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Resonant tori and instabilities in Hamiltonian systems

Anna Litvak-Hinenzon[1,2] **and Vered Rom-Kedar**[1]

[1] The Faculty of Mathematics and Computer Sciences, The Weizmann Institute of Science, Rehovot 76100, Israel

E-mail: litvak@maths.warwick.ac.uk and vered@wisdom.weizmann.ac.il

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Abstract

The existence of lower-dimensional resonant bifurcating tori of *parabolic*, hyperbolic and elliptic normal stability types is proved to be generic and persistent in a class of n degrees of freedom (DOF) integrable Hamiltonian systems with $n \geqslant 3$. *Parabolic resonance* (PR) (respectively, hyperbolic or elliptic resonance) is created when a small Hamiltonian perturbation is added to an integrable Hamiltonian system possessing a resonant torus of the corresponding normal stability. It is numerically demonstrated that PRs cause intricate behaviour and large instabilities. The role of lower-dimensional bifurcating resonant tori in creation of instability mechanisms is illustrated using phenomenological models of near integrable Hamiltonian systems with 3, 4 and 5 DOF. Critical n values for which the system first persistently possesses mechanisms for large instabilities of a certain type are found. Initial numerical studies of the rate and time of development of the most significant instabilities are presented.

Mathematics Subject Classification: 37J35, 70H14, 37J20, 37J40, 70H08

1. Introduction

The study of instabilities in near-integrable n degrees of freedom (DOF) Hamiltonian systems possesses great challenges, especially for the case $n > 2$ [2, 27]. Following Arnold [1], the problem which has attracted most attention (yet see [11, 28]) has been the study of instabilities for a generic initial condition in a generic, near-integrable system, namely the difficult question of the possible appearance of instabilities in *a priori* stable systems (i.e. instabilities in the action variables along an orbit with initial conditions near a resonant n-torus). A typical model for studying this question is n slightly coupled nonlinear oscillators, studied at a typical point, namely away from the oscillator's equilibria. This study naturally leads to the study of the behaviour near resonances, which then naturally leads to the study of *a priori* unstable

[2] Current address: Mathematics Institute, University of Warwick, Coventry CV4 7AL, UK (Anna Litvak-Hinenzon).

systems [1,20,25]. A typical model for the latter is the study of s pendulums coupled to $(n-s)$ nonlinear oscillators. Here, to obtain new results, one studies the influence of the pendulum separatrices on instabilities in the actions direction, namely, one studies specific regions in phase space (see [12, 20, 27, 33, 43, 3, 4, 5, 13, 45] and references therein). The motivation for studying such specific systems and regions is two folded: first, in all applications we are aware of, there exist some unstable motion which may be locally modelled in the above manner; second, the behaviour near resonant tori (in *a priori* stable systems) may be modelled in this way. Hence, understanding these instabilities may lead to a better understanding of the Arnold diffusion instability mechanism which is relevant for most initial conditions in any non-degenerate system.

The main focus of this paper are the instabilities in near integrable Hamiltonian systems which possess, in their integrable part, resonant parabolic lower-dimensional tori[3] as part of some bifurcation scenario. We call such mechanisms *parabolic resonance* (PR) (see below and appendix A for more precise definitions). A typical model here is a system with a nearly conserved angular momentum, so that at least 2 DOF of the integrable Hamiltonian are non-separable. As in the *a priori* unstable systems, we concentrate on specific regions in phase space; here, the instability zones are associated with parabolic resonant tori (which are part of a bifurcation scenario). Numerically, we find that in these regions the actions exhibit strong instabilities (even for small perturbations the bifurcation actions cover the allowed region of motion in observable timescales).

While integrable separable systems (like n uncoupled pendulums) possess generically either hyperbolic or elliptic lower-dimensional tori, general *non-separable* systems (like a rotating pendulum and $n-1$ pendulums/oscillators) possess parabolic tori in a persistent way (see appendix A). The lower-dimensional tori undergo generic Hamiltonian bifurcations as the actions are varied: hyperbolic and elliptic tori coincide leading to the birth of a normally parabolic torus in a saddle-centre bifurcation (or, in the symmetric case, a pitchfork bifurcation) (see [7, 8, 10, 9, 22–24, 34–36] and references therein). Furthermore, for sufficiently large n ($n \geqslant 3$), some of the action values for which the parabolic bifurcation is realized correspond to resonant tori, namely to parabolic resonant tori. For 2 DOF systems it was proved that the appearance of such tori is a co-dimension one phenomenon [40], hence that it is expected to appear in many applications (e.g. [38, 41, 42]). In [31, 32] the dynamics of 3 DOF systems attaining PRs was studied, whereas in [30] we study the structure of the energy surfaces of systems attaining PR, and of their corresponding resonance web. Here, we investigate the dynamics in higher dimensions, where unperturbed degenerate parabolic resonant tori appear persistently and induce various kinds of instabilities (see also [29]).

This paper is ordered as follows: in section 2 we derive the conditions on the integrable part of the Hamiltonian function for which parabolic tori exist. In section 3 we illustrate several mechanisms of instability by introducing models of 4 and 5 DOF near integrable Hamiltonian systems. Finally, in section 4 conclusions and discussion are presented. In appendix 4 we prove persistence and genericity theorems for lower-dimensional resonant tori. In the second appendix, we identify critical dimensions for which certain mechanisms for instabilities become persistent.

2. Formulation

Consider an integrable n DOF Hamiltonian system, $H_0(p, q)$; $(p, q) \in M \subseteq \mathbb{R}^n \times \mathbb{R}^n$, with n independent integrals of motion: $H_0 = F_1, F_2, \ldots, F_n \in C^\infty(M)$, pairwise in

[3] A lower-dimensional torus is an invariant torus with a smaller number of *inner frequencies* than the number of DOF in the system.

involution: $\{F_i, F_j\} = 0$; $i, j = 1, \ldots, n$, where $M \subseteq \mathbb{R}^{2n}$ is a differentiable symplectic manifold, with the standard symplectic structure, and $\dim M = 2n$. Assume that $n \geqslant 3$ and that the Hamiltonian level sets, $M_g = \{(p, q) \in M, F_i = g_i; i = 1, \ldots, n\}$, are closed. By the Liouville–Arnold theorem (see [2,24]), the connected compact components of the Hamiltonian level sets, M_g, on which all of the $\mathrm{d}F_i$ are (pointwise) linearly independent, are diffeomorphic to n-tori and hence a transformation to action-angle coordinates ($H_0 = H_0(I)$) near such level sets is non-singular. Consider a neighbourhood of a possibly singular level set M_g. By the Liouville–Arnold theorem (see [37]), on each such connected and closed Hamiltonian level set there is some neighbourhood D, in which the Hamiltonian $H_0(p, q)$ may be transformed to the form:

$$H_0(x, y, I), \qquad (x, y, \theta, I) \in U \subseteq \mathbb{R}^s \times \mathbb{R}^s \times \mathbb{T}^{n-s} \times \mathbb{R}^{n-s}, \tag{2.1}$$

which does not depend on the angles, θ, of the $(n-s)$-tori. The symplectic structure of the new integrable Hamiltonian (2.1) is $\sum_{j=1}^{s} \mathrm{d}x_j \wedge \mathrm{d}y_j + \sum_{i=1}^{n-s} \mathrm{d}\theta_i \wedge \mathrm{d}I_i$, where (θ, I) are the action-angle variables ($s = 0$ corresponds to the usual n-tori discussed above). The motion on the $(n-s)$-dimensional family (parametrized by the actions I) of $(n-s)$-tori is described by the equations:

$$\dot{I}_i = 0, \qquad \dot{\theta}_i = \omega_i(x, y, I).$$

The geometrical structure of the new Hamiltonian, $H_0(x, y, I)$, is such that for any fixed I an $(n-s)$-torus is attached to every point of the (x, y)-plane (space, for $s > 1$). The (x, y)-plane (space) is called the *normal plane* (*space*) [2,39] of the $(n-s)$-tori, and defines their stability type (in the normal direction to the *family* of tori, see more precise description in [6]). Invariant lower-dimensional tori, of dimension $n - l$, generically exist for each $1 \leqslant l \leqslant n-1$; indeed, for any given s consider an m-resonant value of I. Then, for each such I, there exists an m-dimensional family (corresponding to different initial angles) of $(n-s-m)$-dimensional tori. All these tori belong to the higher $(n-s)$-dimensional resonant torus associated with I. The existence of such lower-dimensional tori is restricted to the $(n-s-m)$-dimensional resonant surface of I values. The lower-dimensional invariant tori we consider here are of different nature—they correspond to the isolated fixed point(s) of the s-dimensional normal space; hence, they appear on an $(n-s)$-dimensional manifold of I values (such a generalized fixed point corresponds to a manifold on which each $\mathrm{d}F_i$, for $i = 1, \ldots, s$, is linearly dependent on $\mathrm{d}I_1, \ldots, \mathrm{d}I_{n-s}$). Locally, one may choose the (x, y, I) coordinate system so that for these tori,

$$\nabla_{(x,y)} H_0(x, y, I)|_{p_f} = 0, \qquad p_f = (x_f, y_f, I_f). \tag{2.2}$$

The invariant $(n-s)$-tori have an $(n-s)$-dimensional vector of inner frequencies, $\dot{\theta} = \omega(p_f)$. The normal stability type of such families of $(n-s)$-tori is determined by the characteristic eigenvalues (respectively, Floquet multipliers for the corresponding Poincaré map) of the linearization of the system about the tori; generically, these tori are either normally elliptic[4], normally hyperbolic[5], or, if $s > 1$, they may be of hyperbolic-elliptic normal stability type. If the torus has at least one pair of zero characteristic eigenvalues in the direction of the normal (x, y) space, it is said to be *normally parabolic*. In addition, the *normal frequencies*[6] [2,39], Ω, of the $(n-s)$-tori are defined as the (non-negative) imaginary parts of the purely imaginary

[4] If all the characteristic eigenvalues of an invariant lower-dimensional torus (with respect to its normal (x, y) space) are purely imaginary (and do not vanish), it is said to be *normally elliptic*.

[5] If all the characteristic eigenvalues of an invariant lower-dimensional torus (with respect to its normal (x, y) space) have a non-zero real part, it is said to be *normally hyperbolic*.

[6] In some references, these are called characteristic frequencies.

characteristic eigenvalues[7]. For more details on the above, see [2, 10, 6, 14–20, 22, 24, 39] and references therein.

Locally, in the (x, y, I) coordinate system, the existence of a normally parabolic invariant torus is equivalent to the existence of an $i = i_0$, $i_0 \in \{i = 1, \ldots, s\}$, for which

$$\det\left(\left.\frac{\partial^2 H_0}{\partial^2 (x_{i_0}, y_{i_0})}\right|_{p_f} \right) = 0. \tag{2.3}$$

where p_f satisfies (2.2). Denote by p_{fp} solutions satisfying both (2.2) and (2.3). Since we are interested in the generic case in which the parabolic torus is attained as a part of a simple bifurcation scenario, we formulate the following transversality condition:

$$\left\| \frac{\partial}{\partial I_1} \det\left(\frac{\partial^2 H_0}{\partial^2 (x_{i_0}, y_{i_0})} \right)\bigg|_{p_f} \right\| + \cdots + \left\| \frac{\partial}{\partial I_{n-s}} \det\left(\frac{\partial^2 H_0}{\partial^2 (x_{i_0}, y_{i_0})} \right)\bigg|_{p_f} \right\| \neq 0. \tag{2.4}$$

Families of Hamiltonians which have a family of invariant $(n - s)$-tori, and satisfy conditions (2.2), (2.3) and (2.4), undergo a bifurcation at p_{fp} (we call the classes of such smooth Hamiltonians $\mathcal{P}_s$, see appendix A for a more precise definition). It is clear that since we require $2s + 1$ local conditions ((2.2) and (2.3)) on the Hamiltonian function, which has $n + s$ variables, one can expect that if $n \geqslant s + 1$, these conditions will be satisfied on an $(n - s - 1)$-dimensional submanifolds of $\{(x, y, I)\} \subseteq \mathbb{R}^{n+s}$ by an open set of Hamiltonians in a persistent way. Indeed, see appendix A for genericity and persistence results.

In [22] Hanßmann proved a KAM result regarding parabolic tori in a quasi-periodic saddle-centre bifurcation, by which Diophantine parabolic tori (and the whole associated Diophantine saddle-centre bifurcation scenario) survive small Hamiltonian perturbations that break the integrability. Indeed, numerically, we observe that small Hamiltonian perturbations of such integrable Hamiltonians (attaining a parabolic torus) do not appear to induce instabilities in the action directions near a non-resonant parabolic torus. In the next section we demonstrate that this observation changes (dramatically) if the parabolic lower-dimensional invariant torus is *m-resonant* (and locally degenerate). The m-resonance condition is simply that the $(n - s)$-dimensional inner frequencies vector $\omega(p_{fp}) = \nabla_I H_0(x, y, I)|_{p_{fp}}$ satisfies additional m constraints:

$$\langle q^i, \omega(p_{fp}) \rangle = 0, \qquad i = 1, \ldots, m, \tag{2.5}$$

for m linearly independent integer vectors $q^i \in \mathbb{Z}^{n-s} \backslash \{0\}$. Denote the parabolic tori satisfying these resonance conditions (2.5) by p_{fpr}; the behaviour of perturbed orbits, starting near an m-resonant parabolic torus p_{fpr}, is called *parabolic m-resonance* (m-PR). *Degeneracy conditions* on a resonant parabolic torus correspond to vanishing of some additional derivatives of $H_0(x, y, I)$ at p_{fpr}; we call the behaviour of perturbed orbits, starting near such degenerate resonant parabolic tori (with secondary tangency), *tangent m-PR* (m-TPR), or if *infinitely* many such derivatives vanish—*flat PR*[8] (see next section and appendix B for examples and details). It follows that if $n \geqslant s + m + 1$, in the space of integrable Hamiltonians, there exists an open set of Hamiltonians with normally parabolic m-resonant invariant $(n - s)$-tori, p_{fpr}. In fact, for these Hamiltonians, there exists an $(n - s - m - 1)$-dimensional submanifold in the (x, y, I) space along which parabolic m-resonant invariant $(n - s)$-tori, p_{fpr}, live. In the first appendix we formulate this statement more precisely and prove it. In the

[7] In some references (e.g. [10]), the normal frequencies are defined as the positive imaginary parts of all the characteristic eigenvalues.

[8] In the literature, the terms 'j-flat', 'infinitely flat' or '∞-flat' are sometimes used in cases of similar nature, i.e. to indicate a certain tangency of jth, or infinite, order, but in a different context (see, for example, [10] and references therein).

second appendix we examine the possible appearance of degeneracies and formulate theorems regarding the minimal dimensions needed for such degeneracies to appear persistently. The above considerations and the theorems proved in the appendices imply that the *appearance of normally parabolic resonant tori is expected to be common and robust in integrable n DOF Hamiltonian systems with* $n > 2$ (e.g. they are certainly more common than the appearance of fixed points which require $2n$ local conditions)[9]. Similarly, such considerations prove that certain degeneracies appear persistently if $n > 3$ (isoenergetic and degenerate resonances) and others (isoenergetic PRs) if $n > 4$.

Next we examine numerically the consequences of the existence of such degenerate parabolic resonant tori.

3. Phenomenological examples

Using normal form techniques, 4 and 5 DOF integrable Hamiltonian systems attaining PRs (the origin satisfies conditions (2.2), (2.3), (2.4) and (2.5)) are constructed, and typical parameter values are set. Then, the initial conditions for which degeneracies appear (tangent PRs, iso-energetic families of resonant hyperbolic tori and iso-energetic families of resonant parabolic tori) are sought, and numerical solutions of nearby initial conditions of the perturbed system are recorded.

We stress that the only conditions put into the models are the local conditions (2.2)–(2.5). The fact that other degeneracies appear in our models is in accordance to the theorems proved in appendix B, which state that such phenomena are expected to robustly appear in 4 and 5 DOF systems.

3.1. Degenerate and iso-energetic resonances in a 4 DOF model

The simplest model of a 4 DOF integrable Hamiltonian, which attains a parabolic resonant torus of dimension 3 ($s = 1$ in (2.2)) and is symmetric with respect to the x- and y-axes, is

$$H_0(x, y, I_1, I_2, I_3; \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6) = \frac{y^2}{2} - \frac{x^2}{2} I_1 + \frac{x^4}{4} + \left(\alpha_1 + \frac{1}{2}\right) \frac{I_1^2}{2} + \frac{I_2^2}{2} + \frac{I_3^2}{2}$$
$$+\alpha_2 I_2 I_3 + \alpha_3 I_1 I_2 + \alpha_4 I_1 I_3 + \alpha_5 \frac{I_1^2}{2} I_3 + \alpha_6 I_3 + \alpha_7 \frac{I_1^3}{3}. \tag{3.1}$$

The real parameters, $\alpha_i; i = 1, \ldots, 7$, are inserted for generality and compatibility with the 3 DOF phenomenological model which was studied in [31, 29, 32] (see equation (3.23)).

We show below that for any fixed non-zero value of the external parameters there exists an energy surface on which an infinitesimal family of 1-resonant elliptic tori, emanating from a 1-resonant parabolic torus in the direction of I_1, resides. The corresponding perturbed system exhibits *tangent parabolic* 1*-resonance* (1-TPR). The existence of this 1-TPR instability mechanism for such generically constructed systems is in accordance with the following theorem 1 and corollary 1, which are proved in appendix B.

Theorem 1. *Let* $m \geqslant 1$, $n = m + 3$. *Smooth integrable n DOF Hamiltonian systems which possess, in the direction of one of the actions, an infinitesimal family of elliptic and/or hyperbolic resonant* $(n-1)$*-tori, containing a parabolic m-resonant* $(n-1)$*-torus,* p_{tpr}*, on some energy surface* $H_0(x, y, I) = h_{tpr}$ *(defined by the point* p_{tpr}*), are* C^1*-persistent in the space of smooth integrable n DOF Hamiltonian systems, for each* $m \geqslant 1$.

Corollary 1. *Let* $n \geqslant 4$, $1 \leqslant m \leqslant n - 3$. *The existence of an m-TPR is* C^1*-persistent in the space of smooth near integrable n DOF Hamiltonian systems, for each* $n \geqslant 4$.

[9] Adding a parameter allows one to observe some of the phenomena associated with PR even for $n = 2$ (see [40]).

An example of a 1-TPR orbit may be seen in figure 1. All figures presented in this section are calculated for the fixed set of parameters: $\alpha_1 = 1.05$, $\alpha_2 = 1$, $\alpha_3 = 2$, $\alpha_4 = 1$, $\alpha_5 = 1$, $\alpha_6 = \sqrt{2}$. This set of parameters is chosen to avoid other resonances, and/or additional non-generic degeneracies in the system. The role of the parameter α_7 is discussed below. Roughly, it serves to distinguish between 1-TPR ($\alpha_7 \neq 0$, see figure 1) and *flat* 1-PR ($\alpha_7 = 0$, hence all higher order terms of $\dot{\theta}_1(I)$ in I_1 vanish, see figure 2 and discussion below).

The integrable Hamiltonian (3.1) is perturbed by

$$\varepsilon H_1(x, y, \theta_1, I_1, \theta_2, I_2, \theta_3, I_3; \varepsilon, k) = \varepsilon_1 \left(1 - \frac{x^2}{2}\right) \cos(k\theta_1) + \varepsilon_2 \cos(k\theta_2) + \varepsilon_3 \cos(k\theta_3). \tag{3.2}$$

Setting $\alpha_4 = \alpha_5 = \alpha_6 = \alpha_7 = \varepsilon_3 \equiv 0$ in the integrable Hamiltonian (3.1) and in the perturbation (3.2), recovers the 3 DOF Hamiltonian (3.23). In all the figures we fix: $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 \equiv \varepsilon = 0.001$ and $k = 3$. The corresponding 4 DOF Hamiltonian system is

$$\begin{aligned}
\dot{x} &= y, \\
\dot{y} &= xI_1 - x^3 + \varepsilon_1 x \cos(k\theta_1), \\
\dot{\theta}_1 &= -\frac{x^2}{2} + \left(\alpha_1 + \frac{1}{2}\right) I_1 + \alpha_3 I_2 + \alpha_4 I_3 + \alpha_5 I_1 I_3 + \alpha_7 I_1^2, \\
\dot{I}_1 &= \varepsilon_1 k \left(1 - \frac{x^2}{2}\right) \sin(k\theta_1), \\
\dot{\theta}_2 &= I_2 + \alpha_2 I_3 + \alpha_3 I_1, \\
\dot{I}_2 &= \varepsilon_2 k \sin(k\theta_2), \\
\dot{\theta}_3 &= I_3 + \alpha_2 I_2 + \alpha_4 I_1 + \alpha_5 \frac{I_1^2}{2} + \alpha_6, \\
\dot{I}_3 &= \varepsilon_3 k \sin(k\theta_3).
\end{aligned} \tag{3.3}$$

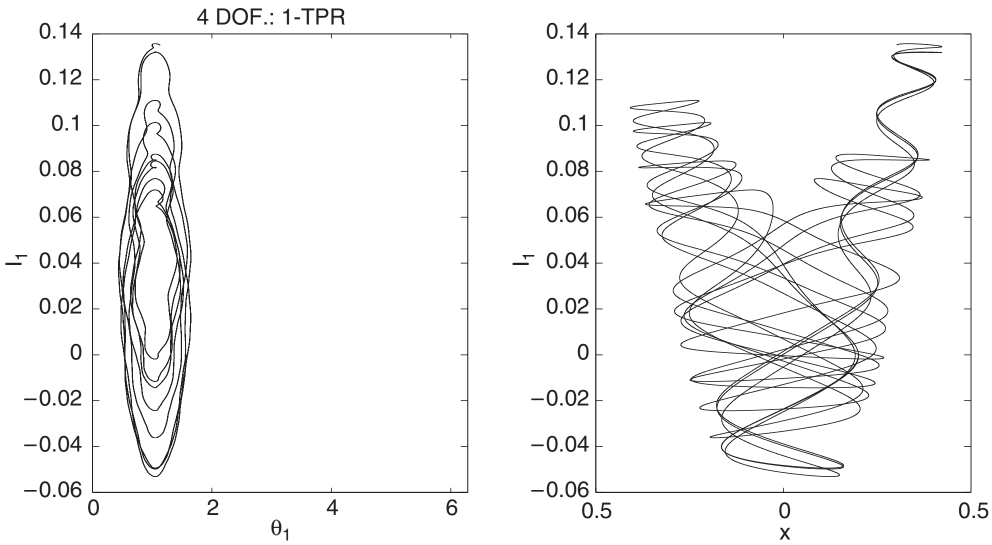

Figure 1. Tangential parabolic 1-resonance in a 4 DOF system. Initial conditions and parameters: $(x, y, I_1, I_2, I_3) = (0.2, 0., 0., 0.525, -1.05)$; $\theta_i = 1.57$, $i = 1, 2, 3$; $H_0 \approx -1.3467$, $\alpha_1 = 1.05$, $\alpha_2 = 1$, $\alpha_3 = 2$, $\alpha_4 = 1$, $\alpha_5 = 1$, $\alpha_6 = \sqrt{2}$, $\alpha_7 = 1$; $k = 3$; $\varepsilon = \varepsilon_i = 1e-3$, $i = 1, 2, 3$; $t = 1000$.

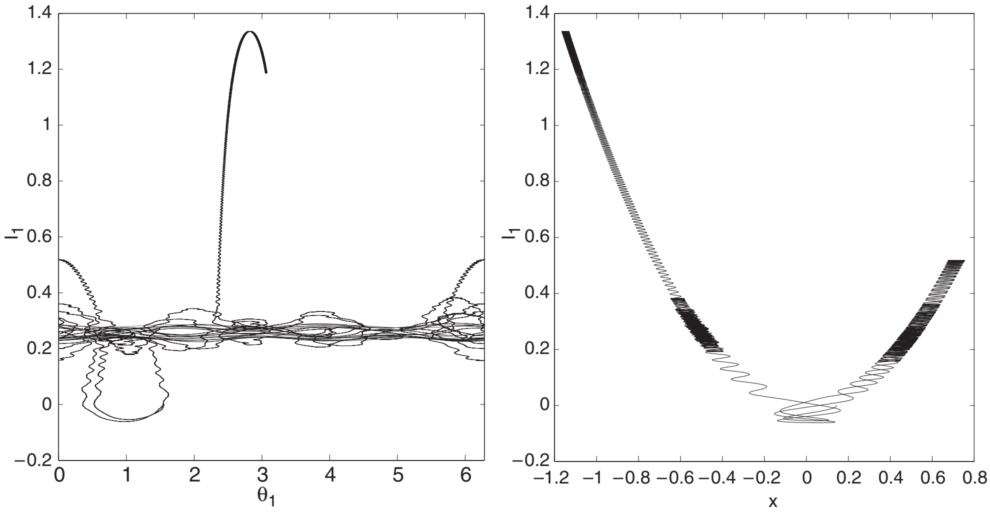

Figure 2. Flat parabolic 1-resonance in a 4 DOF system. Initial conditions and parameters: $(x, y, I_1, I_2, I_3) = (0.1454, 0., 0., 0.525, -1.05)$, $\theta_i = 1.57$, $i = 1, 2, 3$; $H_0 \approx -1.3467$; $\alpha_1 = 1.05$; $\alpha_2 = 1, \alpha_3 = 2, \alpha_4 = 1, \alpha_5 = 1, \alpha_6 = \sqrt{2}, \alpha_7 = 0$; $k = 3$; $\varepsilon = \varepsilon_i = 1e-3$, $i = 1, 2, 3$; $t = 5000$.

The integrable Hamiltonian (3.1) fulfils the fixed point conditions (2.2) on $N_f^0 := \{p_f^0 = (x, y, I_1, I_2, I_3) : (x, y) = (0, 0), I_1, I_2, I_3 \in \mathbb{R}\}$, and on $N_f^\pm := \{p_f^\pm = (x, y, I_1, I_2, I_3) : (x, y) = (\pm\sqrt{I_1}, 0), I_1 \geqslant 0, I_2, I_3 \in \mathbb{R}\}$. Denote: $N_f = N_f^0 \cup N_f^\pm$. The normal stability of the 3-torus p_f^0 may be elliptic ($I_1 < 0$), hyperbolic ($I_1 > 0$) or parabolic ($I_1 = 0$). The family of 3-tori $p_f^\pm$ consists of two families of elliptic tori which meet at a parabolic 3-torus p_{fp}^0 at $I_{1_{fp}} = 0$. Indeed, the parabolicity condition (4.4) for this system is

$$0 = \det \begin{pmatrix} 0 & -I_1 + 3x^2 \\ 1 & 0 \end{pmatrix} \Bigg|_{(0,0,I_{1p},I_{2p},I_{3p})} = I_{1_{fp}}; \tag{3.4}$$

hence it is fulfilled for $I_{1_{fp}} = 0$, for any values of I_2 and I_3, i.e. it is fulfilled for the system (3.1) on the two-dimensional submanifold, $PN_f = \{p_{fp} = (x, y, I_1, I_2, I_3) = (0, 0, 0, I_2, I_3); I_2, I_3 \in \mathbb{R}\} \subset N_f$. Choosing the unit vector $q^1 = e_1^3 = (1, 0, 0)$, the resonance condition (2.5), with $m = 1$, is fulfilled for the system (3.1) on the submanifold $\{p_{fr} = (x, y, I_1, I_2, I_3) = (\pm\sqrt{I_1}, 0, I_1 \geqslant 0, I_2, I_3)\} \subset N_f$, iff

$$\begin{aligned} 0 = \dot{\theta}_1\big|_{(\pm\sqrt{I_{1r}},0,I_{1r},I_{2r},I_{3r})} &= \frac{\partial H_0}{\partial I_1}\bigg|_{(\pm\sqrt{I_{1r}},0,I_{1r},I_{2r},I_{3r})} \\ &= \alpha_1 I_{1r} + \alpha_3 I_{2r} + \alpha_4 I_{3r} + \alpha_5 I_{1r} I_{3r} + \alpha_7 I_{1r}^2. \end{aligned} \tag{3.5}$$

Solutions to (3.5) define the resonance surface $R^{q^1} N_f$. Condition (3.5) is met on a parabolic 3-torus from PN_f, iff $I_{1_{fr}} = I_{1_{fp}} = 0$. Hence, PN_f intersects $R^{q^1} N_f$ along the one-dimensional submanifold $PR^1 = \{p_{fpr} = (x, y, I_1, I_2, I_3) = (0, 0, 0, I_{2_{fpr}}(I_3), I_3)\}$, where

$$I_{2_{fpr}}(I_3) = -\frac{\alpha_4 I_3}{\alpha_3}. \tag{3.6}$$

By (3.5), since

$$\left.\frac{\partial H_0}{\partial I_1}\right|_{N_f} = \left.\frac{\mathrm{d}H_0}{\mathrm{d}I_1}\right|_{N_f},$$

each of the energy surfaces (a one-dimensional family), containing a 1-resonant parabolic 3-torus, $p_{fpr} \in PR^1$, is tangent to N_f at p_{fpr} along I_1 (the resonant direction). One of these energy surfaces has a second-order tangency to N_f along the I_1 direction, if at some point, $p_{tpr} = (x, y, I_1, I_2, I_3) = (0, 0, 0, I_{2_{tpr}}(I_{3_{tpr}}), I_{3_{tpr}})$:

$$\begin{aligned}0 &= \frac{\mathrm{d}}{\mathrm{d}I_1}\left.\left(\frac{\partial H_0(\pm\sqrt{I_1}, 0, I_1, I_2, I_3)}{\partial I_1}\right)\right|_{p_{tpr}} \\ &= (\alpha_1 + \alpha_5 I_3 + 2\alpha_7 I_1)|_{p_{tpr}} = \alpha_1 + \alpha_5 I_{3_{tpr}},\end{aligned} \tag{3.7}$$

which implies

$$I_{3_{tpr}} = -\frac{\alpha_1}{\alpha_5}. \tag{3.8}$$

Substituting (3.8) into (3.6) yields

$$p_{tpr} = (x, y, I_1, I_2, I_3) = \left(0, 0, 0, \frac{\alpha_1\alpha_4}{\alpha_3\alpha_5}, -\frac{\alpha_1}{\alpha_5}\right), \tag{3.9}$$

for *any* fixed pre-chosen set of non-zero parameters, $\alpha_i; i = 1, \ldots, 7$. The point (3.9) defines the energy level $H_0(x, y, I_1, I_2, I_3) = H_0(p_{tpr}) = h_{tpr}$ on which resides an infinitesimal family of elliptic 1-resonant tori emanating from the 1-resonant parabolic torus, p_{tpr}.

In figure 1 an example of a perturbed orbit exhibiting 1-TPR (starting near the point (3.9)) is shown. In the left plot we project the eight-dimensional orbit on the (θ_1, I_1) plane, and in the right plot on the (x, I_1) plane. The orbit performs fast travel through the successive elliptic resonance zones and reaches the top of the perturbed energy surface (in the I_1 direction) in quite a short time (see discussion in section 3.3), then goes back down to the PR zone, reaches the bottom of the energy surface and starts to go up again. The displacements in the action I_1 are of size $\Delta I_1 \approx 0.2$, namely of two orders of magnitude larger than ε (see also the upper plot of figure 7 in section 3.3). We now show that $0.2 = \mathrm{O}(\varepsilon^{1/3})$ is the maximal possible displacement of I_1 on this energy surface as long as I_2 and I_3 stay near their initial values, as is the case here.

The transversality condition which ensures that the existence of the second-order tangency at (3.7) is persistent (while higher-order tangencies are not) is

$$\frac{d^2}{dI_1^2}\left.\left(\frac{\partial H_0(\pm\sqrt{I_1}, 0, I_1, I_2, I_3)}{\partial I_1}\right)\right|_{p_{tpr}} = 2\alpha_7 \neq 0. \tag{3.10}$$

Namely, the term $\alpha_7(I_1^3/3)$ in (3.1) ensures that condition (3.10) is fulfilled for nearby sets of non-zero parameters. More importantly here, the existence of higher-order terms in I_1 with positive coefficients ensures that the energy surfaces are bounded in the positive I_1 direction; the bounds of the energy surfaces (if exist) in I_1 correspond to the elliptic equilibria of the (x, y) system (see [2,29,32]), namely they may be calculated from the Hamiltonian equation by substituting $(x, y) = (x_{ell}, y_{ell})$ into the Hamiltonian function. For the system (3.3), $(x, y) = (0, 0)$ is an elliptic fixed point of the (x, y) plane for $I_1 < 0$, for all ε values. Hence, the lower bound in I_1 of the perturbed energy surfaces, when exists, is obtained from solving the equation

$$\begin{aligned}&\alpha_7\frac{I_1^3}{3} + \left(\alpha_1 + \frac{1}{2} + \alpha_5 I_3\right)\frac{I_1^2}{2} + (\alpha_3 I_2 + \alpha_4 I_3)\, I_1 + \frac{I_2^2}{2} + \frac{I_3^2}{2} \\ &\qquad + \alpha_2 I_2 I_3 + \alpha_6 I_3 + \mathrm{O}\,(\varepsilon_1, \varepsilon_2, \varepsilon_3) - h = 0,\end{aligned} \tag{3.11}$$

with respect to I_1. Similarly, the upper bound in I_1 of the perturbed energy surfaces, when exists, is obtained by substituting $(x, y) = (\pm\sqrt{I_1 + \mathrm{O}(\varepsilon)}, \mathrm{O}(\varepsilon))$ and solving the equation

$$\alpha_7\frac{I_1^3}{3} + (\alpha_1 + \alpha_5 I_3)\frac{I_1^2}{2} + (\alpha_3 I_2 + \alpha_4 I_3)I_1 + \frac{I_2^2}{2} + \frac{I_3^2}{2} + \alpha_2 I_2 I_3 + \alpha_6 I_3 + \mathrm{O}(\varepsilon_1, \varepsilon_2, \varepsilon_3) - h = 0, \tag{3.12}$$

with respect to I_1. Consider first the unperturbed case, $\varepsilon = \varepsilon_i = 0$ ($i = 1, 2, 3$), at p_{tpr} (equation (3.9)), $\alpha_1 + \alpha_5 I_3 = 0$. Hence, this branch of equilibrium manifold has a second-order tangency to the energy surface with the energy value h_{tpr}, along the I_1 axis, corresponding to a 1-TPR in the perturbed system. Furthermore, near $\alpha_1 + \alpha_5 I_3 = 0$, positive α_7 (or the coefficients of higher-order terms in I_1) is a necessary condition for getting an upper finite bound on I_1. For $\alpha_7 = 0$ (as higher-order terms in I_1 are not considered in (3.1)), this branch of equilibrium manifold is parallel to the I_1 axis to all orders, corresponding to a *flat parabolic* 1-*resonance* (flat 1-PR). We claim that the instability in the I_1 direction near these degenerate PRs is maximal; namely, the only barrier for it is the extent of the energy surface in the PR neighbourhood. Indeed, solving equation (3.12) for I_1 with the parameters, initial actions and energy values as in figure 1, and with $\varepsilon = \varepsilon_i = 0$ ($i = 1, 2, 3$), one obtains the upper bound in I_1 of the unperturbed energy surface as $I_{1\mathrm{up}}^{H_0} \approx 0.1$. The difference in the upper bounds in I_1 of the unperturbed and perturbed energy surfaces with $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 \equiv \varepsilon \neq 0$ is of order $\mathrm{O}(\varepsilon^{1/3})$, i.e. $I_{1\mathrm{up}}^{H} = I_{1\mathrm{up}}^{H_0} + \mathrm{O}(\varepsilon^{1/3}) \approx 0.1 + \mathrm{O}(\varepsilon^{1/3})$. Hence, as long as the actions I_2 and I_3 do not vary much, as is the case here, the perturbed orbit in figure 1 indeed reaches the boundaries in I_1 of the perturbed energy surface.

Generally, $\alpha_7 = 0$ corresponds to a third-order tangency in the I_1 direction (2-flat-TPR, see appendix B) (a co-dimension one phenomenon for 4 DOF integrable systems). Since we did not include higher-order terms in (3.1), here this situation corresponds to a flat PR. This flatness implies that the frequency $\dot{\theta}_1(\pm\sqrt{I_1}, 0, I_1, \alpha_1\alpha_4/\alpha_3\alpha_5, -\alpha_1/\alpha_5)$ vanishes identically for all non-negative values of I_1; the unperturbed energy surface which contains the point p_{tpr} (equation (3.9)) is unbounded in the positive direction of I_1 (see equation (3.12)). Perturbed orbits corresponding to this flat 1-PR may exhibit order one instabilities (see figure 2). Compare this figure with figure 1: in both figures the orbit is seen to exhibit fast travels along the successive elliptic resonances, repeatedly passing through the PR zone. The orbit in figure 1 exhibits displacements of size $\Delta I_1 \approx 0.2$ even after a much longer run time than shown ($t \geqslant 5000$). The orbit in figure 2 reaches displacements of size $\Delta I_1 \approx 0.5$ at $t \approx 1000$, occasionally gets trapped in a specific resonance zone, and then, after a longer time ($t = 5000$) it reaches I_1 values which are of order O(1) (see also figures 7 and 6 in section 3.3). To control our simulations, we ensure that the value of the Hamiltonian is preserved at least up to the 12th digit after the decimal point.

These simulations demonstrate the critical role of the extent of the energy surfaces in the action directions, and the connection between this simple geometrical phenomena and the dynamical phenomena of resonances—fast growing extent of the allowed range of actions in the energy surface corresponds to near-flatness in that direction, which corresponds to an infinitesimal family of bifurcating resonant tori.

In [40, 29, 31, 32] qualitatively similar behaviour of perturbed orbits near a flat 1-PR was observed for 2 and 3 DOF systems. We emphasize that in 2 DOF 1-TPR is of co-dimension two, in 3 DOF systems it is of co-dimension one, and only for systems with 4 or more DOF it becomes of co-dimension zero (its existence is persistent). It is worth noting that in [40, 31, 32] simulations of flat (equivalent to looking at p_{tpr} with $\alpha_7 = 0$) and *near*-flat (equivalent to looking *near* p_{tpr} with $\alpha_7 = 0$, i.e. taking $\alpha_7 \ll 1$) orbits were presented, and no TPR orbits were shown.

The example presented here illustrates the fact that $n = 4$ is the first value of n for which integrable n DOF Hamiltonians persistently possess an infinitesimal family of resonant tori, containing a parabolic resonant torus on the same energy surface. In [29] and here, it is the first demonstration that TPR is a strong enough mechanism for inducing instability.

In addition, $n = 4$ is also the first value of n for which the existence of a one-dimensional family of hyperbolic (elliptic) 1-resonant tori on the same unperturbed energy surface becomes generic. This possible mechanism for instabilities occurs for the integrable Hamiltonian (3.1) when the three-dimensional submanifold, $HN_f := \{(x, y, I_1, I_2, I_3) : (x, y) = (0, 0), I_1 > 0, I_2, I_3 \in \mathbb{R}\} \subset N_f$, corresponding to normally hyperbolic 3-tori, intersects transversely the two-dimensional submanifold, $R^1N_f \subset N_f$, which corresponds to 1-resonant 3-tori, on a certain four-dimensional energy surface, defined by the equation

$$H_0(x, y, I_1, I_2, I_3) - h_{hr} = 0,$$

along a one-dimensional curve of hyperbolic 1-resonant 3-tori. For $q^1 = e_1^3 = (1, 0, 0)$, the submanifold R^1N_f is defined by the equation

$$\begin{aligned} 0 = \dot{\theta}_1\big|_{(0,0,I_{1r},I_{2r},I_{3r})} &= \left.\frac{\partial H_0}{\partial I_1}\right|_{(0,0,I_{1r},I_{2r},I_{3r})} \\ &= \left(\alpha_1 + \frac{1}{2}\right) I_{1r} + \alpha_3 I_{2r} + \alpha_4 I_{3r} + \alpha_5 I_{1r} I_{3r} + \alpha_7 I_{1r}^2, \end{aligned} \tag{3.13}$$

which may be solved for I_2 as a function of the remaining actions:

$$I_{2r}(I_1, I_3) = -\frac{(\alpha_1 + 1/2)I_1 + \alpha_4 I_3 + \alpha_5 I_1 I_3 + \alpha_7 I_1^2}{\alpha_3}, \tag{3.14}$$

where for $I_1 > 0$, $I_{2r}(I_1, I_3)$ corresponds to a two-dimensional family of hyperbolic resonant tori. The intersection of this two-dimensional family with a certain energy

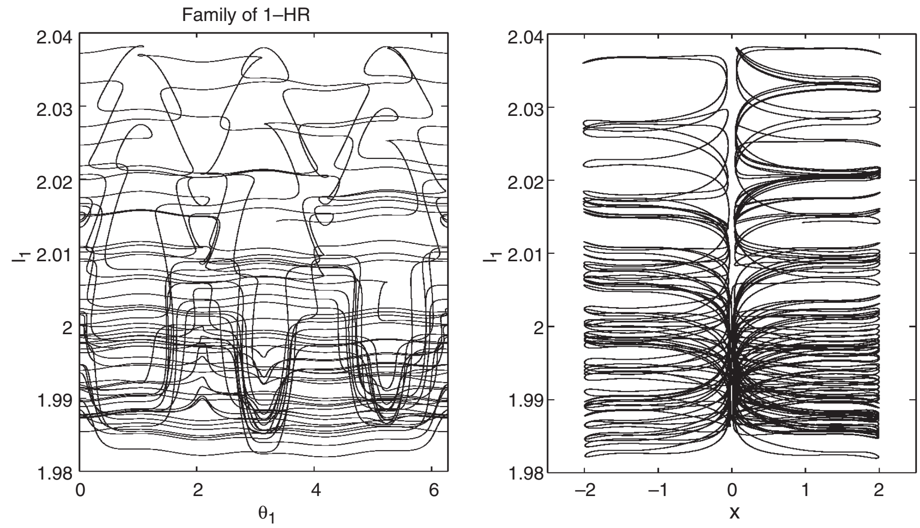

Figure 3. Perturbation of an iso-energetic one-dimensional family of hyperbolic 1-resonant tori in a 4 DOF system: 1-family of 1-HR. Initial conditions and parameters: $(x, y, I_1, I_2, I_3) = (0.0015, 0., 2., -8.318\,29, 4.512\,19)$; $\theta_i = 1.57$, $i = 1, 2, 3$; $H_0 \approx 1.4999$; $\alpha_1 = 1.05$, $\alpha_2 = 1$, $\alpha_3 = 2$, $\alpha_4 = 1$, $\alpha_5 = 1$, $\alpha_6 = \sqrt{2}$, $\alpha_7 = 0$; $k = 3$; $\varepsilon = \varepsilon_i = 1e - 3$, $i = 1, 2, 3$; $t = 1000$.

surface, corresponding to some energy value $h_{hr} = H_0(0, 0, I_{1_{hr}}, I_{2_{hr}}(I_{1_{hr}}, I_{3_{hr}}), I_{3_{hr}})$, along a one-dimensional curve, is defined by the equation

$$H_0(0, 0, I_1, I_{2r}(I_1, I_3), I_3) - h_{hr} = 0, \tag{3.15}$$

which may be solved for $I_3 = I_{3_{hr}}(I_1)$, for $I_1 > 0$. For example, to plot figure 3 we choose the energy value $H_0 = h_{hr} \approx 1.4999$, and $I_1 = 2$ as the initial value for I_1. In this figure we present projections on the planes (θ_1, I_1) (left) and (x, I_1) (right) of an eight-dimensional perturbed orbit travelling near a family of hyperbolic 1-resonant tori of system (3.1) (through a family of 1-HR in the perturbed system (3.3)). Notice that the structure of the orbit in these planes is very different from that of the 1-TPR and flat 1-PR cases (compare figure 3 with figures 1 and 2), and that the instabilities in I_1 seems to be smaller: in contrast to the 1-TPR case, this 1-HR orbit did not reach the boundaries of the perturbed energy surface during the time of integration (up to $t = 10\,000$), and the maximal displacement in I_1 is approximately of size $\Delta I_1 \approx 0.06$ (see also figure 7 in section 3.3).

The occurrence of the iso-energetic one-dimensional family of 1-HR in system (3.3) is in accordance with corollary 5 (see appendix B).

3.2. Iso-energetic PRs in a 5 DOF model

For $n \geqslant 5$ DOF systems, a one-dimensional family of 1-resonant parabolic 4-tori appears persistently on the same energy surface. Indeed, consider

$$\begin{aligned} H_0(x, y, I; \alpha) = {} & \frac{y^2}{2} - \frac{x^2}{2}(I_1 + \beta I_2) + \frac{x^4}{4} + \left(\alpha_1 + \frac{1}{2}\right)\frac{I_1^2}{2} + \frac{I_2^2}{2} + \frac{I_3^2}{2} + \frac{I_4^2}{2} \\ & + \alpha_2 I_2 I_3 + \alpha_3 I_1 I_2 + \alpha_4 I_1 I_3 + \alpha_5 I_3 I_4 + \alpha_6 I_1 I_4 + \alpha_7 I_2 I_4 + \alpha_8 I_4, \end{aligned} \tag{3.16}$$

where the parameters, $\alpha_i; i = 1, \ldots, 8, \beta$, are real, and are fixed to generic values. The introduction of the new parameter β, which appears in the mixed term $(x^2/2)(I_1 + \beta I_2)$ (namely was set to zero in (3.1)), is explained below. We use the perturbation

$$\varepsilon H_1(x, y, \theta, I; \varepsilon, k) = \varepsilon \left[\left(1 - \frac{x^2}{2}\right)\cos(k\theta_1) + \cos(k\theta_2) + \cos(k\theta_3) + \cos(k\theta_4)\right]. \tag{3.17}$$

The corresponding 5 DOF Hamiltonian system of the Hamiltonian equations (3.16) and (3.17) is

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= x(I_1 + \beta I_2) - x^3 + \varepsilon x \cos(k\theta_1), \\ \dot{\theta}_1 &= -\frac{x^2}{2} + \left(\alpha_1 + \frac{1}{2}\right) I_1 + \alpha_3 I_2 + \alpha_4 I_3 + \alpha_6 I_4, \\ \dot{I}_1 &= \varepsilon k \left(1 - \frac{x^2}{2}\right)\sin(k\theta_1), \\ \dot{\theta}_2 &= -\beta\frac{x^2}{2} + I_2 + \alpha_2 I_3 + \alpha_3 I_1 + \alpha_7 I_4, \\ \dot{I}_2 &= \varepsilon k \sin(k\theta_2), \\ \dot{\theta}_3 &= I_3 + \alpha_2 I_2 + \alpha_4 I_1 + \alpha_5 I_4, \\ \dot{I}_3 &= \varepsilon k \sin(k\theta_3), \\ \dot{\theta}_4 &= I_4 + \alpha_5 I_3 + \alpha_6 I_1 + \alpha_7 I_2 + \alpha_8, \\ \dot{I}_4 &= \varepsilon k \sin(k\theta_4). \end{aligned} \tag{3.18}$$

By construction, the 5 DOF integrable Hamiltonian (3.16) fulfils conditions (2.2) for equilibria in the (x, y) plane at $(x, y) = (0, 0)$, for any values of the actions, I, and at $(x, y) =$

$(\pm\sqrt{I_1+\beta I_2}, 0)$, for $I_1 \geqslant -\beta I_2$, and any $I_2, I_3, I_4 \in \mathbb{R}$. The system (3.16) fulfils the parabolicity condition (2.3) for $I_{1p}(I_2, I_3, I_4) = I_{1p}(I_2) = -\beta I_2$ (for any values of the actions: I_2, I_3, I_4). Hence, for the system (3.16), the three-dimensional submanifold corresponding to invariant parabolic 4-tori is $PN_f = \{(x, y, I_1, I_2, I_3, I_4) : (x, y) = (0, 0), I_1 = -\beta I_2; I_2, I_3, I_4 \in \mathbb{R}\}$. Choosing the unit vector $q^1 = e_1^4 = (1, 0, 0, 0)$, the resonance condition (2.5) is fulfilled for the system (3.16) on PN_f, iff

$$0 = \dot{\theta}_1\big|_{(0,0,-\beta I_{2r}, I_{2r}, I_{3r}, I_{4r})} = \left.\frac{\partial H_0}{\partial I_1}\right|_{(0,0,-\beta I_{2r}, I_{2r}, I_{3r}, I_{4r})}$$
$$= \left(-\beta\left(\alpha_1 + \frac{1}{2}\right) + \alpha_3\right) I_{2r} + \alpha_4 I_{3r} + \alpha_6 I_{4r}, \tag{3.19}$$

for any fixed set of parameters, α_i $(i = 1, \ldots, 8), \beta$. Condition (3.19) is met on a two-dimensional submanifold, PR^1, defined by the equation

$$I_{3_{pr}}(I_2, I_4) = -\frac{(-\beta(\alpha_1 + 1/2) + \alpha_3)I_2 + \alpha_6 I_4}{\alpha_4}. \tag{3.20}$$

Fixing an energy value, h_{pr}, imposes one extra condition on the remaining actions:

$$H_0(0, 0, I_{1p}(I_2), I_2, I_{3_{pr}}(I_2, I_4), I_4) - h_{pr} = 0, \tag{3.21}$$

which may be solved for I_4, obtaining an iso-energetic one-dimensional family of parabolic 1-resonant 4-tori on the curve

$$HPR^1 = \{(x, y, I_1, I_2, I_3, I_4) = (0, 0, -\beta I_2, I_2, I_{3_{pr}}(I_2, I_{4_{pr}}(I_2)), I_{4_{pr}}(I_2));\ I_2 \in \mathbb{R}\}. \tag{3.22}$$

Now the role of the parameter β becomes clear: for $\beta = 0$, this iso-energetic one-dimensional family of resonant tori has no component in the I_1 direction, the direction in which the resonance occurs. It suggests that excursions associated with this resonance will lead to instabilities along this family only for non-zero β, which explains the choice of our mixed term. Indeed, when one sets $\beta = 0$, no new dynamical phenomena were numerically detected. This observation falls into a more general framework which distinguishes between different types of PR according to the relation between the actions which govern the bifurcating tori and the resonance directions. While the persistence theorems apply equally well to all cases, the corresponding perturbed orbits exhibit significantly different behaviours as demonstrated in [29,32] for the two possible types of 1-PR in 3 DOF systems.

In figure 4 we present a perturbed orbit of the system (3.18), starting near a 1-resonant parabolic 4-torus of the unperturbed system, which is a part of a family of 1-resonant parabolic tori (1-family 1-PR), all belonging to the same unperturbed energy surface, $h_{pr} = -0.9626$. In the left plot of this figure, the projection of the ten-dimensional orbit on the (θ_1, I_1) plane is presented, and the projection of the orbit on the (x, I_1) plane is seen on the right. The remaining (non-resonant) actions stay near their initial values (performing infinitesimally small oscillations) for all integration times (up to $t = 10\,000$). The structure of the orbit in figure 4 somewhat resembles the structure of the orbit of the 4 DOF system (3.3) exhibiting a 1-family 1-HR, seen in figure 3. The (perturbed) 1-family 1-PR orbit in figure 4 exhibits instabilities only in the direction of the resonant action, I_1, with displacements of size $\Delta I_1 \approx 0.08$, while the value of I_2 suffers a small drift with much weaker oscillation. Hence, the equation $I_1 + \beta I_2 = 0$, which corresponds to parabolic stability type of the tori, does not hold during the whole run time of the orbit. However, the orbit in figure 4 repeatedly passes through PR zones, with a small drift along the parabolic family axis $(I_1 + \beta I_2 = 0)$, and bifurcates to nearby elliptic and hyperbolic stability type behaviour. This scenario is further illustrated in figure 5. In this figure

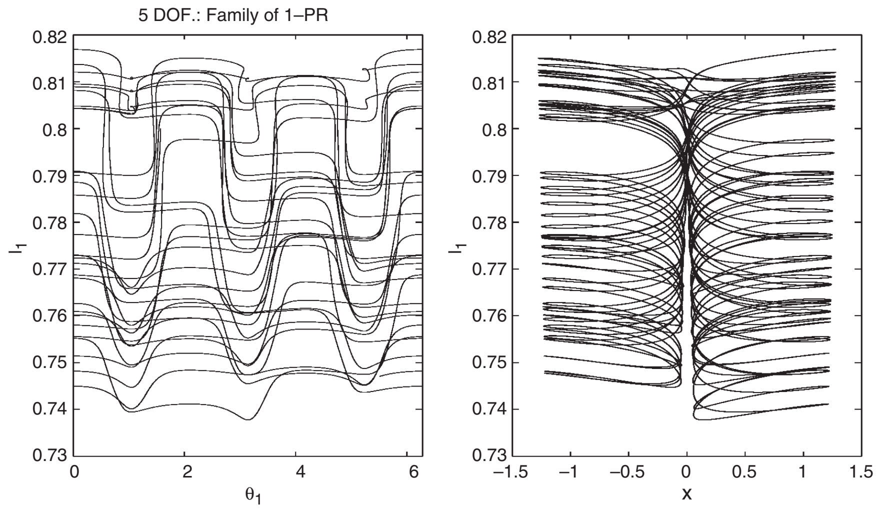

Figure 4. Perturbation of an iso-energetic one-dimensional family of parabolic 1-resonant tori in a 5 DOF system: 1-family of 1-PR. Initial conditions and parameters: $(x, y, I_1, I_2, I_3, I_4) = (0.0002, 0., 0.8, -0.8, 0.4665, -0.999)$, $\theta_i = 1.57$, $i = 1, \ldots, 4$; $H_0 \approx -0.9626$; $\beta = 1$; $\alpha_1 = 1.05$, $\alpha_2 = 1$, $\alpha_3 = 2$, $\alpha_4 = 3.8$, $\alpha_5 = \sqrt{3}$, $\alpha_6 = \sqrt{2}$, $\alpha_7 = 1.5$, $\alpha_8 = \sqrt{2}$; $k = 3$; $\varepsilon = 1e - 3$; $t = 1000$.

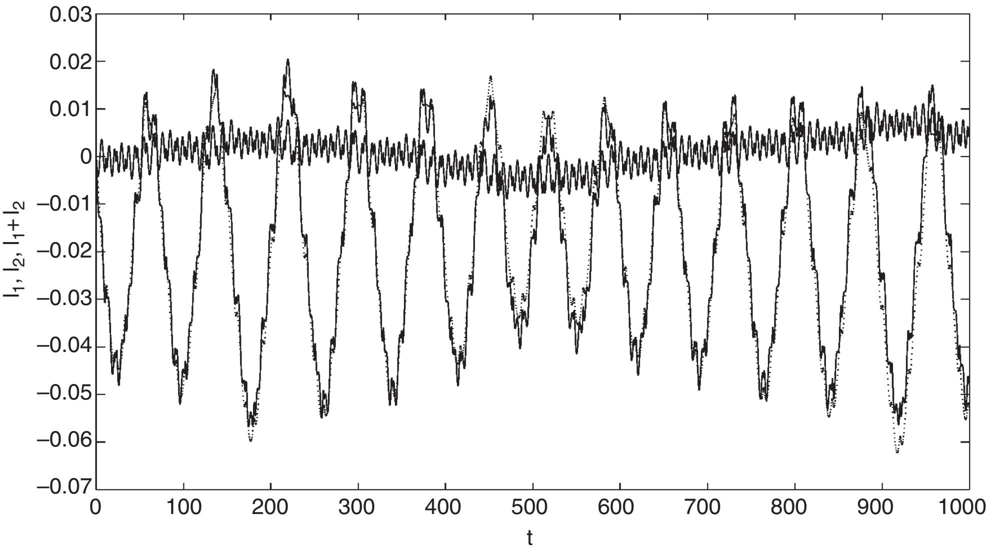

Figure 5. 1-family of 1-PR. The curves (······): $(I_1(t) - I_1(0))$; (- - - -): $(I_2(t) - I_2(0))$ and (——): $(I_1(t) + I_2(t))$ are plotted as functions of the time, t. Initial conditions and parameters: as in figure 4.

we plot $I_1(t) - I_1(0)$ (dotted curve exhibiting large oscillations), $I_2(t) - I_2(0)$ (dashed curve exhibiting small oscillations) and $I_1(t) + \beta I_2(t)$ (solid curve) as functions of the time, t. In this figure, only the area near zero, which is covered by the curve $I_2(t) - I_2(0)$, corresponds to parabolic stability type. Below this region the actions correspond to elliptic stability type, and above it to hyperbolic stability type. See section 3.3 for further discussion, and comparison between all the mechanisms for instabilities proposed here.

When the initial values of I_1 and I_2 are zero, different behaviour of nearby perturbed orbits is observed in simulations: either one of the two orbit structures which are described for the case of isolated 1-PR in [29, 32] is observed, depending on the initial values of the other actions.

The persistent appearance of these 1-family 1-PR behaviours are supported by corollary 6 (see appendix B).

3.3. Instabilities rate

In sections 3.1 and 3.2 we demonstrate that the different instability mechanisms may result in qualitative different orbit structures (compare figures 1–4). Here we present some quantifiers for the resulting differences in the instability rates, which we define as the maximal observed displacement in the resonant action(s) over a certain time interval:

$$\Delta I(T, \varepsilon) = \max_{j;\ 0<t\leqslant T} |I_j(t) - I_j(0)|;$$

we compare these displacements for the different mechanisms described above, for a given fixed values of ε and T: $\varepsilon = 0.001$ and $T = 1000$ (or 5000). An exhaustive study of the dependence of these displacements and the averaged velocities by which they are reached as a function of ε and T for the different types of mechanism is certainly needed, yet it is beyond the scope of this work.

In figure 6 we present $I_1(t)$ for the degenerate case of flat 1-PR, where for fixed values of I_2 and I_3 (at which the flat 1-PR occurs) the energy surface is unbounded in the I_1 direction and large ($\Delta I(5000, 0.001) \approx 1.4$), fast ($t < 5000$, namely, probably not exponentially small in ε) instabilities occur. In all the other cases presented here (figure 7) for any fixed values of I_2 and I_3 the energy surfaces are bounded in I_1 and such large instabilities are prohibited as long as I_2 and I_3 remain close to their initial values.

In figure 7 we present the curves $I_1(t) - I_1(0)$ for time intervals $t \in [0, 1000]$ (in fact, the same instabilities are observed up to $t = 10\,000$). The upper plot of this figure corresponds

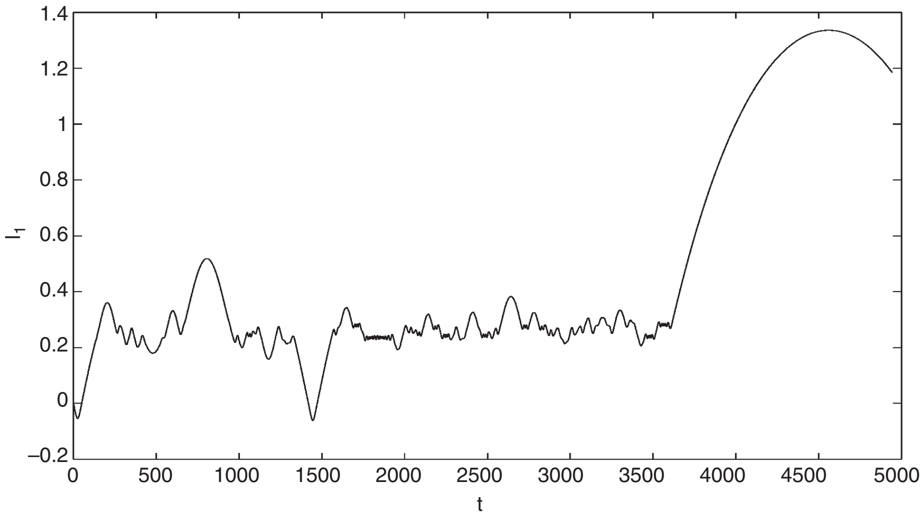

Figure 6. Flat 1-PR in a 4 DOF system. The curve $I_1(t) - I_1(0)$ is plotted as a function of the time t. Initial conditions and parameters as in figure 2.

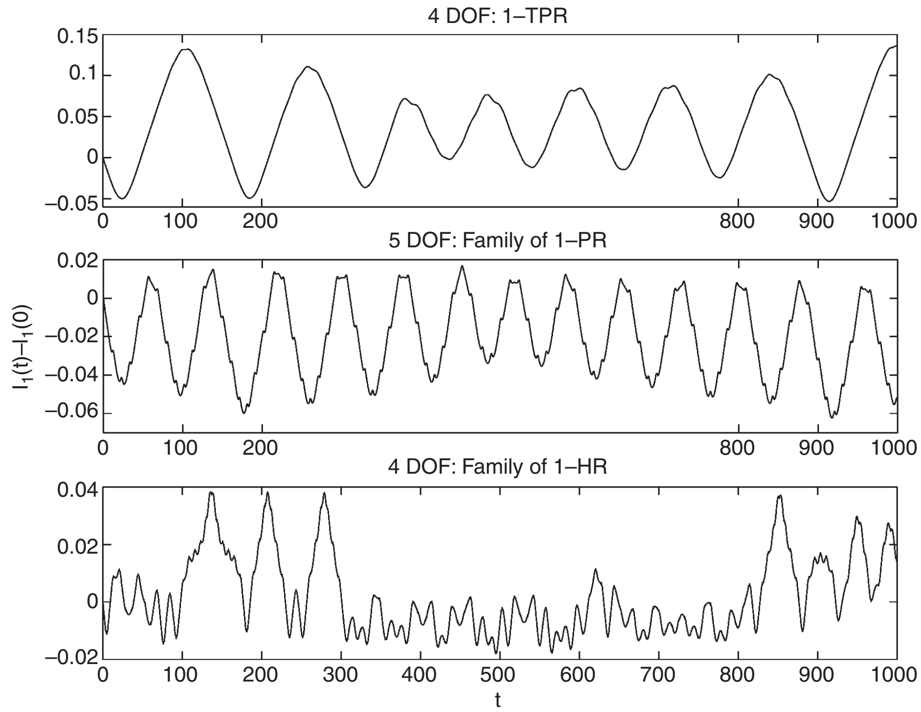

Figure 7. Upper plot: 1-TPR (see figure 1), middle plot: 1-family of 1-PR (see figure 4), lower plot: 1-family of 1-HR (see figure 3).

to the instabilities which are developed for the 4 DOF system (3.3) as a result of a 1-TPR: $\Delta I(1000, 0.001) \approx 0.2$; in the middle plot the instabilities which develop as a result of an iso-energetic 1-family of 1-PR in the 5 DOF system (3.18) are presented: $\Delta I(1000, 0.001) \approx 0.08$; in the lower plot the instabilities resulting from an iso-energetic 1-family of 1-HR in the 4 DOF system 23 are shown: $\Delta I(1000, 0.001) \approx 0.06$. Though relatively ordered, the mechanism of TPR exhibits the largest and the fastest instabilities. Moreover, in this case the displacements in I_1 are maximal; the orbit travels between the lowest and the uppermost values of I_1 on the energy surface at the given I_2 and I_3 values, on time intervals of order O(100). In the other cases such displacements do not occur even on longer time intervals. We suggest that in the 1-TPR case when instabilities in I_1 develop and the orbit bifurcates to elliptic stability type, it goes through successive elliptic resonance zones (as the first frequency in this case does not depend on the largest term in I_1), resulting in long runs on which $\dot{I}_1$ has one sign. In the other two cases the orbit visits near elliptic or hyperbolic tori (of the unperturbed system) which are only near resonance (as in these cases the first frequency does depend on all the terms in I_1) and $\dot{I}_1$ oscillates near zero. Moreover, the fact that the tangency point corresponds to a resonant parabolic torus is crucial, as a parabolic torus admits a bifurcation point near which resonant tori of other stability types reside. The numerical evidence presented here and in [40, 29, 31, 32] indicates that the successive changes in the orbit through different stability types result in larger instabilities even if the zones it visits are only near resonant.

The above statement is further supported when comparing 2-PR (which corresponds to a parabolic torus of fixed points in the integrable system) versus 2-HR (which corresponds to a hyperbolic torus of fixed points in the integrable system) in the 3 DOF system (3.23) (see

[29, 32]):

$$
\begin{aligned}
H(x, y, \theta_1, I_1, \theta_2, I_2; \alpha_1, \alpha_2, \alpha_3, \varepsilon, k) &= H_0(x, y, I_1, I_2) + \varepsilon H_1(x, y, \theta_1, I_1, \theta_2, I_2) \\
&= \frac{y^2}{2} - \frac{x^2}{2} I_1 + \frac{x^4}{4} + \left(\alpha_1 + \frac{1}{2}\right) \frac{I_1^2}{2} + \frac{I_2^2}{2} + \alpha_2 I_2 + \alpha_3 I_1 I_2 \\
&\quad + \varepsilon_1 \left(1 - \frac{x^2}{2}\right) \cos(k\theta_1) + \varepsilon_2 \cos(k\theta_2).
\end{aligned} \tag{3.23}
$$

Figure 8 demonstrates that the instabilities developed in both actions are greater for the PR than for the hyperbolic resonance; the maximal displacements observed for the resonant actions, I_1 and I_2, in the plotted time interval ($t = 1000$) are: $\Delta I_1 \approx 0.25$ and $\Delta I_2 \approx 0.3$ for 2-PR (figures 8 (left) and 9). $\Delta I_1 \approx 0.07$ and $\Delta I_2 \approx 0.08$ for 2-HR (figures 8-(right) and 10). Hence, $\Delta I(1000, 0.001) \approx 0.3$ for 2-PR, and $\Delta I(1000, 0.001) \approx 0.08$ for 2-HR. The different orbit structures for both cases are presented in figures 9 and 10, respectively. The orbit corresponding

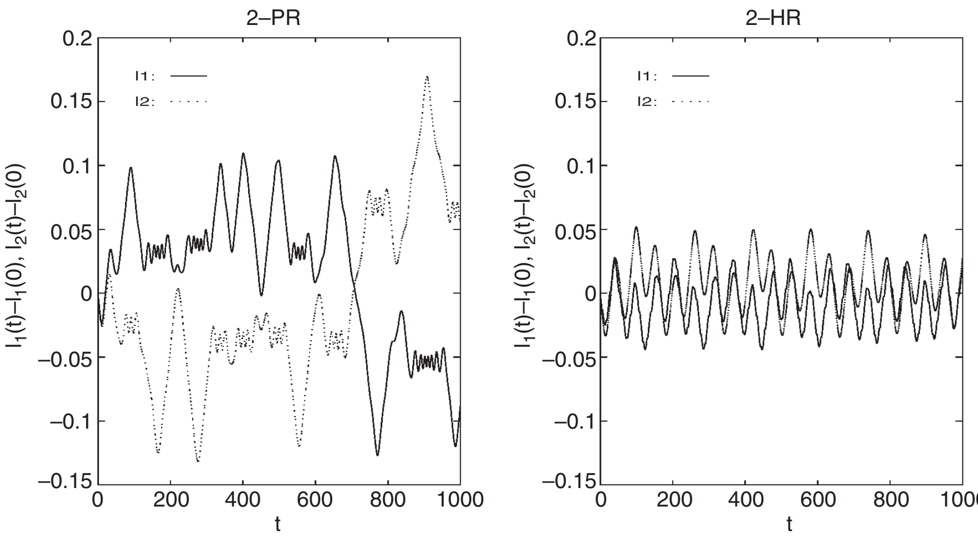

Figure 8. 2-PR versus 2-HR in the 3 DOF system (3.23). Initial conditions and parameters for the 2-PR: $(x, y, I_1, I_2) = (0.02, 0., 0., 0.)$; $(\alpha_1 = 1.05, \alpha_2 = 0., \alpha_3 = 2.)$; $\theta_1 = \theta_2 = 1.57$; $H_0 \approx 4e-4$; $k = 3$; $\varepsilon_1 = \varepsilon_2 = 0.001$ (see figure 9). Initial conditions and parameters for the 2-HR: $(x, y, I_1, I_2) = (0.02, 0., 1., -2.)$; $(\alpha_1 = 1.5, \alpha_2 = 1., \alpha_3 = 1.)$; $\theta_1 = \theta_2 = 1.57$; $H_0 \approx -1.0196$; $k = 3$; $\varepsilon_1 = \varepsilon_2 = 0.001$ (see figure 10).

to 2-PR slides through bifurcations as the actions change in time, while the orbit corresponding to 2-HR does not, and its behaviour corresponds to a hyperbolic stability type behaviour for the whole time of its run.

In 3 DOF systems, 2-PR is a co-dimension one phenomenon; its existence becomes persistent (*without* dependence on external parameters) for $n \geqslant 4$ DOF systems. In higher-dimensional systems the orbit structure and the rate of instability of the resonant actions near such double resonances appear to be similar. We present here the 3 DOF orbits (figures 8, 9 and 10), as $n = 3$ is the first value of n for which such 2-resonances exist (2-HR persistently without dependence on external parameters, and 2-PR persistently as a co-dimension one phenomenon; these statements are formulated and proved in section A.1 of appendix A).

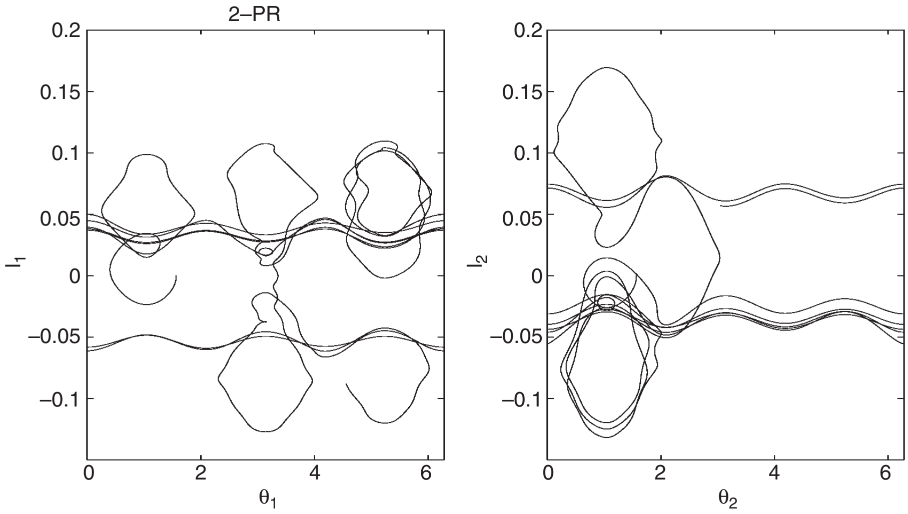

Figure 9. 2-PR in a 3 DOF system. Initial conditions and parameters: $(x, y, I_1, I_2) = (0.02, 0., 0., 0.)$; $(\alpha_1 = 1.05, \alpha_2 = 0., \alpha_3 = 2.)$; $\theta_1 = \theta_2 = 1.57$; $H_0 \approx 4e-4$; $k = 3$; $\varepsilon_1 = \varepsilon_2 = 0.001$; $t = 1000$.

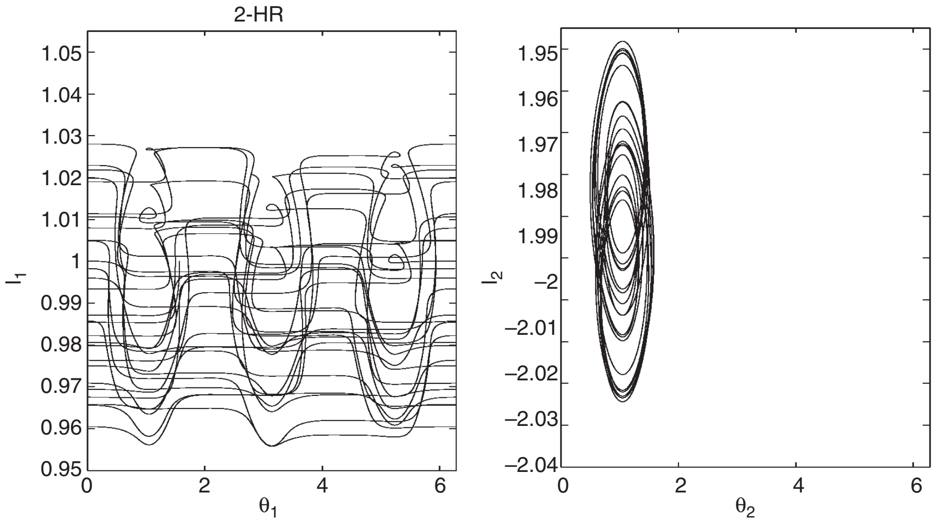

Figure 10. 2-HR in a 3 DOF system. Initial conditions and parameters: $(x, y, I_1, I_2) = (0.02, 0., 1., -2.)$; $(\alpha_1 = 1.5, \alpha_2 = 1., \alpha_3 = 1.)$; $\theta_1 = \theta_2 = 1.57$; $H_0 \approx -1.0196$; $k = 3$; $\varepsilon_1 = \varepsilon_2 = 0.001$; $t = 1000$.

4. Conclusion and discussion

Parabolic resonant tori always correspond to junctions of strong (low-order) resonances in the unperturbed system. Hence, PRs always correspond to cross-resonance diffusion in action variables. Up till now special examples, exhibiting only hyperbolic behaviour, were constructed to study cross-resonance diffusion, as people believed that hyperbolic resonances result in the strongest possible instabilities. Here and in [40, 29, 31, 32], we present numerical evidence demonstrating that the existence of parabolic resonant tori in the unperturbed system does not interfere with the creation of instabilities and may even induce stronger (larger and faster) instabilities than the instabilities exhibited near hyperbolic resonant tori. Clearly these are only preliminary indications and a thorough numerical and analytical studies are needed to quantify such statements. Parabolic tori are *a priori* resonant and are a part of some bifurcation scenario, where near them tori of other stability types reside. We propose these reasons for the observation that PRs result in strong instabilities. Moreover, we prove that the existence of PRs is persistent in the space of near integrable n DOF Hamiltonians for each $n \geqslant 3$. PRs are expected to appear in systems with non-separable integrable skeleton. Such are typical, for example, to unperturbed systems which preserve angular momentum, yet are a-typical to systems which are composed of n slightly coupled oscillators. We suggest that since most higher-dimensional studies have concentrated on studying the latter models, the phenomena of PR and its higher-order degeneracies have not received its proper exposure.

We propose here several mechanisms for instabilities, which are persistent without dependence on external parameters in systems with $n \geqslant 4$ DOF; hence $n = 4$ is a critical n value. In particular, we demonstrate that 1-TPR is a unique mechanism which results in fast strong instabilities (the orbit reaches the boundaries of its energy surface in I_1 on relatively short timescales), and it persistently exists for $n \geqslant 4$ DOF systems. Orbits of near integrable 3, 4 and 5 DOF systems exhibiting PRs are numerically observed to exhibit complicated behaviour, which includes successive slides through bifurcations (similar to the ones observed in dynamical bifurcations, e.g. see [26]) with large variations in those actions which are involved in both the bifurcation and the resonance.

It is numerically demonstrated that orbits exhibiting 2-PR display relatively strong and fast instabilities in the direction of both resonant actions. In $n \geqslant 4$ DOF systems, this phenomenon is persistent and non-degenerate—it may occur on energy surfaces for which the KAM iso-energetically non-degeneracy condition does not fail anywhere.

In all Hamiltonian systems with $n > 2$ DOF, including those studied here, most initial conditions reside near *a priori* stable maximal tori. For such orbits the usual difficult questions regarding Arnold diffusion arise (for all the discussions regarding Arnold diffusion we assume the perturbed Hamiltonian to be analytic). Nonetheless, it is obvious that if there is any nontrivial structure to the unperturbed energy surface (e.g. see [29, 30]), specific regions in phase space may be subject to much faster instabilities than the exponentially slow Arnold diffusion. In all applications we are aware of, such non-trivial structures exist. The whiskers of families of *a priori* unstable tori supply one mechanism of instability in such systems. We demonstrated that the existence of parabolic resonant tori supplies another mechanism for fast phase space instabilities.

Finally, to some speculations. As for the *a priori* unstable situation, one may want to try and relate such mechanisms to the generic resonance web structure which appears near a generic point in phase space. Arnold diffusion mechanism relies on the creation of whiskered non-resonant $(n-1)$-tori (whiskers) by the perturbation, which in turn create the transition chains. However, only sufficiently non-resonant whiskers are preserved under small Hamiltonian perturbations, and in generic systems gaps are created. What structures correspond to the boundaries of the gaps? Since on one side of the gap boundary the tori are hyperbolic and on the other they are not (perhaps they disintegrate to smaller tori with different normal stabilities), it is quite possible that the boundary of the gaps is precisely given by families of parabolic tori, dense set of which is resonant. Furthermore, one expects that in the gaps the lower-dimensional tori will undergo generic bifurcations. For example, since the existence of action values for which parabolic 1-resonant $(n-2)$-tori are created is generic in the class $\mathcal{P}_2$, and is persistent for smooth integrable $n \geqslant 4$ DOF Hamiltonians (see appendix A), we expect such parabolic resonant $(n - 2)$-tori to exist in the gaps. The implications of such a scenario are vast. We have demonstrated numerically that orbits starting in the vicinity of parabolic resonant tori develop complicated behaviour and relatively fast instabilities. It follows that if the various mechanisms of instability associated with PR exist in the gaps and on their boundaries, they supply a much faster mechanism for transport than the instability through the whiskers. For example, recent works, such as [12, 20, 43] (see also references therein), showed that in *a priori* unstable systems, 'diffusion' in action variables exists in the gaps between the non-resonant whiskers (diffusion across resonances). In such systems, the relative instability rate in the gaps appears to be system dependent; in some examples it is faster than the instability rate along the non-resonant whiskers chain, while in other examples it is slower [12,20]; in [43] a transition chain of both non-resonant and resonant hyperbolic tori was constructed, and it was suggested there that the development of instabilities (diffusion) in the action variables is independent of KAM tori. These recent findings are not in contradiction (and may even be in accordance) to our suggestion that parabolic resonant tori may play an important role in traversing the gaps. However, this requires further investigation.

Appendix A. Genericity of lower-dimensional resonant tori

Consider smooth n DOF integrable Hamiltonian systems, with $n \geqslant 3$, which are already transformed to the form $H_0(x, y, I)$ (2.1) in some vicinity of one of their closed and connected level sets, M_g (as in the Liouville–Arnold theorem). We define below the *class* $\mathcal{P}_s$ of such systems as systems in which there exists a lower-dimensional torus which undergoes a change of stability as the actions are varied (namely truly coupled higher-dimensional systems). Then, we prove that among such systems those exhibiting PRs are generic.

Definition 1. *Let $H_0(x, y, I)$ be an n DOF integrable Hamiltonian, which is obtained from some general smooth n DOF integrable Hamiltonian by a non-singular transformation near some level set. The Hamiltonian is said to belong to the class $\mathcal{P}_s$ if $s \geqslant 1$ is the minimal integer for which the following conditions are satisfied:*

(a) $H_0(x, y, I)$ possesses at least one isolated equilibria in its $2s$ -dimensional (x, y) normal space of the $(n - s)$-tori, parameterized by the actions I, where $x = (x_1, \ldots, x_s)$, $y = (y_1, \ldots, y_s) \in \mathbb{R}^s$. Namely, at this equilibrium point p_f:

$$\nabla_{(x,y)} H_0(x, y, I)|_{p_f} = 0, \qquad p_f = (x_f, y_f, I_f). \tag{4.1}$$

(b) There exists an $i = i_0 \in \{1, \ldots, s\}$, for which

$$\det \left(\left. \frac{\partial^2 H_0}{\partial^2 (x_{i_0}, y_{i_0})} \right|_{p_f} \right) = 0. \tag{4.2}$$

(c) At the point p_f, $H_0(x, y, I)$ fulfils the following transversality condition, (4.3), for $i = i_0 \in \{1, \ldots, s\}$, for which condition (4.2) holds:

$$\left\| \frac{\partial}{\partial I_1} \det \left(\frac{\partial^2 H_0}{\partial^2 (x_{i_0}, y_{i_0})} \right) \Bigg|_{p_f} \right\| + \cdots + \left\| \frac{\partial}{\partial I_{n-s}} \det \left(\frac{\partial^2 H_0}{\partial^2 (x_{i_0}, y_{i_0})} \right) \Bigg|_{p_f} \right\| \neq 0. \tag{4.3}$$

Indeed, systems which belong to the class $\mathcal{P}_s$, attain an invariant $(n - s)$-torus which undergoes a bifurcation. Generic integrable Hamiltonian systems which do not belong to this class, may possess families of invariant $(n - s)$-tori (isolated in their normal (x, y) space) of only one stability type: either elliptic or hyperbolic (or, if $s > 1$, partially hyperbolic).

Appendix A.1. Maximal lower-dimensional tori

The *maximal* lower-dimensional tori are the $(n-1)$-tori of the integrable n DOF Hamiltonian system. For this case $s = 1$, and the (x, y) normal space is a two-dimensional plane.

Theorem 2. *Let $n, m \in \mathbb{N}$, where $n \geqslant 3$; $1 \leqslant m \leqslant n-2$, and $1 \leqslant r \leqslant \infty$. Smooth integrable n DOF Hamiltonian systems which possess an $(n-2-m)$-dimensional family of normally parabolic, m-resonant, $(n-1)$-tori, form a C^1-open set in the space of smooth integrable n DOF Hamiltonian systems, for each $n \geqslant 3$. Hence, the existence of such families is C^1-persistent in each such space. Each such normally parabolic $(n-1)$-torus generically consists of an m-dimensional family of non-resonant $(n-1-m)$-tori.*

Proof. To prove theorem 2 we first prove the following proposition.

Proposition 1. *Let $n, m \in \mathbb{N}$, where $n \geqslant 3$; $1 \leqslant m \leqslant n-2$, and $1 \leqslant r \leqslant \infty$. Smooth integrable n DOF Hamiltonian systems which possess an $(n-2-m)$-dimensional family of normally parabolic, m-resonant $(n-1)$-tori form a C^1-open, C^r-dense set in the class $\mathcal{P}_1$, hence are C^1-generic in this class.*

Remark 1. By a zero-dimensional family we mean an isolated invariant torus, or a discrete set of isolated invariant tori.

Proof of proposition. Let $H_0(x, y, I)$ be any integrable $n \geqslant 3$ DOF Hamiltonian from class $\mathcal{P}_1$. Denote by N (here $N = \mathbb{R}^{n+1}$) the $(n+1)$-dimensional smooth manifold, on which the x, y, I variables of the Hamiltonian function $H_0(x, y, I)$ are defined. Each point in N corresponds to an $(n-1)$-torus in the full phase space. If an $(n-1)$-torus is attached to a fixed point of the (x, y) plane, it is an invariant $(n-1)$-torus. The conditions for existence of an equilibrium of the (x, y)-system are (4.1). Since (4.1) defines two equations on the $n+1$ variables, (x, y, I), each of the two conditions in (4.1) is generically fulfilled on a smooth n-dimensional submanifold of N (this is a consequence of a corollary to the Sard theorem, e.g. see [24] and references therein). By the Transversality theorem [24], the n DOF integrable Hamiltonian systems for which these two smooth submanifolds of N are transverse form a C^1-*open*, C^r-*dense* set in the space of n DOF near integrable Hamiltonian systems (with $n \geqslant 2$). By the definition of the class $\mathcal{P}_1$ (definition 1), these two smooth n-dimensional submanifolds must intersect each other. Hence, they intersect each other transversely along an $(n-1)$-dimensional submanifold for a C^1-*open*, C^r-*dense* set of the class $\mathcal{P}_1$. If more than one fixed point exists in the (x, y)-plane, generically each such equilibrium corresponds to one branch of solutions $\{(x_f, y_f, I_f)\}$, each defining a smooth $(n-1)$-dimensional submanifold of N. For a generic integrable Hamiltonian system, the set of such $\{(x_f, y_f, I_f)\}$ branches of isolated equilibria in the (x, y)-plane is countable. Let us denote these smooth submanifolds by N_f^i, $i = 1, 2, \ldots$, or in short, N_f (when appropriate, with some abuse of notation, we will refer to N_f as the union of all the smooth submanifolds N_f^i) . Each such isolated equilibria in the (x, y)-plane corresponds to a family of invariant $(n-1)$-tori, parametrized by the actions I, in the $2n$-dimensional phase space.

Since condition (4.1) and all conditions hereafter are imposed on the Hamiltonian function and its partial derivatives, and not on the jets of the vector field, the standard transversality theorems may be invoked to obtain genericity results in the class $\mathcal{P}_1$. In particular, in this formulation the symplectic structure plays no role.

The generic normal stability type of the invariant $(n-1)$-tori is either hyperbolic or elliptic, and for some action values such a torus may be parabolic. The condition for a fixed point from N_f to be parabolic is

$$\det\left(\frac{\partial^2 H_0}{\partial^2(x,y)}\bigg|_{(x_p,y_p,I_p)}\right)=0. \tag{4.4}$$

The variables which fulfil the (instantaneous) parabolicity condition, (4.4), generically span an n-dimensional submanifold $P \subset N$. For each $i = 1, 2, \ldots$, by the Transversality theorem, the n DOF integrable Hamiltonian systems for which P is transverse to N_f in N form a C^1*-open* C^r*-dense* set in the class $\mathcal{P}_1$, for each fixed $n \geqslant 2$. By the assumptions on the class $\mathcal{P}_1$, P must intersect at least one of the submanifolds N_f, at least in one point. However, if P and N_f intersect, they intersect transversely on a C^1*-open*, C^r*-dense* set of Hamiltonians from the class $\mathcal{P}_1$. The definition of a transverse intersection (e.g. see [24]) implies that if $P \pitchfork N_f$ and $P \cap N_f \neq \emptyset$, then P must intersect N_f transversely along an $(n-2)$-dimensional submanifold of N. Let us denote this $(n-2)$-dimensional submanifold by PN_f. Note that PN_f corresponds to an $(n-2)$-dimensional family of parabolic $(n-1)$-tori. It follows that n DOF integrable Hamiltonian systems which possess an $(n-2)$-dimensional family of invariant normally parabolic $(n-1)$-tori form a C^1*-open* C^r*-dense* set in the class $\mathcal{P}_1$, for each fixed $n \geqslant 2$. Moreover, taking intersections over all $2 \leqslant n \in \mathbb{N}$, yields, by the Baire theorem, that n DOF integrable Hamiltonian systems possessing an $(n-2)$-dimensional family of normally parabolic $(n-1)$-tori form a C^r*-dense*, C^1-G_δ (dense) set in the class $\mathcal{P}_1$. Hence, the existence of normally parabolic $(n-1)$-tori is C^1-generic in this class.

The inner frequencies, $\omega_i(x_f, y_f, I_f)$; $i = 1, \ldots, n-1$, of a resonant invariant $(n-1)$-torus, $(x_f, y_f, I_f) \in N_f$, are rationally dependent; there exists a vector of integers, $q^j \in \mathbb{Z}^{n-1}\backslash\{0\}$, such that

$$\langle q^j, \omega(x_f, y_f, I_r)\rangle = 0, \qquad (x_f, y_f, I_r) \in N_f. \tag{4.5}$$

Equation (4.5) is generically satisfied on an $(n-2)$-dimensional submanifold of N_f, the resonant submanifold $R^{q^j}N_f$. This follows from transversality and the fact that the rational numbers form a dense set in the space of real numbers. Consider the intersection of m such resonant submanifolds: taking m independent vectors of integers, q^j, $j = 1, \ldots, m$, the corresponding m resonant $(n-2)$-dimensional submanifolds of N_f must all intersect (as all resonant surfaces pass through the origin of the frequency space, and the inner frequencies depend smoothly on the actions in a neighbourhood of the $(n-1)$-tori). Hence, by the Transversality theorem, they intersect transversely on an $(n-1-m)$-dimensional submanifold of N_f for a C^1*-open*, C^r*-dense* set of integrable Hamiltonians from the class $\mathcal{P}_1$. Let us denote this $(n-1-m)$-dimensional submanifold by R^mN_f. i.e.

$$R^mN_f = R^{q^1}N_f \cap \cdots \cap R^{q^m}N_f, \qquad 1 \leqslant m \leqslant n-2.$$

R^mN_f consists of m-resonant $(n-1)$-tori, which are generically hyperbolic or elliptic. Note that if $m = n-1$, then $R^{n-1}N_f$ is a discrete set of points, each corresponding to an $(n-1)$-resonant $(n-1)$-torus of fixed points, which is generically hyperbolic or elliptic, and resides on a certain energy surface, which is determined by the point $(x_f, y_f, I_r) \in R^{n-1}N_f$.

By the Transversality theorem, for each fixed $n \geqslant 3$, and for $1 \leqslant m \leqslant n-2$, those n DOF integrable Hamiltonians for which the $(n-1-m)$-dimensional surface R^mN_f of resonant tori is transverse to the $(n-2)$-dimensional surface PN_f of parabolic tori form a C^1*-open*, C^r*-dense* set in the class $\mathcal{P}_1$. However, as PN_f corresponds to a smooth $(n-2)$-dimensional family of parabolic tori in N_f, it contains a dense set of m-resonant parabolic tori, hence it intersects R^mN_f at least in one point. Therefore, by transversality, it must intersect R^mN_f along an

$(n-2-m)$-dimensional submanifold of N_f. Hence, the systems for which this intersection is transverse form a C^1-*open*, C^r-*dense* set in the class $\mathcal{P}_1$. Let us denote this $(n-2-m)$-dimensional submanifold by PR^m, where $PR^m \subset R^m N_f \subset N_f$. Taking intersections over all $n \geqslant 3$, by the Baire theorem, those n DOF integrable Hamiltonian systems for which PR^m is transverse form a C^r-*dense*, C^1-G_δ (dense) set in the class $\mathcal{P}_1$. Hence, they are C^1-*generic* in $\mathcal{P}_1$. □

As transverse intersections is a C^1-open condition, and the condition (4.3) is a transversality condition, it follows that systems from the class $\mathcal{P}_1$ for which the $(n-2-m)$-dimensional manifold, PR^m, of normally parabolic, m-resonant, $(n-1)$-tori is transverse form a C^1-*open* set in the space of n DOF integrable Hamiltonian systems for each $n \geqslant 3$, $1 \leqslant m \leqslant n-2$. The assertion that each such normally parabolic m-resonant $(n-1)$-torus consists of an m-dimensional family of non-resonant, $(n-1-m)$-tori follows from Arnold [2]. Hence, we have established theorem 2. □

Consider a small Hamiltonian perturbation of $H_0(x, y, I)$, $\varepsilon H_1(x, y, \theta, I; \varepsilon)$, with ε small; this perturbation may be considered as the higher-order terms of the original Hamiltonian. The effect of such a perturbation clearly depends on the structure of the integrable flow, as numerically demonstrated in section 3. We define m-PR in the following way (see also [40, 32, 29]).

Definition 2. *m-PR occurs when a small Hamiltonian perturbation $\varepsilon H_1(x, y, \theta, I; \varepsilon)$ is applied to an integrable n DOF Hamiltonian system possessing an m-resonant lower-dimensional normally parabolic torus.*

A m-PR occurs for a set of initial conditions of the perturbed system in the vicinity of the previously existing normally parabolic resonant torus of the integrable system. A near integrable Hamiltonian system is said to belong to the class $\mathcal{P}_s$ if its integrable part is of this class. Hence, considering smooth n DOF near integrable Hamiltonian systems of the form

$$H(x, y, \theta, I; \varepsilon) = H_0(x, y, I) + \varepsilon H_1(x, y, \theta, I; \varepsilon), \tag{4.6}$$

where $\varepsilon H_1(x, y, \theta, I; \varepsilon)$ is a bounded Hamiltonian perturbation of the integrable Hamiltonian $H_0(x, y, I)$, we have the following corollary.

Corollary 2. *Let $n, m \in \mathbb{N}$, where $n \geqslant 3$; $1 \leqslant m \leqslant n-2$, and $1 \leqslant r \leqslant \infty$. Smooth near integrable n DOF Hamiltonian systems of the form (4.6), exhibiting m-PR, form a C^1-open, C^r-dense set in the class $\mathcal{P}_1$, and a C^1-open set in the space of smooth near integrable n DOF Hamiltonian systems, for each fixed $n \geqslant 3$. Hence, the existence of a m-PR is C^1-generic in the class $\mathcal{P}_1$ and is C^1-persistent in the space of smooth near integrable n DOF Hamiltonians, for each $n \geqslant 3$.*

Hereafter, by generic we mean C^1-generic, and by persistent we mean C^1-persistent.

In fact, we have established above the following theorem.

Theorem 3. *Let $3 \leqslant n \in \mathbb{N}$ and $1 \leqslant r \leqslant \infty$.*

(a) Let $1 \leqslant m \leqslant n-2$. Smooth integrable n DOF Hamiltonian systems which possess $(n-1-m)$-dimensional families of both normally hyperbolic and normally elliptic (bifurcating), m-resonant, $(n-1)$-tori, form a C^1-open, C^r-dense set in the class $\mathcal{P}_1$, hence generic in this class. Such systems are persistent in the space of smooth integrable n DOF Hamiltonian systems for each $n \geqslant 3$.

(b) Let $m = n-1$. Smooth integrable n DOF Hamiltonian systems which possess a normally hyperbolic (respectively, normally elliptic), $(n-1)$-resonant, $(n-1)$-torus of fixed points, form a C^1-open, C^r-dense set in the class $\mathcal{P}_1$, hence generic in this class. Such systems

are persistent in the space of smooth integrable n DOF Hamiltonian systems for each $n \geqslant 3$.

For $1 \leqslant m \leqslant n-2$, each such normally hyperbolic (respectively, elliptic) $(n-1)$-torus consists of an m-dimensional family of $(n-1-m)$-tori, where generically these are non-resonant.

We define *hyperbolic (elliptic) m-resonance* in the same manner as m-PR (see [40, 21] for the 2 DOF case).

Definition 3. *Hyperbolic (elliptic) m-resonance occurs when a small Hamiltonian perturbation of the form* $\varepsilon H_1(x, y, \theta, I; \varepsilon)$ *is applied to an integrable n DOF Hamiltonian system possessing an m-resonant lower-dimensional normally hyperbolic (elliptic) torus.*

Hence, considering smooth near integrable n DOF Hamiltonian systems of the form (4.6), with a bounded perturbation, we have the following corollary.

Corollary 3. *Let* $3 \leqslant n \in \mathbb{N}$ *and* $1 \leqslant r \leqslant \infty$.

(a) Let $1 \leqslant m \leqslant n-2$. *Smooth near integrable n DOF Hamiltonian systems of the form (4.6), exhibiting both hyperbolic and elliptic m-resonance, form a* C^1*-open,* C^r*-dense set in the class* $\mathcal{P}_1$*, hence generic in this class. Such systems are persistent in the space of smooth near integrable n DOF Hamiltonian systems for each* $n \geqslant 3$.

(b) Let $m = n-1$. *Smooth near integrable n DOF Hamiltonian systems exhibiting hyperbolic (respectively, elliptic)* $(n-1)$*-resonance, form a* C^1*-open,* C^r*-dense set in the class* $\mathcal{P}_1$*, hence generic in this class. Such systems are persistent in the space of smooth near integrable n DOF Hamiltonian systems for each* $n \geqslant 3$.

The elliptic m-resonance occurs for a set of initial conditions of the perturbed system which are in the vicinity of the previously existing normally elliptic resonant torus. The hyperbolic m-resonance occurs for a set of initial conditions of the perturbed system which are in the vicinity of the previously existing normally hyperbolic resonant torus and its intersecting stable and unstable manifolds. The spatial extent of these manifolds in these higher-dimensional systems is the subject of current research (see, for example, [20, 43] and references therein).

Theorem 2, proposition 1, corollary 2 and the first part of theorem 3 and of corollary 3 apply only for the case $n \geqslant 3$, while the second part of theorem 3 and of corollary 3 apply for the case $n = 2$ as well.

In theorems 2, 3, proposition 1 and corollaries 2, 3, we prove that the existence of resonant tori of the above prescribed properties are generic and persistent—the actions of the unperturbed $(n-1)$-tori serve as internal parameters. Next we prove that the existence of $(n-1)$-resonant normally parabolic $(n-1)$-tori of fixed points is a co-dimension one phenomenon, namely it requires one additional external parameter.

Consider one parameter families of integrable n DOF Hamiltonian systems with Hamiltonians of the form: $H_0(x, y, I; \mu)$, where $\mu \in \mathbb{R}$ and all the assumptions stated in sections 1 and 2, regarding smoothness, manifolds and constants of motion, remain the same. The class $\mathcal{P}_s^*$ is defined in this case (correspondingly to definition 1) as the class of one parameter families of integrable n DOF Hamiltonians of the form $H_0(x, y, I; \mu)$, satisfying the conditions of definition 1 with $s = 1$, where the transversality condition in this case is

$$\left\| \frac{\partial}{\partial \mu} \det\left(\frac{\partial^2 H_0}{\partial^2(x, y)}\right)\bigg|_{p_f} \right\| + \left\| \frac{\partial}{\partial I_1} \det\left(\frac{\partial^2 H_0}{\partial^2(x, y)}\right)\bigg|_{p_f} \right\| + \cdots + \left\| \frac{\partial}{\partial I_{n-1}} \det\left(\frac{\partial^2 H_0}{\partial^2(x, y)}\right)\bigg|_{p_f} \right\| \neq 0. \tag{4.7}$$

Denote by $N \subseteq \mathbb{R}^{n+2}$ the $(n+2)$-dimensional submanifold of M, spanned by the x, y, I, μ variables and parameter of the Hamiltonian. Then, N_f is an n-dimensional submanifold, P is an $(n+1)$-dimensional submanifold and PN_f is an $(n-1)$-dimensional submanifold. The resonance submanifolds, $R^{q^j}N_f$, are each of dimension $(n-1)$, and the submanifold $R^m N_f$, which corresponds to transverse intersections of m-independent resonance conditions in N_f, is $(n-m)$-dimensional. The submanifold PR^{n-2}, which corresponds to normally parabolic $(n-2)$-resonant $(n-1)$-tori, is one-dimensional. Denote by q^{n-1} an $(n-1)$-dimensional vector of integers which is linearly independent of the $(n-2)$-independent $(n-1)$-dimensional vectors of integers used to construct PR^{n-2}.

Remark 2. By symplectic change of coordinates, the set of $(n-1)$-independent $(n-1)$-dimensional vectors of integers, $q^1, \ldots, q^{n-1}$, may be chosen as a set of $(n-1)$ unit vectors (i.e. $q^j = e_j^{n-1}$). Then each independent resonance condition of the form (4.5) corresponds to vanishing of the jth frequency of the $(n-1)$-torus.

The resonance condition (4.5) for q^{n-1} is generically met on an $(n-1)$-dimensional submanifold, $R^{q^{n-1}}N_f \subset N_f$. Using the same reasoning as above, the two submanifolds of N_f, PR^{n-2} (a one-dimensional curve) and $R^{q^{n-1}}N_f$ (a co-dimension one submanifold of N_f), intersect transversely in a C^1-*open*, C^r-*dense* set of the class $\mathcal{P}_s^*$, and for a C^1-*open* set of one parameter families of integrable n DOF Hamiltonian systems, for each $n \geqslant 3$. This transverse intersection occurs on a set of isolated points, each corresponding to a normally parabolic $(n-1)$-resonant $(n-1)$-torus of fixed points. Hence, we have established the following theorem.

Theorem 4. *Let* $3 \leqslant n \in \mathbb{N}$, $\mu \in \mathbb{R}$ *and* $1 \leqslant r \leqslant \infty$. *The existence of one parameter families of smooth integrable n DOF Hamiltonian systems which possess a normally parabolic* $(n-1)$*-resonant* $(n-1)$*-torus of fixed points is generic in the class* $\mathcal{P}_s^*$*, and persistent in the space of smooth integrable n DOF Hamiltonian systems depending on one parameter, for each* $n \geqslant 3$.

Consider smooth near integrable systems of the form

$$H(x, y, \theta, I; \varepsilon, \mu) = H_0(x, y, I; \mu) + \varepsilon H_1(x, y, \theta, I; \varepsilon, \mu), \tag{4.8}$$

where the Hamiltonian perturbation $\varepsilon H_1(x, y, \theta, I; \varepsilon, \mu)$ is bounded. A near integrable system of the form (4.8) is said to be of class $\mathcal{P}_s^*$ if its integrable part is of this class. Then, we have the following corollary.

Corollary 4. *Let* $3 \leqslant n \in \mathbb{N}$, $\mu \in \mathbb{R}$ *and* $1 \leqslant r \leqslant \infty$. *The existence of smooth near integrable n DOF Hamiltonian systems of the form (4.8), exhibiting parabolic* $(n-1)$*-resonance, is generic in the class* $\mathcal{P}_s^*$*, and persistent in the space of n DOF smooth near integrable Hamiltonian systems depending on one parameter, for each* $n \geqslant 3$.

Theorem 4 and corollary 4 apply for the case $n = 2$ as well.

Remark 3. In the theorems and corollaries above the actions (or constants of motion), I, of the integrable Hamiltonian serve as internal parameters, and govern the stability type of the lower-dimensional invariant tori. Hence, (4.7) is a necessary condition for a given integrable Hamiltonian system of the form $H_0(x, y, I; \mu)$ to possess a normally parabolic resonant torus. Generally, systems which conserve angular momentum satisfy (4.7) yet separable Hamiltonian systems (e.g. systems of the form $H_0 = H_{0xy}(x, y) + \sum H_i(I_i; \mu)$) do not, see [40] for details (there only the 2 DOF case is discussed, but the same ideas apply to the higher-dimensional cases).

Appendix A.2. The general case of lower-dimensional tori: $s > 1$

When $2 \leqslant s \leqslant n-2$, an invariant $(n-s)$-torus of an integrable n DOF system is still considered normally parabolic if (at least) one pair of its normal frequencies vanishes. When $s > 1$, an invariant $(n-s)$-torus, with one vanishing normal frequency, has additional $s-1$ pairs of characteristic eigenvalues which may correspond to several possible stability types (elliptic, generically real hyperbolic, complex hyperbolic or parabolic). Analogous statements to theorems and corollaries 2–4, may be formulated, e.g. the following one.

Theorem 5. *Let $n, s, m \in \mathbb{N}$, where $n \geqslant 4$, $2 \leqslant s \leqslant n-2$, $1 \leqslant m \leqslant n-s-1$, and $1 \leqslant r \leqslant \infty$. Smooth integrable n DOF Hamiltonian systems which possess an $(n-s-m-1)$-dimensional family of normally parabolic, m-resonant, $(n-s)$-tori, form a C^1-open, C^r-dense set in the class $\mathcal{P}_s$, and a C^1-open set in the space of smooth integrable n DOF Hamiltonian systems, for each fixed $n \geqslant 3$. Hence, their existence is generic in the class $\mathcal{P}_s$, and persistent in the space of smooth integrable Hamiltonian systems, for each fixed $n \geqslant 3$. Each such normally parabolic $(n-s)$-torus consists of an m-dimensional family of $(n-s-m)$-tori.*

For example, it follows that $n = 4$ appears as a critical dimension for which n DOF integrable Hamiltonian systems first possess a 1-resonant normally parabolic 2-torus persistently.

The behaviour of perturbed orbits near a resonant $(n-s)$-torus might be quite different than the behaviour of perturbed orbits that are studied numerically in sections 3 and deserves a separate study. Hereafter we consider the case $s = 1$.

Appendix B. Underlying degeneracies and critical n values

The existence of families of iso-energetic lower-dimensional resonant tori on one hand, and degenerate PR on the other, may both induce instabilities in the perturbed system, as demonstrated in section 3. Below we prove that the appearance of both is persistent.

Appendixc B.1. Iso-energetic resonances

The appearance of a family of resonant tori on the same energy surface is one possible source of large instabilities in the perturbed flow, as postulated by Arnold. Families of lower-dimensional tori appear on the boundary or singular folds of the energy surface, and are intersected by the resonance planes in the same fashion that the energy surface is intersected by them, with one additional co-dimension. Indeed, for a given $n \geqslant 3$ and $1 \leqslant m \leqslant n-1$ values, an l-dimensional family $(1 \leqslant l \leqslant n-m-1)$ of m-resonant $(n-1)$-tori of the same stability type generically intersects transversely the n-dimensional energy surface

$$H_0(x, y, I) - h = 0, \tag{4.9}$$

where h is some fixed energy value, along an $(l-1)$-dimensional manifold (by the same transversality arguments of the previous section). For $l \geqslant 2$ this leads to the non-trivial observation that families of tori of specific properties (e.g. m-resonant parabolic or hyperbolic) reside on the same energy surface. We have thus established that such a mechanism exists persistently for sufficiently large systems.

Corollary 5. *For $n \geqslant 4$, the existence of an $(n-m-2)$-dimensional family of normally hyperbolic or elliptic, m-resonant, $(n-1)$-tori on a given energy surface is persistent for $0 \leqslant m \leqslant n-3$. For $n = 3$ the existence of only isolated iso-energetic 1-resonant, 2-tori is persistent.*

Corollary 6. *For $n \geqslant 5$, the existence of an $(n - m - 3)$-dimensional family of normally parabolic, m-resonant $(n - 1)$-tori on a given energy surface is persistent for $0 \leqslant m \leqslant n - 4$. For $n = 3, 4$ the existence of only isolated iso-energetic parabolic 1-resonant, $(n - 1)$-tori is persistent.*

Remark 4. In the non-resonant case ($m = 0$), the statements in the above two corollaries apply to $n \geqslant 3$ and $n \geqslant 4$, respectively.

The first corollary is a result from the realization that the critical dimension for having the resonance web on the energy surface boundaries or singular folds is larger by one from the critical dimension for having a resonance web on the energy surface itself. Similar realization regarding the boundary of the singular fold surface (a co-dimension-three surface) implies the second corollary.

Appendix B.2. Tangent resonances

Consider the n-dimensional space (H_0, I) on which energy surfaces are presented as the hyper-planes $H_0 = h$. The family of lower-dimensional invariant tori N_f is presented in this space as a co-dimension-one smooth surface. Notice that N_f is tangent to the energy surfaces containing lower-dimensional resonant tori, at these tori, $R^m N_f^h$ (namely at the m-resonant lower-dimensional tori with energy h) along the m-resonant directions: e.g. let us assume that the coordinates (x, y, I) are chosen so that $\dot{\theta}_j = \partial H_0/\partial I_j|_{p_{fr}} = 0$; $j = 1, \ldots, m$ (see remark 2), where $p_{fr} \in R^m N_f^h$, and $PN_f = \{(x, y, I)|x = y = I_1 = 0\}$. Then, since on N_f, $\mathrm{d}H_0/\mathrm{d}I_j|_{p_f} = \partial H_0/\partial I_j|_{p_f}$ (by (2.2), and assuming that N_f may be expressed as a graph on the action variables), it follows that N_f is tangent to the energy surface in the directions $I_1, \ldots, I_m$ at p_{fr}. If this tangency occurs at the parabolic resonant torus $p_{pr} \in PR^{mh} \subset R^m N_f^h$, and if it is of *higher order*, we expect stronger instability under perturbation. To leading order this occurs along the I_j direction when the next order derivative at p_{pr} vanishes:

$$\frac{\mathrm{d}}{\mathrm{d}I_j}\left(\frac{\partial H_0(x_f(I_{fr}), y_f(I_{fr}), I_{fr}; \mu_{fr})}{\partial I_j}\right)\bigg|_{(x_{pr}, y_{pr}, I_{pr}; \mu_{pr})} = 0, \qquad \text{for some } j \in \{1, \ldots, m\}, \tag{4.10}$$

where $\{(x_f(I_{fr}), y_f(I_{fr}), I_{fr}; \mu_{fr})\} \subset R^m N_f$ denotes the resonant family of tori belonging to a smooth branch N_f^i of N_f which contains the point p_{pr} (corresponding to a parabolic m-resonant $(n - 1)$-torus). In this case an infinitesimal family of m-resonant tori (elliptic or/and hyperbolic) emanating from the parabolic torus belongs to the same energy surface. We thus define the following definition.

Definition 4. *A m-TPR, in the direction of the action I_j (for some $j = 1, \ldots, m$), occurs when a small Hamiltonian perturbation of the form $\varepsilon H_1(x, y, \theta, I; \mu, \varepsilon)$ is applied to an integrable Hamiltonian system, which possesses an energy surface on which an infinitesimal family of resonant tori emanating from a parabolic m-resonant torus exists in the direction of I_j.*

Condition (4.10) is set for C^∞ functions of the form $H_0(x, y, I; \mu)$. Furthermore, the arguments are smooth functions on each smooth branch of N_f, N_f^i. Hence, from the Sard theorem it follows that condition (4.10) is satisfied for almost all systems in the C^1-*open*, C^r-*dense* set of n DOF integrable Hamiltonian systems from the class $\mathcal{P}_s^*$, which possess an $(n - 2 - m + p)$-dimensional family of parabolic m-resonant $(n - 1)$-tori (cf theorem 2), with $n \geqslant m + 3 - p$, $1 \leqslant m \leqslant n - 1$ (i.e. on an C^1-*open* subset of systems of full Lebesgue measure). If $p = 0$ and $n = m + 3$, each point from the discrete set of isolated points, $p_{tpr} = (x_{tpr}, y_{tpr}, I_{tpr}) \in PR^m$, which solves equation (4.10), defines an energy surface,

$H_0(x, y, I) = h_{tpr}$, on which an infinitesimal family of resonant tori, containing a parabolic m-resonant torus, exists. Hence, by theorem 2, we have established[10].

Theorem 6. *Let $m \geqslant 1$, $n = m + 3$. Smooth integrable n DOF Hamiltonian systems which possess, in the direction of one of the actions, an infinitesimal family of elliptic and/or hyperbolic resonant $(n - 1)$-tori, containing a parabolic m-resonant $(n - 1)$-torus, p_{tpr}, on some energy surface $H_0(x, y, I) = h_{tpr}$ (defined by the point p_{tpr}), are C^1-persistent in the space of smooth integrable n DOF Hamiltonian systems, for each $m \geqslant 1$.*

If $n+p \geqslant m+4$, the TPR occurs on each energy surface in an open set of energy values; the condition $H_0(x, y, I; \mu)-h_{pr} = 0$ is generically satisfied on some $(n+p)$-dimensional energy surface, which generically intersects transversely the $(n - m - 2 + p)$-dimensional manifold PR^m along an $(n - m - 3 + p)$-dimensional submanifold, HPR^m (by the same reasoning as given in the proofs above) for a C^1-*open* set of integrable n DOF Hamiltonian systems. Then, condition (4.10) is satisfied on a subset of full measure of the C^1-*open* set of integrable n DOF Hamiltonian systems, for which HPR^m is transverse. Hence, we have established the following theorem.

Theorem 7. *Let $m \geqslant 1$, $n \geqslant m + 4$. Integrable n DOF Hamiltonian systems which possess in the direction of one of the actions an $(n - m - 4)$-dimensional family of infinitesimal families of elliptic and/or hyperbolic resonant $(n-1)$-tori, containing a parabolic m-resonant $(n - 1)$-torus, on each energy surface belonging to an open set of energy values are C^1-persistent in the space of n DOF smooth integrable Hamiltonian systems for each $n \geqslant m+4$.*

Theorem 8. *Let $n \geqslant 3$, $\mu \in \mathbb{R}^2$ (respectively, $\mu \in \mathbb{R}$). The existence of two (respectively, one) parameter families of integrable n DOF Hamiltonian systems which possess in the direction of one of the actions, an infinitesimal family of elliptic and/or hyperbolic resonant $(n - 1)$-tori, containing a parabolic $(n - 1)$-torus of fixed points (respectively, consisting of a family of invariant circles), on the energy surface $H_0(x, y, I; \mu) = H_0(x_{tpr}, y_{tpr}, I_{tpr}; \mu_{tpr}) = h_{tpr}$, is C^1-persistent in the space of smooth integrable n DOF Hamiltonian systems, for each $n \geqslant 3$.*

Now, consider a small Hamiltonian perturbation of the form: $\varepsilon H_1(x, y, I; \varepsilon, \mu)$.

Corollary 7. *Let $n \geqslant 4$, $1 \leqslant m \leqslant n - 3$. The existence of a m-TPR is C^1-persistent in the space of smooth near integrable n DOF Hamiltonian systems, for each $n \geqslant 4$.*

Corollary 8. *Let $n \geqslant 3$, $m \in \{n - 1, n - 2\}$,*

$$p = \begin{cases} 1, & m = n - 2, \\ 2, & m = n - 1. \end{cases}$$

The existence of a m-TPR is C^1-persistent in p parameter families of smooth near integrable n DOF Hamiltonian systems, for each $n \geqslant 3$.

It follows that $n = 4$ is the smallest number of DOF for which the existence of 1-TPR is persistent without assuming dependence of the system on external parameters. For $n = 4 + l$, $l \geqslant 1$, additional l conditions guarantying tangency of order $l + 2$ may be imposed:

$$\frac{\mathrm{d}^{k+1}}{\mathrm{d}I_j^{k+1}} \left(\frac{\partial H_0(x_f(I_{fr}), y_f(I_{fr}), I_{fr}; \mu_{fr})}{\partial I_j} \right)\bigg|_{(x_{pr}, y_{pr}, I_{pr}; \mu_{pr})} = 0, \qquad k = 1, \ldots, l, \quad j \in \{1, \ldots, m\}. \tag{4.11}$$

[10] Note that theorem 6 and corollary 7 are identical to theorem 1 and corollary 1 in section 3, and are stated here again for the convenience of the reader.

These conditions are satisfied on a C^1*-open* set in the space of integrable $4+l$ DOF Hamiltonian systems, which persistently possess a parabolic 1-resonant $(3+l)$-torus. As l grows, higher-order terms of $\dot{\theta}_j$ along some branch N_f^i vanish, causing a 'more flat' situation, which we denote by '$(l+1)$-*flat*'. See section 3 for description of the corresponding numerical results. If condition (4.11) is satisfied for all positive k values ($l = \infty$), we say that the perturbed system has a *flat PR*.

Theorem 9. *Let $m = 1$, $l \geqslant 1$, $n = 4 + l$. Smooth integrable n DOF Hamiltonian systems which possess, in the direction of one of the actions, an $(l + 1)$-flat family of elliptic and/or hyperbolic resonant $(n - 1)$-tori, containing a parabolic m-resonant $(n - 1)$-torus, on some energy surface, $H_0(x, y, I) = h_{fpr}$, are C^1-persistent in the space of smooth integrable $4 + l$ DOF Hamiltonian systems, for each $l \geqslant 1$.*

Corollary 9. *Let $n \geqslant 5$, $m = 1$, $l \geqslant 1$. The existence of an $(l+1)$-flat-TPR is C^1-persistent in the space of smooth near integrable n DOF Hamiltonian systems, with $n = 4+l$, for each $l \geqslant 1$.*

Higher-order tangencies at p_{pr} along all directions of $R^m N_f$ may be similarly formulated. However, such a situation is obviously of non-zero co-dimension for all n values.

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