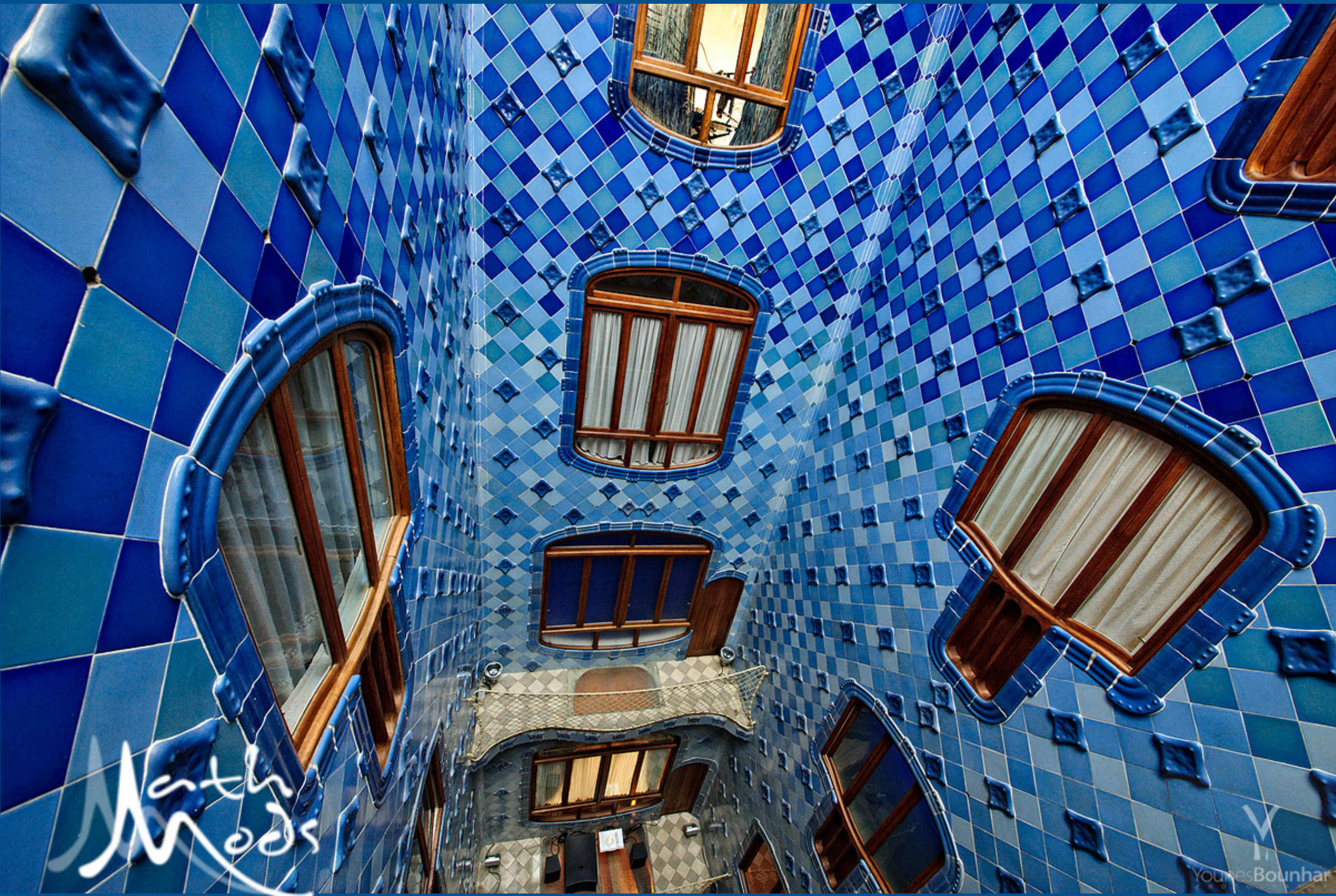
The Optimal Transport Problem

Master Thesis

Oscar Ramirez



The Optimal Transport Problem

Master Thesis

by

Oscar Ramirez

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Preface

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Oscar Ramirez
Barcelona, September 2018

Contents

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Preliminaries.

Notation

$\mathbb{R}$	Real numbers field.
$\bar{\mathbb{R}}$	$\mathbb{R} \cup \{+\infty\}$. That is $[-\infty, \infty]$
$\mathbb{R}_+$	The set of nonnegative real numbers, that is the interval $[0, \infty)$.
$\bar{\mathbb{R}}_+$	The set of nonnegative extended real numbers, that is the interval $[0, \infty]$
δ_x	The Dirac mass at point x.
$\mathbb{R}^d$	The d-dimensional Euclidean space.
$\mathcal{P}(X)$	Space of probabilities on X.
$\mu \ll \nu$	The measure is absolutely continuous with respect to ν.
$\mathbb{1}_\Omega$	Indicator function of a set Ω. If $x \in \Omega$ then $\mathbb{1}_\Omega(x) = 1$. If $x \in \Omega^c$, we have $\mathbb{1}_\Omega(x) = 0$.
$\mu \llcorner A$	A measure μ restricted to a set A.
ω_d	The Measure of the unite ball in $\mathbb{R}^d$.
$\wedge$	The min operator, that is $a \wedge b := \min\{a, b\}$.
$\vee$	The max operator, that is $a \vee b := \max\{a, b\}$.
$T_\# \mu$	The image measure of μ through the map T.
$f_{\mid\Omega}$	The restriction of a function f to a set Ω.
$\Pi(\mu, \nu)$	The set of transport plans from μ to ν.
$\frac{\delta F}{\delta \rho}$	First variation of $F : \mathcal{P}(X) \to \mathbb{R}$, that is $\frac{\mathrm{d}}{\mathrm{d}\epsilon} F(\rho + \epsilon\chi)\Big\vert_{\epsilon=0} = \int \frac{\delta F}{\delta \rho} \mathrm{d}\chi$
W_p	Wasserstein distance of order p.
$\mathbb{W}_p$	Wasserstein space of order p.
γ_T	The transport plan in $\Pi(\mu, \nu)$.
$M(T)$	Monge cost of a map T.
$K(\gamma)$	Kantorovich cost of a plan γ.
$\mu \otimes \nu$	The product measure of μ and ν such that $\mu \otimes \nu(A \times B) = \mu(A)\nu(B)$.
$M^{k \times h}$	The set of real matrices with k rows and h columns.
$M^\top$	Transpose of a matrix M.
i.i.d.	Independent and identical probability distributions.
l.s.c.	Lower semicontinuous.

Definitions.

Definition 1 (Lower Semicontinuity.)**.** *On a complete metric space X, a function $f : X \to \bar{\mathbb{R}}$ is said to be lower semi-continuous (l.s.c.) if for every sequence $(x_n)_{n \in \mathbb{N}}$ converging to $x \in X$, we have*

$$f(x) \leq \liminf_{n \in \mathbb{N}} f(x_n)$$

Definition 2 (Sequentially compact.)**.** *A subset K of a metric space X is said to be compact if from any sequence x_n, we can extract a converging subsequence $x_{n_k} \to x \in K$.*

We can see from the above definition that any continuous function is lower-semicontinuous.

Definition 3 (Compactness.). *A subset K of a metric space X is compact if every open cover of K has a finite subcover.*

Theorem 1. *A subset of a metric space is compact if and only if it is sequentially compact.*

Theorem 2. *Maxima and Minima Let X be a compact metric space and $f : X \to \mathbb{R}$ is continuous, real-valued function. Then f is bounded on X and attains its maximum and minimum. That is, there are x, y belonging to X such that,*

$$f(x) = \inf_{z \in X} f(z) \qquad \text{and} \qquad f(y) = \sup_{z \in X} f(z)$$

Continuity is a strong requirement. Luckily we can assure the existence of a minimizer on lower-semicontinuous functions (or maximizer on upper-semicontinuous). The usual procedure to prove existence of a minimizer is making use of Weierstrass' criterion, we take a minimizer sequence and we prove the space in which we are trying to find a minimizer element is compact.

Theorem 3. *Weierstrass' criterion for existence of minimizers. If $f : X \to \bar{\mathbb{R}}$ is lower semi-continuous and X is compact, then there exists $\hat{x} \in X$.*

Proof. Define $l := \inf\{f(x) : x \in X\} \in \bar{\mathbb{R}}$, notice that $l = +\infty$ only if f is identically $+\infty$, then this case is trivial since any point minimizes f. By compactness there exists a minimizing sequence x_n, that is $f(x_n) \to l$. By compactness we can extract a subsequence converging to some $\hat{x}$ such that $\hat{x} \in X$. By lower-semicontinuity of f, we have that $f(\hat{x}) \liminf_n f(x_n) = l$. Since l is the infimum $l \leq f(\hat{x})$. This proves that $l = f(\hat{x}) \in \mathbb{R}$. □

We can apply the above analysis using a notion of upper-semicontinuity and compactness to find the maximum.

Definition 4. *Topological dual*

Definition 5. *Weak compactness in dual spaces A sequence x_n in a Banach space X is said to be weakly convergin to x, and we write $x_n \rightharpoonup x$, if for every $\xi \in X^*$. We have $\langle \xi, x_n \rangle \to \langle \xi, x \rangle$. A sequence*

Let $(X_1, \mathcal{A}_1, \mu_1)$ and $(X_2, \mathcal{A}_2, \mu_2)$ be two spaces with finite nonnegative measures. On the space $X_1 \times X_2$ we consider sets of the form $\mathcal{A}1 \times \mathcal{A}_2$, where $A_i \in \mathcal{A}_i$, called measurable rectangles. Let $\mu_1 \times \mu_2 (A_1 \times A_2)$:= μ 1 (A 1)μ 2 (A 2). Extending the function μ 1 ×μ 2 by additivity to finite unions of pairwise disjoint measurable rectangles we obtain a finitely additive function on the algebra R generated by such rectangles. We observe that such an extension of μ 1 ×μ 2 to R is well-defined (is independent of partitions of the set into pairwise disjoint measurable rectangles), which is obvious by the additivity of μ 1 and μ 2 . Fi- nally, let A 1 □A 2 denote the σ-algebra generated by all measurable rectangles; this σ-algebra is called the product of the σ-algebras A 1 and A 2 .

Theorem 4. *The set function $\mu_1 \times \mu_2$is countably additive on the algebra generated by all measurable rectangles and uniquely extends to a countably additive measure, denoted by $\mu_1 \otimes \mu_2$, on the Lebesgue completion of this algebra denoted by $\mathcal{A}1 \otimes \mathcal{A}2$*

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Basics in Convex Analysis.

Proposition 1. *Let $f : X \to \bar{\mathbb{R}}$ be a convex and lower-semicontinuous function. Assume that there exists $x_0 \in X$ such that $f(x_0) = -\infty$. Then f is nowhere finite on X.*

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Linear Programming

Interior Methods

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Optimal Transport Theory

Problem 1. *Given two probability measures $\mu \in \mathcal{P}(X)$ and $\nu \in \mathcal{P}(Y)$ and a cost function $c : X \times Y \to \{0, +\infty\}$, solve*

$$\inf\left\{M(T) := \int c(x, T(x))\mathrm{d}\mu(x) : \quad T_{\#}\mu = \nu\right\} \tag{MP}$$

Definition 6. *coupling*

Problem 2. *Given $\mu \in \mathcal{P}(X)$, $\nu \in \mathcal{P}(Y)$, and $c : X \times Y \to [0, +\infty]$, we consider the problem*

$$\inf\left\{K(\gamma) := \int_{X\times Y} c\mathrm{d}\gamma : \gamma \in \Pi(\mu, \nu)\right\} \tag{KP}$$

where $\Pi(\mu, \nu)$ is the set of transport plans.

Kantorovich formulation as relaxation

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Computational Optimal Transport

Linear Programming Formulation.

Sinkhorn-Knopp Algorithm.

Simplex Method Algorithm.

Simulated Annealing.

Continuous Formulation.

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Applications

Nash Equilibrium.
Track of a Dynamic.
Domain Adaptation.
Isoperimetric Inequality.
Barycenter of a Fourier Power Spectrum.

Bibliography