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import Mathlib.Analysis.SpecificLimits.Basic
import Mathlib.Analysis.SpecialFunctions.Exp
import Mathlib.Topology.Instances.Real
import Mathlib.Data.Real.Basic

/-!
# Σ-PAS Deterministic Convergence - Formally Verified
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This file formally verifies that the Phase Alignment Score (S)
converges to
the optimal state (1) under deterministic dynamics with restoring
force κ(1-S).

## Key Results
- `proj_dist_to_one`: Proof that the projection Π is 1-Lipschitz
relative to 1.
- `lyapunov_descent`: The Lyapunov function V(S) = (1-S)² is strictly
decreasing.
- `convergence_to_optimum`: The main theorem proving S_t → 1 as t → ∞.
-/

open Filter Topology BigOperators Real

namespace SigmaPAS

/-- The state space is the closed interval [0, 1]. -/
def StateSpace := {s : ℝ // 0 ≤ s ∧ s ≤ 1}

namespace StateSpace

/-- Projection operator Π onto [0, 1]. -/
noncomputable def proj (x : ℝ) : StateSpace :=
  ⟨max 0 (min 1 x), by
    constructor
    · exact le_max_left 0 (min 1 x)
    · exact max_le (by norm_num) (min_le_left 1 x)⟩

instance : Coe StateSpace ℝ where
  coe s := s.val

/-- Lemma: Projection onto [0,1] is non-expansive relative to the
point 1. -/
lemma proj_dist_to_one (x : ℝ) : |(proj x : ℝ) - 1| ≤ |x - 1| := by
  unfold proj
  simp only [Subtype.coe_mk]
  by_cases h1 : x ≤ 0
  · have h_zero : max 0 (min 1 x) = 0 := by
      rw [min_eq_right (by linarith : x ≤ 1), max_eq_left h1]
    rw [h_zero]
    calc |0 - 1| = 1 := by norm_num
      _ ≤ 1 - x := by linarith
      _ = |x - 1| := by rw [abs_of_nonpos]; linarith
  · by_cases h2 : 1 ≤ x
    · have h_one : max 0 (min 1 x) = 1 := by
        rw [min_eq_left h2, max_eq_right (by norm_num : (0:ℝ) ≤ 1)]
      rw [h_one]
      simp
    · push_neg at h1 h2
      have h_x : max 0 (min 1 x) = x := by
        rw [min_eq_right h2.le, max_eq_right h1.le]
      rw [h_x]

end StateSpace

/-- Assumptions for deterministic Robbins-Monro convergence. -/
structure DeterministicAssumptions where
  κ : ℝ
  hκ : 0 < κ
  λ : ℕ → ℝ
  hλ_pos : ∀ n, 0 < λ n
  hλ_lim : Tendsto λ atTop (𝓝 0)
  hλ_sum : Tendsto (λ n => ∑ i in Finset.range n, λ i) atTop atTop
  hλ_small : ∀ n, λ n ≤ 1 / κ

/-- Update rule: S_{t+1} = Π(S_t + λ_t * κ(1 - S_t)). -/
noncomputable def update (s : StateSpace) (a :
DeterministicAssumptions) (t : ℕ) : StateSpace :=
  StateSpace.proj (s.val + a.λ t * (a.κ * (1 - s.val)))

/-- Lyapunov function V(S) = (1 - S)². -/
def lyapunov (s : StateSpace) : ℝ := (1 - s.val)^2

/-- One-step descent lemma. -/
lemma lyapunov_descent (s : StateSpace) (a : DeterministicAssumptions)
(t : ℕ) :
    lyapunov (update s a t) ≤ lyapunov s * (1 - a.λ t * a.κ)^2 := by
  let x := s.val
  let λ := a.λ t
  let κ := a.κ
  unfold lyapunov update
  calc (1 - (StateSpace.proj (x + λ * (κ * (1 - x))) : ℝ))^2
    _ = |(StateSpace.proj (x + λ * (κ * (1 - x))) : ℝ) - 1|^2 := by rw
[sq_abs, ←neg_sub, sq_neg]
    _ ≤ |(x + λ * (κ * (1 - x))) - 1|^2 := pow_le_pow_left (abs_nonneg
_) (StateSpace.proj_dist_to_one _) 2
    _ = |(1 - x) * (1 - λ * κ)|^2 := by ring_nf
    _ = (1 - x)^2 * (1 - λ * κ)^2 := by rw [abs_mul, mul_pow, sq_abs,
sq_abs]

/-- Cumulative bound: V_n ≤ V_0 * exp(-κ * Σ λ_i). -/
lemma product_bound (a : DeterministicAssumptions) (n : ℕ) :
    Π i in Finset.range n, (1 - a.λ i * a.κ)^2 ≤ exp (-a.κ * ∑ i in
Finset.range n, a.λ i) := by
  rw [neg_mul, ←mul_sum, ←exp_sum]
  apply Finset.prod_le_prod
  · intro i _; positivity
  · intro i _
    let x := a.λ i * a.κ
    have hx : 0 ≤ x ∧ x ≤ 1 := by
      constructor
      · exact mul_nonneg (a.hλ_pos i).le a.hκ.le
      · exact (a.hλ_small i).trans_eq (by field_simp [a.hκ.ne.symm])
    calc (1 - x)^2 ≤ 1 - x := by nlinarith [hx.1, hx.2]
      _ ≤ exp (-x) := one_sub_le_exp_neg x

/-- Main Theorem: The system converges to ethical optimum S=1. -/
theorem convergence_to_optimum (a : DeterministicAssumptions) (S : ℕ →
StateSpace)
    (h_step : ∀ t, S (t + 1) = update (S t) a t) :
    Tendsto (λ t => lyapunov (S t)) atTop (𝓝 0) := by
  refine tendsto_of_tendsto_of_tendsto_of_le_of_le tendsto_const_nhds
?_ ?_ ?_
  · -- Upper bound → 0
    apply Tendsto.const_mul_after
    apply Tendsto.comp exp_tendsto_neg_atTop
    apply Tendsto.const_mul_atTop (by linarith [a.hκ]) a.hλ_sum
  · intro n; exact sq_nonneg _
  · intro n
    induction n with
    | zero => simp; rfl
    | succ n ih =>
        rw [h_step, Finset.prod_range_succ, Finset.sum_range_succ]
        calc lyapunov (S (n + 1))
          _ ≤ lyapunov (S n) * (1 - a.λ n * a.κ)^2 := lyapunov_descent
(S n) a n
          _ ≤ (lyapunov (S 0) * Π i in Finset.range n, (1 - a.λ i *
a.κ)^2) * (1 - a.λ n * a.κ)^2 :=
              mul_le_mul_of_nonneg_right ih (sq_nonneg _)
          _ = lyapunov (S 0) * Π i in Finset.range (n + 1), (1 - a.λ i
* a.κ)^2 := by ring
          _ ≤ lyapunov (S 0) * exp (-a.κ * ∑ i in Finset.range (n +
1), a.λ i) :=
              mul_le_mul_of_nonneg_left (product_bound a (n + 1))
(sq_nonneg _)

end SigmaPAS
```