The Optimal Transport Problem

Master Thesis

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by

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Preface

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Notation Table.

$\emptyset$	Empty set
$\mathbb{R}$	Real numbers field.
$\bar{\mathbb{R}}$	$\mathbb{R} \cup \{+\infty\}$. That is $[-\infty, \infty]$
$\mathbb{R}_+$	The set of nonnegative real numbers, that is the interval $[0, \infty)$.
$\bar{\mathbb{R}}_+$	The set of nonnegative extended real numbers, that is the interval $[0, \infty]$
δ_x	The Dirac mass at point x.
$\mathbb{R}^d$	The d-dimensional Euclidean space.
id	Identity map.
$\mathcal{P}(X)$	Space of probabilities on X.
$\mu \ll \nu$	The measure is absolutely continuous with respect to ν.
$\mathbb{1}_\Omega$	Indicator function of a set Ω. If $x \in \Omega$ then $\mathbb{1}_\Omega(x) = 1$. If $x \in \Omega^c$, we have $\mathbb{1}_\Omega(x) = 0$.
$\mu \llcorner A$	A measure μ restricted to a set A.
ω_d	The Measure of the unite ball in $\mathbb{R}^d$.
$\wedge$	The min operator, that is $a \wedge b := \min\{a, b\}$.
$\vee$	The max operator, that is $a \vee b := \max\{a, b\}$.
$\mathrm{D}T(x)$	Jacobian matrix of a map $T(x)$.
$T_\# \mu$	The image measure of μ through the map T.
$f_{\vert\Omega}$	The restriction of a function f to a set Ω.
$\Pi(\mu, \nu)$	The set of transport plans from μ to ν.
$\frac{\delta F}{\delta \rho}$	First variation of $F : \mathcal{P}(X) \to \mathbb{R}$, that is $\frac{\mathrm{d}}{\mathrm{d}\epsilon} F(\rho + \epsilon\chi)\Big\vert_{\epsilon=0} = \int \frac{\delta F}{\delta \rho} \mathrm{d}\chi$
W_p	Wasserstein distance of order p.
$\mathbb{W}_p$	Wasserstein space of order p.
γ_T	The transport plan in $\Pi(\mu, \nu)$ induced by a map T. That is $\gamma_T = (\mathrm{id}, T)_\# \mu$ and $T_\# \mu = \nu$.
$M(T)$	Monge cost of a map T.
$K(\gamma)$	Kantorovich cost of a plan γ.
$\mu \otimes \nu$	The product measure of μ and ν such that $\mu \otimes \nu(A \times B) = \mu(A)\nu(B)$.
$M^{k \times h}$	The set of real matrices with k rows and h columns.
$M^\top$	Transpose of a matrix M.
i.i.d.	Independent and identical probability distributions.
l.s.c.	Lower semicontinuous.

1

Preliminaries.

Definitions and important theorems to remember.

We start this chapter reminding the basic definitions and theorems in topology and measure theory, since they are needed to have a suitable framework to discuss the optimal transport problem and its applications.

We start with the definition of topology that is needed to introduce a notion of continuity. We refer to [1] for more details.

Definition 1.1 (Topology)**.** *A topology on a set X is a collection $\mathcal{T}$ of subsets of X having the following properties*

- *The space X itself and $\emptyset$ are in $\mathcal{T}$.*
- *The union of the elements of any sub-collection of $\mathcal{T}$ is in $\mathcal{T}$.*
- *The intersection of the elements of any finite sub-collection of $\mathcal{T}$ is in $\mathcal{T}$.*

A pair $(X, \mathcal{T})$ is called a topological space. The elements of $\mathcal{T}$ are called open sets. The complements of the sets of $\mathcal{T}$ are called closed sets. We call a neighborhood of x an element of $\mathcal{T}$ containing x.

Definition 1.2 (Topological Basis.)**.** *Give a set X endowed with a topology $\mathcal{T}$. We call a basis for $\mathcal{T}$ is a collection $\mathcal{B}$ of subsets of X (called basis elements), such that,*

1. *For each $x \in X$, there is at least one basis element B containing x.*
2. *If x belongs to the intersection of two basis elements B_1 and B_2, then there is a basis element B_3 containing x such that $B_3 \in B_1 \cap B_2$.*

Definition 1.3 (Dense set)**.** *A subset D of a topological space X is dense in X if for any point x in X, any neighborhood of x contains at least one point from D.*

Definition 1.4 (Separable space)**.** *A topological space is called separable if it contains a countable, dense subset.*

Definition 1.5. *Let X be a Hausdorff space. Let $\mathcal{V}(x_0)$ be a topological basis of X, such that all $V \in \mathcal{V}$ contains x_0. Let $f : X \to \bar{\mathbb{R}}$ a functional valued in $\bar{\mathbb{R}}$. We define,*

$$\liminf_{x \to x_0} f(x) = \sup_{V \in \mathcal{V}(x_0)} \inf_{s \in V} f(s)$$

$$\limsup_{x \to x_0} f(x) = \inf_{V \in \mathcal{V}(x_0)} \sup_{s \in V} f(s)$$

The above definitions can be expressed in terms of sequences of real numbers. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X, the above formulation is equivalent to say.

$$\liminf_{n\in\mathbb{N}} x_n := \lim_{n\to\infty}\left(\inf_{m\geq n} x_m\right)$$

Equivalently for $\limsup$,

$$\liminf_{n\in\mathbb{N}} x_n := \lim_{n\to\infty}\left(\sup_{m\geq n} x_m\right)$$

Please note that the convergence to some point x_0, $(x_n)_{n\in\mathbb{N}} \to x_0$ is not required in the last definitions.

Definition 1.6 (Lower Semicontinuity)**.** *On a complete metric space X, a function $f : X \to \bar{\mathbb{R}}$ is said to be lower semi-continuous (l.s.c.) if for every sequence $(x_n)_{n\in\mathbb{N}}$ converging to $x \in X$, we have*

$$f(x) \leq \liminf_{n\in\mathbb{N}} f(x_n)$$

Definition 1.7 (Sequentially compact)**.** *A subset K of a metric space X is said to be compact if from any sequence x_n, we can extract a converging subsequence $x_{n_k} \to x \in K$.*

We can see from the above definition that any continuous function is lower-semicontinuous.

Definition 1.8 (Compactness)**.** *A subset K of a metric space X is compact if every open cover of K has a finite subcover.*

Theorem 1.1. *A subset of a metric space is compact if and only if it is sequentially compact.*

Theorem 1.2 (Maxima and Minima)**.** *Let X be a compact metric space and $f : X \to \mathbb{R}$ is continuous, real-valued function. Then f is bounded on X and attains its maximum and minimum. That is, there are x, y belonging to X such that,*

$$f(x) = \inf_{z\in X} f(z) \qquad \text{and} \qquad f(y) = \sup_{z\in X} f(z)$$

Continuity is a strong requirement. Luckily we can assure the existence of a minimizer on lower-semicontinuous functions (or maximizer on upper-semicontinuous). The usual procedure to prove existence of a minimizer is making use of Weierstrass' criterion. We take a minimizing sequence and then we prove that the space in which we are trying to find a minimizer element is compact.

Theorem 1.3 (Weierstrass' criterion for existence of minimizers)**.** *If $f : X \to \bar{\mathbb{R}}$ is lower semi-continuous and X is compact, then there exists $\hat{x} \in X$.*

Proof. Define $l := \inf\{f(x) : x \in X\} \in \bar{\mathbb{R}}$, notice that $l = +\infty$ only if f is identically $+\infty$, then this case is trivial since any point minimizes f. By compactness there exists a minimizing sequence x_n, that is $f(x_n) \to l$. By compactness we can extract a subsequence converging to some $\hat{x}$ such that $\hat{x} \in X$. By lower-semicontinuity of f, we have that $f(\hat{x}) \liminf_n f(x_n) = l$. Since l is the infimum $l \leq f(\hat{x})$. This proves that $l = f(\hat{x}) \in \mathbb{R}$. □

We can apply the above analysis using a notion of upper-semicontinuity and compactness to find the maximum. Explanation about notions of convergence with bounded functionals and vanishing in infinity functions. If X is compact we have $C_0(X) = C_b(X) = C(X)$ if X and both notions of convergence coincide.

Definition 1.9 (Topological Dual)**.** *If X is a normed space, the dual space $X^* = \mathcal{B}(X, \mathbb{R})$. Consists of all linear and bounded functionals mapping from X to $\mathbb{R}$.*

Definition 1.10 (Weak compactness in dual spaces)**.** *A sequence x_n in a Banach space X is said to be weakly converging to x, and we write $x_n \rightharpoonup x$, if for every $\xi \in X^*$. We have $\langle \xi, x_n\rangle \to \langle \xi, x\rangle$. A sequence $\xi_n \in X^*$ is said to be weakly-* converging to ξ, and we write $\xi_n \rightharpoonup \xi$, if for every $x \in X$ we have $\langle \xi_n, x\rangle \to \langle \xi, n\rangle$.*

Theorem 1.4 (Banach-Alaouglu). *If X is separable and ξ_n is a bounded sequence in X^*, then there exists a subsequence ξ_{n_k} weakly converging to some $\xi \in X^*$.*

The Banach-Alaouglu's theorem is a well known result in functional analysis, an equivalent formulation is saying the closed unit ball in X^* is weak-* compact.

Definition 1.11. *A sequence μ_n probability measures over X is said to be tight if for every $\epsilon > 0$, there exists a compact subset $K \subset X S$ such that $\mu_n (X \backslash K) < \epsilon$ for every n.*

Definition 1.12 (Tightness). *Let $(\mathcal{X}, \mathcal{T})$ a topological space, and let $\mathcal{A}$ a σ-algebra on $\mathcal{X}$ that contains the topology $\mathcal{T}$. Let M be a collection of measures defined on $\mathcal{A}$. The collection M is called tight if for every $\epsilon > 0$ there is a compact subset K_ϵ of $\mathcal{X}$ such that, for all measures $\mu \in M$ we have,*

$$|\mu| (\mathcal{X} \backslash K_\epsilon) < \epsilon$$

Theorem 1.5 (Prokhorov). *Suppose that μ_n is a tight sequence of probability measures over a Polish space $\mathcal{X}$. Then there exists $\mu \in \mathcal{P}(X)$ and a subsequence μ_{n_k} such that $\mu_{n_k} \rightharpoonup \mu$, in duality with $C_b(X)$. Conversely, every sequence $\mu_{n_k} \rightharpoonup \mu$ is tight.*

Definition 1.13. *Let $F : X \to \bar{\mathbb{R}}$ be a given functional bounded from below on a metric space X. Let $\mathcal{G}$ be the set of lower semicontinuous functions $G : X \to \bar{\mathbb{R}}$, such that $G \leq F$. We call a relaxation the supremum of $\mathcal{G}$. This functional does exist since the supremum of an arbitrary family of lower semicontinuous functions is also lower semicontinuous. It is possible to have a representation formula as follows:*

$$\bar{F}(x) = \inf\left\{\liminf_{n \in \mathbb{N}} F(x_n) : x_n \to x\right\}. \tag{1.1}$$

As consequence of this definition we see that $F \geq \bar{F}$ implies $\inf F \geq \inf \bar{F}$. Let $l = \inf F$ then $F \geq l$. A constant function is lower semicontinuous. Therefore, $\bar{F} \geq l$ and $\inf \bar{F} \geq \inf F$. Implying that the infimum of both F and its regularization $\bar{F}$ coincide, i.e. $\inf \bar{F} = \inf F$.

Definition 1.14. *A measure over a set Ω is atomless if $\forall x \in \Omega$, we have $\mu(\{x\}) = 0$.*

Definition 1.15 (Proper convex function). *Let $f : X \to \bar{\mathbb{R}}$, a function taking values in the extended real number line. We call it proper convex function if $\exists x \in X$ such that*

$$f(x) < \infty$$

And $\forall x \in X$,

$$f(x) > -\infty \tag{1.2}$$

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Basics in Convex Analysis.

Proposition 2.1. *Let $f : X \to \bar{\mathbb{R}}$ be a convex and lower-semicontinuous function. Assume that there exists $x_0 \in X$ such that $f(x_0) = -\infty$. Then f is nowhere finite on X.*

Theorem 2.1. *If f_α is an arbitrary family of lower semi-continuous functions on X, then $f = \sup_\alpha f_\alpha$ is also lower-semicontinuous.*

Definition 2.1 (Convex conjugate function)**.** *Let X be a Banach space, let $f : X \to \bar{\mathbb{R}}$ be a functional over X. We call the convex conjugate to the function $f^* : X^* \to \bar{\mathbb{R}}$, defined as*

$$f^*(x^*) = \sup_{x\in X}\{\langle x^*, x\rangle - f(x)\}$$

Proposition 2.2. *The convex conjugate $f^* : X^* \to \bar{\mathbb{R}}$ of a function $f : X \to \bar{\mathbb{R}}$ is convex.*

Proof. Let x^*, y^* elements of the dual space X^*, and $t \in [0, 1]$,

$$\begin{aligned}
f^*(tx^* + (1-t)y^*) &= \sup_{x\in X}\{\langle tx^* + (1-t)y^*, x\rangle - f(x)\} \\
&= \sup_{x\in X}\{\langle tx^* + (1-t)y^*, x\rangle - tf(x) - (1-t)f(x)\} \\
&= \sup_{x\in X}\{t\langle x^*, x\rangle + (1-t)\langle y^*, x\rangle - tf(x) - (1-t)f(x)\} \\
&\le \sup_{x,y\in X}\{t\langle x^*, x\rangle + (1-t)\langle y^*, y\rangle - tf(x) - (1-t)f(y)\} \\
&= t\sup_{x\in X}\{\langle x^*, x\rangle - f(x)\} + (1-t)t\sup_{y\in X}\{\langle y^*, y\rangle - f(y)\} \\
&= tf^*(x^*) + (1-t)f^*(y^*).
\end{aligned}$$

Therefore f^* is convex regardless the convexity of f. □

Theorem 2.2. *A function $f : \mathbb{R}^d \to \bar{\mathbb{R}}$ is convex and lower-semicontinuous if and only if $f^{**} = f$.*

Lemma 2.1 (Convex envelope theorem)**.** *Let X be a reflexive Banach Space. Then the convex conjugate function f^* is the maximum convex functional below f (also called convex envelope), i.e. if ϱ is convex functional and $\varrho(x) \le f(x)$, $\forall x \in X$. Then, $f^{**}(x) \le f(x)$, and $\varrho(x) \le f^{**}(u)$, $\forall x \in U$. In particular $f^{**} = f$ if and only if J is convex.*

Definition 2.2 (Legendre Transform)**.** *Let $f : \mathbb{R}^d \to \bar{\mathbb{R}}$ be a convex function, we call the Legendre transform f^**

$$f^*(y) = \sup_{x\in\mathbb{R}^d}\{x \cdot y - f(x)\}$$

Corollary 2.1. *A function $f : \mathbb{R}^d \to \bar{\mathbb{R}}$ is convex and l.s.c. if and only if $f^{**} = f$.*

Definition 2.3 (Subdifferential)**.** *Given a proper convex function $f : X \to (-\infty, \infty]$, the subdifferential of such a function is the mapping $\partial f : X \to X^*$ defined by,*

$$\partial f(x) = \{x^* \in X^*; f(x) - f(y) \leq \langle x^*, x - y \rangle,\ \forall y \in X\}$$

Theorem 2.3 (Geometrical version of Hahn-Banach Theorem)**.** *Let M be a vector subspace of the topological vector space X. Suppose K is a non-empty convex open subset of X with $K \cap M = \emptyset$. Then there is a closed hyperplane $N \in X$ containing M with $K \cap N = \emptyset$.*

Theorem 2.4. *The epigraph of a convex and lower semicontinuous function is a closed convex set in $\mathbb{R}^d \times \mathbb{R}$, and can be written as the intersection of the half-spaces which contain it.*

3

Linear Programming

Linear programming is a well studied branch of the mathematics that studies the optimization of linear functions under linear constrains. The study of linear programming started during the second part of the 1940s, as a technique military oriented problems.

We can formulate the problem in its general form as follows:

Problem 1. *Given a cost vector* $\mathbf{c} \in \mathbb{R}^n$*, a linear operator* $\mathbf{A} \in M^{m\times n}$

$$\begin{aligned} \text{min} \quad & \mathbf{c}^\top \mathbf{x} && (3.1)\\ \text{subject to} \quad & \mathbf{A}\mathbf{x} = \mathbf{b} && (3.2)\\ & \mathbf{x} \geq 0 && (3.3) \end{aligned}$$

Where . $\mathbf{A}$ is a $m \times n$ matrix, and $\mathbf{b}$ is an m-dimensional column vector. The vector inequality means $\mathbf{x} \geq 0$ means that each component is nonnegative. This problem has a solution if $n > m$.

Consider the system of equalities (3.2), the vector $\mathbf{b}$ belongs to $\mathbb{R}^m$.

Definition 3.1. *Given the set of* m *simultaneous linear equations* (3.2) *with* n *unknowns, let* $\mathbf{B}$ *be any nonsingular* $m \times m$ *submatrix made up of columns of* $\mathbb{A}$*. Then if all* $n - m$

Definition 3.2. *If one or more of the basic variables in a basic solution has value zero, that solution is said to be degenerate solution basic solution*

Theorem 3.1 (Fundamental theorem of linear programming.)**.** *Given a linear program in*

Simplex Method.

Interior Method.

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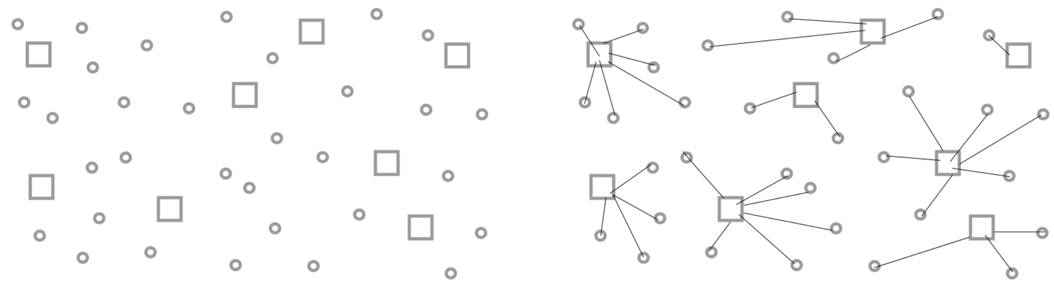
Optimal Transport Theory

To introduce the optimal transport problem please imagine we are asked by a consortium of factories to design a plan for distributing their products among its many costumers in such a way that the transportation costs are minimal.

We can start the approach of this problem considering the costumers as members of the set X and the factories as members of a set Y. We want to know which factory $y \in Y$ is going to supply a costumer $x \in X$, i.e. $y = T(x) \in Y$. Therefore, we can estimate the transportation cost $c(x, T(x))$ of supplying a costumer x with a factory $y = T(x)$. We see that our problem is reduced to find an assigning map from the set of costumers to the set of factories in such a way that the total cost $C(X,Y) = \sum_{x \in X} c(x, T(x))$ is minimal.

Figure 4.1: Illustration of the problem of Factories supplying Costumers.



(a) Factories represented by squares. Costumers represented by circles.

(b) Factories represented by squares. Costumers represented by circles. Assignation of a factory to a costumer represented by a line.

Gaspard Monge was a French mathematician who introduced for the first time the optimal transport problem as *déblais et remblais* in 1781. Monge was interested in finding a map that distributes an amount of sand or soil extracted from the earth or a mine distributed according to a density f, onto a new construction whose density of mass is characterized by a density g, in such a way the average displacement is minimal. In order to give a more precise idea of the problem, we make use of modern mathematical language and notation to state it as follows: Given two densities of mass f and g, Monge was interested in finding a map $T : \mathbb{R}^3 \to \mathbb{R}^3$ pushing the one onto the other,

$$\int_A g(y)\mathrm{dy} = \int_{T^{-1}(A)} f(x)\mathrm{dx}$$

For any Borel subset $A \subset \mathbb{R}^3$. And the transport also should minimize the quantity,

$$\int_{\mathbb{R}^3} \left|x - T(x)\right| f(x)\mathrm{dx}$$

We need to mention that given the context for which the problem was formulated, originally it was bind to $\mathbb{R}^3$ or $\mathbb{R}^2$ but we can consider the general case in $\mathbb{R}^d$. Notice that the problem requires a notion of measure. Then we find convenient to consider the measures μ on $X \subset \mathbb{R}^d$ and ν on $Y \subset \mathbb{R}^d$ induced by the densities f and g respectively. Therefore, we need to search for the optimum in the set of measurables maps $T : X \to Y$ such that the condition (4) is translated to,

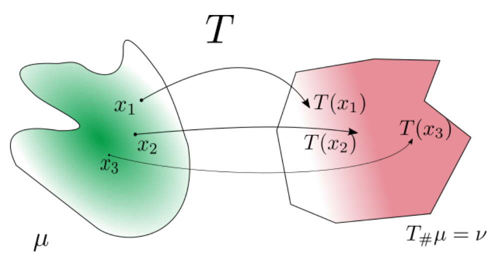
$$(T_{\#}\mu)(A) = \mu(T^{-1}(A)) \tag{4.1}$$

for every measurable set A. In other words, we need $T_{\#}\mu = \nu$. In the Euclidean frameworks if we assume f, g and T regular enough and T also injective, this equality implies,

$$g(T(x)) \det(\mathrm{D}T(x)) = f(x) \tag{4.2}$$

The equation (4.2) is nonlinear in T making difficult the analysis of the Monge Problem. Moreover, proving existence of a minimizer under this setting.

Figure 4.2: Monge problem. Finding a map.



Problem 2. *Given two probability measures $\mu \in \mathcal{P}(X)$ and $\nu \in \mathcal{P}(Y)$ and a cost function $c : X \times Y \to \{0, +\infty\}$, the Monge's problem consists in finding a map $T : X \to Y$*

$$\inf\left\{M(T) := \int_X c(x, T(x))\mathrm{d}\mu(x) : \quad T_{\#}\mu = \nu\right\} \tag{MP}$$

Definition 4.1. *The space of Borel probability measures on $\mathcal{X}$ is denoted by $\mathcal{P}(\mathcal{X})$. The weak topology on $\mathcal{P}(\mathcal{X})$ is induced by convergence against bounded continuous test functions on $\mathcal{X}$, that is $C_b(\mathcal{X})$.*

Definition 4.2 (Coupling)**.** *Let $(\mathcal{X}, \mu)$ and $(\mathcal{Y}, \nu)$ be two probability spaces. Coupling μ and ν means constructing two random variables X and Y on some probability space $(\Omega, \mathcal{P})$ such that $\mathrm{law}(X) = \mu$, $\mathrm{law}(Y) = \nu$. The couple (X, Y) is called a coupling of (μ, ν).*

Theorem 4.1. *Let $(\mathcal{X}_i, \mu_i)$, $i = 1, 2, 3$, be Polish probability spaces. If (X_1, X_2) is a coupling of (μ_1, μ_2) and $(\mathcal{Y}_2, Y_3)$ is a coupling of (μ_2, μ_3), then it is possible to construct a triple of random variables (Z_1, Z_2, Z_3) such that (Z_1, Z_2) has the same law as (X_1, X_2) and (Z_2, Z_3) has the same law as (Y_2, Y_3).*

Problem 3. *Given $\mu \in \mathcal{P}(X)$, $\nu \in \mathcal{P}(Y)$, and $c : X \times Y \to [0, +\infty]$, we consider the problem*

$$\inf\left\{K(\gamma) := \int_{X\times Y} c\mathrm{d}\gamma : \gamma \in \Pi(\mu, \nu)\right\} \tag{KP}$$

where $\Pi(\mu, \nu)$ is the set of transport plans.

Kantorovich formulation as relaxation

Theorem 4.2. *Let X and Y be compact metric spaces, $\mu \in \mathcal{P}(X)$, $\nu \in \mathcal{P}$ and $c : X \times Y \to \mathbb{R}$ a continuous function. Then* (KP) *admits a solution.*

Cyclical Monotonicity.

Definition 4.3 (c-transform). *Let X and Y be sets, and $c : X \times Y \to (-\infty, \infty]$. A function $\psi : X \to \bar{\mathbb{R}}$ is said to be c-convex if it is not identically to $+\infty$ and there exists $\psi^c : Y \to \bar{\mathbb{R}}$*

$$\psi^c(y) = \inf_{x \in X} c(x, y) - \psi(x). \tag{4.3}$$

Definition 4.4. *Let X, Y be arbitrary sets, and $c : X \times Y \to (-\infty, \infty]$ be a cost function. A subset $\Gamma \subset X \times Y$ is said to be c-cyclically monotone if, for any $N \in \mathbb{N}$, and any family of points $(x_1, y_1), (x_2, y_2), \ldots (x_N, y_N)$ of Γ, the inequality*

$$\sum_{i=1}^{N} c(x_i, y_i) \leq \sum_{i=1}^{N} c(x_i, y_{i+1})$$

considering $N + 1 = 1$.

Since any permutation σ over the set $\{1, \ldots, N\}$ can be written as a product of disjoint cycles, we have that this property satisfies,

$$\sum_{i=1}^{N} c(x_i, y_i) \leq \sum_{i=1}^{N} c(x_i, y_{\sigma(i)}) \tag{4.4}$$

Definition 4.5 (Support of transport plan.). *Given a separable metric space X, the support of a measure γ is defined as the smallest closed set on which γ is concentrated,*

$$\operatorname{spt}(\gamma) := \bigcap_{\substack{\gamma(X \backslash A) = 0 \\ A = \bar{A}}} A \tag{4.5}$$

We can fix a point $(x_0, y_0) \in \operatorname{spt}(\gamma)$,

Theorem 4.3.

Wasserstein Spaces. $\mathbb{W}_p$

5

Computation of an Optimal Transport

The approximation of an optimal transport is a challenging problem, computationally speaking. We have found a rich literature on it, and many recent advances in this topic have arisen in the very last years.

Linear Programming Formulation.

Simplex Method Algorithm and Duality.

Sinkhorn-Knopp Algorithm.

Simulated Annealing.

Continuous Formulation.

Beckmann problem and

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Applications

Nash Equilibrium.
Track of a Dynamic.
Domain Adaptation.
Isoperimetric Inequality.
Dynamical Optimal transport.
Barycenter of a Fourier Power Spectrum.****

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