Lattice Differential Equations

An overview

Sjoerd M. Verduyn Lunel

Universiteit Leiden

Abstract

Coupled lattice maps are systems that model spatial structures where the state of a site is determined dynamically by the previous state at that site and that of its neighbours.

Such models have a wide range of applications to physics (crystals), biology (nervous systems, population dynamics), economics (interaction of different markets) and reaction-diffusion equations. Numerically, one can observe the formation of waves, patterns, synchronization in which coupling plays an important role.

Lattice differential equations (LDE's) are continuous-time dynamical systems, usually infinite-dimensional, which possess a discrete spatial structure modeled on a lattice.

Travelling waves in discrete spatial media such as lattices leads to differential equations with retarded and advanced arguments.

Three examples

- Lattice gass celular automata
  - Basic principle:
    different microscopic interactions can lead to the same form of macroscopic equations (e.g. Navier-Stokes equations)
- Coupled map lattices (more structure)
  - Basic principle:
    networks of interacting elements whose evolution in time is discrete. See [2]
- Lattice differential equations (even more structure)
  - Basic principle:
    large systems of ordinary differential equations in which the state vector is coordinatized by a lattice. See [3].

Lattice Gass Celular Automata (LGCA)

- (artificial) micro-worlds of particles 'living' on lattices with interactions that conserve mass and momentum.
- cells may be empty or occupied by at most one particle with unit mass (i.e., each cell has two states). Velocity and momentum can be assigned to each particle by the vector connecting the node to its next neighbour along the link where the particle is located (the so-called lattice velocities).
  - microscopic interaction is strictly local.
  - particles exchange momentum while conserving the mass and the momentum summed up over each node.
  - after this collision each particle propagates along its associated link to its next neighbour node.

LGCA II

- microdynamics: repetition of collision and propagation.
- macroscopic: mass and momentum are calculated by mean values of large spatial regions (100.000 nodes).

Basic question:

Do the mean values obey the NS-equation?

No, there is a third essential condition needed (in addition to mass and momentum conservation): the lattice needs symmetry.

[Frisch, Hasslacher and Pomeau [1986]].

Lattice Boltzman Models (LBM)

LBMs are based on LGCA, but instead of particles, LBMs deal with continuous distribution functions which interact locally.

- only distributions at a single mode are involved.
- no coarse-graining (i.e., mean values of large spatial regions) is necessary.

Molecular Dynamics (MD)

- one simulates macroscopic behaviour as good as possible by setting up a model which describes the microscopic interactions as good as possible.
- the complexity of the interactions in MD restricts the number of particles and the time of integration.

Coupled Map Lattices (CML)

CMLs are used to model:

- neural nets.
- dynamics of interacting pressure and magnetic waves in plasma's (Denman).
- theory of transmission lines.
- phenomenological models to study the behaviour of large collections of coupled chaotic elements (Kaneko).

The general form

$$x_{t+1} = \Phi(x_t), \qquad t = 0, 1, 2, 3, \dots ,$$

where the state x_t is given by

$$x_t = (x_t^1, x_t^2, \dots , x_t^N) \in \mathbb{R}^N$$

and

$$\Phi : \mathbb{R}^N \to \mathbb{R}^N.$$

More specifically,

$$x_{t+1}^{(i)} = \Phi_{local}^{(i)}(x_t^{(i)}) + \Phi_{neigh}(x_t).$$

A further simplification

$$x_{t+1}^{(i)} = (1-\epsilon)S(x_t^{(i)}) + \frac{\epsilon}{p} \sum_{j \in neigh} S(x_t^{(j)}),$$

where $\epsilon \in [0,1]$ is the coupling term.

Example (two special dimensions)

$$x_{t+1}^{(kl)} = (1-\varepsilon)S(x_t^{(kl)}) + \frac{\varepsilon}{p} \sum_{\substack{p \text{ nearest} \\ \text{neighbours}}} S(x_t^{(ij)}),$$

where $\varepsilon \in [0,1]$ and

$$S : [0,1] \to [0,1]$$

describes the local dynamics.

When $p = 4$, the coupling mimics a discrete version of the diffusion operator.

When the p-neighbourhood encompasses the entire lattice, the coupling is known as mean-field.

Models framed as CMLs

Biological applications

- population dynamics
- neural behaviour

An outstanding problem is the identification of organizing principles to explain the synchronization of large populations of neurons possessing individually complex dynamics.

- Andersen and Anderson: synchronized activity of the reticular thalamic nucleus (RTN) actes as a pacemaker for the so-called 'spindle oscillations' observed during various sleep stages.

At the molecular level Cocho et al. used CMLs to model the evolution of genetic sequences.

Models framed as CMLs II

Image processing

- 'shape from shading problem': algorithms use CMLs. Idea: shape of an object is thought as a function which minimizes a given functional.

Phenomenological models

- fluid dynamics
  - CMLs are constructed to capture the features of a fluid undergoing the transition from laminar to turbulent flow.
- reaction diffusion models
  - CMLs are constructed by nearest neighbourhood coupling.
  - simulate interfacial phenomena in reaction diffusion systems.
- arrays of globally coupled oscillators
  - CMLs are used to describe magnetic properties of spin systems

A simple illustration

Let p_n denote the size of the population at the start of the n^{th} time period. A balance equation yields a mathematical model

$$p_{n+1} - p_n = f(p_n).$$

The function f describes the growth of the population and can be linear (proportional growth) $f(p_n) = ap_n$ or quadratic (logistic growth)

$$f(p_n) = a(1 - \frac{p_n}{N})p_n$$

or even more general.

If the function f is proportional with the length of the time period and $p(t)$ denotes the size at time t, then we can divide by the length of the time period and let the length tend to zero to arrive at a differential equation for p:

$$\frac{dp}{dt}(t) = g(p(t)),$$

A simple illustration II

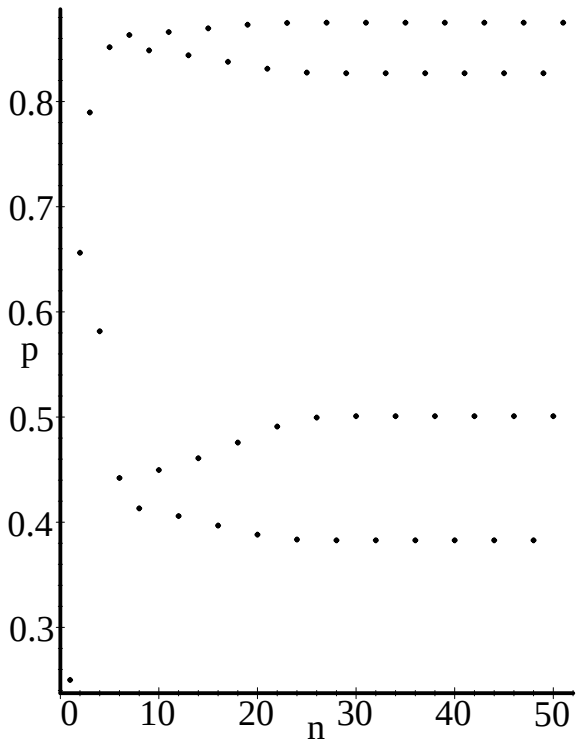

Figure 1: Voor de startwaarde $p_0 = .250$ is de oplossing van $p_{n+1} = 3.50 p_n(1 - p_n)$ getekend.

A simple illustration III

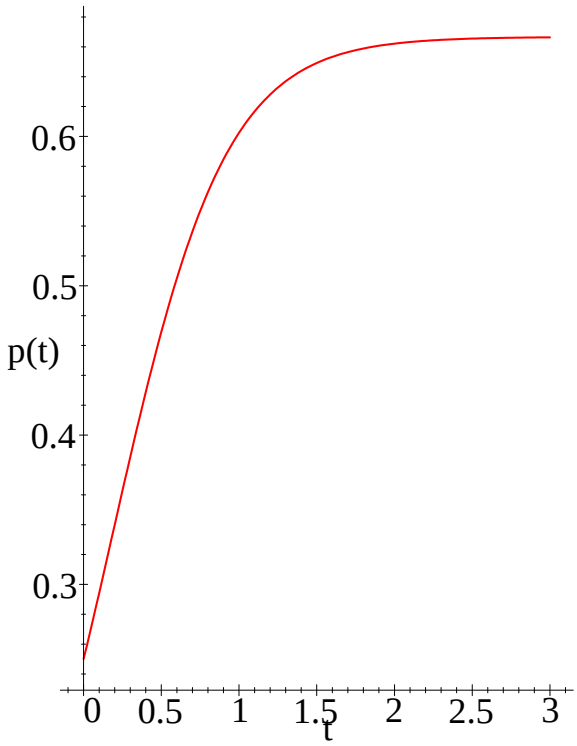

Figure 2: Voor de startwaarde $p(0)$ gelijk aan .250 is de oplossing van $\frac{dp}{dt}(t) = 2.75p(t)(1-p(t))$ getekend.

A simple illustration IV

Using the Euler approximation of the continuous logistic equation, we arrive at

$$\frac{1}{h}[p_{n+1} - p_n] = ap_n(1 - p_n)$$

or

$$p_{n+1} = (1 + ha)p_n(1 - \frac{ha}{1 + ha}p_n).$$

After the transformation

$$q_n = \frac{ha}{1 + ha}p_n$$

we arrive at

$$q_n = (1 + ha)q_n(1 - q_n).$$

Away from the parameter values where the difference equation is a good approximation of the continuous equation ($ah > 2.75$) arises complex dynamics.

A simple illustration V

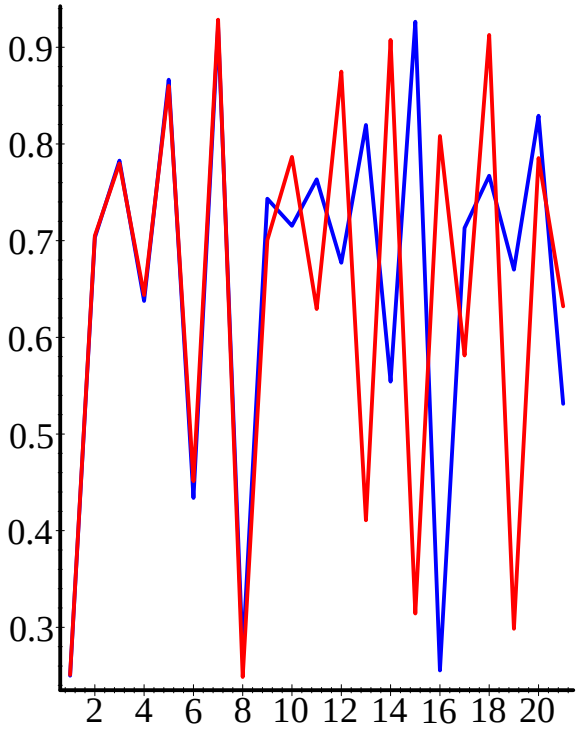

Figure 3: Voor de startwaarden $p_0 = .250$ en $p_0 = .251$ zijn de oplossingen van $p_{n+1} = 3.75p_n(1 - p_n)$ getekend.

Lattice differential equations

A general autonomous LDE on a lattice Λ can be written as

$$\dot{u}_\eta = g_\eta(u), \qquad \eta \in \Lambda.$$

The state vector $u = \{u_\eta\}_{\eta\in\Lambda}$ is coordinatized by the set Λ, the *lattice*, which possesses some underlying spatial structure.

It is also typical to impose some growth or boundedness condition on u_η (as a function of η) in order that the initial value problem be well-posed.

This restricts u to some Banach space X, for example

$$u \in X = l^p(\Lambda).$$

Lattice differential equations are of particular interest in modeling a variety of applications in which spatial structure plays a role.

Lattice differential equations II

Models are to be found in

- chemical reaction theory.
- image processing and pattern recognition.
- material science.
- biology.

The numerical and experimental work of Leon Chua and his collaborators, and Martin Hasler and his collaborators, are strong motivations for the study of LDE.

They are developing algorithms based on LDEs which identify various prescribed patterns, for example edges, or corners, in a digitized image.

Their equations are more difficult then the standard discretization of the Laplacian and contain a separate coupling coefficient for each of the eight neighbors in the 3×3 square in $\mathbb{Z}^2$ centered at (i, j).

Lattice differential equations III

Of course partial differential equations are extensively used to model spatial structures in systems, and one naturally obtains an LDE upon making a spatial discretization of the PDE. For example, let $f : \mathbb{R} \to \mathbb{R}$ and $\alpha \in \mathbb{R}$, and consider the system

$$\dot{u}_i = \alpha(u_{i+1} + u_{i-1} - 2u_i) - f(u_i), \qquad i \in \mathbb{Z}.$$

For large α, the system arises as a discretization of the so-called Allen-Cahn, or Chafee-Infante, or Nagumo equation,

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - f(u), \qquad x \in \mathbb{R},$$

where $\alpha = h^{-2}$, with $0 < h \ll 1$ the grid size.

On the other hand, much theoretical work in lattice differential equations concerns one-dimensional lattices with weak coupling $0 < \alpha \ll 1$, between lattice sites.

Lattice differential equations IV

To understand one mechanism which can generate patterns, let us consider for $(i,j) \in \mathbb{Z}^2$

$$\dot{u}_{i,j} = \alpha^+(\Delta^+ u)_{i,j} + \alpha^\times(\Delta^\times u)_{i,j} - f(u_{i,j}).$$

Here Δ^+ and $\Delta^\times$ correspond to discrete Laplace operators based on $+$- and $\times$-shaped stencils, given by

$$(\Delta^+ u)_{i,j} = \left(\sum_{|a-i|+|b-j|=1} u_{a,b} \right) - 4u_{i,j},$$

and

$$(\Delta^\times u)_{i,j} = \left(\sum_{|a-i|=|b-j|=1} u_{a,b} \right) - 4u_{i,j}.$$

The parameters $\alpha^+, \alpha^\times \in \mathbb{R}$ are the so-called *coupling coefficients*, and they can be of either sign or of any magnitude.

Lattice differential equations V

In numerical simulations,

$$\dot{u}_{i,j} = (\Delta^+ + \Delta^\times)\Big(\beta^+(\Delta^+ u)_{i,j} + \beta^\times(\Delta^\times u)_{i,j} - (\gamma - 2)u_{i,j} + \log\big(\frac{1+u_{i,j}}{1-u_{i,j}}\big)\Big)$$

a wide variety of patterns, including stripes and checks, were observed.

Beginning with a random choice of values $u^0_{i,j} \in (-1, 1)$ for the initial condition, on a large rectangular sublattice with standard boundary conditions, it was observed that an initial rapid coarsening (as with the corresponding PDE) took place, in which spatial patterns sometimes emerged.

Often different patterns, separated by interfaces, appeared in different regions of the lattice. Then, in many cases, the interfaces slowly moved as the solution $u(t)$ tended toward an equilibrium state.

Lattice differential equations VI

Figure 4: $(\beta^+, \beta^\times) = (\mathbf{0.25}, \mathbf{-0.5}),\ \gamma = \mathbf{1.0}$

Lattice differential equations VII

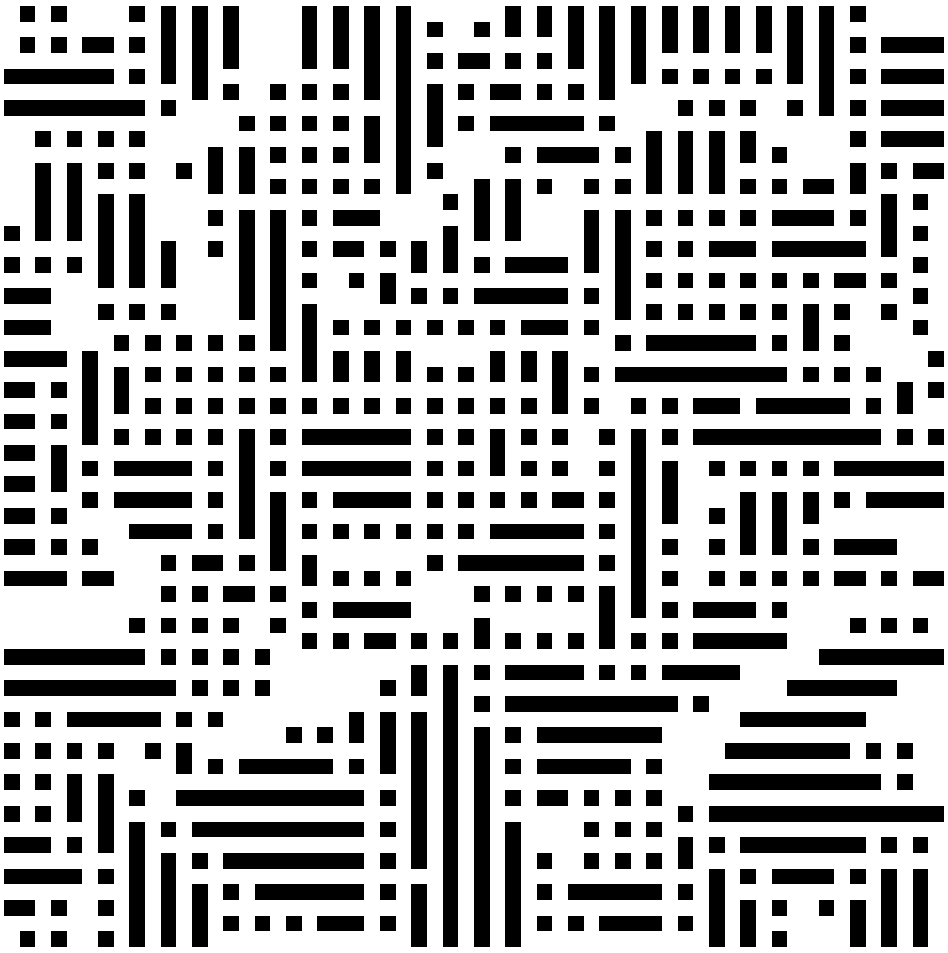

Figure 5: $(\beta^+, \beta^\times) = (-\mathbf{1.0}, \mathbf{2.0}),\ \gamma = \mathbf{6.0}$

Lattice differential equations VIII

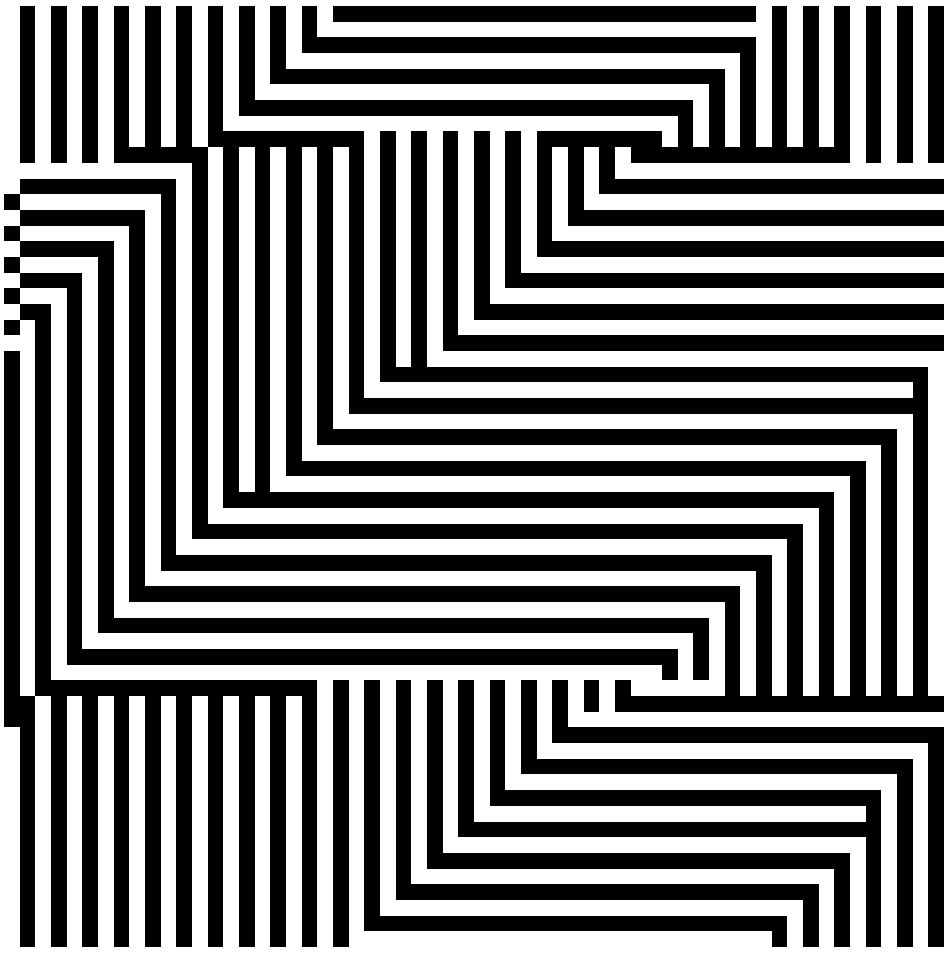

Figure 6: $(\beta^+, \beta^\times) = (-\mathbf{0.5}, \mathbf{2.0}),\ \gamma = \mathbf{10.0}$

Lattice differential equations IX

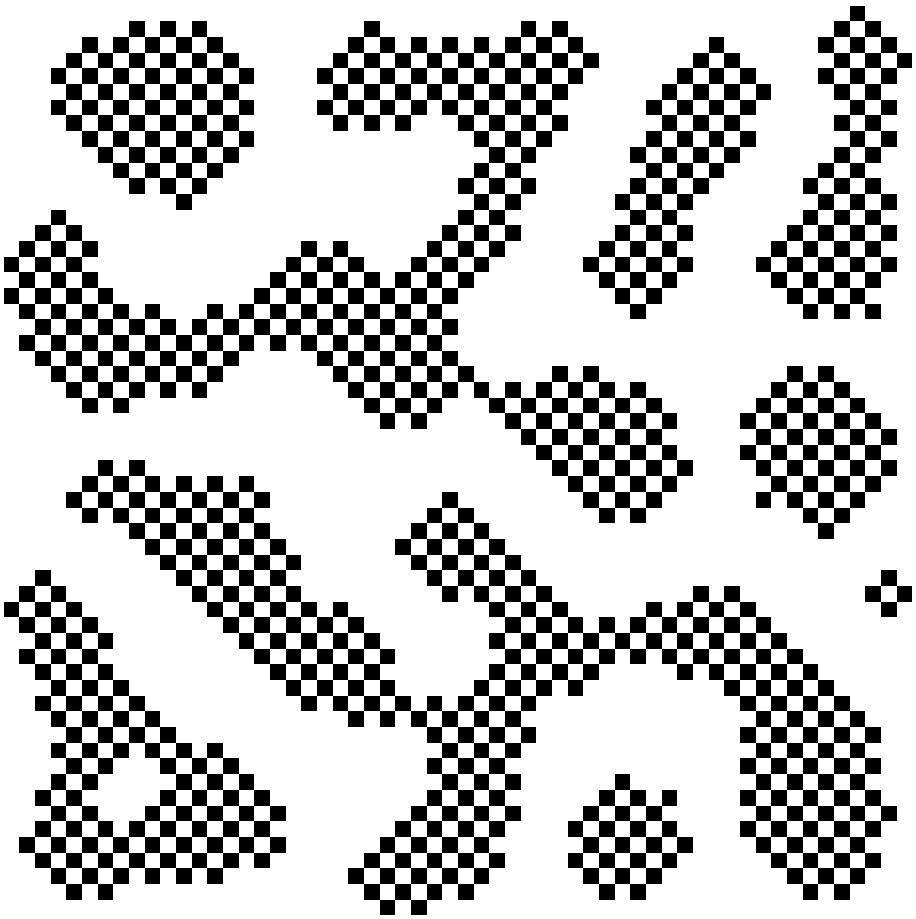

Figure 7: $(\beta^+, \beta^\times) = (\mathbf{0.25}, \mathbf{-0.5}),\ \gamma = \mathbf{-0.1}$

Traveling waves

While there is a vast literature on traveling wave solutions of PDE's, very little is known for LDE's. For the PDE,

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - f(u), \qquad x \in \mathbb{R},$$

a traveling wave solution takes the form $u(t,x) = \varphi(x - ct)$ for some function $\varphi : \mathbb{R} \to \mathbb{R}$ and a quantity $c \in \mathbb{R}$.

Substitution into the equation yields

$$-c\varphi'(\xi) = \varphi''(\xi) - f(\varphi(\xi)).$$

One typically imposes boundary conditions for φ, of the form

$$\varphi(-\infty) = q_-, \qquad \varphi(\infty) = q_+,$$

where $f(q_\pm) = 0$, generally with $z = q_\pm$ as stable equilibria for the associated ODE $\dot{z} = -f(z)$.

Both φ as well as c are unknown, and must be sought as part of the solution.

Traveling waves II

For an LDE, say on the one-dimensional lattice $\mathbb{Z}$, a traveling wave solution takes the form

$$u_i(t) = \varphi(i - ct), \qquad i \in \mathbb{Z},$$

for some $c \in \mathbb{R}$. Substitution into

$$\dot{u}_i = \alpha(u_{i+1} + u_{i-1} - 2u_i) - f(u_i), \qquad i \in \mathbb{Z}.$$

yields the equation

$$\begin{aligned} -c\varphi'(\xi) \quad = \quad & \alpha(\varphi(\xi + 1) + \varphi(\xi - 1) - 2\varphi(\xi)) \\ & -f(\varphi(\xi)). \end{aligned}$$

This is a differential-difference equation if $c \neq 0$, and is a difference equation if $c = 0$ (the wave speed c is unknown and part of the solution).

This forward-backward equation is a much more difficult system to analyze than the corresponding ODE for the travelling wave of PDE. See [4].

References

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[3] J. Mallet-Paret, Spatial patterns, spatial chaos, and traveling waves in lattice differential equations, in: [2] pp. 105–129.

[4] J. Mallet-Paret, S.M. Verduyn Lunel, Exponential Dichotomies and Wiener-Hopf Factorizations for Mixed-Type Functional Differntial Equations, Report Universiteit Leiden, MI 2001-17.