{a^2}\int \frac{1}{\left(x^2+a^2\right)^{n-1}} - \frac{a^2}{(x^2+a^2)^n}\ dx \\ I_n &= \frac{x}{a^2\left(x^2+a^2\right)^{n-1}} + \frac{(2n-3)}{a^2}I_{n-1} - (2n-3)I_n \end{aligned}$$

Combine the I_n terms

$$\begin{aligned} (2n-2)I_n &= \frac{x}{a^2\left(x^2+a^2\right)^{n-1}} + \frac{(2n-3)}{a^2}I_{n-1} \\ 2a^2(n-1)I_n &= \frac{x}{\left(x^2+a^2\right)^{n-1}} + (2n-3)I_{n-1} \end{aligned}$$

100. Show that if $I_n = \int x^n e^{ax}\, dx$ then

$$I_n = \frac{x^n e^{ax}}{a} - \frac{n}{a} I_{n-1}$$

Use integration by parts where

$$\left\{\begin{array}{ll} u = x^n & v' = e^{ax} \\ u' = nx^{n-1} & v = \dfrac{e^{ax}}{a} \end{array}\right\}$$

$$\begin{aligned} I_n &= [uv] - \int u'v\, dx \\ I_n &= \frac{x^n e^{ax}}{a} - \frac{n}{a}\int x^{n-1} e^{ax}\, dx \\ I_n &= \frac{x^n e^{ax}}{a} - \frac{n}{a} I_{n-1} \end{aligned}$$