

Homework Assignment 1

FE 621: Computational Methods in Finance

Instructor: Ionut Florescu

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Rukmal Weerawarana

rweerawa@stevens.edu | 104-307-27

Department of Financial Engineering

Stevens Institute of Technology

Overview

In this Homework Assignment, we explore various numerical optimization methods through the lens of the Black-Scholes-Merton Option pricing model (Shreve 2004). Using this, we calculate explore the implied volatility of options for various assets traded on the market. Furthermore, we also explore numeric methods of differential calculation to compute the Greeks of these candidate options. Finally, we explore numeric integration and the behavior of various quadrature methods.

Unless otherwise stated, the following shorthand notation is used to distinguish between dates:

- **DATA1** - Wednesday, February 6 2019 (2/6/19)
- **DATA2** - Thursday, February 7 2019 (2/7/19)

The content of this Homework Assignment is divided into three sections; the first discusses data gathering, formatting, and a discussion of the assets being examined. The second contains data analysis, and an exploration of implied volatility through the Black-Scholes-Merton pricing framework and related computations. Finally, the third section discusses numerical integration and the convergence of various quadrature rules.

See Appendix A for specific question implementations, and (Weerawarana 2019) for full source code of the fe621 Python package.

1 Data Overview

1.1 Asset Descriptions

1.1.1 SPY - SPDR S&P 500 ETF (State Street Global Advisors 2019)

The S&P 500 (i.e. *Standard & Poor's 500*) is a stock market index tracking the 500 largest companies on the American Stock Exchange by Market Capitalization. In this case, the market capitalization is defined as the number of outstanding shares, multiplied by the current share price. A stock market index is designed to be a metric that can be used by market observers as a benchmark to gauge the relative health of the stock market, by analyzing the aggregate performance of its largest components.

However, this index is not the same as the SPY ETF. An ETF (*Exchange Traded Fund*) is a basket of stocks that is designed to track a specific index or benchmark. That is, it provides investors with exposure to a index or benchmark, without having to own all of the underlying assets that constitute a composite asset. In addition to more liquidity, this type of investment also provides lower transaction costs and required minimum investment to gain exposure to a given index or benchmark. It is traded on an exchange, akin to a typical traded asset.

1.1.2 VIX - CBOE Volatility Index (CBOE (Chicago Board Options Exchange) 2019)

The CBOE (*Chicago Board Options Exchange*) volatility index, VIX is an exchange traded product (ETP) designed to give investors exposure to the market's expectation of 30-day volatility. It is priced using a large set of implied volatility of put and call options on the S&P 500 index to gauge investor sentiment. Typically, the price of the VIX has an inverse relationship to the price of the S&P 500 index. Similar to an ETF, an ETP is also traded on an exchange as a typical traded asset.

1.2 Data Gathering

For the assignment, we downloaded monthly options on *Amazon Inc.* (ticker: AMZN) and *S&P 500 ETF* (ticker: SPY) at various strike prices for the following dates:

- 02/15/19 - Friday, February 15 2019;
- 03/15/19 - Friday, March 15 2019;
- 04/18/19 - Thursday, April 18 2019.

A wide variety of option strike prices were considered, with the following ranges:

- AMZN - \$1555 to \$1725 in increments of \$5 (35 strike prices);
- SPY - \$256 to \$284 in increments of \$1 (29 strike prices).

Intra-day minute closing price data was gathered for both put and call options with expiration dates and strike prices detailed above. This intra-day data was gathered for the trading day 2/6/19 (February 6 2019; **DATA1**). Additionally, intra-day minute closing price data was also downloaded for each of the underlying assets. This data was downloaded for both 2/6/19 (February 6 2019; **DATA1**), and 2/7/19 (February 7 2019; **DATA2**).

This data detailed above was gathered utilizing *Rblpapi* (Armstrong et al. 2018), which provides an R interface to data on the Bloomberg Terminal (Bloomberg L.P. 2019). The data download was automated, and corresponding intra-day prices for each of the options were output to individual files. The source code for this implementation is available in Appendix A.1.1.

Furthermore, as a proxy for the *risk-free rate*, we chose to utilize the effective Federal Funds Rate (FFR). This is the interest rate at which depository institutions in the United States lend reserve balances to other depository institutions overnight. This data was gathered for both dates, and correspond to **DATA1** and **DATA2**. The effective FFR is published daily by the US Federal Reserve Board of Governors, and are expressed as yields per annum (Board of Governors of the Federal Reserve System 2019).

1.2.1 Data Cleaning

For easier programmatic access, the data was placed in a hierarchical structure, corresponding to the **DATA1**, **DATA2** data division. Each of the option and asset prices for the corresponding days were placed in the requisite sub-folders. This directory structure is reproduced below.

```
data
├── DATA1
│   ├── AMZN
│   ├── SPY
│   └── VIX
└── DATA2
    ├── AMZN
    ├── SPY
    └── VIX
```

8 directories

Option price filenames were changed to OOC format option names, discussed further below. This was done utilizing a cleaning script, written in Python. This script employs utility functions from the *fe621* Python package (Weerawarana 2019). The cleaning script is reproduced in Appendix A.1.2.

1.3 Option Naming Convention

A modern convention for naming option contracts was proposed by the Options Clearing Commission (OCC) in 2008 (Options Symbology Initiative Working Group 2008), and adopted in 2010. The OCC is an organization that acts as both the issuer and guarantor for option and future contracts. The OCC is governed by the Securities and Exchange Commission (SEC) and the Commodities Futures Trading Commission (CFTC). The current convention for option naming is best explained by example.

Consider the option code, *AMZN190215C01960000*. This corresponds to a **Call Option** on **Amazon Inc. (AMZN)**, with a strike price of **\$1960.00** and an expiration date of **2/15/19** (February 15 2019).

The methodology of this nomenclature is explained in detail below:

AMZN190215C01960000

- **AMZN** - Ticker of the company (arbitrary length; always first sequence of characters)
- **19** - Expiration year of the contract (shortened to two digits, i.e. 2019 → 19)
- **02** - Expiration month of the contract
- **15** - Expiration day of the contract
- **C** - Type of option (*C* for call, *P* for put)
- **01960** - Dollar component of strike price (in \$; always 5 digits)

- **000** - $\frac{1}{1000}$ th Dollar component of strike price (in $\frac{1}{1000}$ \$; always 3 digits)

Similarly, the following option code corresponds to a **Put Option** on **SPDR S&P 500 ETF (SPY)**, with a strike price of **\$287.50** and an expiration date of **3/15/19** (March 15 2019):

SPY190315P00287500

Finally, the following option code corresponds to a **Call Option** on **CBOE Volatility Index (VIX)**, with a strike price of **\$16.35** and an expiration date of **4/18/19** (February 18 2019):

VIX190418C00016350

2 Data Analysis

Note: All Python scripts reproduced in this section are extracted from the fe621 (Weerawarana 2019) package created for this class.

2.1 Black-Scholes Model Formulas

With the probabilities d_1 and d_2 defined as:

$$d_1 = \frac{\log\left(\frac{S_t}{K}\right) + \left(r - \frac{\sigma^2}{2}\right)(T - t)}{\sigma\sqrt{T - t}}$$

$$d_2 = d_1 - \sigma\sqrt{T - t}$$

$$\Phi(x) = \int_{-\infty}^x \phi(z)dz = \int_{-\infty}^x \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz$$

```

1 from typing import Tuple
2
3 import numpy as np
4
5
6 def computeD1D2(current: float, volatility: float, ttm: float, strike: float,
7                 rf: float) -> Tuple[float, float]:
8     """Helper function to compute the risk-adjusted priors of exercising the
9     option contract, and keeping the underlying asset. This is used in the
10    computation of both the Call and Put options in the
11    Black-Scholes-Merton framework.
12
13    Arguments:
14        current {float} -- Current price of the underlying asset.
15        volatility {float} -- Volatility of the underlying asset price.
16        ttm {float} -- Time to expiration (in years).
17        strike {float} -- Strike price of the option contract.
18        rf {float} -- Risk-free rate (annual).
19
20    Returns:
21        Tuple[float, float] -- Tuple with d1, and d2 respectively.
22    """
23
24    d1 = (np.log(current / strike) + (rf - (volatility / 2)) * ttm) \
25          / (volatility * np.sqrt(ttm))
26    d2 = d1 - (volatility * np.sqrt(ttm))
27
28    return (d1, d2)

```

../fe621/black_scholes/util.py

Note: The following assumes the dividend rate, $q = 0$.

2.1.1 Put Option

The Black-Scholes Option price for a European Put ($P(S_t)$) option is defined as:

$$P(S_t) = Ke^{-r(T-t)}\Phi(-d_2) - S_t\Phi(-d_1)$$

```

1 from .util import computeD1D2
2
3 from scipy.stats import norm
4 import numpy as np
5
6
7 def blackScholesPut(current: float, volatility: float, ttm: float,
8                     strike: float, rf: float) -> float:
9     """Function to compute the Black-Scholes-Merton price of a European Put
10    Option, parameterized by the current underlying asset price, volatility,
11    time to expiration, strike price, and risk-free rate.
12
13    Arguments:
14        current {float} -- Current price of the underlying asset.
15        volatility {float} -- Volatility of the underlying asset price.
16        ttm {float} -- Time to expiration (in years).
17        strike {float} -- Strike price of the option contract.
18        rf {float} -- Risk-free rate (annual).
19
20    Returns:
21        float -- Price of a European Put Option contract.
22    """
23
24    d1, d2 = computeD1D2(current, volatility, ttm, strike, rf)
25
26    put = (strike * np.exp(-1 * rf * ttm) * norm.cdf(-1 * d2)) \
27          - (strike * norm.cdf(-1 * d1))
28
29    return put

```

../fe621/black_scholes/put.py

2.1.2 Call Option

The Black-Scholes Option price for a European Call ($C(S_t)$) option is defined as:

$$C(S_t) = S_t \Phi(d_1) - Ke^{-r(T-t)} \Phi(d_2)$$

```

1 from .util import computeD1D2
2
3 from scipy.stats import norm
4 import numpy as np
5
6
7 def blackScholesCall(current: float, volatility: float, ttm: float,
8                      strike: float, rf: float) -> float:
9     """Function to compute the Black-Scholes-Merton price of a European Call
10    Option, parameterized by the current underlying asset price, volatility,
11    time to expiration, strike price, and risk-free rate.
12
13    Arguments:
14        current {float} -- Current price of the underlying asset.
15        volatility {float} -- Volatility of the underlying asset price.
16        ttm {float} -- Time to expiration (in years).
17        strike {float} -- Strike price of the option contract.
18        rf {float} -- Risk-free rate (annual).

```

```

19
20 Returns:
21     float -- Price of a European Call Option contract.
22     """
23
24     d1, d2 = computeD1D2(current, volatility, ttm, strike, rf)
25
26     call = (current * norm.cdf(d1)) \
27         - (strike * np.exp(-1 * rf * ttm) * norm.cdf(d2))
28
29     return call

```

../fe621/black_scholes/call.py

2.1.3 Put-Call Parity

The relationship between the price of a Call and Put option is governed by Put-Call parity:

$$P(S_t) = C(S_t) - S_t + Ke^{-r(T-t)}$$

```

1 import numpy as np
2
3
4 def call(put: float, current: float, strike: float, ttm: float,
5         rf: float) -> float:
6     """Function to compute the price of a European Call option contract from a
7     European Put option contract price using Put-Call parity.
8
9     Arguments:
10         put {float} -- Price of the put option.
11         current {float} -- Current price of the underlying asset.
12         strike {float} -- Strike price of the option contract.
13         ttm {float} -- Time to expiration (in years).
14         rf {float} -- Risk-free rate (annual).
15
16     Returns:
17         float -- Price of a European Call Option contract.
18     """
19
20     return put + current - (strike * np.exp(-1 * rf * ttm))
21
22
23 def put(call: float, current: float, strike: float, ttm: float,
24        rf: float) -> float:
25     """Function to compute the price of a European Put option contract from a
26     European Call option contract price using Put-Call parity.
27
28     Arguments:
29         call {float} -- Price of the call option.
30         current {float} -- Current price of the underlying asset.
31         strike {float} -- Strike price of the option contract.
32         ttm {float} -- Time to expiration (in years).
33         rf {float} -- Risk-free rate (annual).
34
35     Returns:
36         float -- Price of a European Put Option contract.
37     """
38
39     return call - current + (strike * np.exp(-1 * rf * ttm))

```

../fe621/black_scholes/parity.py

2.1.4 The Greeks

The Greeks are the quantities representing the sensitivity of the price of a derivative with respect to changes in the underlying parameters. The following formulas are implemented to calculate each of the Greeks using the Black-Scholes option pricing formula. These formulas are derived in full in Stefanica 2011 and Weerawarana 2016.

Note: The following assumes the dividend rate, $q = 0$.

Delta

The Delta (Δ) of an option is the first derivative of an option with respect to the price of the underlying asset at time t , S_t .

$$\Delta(C) = \frac{\partial C(S_t)}{\partial S_t} = \Phi(d_1)$$

Gamma

The Gamma (Γ) of an option is the second derivative of an option with respect to the price of the underlying asset at time t , S_t .

$$\Gamma(C) = \frac{\partial^2 C(S_t)}{\partial S_t^2} = \frac{\phi(d_1)}{S_t \sigma \sqrt{T-t}}$$

Vega

The Vega (ν) of an option is the first derivative of an option with respect to the volatility of the underlying asset at time t , σ .

$$\nu(C) = \nu(P) = \frac{\partial C(S_t)}{\partial \sigma} = S_t \sqrt{T-t} \phi(d_1)$$

```

1 from .util import computeD1D2
2
3 from scipy.stats import norm
4
5 import numpy as np
6
7
8 def callDelta(current: float, volatility: float, ttm: float, strike: float,
9               rf: float) -> float:
10     """Function to compute the Delta of a call option using the Black-Scholes
11     formula.
12
13     Arguments:
14         current {float} -- Current price of the underlying asset.
15         volatility {float} -- Volatility of the underlying asset price.
16         ttm {float} -- Time to expiration (in years).
17         strike {float} -- Strike price of the option contract.
18         rf {float} -- Risk-free rate (annual).

```



```

19
20     Returns:
21         float -- Delta of a European Call Option contract.
22     """
23
24     d1, d2 = computeD1D2(current, volatility, ttm, strike, rf)
25
26     return np.cdf(d1)
27
28
29 def callGamma(current: float, volatility: float, ttm: float, strike: float,
30              rf: float) -> float:
31     """Function to compute the Gamma of a Call option using the Black-Scholes
32     formula.
33
34     Arguments:
35         current {float} -- Current price of the underlying asset.
36         volatility {float} -- Volatility of the underlying asset price.
37         ttm {float} -- Time to expiration (in years).
38         strike {float} -- Strike price of the option contract.
39         rf {float} -- Risk-free rate (annual).
40
41     Returns:
42         float -- Delta of a European Call Option contract.
43     """
44
45     d1, d2 = computeD1D2(current, volatility, ttm, strike)
46
47     return norm.pdf(d1) * (1 / (current * volatility * np.sqrt(ttm)))
48
49
50 def vega(current: float, volatility: float, ttm: float, strike: float,
51         rf: float) -> float:
52     """Function to compute the Vega of an option using the Black-Scholes formula.
53
54     Arguments:
55         current {float} -- Current price of the underlying asset.
56         volatility {float} -- Volatility of the underlying asset price.
57         ttm {float} -- Time to expiration (in years).
58         strike {float} -- Strike price of the option contract.
59         rf {float} -- Risk-free rate (annual).
60
61     Returns:
62         float -- Vega of a European Option contract.
63     """
64
65     d1, d2 = computeD1D2(current, volatility, ttm, strike, rf)
66
67     return current * np.sqrt(ttm) * norm.pdf(d1)

```

../fe621/black_scholes/greeks.py

References

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A Appendix

A.1 Question 1 Implementation

A.1.1 Bloomberg Data Download

```

1 library("Rblpapi")
2
3 # Connect to Bloomberg Terminal backend service
4 blpConnect(host = "localhost", port = 8194)
5
6
7 #-----
8 # Data Download Functionality
9 #-----
10
11
12 getPrice <- function(security, startTime, endTime, timeZone) {
13   # Downloads and returns the closing price of a given security
14   # for each minute in the trading day.
15   #
16   # Args:
17   #   security: Name of the security to be downloaded.
18   #   startTime: Datetime object with the start time.
19   #   endTime: Datetime object with the end time.
20   #   timeZone: Time zone of the target start and end times.
21   #
22   # Returns:
23   #   DataFrame with the closing price for each minute in the
24   #   trading day.
25
26   # Getting price data
27   data <- getBars(security = security, barInterval = 1,
28                   startTime = startTime, endTime = endTime,
29                   tz = timeZone)
30
31   # Isolate time and closing price
32   data <- data[c("times", "close")]
33
34   # Rename columns
35   colnames(data) <- c("Dates", "Close")
36
37   # Return
38   data
39 }
40
41
42 createOptionName <- function(security, dates, prices, type, suffix) {
43   # Creates the Bloomberg-standard option name, given a security, date, price,
44   # option type and suffix.
45   #
46   # Args:
47   #   security: Name of the security to be included in the option price.
48   #   dates: Dates to be included in option name.
49   #   prices: Prices to be included in the option name.
50   #   type: Type of the option ("C" or "P").
51   #   suffix: Suffix for option name (typically "Index" or "Equity").
52   #
53   # Returns:
54   #   Vector of Bloomberg-compatible option names.

```

```

55
56 # Empty vector to store names
57 names <- c()
58
59 # Iterate over each date and price
60 for (date in dates) {
61   for (price in prices) {
62     # Building option name
63     name <- paste(security, date, paste(type, price, sep = ""), suffix)
64
65     # Appending to list of option names
66     names <- c(names, name)
67   }
68 }
69
70 # Returning names
71 names
72 }
73
74
75 #-----
76 # DATA1
77 #-----
78
79
80 # Define Start and End times (DATA1)
81 data1Start <- ISOdatetime(year = 2019, month = 2, day = 6,
82                           hour = 9, min = 30, sec = 0)
83 data1End <- ISOdatetime(year = 2019, month = 2, day = 6,
84                         hour = 16, min = 0, sec = 0)
85
86 # Defining time zone
87 timeZone = "America/New_York"
88
89 # Defining top-level securities
90 securities <- c("SPY US Equity", "AMZN US Equity", "VIX Index")
91
92 # Getting prices for each of the top-level securities
93 for (security in securities) {
94   data <- getPrice(security, data1Start, data1End, timeZone)
95   write.csv(data, file = paste(security, "DATA1", "csv", sep = "."),
96            row.names = FALSE)
97 }
98
99 # Expiration dates
100 expDates <- c("2/15/19", "3/15/19", "4/18/19")
101
102 # Defining put and call prices for SPY and AMZN options
103 # Grabbing prices for 5% +/- current price
104
105 # Current SPY price
106 spyCurrent <- 270
107 spyPrices <- c(floor(0.95 * spyCurrent):ceiling(1.05 * spyCurrent))
108
109 # Current AMZN price (need to do this manually because of option strikes)
110 # Closest option price to 95% of price is at $1557.50 and 105% is $1722.50
111 amznCurrent <- 1640
112 amznPrices <- seq(1555, 1725, by=5)
113
114 # Creating option names for SPY and AMZN
115 spyOptions <- createOptionName("SPY", expDates, spyPrices, "C", "Equity")

```

```

116 spyOptions <- c(spyOptions, createOptionName("SPY", expDates, spyPrices,
117                                               "P", "Equity"))
118
119 amznOptions <- createOptionName("AMZN", expDates, amznPrices, "C", "Equity")
120 amznOptions <- c(amznOptions, createOptionName("AMZN", expDates, amznPrices,
121                                               "P", "Equity"))
122
123 # Getting prices for each of the options
124 for (option in c(amznOptions, spyOptions)) {
125   data <- getPrice(option, data1Start, data1End, timeZone)
126   # Only print to file if option exists
127   if (all(dim(data) > 0)) {
128     optionFileName <- gsub("/", "-", option) # Need to do this for Windows
129     write.csv(data, file = paste(optionFileName, "csv", sep = "."),
130              row.names = FALSE)
131   }
132 }
133
134
135 # -----
136 # DATA2
137 # -----
138
139 # Define Start and End times (DATA2)
140 data2Start <- ISOdatetime(year = 2019, month = 2, day = 7,
141                           hour = 9, min = 30, sec = 0)
142 data2End <- ISOdatetime(year = 2019, month = 2, day = 7,
143                         hour = 16, min = 0, sec = 0)
144
145 # Getting prices for each of the top-level securities
146 for (security in securities) {
147   data <- getPrice(security, data2Start, data2End, timeZone)
148   write.csv(data, file = paste(security, "DATA2", "csv", sep = "."),
149            row.names = FALSE)
150 }

```

question_solutions/question_1.R

A.1.2 Data Cleaning

```

1 # Script to rename option files, from the R data download script format to
2 # OOC-compliant names.
3
4 from context import fe621
5
6 import os
7
8 option_file_paths = [os.getcwd() + i for i in ['/Homework 1/data/DATA1/AMZN',
9                                               '/Homework 1/data/DATA1/SPY']]
10
11 for option_file_path in option_file_paths:
12     fe621.util.renameOptionFiles(folder_path=option_file_path)

```

data_cleaning.py