# Model Reduction for Linear Systems

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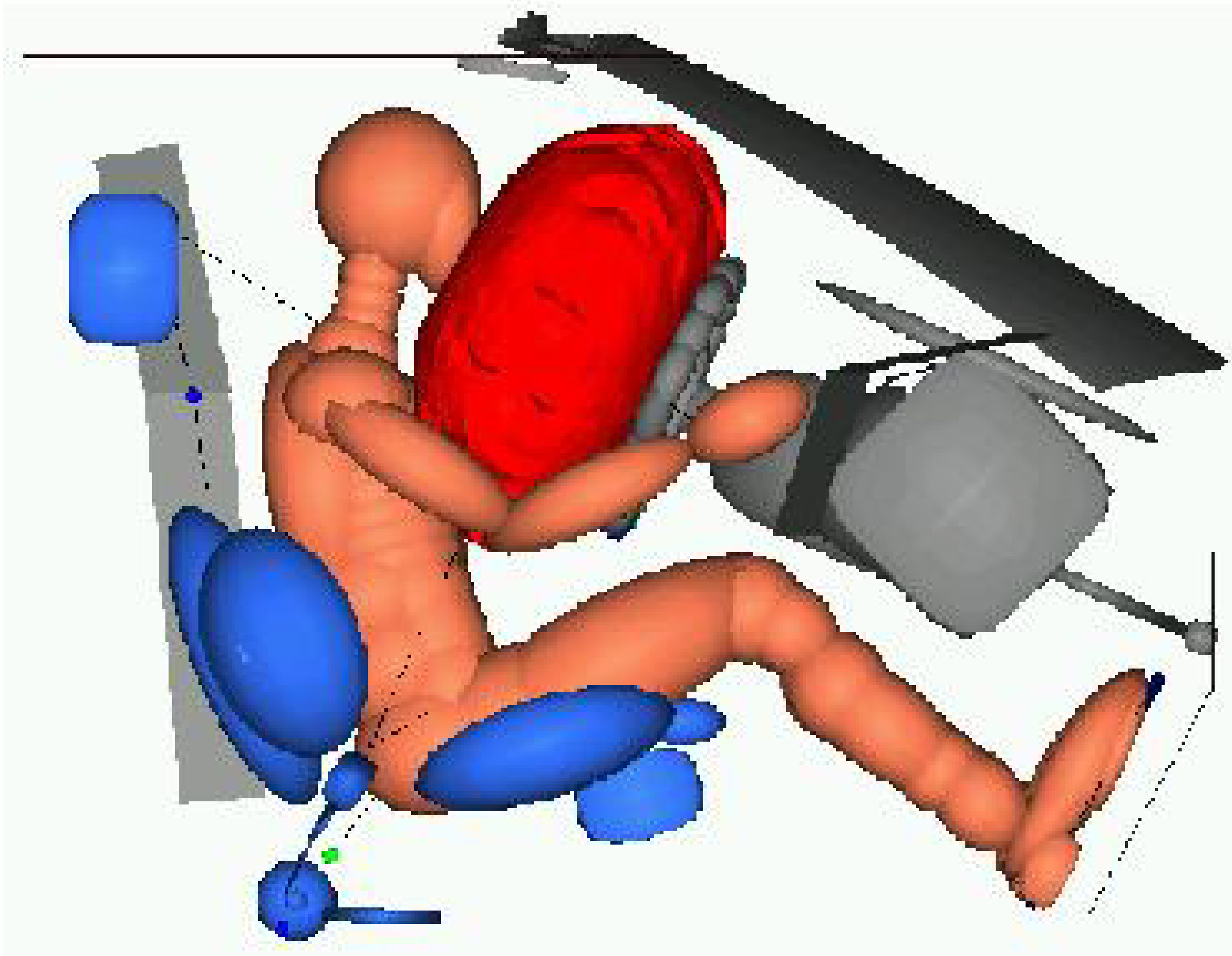
# Contents

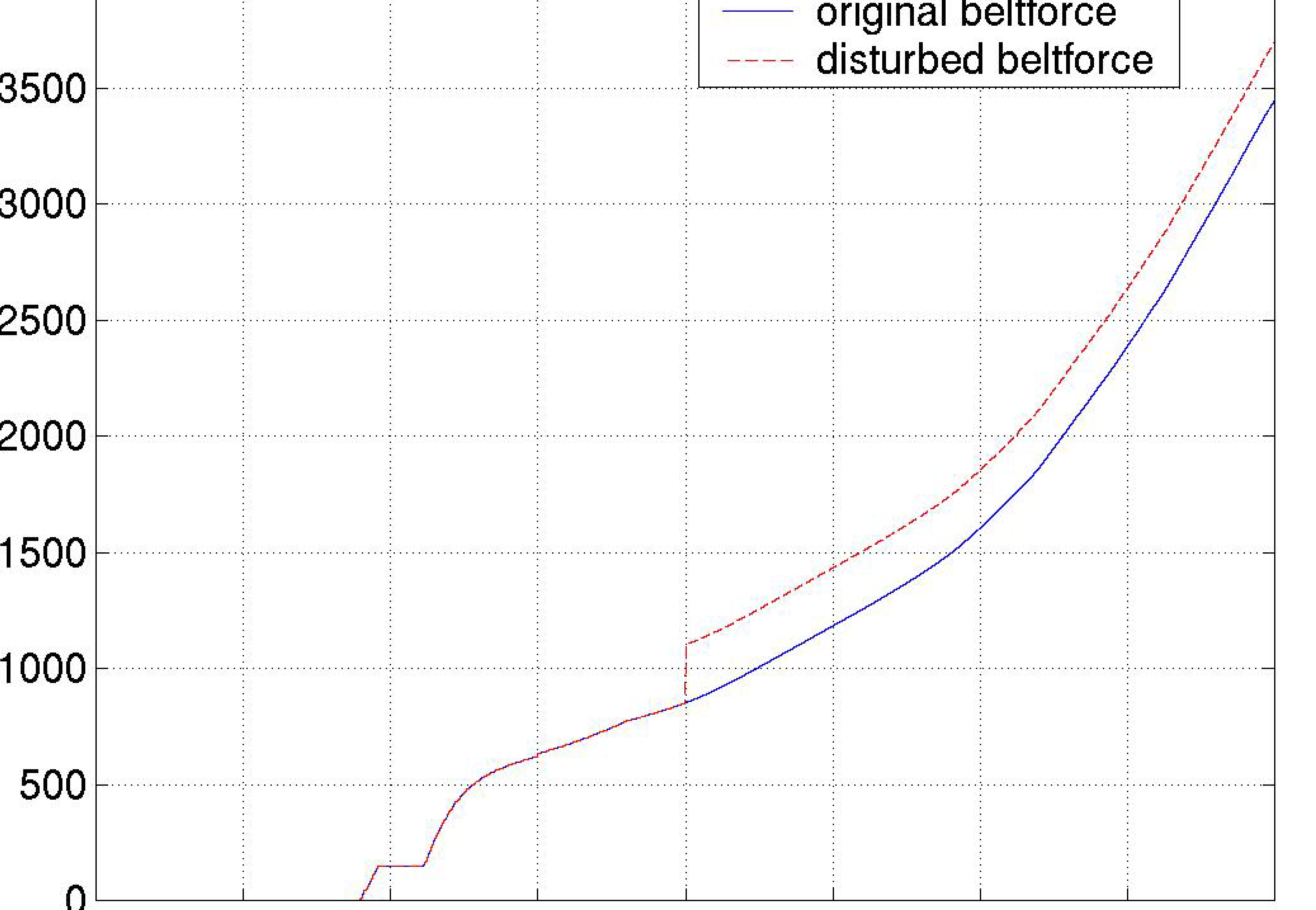
- reduction as part of control design process
- broad scope
- linear systems
- reduction based on projections
  - modal, balancing, weighting
- conclusions

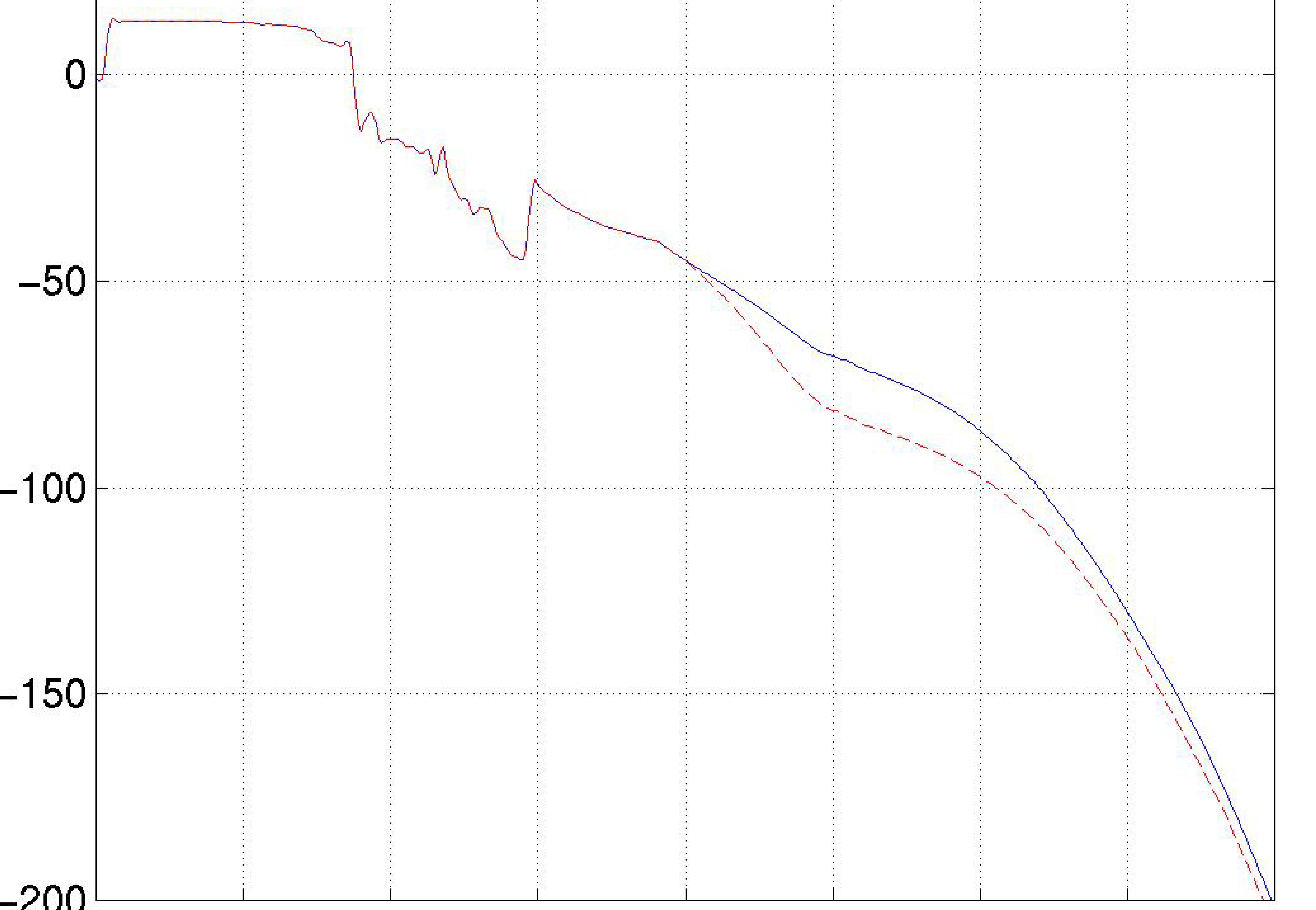
# Control Paradigm

- modelling plant, specs and disturbances
- robust control design problem
- iterative reduction - controller design
- use Matlab tools during design

# Automotive Safety Restraint Systems




original beltforce
disturbed beltforce
3500
3000
2500
2000
1500
1000
500
0


0
−50
−100
−150
−200

# Compact Disc Player

- FEM, reduction down to 60 modes (120 states)
- I/O behaviour?
- Control relevant reduction down to 9 states

# Model Reduction Problem

Given a I/O map $S_{n,}$ with 'complexity' n

find a $S_r$ with r << n such that

$|| S_n - S_r ||$ 'small'

# Control oriented motivation

- complexcity of controller = complexcity of design model
- computational aspects

## Requirements

- Preserve **relevant** properties (dynamics, stability)
- computational attractive
- error bounds?
- I/O oriented

# Model Reduction possibilities

- Identification (black-box approach)
- Model-based:
  - moments matching (Pade, moments, Volterra)
  - optimization based (H2/H$\propto$, NP-hard)
  - projection based:
    - modal truncation
    - balancing
    - SVD approaches (extension to nonlinear)
    - optimal Hankel Norm approximation

# Linear Systems

$$\dot{x} = Ax + Bu$$

$$y = Cx$$

$$S_n = \begin{pmatrix} A & B \\ C & 0 \end{pmatrix} \in R^{(n+m)\times(n+l)}$$

Problem: find approximation with state dimension r << n:

$$S_r = \begin{pmatrix} A_r & B_r \\ C_r & 0 \end{pmatrix} \in R^{(r+m)\times(r+l)}$$

# State transformation

define state transformation $z = Tx$,

with T $n \times n$ nonsingular then

$$\dot{z} = TAT^{-1}x + TBu$$

$$y = CT^{-1}x$$

$$\overline{A} = TAT^{-1} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \overline{B} = TB = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$$

$$\overline{C} = CT^{-1} = \begin{bmatrix} C_1 & C_2 \end{bmatrix}$$

# Reduced Model

$$S_r = \begin{bmatrix} A_{11} & B_1 \\ C_1 & 0 \end{bmatrix}$$ corresponding to states $z_1$

Alternative: singular perturbation:

Alternative: singular perturbation:

$$\dot{z}_1 = A_{11} z_1 + A_{12} z_2 + B_1 u$$

$$0 = A_{21} z_1 + A_{22} z_2 + B_2 u, \quad \text{so}$$

$$z_2 = -A_{22}^{-1} A_{21} z_1 - A_{22}^{-1} B_2 u$$

then

$$S_r = \begin{bmatrix} (A_{11} - A_{12} A_{22}^{-1} A_{21}) & (B_1 - A_{12} A_{22}^{-1} B_2) \\ (C_1 - C_2 A_{22}^{-1} A_{21}) & -C_2 A_{22}^{-1} B_2 \end{bmatrix}$$

# Question: Which T???

- Modal reduction: Eigenvalue decomposition
- Balanced reduction: Balancing transformation

# Modal Truncation

$T$ eigenvectormatrix of $A$, then

$$\bar{A} = TAT^{-1} = \left[\begin{array}{c|c} \Lambda_{11} & 0 \\ \hline 0 & \Lambda_{22} \end{array}\right] = \left[\begin{array}{ccc|cc} \lambda_1 & & & & \\ & \lambda_2 & & & \\ & & \cdot & & \\ \hline & & & \cdot & \\ & & & & \lambda_n \end{array}\right]$$

with

$$0 \rangle \lambda_1 \rangle \ldots \rangle \lambda_n$$

# Balancing

$$h(t) = Ce^{At}B, t \geq 0$$ impulsresponse

$$x(t) = e^{At}B\delta(t)$$ input to state map

$$y(t) = Ce^{At}x(0)$$ state to output map

*Grammians*

$$P = \int_0^\infty e^{At}BB^T e^{A^T t}dt$$ controllability

$$Q = \int_0^\infty e^{A^T t}C^T Ce^{At}dt$$ observability

$z = Tx,$ then

$$\overline{P} = TPT^T$$

$$\overline{Q} = T^{-T} Q T^{-1}$$

$$\lambda_{\mathrm{i}}(\overline{P}\,\overline{Q}) = \lambda_{\mathrm{i}}(TPT^T T^{-T} Q T^{-1}) = \lambda_{\mathrm{i}}(TPQT^{-1}) = \lambda_{\mathrm{i}}(PQ)$$

are I/O invariants,

$\sigma_i = \sqrt{\lambda_{\mathrm{i}}(PQ)}$ are called Hankel Singular Values the Hankel Singular Values $\sigma_i(S)$ of $S = \begin{bmatrix} A & B \\ C & 0 \end{bmatrix}$

$S$ stable, are computed by solving Lyapunov Eqs :

$$AP + PA^T + BB^T = 0$$

$$A^T Q + QA + C^T C = 0$$

let $P = UU^T$, and $Q = LL^T$

compute SVD $\quad U^T L = W\Sigma^2 V^T$

and take $\quad T = \Sigma^{1/2} W U^{-1} = \Sigma^{-1/2} V^T L^T$

then after balancing:

$$\bar{P} = \bar{Q} = \Sigma = diag(\sigma_1, \ldots, \sigma_n)$$

# Balanced Truncation

$$S_r = \begin{bmatrix} A_{11} & B_1 \\ C_1 & 0 \end{bmatrix}, \quad \text{according to } \Sigma = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix}$$

## properties

$$S_r = \begin{bmatrix} A_{11} & B_1 \\ C_1 & 0 \end{bmatrix}$$

and

$$A_{11}\Sigma_1 + \Sigma_1 A_{11}^T + B_1 B_1^T = 0$$

$$\text{i.e. stable, HSV } \sigma_1..\sigma_r$$

$$\text{error bound:}$$

$$\left\| S_n - S_r \right\|_\infty \leq 2(\sigma_{r+1} + ... + \sigma_n)$$

# extensions

- Frequency weighting
- Closed-loop balancing
- LQG balancing
- Stochastic balancing
- uncertainty reduction

# Final Statements

- Balancing is beautiful!
- Error bounds are not used
- Iteration is key
- Reduction is like control: it is all about design!