Introduction to Smooth Manifolds: Chapter #5

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Problem 5.1

Consider the map $\Phi : \mathbb{R}^4 \mapsto \mathbb{R}^2$ defined by

$$\Phi(x, y, s, t) = (x^2 + y, x^2 + y^2 + s^2 + t^2 + y)$$

Show that (0,1) is a regular value of Φ, and that the level set $\Phi^{-1}(0, 1)$ is diffeomorphic to $\mathbb{S}^2$.

First note that

$$d\Phi|_p = \begin{pmatrix} 2x_p & 1 & 0 & 0 \\ 2x_p & 2y_p + 1 & 2s_p & 2t_p \end{pmatrix}$$

Note that if $d\Phi|_p$ has rank ≤ 2, then $x_p = s_p = t_p = 0$.
However, at the value $\Phi(p^*) = (0, 1)$, $x_p = s_p = t_p = 0$ implies that $y^2 + y = 1$ therefore $y = \frac{1+\pm\sqrt{6}}{2}$, therefore $2y + 1 \neq 0$. Thus $d\Phi$ has rank 2 for all points on the level curve $\Phi^{-1}(0, 1)$, and thus $(0, 1)$ is a regular value.
Now, the manifold given by $\Phi^{-1}(0, 1)$ must satisfy the system of equations

$$\begin{aligned} x^2 + y &= 0 \\ x^2 + y + y^2 + s^2 + t^2 &= 1 \\ &\iff \\ y^2 + s^2 + t^2 &= 1 \end{aligned}$$

TODO: but why diffeomorphic?

Problem 5.2

Prove Theorem 5.11: If M is a smooth n-manifold with boundary, then with the subspace topology, ∂M is a topological $(n - 1)$-dimensional manifold (without boundary), and has a smooth structure such that it is a properly embedded submanifold of M.

Let $p \in \partial M$ and let (U, ϕ) be a boundary chart, and let $\pi : \mathbb{H}^n \mapsto \mathbb{R}^{n-1}$ be the projection onto the first $n - 1$ coordinates. Define

$$\partial U = \partial M \cap U \qquad \hat{\partial U} = \phi(\partial U) \qquad \psi = \pi \circ \phi|_{\partial U} : \partial U \mapsto \hat{\partial U}$$

Following the argument of Theorem 5.8, $(\partial U, \psi)$ is a smooth chart for ∂M as an $(n - 1)$-dimensional manifold, and the collection of all such charts form a smooth atlas.
Note that ∂M is a closed subset of M and an $(n - 1)$- dimensional topological manifold by Proposition 1.38. Therefore by Proposition 5.5 ∂M is properly embedded in M.

Problem 5.3

Prove Proposition 5.21: Suppose M is a smooth manifold with or without boundary, and $S \subseteq M$ is an immersed submanifold. If any of the following holds, the S is embedded.

Problem 5.3 continued on next page...

a) S **has codimension** 0 **in** M**.**

b) **The inclusion map** $S \subseteq M$ **is proper.**

c) S **is compact.**

First note that the inclusion map is an injective smooth immersion. Then

a) Consequence of Proposition 4.22d applied to the inclusion map. In the case where both S and M have nonempty boundary, 4.22d can be applied to $\mathrm{Int}S$ and ∂S separately.

b) Consequence of Proposition 4.22b applied to the inclusion map.

c) Consequence of Proposition 4.22C applied to the inclusion map.

Problem 5.4

Show that the image of a curve $(-\pi, \pi) \mapsto \mathbb{R}^2$ of Example 4.19 is not an embedded submanifold of $\mathbb{R}^2$. [Be careful: this is not the same as showing that β is not an embedding.

Problem 5.5

Let $\gamma : \mathbb{R} \mapsto \mathbb{T}^2$ be the curve of Example 4.20. Show that$\gamma(\mathbb{R})$ is not an embedded submanifold of the torus. [Remark: the warning in Problem 5-4 applies in this case as well.]

In problem 4-4 it was shown that $\gamma(\mathbb{R})$ is dense in $\mathbb{T}^2$.
Because $\gamma(\mathbb{R})$ is dense in $\mathbb{T}^2$ and $\mathbb{T}^2 \backslash \gamma(\mathbb{R}) \neq \emptyset$, $\gamma(R)$ is not closed in the subspace topology. Therefore the inclusion map

Problem 5.6

Suppose $M \subseteq \mathbb{R}^n$ is an embedded m-dimensional submanifold, and let $UM \subseteq T\mathbb{R}^n$ be the set of all unit tangent vectors to M:

$$UM = \{(x, v) \in T\mathbb{R}^n : x \in M, v \in T_x M, |v| = 1\}.$$

It is called the unit tanget bundle of M. Prove that UM is an embedded $(2m-1)$-dimensional submanifold of $T\mathbb{R}^n \approx \mathbb{R} \times bbR$.

Problem 5.7 continued on next page. . .

Problem 5.7

Let $F : \mathbb{R}^2 \mapsto \mathbb{R}$ be defined by $F(x, y) = x^3 + xy + y^3$. Which level sets of F are embedded submanifolds of $\mathbb{R}^2$? For each level set, prove either that it is or that it is not an embedded submanifold.

Problem 5.8

Suppose M is a smooth n-dimensional manifold and $B \subseteq M$ is a regular coordinate ball. Show that $M \setminus B$ is a smooth manifold with boundary, whose boundary is diffeomorphic to $\mathbb{S}^{n-1}$.

Problem 5.9

Let $S \subseteq \mathbb{R}^2$ be the boundary of the square of side 2 centered at the origin (see Problem 3-5). Show that S does not have a topology and smooth structure in which it is an immeresed submanifold of $\mathbb{R}^2$.

Problem 5.10

For each $a \in \mathbb{R}$, let M_a be the subset of $\mathbb{R}^2$ defined by

$$M_a = \left\{(x, y) : y^2 = x(x-1)(x-a)\right\}.$$

For which values of a is M_a an embedded submanifold of $\mathbb{R}^2$? For which values can M_a be given a topology and smooth structure making it into an immersed submanifold?

Problem 5.11

Let $\Phi : \mathbb{R}^2 \mapsto \mathbb{R}$ be defined by $\Phi(x, y) = x^2 - y^2$.

- **a) Show that $\phi^{-1}(0)$ is not an embedded submanifold of $\mathbb{R}^2$.**
- **b) Can $\phi^{-1}(0)$ be given a topology and smooth structure making it into an immersed submanifold of $\mathbb{R}^2$?**
- **c) Answer the same two questions for $\Psi : \mathbb{R}^2 \mapsto \mathbb{R}$ defined by $\Psi(x, y) = x^2 - y^3$.**

Problem 5.12 continued on next page…

Problem 5.12

Suppose E and M are smooth manifolds with boundary, and $\pi : E \mapsto M$ is a smooth covering map. Show that the restriction of π to each connected component of ∂E is a smooth covering map onto a component of ∂M.

Problem 5.13

Prove that the image of the dense curve on the torus described in Example 4.20 is a weakly embedded submanifold of $\mathbb{T}^2$.

Problem 5.14

Prove Theorem 5.32 (uniqueness of the smooth structure on an immersed submanifold once the topology is given).

Problem 5.15

Show by example that an immersed submanifold $S \subseteq M$ might have more than one topology and smooth structure with respect to which it is an immersed submanifold.

Problem 5.16

Prove Theorem 5.33: If M is a smooth manifold and $S \subseteq M$ is a weakly embedded submanifold, the S has only one topology and smooth structure with respect to which it is an immersed submanifold.

Problem 5.17

Prove Lemma 5.34: Suppose M is a smooth manifold, $S \subseteq M$ is a smooth submanifold, and $f \in C^\infty(S)$.

a) If S is embedded, then there exist a neighborhood U of S in M and a smooth function $\tilde{f} \in C^\infty(U)$ such that $\tilde{f}|_S = f$.

b) If S is properly embedded, then the neighborhood U in part (a) can be taken to be all of M.

Problem 5.17 continued on next page...

Problem 5.18

Suppose M is a smooth manifold and $S \subseteq M$ is a smooth submanifold.

a) **Show that S is embedded if and only if every $f \in C^\infty(S)$ has a smooth extension to a neighborhood of S in M. [Hint: if S is not embedded, let $p \in S$ be a point that is not in the domain of any slice chart. Let U be a neighborhood of p in S that is embedded, and consider a function $f \in C^\infty(S)$ that is supported in U and equal to 1 at p.]**

b) **Show that S is properly embedded if and only if every $f \in C^\infty(S)$ has a smooth extension to all of M.**

Problem 5.19

Suppose $S \subset M$ is an embedded submanifold and $\gamma : J \mapsto M$ is a smooth curve whose image happens to lie in S. Show that $\gamma'(t)$ is in the subspace $T_{\gamma(t)}S$ of $T_{\gamma(t)}M$ for all $t \in J$. Give a counterexample if S is not embedded.

Problem 5.20

Show by giving a counterexample that the conclusion of Proposition 5.37 may be false if S is merely immersed.

Problem 5.21

Prove Proposition 5.47: Suppose M is a smooth manifold and $f \in C^\infty(M)$.

a) **For each regular value b of f, the sublevel set $f^{-1}\big((-\infty, b]\big)$ is a regular domain in M.**

b) **If a and b are two regular values of f with $a < b$, then $f^{-1}\big([a, b]\big)$ is a regular domain in M.**

Problem 5.22

Prove Theorem 5.48: If M is a smooth manifold and $D \subseteq M$ is a regular domain, then there exists a defining function for D. If D is compact, then f can be taken to be a smooth exhaustion function for M.

Problem 5.22 continued on next page...

Problem 5.23

Suppose M is a smooth manifold with boundary, N is a smooth manifold, and $F : M \mapsto N$ is a smooth map. Let $S = F^{-1}(c)$, where $c \in N$ is a regular value for both F and $F|_{\partial M}$. Prove that S is a smooth submanifold with boundary in M, with $\partial S = S \cap \partial M$.