Introduction to Smooth Manifolds: Chapter #4

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Problem 1

Use the inclusion map $\mathbb{H}^n \hookrightarrow \mathbb{R}^n$ to show that Theorem 4.5 does not extend to the case in which M is a manifold with boundary.

Let $p \in \partial\mathbb{H}^n$ and note that:

1. The inclusion $\mathbb{H}^n \hookrightarrow \mathbb{R}^n$ is smooth.
2. $d\iota_p$ is invertible. **[NOTE: why? look up definition]**

However, there are no neighborhoods $U \ni p, V \ni F(P)$, s.t. $\iota|_U : U \mapsto V$ is a diffeomorphism. To see this, note that every U is open in $\mathbb{H}^N$ but each $V \ni F(p)$ is not open in $\mathbb{R}^n$.

Problem 2

Suppose M is a smooth manifold (without boundary), N is a smooth manifold with boundary, and $F : M \mapsto N$ is smooth. SHow that if $p \in M$ is a point such that dF_p is nonsingular, then $F(p) \in \mathbf{Int}N$.

Problem 3

Formulate and prove a version of the rank theorem for a map of constant rank whose domain is a smooth manifold with boundary.

Problem 4

Let $\gamma : \mathbb{R} \mapsto \mathbb{T}^2$ be the curve of Example 4.20. Show that the image set $\gamma(R)$ is dense in $\mathbb{T}^2$.

Let $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1 \subseteq \mathbb{C}^2$ denote the torus and let α be any irrational number and consider the map $\gamma : \mathbb{R} \mapsto \mathbb{T}^2$ given by

$$\gamma(t) = \left(e^{2\pi i t}, e^{2\pi i \alpha t}\right)$$

Let $(a, b) \in \mathbb{T}^2$. To show that the $\gamma(\mathbb{R})$ is dense in $\mathbb{T}^2$, we demonstrate a sequence in $\gamma(\mathbb{R})$ that converges to (a, b). L

Let $\beta \in \mathbb{R}$ s.t. $a = e^{2\pi i \beta}$. Consider the set

Problem 4 continued on next page. . .

$$
\begin{aligned}
\{\gamma(k+\beta)\}_{k\in\mathbb{Z}} &= \left\{\left(e^{2\pi i(k+\beta)}, e^{2\pi i\alpha(k+\beta)}\right)\right\}_{k\in\mathbb{Z}} \\
&= \left\{\left(a, e^{2\pi i\alpha\beta} e^{2\pi i\alpha k}\right)\right\}_{k\in\mathbb{Z}}
\end{aligned}
$$

$\alpha k \mod 1$ is dense in $[0, 1]$ because it is an irrational rotation rotation of the circle, therefore $e^{2\pi i\alpha k}$ is dense in $\mathbb{C}$, therefore there is a subsequence $e^{2\pi i\alpha k_j} \to be^{-2\pi i\alpha\beta}$, therefore

$$
\{\gamma(k+\beta)\}_{k\in\mathbb{Z}} = \left\{\left(a, e^{2\pi i\alpha\beta} e^{2\pi i\alpha k}\right)\right\}_{k\in\mathbb{Z}} \to (a, b)
$$

Problem 5

Let $\mathbb{CP}^n$ denote the n- dimensional complex projective space, as defined in Problem 1-9.

a) **Show that the quotient map $\pi : \mathbb{C}^{n+1} \setminus \{0\} \mapsto \mathbb{CP}^n$ is a surjective smooth submersion.**

b) **Show that $\mathbb{CP}^1$ is diffeomorphic to $\mathbb{S}^2$.**

Problem 6

Let M be a nonempty smooth compact manifold. Show that there is no smooth submersion $F : M \mapsto \mathbb{R}^k$ for any $k > 0$.

Let M be compact and $F : M \mapsto \mathbb{R}^k$ be smooth.
Note that

1. $F(M)$ is compact because M is compact and F is continuous.
2. $F(M)$ is open because F is a smooth submersion.

Because all non-empty compact sets in $\mathbb{R}^k$ are closed, this is a contradiction and such a map cannot exist.

Problem 7

TODO

Problem 8 continued on next page...

Problem 8

This problem shows that the converse of Theorem 4.29 is false. Let $\pi : \mathbb{R}^2 :\mapsto \mathbb{R}$ be defined by $\pi(x, y)_= xy$. Show that π is surjective and smooth, and for each smooth manifold P, a map $F : \mathbb{R} \mapsto P$ is smooth if and only if $F \circ \pi$ is smooth; but π is not a smooth submersion.

Problem 9

Let M be a connected smooth manifold, and let $\pi : E \mapsto M$ be a topological covering map. Complete the proof of Proposition 4.40 by showing that there is only one smooth structure on E such that pi is a smooth covering map. [Hint: use the existence of smooth local sections]

Problem 10

Show that the map $q : \mathbb{S}^n \mapsto \mathbb{RP}^n$ defined in Example 2.13(f) is a smooth covering map.

Define $q : \mathbb{S}^n \mapsto \mathbb{RP}^n$ as the restriction of $\pi : \mathbb{R}^n \setminus \{0\} \mapsto \mathbb{RP}^n$ to $\mathbb{S}^n \subseteq \mathbb{R}^n \setminus \{0\}$.
TODO: Complete.

Problem 11

Show that a topological covering map is proper iff its fibers are finite, and therefore the converse of Proposition 4.46 is fale.

Suppose $\pi : E \mapsto X$ is a topological covering map.

$\Longrightarrow$

Suppose π is proper.

Let $p \in X$.

Suppose $\pi^{-1}(p)$ is infinite, then it admits an infinite cover (the open ball around each point in the fiber) with no finite subcover, and therefore is not compact. Since $\{p\}$ is compact, this is a contradiction.

$\Longleftarrow$

Suppose π has finite fibers.

Let $K \subseteq X$ be compact, let $L = \pi^{-1}(K)$, $D \subseteq L \subset E$ be an infinite set, and $Y = \pi(D)$.

Because $D = \cup_{y \in Y} \pi^{-1}(y)$ is infinite and $\pi^{-1}(y)$ is finite, Y must be infinite. Because $K \supseteq Y$ is compact, Y must contain a limit point $\bar{y}$. We show that $\pi^{-1}(\bar{y})$ is a limit point of D.

Let U be an evenly covered neighborhood of y_i and $U \supseteq \{y_i\} \to \bar{y}$. Because the fiber$\pi_{-1}(\bar{y})$ is finite, π^U contains components V_j mapped homeomorphically onto U by π and therefore, for each i, j there exist $e_{i,j} \in V_j$ s.t. $\pi(e_{i,j}) = y_i$. Because $\{y_i\} \subseteq \pi(D)$, for each i, there is at least one j s.t. $e_{i,j} \in D$. Because there are finitely many j and infinitely many i, there must be a $\hat{j}$ s.t. $e_{i,\hat{j}} \in D$ for infinitely many i. Because $e_{i,\hat{j}}$ converges, $\lim_i e_{i,\hat{j}}$ is a limit point of D.

Therefore π is proper.

Problem 12

Using the covering map $\epsilon^2 : \mathbb{R}^2 \mapsto \mathbb{T}^2$ (see Example 4.35), show that the immersion $X : \mathbb{R}^2 \mapsto \mathbb{R}^3$ defined in example 4.2(d) descends to a smooth embedding of $\mathbb{T}^2$ into $\mathbb{R}^3$. Specifically, show that X passes to the quotient to define a smooth map $\tilde{X} : \mathbb{T}^2 \mapsto \mathbb{R}^3$, and then show that $\tilde{X}$ is a smooth embedding whose image is the given surface of revolution.

Let $\epsilon^2 : \mathbb{R}^2 \mapsto \mathbb{T}^2$, $X : \mathbb{R}^2 \mapsto \mathbb{R}^3$ be given by

$$\begin{aligned} \epsilon^2\left(x^1, x^2\right) &= \left(e^{2\pi i x^1}, e^{2\pi i x^2}\right) \\ X(u, v) &= \Big((2 + \cos 2\pi u)\cos 2\pi v, (2 + \cos 2\pi u)\sin 2\pi v, \sin 2\pi u\Big) \end{aligned}$$

Problem 13

Define a map $F : \mathbb{S}^2 \mapsto \mathbb{R}^4$ by $F(x, y, z) = (x^2 - y^2, xy, xz, yz)$. Using the smooth covering map of Example 2.13(f) and Problem 4-10, show that F descends to a smooth embedding of $\mathbb{RP}^2$ into $\mathbb{R}^4$.