Wave transmission – MATLAB project 3

Goal: The goals of this project are:

- to explore the effect of dispersion on pulse propagation,
- to understand the approximate analytic model of "group velocity
- to compare the "group velocity" model to exact numerical calculations
- to learn how to perform these calculations effectively.

Problem Formulation: We consider the propagation of a narrowband pulse (eq. (2)) in a dispersive line with dispersive relation $k = k(\omega)$. The line is excited by a signal $V_G(t)$ at $z = 0$

A) Exact expression: Show that the voltage along the line is expressed explicitly as

(1) $$V(z,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, e^{j(\omega t - k(\omega)z)}\tilde{V}_G(\omega) = \frac{1}{\pi}\mathrm{Re}\int_0^{\infty} d\omega\, e^{j(\omega t - k(\omega)z)}\tilde{V}_G(\omega)$$

where $\tilde{V}_G(\omega)$ is the spectrum of the signal $V_G(t)$.

B) The source is a modulate pulse of the form

(2) $$V_G(t) = F(t)\sin(2\pi f_0 t), \quad F(t) = 1 \quad \text{for } -\frac{T}{2} < t < \frac{T}{2} \quad \text{and 0 otherwise.}$$

where $F(t)$ is a square pulse of length $T = 1n\sec$ (the "data") and the modulation frequency is $f_0 = 10\,\mathrm{GHz}$ (the "RF carrier frequency").

B1) Calculate and plot the signal $V_G(t)$ and its spectrum $\tilde{V}_G(\omega)$ and show that the fractional bandwidth is small. Specifically, show that the fractional bandwidth satisfies the "thumb rule"

(3) $$\text{fractional bandwidth} \triangleq \frac{(\Delta f)_F}{f_0} = \frac{2}{N}$$

where

(4) $$(\Delta f)_F = \frac{2}{T}$$

is the bandwidth between the two zeros of the spectrum and

(5) $$N \triangleq \frac{T}{1/f_0} = \text{number of RF oscilations within } F(t).$$

C) Approximate propagation model: Based on the lectures, show that the approximate pulse propagation mode

(6) $$V(z,t) \approx F(t - z/v_g)\sin(\omega_0[t - z/v_p])$$

where

(7) $$v_g = \left.\frac{1}{dk/d\omega}\right|_{\omega_0} = \text{group velocity} \qquad v_p = \left.\frac{\omega}{k}\right|_{\omega_0} = \text{phase velocity}$$

C1) We are given a waveguide dispersion

(8) $$k(\omega) = \sqrt{\left(\frac{\omega}{c}\right)^2 - \left(\frac{\pi}{a}\right)^2}, \quad a = 20\cdot 10^{-3}, \quad c = \text{speed of light}, \qquad f > f_{\text{cutoff}} = \frac{c}{2a} = 7.5\ \text{GHz}$$

Plot the dispersion relation $k(\omega)$ as a function of ω/c. Show that

(9) $$v_p = c\Big/\sqrt{1 - \left(f_{\text{cutoff}}/f_0\right)^2} > c, \qquad v_g = c\sqrt{1 - \left(f_{\text{cutoff}}/f_0\right)^2} < c$$

and calculate the values at $f_0 = 10\,\mathrm{GHz}$

C2) We are given a waveguide dispersion

Show that the distance $z_{1/4}$ where the pulse $F(t - z / v_g)$ lags by quarter of a cycle behind the carrier $\sin(\omega_0[t - z / v_p])$ is given by

(9) $$2\pi f_0(\frac{z}{v_g} - \frac{z}{v_p}) = \frac{\pi}{4}$$

C3) Calculate the signal (6): at $z = z_{1/4}, 2z_{1/4}, 3z_{1/4}, 4z_{1/4},$ as well as at $z = 1$ and $z = 200$. Discuss the results. Compare with the calculations in (11).

D) Exact numerical evaluation of (1) using FFT and compare with the results of Part C3.
Instruction: For the calculation it is convenient to define the time relative to the pulse center

(10) $$t = \frac{z}{v_g} + \tau$$

(11) $$\Rightarrow \quad V(z,\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega e^{j\omega\tau} e^{-j[k(\omega) - \frac{\omega}{v_g}]z} \hat{V}_G(\omega)$$

The integral in (11) is more convenient for numerical calculations since:

- The signal is concentrated all the time about $\tau = 0$
- The fast variation of the phase $e^{-jk(\omega)z}$ in the integral of (1), in particular for large *z*, is canceled by the term $e^{j(\omega/v_g)z}$

Submission Guidelines

1. The project is performed and submitted in couples, but students may also do it by their own.
2. All answers should be explained.
3. Attach your code at the end of the report.
4. The couples will be examined on their work. Please bring a hard copy of the report to the exam.
5. Submission via the MODEL.