Eigenvectors in Cross Data Matrix Method

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February 1, 2019

PCA in HDLSS

- High-dimension, low-sample -size (HDLSS) data occurs in many modern science fields such as genetic microarrays, medical imaging, text recognition, finance, chemometrics, and so on.

- Principal Component Analysis (PCA) is an important tool for dimension reduction especially when the dimension is very high.

Literature

- $d \to \infty$: see Johnstone (2001), Baik (2005), and Pual (2007) for Gaussian assumptions.
- $d/n \to c > 0$: see Baik and Silverstein (2006) for non-Gaussian assumptions.
- $d/n \to 0$: see Johnstone and Lu (2009).
- $d \to \infty$ and n is fixed: see Hall et al. (2005) and Ahn et al. (2007).

Literature

- Yata and Aoshima (2009): more general settings without assuming either the normality or ρ-mixing condition

Literature

ELSEVIER

Journal of Multivariate Analysis

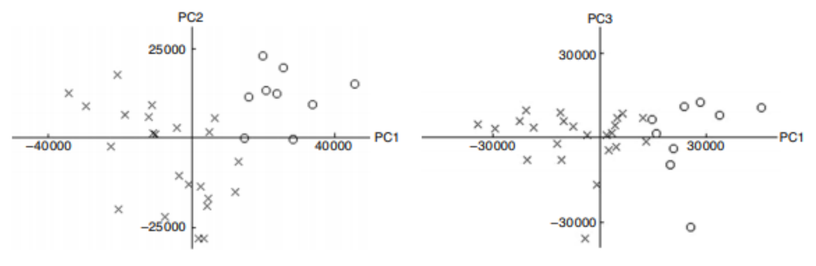
Volume 101, Issue 9, October 2010, Pages 2060-2077

Effective PCA for high-dimension, low-sample-size data with singular value decomposition of cross data matrix

Kazuyoshi Yata [a], Makoto Aoshima [b]

Example

The dataset contains 34 patients with 12600 genes. There are 9 Normal Prostate (plotted as o) and 25 Prostate Tumors (plotted x)



(PCA)

Example

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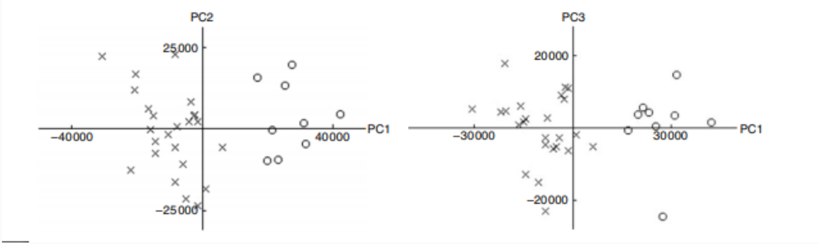


(cross-PCA)

Outline

- Yata and Aoshima (2010)'s general setting
- PCA vs cross-PCA

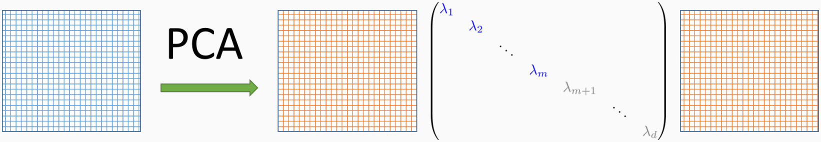
    1. Consistency for eigenvectors (Yata and Aoshima)
    2. Asymptotic normality for eigenvectors (Our work)

- Simulation Results

Yata and Aoshima (2010)'s general setting

$$\lambda_i = a_i d^{\alpha_i} \ (i = 1, \ldots, m) \text{ and } \lambda_j = c_j \ (j = m+1, \ldots, d).$$

Here $a_i > 0$, $c_j \geq 0$ and $\alpha_1 > \alpha_2 > \cdots > \alpha_m$ preserving $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_d$.



Yata and Aoshima (2010)'s general setting

$$
\begin{pmatrix}
\overbrace{\lambda_1 \qquad \lambda_2}^{s_1} & & & & & \\
& \ddots & & & & \\
& & \lambda_m & & & \\
& & & \overbrace{\lambda_{m+1} \qquad \ddots \qquad \lambda_d}^{d-m} & & \\
\end{pmatrix}
$$

$$s_1 + \cdots + s_u + \cdots s_J = m$$

Some interesting facts

- PCA:

$$\boldsymbol{X}\boldsymbol{X}^\top \text{ and its dual matrix } \boldsymbol{X}^\top\boldsymbol{X}$$

(a) $d \times d$ vs $n \times n$ (In general, one of them needs to be small).

(b) Share the same eigenvalues in the sample covariance matrix.

(c) Not share the same eigenvalues in the population covariance matrix.

- cross-PCA:

$$\boldsymbol{X}_{(1)}^\top\boldsymbol{X}_{(2)} \text{ and } \boldsymbol{X}_{(2)}^\top\boldsymbol{X}_{(1)} \text{ (cross data matrix)}$$

(a) Share the same eigenvalues in the sample covariance matrix.

(b) Not share the same eigenvalues in the population covariance matrix.

Yata and Aoshima

Theorem (PCA) Suppose that

(i) $d \to \infty$ and $n \to \infty$ for i such that $\alpha_i > 1$.

(ii) $d \to \infty$ and $d^{2-2\alpha_i}/n \to 0$ for i such that $\alpha_i \in (0, 1]$.

For $1 = 1, \ldots, m$, as $d \to \infty$. Then

(a) $\mathrm{Angle}(\widehat{\boldsymbol{h}}_i, \boldsymbol{h}_i) \to 0$ in probability.

(b) $\mathrm{Angle}(\widehat{\boldsymbol{h}}_i, \boldsymbol{h}_k) \to \frac{\pi}{2},\ k = m+1, \ldots, d$ in probability.

Theorem (cross-PCA) Suppose that

(i) $d \to \infty$ and $n \to \infty$ for i such that $\alpha_i > 1$.

(ii) $d \to \infty$ and $d^{1-\alpha_i}/n \to 0$ for i such that $\alpha_i \in (1/2, 1]$.

(iii) $d \to \infty$ and $d^{2-2\alpha_i}/n \to 0$ for i such that $\alpha_i \in (0, 1/2]$.

For $1 = 1, \ldots, m$, as $d \to \infty$. Then

(a) $\mathrm{Angle}(\widetilde{\boldsymbol{h}}_i^*, \boldsymbol{h}_i) \to 0$ in probability.

(b) $\mathrm{Angle}(\widetilde{\boldsymbol{h}}_i^*, \boldsymbol{h}_k) \to \frac{\pi}{2},\ k = m+1, \ldots, d$ in probability.

Our work (PCA)

Theorem (PCA) Suppose that Yata and Aoshima's PCA conditions hold. Assume that the first m population eigenvalues are distinct. Consider the jth eigenvector, where $s_{u-1} < j \leq s_u$. Then, we have (a)

$$\sqrt{n}\,\mathrm{vec}\left[\ \left(\widehat{\boldsymbol{h}}_{j(1)} - \boldsymbol{h}_j, \widehat{\boldsymbol{h}}_{j(2)} - \boldsymbol{h}_j\right)^\top \boldsymbol{H}_{[1:m]}\ \right] \Rightarrow N\left(\mathbf{0}, \boldsymbol{\psi}^{[j]}\right),$$

where $\boldsymbol{\psi}^{[j]}$ is a block matrix consisting of $m \times m$ blocks, and each block is of size 2×2. Precisely, $\boldsymbol{\psi}^{[j]}$ takes the following form: its $(k,k)^{\mathrm{th}}$ block is given by

$$\boldsymbol{\psi}^{[j]}_{kk} = \begin{cases} \frac{a_k a_j}{(a_k - a_j)^2}\mathrm{E}[z^2_{11(1)} z^2_{21(1)}], & k \neq j \\ 0, & k = j \end{cases}$$

and its $(k,k')^{\mathrm{th}}$ block, where $s_{u-1} < k, k' \leq s_u$, is given by

$$\boldsymbol{\psi}^{[j]}_{kk'} = \frac{a_k a_{k'} a_j^2}{(a_k - a_j)(a_{k'} - a_j)}\mathrm{E}[z_{11(1)} z_{21(1)} z^2_{31(1)}],$$

and all other blocks are zero matrices, i.e., $\boldsymbol{\psi}^{[j]}_{kk'} = \mathbf{0}_{2\times 2}$ for k, k' not being in the same eigen-groups.

(b)

$$\left\| \left(\widehat{\boldsymbol{h}}_{j(1)} - \boldsymbol{h}_j, \widehat{\boldsymbol{h}}_{j(2)} - \boldsymbol{h}_j \right)^\top \boldsymbol{H}_{\text{tail}} \right\|_2 = o_p \left(n^{-1/4} \right).$$

Corollary (full-PCA)

Assume that the first s_1 population eigenvalues are distinct. Consider the jth eigenvector, $1 \leq j \leq s_1$. Then, we have, for $1 \leq k \leq s_1$

$$\sqrt{2n}\boldsymbol{h}_k^\top (\widehat{\boldsymbol{h}}_j - \boldsymbol{h}_j) \Rightarrow N\left(0, \frac{a_k a_j}{(a_k - a_j)^2} \mathrm{E}[z_{11(1)}^2 z_{21(1)}^2]\right), \quad k \neq j,$$

$$\sqrt{2n}\boldsymbol{h}_j^\top (\widehat{\boldsymbol{h}}_j - \boldsymbol{h}_j) = o_p(1).$$

Our work (cross-PCA)

Theorem (cross-PCA) Suppose that Yata and Aoshima's cross-PCA conditions hold. Assume that the first m population eigenvalues are distinct. Consider the jth eigenvector, where $s_{u-1} < j \leq s_u$. Then, we have (a)

$$\sqrt{n}\,\mathrm{vec}\left[\ \left(\widetilde{\boldsymbol{h}}_{j(1)} - \boldsymbol{h}_j, \widetilde{\boldsymbol{h}}_{j(2)} - \boldsymbol{h}_j\right)^\top \boldsymbol{H}_{[1:m]}\ \right] \Rightarrow N\left(\boldsymbol{0}, \boldsymbol{M}^{[j]}\right),$$

where $\boldsymbol{M}^{[j]}$ is a block matrix consisting of $m \times m$ blocks, and each block is of size 2×2. Precisely, $\boldsymbol{M}^{[j]}$ takes the following form: its $(k,k)^{\mathrm{th}}$ block is given by

$$\boldsymbol{M}^{[j]}_{kk} = \begin{cases} \dfrac{a_k a_j}{(a_k - a_j)^2} \mathrm{E}[z_{11(1)}^2 z_{21(1)}^2] \begin{pmatrix} \frac{a_k^2 + a_j^2}{(a_k + a_j)^2} & \frac{2a_k a_j}{(a_k + a_j)^2} \\ \frac{2a_k a_j}{(a_k + a_j)^2} & \frac{a_k^2 + a_j^2}{(a_k + a_j)^2} \end{pmatrix}, & k \neq j \\ 0, & k = j \end{cases}$$

and its $(k,k')^{\mathrm{th}}$ block, where $s_{u-1} < k, k' \leq s_u$, is given by

$$\boldsymbol{M}^{[j]}_{kk'} = \begin{cases} \dfrac{a_k a_{k'} a_j^2}{(a_k - a_j)(a_{k'} - a_j)} \mathrm{E}[z_{11(1)} z_{21(1)} z_{31(1)}^2] \begin{pmatrix} \frac{a_j^2 + a_j a_k}{(a_j + a_k)(a_j + a_{k'})} & \frac{a_j(a_k + a_{k'})}{(a_j + a_k)(a_j + a_{k'})} \\ \frac{a_j(a_k + a_{k'})}{(a_j + a_k)(a_j + a_{k'})} & \frac{a_j^2 + a_j a_{k'}}{(a_j + a_k)(a_j + a_{k'})} \end{pmatrix}, & k, k' \neq j \\ 0 & k = j \text{ or } k' = j \end{cases}$$

and all other blocks are zero matrices, i.e., $\boldsymbol{M}^{[j]}_{kk'} = \boldsymbol{0}_{2\times 2}$ for k, k' not being in the same eigen-groups.

Our work(cross-PCA)

(b)

$$\left\| \left(\widetilde{\boldsymbol{h}}_{j(1)} - \boldsymbol{h}_j, \widetilde{\boldsymbol{h}}_{j(2)} - \boldsymbol{h}_j \right)^\top \boldsymbol{H}_{\mathrm{tail}} \right\|_2 = o_p\left(n^{-1/4} \right).$$

Corollary (cross-PCA)

Let $\widetilde{\boldsymbol{h}}_j = \{\widetilde{\boldsymbol{h}}_{j(1)} + \mathrm{sign}(\widetilde{\boldsymbol{h}}_{j(1)}^\top \widetilde{\boldsymbol{h}}_{j(2)})\widetilde{\boldsymbol{h}}_{j(2)}\}$ and $\widetilde{\boldsymbol{h}}_j^* = \widetilde{\boldsymbol{h}}_j / \|\widetilde{\boldsymbol{h}}_j\|$. Assume that the first s_1 population eigenvalues are distinct. Consider the jth eigenvector, $1 \leq j \leq s_1$. Then, we have, for $1 \leq k \leq s_1$

$$\sqrt{2n}\boldsymbol{h}_k^\top (\widetilde{\boldsymbol{h}}_j^* - \boldsymbol{h}_j) \Rightarrow N\left(0, \frac{a_k a_j}{(a_k - a_j)^2} \mathrm{E}[z_{11(1)}^2 z_{21(1)}^2]\right), \quad k \neq j,$$

$$\sqrt{2n}\boldsymbol{h}_j^\top (\widetilde{\boldsymbol{h}}_j^* - \boldsymbol{h}_j) = o_p(1).$$

Asymptotics results

1. Based on the inequality $1 \leq 2(a_j^2 + a_k^2)/(a_j + a_k)^2 \leq 2$, for $k, j = 1, \ldots, s_1$

$$\lim_{n\to\infty} \mathrm{var}(\boldsymbol{h}_k^\top \widehat{\boldsymbol{h}}_j) \leq \lim_{n\to\infty} \mathrm{var}(\boldsymbol{h}_k^\top \widetilde{\boldsymbol{h}}_{j(i)}) \leq \lim_{n\to\infty} \mathrm{var}(\boldsymbol{h}_k^\top \widehat{\boldsymbol{h}}_{j(i)}).$$

2. Let $\widetilde{\boldsymbol{h}}_j^w = \widetilde{\boldsymbol{h}}_j/\sqrt{2}$. Then for $k, j = 1, \ldots, s_1$

$$\lim_{n\to\infty} \mathrm{var}(\sqrt{2n}\boldsymbol{h}_k^\top \widehat{\boldsymbol{h}}_j) = \lim_{n\to\infty} \mathrm{var}(\sqrt{n}\boldsymbol{h}_k^\top \widetilde{\boldsymbol{h}}_j^w) = \lim_{n\to\infty} \mathrm{var}(\sqrt{2n}\boldsymbol{h}_k^\top \widetilde{\boldsymbol{h}}_j^*).$$

3. Define $\mathrm{ave}_{\mathrm{tail}}\boldsymbol{h}_\ell^\top \widetilde{\boldsymbol{h}}_{j(i)} = \sum_{\ell=m+1}^d \boldsymbol{h}_\ell^\top \widetilde{\boldsymbol{h}}_{j(i)}/(d-m+1)$. Then

$$\mathrm{ave}_{\mathrm{tail}}\boldsymbol{h}_\ell^\top \widetilde{\boldsymbol{h}}_{j(i)} = o_p(n^{-1/2}).$$

   Further, $\mathrm{ave}_{\mathrm{tail}}\boldsymbol{h}_\ell^\top \widetilde{\boldsymbol{h}}_{j(i)}/\boldsymbol{h}_k^\top \widetilde{\boldsymbol{h}}_{j(i)} = o_p(1),\ k = 1, \ldots, m.$

cross-PCA: A sketch of proof

(S1) Starting with

$$\boldsymbol{S}_{D(1)}\boldsymbol{S}_{D(1)}^{\top}\widetilde{\boldsymbol{u}}_{j(1)} = \widetilde{\lambda}_j^2\widetilde{\boldsymbol{u}}_{j(1)},\ j = 1, \ldots, n. \tag{1}$$

Let $\boldsymbol{W} = n^{-2}\boldsymbol{X}_{(1)}\boldsymbol{X}_{(1)}^{\top}\boldsymbol{X}_{(2)}\boldsymbol{X}_{(2)}^{\top}$. Thus, multiplying both sides of (1) by $\boldsymbol{X}_{(1)}$,

$$\boldsymbol{W}(\boldsymbol{X}_{(1)}\widetilde{\boldsymbol{u}}_{j(1)}) = \widetilde{\lambda}_j^2(\boldsymbol{X}_{(1)}\widetilde{\boldsymbol{u}}_{j(1)}) \text{ and } \boldsymbol{W}\widetilde{\boldsymbol{h}}_{j(1)} = \widetilde{\lambda}_j^2\widetilde{\boldsymbol{h}}_{j(1)},\ j = 1, \ldots, n$$

Note that $\widetilde{\boldsymbol{h}}_{j(1)} = (n\widetilde{\lambda}_j)^{-1/2}\boldsymbol{X}_{(1)}\widetilde{\boldsymbol{u}}_{j(1)},\ j = 1, \ldots, n$. Based on the independence of $\boldsymbol{X}_{(1)}$ and $\boldsymbol{X}_{(2)}$, $\mathrm{E}\boldsymbol{W} = \boldsymbol{\Sigma}^2 = \boldsymbol{H}\boldsymbol{\Lambda}^2\boldsymbol{H}^{\top}$. Let $\boldsymbol{\Sigma}^* = \boldsymbol{\Sigma}^2$ and $\boldsymbol{\Lambda}^* = \boldsymbol{\Lambda}^2$. Let $\lambda_j^* = \lambda_j^2$ and $\widetilde{\lambda}_j^* = \widetilde{\lambda}_j^2$. Thus we have $\boldsymbol{\Sigma}^*\boldsymbol{h}_j = \boldsymbol{\lambda}_j^*\boldsymbol{h}_j$.

Proposition (perturbation method). For fixed m,

$$(\boldsymbol{\Sigma}_{[m]}^* + \epsilon\dot{\boldsymbol{\Sigma}}_{[m]}^*)(\boldsymbol{h}_{j[m]} + \epsilon\dot{\boldsymbol{h}}_{j[m]}) = (\boldsymbol{\lambda}_{j[m]}^* + \epsilon\dot{\boldsymbol{\lambda}}_{j[m]}^*)(\boldsymbol{h}_{j[m]} + \epsilon\dot{\boldsymbol{h}}_{j[m]}),$$

implying $\dot{\boldsymbol{h}}_{j[m]} = (\lambda_{j[m]}^*\boldsymbol{I}_m - \boldsymbol{\Sigma}_{[m]}^*)^{+}\dot{\boldsymbol{\Sigma}}_{[m]}^*\boldsymbol{h}_{j[m]},\ j = 1, \ldots, n$. Here, $\epsilon\dot{\boldsymbol{\Sigma}}_{[m]}^* = \widetilde{\boldsymbol{\Sigma}}_{[m]}^* - \boldsymbol{\Sigma}_{[m]}^*$, $\epsilon\dot{\boldsymbol{h}}_{j[m]} = \widetilde{\boldsymbol{h}}_{j(i)[m]} - \boldsymbol{h}_{j[m]}$, and $\epsilon\dot{\boldsymbol{\lambda}}_{j[m]}^* = \widetilde{\boldsymbol{\lambda}}_{j[m]}^* - \boldsymbol{\lambda}_{j[m]}^*$.

cross-PCA: A sketch of proof

(S2) By the consistency results from Yata and Aoshima (2010), the perturbation method allows us to consider the jth eigenvector and the respective matrix $H_{1\backslash j}$, $1 \le j \le s_1$. It implies that

$$
\begin{aligned}
\boldsymbol{H}_{1\backslash j}^{\top} \dot{\boldsymbol{h}}_j &= \boldsymbol{H}_{1\backslash j}^{\top} (\lambda_j^* \boldsymbol{I}_d - \boldsymbol{\Sigma}^*)^{+} \dot{\boldsymbol{\Sigma}}^* \boldsymbol{h}_j \\
&= \left(\frac{1}{\lambda_j^* - \lambda_k^*} \boldsymbol{h}_k^{\top} \dot{\boldsymbol{\Sigma}}^* \boldsymbol{h}_j \right)_{k \in \{1,\ldots,s_1\} \backslash j} = \left(\frac{1}{\lambda_j^* - \lambda_k^*} (\boldsymbol{h}_j^{\top} \otimes \boldsymbol{h}_k^{\top}) \mathrm{vec} \dot{\boldsymbol{\Sigma}}^* \right)_{k \in \{1,\ldots,s_1\} \backslash j}, \\
\boldsymbol{h}_j^{\top} \dot{\boldsymbol{h}}_j &= 0. \qquad (2)
\end{aligned}
$$

As for the right hand side of (2), we consider the perturbation of its component with the index k:

$$
\begin{aligned}
\frac{1}{\lambda_j^* - \lambda_k^*} (\boldsymbol{h}_j^{\top} \otimes \boldsymbol{h}_k^{\top}) (\mathrm{vec} \boldsymbol{W} - \mathrm{vec} \boldsymbol{\Sigma}^*) &= \frac{1}{\lambda_j^* - \lambda_k^*} (\boldsymbol{h}_j^{\top} \otimes \boldsymbol{h}_k^{\top}) \{ (\boldsymbol{I}_d \otimes \boldsymbol{S}_{(1)}) \mathrm{vec} \boldsymbol{S}_{(2)} - (\boldsymbol{I}_d \otimes \boldsymbol{\Sigma}) \mathrm{vec} \boldsymbol{\Sigma} \} \\
&= \frac{1}{\lambda_j^* - \lambda_k^*} (\boldsymbol{h}_j^{\top} \otimes \boldsymbol{h}_k^{\top}) \{ \boldsymbol{I}_d \otimes (\boldsymbol{S}_{(1)} - \boldsymbol{\Sigma}) \} \mathrm{vec} \boldsymbol{S}_{(2)} \\
&+ \frac{1}{\lambda_j^* - \lambda_k^*} (\boldsymbol{h}_j^{\top} \otimes \boldsymbol{h}_k^{\top}) (\boldsymbol{I}_d \otimes \boldsymbol{\Sigma}) (\mathrm{vec} \boldsymbol{S}_{(2)} - \mathrm{vec} \boldsymbol{\Sigma}), \qquad (3)
\end{aligned}
$$

where $\boldsymbol{S}_{(1)} = n^{-1} \boldsymbol{X}_{(1)} \boldsymbol{X}_{(1)}^{\top}$ and $\boldsymbol{S}_{(2)} = n^{-1} \boldsymbol{X}_{(2)} \boldsymbol{X}_{(2)}^{\top}$.

cross-PCA: A sketch of proof

(S3)

$$
\begin{aligned}
&\frac{\sqrt{n}}{\lambda_j^* - \lambda_k^*}(\boldsymbol{h}_j^\top \otimes \boldsymbol{h}_k^\top)(\mathrm{vec}\boldsymbol{W} - \mathrm{vec}\boldsymbol{\Sigma}^*) \\
&= \frac{\sqrt{\lambda_j \lambda_k}}{\lambda_j - \lambda_k} \frac{\sqrt{n}}{n} \sum_{s=1}^{n} \left\{ \frac{\lambda_j}{\lambda_j + \lambda_k} z_{ks(1)} z_{js(1)} + \frac{\lambda_k}{\lambda_j + \lambda_k} z_{ks(2)} z_{js(2)} \right\} + o_p(1).
\end{aligned}
$$

$$
\begin{aligned}
&\sqrt{n}\boldsymbol{H}_{1\setminus j}^\top(\widetilde{\boldsymbol{h}}_{j(1)} - \boldsymbol{h}_j) \\
&= \frac{\sqrt{n}}{n} \sum_{s=1}^{n} \left(\frac{\sqrt{a_j a_k}}{a_j - a_k} \left\{ \frac{a_j}{a_j + a_k} z_{(1)ks} z_{(1)js} + \frac{a_k}{a_j + a_k} z_{(2)ks} z_{(2)js} \right\} \right)_{k \in \{1,\ldots,s_1\} \setminus j} (1 + o_p(1)) \\
&+ o_p(1)
\end{aligned}
$$

and

$$
\sqrt{n}\boldsymbol{h}_j^\top \widetilde{\boldsymbol{h}}_{j(1)} = o_p(1).
$$

Simulation Results

- Sample size $n = 100$
- data_set = 'truncated-t'
- $d = 10^5$
- The first four eigenvalues $16d^{2/3}, 4d^{2/3}, d^{1/3}, d^{1/3}$ with $s_1 = 2$.
- Run 1000 times

	Half-PCA		Full-PCA		Cross-PCA	
	$\sqrt{n}\boldsymbol{h}_1^\top\widehat{\boldsymbol{h}}_{2(1)}$	$\sqrt{n}\boldsymbol{h}_2^\top\widehat{\boldsymbol{h}}_{1(1)}$	$\sqrt{2n}\boldsymbol{h}_1^\top\widehat{\boldsymbol{h}}_2$	$\sqrt{2n}\boldsymbol{h}_2^\top\widehat{\boldsymbol{h}}_1$	$\sqrt{2n}\boldsymbol{h}_1^\top\widetilde{\boldsymbol{h}}_2^*$	$\sqrt{2n}\boldsymbol{h}_2^\top\widetilde{\boldsymbol{h}}_1^*$
$mean$	0.015	-0.069	0.025	-0.007	0.013	0.004
var	1.144	1.140	0.670	0.671	0.600	0.606
	Half-PCA		Full-PCA		Cross-PCA	
	$\sqrt{n}(\boldsymbol{h}_1^\top\widehat{\boldsymbol{h}}_{1(1)} - 1)$	$\sqrt{n}(\boldsymbol{h}_2^\top\widehat{\boldsymbol{h}}_{2(1)} - 1)$	$\sqrt{2n}(\boldsymbol{h}_1^\top\widehat{\boldsymbol{h}}_1 - 1)$	$\sqrt{2n}(\boldsymbol{h}_2^\top\widehat{\boldsymbol{h}}_2 - 1)$	$\sqrt{2n}(\boldsymbol{h}_1^\top\widetilde{\boldsymbol{h}}_1^* - 1)$	$\sqrt{2n}(\boldsymbol{h}_2^\top\widetilde{\boldsymbol{h}}_2^* - 1)$
$mean$	-0.090	-0.090	-0.036	-0.037	-0.031	-0.031
var	0.2205	0.2204	0.049	0.049	0.008	0.007