Introduction to Smooth Manifolds: Chapter #3

Chris Phelps

Problem 1

Suppose M and N are smooth manifolds with or without boundary, and $F : M \mapsto N$ is a smooth map. Show that $dF_p : T_pM \mapsto T_{F(p)}N$ is the zero map for each $p \in M$ if and only if F is constant on each component of M.

$\implies$
Assume dF_p is the zero map, let (U, ϕ) be a coordinate chart on M containing p and (V, ψ) be a coordinate chart on N containing $F(p)$.

$\phi^{-1} \circ F \circ \psi$ is a map from $\phi(U) \mapsto \phi(V)$ and $d(\phi^{-1} \circ F \circ \psi) = 0$, so $\phi^{-1} \circ F \circ \psi$ is constant on $\phi^{-1}(U)$. Since ψ is a diffeomorphism this means $\phi^{-1} \circ F$ is and ϕ^{-1} is a diffeomorphism this means F is constant on $U \cap V$. Then we can use the fact that F is constant on every coordinate chart (U, ϕ) to determine that F is constant on M. By the gluing lemma, there is a unique smooth map that agree with this construction on all intersections of smooth charts, therefore it is the constant map.
$\impliedby$
Assume F be constant and let $f \in C^\infty(N)$, then $dF_p(v)(f) = v(f \circ F)$. Note that $f \circ F$ is constant. Therefore by Lemma 3.4a, $v(f \circ F) = 0$. Since $dF_p(v)(f) = 0$ for all f, $dF_p(v) = 0$ for all $v \in T_pM$ and dF_p is the zero map.

Problem 2

Prove Proposition 3.14: Let $M_1, \ldots, M_k$ be smooth manifolds, and for each j , let $\pi_j : M_1 \times \ldots \times M_k \mapsto M_j$ be the projection onto the M_j factor. For any point $p = (p_1, \ldots, p_k) \in M_1 \times \ldots \times M_k$, the map

$$\alpha : T_P(M_1 \times \ldots \times M_k) \mapsto T_{P_1}M_1 \oplus \ldots \oplus T_{p_k}M_k$$

defined by

$$\alpha(v) = (d(\pi_1)_p(v), \ldots, d(\pi_k)_p(v))$$

is an isomorphism.

Problem 3

Prove that if M and N are smooth manifolds, then $T(M \times N)$ is diffeomorphic to $TM \times TN$.

Problem 4

Show that TS_1 is diffeomorphic to $S_1 \times \mathbb{R}$.

Problem 4 continued on next page...

First, it is clear that $T\mathbb{S}^1 = \mathbb{S}^1 \times \mathbb{R}$ as a set. Will attempt to brute force this for learning porpoises.
Pick two points a, b in $\mathbb{S}^1$ and two coordinate charts $(U_\alpha, \phi_\alpha), (U_\beta, \phi_\beta)$ where $U_\alpha = \mathbb{S}^1 \setminus b, U_\beta = \mathbb{S}^1 \setminus a, \phi_\alpha(U_\alpha) = [-\pi, \pi), \phi_\beta(U_\beta) = [0, 2\pi)$. It is clear that this is possible by taking the normal identification of $\mathbb{S}^1$ with the unit circle and letting $a = (1, 0), b = (-1, 0)$. In addition, note that this choice satisfies the smooth manifold chart lemma.
Now take the standard $(\pi^{-1}(U_\alpha), \tilde{\phi_\alpha})$ coordinate maps. Then $\tilde{\phi}_\alpha \circ \tilde{\phi}_\beta^{-1}(x, v) =$
TODO: review and complete

Problem 5

Let $\mathbb{S}^1 \subset \mathbb{R}^2$ be the unit circle, and let $K \subset \mathbb{R}^2$ be the boundary of the square of side 2 centered at the origin. Show that there is a homeomorphism $F : \mathbb{R}^2 \mapsto \mathbb{R}^2$ such that $F(\mathbb{S}^1) = K$ but no diffeomorphism with the same property.

Note that $K = \{(x, y) : \max |x|, |y| = 1\}$.
Let $G : \mathbb{R}^2 \setminus 0 \mapsto \mathbb{R}^2 \setminus 0$ st $G(x, y) = \frac{x^2+y^2}{\max |x|,|y|}$ and let $F(x, y) = \begin{cases} (x \cdot G(x, y), y \cdot G(x, y)) & (x, y) \neq (0, 0) \\ 0 & \text{otherwise} \end{cases}$.
Then on the unit circle

$$F(x, y) = \begin{cases} (\frac{x}{|x|}, \frac{y}{|x|}) & |x| >= |y| \\ (\frac{x}{|y|}, \frac{y}{|y|}) & |x| < |y| \end{cases}$$

So that $F(\mathbb{S}^1) = K$. Proof that this is a homeomorphism is omitted, but basically divide plane on lines $y = x$ and $y = -x$, it is a homeomorphism on each, and stitch back together.
Then observe that $\lim_{x \to 0, y \to 0} F(x, y) = 0$.
To show that there is no diffeomorpishm, consider a smooth curve $\gamma : J \mapsto \mathbb{S}^1$. Suppose that such a diffeomorphism F exists, and let $t_0 \in J$ s.t. $F(\gamma(t_0)) = (1, 1)$.
we need a way to show that t < t0 is on one leg and t0 > j is on the other
Do the normal thing of identifying $\gamma \circ \iota = \gamma$.
Then $F \circ \gamma$ is a diffeomorphism so that $dF \circ d\gamma : TJ \to \mathbb{R}^2$. Taking a limit from $t < t_0$ and $t > t_0$ shows that this map cannot even be continuous.
TODO: review and complete

Problem 6

Consider $\mathbb{S}^3$ as the unit sphere in $\mathbb{C}^2$ under the usual identification. For each $z = (z^1, z^2) \in \mathbb{S}^3$, define a curve $\gamma_z : \mathbb{R} \mapsto \mathbb{S}^3$ by $\gamma_z = e^{it}z^1, e^{it}z^2)$. Show that γ_z is a smooth curve whose velocity is never zero.

Problem 7 continued on next page...

Problem 7

Let M be a smooth manifold with or without boundary and p be a point of M. Let $C_p^\infty(M)$ denote the algebra of germs of smooth real-valued functions at p, and let $\mathcal{D}_p M$ denote the vector space of derivations of $C_p^\infty(M)$. Define a map $\Phi : \mathcal{D}_p M \mapsto T_p M$ by $(\Phi v) f = v([f]_p)$. Show that Φ is an isomorphism.

Problem 8

Let M be a smooth manifold with or without boundary and $p \in M$. Let $\mathcal{V}_p M$ denote the set of equivalence classes of smooth curves starting at p under the relation $\gamma_1 = \gamma_2$ if $(f \circ \gamma)'(0) = (f \circ \gamma_2)'(0)$ for every smooth real valued function f defined in a neighborhood of p. Show that the map $\Psi : \mathcal{V}_p M \mapsto T_p M$ defined by $\Psi[\gamma] = \gamma'(0)$ is well-defined and bijective.