

Introduction to Smooth Manifolds: Chapter #5

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Problem 1

Consider the map $\Phi : \mathbb{R}^4 \mapsto \mathbb{R}^2$ defined by

$$\Phi(x, y, s, t) = (x^2 + y, x^2 + y^2 + s^2 + t^2 + y)$$

Show that $(0, 1)$ is a regular value of Φ , and that the level set $\Phi^{-1}(0, 1)$ is diffeomorphic to \mathbb{S}^2 .

First note that

$$d\Phi|_p = \begin{pmatrix} 2x_p & 1 & 0 & 0 \\ 2x_p & 2y_p + 1 & 2s_p & 2t_p \end{pmatrix}$$

Note that if $d\Phi|_p$ has rank ≤ 2 , then $x_p = s_p = t_p = 0$.

However, at the value $\Phi(p^*) = (0, 1)$, $x_p = s_p = t_p = 0$ implies that $y^2 + y = 1$ therefore $y = \frac{1 \pm \sqrt{5}}{2}$, therefore $2y + 1 \neq 0$. Thus $d\Phi$ has rank 2 for all points on the level curve $\Phi^{-1}(0, 1)$, and thus $(0, 1)$ is a regular value.

Now, the manifold given by $\Phi^{-1}(0, 1)$ must satisfy the system of equations

$$\begin{aligned} x^2 + y &= 0 \\ x^2 + y + y^2 + s^2 + t^2 &= 1 \\ &\iff \\ y^2 + s^2 + t^2 &= 1 \end{aligned}$$

TODO: but why diffeomorphic?

Problem 2

Prove Theorem 5.11: If M is a smooth n -manifold with boundary, then with the subspace topology, ∂M is a topological $(n - 1)$ -dimensional manifold (without boundary), and has a smooth structure such that it is a properly embedded submanifold of M .

Problem 3

Prove Proposition 5.21: Suppose M is a smooth manifold with or without boundary, and $S \subseteq M$ is an immersed submanifold. If any of the following holds, the S is embedded.

- a) S has codimension 0 in M .
- b) The inclusion map $S \subseteq M$ is proper.
- c) S is compact.

Problem 4

Show that the image of a curve $(-\pi, \pi) \mapsto \mathbb{R}^2$ of Example 4.19 is not an embedded submanifold of \mathbb{R}^2 . [Be careful: this is not the same as showing that β is not an embedding.]

Problem 5

Let $\gamma : \mathbb{R} \mapsto \mathbb{T}^2$ be the curve of Example 4.20. Show that $\gamma(\mathbb{R})$ is not an embedded submanifold of the torus. [Remark: the warning in Problem 5-4 applies in this case as well.]

Problem 6

Suppose $M \subseteq \mathbb{R}^n$ is an embedded m -dimensional submanifold, and let $UM \subseteq T\mathbb{R}^n$ be the set of all unit tangent vectors to M :

$$UM = \{(x, v) \in T\mathbb{R}^n : x \in M, v \in T_x M, |v| = 1\}.$$

It is called the unit tangent bundle of M . Prove that UM is an embedded $(2m - 1)$ -dimensional submanifold of $T\mathbb{R}^n \approx \mathbb{R} \times \mathbb{R}^n$.

Problem 7

Let $F : \mathbb{R}^2 \mapsto \mathbb{R}$ be defined by $F(x, y) = x^3 + xy + y^3$. Which level sets of F are embedded submanifolds of \mathbb{R}^2 ? For each level set, prove either that it is or that it is not an embedded submanifold.

Problem 8

Suppose M is a smooth n -dimensional manifold and $B \subseteq M$ is a regular coordinate ball. Show that $M \setminus B$ is a smooth manifold with boundary, whose boundary is diffeomorphic to \mathbb{S}^{n-1} .

Problem 9

Let $S \subseteq \mathbb{R}^2$ be the boundary of the square of side 2 centered at the origin (see Problem 3-5). Show that S does not have a topology and smooth structure in which it is an immersed submanifold of \mathbb{R}^2 .

Problem 10

For each $a \in \mathbb{R}$, let M_a be the subset of \mathbb{R}^2 defined by

$$M_a = \{(x, y) : y^2 = x(x-1)(x-a)\}.$$

For which values of a is M_a an embedded submanifold of \mathbb{R}^2 ? For which values can M_a be given a topology and smooth structure making it into an immersed submanifold?

Problem 11

Let $\Phi : \mathbb{R}^2 \mapsto \mathbb{R}$ be defined by $\Phi(x, y) = x^2 - y^2$.

- Show that $\phi^{-1}(0)$ is not an embedded submanifold of \mathbb{R}^2 .
- Can $\phi^{-1}(0)$ be given a topology and smooth structure making it into an immersed submanifold of \mathbb{R}^2 ?
- Answer the same two questions for $\Psi : \mathbb{R}^2 \mapsto \mathbb{R}$ defined by $\Psi(x, y) = x^2 - y^3$.

Problem 12

Suppose E and M are smooth manifolds with boundary, and $\pi : E \mapsto M$ is a smooth covering map. Show that the restriction of π to each connected component of ∂E is a smooth covering map onto a component of ∂M .

Problem 13

Prove that the image of the dense curve on the torus described in Example 4.20 is a weakly embedded submanifold of \mathbb{T}^2 .

Problem 14

Prove Theorem 5.32 (uniqueness of the smooth structure on an immersed submanifold once the topology is given).

Problem 15

Show by example that an immersed submanifold $S \subseteq M$ might have more than one topology and smooth structure with respect to which it is an immersed submanifold.

Problem 16

Prove Theorem 5.33: If M is a smooth manifold and $S \subseteq M$ is a weakly embedded submanifold, the S has only one topology and smooth structure with respect to which it is an immersed submanifold.

Problem 17

Prove Lemma 5.34: Suppose M is a smooth manifold, $S \subseteq M$ is a smooth submanifold, and $f \in C^\infty(S)$.

- If S is embedded, then there exist a neighborhood U of S in M and a smooth function $\tilde{f} \in C^\infty(U)$ such that $\tilde{f}|_S = f$.
- If S is properly embedded, then the neighborhood U in part (a) can be taken to be all of M .

Problem 18

Suppose M is a smooth manifold and $S \subseteq M$ is a smooth submanifold.

- Show that S is embedded if and only if every $f \in C^\infty(S)$ has a smooth extension to a neighborhood of S in M . [Hint: if S is not embedded, let $p \in S$ be a point that is not in the domain of any slice chart. Let U be a neighborhood of p in S that is embedded, and consider a function $f \in C^\infty(S)$ that is supported in U and equal to 1 at p .]
- Show that S is properly embedded if and only if every $f \in C^\infty(S)$ has a smooth extension to all of M .

Problem 19

Suppose $S \subset M$ is an embedded submanifold and $\gamma : J \mapsto M$ is a smooth curve whose image happens to lie in S . Show that $\gamma'(t)$ is in the subspace $T_{\gamma(t)}S$ of $T_{\gamma(t)}M$ for all $t \in J$. Give a counterexample if S is not embedded.

Problem 20

Show by giving a counterexample that the conclusion of Proposition 5.37 may be false if S is merely immersed.

Problem 21

Prove Proposition 5.47: Suppose M is a smooth manifold and $f \in C^\infty(M)$.

- a) For each regular value b of f , the sublevel set $f^{-1}((-\infty, b])$ is a regular domain in M .
- b) If a and b are two regular values of f with $a < b$, then $f^{-1}([a, b])$ is a regular domain in M .

Problem 22

Prove Theorem 5.48 (existence of defining functions for regular domains).

Problem 23

Suppose M is a smooth manifold with boundary, N is a smooth manifold, and $F : M \mapsto N$ is a smooth map. Let $S = F^{-1}(c)$, where $c \in N$ is a regular value for both F and $F|_{\partial M}$. Prove that S is a smooth submanifold with boundary in M , with $\partial S = S \cap \partial M$.