INPUT-OUTPUT REPRESENTATION OF NONLINEAR SYSTEMS

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1 INTRODUCTION

- INPUT-OUTPUT SYSTEMS

INPUTS: u	dim m	($m = 1$)
OUTPUTS: y	dim p	($p = 1$)
STATE: x	dim n	

- STATIC RELATION

$$y = \alpha u \qquad \text{(linear)}$$

$$\alpha : (p, m) \text{ - matrix}$$

*** What are the dominant components u_i?**

$$y = \alpha(u) \qquad \text{(nonlinear)}$$

$$\alpha : \mathbb{R}^m \to \mathbb{R}^p \qquad \text{smooth} \quad \alpha(0) = 0$$

Taylor series expansion : $\alpha(u) = \alpha_0 u + u^T \alpha_1 u + \dots$

- DYNAMIC RELATION

$$\begin{cases} \dot{x} = Ax + Bu & \text{(linear)} \\ x(0) = x_0 \\ y = Cx \end{cases}$$

$$y(t) = Ce^{At}x_0 + \int_0^t Ce^{A(t-s)}Bu(s)ds \qquad (*)$$

$$W(t) \triangleq Ce^{At}B \qquad \text{Impulse response function}$$

$$(*) \quad W(s) = \mathcal{L}(Ce^{At}B) \qquad \text{Transfer function}$$

$$= C(sI - A)^{-1}B$$

$$Y(s) = W(s)U(s) \qquad (x_0 = 0)$$

Basic question (again) :

What are the dominant 'modes' in the input-output description (*) ?

2 WIENER-VOLTERRA SERIES

$$\begin{cases} \dot{x} = f(x) + g(x)\,u \\ x(0) = x_0 \\ y = h(x) \end{cases} \qquad \text{(nonlinear)}$$

Assume :

- $\dim u = \dim y = 1$
- $f(x_0) = 0, \quad h(x_0) = 0$
- $f, g, h \quad analytic$

$$y(t) = \int_0^t W_1(t,s)u(s)ds + \int_0^t \int_0^{s_1} W_2(t,s_1,s_2)u(s_1)u(s_2)ds_2 ds_1 + \int_0^t \int_0^{s_1} \int_0^{s_2} W_3(t,s_1,s_2,s_3)u(s_1)u(s_2)u(s_3)ds_3 ds_2 ds_1 + \ldots \qquad (**)$$

* Infinite Volterra series, $k-th$ kernel $W_k(\cdot,\cdot,\cdot)$
* Truncation of order r gives error of $o(\|u\|_1^r)$,

  $\|\cdot\|_1$ standard norm on $L^1([0,T],\mathbb{R})$
* If $f(x_0) \neq 0$ extra kernel $W_0(t)$
* Alternative, input - output representation (**)

  viewed as input - output map defined on $(-\infty, T)$

$$\dot{x} = f(x) + g(x)u, \qquad y = h(x)$$

Solution $\gamma_u(t,s,x) \overset{\Delta}{=} x(t,s,x,u)$ at time t with initial condition $\gamma_u(s,s,x) = x$

Let $\rho(s) \overset{\Delta}{=} \gamma_0(t,s,\gamma_u(s,0,x_0))$, then

$$h(\gamma_u(t,0,x_0)) = h(\gamma_0(t,0,x_0)) + \int_0^t \frac{d}{ds} h(\rho(s))ds$$

$$\frac{d}{ds} h(\rho(s)) = u(s)\left[\frac{\partial h(\gamma_0(t,s,x))}{\partial x} g(x)\right]_{x=\gamma_u(s,0,x_0)}$$

Define $W_0(t) \overset{\Delta}{=} h(\gamma_0(t,0,x_0))$

$$\overline{W}_1(t,s,x) \overset{\Delta}{=} \frac{\partial h(\gamma_0(t,s,x))}{\partial x} g(x)$$

Thus

$$h(\gamma_u(t,0,x_0)) = W_0(t) + \int_0^t u(s)\overline{W}_1(t,s,\gamma_u(s,0,x_0))ds$$

Repeating this procedure with $h(\cdot)$ replaced by $\overline{W}_1(t,s,\cdot)$ yields

$$h(\gamma_u(t,0,x_0)) = W_0(t) + \int_0^t W_1(t,s)u(s)ds +$$

$$\int_0^t \int_0^s \overline{W}_2(t,s,r,\gamma_u(r,0,x_0))u(s)u(r)drds$$

$$
\begin{aligned}
y(t) = W_0(t) + \int_0^t W_1(t,s)u(s)ds\, + \\
\int_0^t \int_0^{s_1} W_2(t,s_1,s_2)u(s_1)u(s_2)ds_2 ds_1 + \\
\int_0^t \int_0^{s_1} \int_0^{s_2} W_3(t,s_1,s_2,s_3)u(s_1)u(s_2)u(s_3)ds_3 ds_2 ds_1 + \ldots
\end{aligned}
$$

Volterra series expansion on finite interval $[0,T]$

and with $|u(t)| < K$, for some $K > 0$

$$\begin{cases} \dot{x} = Ax + (Bx)u \\ x(0) = x_0 \\ y = Cx \end{cases} \qquad \text{(bilinear)}$$

$$W_0(t) = Ce^{At}x_0$$

$$W_1(t,s) = Ce^{A(t-s)}Be^{As}x_0$$

$$W_2(t,s_1,s_2) = Ce^{A(t-s_1)}Be^{A(s_1-s_2)}Be^{As_2}x_0$$

$$\vdots$$

Remark

In general the kernel W_i can be determined as:

$$W_0(t) = h(\varphi(t))$$
$$W_1(t, s_1) = L_g[h(\varphi(t - s_1))] \cdot \varphi(s_1)$$
$$W_2(t, s_1, s_2) = L_g[L_g[h(\varphi(t - s_1))] \cdot \varphi(s_1 - s_2)] \cdot \varphi(s_2)$$

$\varphi(t)$ solution of $\dot{x} = f(x),\ x(0) = x_0$

Basic question :

In order to ‘simplify’, is there a way to truncate the Wiener-Volterra series?

Note: Depends on T, K (size of inputs)

Remark

A truncated Volterra-series does not necessarily have a lower dimensional state space realization

$$\dot{\tilde{x}} = \tilde{f}(\tilde{x}) + \tilde{g}(\tilde{x})u, \quad y = \tilde{h}(\tilde{x})$$

3 CHEN-FLIESS SERIES

* Alternative functional expansion
* Reveals underlying algebraic structure

Setting:

$$\begin{cases} \dot{x} = f(x) + g(x)\,u, & x(0) = x_0 \\ y = h(x) \end{cases}$$

Wiener - Volterra series $W_i(t, s_1, \ldots, s_i)$, $i = 0, 1, 2, \ldots$

Key idea $(W_i(\cdots)$ analytic!$)$

Expand $W_i(t, s_1, \ldots, s_i)$ as Taylor series in

$t - s_1,\; s_1 - s_2,\; s_2 - s_3,\; \ldots,\; s_i$ (See bilinear example)

$$W_i(t, s_1, \ldots, s_i) = \sum_{k_0, k_1, \ldots, k_i = 0}^{\infty} c_{i,k_0 k_1 \ldots k_i} \cdot \frac{(t - s_1)^{k_0} (s_1 - s_2)^{k_1} \ldots (s_i)^{k_i}}{k_0! \, k_1! \ldots k_i!}$$

Where the coefficients $c_{i,k_0 k_1 \ldots k_i}$ depend on x_0

Consider the iterated integrals

$$\int_0^t \int_0^{s_1} \dots \int_0^{s_i} \frac{(t-s_1)^{k_0}}{k_0!} u(s_1) \dots u(s_i) \frac{(s_i)^{k_i}}{k_i!} ds_i \dots ds_1$$

Set $\xi_0(t) \triangleq t\,;$ $\qquad \int_0^t d\xi_0 = t$

$\xi_1(t) \triangleq \int_0^t u(s)ds\,;$ $\qquad \int_0^t d\xi_1 = \xi_1$

$$\int_0^t d\xi_{i_k} \dots d\xi_{i_0} \triangleq \int_0^t d\xi_{i_k}(s) \int_0^{s_1} d\xi_{i_{k-1}} \dots d\xi_{i_0}$$

with $\quad i_0, \dots, i_k \in \{0,1\}$

Thus

$$\int_0^t \int_0^{s_1} \dots \int_0^{s_i} \frac{(t-s_1)^{k_0}}{k_0!} u(s_1) \dots u(s_i) \frac{(s_i)^{k_i}}{k_i!} ds_i \dots ds_1 =$$

$$\int_0^t (d\xi_0)^{k_0} d\xi_1 \dots (d\xi_0)^{k_{i-1}} d\xi_1 (d\xi_0)^{k_i}$$

Also

$$c_{i,k_0 k_1 \dots k_i}(x_0) = L_f^{\ k_i} L_g L_f^{\ k_{i-1}} L_g \dots L_f^{\ k_0} h(x_0)$$

$$y(t) = h(x_0) + \sum_{k=0}^{\infty} \sum_{i_0,\dots,i_k=0}^{m} L_{g_{i_0}} \dots L_{g_{i_k}} h(x_0) \int_0^t d\xi_{i_k} \dots d\xi_{i_0}$$

(Here $g_0 \triangleq f$)

Chen-Fliess functional expansion

- Non commuting
- Truncation of Chen-Fliess series?

$$\int_0^t d\xi_0 d\xi_0 = \int_0^t d\xi_0(s)\int_0^s d\xi_0 = \int_0^t d\xi_0(s)s$$

$$= \int_0^t s ds = \frac{1}{2}t^2$$

$$\int_0^t d\xi_0 d\xi_1 = \int_0^t d\xi_0(s)\int_0^s d\xi_1 = \int_0^t d\xi_0(s)u(s)$$

$$= \int_0^t u(s)ds$$

$$\int_0^t d\xi_1 d\xi_0 = \int_0^t d\xi_1(s)\int_0^s d\xi_0 = \int_0^t s d\xi_1(s)$$

$$= \int_0^t s u_1(s)ds$$

4 CONCLUSIONS

* Volterra / Fliess series ↔ multivariable Laplace transform ↔ multivariable transfer function for certain class of systems

* Bilinearization / Carleman linearization :

  "Natural" approximation technique

* Truncation of Volterra series + state space realization of the truncated series has no physical interpretation

* Same with respect to bilinearization of some order

* Other input-output representations:

  Input-output differential equations

* Identification