Introduction to Smooth Manifolds: Chapters 1-9

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CHAPTER 1

CHAPTER 2

Problem 2.1

Define $f : \mathbb{R} \mapsto \mathbb{R}$ by (Heaviside function)
Show that for every $x \in \mathbb{R}$, there are smooth coordinate charts (U, ϕ) containing x and (V, ψ) containing $f(x)$ such that $\psi \cdot f \cdot \phi^{-1}$ is smooth as a map from $(\phi(U \cap f^{-1}(V)))$ to $\psi(V)$ but f is not smooth in the sense we have defined in this chapter.

For all $x \neq 0$ this is obvious by selecting and U not containing 0, then f_U is linear thus smooth.
For $x = 0$, we have $f(x) = 1$, so we can select the neighborhood $U = (-\delta, \delta)$ and the neighborhood $V = (-\epsilon, \epsilon)$, and let ϕ, ψ be the identity.
Then $f^-1(V) = (-\infty, 0]$ so that $U \cap f^{-1}(V) = (-\epsilon, 0]$ and $(\phi \circ f \circ \phi^{-1})_{U \cap f^{-1}(V)} = 0$, so it is smooth. However, it is not $\phi \circ f \circ \phi^{-1}$ is not smooth on the open set U, so is not smooth in the context defined in the chapter.
As an aside, this is consistent with Proposition 2.5 because

1. In condition (a), it violates the openness part of the statement.
2. In condition (b), it violates the continuous.

Problem 2.2

Prove Proposition 2.12: Suppose $M_1 \times \ldots \times M_k$ and N are smooth manifolds with or without boundary, such that at most one of $M_1, \ldots, M_k$ has nonempty boundary. For each i, let $pi_i : M_1 \times \ldots \times M_k \mapsto M_i$ denote the projection onto the M_i factor. A map $F : N \mapsto M_1 \times \ldots \times M_k$ is smooth if and only if each of the component maps $F_i = \pi_i \circ F : N\ M_i$ is smooth.

Observation: A function $f : \mathbb{R}^n \mapsto \mathbb{R}^m$ is smooth if and only if its coordinate functions are smooth. (p. 11).
$\implies$
Trivial.

$\impliedby$
Let $p \in N$. By Proposition 2.5 (a), there exists smooth chart $(U_i, \psi_i), (V_i, \phi_i)$ such that
1. $F_i(p) \in V_i$ 2. $U_i \cap F_i^{-1}$ is open in N 3. $\psi_i \circ F_i \circ \phi_i^{-1}$ is smooth on from $\phi_i(U_i \cap F_i^{-1})$ to $\psi_i(V_i)$
Let $U = \cap_i U_i$ and $V = \Pi_i V_i$. First observe that for each i
a. $\phi_i|_U = \phi$

b. $F_i|_U$ is smooth for each i.
Then let ψ_i^j be the j th coordinate map of ψ_i, and note that $\psi|_U = (\psi_1^1, \dots, \psi_k^{n_k})$ is a smooth map with ϕ_i^j the coordinate functions.
We then establish the following properties
1. $F(p) \in V$: clear from definition of V 2. $U \cap F^{-1}(V)$ is open in N: note that $F^{-1}(V) = F^{-1}(\Pi_i V_i) = F^{-1}(\cap_i (V_i \times \Pi_{j \neq i} M_j))$ 3. $\psi \circ F \circ \phi^{-1}$
By a-b above and the definition of ψ
$\psi \circ F \circ \phi^{-1} = (\psi_1 \circ F_1 \circ \phi_1^{-1}, \dots, \psi_k \circ F_k \circ \phi_k^{-1})$ is a function $mathbbR^n :\mapsto \mathbb{R}^m$, and is smooth in each coordinate, therefore it is smooth.
Because 1-3 are satisfied, the statement follows from Proposition 2.5

Problem 2.3

For each of the following maps between spheres, compute sufficiently many coordinate representations to prove that it is smooth.

1. $p_n : \mathbb{S}^1 \mapsto \mathbb{S}^1$ **is the nth power map (**$p_n(z) = z^n$**)**
2. $\alpha : \mathbb{S}^n \mapsto \mathbb{S}^n$ **is the antipodal map (**$\alpha(x) = -x$**)**
3. $F : \mathbb{S}^3 \mapsto \mathbb{S}^2$ **is given by** $F(w, z) = (z\bar{w} + w\bar{z}, iw\bar{z} - iz\bar{w}, z\bar{z} - w\bar{w})$

Problem 2.4

Show that the inclusion map $\bar{\mathbb{B}}^n \mapsto \mathbb{R}^n$ is smooth when $\bar{\mathbb{B}}^n$ is regarded as a manifold with boundary.

Problem 2.5

Let $\mathbb{R}$ be the real line with its standard smooth structure, and let $\tilde{\mathbb{R}}$ denote the same topological manifold with the smooth structure defined in Example 1.23. Let $f : \mathbb{R} \mapsto \mathbb{R}$ be a function that is smooth in the usual sense.

1. **Show that f is also smooth as a map from $\mathbb{R}$ to $\tilde{\mathbb{R}}$.**
2. **Show that f is smooth as a map from $\tilde{\mathbb{R}}$ to $\mathbb{R}$ if and only if $f^(n)(0) = 0$ whenever n is not an integral multiple of 3.**

Problem 2.6

Let $P : \mathbb{R}^n \setminus \{0\} \mapsto \mathbb{R}^{k+1} \setminus \{0\}$ be a smooth function, and suppose that for some $d \in \mathbb{Z}, P(\gamma x) = \gamma^d P(x)$ for all $\lambda \in \mathbb{R} \setminus \{0\}$ and $x \in \mathbb{R}^n \setminus \{0\}$. Show that the map $\tilde{P} : \mathbb{RP}^n \mapsto \mathbb{RP}^k$ defined by $([x]) = [P(x)]$ is well-defined and smooth.

Problem 2.7

Let M be a nonempty smooth n-manifold with or without boundary, and suppose $n \geq 1$. Show that the vector space $C^\infty(M)$ is infinite-dimensional.

Problem 2.8

Define $F : \mathbb{R}^n \mapsto \mathbb{RP}^n$ by $F(x^1, \ldots, x^n) = [x^1, \ldots, x^n, 1]$. Show that F is a diffeomorphism onto a dense open subset of $\mathbb{RP}^n$ Do the same for $G : \mathbb{C}' \to \mathbb{CP}^n$ defined by $G(z^1, \ldots, z^n) = [z^1, \ldots, z^n, 1]$.

Problem 2.9

Problem 2.10

For any topological space M, let $C(M)$ denote the algebra of continuous functions $f : M \mapsto \mathbb{R}$. Given a continuous map $F : M \mapsto \mathbb{N}$, define $F^* : C(N) \mapsto C(M)$ by $F^*(f) = f \circ F$.

1. **Show that F^* is a linear map.**
2. **Suppose that M and N are smooth manifolds. Show that $F : M \mapsto N$ is smooth if and only if $F^*(C^\infty(N)) \subseteq C^\infty(M)$.**
3. **Suppose $F : M \mapsto N$ is a homeomorphism between smooth manifolds. Show that it is a diffeomorphism if and only if F^* restricts to an isomorphism from $C^\infty(N)$ to $C^\infty(M)$.**

Problem 2.11 continued on next page...

Problem 2.11

Suppose V is a real vector space of dimension $n \geq 1$. Define the projectivization of V, denoted $\mathbb{P}(V)$, to be the set of 1-dimensional linear subspaces of V, with the quotient topology induced by the map $\pi : V \setminus \{0\} \mapsto \mathbb{P}(V)$ that sends x to its span. Show that $\mathbb{P}(V)$ is a topological $(n-1)$-submanifold, and has a unique smooth structure with the property that for each basis $(E_1, \ldots, E_n)$ for V, the map $E : \mathbb{RP}^{n-1} \mapsto \mathbb{P}(V)$ defined by $E[v^1, \ldots, v^n] = [v^1 E_i]$ (where brackes denote equivalence classes) is a diffeomorphism.

Problem 2.12

State and prove an anology of 2-11 in complex vector spaces.

Problem 2.13

Suppose M is a topological space with the property that for every indexed open cover $\mathcal{X}$ of M, there exists a partition of unity subordinate to $\mathcal{X}$. Show that M is paracompact.

Problem 2.14

Suppose that A and B are disjoint closed subsets of a smooth manifold M. Show that there exists $f \in C^\infty(M)$ such that $0 \leq f(x) \leq 1$ for all $x \in M$, $f^{-1}(0) = A$, and $f^{-1}(1) = B$.

We use without proof the fact that topological manifolds are T_4.
Therefore for A, B disjoint closed sets there exist disjoint neighborhoods $U_A \supseteq A, U_B \supseteq B$.
By Proposition 2.2.5 there exist smooth bump fuctinos $\psi_A, \psi_B : M \mapsto \mathbb{R}$ s.t.

1. $\phi_A^{-1}(1) = A$
2. $\phi_B^{-1}(1) = B$
3. $\text{supp}\phi_A \subseteq U_A$
4. $\text{supp}\phi_B \subseteq U_B$

Note that in particular this implies

1. $\phi_A(p) = 0$ for all $p \in B$
2. $\phi_B(p) = 0$ for all $p \in A$

Let $f = \frac{1}{2}(1 - \phi_A + \phi_B)$. f is smooth because it is a linear combination of smooth functions, and clearly $0 \leq f(x) \leq 1$ for all $x \in M$.

Problem 2.14 continued on next page...

Then for all $x \in A$

$$\begin{aligned} f(x) &= \frac{1}{2}\left(1 - \phi_A(x) + \phi_B(x)\right) \\ &= \frac{1}{2}\left(1 - 1 + 0\right) \\ &= 0. \end{aligned}$$

Similarly, for all $x \in B$

$$\begin{aligned} f(x) &= \frac{1}{2}\left(1 - \phi_A(x) + \phi_B(x)\right) \\ &= \frac{1}{2}\left(1 - 0 + 1\right) \\ &= 1. \end{aligned}$$

CHAPTER 3

Problem 3.1

Suppose M and N are smooth manifolds with or without boundary, and $F : M \mapsto N$ is a smooth map. Show that $dF_p : T_pM \mapsto T_{F(p)}N$ is the zero map for each $p \in M$ if and only if F is constant on each component of M.

$\Longrightarrow$
Assume dF_p is the zero map, let (U, ϕ) be a coordinate chart on M containing p and (V, ψ) be a coordinate chart on N containing $F(p)$.

$\phi^{-1} \circ F \circ \psi$ is a map from $\phi(U) \mapsto \phi(V)$ and $d(\phi^{-1} \circ F \circ \psi) = 0$, so $\phi^{-1} \circ F \circ \psi$ is constant on $\phi^{-1}(U)$. Since ψ is a diffeomorphism this means $\phi^{-1} \circ F$ is and ϕ^{-1} is a diffeomorphism this means F is constant on $U \cap V$. Then we can use the fact that F is constant on every coordinate chart (U, ϕ) to determine that F is constant on M. By the gluing lemma, there is a unique smooth map that agree with this construction on all intersections of smooth charts, therefore it is the constant map.
$\Longleftarrow$
Assume F be constant and let $f \in C^\infty(N)$, then $dF_p(v)(f) = v(f \circ F)$. Note that $f \circ F$ is constant. Therefore by Lemma 3.4a, $v(f \circ F) = 0$. Since $dF_p(v)(f) = 0$ for all f, $dF_p(v) = 0$ for all $v \in T_pM$ and dF_p is the zero map.

Problem 3.2

Prove Proposition 3.14: Let $M_1, \ldots, M_k$ be smooth manifolds, and for each j, let $\pi_j : M_1 \times \ldots \times M_k \mapsto M_j$ be the projection onto the M_j factor. For any point $p = (p_1, \ldots, p_k) \in M_1 \times \ldots \times M_k$, the map

$$\alpha : T_P(M_1 \times \ldots \times M_k) \mapsto T_{P_1}M_1 \oplus \ldots \oplus T_{p_k}M_k$$

defined by

$$\alpha(v) = (d(\pi_1)_p(v), \ldots, d(\pi_k)_p(v))$$

is an isomorphism.

Problem 3.3

Prove that if M and N are smooth manifolds, then $T(M \times N)$ is diffeomorphic to $TM \times TN$.

Problem 3.3 continued on next page...

Problem 3.4

Show that TS_1 is diffeomorphic to $S_1 \times \mathbb{R}$.

First, it is clear that $T\mathbb{S}^1 = \mathbb{S}^1 \times \mathbb{R}$ as a set. Will attempt to brute force this for learning porpoises.
Pick two points a, b in $\mathbb{S}^1$ and two coordinate charts $(U_\alpha, \phi_\alpha), (U_\beta, \phi_\beta)$ where $U_\alpha = \mathbb{S}^1 \setminus b, U_\beta = \mathbb{S}^1 \setminus a, \phi_\alpha(U_\alpha) = [-\pi, \pi), \phi_\beta(U_\beta) = [0, 2\pi)$. It is clear that this is possible by taking the normal identification of $\mathbb{S}^1$ with the unit circle and letting $a = (1, 0), b = (-1, 0)$. In addition, note that this choice satisfies the smooth manifold chart lemma.
Now take the standard $(\pi^{-1}(U_\alpha), \tilde{\phi_\alpha})$ coordinate maps. Then $\tilde{\phi}_\alpha \circ \tilde{\phi}_\beta^{-1}(x, v) =$
TODO: review and complete

Problem 3.5

Let $\mathbb{S}^1 \subset \mathbb{R}^2$ be the unit circle, and let $K \subset \mathbb{R}^2$ be the boundary of the square of side 2 centered at the origin. Show that there is a homeomorphism $F : \mathbb{R}^2 \mapsto \mathbb{R}^2$ such that $F(\mathbb{S}^1) = K$ but no diffeomorphism with the same property.

Note that $K = \{(x, y) : \max |x|, |y| = 1\}$.
Let $G : \mathbb{R}^2 \setminus 0 \mapsto \mathbb{R}^2 \setminus 0$ st $G(x, y) = \frac{x^2+y^2}{\max |x|, |y|}$ and let $F(x, y) = \begin{cases} (x \cdot G(x, y), y \cdot G(x, y)) & (x, y) \neq (0, 0) \\ 0 & \text{otherwise} \end{cases}$.
Then on the unit circle

$$F(x, y) = \begin{cases} (\frac{x}{|x|}, \frac{y}{|x|}) & |x| >= |y| \\ (\frac{x}{|y|}, \frac{y}{|y|}) & |x| < |y| \end{cases}$$

So that $F(\mathbb{S}^1) = K$. Proof that this is a homeomorphism is omitted, but basically divide plane on lines $y = x$ and $y = -x$, it is a homeomorphism on each, and stitch back together.
Then observe that $\lim_{x \to 0, y \to 0} F(x, y) = 0$.
To show that there is no diffeomorpishm, consider a smooth curve $\gamma : J \mapsto \mathbb{S}^1$. Suppose that such a diffeomorphism F exists, and let $t_0 \in J$ s.t. $F(\gamma(t_0)) = (1, 1)$.
we need a way to show that t < t0 is on one leg and t0 > j is on the other
Do the normal thing of identifying $\gamma \circ \iota = \gamma$.
Then $F \circ \gamma$ is a diffeomorphism so that $dF \circ d\gamma : TJ \to \mathbb{R}^2$. Taking a limit from $t < t_0$ and $t > t_0$ shows that this map cannot even be continuous.
TODO: review and complete

Problem 3.6

Consider $\mathbb{S}^3$ as the unit sphere in $\mathbb{C}^2$ under the usual identification. For each $z = (z^1, z^2) \in \mathbb{S}^3$, define a curve $\gamma_z : \mathbb{R} \mapsto \mathbb{S}^3$ by $\gamma_z = e^{it}z^1, e^{it}z^2)$. Show that γ_z is a smooth curve whose velocity is never zero.

Problem 3.7

Let M be a smooth manifold with or without boundary and p be a point of M. Let $C_p^\infty(M)$ denote the algebra of germs of smooth real-valued functions at p, and let $\mathcal{D}_pM$ denote the vector space of derivations of $C_p^\infty(M)$. Define a map $\Phi : \mathcal{D}_pM \mapsto T_pM$ by $(\Phi v)f = v([f]_p)$. Show that Φ is an isomorphism.

Problem 3.8

Let M be a smooth manifold with or without boundary and $p \in M$. Let $\mathcal{V}_pM$ denote the set of equivalence classes of smooth curves starting at p under the relation $\gamma_1 = \gamma_2$ if $(f \circ \gamma)'(0) = (f \circ \gamma_2)'(0)$ for every smooth real valued function f defined in a neighborhood of p. Show that the map $\Psi : \mathcal{V}_pM \mapsto T_pM$ defined by $\Psi[\gamma] = \gamma'(0)$ is well-defined and bijective.

CHAPTER 4

Problem 4.1

Use the inclusion map $\mathbb{H}^n \hookrightarrow \mathbb{R}^n$ to show that Theorem 4.5 does not extend to the case in which M is a manifold with boundary.

Let $p \in \partial\mathbb{H}^n$ and note that:

1. The inclusion $\mathbb{H}^n \hookrightarrow \mathbb{R}^n$ is smooth.
2. $d\iota_p$ is invertible. **[NOTE: why? look up definition]**

However, there are no neighborhoods $U \ni p, V \ni F(P)$, s.t. $\iota|_U : U \mapsto V$ is a diffeomorphism. To see this, note that every U is open in $\mathbb{H}^N$ but each $V \ni F(p)$ is not open in $\mathbb{R}^n$.

Problem 4.2

Suppose M is a smooth manifold (without boundary), N is a smooth manifold with boundary, and $F : M \mapsto N$ is smooth. SHow that if $p \in M$ is a point such that dF_p is nonsingular, then $F(p) \in \mathbf{Int} N$.

Problem 4.3

Formulate and prove a version of the rank theorem for a map of constant rank whose domain is a smooth manifold with boundary.

Problem 4.4

Let $\gamma : \mathbb{R} \mapsto \mathbb{T}^2$ be the curve of Example 4.20. Show that the image set $\gamma(R)$ is dense in $\mathbb{T}^2$.

Let $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1 \subseteq \mathbb{C}^2$ denote the torus and let α be any irrational number and consider the map $\gamma : \mathbb{R} \mapsto \mathbb{T}^2$ given by

$$\gamma(t) = \left(e^{2\pi it}, e^{2\pi i\alpha t}\right)$$

Let $(a, b) \in \mathbb{T}^2$. To show that the $\gamma(\mathbb{R})$ is dense in $\mathbb{T}^2$, we demonstrate a sequence in $\gamma(\mathbb{R})$ that converges to (a, b). L

Let $\beta \in \mathbb{R}$ s.t. $a = e^{2\pi i\beta}$. Consider the set

$$\begin{aligned}\{\gamma(k+\beta)\}_{k\in\mathbb{Z}} &= \left\{\left(e^{2\pi i(k+\beta)}, e^{2\pi i\alpha(k+\beta)}\right)\right\}_{k\in\mathbb{Z}} \\ &= \left\{\left(a, e^{2\pi i\alpha\beta}e^{2\pi i\alpha k}\right)\right\}_{k\in\mathbb{Z}}\end{aligned}$$

$\alpha k \mod 1$ is dense in $[0, 1]$ because it is an irrational rotation rotation of the circle, therefore $e^{2\pi i\alpha k}$ is dense in $\mathbb{C}$, therefore there is a subsequence $e^{2\pi i\alpha k_j} \to be^{-2\pi i\alpha\beta}$, therefore

$$\{\gamma(k+\beta)\}_{k\in\mathbb{Z}} = \left\{\left(a, e^{2\pi i\alpha\beta}e^{2\pi i\alpha k}\right)\right\}_{k\in\mathbb{Z}} \to (a, b)$$

Problem 4.5

Let $\mathbb{CP}^n$ denote the n- dimensional complex projective space, as defined in Problem 1-9.

a) Show that the quotient map $\pi : \mathbb{C}^{n+1} \setminus \{0\} \mapsto \mathbb{CP}^n$ is a surjective smooth submersion.

b) Show that $\mathbb{CP}^1$ is diffeomorphic to $\mathbb{S}^2$.

Problem 4.6

Let M be a nonempty smooth compact manifold. Show that there is no smooth submersion $F : M \mapsto \mathbb{R}^k$ for any $k > 0$.

Let M be compact and $F : M \mapsto \mathbb{R}^k$ be smooth.
Note that

1. $F(M)$ is compact because M is compact and F is continuous.
2. $F(M)$ is open because F is a smooth submersion.

Because all non-empty compact sets in $\mathbb{R}^k$ are closed, this is a contradiction and such a map cannot exist.

Problem 4.7

TODO

Problem 4.8

This problem shows that the converse of Theorem 4.29 is false. Let $\pi : \mathbb{R}^2 :\mapsto \mathbb{R}$ be defined by $\pi(x, y)_{=}xy$. Show that π is surjective and smooth, and for each smooth manifold P, a map $F : \mathbb{R} \mapsto P$ is smooth if and only if $F \circ \pi$ is smooth; but π is not a smooth submersion.

Problem 4.9

Let M be a connected smooth manifold, and let $\pi : E \mapsto M$ be a topological covering map. Complete the proof of Proposition 4.40 by showing that there is only one smooth structure on E such that pi is a smooth covering map. [Hint: use the existence of smooth local sections]

Problem 4.10

Show that the map $q : \mathbb{S}^n \mapsto \mathbb{RP}^n$ defined in Example 2.13(f) is a smooth covering map.

Define $q : \mathbb{S}^n \mapsto \mathbb{RP}^n$ as the restriction of $\pi : \mathbb{R}^n \setminus \{0\} \mapsto \mathbb{RP}^n$ to $\mathbb{S}^n \subseteq \mathbb{R}^n \setminus \{0\}$.
TODO: Complete.

Problem 4.11

Show that a topological covering map is proper iff its fibers are finite, and therefore the converse of Proposition 4.46 is fale.

Suppose $\pi : E \mapsto X$ is a topological covering map.

$\Longrightarrow$

Suppose π is proper.

Let $p \in X$.

Suppose $\pi^{-1}(p)$ is infinite, then it admits an infinite cover (the open ball around each point in the fiber) with no finite subcover, and therefore is not compact. Since $\{p\}$ is compact, this is a contradiction.

$\Longleftarrow$

Suppose π has finite fibers.

Let $K \subseteq X$ be compact, let $L = \pi^{-1}(K)$, $D \subsetneq L \subset E$ be an infinite set, and $Y = \pi(D)$. Because $D = \cup_{y \in Y} \pi^{-1}(y)$ is infinite and $\pi^{-1}(y)$ is finite, Y must be infinite. Because $K \supseteq Y$ is compact, Y must contain a limit point $\bar{y}$. We show that $\pi^{-1}(\bar{y})$ is a limit point of D.

Let U be an evenly covered neighborhood of y_i and $U \supseteq \{y_i\} \to \bar{y}$. Because the fiber$\pi_{-1}(\bar{y})$ is finite, π^U contains components V_j mapped homeomorphically onto U by π and therefore, for each i, j there exist $e_{i,j} \in V_j$ s.t. $\pi(e_{i,j}) = y_i$. Because $\{y_i\} \subseteq \pi(D)$, for each i, there is at least one j s.t. $e_{i,j} \in D$. Because there are finitely many j and infinitely many i, there must be a $\hat{j}$ s.t. $e_{i,\hat{j}} \in D$ for infinitely many i. Because $e_{i,\hat{j}}$ converges, $\lim_i e_{i,\hat{j}}$ is a limit point of D.

Therefore π is proper.

Problem 4.12

Using the covering map $\epsilon^2 : \mathbb{R}^2 \mapsto \mathbb{T}^2$ (see Example 4.35), show that the immersion $X : \mathbb{R}^2 \mapsto \mathbb{R}^3$ defined in example 4.2(d) descends to a smooth embedding of $\mathbb{T}^2$ into $\mathbb{R}^3$. Specifically, show that X passes to the quotient to define a smooth map $\tilde{X} : \mathbb{T}^2 \mapsto \mathbb{R}^3$, and then show that $\tilde{X}$ is a smooth embedding whose image is the given surface of revolution.

Let $\epsilon^2 : \mathbb{R}^2 \mapsto \mathbb{T}^2$, $X : \mathbb{R}^2 \mapsto \mathbb{R}^3$ be given by

$$\begin{aligned} \epsilon^2\left(x^1, x^2\right) &= \left(e^{2\pi i x^1}, e^{2\pi i x^2}\right) \\ X(u, v) &= \Big((2 + \cos 2\pi u)\cos 2\pi v, (2 + \cos 2\pi u)\sin 2\pi v, \sin 2\pi u\Big) \end{aligned}$$

Problem 4.13

Define a map $F : \mathbb{S}^2 \mapsto \mathbb{R}^4$ by $F(x, y, z) = (x^2 - y^2, xy, xz, yz)$. Using the smooth covering map of Example 2.13(f) and Problem 4-10, show that F descends to a smooth embedding of $\mathbb{RP}^2$ into $\mathbb{R}^4$.

CHAPTER 5

Problem 5.1

Consider the map $\Phi : \mathbb{R}^4 \mapsto \mathbb{R}^2$ defined by

$$\Phi(x, y, s, t) = (x^2 + y, x^2 + y^2 + s^2 + t^2 + y)$$

Show that (0,1) is a regular value of Φ, and that the level set $\Phi^{-1}(0,1)$ is diffeomorphic to $\mathbb{S}^2$.

First note that

$$d\Phi|_p = \begin{pmatrix} 2x_p & 1 & 0 & 0 \\ 2x_p & 2y_p + 1 & 2s_p & 2t_p \end{pmatrix}$$

Note that if $d\Phi|_p$ has rank ≤ 2, then $x_p = s_p = t_p = 0$.
However, at the value $\Phi(p^*) = (0, 1)$, $x_p = s_p = t_p = 0$ implies that $y^2 + y = 1$ therefore $y = \frac{1+\pm\sqrt{6}}{2}$, therefore $2y + 1 \neq 0$. Thus $d\Phi$ has rank 2 for all points on the level curve $\Phi^{-1}(0, 1)$, and thus $(0, 1)$ is a regular value.
Now, the manifold given by $\Phi^{-1}(0, 1)$ must satisfy the system of equations

$$\begin{aligned} x^2 + y &= 0 \\ x^2 + y + y^2 + s^2 + t^2 &= 1 \\ &\iff \\ y^2 + s^2 + t^2 &= 1 \end{aligned}$$

TODO: but why diffeomorphic?

Problem 5.2

Prove Theorem 5.11: If M is a smooth n-manifold with boundary, then with the subspace topology, ∂M is a topological $(n-1)$-dimensional manifold (without boundary), and has a smooth structure such that it is a properly embedded submanifold of M.

Problem 5.3

Prove Proposition 5.21: Suppose M is a smooth manifold with or without boundary, and $S \subseteq M$ is an immersed submanifold. If any of the following holds, the S is embedded.

Problem 5.3 continued on next page. . .

a) S **has codimension** 0 **in** M**.**

b) **The inclusion map** $S \subseteq M$ **is proper.**

c) S **is compact.**

Problem 5.4

Show that the image of a curve $(-\pi, \pi) \mapsto \mathbb{R}^2$ of Example 4.19 is not an embedded submanifold of $\mathbb{R}^2$. [Be careful: this is not the same as showing that β is not an embedding.

Problem 5.5

Let $\gamma : \mathbb{R} \mapsto \mathbb{T}^2$ be the curve of Example 4.20. Show that$\gamma(\mathbb{R})$ is not an embedded submanifold of the torus. [Remark: the warning in Problem 5-4 applies in this case as well.]

Problem 5.6

Suppose $M \subseteq \mathbb{R}^n$ is an embedded m-dimensional submanifold, and let $UM \subseteq T\mathbb{R}^n$ be the set of all unit tangent vectors to M:

$$UM = \{(x, v) \in T\mathbb{R}^n : x \in M, v \in T_x M, |v| = 1\}.$$

It is called the unit tanget bundle of M. Prove that UM is an embedded $(2m-1)$-dimensional submanifold of $T\mathbb{R}^n \approx \mathbb{R} \times bbR$.

Problem 5.7

Let $F : \mathbb{R}^2 \mapsto \mathbb{R}$ be defined by $F(x, y) = x^3 + xy + y^3$. Which level sets of F are embedded submanifolds of $\mathbb{R}^2$? For each level set, prove either that it is or that it is not an embedded submanifold.

Problem 5.8

Suppose M is a smooth n-dimensional manifold and $B \subseteq M$ is a regular coordinate ball. Show that $M \setminus B$ is a smooth manifold with boundary, whose boundary is diffeomorphic to $\mathbb{S}^{n-1}$.

Problem 5.9

Let $S \subseteq \mathbb{R}^2$ be the boundary of the square of side 2 centered at the origin (see Problem 3-5). Show that S does not have a topology and smooth structure in which it is an immeresed submanifold of $\mathbb{R}^2$.

Problem 5.10

For each $a \in \mathbb{R}$, let M_a be the subset of $\mathbb{R}^2$ defined by

$$M_a = \left\{(x, y) : y^2 = x(x-1)(x-a)\right\}.$$

For which values of a is M_a an embedded submanifold of $\mathbb{R}^2$? For which values can M_a be given a topology and smooth structure making it into an immersed submanifold?

Problem 5.11

Let $\Phi : \mathbb{R}^2 \mapsto \mathbb{R}$ be defined by $\Phi(x, y) = x^2 - y^2$.

a) Show that $\phi^{-1}(0)$ is not an embedded submanifold of $\mathbb{R}^2$.

b) Can $\phi^{-1}(0)$ be given a topology and smooth structure making it into an immersed submanifold of $\mathbb{R}^2$?

c) Answer the same two questions for $\Psi : \mathbb{R}^2 \mapsto \mathbb{R}$ defined by $\Psi(x, y) = x^2 - y^3$.

Problem 5.12

Suppose E and M are smooth manifolds with boundary, and $\pi : E \mapsto M$ is a smooth covering map. Show that the restriction of π to each connected component of ∂E is a smooth covering map onto a component of ∂M.

Problem 5.13 continued on next page...

Problem 5.13

Prove that the image of the dense curve on the torus described in Example 4.20 is a weakly embedded submanifold of $\mathbb{T}^2$.

Problem 5.14

Prove Theorem 5.32 (uniqueness of the smooth structure on an immersed submanifold once the topology is given).

Problem 5.15

Show by example that an immersed submanifold $S \subseteq M$ might have more than one topology and smooth structure with respect to which it is an immersed submanifold.

Problem 5.16

Prove Theorem 5.33: If M is a smooth manifold and $S \subseteq M$ is a weakly embedded submanifold, the S has only one topology and smooth structure with respect to which it is an immersed submanifold.

Problem 5.17

Prove Lemma 5.34: Suppose M is a smooth manifold, $S \subseteq M$ is a smooth submanifold, and $f \in C^\infty(S)$.

a) If S is embedded, then there exist a neighborhood U of S in M and a smooth function $\tilde{f} \in C^\infty(U)$ such that $\tilde{f}|_S = f$.

b) If S is properly embedded, then the neighborhood U in part (a) can be taken to be all of M.

Problem 5.18

Suppose M is a smooth manifold and $S \subseteq M$ is a smooth submanifold.

Problem 5.18 continued on next page. . .

a) Show that S is embedded if and only if every $f \in C^\infty(S)$ has a smooth extension to a neighborhood of S in M. [Hint: if S is not embedded, let $p \in S$ be a point that is not in the domain of any slice chart. Let U be a neighborhood of p in S that is embedded, and consider a function $f \in C^\infty(S)$ that is supported in U and equal to 1 at p.]

b) Show that S is properly embedded if and only if every $f \in C^\infty(S)$ has a smooth extension to all of M.

Problem 5.19

Suppose $S \subset M$ is an embedded submanifold and $\gamma : J \mapsto M$ is a smooth curve whose image happens to lie in S. Show that $\gamma'(t)$ is in the subspace $T_{\gamma(t)}S$ of $T_{\gamma(t)}M$ for all $t \in J$. Give a counterexample if S is not embedded.

Problem 5.20

Show by giving a counterexample that the conclusion of Proposition 5.37 may be false if S is merely immersed.

Problem 5.21

Prove Proposition 5.47: Suppose M is a smooth manifold and $f \in C^\infty(M)$.

a) For each regular value b of f, the sublevel set $f^{-1}\big((-\infty, b]\big)$ is a regular domain in M.

b) If a and b are two regular values of f with $a < b$, then $f^{-1}\big([a, b]\big)$ is a regular domain in M.

Problem 5.22

Prove Theorem 5.48: If M is a smooth manifold and $D \subseteq M$ is a regular domain, then there exists a defining function for D. If D is compact, then f can be taken to be a smooth exhaustion function for M.

Problem 5.23 continued on next page...

Problem 5.23

Suppose M is a smooth manifold with boundary, N is a smooth manifold, and $F : M \mapsto N$ is a smooth map. Let $S = F^{-1}(c)$, where $c \in N$ is a regular value for both F and $F|_{\partial M}$. Prove that S is a smooth submanifold with boundary in M, with $\partial S = S \cap \partial M$.