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THE EXIT DISTRIBUTION ON THE STOCHASTIC SEPARATRIX IN KRAMERS' EXIT PROBLEM*

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Abstract. Kramers' exit problem is concerned with noise activated escape from a potential well. In the case when the noise strength, ε (temperature measured in units of potential barrier height), is small, this becomes a singular perturbation problem. Unexpectedly, the noise induced escape from the domain of attraction of the stable equilibrium at the bottom of the well does not occur at points on the separatrix where the energy is minimal. Rather, the distribution of exit points on the separatrix (in the phase plane) is shown to be spread away from the saddle point (where the energy is minimal). It is shown that, actually, most of the probability of the exit points on the separatrix is located at a distance $O(\sqrt{\varepsilon})$ from the saddle point, and the probability vanishes altogether at the saddle point. In this problem, large deviations theory fails to predict the distribution of the exit point for finite noise. We construct an asymptotic approximation to the solution of the Fokker–Planck equation with a source in the domain and absorption at the separatrix and find the exit density as the normal component of the probability flux density on the boundary. We resolve the singularity at the saddle point.

Key words. stochastic differential equations, Langevin's equation, the exit problem, first passage time

AMS subject classifications. 60H10, 82C31, 46N55, 47N55

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1. Introduction.

The exit problem in the theory of stochastic differential equations concerns the escape of the random trajectories of a dynamical system driven by noise from the domain of attraction of the underlying noiseless dynamics [1], [2], [3], [4], [5], [6], [7], [8] (and references therein). Large deviations theory (LDT) [5], [9] predicts that in the limit of vanishing noise escapes are concentrated at the absolute minima of an action functional on the separatrix (the boundary of the domain of attraction of an attractor of the noiseless dynamics). However, it has been observed in numerical simulations [2], [4], [10], [11] that, for finite noise strength, this is not the case, and, actually, escaping trajectories avoid the absolute minimum so that the escape distribution is spread on the separatrix away from the points predicted by LDT. Some analytical results concerning this saddle point avoidance phenomenon were given in [2], [4], [12], [13], [14], [15], [16], [17], [18], [19], [20], [21], [22], and, more recently, [23].

Kramers' model of activated escape [1] has become a cornerstone in statistical physics, with applications in many branches of science and mathematics [1], [2], [3], [4], [5], [6], [7], [8], [9], [10], [11], [12], [13], [14], [15], [16], [17], [18], [19], [20], [21], [22], [23], [24], [25], [26], [27], [28], [29], [30], [31], [32], [33], [34]. It has important applications in diverse areas such as communications theory [4], [24], stochastic stability of structures [10], [25], and even the modern theory of finance [26].

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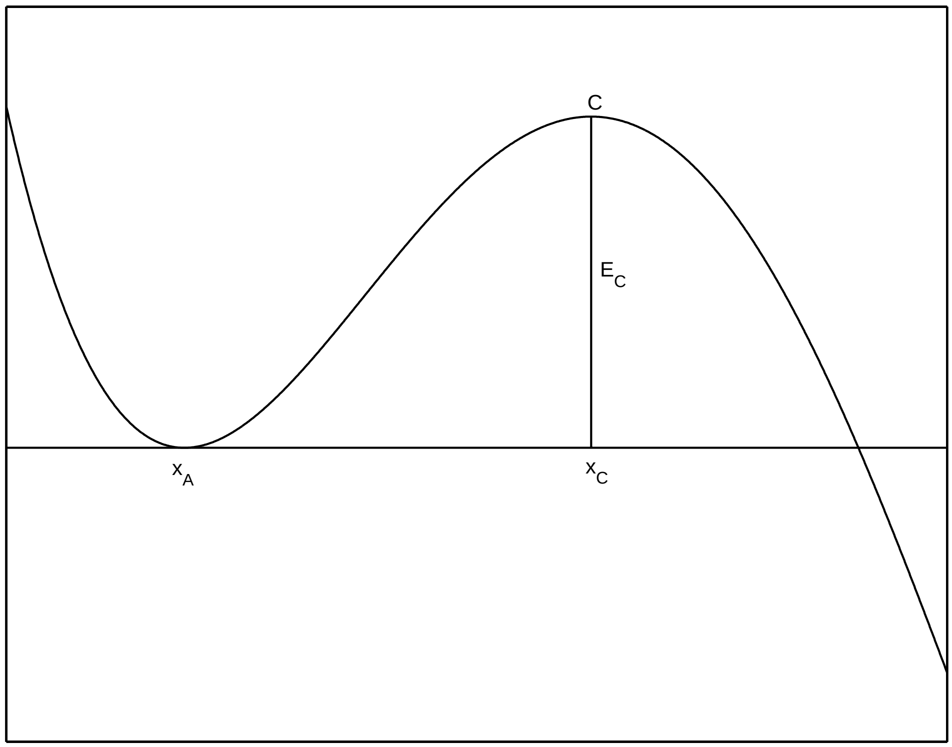

FIG. 1.1. *The potential well.*

Vast literature on exit problems has been accumulated [8], and the problem is still an active area of physical, chemical, biological, and mathematical research. The problem of distribution of the exit points is related to the distribution of energies of the escaping particles [6], [27], to the phenomenon of saddle point avoidance elaborated by Berezhkovskii et al. (see the review [28] and references therein), and to numerical simulations of escape problems (see, e.g., [16]). The problem of determining the distribution of exit points has been studied in different contexts, under various assumptions, and by a variety of analytical, numerical, and experimental methods in [2], [3], [4], [5], [6], [7], [8], [9], [10], [11], [12], [13], [14], [15], [16], [17], [18], [19], [20], [21], [22]. The analytical study of this problem in Kramers' setting, however, is not covered by the cited results and requires separate analysis, as attempted in [23].

In this paper, an alternative approach to that of [23] is proposed, and issues left unresolved in [34] and [23] are clarified. Specifically, we construct an asymptotic approximation to the stationary solution of the Fokker–Planck equation (FPE) in the domain of attraction of the stable equilibrium point, with a source inside the domain and with absorbing boundary conditions. We calculate the exit distribution as the normal component of the absorption probability flux density on the boundary. We resolve the singularity of the solution near the saddle point, an issue that was left unresolved in [34] and [23]. Our result differs from that of [23], though both are qualitatively similar. Specifically, we find that the exit distribution on the separatrix vanishes linearly with arc length near the saddle point (not as another power of the arc length, as claimed in [23]). We show that the most likely exit point is removed

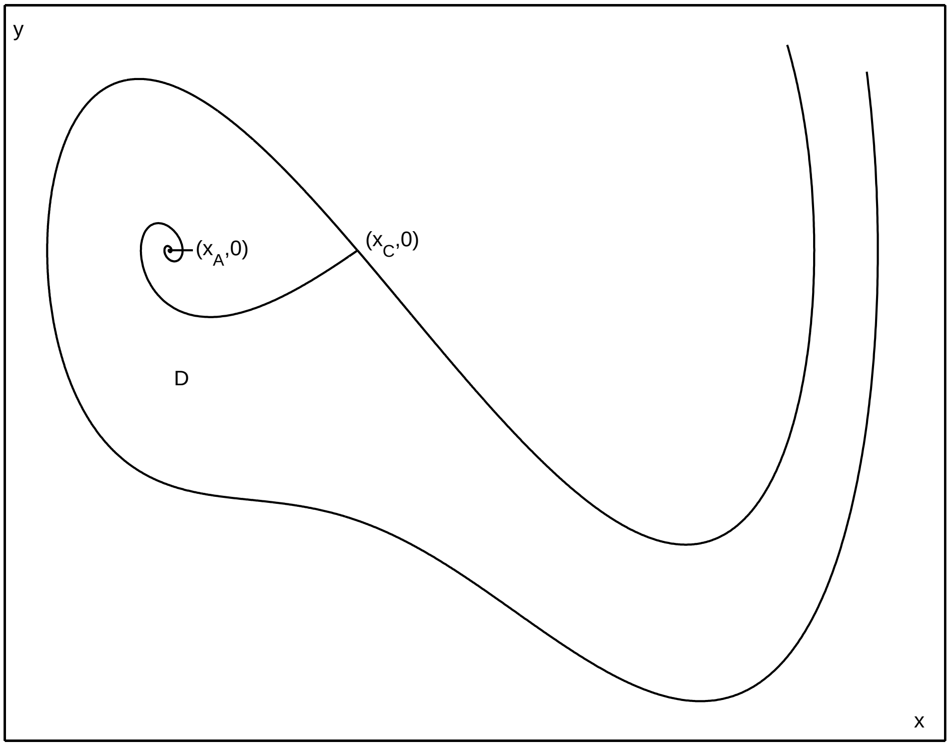

FIG. 1.2. *The domain of attraction in phase space, D, is bounded by the separatrix* Γ.

$O(\sqrt{\varepsilon})$ away from the saddle point, as claimed in [23], where ε is the dimensionless measure of the noise variance (and temperature is measured in units of the barrier height).

Kramers' problem of activated escape [1] is concerned with the motion of a Brownian particle in a field of force. The motion is described by the dimensionless Langevin equation

$$\ddot{x} + \beta\dot{x} + U'(x) = \sqrt{2\varepsilon\beta}\,\dot{\mathrm{w}}, \tag{1.1}$$

where $U(x)$ is a potential that forms a well with barrier height normalized to 1 (see Figure 1.1), β is the dissipation constant normalized by the frequency of vibration at the bottom of the well, ε is dimensionless temperature normalized by the barrier height, and $\dot{\mathrm{w}}$ is standard Gaussian white noise [1]. If ε is a small parameter (e.g., if the barrier of the well is high), the stochastic trajectories of (1.1) in configuration space $x(t)$ stay inside the well for a long time but ultimately escape [1], [2], [3], [4], [5], [6], [7], [8]. To describe the escape process, the Langevin equation is converted to the phase plane system

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -\beta y - U'(x) + \sqrt{2\varepsilon\beta}\,\dot{\mathrm{w}}. \end{aligned} \tag{1.2}$$

The domain of attraction of the stable equilibrium point of the noiseless dynamics in phase space,

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -\beta y - U'(x), \end{aligned} \tag{1.3}$$

located at the equilibrium point at the bottom of the potential well, is denoted by D and is bounded by a separatrix, Γ, which for small ε is also the *stochastic separatrix* [29], [30], [31], [32], [33], that is, the locus of points where the random trajectories of (1.2) are equally likely to escape or to return to the well (see Figure 1.2 and also [35]).

The specific exit problem for (1.2) is to determine the probability density function (pdf) of the points where escaping trajectories hit Γ. In addition to the asymptotic expressions for the solution of the FPE and to the exit density, we compare the graph of the asymptotic expression for the exit pdf with a normalized histogram of exit points obtained by numerical simulation of the original stochastic dynamics equation (1.2) (see Figure 6.1). The numerical result indicates that this pdf is asymmetric about its maximum, which is achieved at a point on Γ whose distance from the saddle point is $O\left(\sqrt{\varepsilon}\right)$, in agreement with the above-mentioned asymptotic result.

2. Notation and formulation. The notation is the same as in [23] and is reproduced here for the sake of completeness. The drift vector and the noise matrix of the stochastic system (1.2), corresponding to the Langevin equation (1.1) in the phase plane, are denoted by

$$\boldsymbol{b}(x,y) = \begin{pmatrix} b_1(x,y) \\ b_2(x,y) \end{pmatrix} = \begin{pmatrix} y \\ -\beta y - U'(x) \end{pmatrix}, \quad \boldsymbol{\sigma}(x,y) = \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{2\varepsilon\beta} \end{pmatrix}. \tag{2.1}$$

The underlying deterministic dynamics of the system are governed by (1.3). Its phase plane portrait is given in Figure 1.2, where the point $(x_A, 0)$ is an attractor, while the point $(x_C, 0)$ is a saddle point. We assume for simplicity that $x_C = 0$. The domain of attraction of the attractor and its boundary are denoted by D and Γ, respectively. The curve Γ is the separatrix of the noiseless dynamics, and in the limit $\varepsilon \to 0$ it becomes the stochastic separatrix. We denote by $\omega_C = \sqrt{-U''(x_C)}$ the imaginary frequency at the top of the barrier.

The separatrix Γ is given by $y = y_\Gamma(x)$, where $y_\Gamma(x)$ is the solution of the initial value problem [36]

$$y_\Gamma'(x) = -\beta - \frac{U'(x)}{y_\Gamma(x)}, \qquad y_\Gamma(x_C) = 0. \tag{2.2}$$

Of the two solutions of (2.2), $y_\Gamma(x)$ is the one with

$$y_\Gamma'(x_C) = -\frac{\beta + \sqrt{\beta^2 + 4\omega_C^2}}{2} = -\lambda. \tag{2.3}$$

We denote by

$$\boldsymbol{\nu}(x, y_\Gamma(x)) = \begin{pmatrix} \nu_1(x, y_\Gamma(x)) \\ \nu_2(x, y_\Gamma(x)) \end{pmatrix}$$

the unit outer normal on Γ. Obviously,

$$\boldsymbol{\nu}(x, y_\Gamma(x)) = \frac{1}{\sqrt{\left[\beta y_\Gamma(x) + U'(x)\right]^2 + y_\Gamma^2(x)}} \begin{pmatrix} \beta y_\Gamma(x) + U'(x) \\ y_\Gamma(x) \end{pmatrix}. \tag{2.4}$$

The pdf of exit points on Γ is determined from the solution of the stationary FPE in D corresponding to the system (1.2), with a unit source at a point $(x_0, y_0) \in D$ and absorbing boundary condition on Γ [3], [34]. Specifically, the solution of the FPE

$$-y\frac{\partial p}{\partial x} + \frac{\partial}{\partial y}\left[(\beta y + U'(x))\, p\right] + \varepsilon\beta = \frac{\partial^2 p}{\partial y^2} = -\delta(x - x_0, y - y_0) \tag{2.5}$$

in D is a function $p_\varepsilon(x, y \,|\, x_0, y_0)$ which satisfies the boundary condition

$$p_\varepsilon(x,\, y_\Gamma(x)|\, x_0, y_0) = 0 \tag{2.6}$$

and can be written as the conservation law

$$\nabla \cdot \boldsymbol{J}(x, y \,|\, x_0, y_0) = \delta(x - x_0, y - y_0), \tag{2.7}$$

where

$$\boldsymbol{J}(x, y \,|\, x_0, y_0) = \begin{pmatrix} y p_\varepsilon(x, y \,|\, x_0, y_0) \\ \\ -\left[\beta y + U'(x)\right] p_\varepsilon(x, y \,|\, x_0, y_0) - \varepsilon\beta \dfrac{\partial p_\varepsilon(x, y \,|\, x_0, y_0)}{\partial y} \end{pmatrix}. \tag{2.8}$$

Due to the boundary condition (2.6), the flux density vector on Γ reduces to

$$\boldsymbol{J}(x, y_\Gamma(x) \,|\, x_0, y_0) = \begin{pmatrix} 0 \\ \\ -\varepsilon\beta \dfrac{\partial p_\varepsilon(x, y_\Gamma(x) \,|\, x_0, y_0)}{\partial y} \end{pmatrix}. \tag{2.9}$$

The pdf per unit arc length of the exit points on Γ is given by

$$\tilde{p}_\Gamma(s \,|\, x_0, y_0)\, ds = \mathcal{N}^{-1} \boldsymbol{J}(x, y_\Gamma(x) \,|\, x_0, y_0) \cdot \boldsymbol{\nu}(x, y_\Gamma(x))\, ds, \tag{2.10}$$

where the normalization constant is

$$\mathcal{N} = \oint_\Gamma \boldsymbol{J}(x, y \,|\, x_0, y_0) \cdot \boldsymbol{\nu}(x, y)\, ds. \tag{2.11}$$

This pdf can be converted to a pdf per unit x by the identities

$$\begin{aligned} \tilde{p}_\Gamma(s \,|\, x_0, y_0)\, ds &= -\mathcal{N}^{-1} \varepsilon\beta \frac{\partial p_\varepsilon(x, y_\Gamma(x) \,|\, x_0, y_0)}{\partial y} \nu_2(x, y_\Gamma(x))\, ds \\ &= -\mathcal{N}^{-1} \varepsilon\beta \frac{\partial p_\varepsilon(x, y_\Gamma(x) \,|\, x_0, y_0)}{\partial y}\, dx \\ &= p_\Gamma(x \,|\, x_0, y_0)\, dx, \end{aligned} \tag{2.12}$$

and the normalization constant can be written as

$$\mathcal{N} = -\oint_\Gamma \varepsilon\beta \frac{\partial p_\varepsilon(x, y_\Gamma(x) \,|\, x_0, y_0)}{\partial y}\, dx.$$

Thus the exit density per unit x, denoted above by $p_\Gamma(x \,|\, x_0, y_0)$, is the normal component of the normalized flux density of the stationary pdf $p_\varepsilon(x, y \,|\, x_0, y_0)$ on Γ.

3. The outer WKB solution. We follow the method of [34] for the asymptotic construction of the solution, with the necessary modifications. The asymptotic solution of the FPE (2.5) is constructed in the form of a WKB outer solution multiplied by a boundary layer function. A solution of the form

$$p_\varepsilon(x,y\,|\,x_0,y_0) = e^{-E(x,y)/\varepsilon} q_\varepsilon(x,y\,|\,x_0,y_0), \tag{3.1}$$

where

$$E(x,y) = \frac{y^2}{2} + U(x),$$

is substituted into the FPE (2.5), and the resulting equation for $q_\varepsilon(x,y\,|\,x_0,y_0)$ is given by

$$-y\frac{\partial q_\varepsilon}{\partial x} + (-\beta y + U'(x))\frac{\partial q_\varepsilon}{\partial y} + \varepsilon\beta\frac{\partial^2 q_\varepsilon}{\partial y^2} = -e^{E(x,y)/\varepsilon}\delta(x-x_0,y-y_0). \tag{3.2}$$

The boundary condition for (3.2) is

$$q_\varepsilon(x,y_\Gamma(x)\,|\,x_0,y_0) = 0 \quad \text{for } (x_0,y_0)\in D. \tag{3.3}$$

To derive a boundary layer equation for $q_\varepsilon(x,y\,|\,x_0,y_0)$, we note that, since the right-hand side of (3.2) can be written as $e^{E(x_0,y_0)/\varepsilon}\delta(x-x_0,y-y_0)$, the constant $e^{E(x_0,y_0)/\varepsilon}$ is only a scaling factor for $q_\varepsilon(x,y\,|\,x_0,y_0)$. Thus this constant can be replaced by an arbitrary nonzero constant which may depend on ε. To simplify the calculations, we choose a constant $C(\varepsilon)$ that is transcendentally small in ε. This does not affect the evaluation of the exit density, because this constant cancels out in the numerator and denominator of (3.7) below. Thus we rescale $q_\varepsilon(x,y\,|\,x_0,y_0)$ so that (3.2) becomes

$$-y\frac{\partial q_\varepsilon}{\partial x} + (-\beta y + U'(x))\frac{\partial q_\varepsilon}{\partial y} + \varepsilon\beta\frac{\partial^2 q_\varepsilon}{\partial y^2} = -C(\varepsilon)\delta(x-x_0,y-y_0). \tag{3.4}$$

Keeping this in mind, we find that the outer expansion of $q_\varepsilon(x,y\,|\,x_0,y_0)$,

$$q_\varepsilon(x,y\,|\,x_0,y_0) = q_0(x,y\,|\,x_0,y_0) + \varepsilon q_1(x,y\,|\,x_0,y_0) + \cdots, \tag{3.5}$$

gives

$$-y\frac{\partial q_0}{\partial x} + (-\beta y + U'(x))\frac{\partial q_0}{\partial y} = 0.$$

This is a partial differential equation of first order whose characteristics converge to the stable equilibrium point $(x_A,0)$. It follows that $q_0(x,y\,|\,x_0,y_0) = \text{const}$ (see [37]). The scaling constant $C(\varepsilon)$ is chosen so that $q_0(x,y\,|\,x_0,y_0)=1$. It follows that

$$\lim_{\varepsilon\to 0} q_\varepsilon(x,y\,|\,x_0,y_0) = 1 \quad \text{for all} \quad (x,y)\,,(x_0,y_0)\in D. \tag{3.6}$$

The exit pdf can be written in terms of the function $q_\varepsilon(x,y\,|\,x_0,y_0)$ as

$$p_\Gamma(x\,|\,x_0,y_0)\,dx = \frac{\exp\left\{-\dfrac{E(x,y_\Gamma(x))}{\varepsilon}\right\}\dfrac{\partial q_\varepsilon(x,y_\Gamma(x)\,|\,x_0,y_0)}{\partial y}\,dx}{\displaystyle\oint_\Gamma \exp\left\{-\dfrac{E(x,y_\Gamma(x))}{\varepsilon}\right\}\dfrac{\partial q_\varepsilon(x,y_\Gamma(x)\,|\,x_0,y_0)}{\partial y}\,dx}. \tag{3.7}$$

Note that $p_\Gamma(x\,|\,x_0,y_0)$ in (3.7) is independent of the scaling factor $C(\varepsilon)$.

4. The boundary layer.

4.1. Boundary layer analysis away from the saddle point. The boundary Γ is noncharacteristic for the boundary value problem (3.2), (3.3), (3.6) in the sense that the drift vector

$$\tilde{\boldsymbol{b}}(x,y) = \begin{pmatrix} -y \\ \\ -\beta y + U'(x) \end{pmatrix}$$

points into D at Γ. Specifically, away from the saddle point $(0,0)$, the normal component of the drift vector is negative; that is,

$$\tilde{\boldsymbol{b}}(x, y_\Gamma(x)) \cdot \boldsymbol{\nu}(x, y_\Gamma(x)) = \frac{-2y_\Gamma^2(x)\,\beta}{\sqrt{(-\beta y_\Gamma(x) + U'(x))^2 + y_\Gamma^2(x)}} < 0.$$

The noncharacteristic boundary layer variables are x and

$$\zeta = \frac{\rho(x,y)}{\varepsilon},$$

where, for $(x,y) \in D$,

$$\rho(x,y) = -\mathrm{dist}\,\{(x,y), \Gamma\}.$$

Note that

$$\boldsymbol{\nu}(x, y_\Gamma(x)) = \nabla\rho(x, y_\Gamma(x)).$$

We define

$$q_\varepsilon(x, y \,|\, x_0, y_0) = Q(x,\zeta),$$

and, as remarked at the end of section 3, we change the constant $e^{E(x_0,y_0)/\varepsilon}$ to a transcendentally small constant $C(\varepsilon)$. Then (3.2) becomes

$$\begin{aligned}\frac{\beta\rho_y^2}{\varepsilon}Q_{\zeta\zeta} + \beta\rho_{yy}Q_\zeta - yQ_x - \frac{y\rho_x}{\varepsilon}Q_\zeta + [-\beta y + U'(x)]\,\frac{\rho_y}{\varepsilon}Q_\zeta \\ = C(\varepsilon)\delta(x-x_0, y-y_0).\end{aligned} \tag{4.1}$$

Expanding

$$Q(x,\zeta) = Q^0(x,\zeta) + \varepsilon Q^1(x,\zeta) + \cdots, \tag{4.2}$$

we obtain the leading order boundary layer equation

$$\begin{aligned}&\beta\rho_y^2(x, y_\Gamma(x))\,Q^0_{\zeta\zeta}(x,\zeta) - y_\Gamma(x)\,\rho_x(x, y_\Gamma(x))\,Q^0_\zeta(x,\zeta) \\ &+ [-\beta y_\Gamma(x) + U'(x)]\,\rho_y(x, y_\Gamma(x))\,Q^0_\zeta(x,\zeta) = 0.\end{aligned} \tag{4.3}$$

Note that $Q_x(x,\zeta)$ does not appear in (4.3) so that $x =$ is a parameter in this equation. The boundary condition (3.3) and matching condition (3.6) give

$$Q(x,0) = 0, \quad \lim_{\zeta\to-\infty} Q(x,\zeta) = 1. \tag{4.4}$$

The solution of (4.3) that satisfies the boundary and matching conditions (4.4) is given by

$$Q^0(x,\zeta) = 1 - \exp\left\{\frac{y_\Gamma(x)\rho_x(x,y_\Gamma(x)) + [\beta y_\Gamma(x) - U'(x)]\rho_y(x,y_\Gamma(x))}{\beta\rho_y^2(x,y_\Gamma(x))}\zeta\right\}. \tag{4.5}$$

We denote

$$r(x) = \frac{y_\Gamma(x)\rho_x(x,y_\Gamma(x)) + [\beta y_\Gamma(x) - U'(x)]\rho_y(x,y_\Gamma(x))}{\beta\rho_y^2(x,y_\Gamma(x))} \tag{4.6}$$

and note that $r(x) > 0$ for $x < 0$ so that the exponent is negative for negative ζ. In particular, we use near the saddle point the linearized expressions

$$y_\Gamma(x) \sim -\lambda x, \quad \rho_x \sim \frac{\lambda}{\sqrt{1+\lambda^2}}, \quad \rho_y \sim \frac{1}{\sqrt{1+\lambda^2}}, \quad U'(x) \sim -\omega_C^2 x.$$

It follows that, near the saddle point, the boundary layer function is

$$Q^0(x,\zeta) \sim 1 - \exp\left\{-2\lambda\sqrt{1+\lambda^2}x\zeta\right\}.$$

4.2. Boundary layer analysis near the saddle point. Since

$$\lim_{x\to 0} \boldsymbol{b}(x,y_\Gamma(x)) \cdot \boldsymbol{\nu}(x,y_\Gamma(x)) = 0,$$

the noncharacteristic boundary layer expansion fails in the $\sqrt{\varepsilon}$-neighborhood of the saddle point. Specifically, in this neighborhood, the terms neglected in the powers series expansion (4.2) of the solution of (4.1) are of the same order of magnitude as the ones retained. Therefore, a separate boundary layer analysis is required in this neighborhood. This expansion has to resolve the singularity of the boundary layer equation (4.1) at the saddle point.

To do so, we introduce in (3.2) the stretched variables

$$(\xi,\eta) = \frac{(x,\rho)}{\sqrt{\varepsilon}}$$

and set

$$q_\varepsilon(x,y \,|\, x_0,y_0) = R_0(\xi,\eta) + \varepsilon R_1(\xi,\eta) + \cdots.$$

Then (3.2) becomes

$$\frac{\partial^2 R_0}{\partial\eta^2} + (a\xi + b\eta)\frac{\partial R_0}{\partial\eta} + (c\xi + d\eta)\frac{\partial R_0}{\partial\xi} = 0, \tag{4.7}$$

where

$$a = 2\lambda^2\frac{\sqrt{(1+\lambda^2)}}{\beta}, \quad b = \frac{-\lambda+\beta}{\beta} < 0, \quad c = \frac{\lambda}{\beta}\left(1+\lambda^2\right) > 0, \quad d = -\frac{\sqrt{(1+\lambda^2)}}{\beta} < 0,$$

and

$$\lambda = \frac{\beta + \sqrt{\beta^2 + 4\omega_C^2}}{2} > 0.$$

The domain D is mapped onto the third quadrant in the (ξ,η)-plane, and the separatrix Γ is mapped onto the line $\eta=0$, $\xi<0$. The boundary and matching conditions (4.4) are now

$$R_0\left(\xi,0\right)=0,\qquad \lim_{\eta\to-\infty}R_0\left(\xi,\eta\right)=1.$$

We seek a solution to (4.7) in the form

$$R_0\left(\xi,\eta\right)=\sqrt{\frac{2}{\pi}}\int_0^{\chi(\xi,\eta)}e^{-z^2/2}\,dz, \tag{4.8}$$

where

$$\chi(\xi,0)=0,\qquad \lim_{\eta\to-\infty}\chi(\xi,\eta)=\infty\quad\text{for }\xi<0.$$

The function χ satisfies the equation

$$\frac{\partial^2\chi}{\partial\eta^2}-\chi\left(\frac{\partial\chi}{\partial\eta}\right)^2+\left(a\xi+b\eta\right)\frac{\partial\chi}{\partial\eta}+\left(c\xi+d\eta\right)\frac{\partial\chi}{\partial\xi}=0. \tag{4.9}$$

We solve (4.9) by the power series expansion

$$\chi=\sum_{n=1}^{\infty}\frac{A_n\left(\xi\right)}{n!}\eta^n$$

and obtain

$$\sum_{n=2}^{\infty}\eta^{n-2}\frac{A_n\left(\xi\right)}{(n-2)!}-\left(\sum_{n=1}^{\infty}\frac{A_n\left(\xi\right)}{n!}\eta^n\right)\left(\sum_{n=1}^{\infty}\frac{A_n\left(\xi\right)}{(n-1)!}\eta^{n-1}\right)^2$$
$$+\left(a\xi+b\eta\right)\sum_{n=1}^{\infty}\frac{A_n\left(\xi\right)}{(n-1)!}\eta^{n-1}+\left(c\xi+d\eta\right)\sum_{n=1}^{\infty}\eta^n\frac{A_n'\left(\xi\right)}{n!}=0.$$

This leads to the hierarchy of equations

$$A_2\left(\xi\right)+a\xi A_1\left(\xi\right)=0, \tag{4.10}$$

$$A_3\left(\xi\right)-A_1^3\left(\xi\right)+bA_1\left(\xi\right)+a\xi A_2\left(\xi\right)+c\xi A_1'\left(\xi\right)=0, \tag{4.11}$$

and so on. Substituting from (4.10), we obtain (4.11) in the form

$$A_3\left(\xi\right)-A_1^3\left(\xi\right)+bA_1\left(\xi\right)-a^2\xi^2A_1\left(\xi\right)+c\xi A_1'\left(\xi\right)=0. \tag{4.12}$$

The function $A_3\left(\xi\right)$ is determined by matching the solution (4.8) with the boundary layer function $Q^0\left(x,\zeta\right)$ in the matching region. In the matching region, we express the boundary layer function $Q^0(x,\zeta)$ in terms of the variables (ξ,η),

$$Q^0\left(x,\zeta\right)=1-\exp\left\{-2\lambda\sqrt{1+\lambda^2}\xi\eta\right\}=2\lambda\sqrt{1+\lambda^2}\xi\eta+O\big(\left(\xi\eta\right)^2\big)\quad\text{for small }\eta.$$

On the other hand, we have

$$R_0=\sqrt{\frac{2}{\pi}}\left\{\chi+O\left(\chi^2\right)\right\}=\sqrt{\frac{2}{\pi}}\sum_{n=1}^{\infty}\frac{A_n\left(\xi\right)}{n!}\eta^n+O\left(\chi^2\right).$$

Matching $2\lambda\sqrt{1+\lambda^2}\xi\eta + O((\xi\eta)^2)$ and $\sqrt{\frac{2}{\pi}}\sum_{n=1}^{\infty}\frac{A_n(\xi)}{n!}\eta^n + O\left(\chi^2\right)$ as functions for small $\xi\eta$, we find that $A_3\left(\xi\right)$ has to be chosen so that

$$\sqrt{\frac{2}{\pi}}A_1\left(\xi\right) = \frac{r(\sqrt{\varepsilon}\xi)}{\sqrt{\varepsilon}}. \tag{4.13}$$

Expanding

$$\sqrt{\frac{2}{\pi}}A_1\left(\xi\right) = \frac{r(\sqrt{\varepsilon}\xi)}{\sqrt{\varepsilon}} = -2\lambda\sqrt{1+\lambda^2}\xi + O\left(\xi^2\right), \tag{4.14}$$

we see that this can be accomplished by choosing

$$A_3\left(\xi\right) = A_1^3\left(\xi\right) + a^2\xi^2 A_1\left(\xi\right) - \left(b+c\right)A_1\left(\xi\right) \tag{4.15}$$

so that (4.12) is reduced to

$$\xi A_1'\left(\xi\right) - A_1\left(\xi\right) = 0.$$

Thus (4.14) follows.

This matching gives a leading order approximation to $\chi(\xi,\eta)$ as a power series in both ξ and η. In higher order matching, the coefficients in the boundary layer equation have to be expanded as power series in ξ and η, and the matching condition in (4.13) has to be satisfied at all orders.

5. The flux and exit density. The exit density on Γ, as given in (3.7), can now be calculated from the boundary layer expansions (4.5) and (4.8). The former gives outside the $\sqrt{\varepsilon}$-neighborhood of the saddle point

$$\left.\frac{\partial p}{\partial y}\right|_{\Gamma} \sim -e^{-E/\varepsilon}\frac{r(x)\rho_y\left(x, y_\Gamma\left(x\right)\right)}{\varepsilon},$$

while the latter gives inside this neighborhood

$$\left.\frac{\partial p}{\partial y}\right|_{\Gamma} \sim -e^{-E/\varepsilon}2\lambda\sqrt{1+\lambda^2}\frac{x\rho_y\left(x, y_\Gamma\left(x\right)\right)}{\varepsilon},$$

which match to leading order as functions in the matching region. It follows that the leading order approximation to the exit pdf on Γ is given by

$$p_\Gamma(x)\,dx \sim \frac{e^{-E(x,y_\Gamma(x))/\varepsilon}r(x)\,dx}{\displaystyle\oint_\Gamma e^{-E(x,y_\Gamma(x))/\varepsilon}r(x)\,dx},$$

where $E\left(x, y_\Gamma(x)\right) = \frac{1}{2}y_\Gamma^2(x) + U(x)$ and $r(x)$ is defined in (4.6).

The most likely exit point is obtained from the equation $p_\Gamma'(x) = 0$ as

$$x_M = -\sqrt{\frac{\varepsilon}{\beta\lambda}}\left(1 + O\left(\sqrt{\frac{\varepsilon}{\beta\lambda}}\right)\right).$$

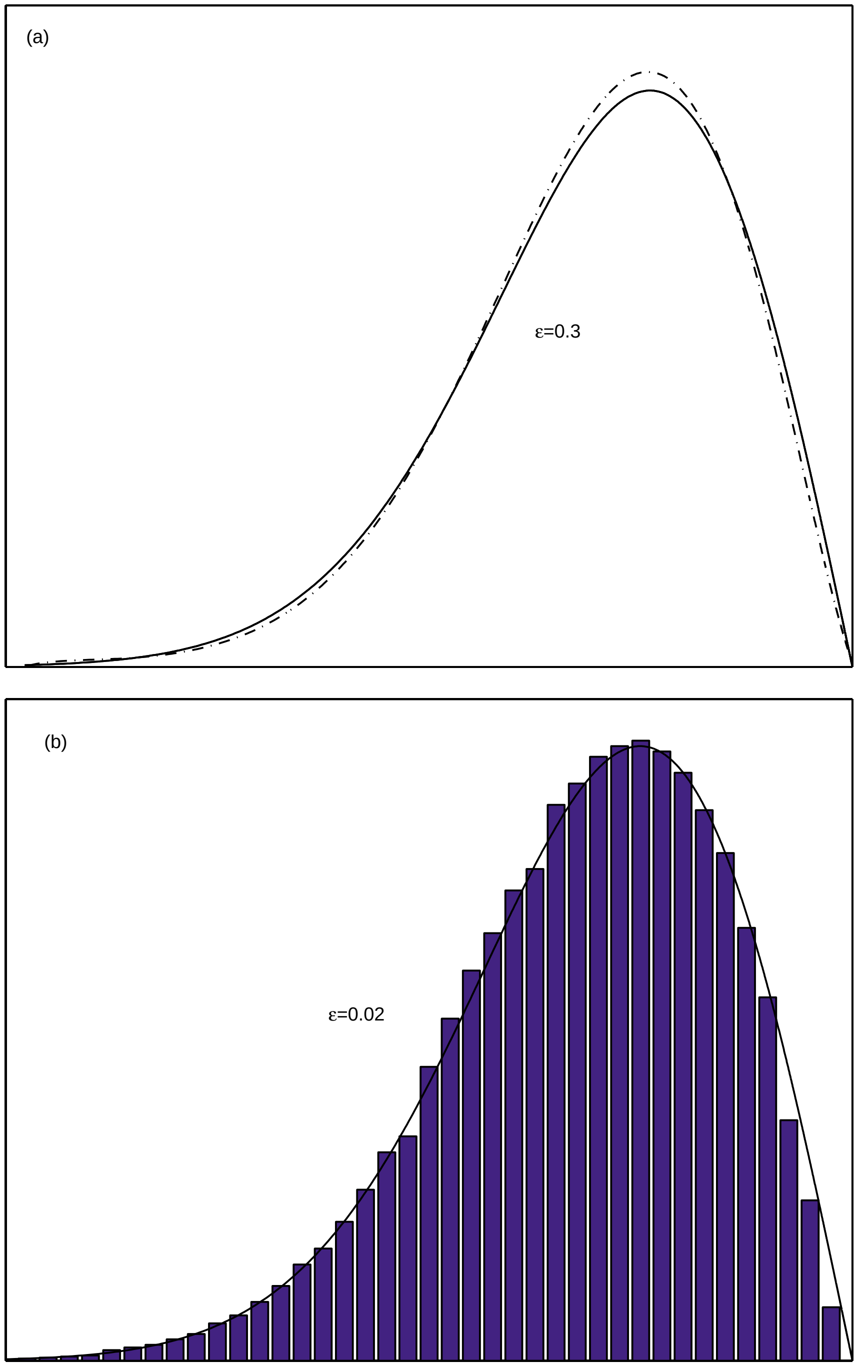

Fig. 6.1. (a) *The graph of the analytic equation* (5.1) *(solid line) and the interpolated histogram (broken line) for* $\epsilon = 0.3$*, both normalized so that underlying area is* 1*.* (b) *Normalized histogram and* (5.1) *for* $\epsilon = 0.02$*.*

6. Discussion. This paper corrects errors of previous attempts at solving Kramers' exit problem. Specifically, the result of [34, p.621, (5.64)] is incorrect. The error leading to it is on page 616, below (5.22); namely, changing y to $-y$ not only changes the forward equation into the backward equation but also reflects the domain in the x-axis. Thus the expansion given in [34] is not on Γ but on its reflection in the x-axis.

Note that all expressions for the exit pdf that have the form

$$p_\varepsilon(x) = C|x|^B \exp\left\{-\frac{E(x, y_\Gamma(x))}{\epsilon}\right\}, \quad B > 0, \tag{6.1}$$

give the most likely exit point at $x = O\left(\sqrt{\epsilon}\right)$. This estimate, therefore, does not distinguish between correct and incorrect results.

The result of the present paper and of [23] are quite different, although both are of the form (6.1). While the exponent B in [23, (7.10)] is given by

$$B = \frac{2\beta[1 + 2\gamma(x_C)\rho_y^2(x_C, 0)]}{\beta + \sqrt{\beta^2 + 4\omega_C^2}}, \tag{6.2}$$

the exponent in the present paper is 1. The result of the present paper fits simulations quite well on the entire separatrix, as can be seen in Figure 6.1, and does so much better than the result of [23], especially in the saddle point region. The approximation becomes more accurate as ε decreases, though it is quite good even at $\varepsilon = 0.3$ (when $\sqrt{\varepsilon} = 0.547$).

In contrast to [23], this paper does not consider the conditional dynamics of the tails of escaping trajectories but rather concerns the FPE. The boundary layer equation is derived from the transport equation (3.2) for the pre-exponential factor in this case.

The unexpected phenomenon of saddle point avoidance was first observed in a class of noise driven dynamical systems lacking detailed balance [4]. It was observed that the pdf of the exit points on the boundary of the domain of attraction of the stable equilibrium point is not necessarily peaked at the saddle point. This phenomenon, not being related to anisotropy in the noise or the dynamics [28], is counterintuitive and requires explanation. It was studied under a variety of assumptions in [12], [13], [14], [15], [16], [17], [18], [19], [20], [21], [22], [28]. The significance of the problem in models of electronic signal tracking devices, such as RADAR, spread spectrum communications (as in cellular phones), and various synchronization devices, is that the determination of the exit distribution on the boundary of the domain of attraction indicates where to tune the lock detector that determines if the signal is lost and has to be acquired afresh.

The realization that the exit point on the separatrix in the classical Kramers problem is not at the saddle point, even for large values of the damping coefficient, came as a surprise. This phenomenon in the Kramers problem was first observed in numerical simulations of the Langevin dynamics [10], [11] and was initially interpreted as a numerical instability of the simulation scheme.

In summary, this paper extends and corrects the analysis of [23]. Specifically, the boundary layer analysis of section 4 corrects the results of [23]. In addition, numerical and graphical representations of the results are given, and the analytical predictions are compared with results of simulation with good agreement. The results indicate that the exit point distribution is wide, spread on a scale of $O\left(\sqrt{\epsilon}\right)$, and the most likely exit point is located on the separatrix at a distance of the same order of magnitude away from the saddle point. The exit pdf decays in the direction of the saddle point and actually vanishes there.

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