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Journal of Process Control 14 (2004) 627–634
JOURNAL OF PROCESS CONTROL
www.elsevier.com/locate/jprocont

H_2 approximation of multiple input/output delay systems

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Received 21 August 2003; received in revised form 17 December 2003; accepted 7 January 2004

Abstract

Many physical multivariable processes can be sufficiently described as linear models with multiple input/output delays. To simplify the synthesis and analysis of control problem, a reduced-complexity model is often desired. In this paper, an H_2 model reduction scheme is introduced for stable linear systems with multiple input/output delays. The reduced model can be a finite dimensional linear model, or a linear model with a time delay. In the latter case the approximation can be improved drastically without increasing the order of the finite dimensional part. The stability is preserved in the approximating models by employing a parametrization of linear stable systems. The optimal parameters can be obtained by solving an optimization problem using a gradient-based method. Two chemical numerical examples are used to show the effectiveness of the proposed scheme.

Keywords: Input/output delay; Multiple; Model reduction; Gradient flow; Optimum

1. Introduction

Model reduction is a problem of deriving low order mathematical models from high order ones according to certain approximation criteria. In many engineering applications, high or even infinite order mathematical models are used to describe physical systems. For practical reasons, it is desirable to replace these high or infinite order models by low order approximations. Model reduction has been a popular research area and has attracted a lot of attention in the last few decades. Various reduction methods have been proposed and algorithms of diverse computational complexity have been presented. Optimal H_2 approximation is an important approach to derive low order models by minimizing the H_2 norm of the error between the original and the reduced order models [1,4,7,8,17,20]. The importance of the problem is that the H_2 norm of a system is the expected root-mean-square value of the output when the input is a unit variance white noise process.

In many engineering applications, control systems cannot be described accurately without the introduction of delay element(s). In an input–output sense, there are two classes of delay systems: one class takes the form of $\mathscr{G}(s) = \mathrm{e}^{-sT}G(s)$, where $G(s)$ is a stable strictly proper real rational transfer function matrix, and T is the value of time delay; and the other class has the transfer matrix $\mathscr{G}(s)$ with the (i, j) element of $\mathrm{e}^{-t_{ij}s}g_{ij}(s)$, where $g_{ij}(s)$ is a stable and strictly proper real rational transfer function, and t_{ij} is the time delay from the jth input to the ith output. Since in the first class of delay systems, each pair of input and output has the same delay, we refer to them as single delay systems. On the contrary, in the latter class of delay systems, each pair of input and output may have different delays, and they are here referred to as multiple delay systems. However, both single delay and multiple input/output delay systems have the irrational function e^{-st} in their transfer functions, which does not admit a finite dimensional realization. For analysis and synthesis purpose, it is often useful to have a finite dimensional approximation of delay systems.

Along this line, many methods have been proposed to find finite dimensional approximation for single delay systems. One class of such methods are based on the Padé approximations of $\exp(-st)$, for example, Johnson et al. [9], Marshak et al. [11], and recently Lam [10]. Another class of methods are to find the finite dimensional approximation of $\mathrm{e}^{-sT}G(s)$ by minimizing the H_2 approximation error [3,22–24]. Despite the wide applications of multiple input/output delay

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doi:10.1016/j.jprocont.2004.01.006

systems in chemical processes, the finite dimensional approximation of multiple input/output delay systems has not been well addressed. One feasible method to approximate the multiple input/output delay systems is through a two-stage method. The first stage is to approximate the transfer function of each input and output pair one by one using the earlier mentioned methods for single delay systems so that a finite dimensional model is obtained. However, approximating multiple input/output delays may result in a very high order model. To obtain a model with desired order, the second stage is used to employ an existing model reduction method, such as, balanced truncation [12,15] to reduce the order to the desired one. It is noticed that the two-stage methods can cause large approximation error especially due to the Padé approximation. This motivates us to find a one-stage approximation method to obtain the desired order model.

It is noticed that when delay systems are approximated by finite dimensional models, the orders of the approximating models have to be high for good approximations. On the other hand, it has been well recognized in the process control community that a high order model can be effectively represented by a low order one with a time delay. That is, if a time delay is introduced into the approximating model, the approximation might be substantially improved [4,7,19,22,24]. For single delay systems, such works can be seen in [22,24], and for multiple input/output delay systems, this problem still remains open.

In this paper, the H_2 approximation of multiple input/output delay systems is investigated. The proposed method is used to find a linear model with or without a time delay such that the H_2 approximation error between the approximating system and the original system is minimized. A recent parametrization method [16] is employed to parametrize the approximating model. This parametrization has the advantage that it involves less parameters. A gradient-based method is proposed to find the parameters of the approximating model. Numerical examples are used to illustrate the efficiency of the proposed method by comparing it with a two-stage method.

2. H_2 approximation scheme

2.1. Canonical parametrization of stable systems

Consider a stable linear nth order time-invariant system $G(s)$. Theorem 1 in the following gives a parametrization of all stable proper systems.

Theorem 1. *For time-invariant system $G(s)$, it is stable if and only if one of the minimal realizations of $G(s)$, denoted as (A,B,C,D), can be parametrized as*

$$A = A_{sk} - \frac{1}{2}C^{\mathrm{T}}C, \quad b_{j1} = 0, \ j > 1, \tag{1}$$

and $A_{sk} - \frac{1}{2}C^{\mathrm{T}}C$ has no eigenvalues on the imaginary axis, where A_{sk} is given by

$$A_{sk} = \begin{bmatrix} 0 & -a_1 & 0 & \cdots & 0 \\ a_1 & 0 & -a_2 & \cdots & 0 \\ 0 & a_2 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 & -a_{n-1} \\ 0 & \cdots & 0 & a_{n-1} & 0 \end{bmatrix},$$

and b_{j1} is the $(j,1)$ element of B, a_i $(i = 1,\dots,n-1)$ are scalar parameters.

Proof. The proof can be obtained from Theorem 1 in [16] by simply letting $\gamma \to \infty$. □

2.2. Problem statement

Consider a system with multiple input/output delays:

$$\mathscr{G}(s) = \begin{bmatrix} g_{11}(s)e^{-st_{11}} & g_{12}(s)e^{-st_{12}} & \cdots & g_{1p}(s)e^{-st_{1p}} \\ g_{21}(s)e^{-st_{21}} & g_{21}(s)e^{-st_{22}} & \cdots & g_{2p}(s)e^{-st_{2p}} \\ \vdots & \vdots & \vdots & \vdots \\ g_{q1}(s)e^{-st_{q1}} & g_{q2}(s)e^{-st_{q2}} & \cdots & g_{qp}(s)e^{-st_{qp}} \end{bmatrix}, \tag{2}$$

where $g_{jk}(s)$ $(j = 1,\dots,q$ and $k = 1,\dots,p)$ are stable and strictly proper scalars, and $t_{jk} \geqslant 0$ $(j = 1,\dots,q$ and $k = 1,\dots,p)$ are time delays. The H_2 approximation problem for delay system $\mathscr{G}(s)$ can be stated as: For $\mathscr{G}(s)$, find an approximating model of the form $G_r(s)e^{-s\tau^2}$ of p inputs and q outputs, where $G_r(s)$ is an rth order system with the form of

$$G_r(s) = \begin{bmatrix} g_{11}^r(s) & g_{12}^r(s) & \cdots & g_{1p}^r(s) \\ g_{21}^r(s) & g_{21}^r(s) & \cdots & g_{2p}^r(s) \\ \vdots & \vdots & \vdots & \vdots \\ g_{q1}^r(s) & g_{q2}^r(s) & \cdots & g_{qp}^r(s) \end{bmatrix}, \tag{3}$$

and $g_{jk}^r(s)$ $(j = 1,\dots,q$ and $k = 1,\dots,p)$ are stable and strictly proper scalar transfer functions, such that the H_2 approximation error:

$$E = \left\| \mathscr{G}(s) - G_r(s)e^{-s\tau^2} \right\|_2 = \sqrt{\frac{1}{2\pi}\int_{-\infty}^{\infty} \operatorname{trace}\left[\left(\mathscr{G}^*(j\omega) - G_r^*(j\omega)e^{j\omega\tau^2}\right)\left(\mathscr{G}(j\omega) - G_r(j\omega)e^{-j\omega\tau^2}\right)\right]d\omega} \tag{4}$$

is minimized. Notice that when τ is set to zero, the approximating model reduces to a finite dimensional model.

2.3. Error expression

To solve the proposed model reduction problem, we first need to obtain the expression of the H_2 approximation error. Assume that $g_{jk}(s)$ has the state-space realization of

$$g_{jk}(s) = \begin{bmatrix} A & b_k \\ c_j & 0 \end{bmatrix},$$

and $G_r(s)$ has the state-space realization of

$$G_r(s) = \begin{bmatrix} A_r & B_r \\ C_r & 0 \end{bmatrix}$$

with $C_r = \begin{bmatrix} (c_1^r)^{\mathrm{T}} & (c_2^r)^{\mathrm{T}} & \cdots & (c_q^r)^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}$ and $B_r = \begin{bmatrix} b_1^r & b_2^r & \cdots & b_p^r \end{bmatrix}$. We give the formula for calculating E^2 in the following theorem.

Theorem 2. *The square of the H_2 approximation error between $\mathscr{G}(s)$ and $G_r(s)$ is given by*

$$E^2 = \sum_{j=1}^{q} \sum_{k=1}^{p} E_{jk}, \tag{5}$$

where

$$E_{jk} = \begin{cases} c_j P_k c_j^{\mathrm{T}} - 2 c_j Q_k \mathrm{e}^{A_r^{\mathrm{T}}(t_{jk}-\tau^2)} (c_j^r)^{\mathrm{T}} + c_j^r R_k (c_j^r)^{\mathrm{T}}, \\ \quad \text{if } t_{jk} \geqslant \tau^2, \\ c_j P_k c_j^{\mathrm{T}} - 2 c_j Q_k \mathrm{e}^{A_r^{\mathrm{T}}(\tau^2-t_{jk})} (c_j^r)^{\mathrm{T}} + c_j^r R_k (c_j^r)^{\mathrm{T}}, \\ \quad \text{if } t_{jk} < \tau^2. \end{cases}$$

P_k, Q_k and R_k are respectively the solutions of

$$A P_k + P_k A^{\mathrm{T}} + b_k b_k^{\mathrm{T}} = 0, \tag{6}$$

$$A Q_k + Q_k A_r^{\mathrm{T}} + b_k (b_k^r)^{\mathrm{T}} = 0, \tag{7}$$

$$A_r R_k + R_k A_r^{\mathrm{T}} + b_k^r (b_k^r)^{\mathrm{T}} = 0. \tag{8}$$

Proof. From (4), we have

$$\begin{aligned} E^2 &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{trace}\Big[\Big(\mathscr{G}^*(j\omega) - G_r^*(j\omega)\mathrm{e}^{j\omega\tau^2}\Big)\Big(\mathscr{G}(j\omega) \\ &\quad - G_r(j\omega)\mathrm{e}^{-j\omega\tau^2}\Big)\Big]\mathrm{d}\omega \\ &= \sum_{j=1}^{q} \sum_{k=1}^{p} \left\| g_{jk}(s)\mathrm{e}^{-st_{jk}} - g_{jk}^r(s)\mathrm{e}^{-s\tau^2} \right\|_2^2 \end{aligned}$$

with $g_{jk}^r(s) = c_j^r (sI - A_r)^{-1} b_k^r$. Notice that $g_{jk}(s)\mathrm{e}^{-st_{jk}}$ and $g_{jk}^r(s)\mathrm{e}^{-s\tau^2}$ are single input and single output (SISO) systems. Following the proof of Theorem 2 in [24], our result can be obtained. □

2.4. Model reduction method

In this section, the parametrization given earlier will be applied to formulate an H_2 approximation of multiple delay systems as an optimization problem. By Theorem 1, all stable $G_r(s)$ can be parametrized as

$$A_r = A_{sk}^r - \frac{1}{2} C_r^{\mathrm{T}} C_r, \tag{9}$$

where

$$A_{sk}^r = \begin{bmatrix} 0 & -\alpha_1 & 0 & \cdots & 0 \\ \alpha_1 & 0 & -\alpha_2 & \cdots & 0 \\ 0 & \alpha_2 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 & -a_{r-1} \\ 0 & \cdots & 0 & \alpha_{r-1} & 0 \end{bmatrix}, \tag{10}$$

$$b_1^r = [\beta \quad 0 \quad \cdots \quad 0]^{\mathrm{T}} \tag{11}$$

under the condition of

$F: A_r = A_{sk}^r - \frac{1}{2} C_r^{\mathrm{T}} C_r$ has no eigenvalues on the imaginary axis.

Remark 1. It is noticed that this parametrization only involves $(p+q)r$ parameters. Compared with the existing H_2 model reduction method for MIMO stable systems based on parameter optimization [2,6,20,25], the parameters used here are much fewer. For example, in [20], the number of the parameters is nr, which can be very high since it depends on the order of the original models, and in [2,6,25], for linear systems, the number of the parameters is $(r+p+q)r$, which is r^2 more than the number of the parameters used here.

With this parametrization, the H_2 model reduction problem is changed to the following constrained optimization problem

$$\min_{\alpha_i, \beta, B_2^r, C_r} \left\{ E^2 : F \text{ is satisfied} \right\} \tag{12}$$

with $B_2^r = \begin{bmatrix} b_2^r & b_3^r & \cdots & b_p^r \end{bmatrix}$. Since the H_2 approximation problem for multiple input/output delay system $\mathscr{G}(s)$ has been changed to a constrained optimization problem, the gradient flow method will be applied to solve the constrained optimization problem.

By introducing a penalty function, consider an unconstrained optimization problem

$$\min \overline{E}^2 = E^2 + E_\varepsilon^2, \tag{13}$$

where

$$E_\varepsilon^2 = \varepsilon \left(\sum_{i=1}^{r-1} \alpha_i^2 + \beta^2 + \mathrm{trace}(B_2^r (B_2^r)^{\mathrm{T}} + C_r^{\mathrm{T}} C_r + \tau^2) \right).$$

For optimization problem (13), the gradient flow method can be applied. This optimization technique has been widely used to solve the dynamic control problems, such as, the computation of the optimal feedback gain [21], model reduction [2,20,25], and the computation of singular value decomposition and balanced realization [5]. One of the important properties of the gradient flow technique is that the objective function is non-increasing along the optimization parameters. Consequently, it is impossible for E_ε to become unbounded. This means that α_i, β, B_2^r, C_r are bounded. Furthermore, we will show that while solving problem (13), if the initial choices of α_i, β, B_2^r, C_r are such that condition F is satisfied, then condition F is satisfied along the optimization process.

Theorem 3. *Consider the system $G_r(s)$ with realization (A_r, B_r, C_r), where A_r is parametrized by (9). If the initial values of α_i, β, B_{2r}, C_r, τ are bounded and satisfy condition F, the objective function $\overline{E}^2$ given in (13) tends to unbounded if one of the eigenvalues of A_r tends to the imaginary axis.*

Proof. Let vec(X) denote the vector formed by stacking the columns of X into one long vector. Then (8) gives

$$\mathrm{vec}(A_r R_k + R_k A_r^{\mathrm{T}}) = -\mathrm{vec}(b_k^r (b_k^r)^{\mathrm{T}})$$

or

$$(A_r \otimes I + I \otimes A_r)\mathrm{vec}(R_k) = -\mathrm{vec}(b_k^r (b_k^r)^{\mathrm{T}}) \quad (\text{see}[26]).$$

It is noticed that the eigenvalues of $A_r \otimes I + I \otimes A_r$ are $\lambda_i(A_r) + \lambda_j(A_r)$ with $i = 1, \ldots, r$ and $j = 1, \ldots, r$. Thus, if one eigenvalue of A_r tends to the imaginary axis, $A_r \otimes I + I \otimes A_r$ tends to non-singular, which implies that R_k tends to unbounded. From the expression of E^2 in (8), E^2 also becomes unbounded, which contradicts with the finiteness of E^2. □

Since it is impossible for $\overline{E}^2$ to become unbounded, from the above theorem, it can be seen that none of the eigenvalues of A_r can tend to the imaginary axis. That is, the condition F should be satisfied. In order to solve the optimization problem (13) using the gradient flow method, the partial derivatives of $\overline{E}^2$ with respect to α_i, β, B_2^r, C_r are required to be computed. The gradient expressions for problem (13) are straightforward to obtain, which are given in the Appendix A. With the expressions of the partial derivatives of $\overline{E}^2$ with respect to α_i, β, B_2^r, C_r, the gradient flow of $\overline{E}^2$ can be formed by the ordinary differential equations (ODEs),

$$\dot{\alpha}_i = \frac{\mathrm{d}\alpha_i}{\mathrm{d}t} = -\frac{\partial \overline{E}^2}{\partial \alpha_i}, \quad 1 \leqslant i \leqslant r-1, \tag{14}$$

$$\dot{\beta} = \frac{\mathrm{d}\beta}{\mathrm{d}t} = -\frac{\partial \overline{E}^2}{\partial \beta}, \tag{15}$$

$$\dot{b}_{ik}^r = \frac{\mathrm{d}b_{ij}^r}{\mathrm{d}t} = -\frac{\partial \overline{E}^2}{\partial b_{ij}^r}, \quad 1 < k \leqslant p, \tag{16}$$

$$\dot{c}_{ji}^r = \frac{\mathrm{d}c_{ki}^r}{\mathrm{d}t} = -\frac{\partial \overline{E}^2}{\partial c_{ki}^r}, \quad 1 \leqslant j \leqslant q, \tag{17}$$

$$\dot{\tau} = \frac{\mathrm{d}\tau}{\mathrm{d}t} = -\frac{\partial \overline{E}^2}{\partial \tau}, \tag{18}$$

where b_{ik}^r, c_{ji}^r for $1 \leqslant i \leqslant r$, $1 < k \leqslant p$, $1 \leqslant j \leqslant q$ are the elements of B_2^r and C_r respectively.

It can be seen that the optimization based on gradient flow idea only requires solving ODEs, which are easy to implement in digital computer using Runge–Kutta method etc., and it avoids the intrinsic step-size adjustment as required in many other gradient decent methods. Some important properties will be stated in the following theorem with proof omitted as it can be obtained in a similar way as in [2,6].

Theorem 4. *Given a set of initial parameters α_i^0, β^0, B_2^{r0}, C_r^0, τ^0 which satisfies condition F, the following properties hold*:

(i) *The set of ODEs* (14)–(18) *has a unique solution* $(\alpha_i(t), \beta(t), B_2^r(t), C_r(t))$ *on* $[0, \infty)$;
(ii) *The objective function $\overline{E}^2$ is non-increasing along the solution* $(\alpha_i(t), \beta(t), B_2^r(t), C_r(t))$;
(iii) *The solution* $(\alpha_i(t), \beta(t), B_2^r(t), C_r(t))$ *is such that condition F is satisfied*;
(iv) $\lim_{t\to\infty} \dot{\alpha}_i(t) = 0$; $\lim_{t\to\infty} \dot{\beta}(t) = 0$; $\lim_{t\to\infty} \dot{B}_2^r(t) = 0$; $\lim_{t\to\infty} \dot{C}_r(t) = 0$; $\lim_{t\to\infty} \dot{\tau}(t) = 0$.

In practice, the gradient flow model reduction procedure should be carried out stagewise for t. From the above theorem, a local optimum can be achieved as $t \to \infty$. At each stage, ε is fixed. Its value is reduced at each of the subsequent stages. From the expressions of $\frac{\partial \overline{E}^2}{\partial \alpha_i}$, $\frac{\partial \overline{E}^2}{\partial \beta}$, $\frac{\partial \overline{E}^2}{\partial b_{ik}^r}$ $\frac{\partial \overline{E}^2}{\partial c_{ii}^r}$ and $\frac{\partial \overline{E}^2}{\partial \tau}$ in the Appendix A, it can be seen that when $\varepsilon \to 0$, the H_2 optimality conditions of $\overline{E}^2$ converge to those of E^2. Thus, the solutions of the unconstrained optimization problem (13) are in fact those of the constrained optimization problem (12) as $\varepsilon \to 0$.

Like other gradient-based methods, gradient flow method guarantees to produce a local optimum, but may not produce the global optimum. How close the local optimum is to the global optimum depends on the choice of the initial parameters. Thus, a reasonably good initial model, such as, the reduced order model obtained by Padé approximation and balanced truncation, is crucial to the effectiveness of the proposed method. The algorithm about how to find the corresponding initial parameters α_i, β, B_2^r, C_r, if an initial reduced order model $\hat{A}_r$, $\hat{B}_r$, $\hat{C}_r$ is given, is presented in the Appendix A (see Algorithm 1).

3. Numerical examples

Example 1. Luyben and Vinate studied a 24-tray distillation column used for separating methanol and water [14]. Considering the temperatures on trays 17 and 4 as the output variables, they studied their responses to changes in the reflux flowrate and steam flowrate, and obtained the following transfer function matrix model:

$$\begin{bmatrix} y_1(s) \\ y_2(s) \end{bmatrix} = \begin{bmatrix} \frac{-21.6e^{-s}}{8.5s+1} & \frac{1.26e^{-0.3s}}{7.05s+1} \\ \frac{-2.75e^{-1.8s}}{8.2s+1} & \frac{4.28e^{-0.35s}}{9.0s+1} \end{bmatrix} \begin{bmatrix} u_1(s) \\ u_2(s) \end{bmatrix}. \tag{19}$$

We will use the proposed method to find a 2nd order finite dimensional model with and without delay to approximate this process model. It is well known that the often used method to approximate time delays is Padé approximation. If we use 3rd order Padé approximations to approximate e^{-s} and $e^{-1.8s}$, and 2nd order approximations to approximate $e^{-0.3s}$ and $e^{-0.35s}$, then a 14th order finite-dimensional model is obtained. In order to obtain a lower order finite-dimensional model, a certain model reduction method, such as, balanced truncation, is often resorted to reduce the order further. By balanced truncation, a 2nd order model is obtained, which has the following transfer function matrix:

$$\begin{bmatrix} y_1(s) \\ y_2(s) \end{bmatrix} = \begin{bmatrix} \frac{-1.865s-0.2093}{s^2+0.1844s+0.009056} & \frac{0.2595s+0.001697}{s^2+0.1844s+0.009056} \\ \frac{-0.09638s-0.03811}{s^2+0.1846s+0.009056} & \frac{0.431s+0.03401}{s^2+0.1844s+0.009056} \end{bmatrix} \begin{bmatrix} u_1(s) \\ u_2(s) \end{bmatrix},$$

and the H_2 approximation error is 2.2482. By the proposed method, the 2nd order approximation has the transfer function matrix:

$$\begin{bmatrix} y_1(s) \\ y_2(s) \end{bmatrix} = \begin{bmatrix} \frac{1.024s-3.008}{s^2+1.239s+0.1418} & \frac{0.1161s+0.2678}{s^2+1.239s+0.1418} \\ \frac{0.2635s-0.4168}{s^2+1.239s+0.1418} & \frac{0.01276s+0.03793}{s^2+1.239s+0.1418} \end{bmatrix} \begin{bmatrix} u_1(s) \\ u_2(s) \end{bmatrix}$$

with H_2 error of 1.4698. If a time delay is introduced in the approximating model, the transfer function matrix is

$$\begin{bmatrix} y_1(s) \\ y_2(s) \end{bmatrix} = e^{-s} \begin{bmatrix} \frac{-2.542s-0.2299}{s^2+0.2082s+0.01064} & \frac{0.1426s+0.0143}{s^2+0.2082s+0.01064} \\ \frac{-0.2603s-0.03175}{s^2+0.2082s+0.01064} & \frac{0.4013s+0.04747}{s^2+0.2082s+0.01064} \end{bmatrix} \begin{bmatrix} u_1(s) \\ u_2(s) \end{bmatrix},$$

and the H_2 error is 0.4591. For comparison, Fig. 1 depicts the quantity of

$$\sqrt{\operatorname{trace}\left[\left(\mathscr{G}^*(j\omega) - G_r^*(j\omega)e^{j\omega\tau^2}\right)\left(\mathscr{G}(j\omega) - G_r(j\omega)e^{-j\omega\tau^2}\right)\right]} \tag{20}$$

along the frequency ω when $G_r(s)$ is an approximating model obtained by the two-stage method (Padé and balanced truncation (BT)), finite dimensional (FD) approximation or FD approximation with a time delay, where τ may be zero or non-zero. It can be observed that the two-stage method approximates the original model better at the low frequency, which is also the characteristic of Padé approximation, and the FD approximation works better at the high frequency. Among these three methods, the FD approximation with a delay approximates the original model the best.

Example 2. Consider the well-known chemical process Wood–Barry Column [18], which has the following input–output relationship

$$\begin{bmatrix} y_1(s) \\ y_2(s) \end{bmatrix} = \begin{bmatrix} \frac{12.8e^{-s}}{16.7s+1} & \frac{-18.9e^{-3s}}{21.0s+1} \\ \frac{6.6e^{-7s}}{10.9s+1} & \frac{-19.4e^{-3s}}{14.4s+1} \end{bmatrix} \begin{bmatrix} u_1(s) \\ u_2(s) \end{bmatrix},$$

where y_1, y_2 are outputs, and u_1, u_2 are inputs. If we use 6th order Padé approximations to approximate e^{-7s}, and 3rd order to approximate e^{-s}, and 4th order to approximate e^{-3s}, then a 21st order approximating model is resulted. In order to obtain a 2nd order approximating model, balanced truncation is applied to further reduce the model, and obtain the following model:

$$\begin{bmatrix} y_1(s) \\ y_2(s) \end{bmatrix} = \begin{bmatrix} \frac{0.1617s+0.1582}{s^2+0.3023s+0.0179} & \frac{0.07021s-0.3187}{s^2+0.3023s+0.0179} \\ \frac{0.0432s+0.171}{s^2+0.3023s+0.0179} & \frac{0.4724s-0.3361}{s^2+0.3023s+0.0179} \end{bmatrix} \begin{bmatrix} u_1(s) \\ u_2(s) \end{bmatrix},$$

and the H_2 approximation error is 1.9165. By the proposed method, the 2nd order approximation has the transfer function matrix

$$\begin{bmatrix} y_1(s) \\ y_2(s) \end{bmatrix} = \begin{bmatrix} \frac{-0.04282s+0.221}{s^2+0.4102s+0.02313} & \frac{0.3364s-0.4232}{s^2+0.4102s+0.02313} \\ \frac{-0.1112s+0.2277}{s^2+0.4102s+0.02313} & \frac{0.5049s-0.4344}{s^2+0.4102s+0.02313} \end{bmatrix} \begin{bmatrix} u_1(s) \\ u_2(s) \end{bmatrix}$$

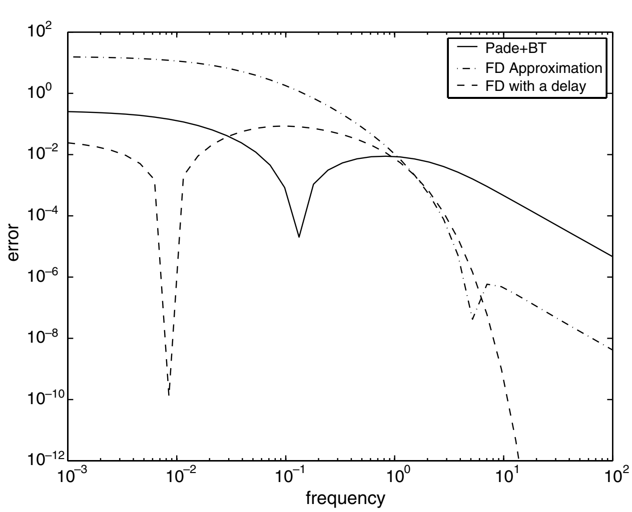

Fig. 1. Error comparison: Example 1.

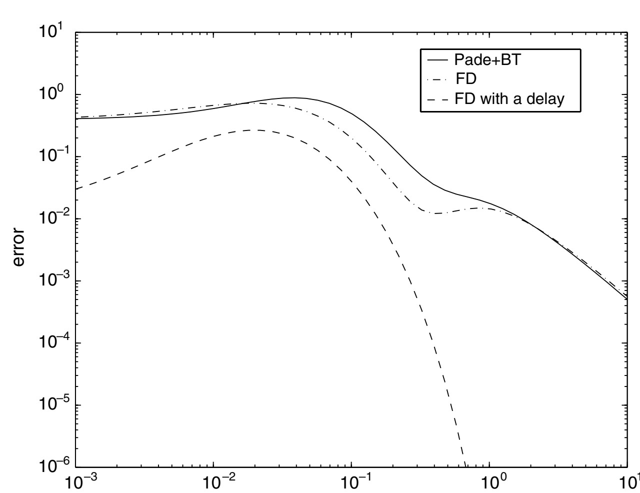

Fig. 2. Error comparison: Example 2.

with H_2 error of 1.8245. If a time delay is introduced in the approximating model, the transfer function matrix is

$$\begin{bmatrix} y_1(s) \\ y_2(s) \end{bmatrix} = e^{-3s} \begin{bmatrix} \frac{0.1101s+0.2461}{s^2+0.4399s+0.02299} & \frac{-1.136s-0.4083}{s^2+0.4399s+0.02299} \\ \frac{-0.391s+0.2343}{s^2+0.4399s+0.02299} & \frac{-1.281s-0.4435}{s^2+0.4399s+0.02299} \end{bmatrix} \begin{bmatrix} u_1(s) \\ u_2(s) \end{bmatrix},$$

and the H_2 error is 1.1330. Fig. 2 also gives the depiction of the quantity (20) along the frequency ω for different approximating models. It also shows that the FD approximation with a delay works the best among these three methods.

4. Conclusions

In this paper, a model reduction method for linear stable systems with multiple input/output delays is proposed. The approximating model may be a finite dimensional model with or without a time delay. Based on a parametrization method of linear stable systems, the approximating model can be obtained by minimizing the H_2 approximation error. A gradient method is proposed to find the optimal parameters. Numerical examples show that the finite dimensional reduced model approximates the original model better than the method based on Padé approximation, and the reduced model with desired order can be obtained in one stage by the users. It is also shown that if a single delay is introduced in the reduced model, the approximation can be improved drastically.

Acknowledgements

Authors would like to acknowledge National Sciences and Engineering Research Council of Canada (NSERC) and Petro-Canada Young Innovator Award for the support of this work.

Appendix A

Gradient flow analysis: The following lemma gives a formula to calculate the derivative of an exponential function.

Lemma 1 ([13, Proposition 4.10]). Suppose $X(t)$ is a differentiable matrix function of t. The derivative of $e^{X(t)}$, denoted as $\frac{d}{dt}(e^{X(t)})$ can be computed from the following equation:

$$\exp(M) = \begin{bmatrix} e^{X(t)} & \frac{d}{dt}\left(e^{X(t)}\right) \\ 0 & e^{X(t)} \end{bmatrix}, \tag{21}$$

where

$$M = \begin{bmatrix} X(t) & \frac{dX(t)}{dt} \\ 0 & X(t) \end{bmatrix}.$$

In (21), if $X(t)$ and $\frac{dX(t)}{dt}$ are known, $\frac{d}{dt}(e^{X(t)})$ can be computed. From the expression of $\overline{E}^2$, the partial derivatives of $\overline{E}^2$ with respect to α_i $(i = 1, 2, \ldots, r-1)$ are

$$\frac{\partial \overline{E}^2}{\partial \alpha_i} = \sum_{j=1}^{q} \sum_{k=1}^{p} \frac{\partial E_{jk}}{\partial \alpha_i} + \frac{\partial E_\varepsilon^2}{\partial \alpha_i},$$

where $\frac{\partial E_{jk}}{\partial \alpha_i}$ can be expressed as

$$\frac{\partial E_{jk}}{\partial \alpha_i} = \begin{cases} -2c_j \frac{\partial Q_k}{\partial \alpha_i} e^{A_r^T(t_{jk}-\tau^2)} (c_j^r)^T - 2c_j Q_k \frac{\partial}{\partial \alpha_i}\left(e^{A_r^T(t_{jk}-\tau^2)}\right)(c_j^r)^T \\ \quad + c_j^r \frac{\partial R_k}{\partial \alpha_i}(c_j^r)^T, \quad \text{if } t_{jk} \geqslant \tau^2 \\ -2c_j \frac{\partial Q_k}{\partial \alpha_i} e^{A_r^T(\tau^2-t_{jk})} (c_j^r)^T - 2c_j Q_k \frac{\partial}{\partial \alpha_i}\left(e^{A_r^T(\tau^2-t_{jk})}\right)(c_j^r)^T \\ \quad + c_j^r \frac{\partial R_k}{\partial \alpha_i}(c_j^r)^T, \quad \text{if } t_{jk} < \tau^2 \end{cases}$$

and $\frac{\partial}{\partial \alpha_i}(e^{A_r^T(t_{jk}-\tau^2)})$ and $\frac{\partial}{\partial \alpha_i}(e^{A_r^T(\tau^2-t_{jk})})$ can be computed from Lemma 1, and $\frac{\partial Q_k}{\partial \alpha_i}$ and $\frac{\partial R_k}{\partial \alpha_i}$ can be solved from the following Lyapunov equations

$$A \frac{\partial Q_k}{\partial \alpha_i} + \frac{\partial Q_k}{\partial \alpha_i} A_r^T + Q_k \frac{\partial (A_{sk}^r)^T}{\partial \alpha_i} = 0, \tag{22}$$

$$A_r \frac{\partial R_k}{\partial \alpha_i} + \frac{\partial R_k}{\partial \alpha_i} A_r^T + \frac{\partial A_{sk}^r}{\partial \alpha_i} R_k + R_k \frac{\partial (A_{sk}^r)^T}{\partial \alpha_i} = 0 \tag{23}$$

with $\frac{\partial A_{sk}^r}{\partial \alpha_i} = e_{i+1}e_i^T - e_i e_{i+1}^T$. Here e_i is the ith standard basis vector of $\mathbb{R}^r$. For $\frac{\partial E_\varepsilon^2}{\partial \alpha_i}$, we have

$$\frac{\partial E_\varepsilon^2}{\partial \alpha_i} = 2\varepsilon\alpha_i.$$

For the partial derivative of $\overline{E}^2$ with respect to β, we have

$$\frac{\partial \overline{E}^2}{\partial \beta} = \sum_{j=1}^{q} \sum_{k=1}^{p} \frac{\partial E_{jk}}{\partial \beta} + \frac{\partial E_\varepsilon^2}{\partial \beta},$$

where $\frac{\partial E_{jk}}{\partial \beta}$ can be expressed as

$$\frac{\partial E_{jk}}{\partial \beta} = \begin{cases} -2c_j \frac{\partial Q_k}{\partial \beta} e^{A_r^T(t_{jk}-\tau^2)} (c_j^r)^T + c_j^r \frac{\partial R_k}{\partial \beta} c_j^{rT}, \\ \quad \text{if } t_{jk} \geqslant \tau^2 \\ -2c_j \frac{\partial Q_k}{\partial \beta} e^{A_r^T(\tau^2-t_{jk})} (c_j^r)^T + c_j^r \frac{\partial R_k}{\partial \beta} c_j^{rT}, \\ \quad \text{if } t_{jk} < \tau^2 \end{cases}$$

and $\frac{\partial Q_k}{\partial \beta}$ and $\frac{\partial R_k}{\partial \beta}$ can be computed as $\frac{\partial Q_k}{\partial \alpha_i}$ and $\frac{\partial R_k}{\partial \alpha_i}$ in (22) and (23), and

$$\frac{\partial E_\varepsilon^2}{\partial \beta} = 2\varepsilon\beta$$

with $\frac{\partial b_1^r}{\partial \beta} = [1 \quad 0 \quad \cdots \quad 0]^T$ and $\frac{\partial b_k^r}{\partial \beta} = 0$ for $k = 2, \ldots, p$. Now consider the partial derivatives of $\overline{E}^2$ with respect to $B_2^r = [b_{im}^r]$ with $1 \leqslant i \leqslant r$ and $1 < m \leqslant p$. It can be seen that

$$\frac{\partial \overline{E}^2}{\partial b_{im}^r} = \sum_{j=1}^{q} \sum_{k=1}^{p} \frac{\partial E_{jk}}{\partial b_{im}^r} + \frac{\partial E_\varepsilon^2}{\partial b_{im}^r},$$

where $\frac{\partial \overline{E}^2}{\partial b_{im}^r}$ can be expressed as

$$\frac{\partial E_{jk}}{\partial b^r_{im}} = \begin{cases} -2c_j \frac{\partial Q_k}{\partial b^r_{im}} e^{A_r^T(t_{jk}-\tau^2)}(c^r_j)^T + c^r_j \frac{\partial R_k}{\partial b^r_{im}} c^{rT}_j, & \text{if } t_{jk} \geqslant \tau^2 \\ -2c_j \frac{\partial Q_k}{\partial b^r_{im}} e^{A_r^T(\tau^2-t_{jk})}(c^r_j)^T + c^r_j \frac{\partial R_k}{\partial b^r_{im}} c^{rT}_j, & \text{if } t_{jk} < \tau^2 \end{cases}$$

and $\frac{\partial Q_k}{\partial b^r_{im}}$, $\frac{\partial R_k}{\partial b^r_{im}}$ can be computed as $\frac{\partial Q_k}{\partial \alpha_i}$ and $\frac{\partial R_k}{\partial \alpha_i}$ in (22) and (23), and

$$\frac{\partial E^2_\varepsilon}{\partial b^r_{im}} = 2\varepsilon\,\mathrm{trace}\left(\frac{\partial B^r_2}{\partial b^r_{im}}(B^r_2)^T\right)$$

with $\frac{\partial B^r_2}{\partial b^r_{im}} = e_i\eta^T_m$ and $\frac{\partial b^k_r}{\partial b^r_{im}} = \begin{cases} e_i, & \text{if } m = k \\ 0, & \text{otherwise} \end{cases}$ for $1 \leqslant i \leqslant r$ and $1 < m \leqslant p$. Here, η_m is the mth standard basis vector of $\mathbb{R}^p$.

For the partial derivatives of $\overline{E}^2$ with respect to $C_r = [c^r_{li}]$ for $1 \leqslant l \leqslant q$, we have

$$\frac{\partial \overline{E}^2}{\partial c^r_{li}} = \sum_{j=1}^{q}\sum_{k=1}^{p} \frac{\partial E_{jk}}{\partial c^r_{li}} + \frac{\partial E^2_\varepsilon}{\partial c^r_{li}}$$

for $i = 1, 2, \ldots, r$, where $\frac{\partial E_{jk}}{\partial c^r_{li}}$ can be expressed as

$$\frac{\partial E_{jk}}{\partial c^r_{li}} = \begin{cases} -2c_j \frac{\partial Q_k}{\partial c^r_{li}} e^{A_r^T(t_{jk}-\tau^2)}(c^r_j)^T - 2c_jQ_k \frac{\partial}{\partial c^r_{li}}\left(e^{A_r^T(t_{jk}-\tau^2)}\right)(c^r_j)^T \\ -2c_jQ_k e^{A_r^T(t_{jk}-\tau^2)} \frac{\partial (c^r_j)^T}{\partial c^r_{li}} + 2\frac{\partial c^r_j}{\partial c^r_{li}} R_k (c^r_j)^T + c^r_j \frac{\partial R_k}{\partial c^r_{li}}(c^r_j)^T, \\ \quad \text{if } t_{jk} \geqslant \tau^2, \\ -2c_j \frac{\partial Q_k}{\partial c^r_{li}} e^{A_r^T(\tau^2-t_{jk})}(c^r_j)^T - 2c_jQ_k \frac{\partial}{\partial c^r_{li}}\left(e^{A_r^T(\tau^2-t_{jk})}\right)(c^r_j)^T \\ -2c_jQ_k e^{A_r^T(\tau^2-t_{jk})} \frac{\partial (c^r_j)^T}{\partial c^r_{li}} + 2\frac{\partial c^r_j}{\partial c^r_{li}} R_k (c^r_j)^T + c^r_j \frac{\partial R_k}{\partial c^r_{li}}(c^r_j)^T, \\ \quad \text{if } t_{jk} < \tau^2, \end{cases}$$

and $\frac{\partial}{\partial c^r_{li}}(e^{A_r^T(t_{jk}-\tau^2)})$, $\frac{\partial}{\partial c^r_{li}}(e^{A_r^T(\tau^2-t_{jk})})$ can be computed from Lemma 1, and $\frac{\partial Q_k}{\partial c^r_{li}}$, $\frac{\partial R_k}{\partial c^r_{li}}$ can be obtained as $\frac{\partial Q_k}{\partial \alpha_i}$ and $\frac{\partial R_k}{\partial \alpha_i}$ in (22) and (23). For $\frac{\partial E^2_\varepsilon}{\partial c^r_{li}}$, we have

$$\frac{\partial E^2_\varepsilon}{\partial c^r_{li}} = 2\varepsilon\,\mathrm{trace}\left(\frac{\partial C_r}{\partial c^r_{li}} C^T_r\right)$$

with

$$\frac{\partial A_r}{\partial c^r_{li}} = -\frac{1}{2}\left(\frac{\partial C^T_r}{\partial c^r_{li}} C_r + C^T_r \frac{\partial C_r}{\partial c^r_{li}}\right), \quad \frac{\partial C_r}{\partial c^r_{li}} = \xi_l e^T_i,$$

$$\frac{\partial c^r_j}{\partial c^r_{li}} = \begin{cases} e^T_i, & \text{if } l = j \\ 0, & \text{otherwise} \end{cases}$$

for $1 \leqslant i \leqslant r$ and $1 \leqslant l \leqslant q$. Here, ξ_l is the lth standard basis vector of $\mathbb{R}^q$.

For the partial derivatives of $\overline{E}^2$ with respect to τ, we have

$$\frac{\partial \overline{E}^2}{\partial \tau} = \sum_{j=1}^{q}\sum_{k=1}^{p} \frac{\partial E_{jk}}{\partial \tau},$$

where $\frac{\partial E_{jk}}{\partial \tau}$ can be expressed as

$$\frac{\partial E_{jk}}{\partial \tau} = \begin{cases} 4\tau c_jQ_k \frac{\partial}{\partial \tau} e^{A_r^T(t_{jk}-\tau^2)}(c^r_j)^T, & \text{if } t_{jk} \geqslant \tau^2 \\ -4\tau c_jQ_k \frac{\partial}{\partial \tau} e^{A_r^T(\tau^2-t_{jk})}(c^r_j)^T, & \text{if } t_{jk} < \tau^2 \end{cases},$$

and $\frac{\partial}{\partial \tau} e^{A_r^T(t_{jk}-\tau^2)}$, $\frac{\partial}{\partial \tau} e^{A_r^T(\tau^2-t_{jk})}$ can be computed from Lemma 1.

Algorithm 1. For a given rth order state space model $(\hat{A}_r, \hat{B}_r, \hat{C}_r)$ which is stable and minimal, find α_i, β, B_{2r}, C_r such that $(\hat{A}_r, \hat{B}_r, \hat{C}_r)$ is equivalent to (A_r, B_r, C_r) satisfying (9)–(11).

(i) Solve $X > 0$ from the following Lyapunov equation

$$X\hat{A}_r + \hat{A}^T_r X + \hat{C}^T_r \hat{C}_r = 0.$$

(ii) By the Cholesky decomposition, find the non-singular matrix L such that $X = L^TL$.

(iii) Let $\widetilde{A}_r = L\hat{A}_rL^{-1}$, $\widetilde{B}_r = L\hat{B}_r$ and $\widetilde{C}_r = \hat{C}_rL^{-1}$; form a skew-symmetric matrix $\widetilde{A}^r_{sk} = \frac{1}{2}(\widetilde{A}_r - \widetilde{A}^T_r)$.

(iv) Denote b as the first column of $\widetilde{B}_r$. For any $x_b \in \mathbb{R}^{1\times 1}, x_A \in \mathbb{R}^{1\times r}$, find the orthogonal matrix $\widetilde{U} = \begin{bmatrix} 1 & 0 \\ 0 & U \end{bmatrix}$ such that the matrix $\begin{bmatrix} x_b & x_A \\ b & \widetilde{A}^r_{sk} \end{bmatrix}$ is changed to its upper Hessenburg form. That is,

$$\widetilde{U}\begin{bmatrix} x_b & x_A \\ b & \widetilde{A}^r_{sk} \end{bmatrix}\widetilde{U}^T = \begin{bmatrix} * & * & * & \cdots & * \\ h_1 & * & * & \cdots & * \\ 0 & h_2 & * & \cdots & * \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & h_r & * \end{bmatrix}.$$

(v) Let $\alpha_i = h_{i+1}$ $(i = 1, 2, \ldots, r-1)$, $\beta = h_1$, $C_r = \widetilde{C}_rU^T$, B_{2r} is one block of $B_r = U\widetilde{B}_r = [B_{1r} \quad B_{2r}]$ with $B_{1r} = [\beta \quad 0 \quad \cdots \quad 0]^T$.

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