Introduction to Smooth Manifolds: Chapter #5

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Problem 1

Consider the map $\Phi : \mathbb{R}^4 \mapsto \mathbb{R}^2$ defined by

$$\Phi(x, y, s, t) = (x^2 + y, x^2 + y^2 + s^2 + t^2 + y)$$

Show that (0,1) is a regular value of Φ, and that the level set $\Phi^{-1}(0, 1)$ is diffeomorphic to $\mathbb{S}^2$.

First note that

$$d\Phi|_p = \begin{pmatrix} 2x_p & 1 & 0 & 0 \\ 2x_p & 2y_p + 1 & 2s_p & 2t_p \end{pmatrix}$$

Note that if $d\Phi|_p$ has rank ≤ 2, then $x_p = s_p = t_p = 0$.
However, at the value $\Phi(p^*) = (0, 1)$, $x_p = s_p = t_p = 0$ implies that $y^2 + y = 1$ therefore $y = \frac{1 + \pm\sqrt{6}}{2}$, therefore $2y + 1 \neq 0$. Thus $d\Phi$ has rank 2 for all points on the level curve $\Phi^{-1}(0, 1)$, and thus $(0, 1)$ is a regular value.
Now, the manifold given by $\Phi^{-1}(0, 1)$ must satisfy the system of equations

$$\begin{aligned} x^2 + y &= 0 \\ x^2 + y + y^2 + s^2 + t^2 &= 1 \\ &\iff \\ y^2 + s^2 + t^2 &= 1 \end{aligned}$$

TODO: but why diffeomorphic?

Problem 2

Prove Theorem 5.11: If M is a smooth n-manifold with boundary, then with the subspace topology, ∂M is a topological $(n - 1)$-dimensional manifold (without boundary), and has a smooth structure such that it is a properly embedded submanifold of M.

Problem 3

Prove Proposition 5.21: Suppose M is a smooth manifold with or without boundary, and $S \subseteq M$ is an immersed submanifold. If any of the following holds, the S is embedded.

a) **S has codimension 0 in M.**

b) **The inclusion map $S \subseteq M$ is proper.**

c) **S is compact.**

Problem 4 continued on next page...

Problem 4

Show that the image of a curve $(-\pi, \pi) \mapsto \mathbb{R}^2$ of Example 4.19 is not an embedded submanifold of $\mathbb{R}^2$. [Be careful: this is not the same as showing that β is not an embedding.

Problem 5

Let $\gamma : \mathbb{R} \mapsto \mathbb{T}^2$ be the curve of Example 4.20. Show that$\gamma(\mathbb{R})$ is not an embedded submanifold of the torus. [Remark: the warning in Problem 5-4 applies in this case as well.]

Problem 6

Suppose $M \subseteq \mathbb{R}^n$ is an embedded m-dimensional submanifold, and let $UM \subseteq T\mathbb{R}^n$ be the set of all unit tangent vectors to M:

$$UM = \{(x, v) \in T\mathbb{R}^n : x \in M, v \in T_xM, |v| = 1\}.$$

It is called the unit tanget bundle of M. Prove that UM is an embedded $(2m-1)$-dimensional submanifold of $T\mathbb{R}^n \approx \mathbb{R} \times bbR$.

Problem 7

Let $F : \mathbb{R}^2 \mapsto \mathbb{R}$ be defined by $F(x, y) = x^3 + xy + y^3$. Which level sets of F are embedded submanifolds of $\mathbb{R}^2$? For each level set, prove either that it is or that it is not an embedded submanifold.

Problem 8

Suppose M is a smooth n-dimensional manifold and $B \subseteq M$ is a regular coordinate ball. Show that $M \setminus B$ is a smooth manifold with boundary, whose boundary is diffeomorphic to $\mathbb{S}^{n-1}$.

Problem 9

Let $S \subseteq \mathbb{R}^2$ be the boundary of the square of side 2 centered at the origin (see Problem 3-5). Show that S does not have a topology and smooth structure in which it is an immeresed submanifold of $\mathbb{R}^2$.

Problem 9 continued on next page...

Problem 10

For each $a \in \mathbb{R}$, let M_a be the subset of $\mathbb{R}^2$ defined by

$$M_a = \left\{(x, y) : y^2 = x(x-1)(x-a)\right\}.$$

For which values of a is M_a an embedded submanifold of $\mathbb{R}^2$? For which values can M_a be given a topology and smooth structure making it into an immersed submanifold?

Problem 11

Let $\Phi : \mathbb{R}^2 \mapsto \mathbb{R}$ be defined by $\Phi(x, y) = x^2 - y^2$.

a) Show that $\phi^{-1}(0)$ is not an embedded submanifold of $\mathbb{R}^2$.

b) Can $\phi^{-1}(0)$ be given a topology and smooth structure making it into an immersed submanifold of $\mathbb{R}^2$?

c) Answer the same two questions for $\Psi : \mathbb{R}^2 \mapsto \mathbb{R}$ defined by $\Psi(x, y) = x^2 - y^3$.

Problem 12

Suppose E and M are smooth manifolds with boundary, and $\pi : E \mapsto M$ is a smooth covering map. Show that the restriction of π to each connected component of ∂E is a smooth covering map onto a component of ∂M.

Problem 13

Prove that the image of the dense curve on the torus described in Example 4.20 is a weakly embedded submanifold of $\mathbb{T}^2$.

Problem 14

Prove Theorem 5.32 (uniqueness of the smooth structure on an immersed submanifold once the topology is given).

Problem 14 continued on next page...

Problem 15

Show by example that an immersed submanifold $S \subseteq M$ might have more than one topology and smooth structure with respect to which it is an immersed submanifold.

Problem 16

Prove Theorem 5.33: If M is a smooth manifold and $S \subseteq M$ is a weakly embedded submanifold, the S has only one topology and smooth structure with respect to which it is an immersed submanifold.

Problem 17

Prove Lemma 5.34: Suppose M is a smooth manifold, $S \subseteq M$ is a smooth submanifold, and $f \in C^\infty(S)$.

a) **If S is embedded, then there exist a neighborhood U of S in M and a smooth function $\tilde{f} \in C^\infty(U)$ such that $\tilde{f}|_S = f$.**

b) **If S is properly embedded, then the neighborhood U in part (a) can be taken to be all of M.**

Problem 18

Suppose M is a smooth manifold and $S \subseteq M$ is a smooth submanifold.

a) **Show that S is embedded if and only if every $f \in C^\infty(S)$ has a smooth extension to a neighborhood of S in M. [Hint: if S is not embedded, let $p \in S$ be a point that is not in the domain of any slice chart. Let U be a neighborhood of p in S that is embedded, and consider a function $f \in C^\infty(S)$ that is supported in U and equal to 1 at p.]**

b) **Show that S is properly embedded if and only if every $f \in C^\infty(S)$ has a smooth extension to all of M.**

Problem 19 continued on next page...

Problem 19

Suppose $S \subset M$ is an embedded submanifold and $\gamma : J \mapsto M$ is a smooth curve whose image happens to lie in S. Show that $\gamma'(t)$ is in the subspace $T_{\gamma(t)}S$ of $T_{\gamma(t)}M$ for all $t \in J$. Give a counterexample if S is not embedded.

Problem 20

Show by giving a counterexample that the conclusion of Proposition 5.37 may be false if S is merely immersed.

Problem 21

Prove Proposition 5.47: Suppose M is a smooth manifold and $f \in C^\infty(M)$.

- **a) For each regular value b of f, the sublevel set $f^{-1}\big((-\infty, b]\big)$ is a regular domain in M.**
- **b) If a and b are two regular values of f with $a < b$, then $f^{-1}\big([a, b]\big)$ is a regular domain in M.**

Problem 22

Prove Theorem 5.48: If M is a smooth manifold and $D \subseteq M$ is a regular domain, then there exists a defining function for D. If D is compact, then f can be taken to be a smooth exhaustion function for M.

Problem 23

Suppose M is a smooth manifold with boundary, N is a smooth manifold, and $F : M \mapsto N$ is a smooth map. Let $S = F^{-1}(c)$, where $c \in N$ is a regular value for both F and $F|_{\partial M}$. Prove that S is a smooth submanifold with boundary in M, with $\partial S = S \cap \partial M$.