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Slow Motion Manifolds Far from the Attractor in Multistable Reaction–Diffusion Equations

João T. Pinto[1]

Center for Dynamical Systems and Nonlinear Studies, Georgia Institute of Technology, Atlanta, Georgia 30332; and Instituto Superior Técnico, 1096 Lisbon, Portugal
E-mail: jpinto@math.ist.utl.pt

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We consider a scalar reaction–diffusion equation with multistable nonlinearity with a particular symmetry. By reduction to a family of transmission problems in **R**, and by contraction arguments, a manifold close to an invariant manifold formed by functions exhibiting a pattern of transition layers is constructed. An approximation for the associated vector field is also provided. This shows that the motion on those manifolds is exponentially slow, as in the well-known case of the bistable equation. However, in opposition to the bistable case, some of these manifolds are far from the attractor. Since these manifolds correspond to metastable patterns, this shows the importance of the transient motion toward the attractor and the importance of these manifolds in organizing that motion. It is also shown that by a suitable perturbation we can obtain new equilibria on those manifolds.

1. INTRODUCTION

The study of the slow evolution of patterns associated to singularly perturbed parabolic equations (metastable patterns) received an important contribution when [12] and [8] brought to this field geometrical arguments from the qualitative theory of differential equations on infinite dimensional spaces. In these works, the authors considered the well-known scalar reaction–diffusion equation of Chaffee–Infante type,

$$u_t = \varepsilon^2 u_{xx} + f(u), \qquad u_x(0) = u_x(1) = 0,$$

in $(0, 1)$, where f is a cubic-like nonlinearity, such that $f = -F'$, where F is a bistable potential with equal wells and $\varepsilon > 0$ is small. It was known (from numerical results, for example) that, after a first fast transient stage of a typical solution, it seems to settle down on a configuration which is

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characterized by regions where u is almost constant and near each one of the two minimizers of F. These regions are separated by steep transition layers with width of $O(\varepsilon)$. After a very long period which corresponds to a very slow motion of these layers (with speed of the order $O(e^{-C/\varepsilon})$), a short period of fast motion takes place. During this fast episode, the solution experiences a dramatic change, corresponding to a very steep decrease in the energy function, corresponding to a decrease of the number of the transition layers. This is, in turn, followed by another slow motion stage similar to the previous one, and the cycle continues until the solution loses all its transition layers and becomes almost constant.

In [12], an interpretation for this slow–fast motion cascade based on the Morse–Smale property is given, on the structure of the elements forming the unstable manifolds of equilibria and on the way a typical solution approaches the attractor in $X = L^2(0, 1)$. When the solution is very close to the attractor, it typically gets very close to the unstable manifold of an equilibrium, $W^u(p_o)$, say, synchronizing with a solution there, by the asymptotic phase property. If p_o is a layered equilibrium, the evolution in $M(p_o) = W^u(p_o) \backslash V_\delta$, where V_δ is a small neighborhood of $\partial W^u(p_o)$, is like one of the described slow motion stages. Let $i(p_o)$ be the instability index of p_o and define P as the set of equilibria such that $p \in P \Rightarrow i(p) < i(p_o)$. It is well known that, if $n(p_o)$ is the number of transition layers of p_o, then $n(p_o) = i(p_o)$. Hence, the motion in the $n(p_o)$-dimensional local invariant manifold $M(p_o)$ can be given by the ODE in $\mathbf{R}^{n(p_o)}$, $\dot{\xi} = c(\xi)$, satisfied by the $n(p_o)$ transition layers positions vector, ξ. The following fast stage corresponds to the motion inside V_δ, until the solution settles down very close to some $M(p_1)$, with $p_1 \in P$, and hence, $n(p_1) < n(p_o)$. Recall that $\partial W^u(p_o) = \overline{\bigcup_{p \in P} W^u(p)}$. Then a new slow motion stage follows, and the cycle continues until the trajectory defined by the solution is very close to one of the two constant stable equilibria.

As seen above, the manifolds $M(p)$ assume the role of organizing the slow motion stages of evolution. By studying the vector field c on each one, we can understand how a typical solution is going to behave during its slow motion stages. This organizing role is one of the main features we retain from this problem and motivates us for the search of similar organizing manifolds in other problems where eventually they are not associated with unstable manifolds of equilibria, but to more complex invariant sets or even with none at all. Manifolds of this type we call *slow manifolds*.

In [12] and [8], following different methods, the authors constructed manifolds of layered functions in X, parameterized by the transition layers' positions vector, ξ, together with exponentially small vector fields $c(\xi)$. Those manifolds are not really locally invariant but were conjectured to be very close to the true locally invariant manifolds of the type $M(p)$ referred to above. The vector fields c were assumed to give good approximations of the ones giving the motion of the transition layers. In those two works, no rigorous proofs were given about the extent to which these were good approximations of true locally invariant manifolds and of the flow on them. In [8] the authors actually proved that there is a slow channel around the approximating manifold where each orbit, once entered in it, stays for a long period of time and the motion of the transition layers is a small perturbation of the one given by the vector field approximation.

In [12] the connection between approximations and true objects was suggested by construction. One of the key ideas is that, for each function in the slow motion manifold M, the profile of each transition layer is essentially given by a heteroclinic solution of the reaction–diffusion equation on the whole line. This is a natural idea since the restriction to $[0, 1]$ of a heteroclinic solution U translated so its zero is at a point $\xi \in (\delta, 1-\delta)$, for a small $\delta > 0$, satisfies $\varepsilon^2 U_{xx} + f(U) = 0$, and by standard arguments, it fails to satisfy the Newmann boundary conditions only by an $O(e^{-C/\varepsilon})$ error, as $\varepsilon \downarrow 0$, for some $C > 0$. This suggests that, for ε small, U is exponentially close in X to a metastable state satisfying the imposed boundary conditions and having a transition layer centered at ξ. For more than one transition layer, the idea is similar if we subdivide $[0, 1]$ so that each subinterval corresponds to one transition layer and the centers of these layers are uniformly far from the endpoints. Then the above arguments are similar in each subinterval, considering, for the "interior" endpoints, smoothness conditions instead of Newmann boundary conditions.

Therefore, following [12], one may consider the elements of M as perturbations in X, of functions constructed by just pasting together pieces of heteroclinics solutions in a convenient way. These first approximations form a parameterized manifold M_o in X. The final manifold approximation is obtained by first formally writing the reaction–diffusion equation in the normal bundle of the conjectured manifold M, written as a perturbation of M_o. Then, by dropping some higher order terms, one obtains an explicitly solvable system whose unknowns are the approximating vector field, c, and the perturbation defining a new, better approximating manifold, M_1.

Since then, other studies about the slow motion stages of (1) have included, in particular, spectral properties and invariant manifold theory [2, 9, 11] and an energy approach [7]. A invariant manifold proof, making rigorous the arguments above, was given in [11], and, with respect to the one dimensional Cahn–Hilliard equation, in [2]. In these works, only solutions with one layer for the reaction–diffusion equation and two layers for the Cahn–Hilliard equation were considered. Both correspond to just one dimensional slow motion manifolds. In fact, the proof depends heavily on this manifold dimension. This method uses an approximation (M_1, c) of the type considered in [12]. Then, by using a spectral gap result already obtained in [8], together with some careful estimates in terms of ε, and an extension of invariant manifold methods, a truly invariant manifold was constructed as the fixed point of a contraction operator acting on a metric space of sections corresponding to the normal bundle of M_1. The construction of that operator follows ideas close to those in [14]. Estimates show that, at the level of the leading exponentially small scales of motion, the approximation c and the true vector field coincide.

In [9], a different method was used. Here, Carr and Pego consider the approximation obtained previously in [8]. They prove that this approximation is very close to the unstable manifold of the corresponding equilibrium and this last one is inside the slow channel. This proof is independent of the number of transition layers and therefore of the manifold dimension. This procedure has also been applied to the n layers' metastable patterns, with $n > 2$, of the one dimensional Cahn–Hilliard equation in a bounded interval, in [5] and [6].

Many works have also appeared on other related problems and types of methods. Among them, we can refer to [18–21]. These show a large variety of possible asymptotic behaviors near the singular limits, depending in a very sensitive way on features like the type of nonlinearity and dimension and shape of the domain. Other related work concerning systems of reaction–diffusion equations, [15], for example, introduces the possibility of the existence of slow oscillating patterns. This suggests that invariant manifold theory could provide a reduction principle in the study of slow dynamics associated to problems near singular limits, even with more complex dynamics, in the spirit of [10]. On the other hand, the transient motion towards the global attractor may exhibit metastable stages far from the attractor itself. We will see below that a case like this arises in the study of scalar reaction–diffusion equations with multistable potentials, as suggested by some formal computations in [20] and by the knowledge we have of the global attractors.

Taking into account the above considerations, it would be useful to have invariant manifold methods not relying on the particular singular limit dynamics in study, but rather on features like fast local transversal attractivity and slow tangential vector field. We observe that the methods followed in [11] and [1] could provide a basis for such methods if they could be generalized for more than one dimensional dynamics and for some non-self-adjoint cases. In [4], the Alikakos and Fusco, in fact, construct a two dimensional manifold in the spirit of [12], [11], and [1] and, based on similar estimates, they show that one can prove the existence of a slow channel around the manifold even without proving the existence of a true locally invariant manifold.

In this study we apply the approach followed by [12], [11], and [1], to (1) but with an N-stable potential, with $N > 2$, which we define in Section 2. This equation was considered in [20], where transition layers' speeds were derived by some formal approximations. In this work, we construct approximating manifolds and vector fields for the transition layers motion, in the spirit of [12] and [11]. With our approach we obtain the same type of estimates that allowed the slow channel proof in [4] and the invariant manifold proof in [11] for the one layer case.

All elements of each of our manifolds are characterized by a specific transition layer pattern, in a way to be defined in Section 2, and each different element is characterized by a vector of *transition layers positions*, ξ. Our main result is Theorem 3.5. It shows that there can be slow motion manifolds far away from the attractor and, therefore, not associated to any equilibria. As we saw above, this new possibility is in opposition with the bistable equation. Hence, the transient motion towards the global attractor can be through metastable states. In Section 4 we show that by perturbing the potential with a $O(e^{-C/\varepsilon})$ perturbation, for ε small, we can create an equilibrium with a pattern, for which no equilibria existed in the unperturbed case. This new equilibrium lies in a slow motion manifold which is very close to an unperturbed one, where no equilibrium existed. We remark that, in [20], the authors gave some formal asymptotic analysis of the speed, with respect to a potential perturbation, suggesting the creation of new equilibrium by stopping the motion of the transition layers.

The conclusions above could be rigorously extended to true locally invariant manifolds and true speed, by a generalization of the proof of [11] to higher dimensions. Since here the slow motion is not associated with unstable manifolds of equilibria, no other proof is currently provided in the literature for our case. With this in mind, with our approach, we obtain estimates that, together with known spectral results on the linearization (see Section 5), allow us to entirely reproduce (in a easier version) the proof given in [4] for the existence of the slow channel. For the one layer case, these same results are sufficient for the proof of the invariant manifold theorem. In another work, we will show that these are also sufficient to prove a similar invariant manifold result for the n layers case, with $n > 1$. This will provide us with a generalization of previous proofs that could be useful for other situations such as some of the problems in the works referred to above.

2. THE N-STABLE PROBLEM

We consider the scalar reaction–diffusion equation with Newmann boundary conditions in $[0, 1]$,

$$u_t = \mathscr{L}(u), \tag{2}$$

where $\mathscr{L}$ is the nonlinear operator with domain $D(\mathscr{L})=\{u\in H^2(0,1)\mid u_x=0$, at $x=0,1\}$, defined by

$$\mathscr{L}(u)=\varepsilon^2 u_{xx}+f(u).$$

Here, $f=-W'$, where $W\in C^{k_o+1}(\mathbf{R})$, with $k_o\geqslant 2$, is a potential of a particular N-stable type we now describe.

The function W has a minimum, attained at exactly N equally spaced minimizers $m_1, ..., m_N$. Let $\mu=(m_i-m_{i-1})/2$, $i=2, ..., N$. For $u\in[m_{i-1}, m_i]$, with $i=2, ..., N$,

$$W(u)=W_o\left(u-\frac{m_{i-1}+m_i}{2}\right),$$

where W_o is a fixed symmetric bistable potential; that is,

(i) W_o is an even function;

(ii) W_o has exactly three critical points at $u=-\mu, 0, \mu$, with 0 being a relative maximizer and $-\mu$ and μ absolute minimizers;

(iii) $W_o''(u)=0$ at exactly two points in $(-\mu,\mu)$.

Therefore, for $u\in[m_{i-1}, m_i]$, $f(u)=f_o(u-(m_{i-1}+m_i)/2)$, where $f_o=-W_o'$ is like a cubic nonlinearity. Observe also that (i)–(iii) imply that $f_o'(\mu)=f_o'(-\mu)<0$.

For all $u\in(-\infty, m_1]\cup[m_N, +\infty)$, we assume that $W''(u)\geqslant c$, for some positive constant c. This implies condition (3.1) in Section 4.3.1 of [13] and, therefore, the global existence of the solutions, as well as the existence of the connected global attractor. It also implies that the supremum of the distances from the values attained by the the solution at each instant $t>0$ to the interval $[m_1, m_N]$ is $O(e^{-\omega t})$, as $\varepsilon\downarrow 0$, for some $\omega>0$, not depending on the initial condition. This assumption, together with the hypothesis on W in $[m_1, m_N]$, is important for a quick initial development of a layer pattern as we describe below.

We recall the well-known fact that there exists $\Phi\in C^{k_o+1}(\mathbf{R})$, such that

$$\Phi''+f_o(\Phi)=0,$$

$$\lim_{x\to-\infty}\Phi(x)=-\mu,\qquad \lim_{x\to+\infty}\Phi(x)=\mu,\qquad \Phi(0)=0.$$

This solution of the equation $\phi''+f_o(\phi)=0$ corresponds to the heteroclinic connecting the critical point $(-\mu,0)$ to the critical point $(\mu,0)$, in the (ϕ,ϕ') phase plane. The solution $-\Phi$ corresponds to the heteroclinic connecting $(\mu,0)$ to $(-\mu,0)$. It is also well known that, for $x>0$,

$$\Phi(x)=\mu-Ke^{-\nu x}(1+\hat{\tau}(x)), \tag{3}$$

where $\nu = \sqrt{-f_o'(\mu)}$, $K > 0$ is independent of x, and $\hat{\tau}$ is a C^{k_o+2} function such that $\bar{\tau}^{(i)}(x) = O(e^{-\nu x})$, as $x \to +\infty$, for $i = 0, 1, ..., k_o + 1$. For $x < 0$, it is enough to note that Φ is an odd function.

We fix a positive integer n. We now construct functions modeling a pattern with n transition layers. Define $\Gamma_o = \{\xi \in \mathbf{R}^n \mid 0 < \xi_1 < \cdots < \xi_n < 1\}$. For each $\xi \in \Gamma_o$, it is also convenient to define $\xi_o = -\xi_1$ and $\xi_{n+1} = 2 - \xi_n$. For $\rho > 0$ small define also $\Gamma = \{\xi \in \Gamma_o \mid \xi_j - \xi_{j-1} > 2\rho,\ j = 1, ..., n+1\}$. Fix an $(n+1)$-tuple,

$$(\sigma_1, ..., \sigma_{n+1}) \in \{1, ..., N\}^{n+1},$$

such that, for each $j = 1, ..., n$, $|\sigma_{j+1} - \sigma_j| = 1$. We consider functions which are close, in $L^2(0, 1)$, to step functions given, for each $x \in [0, 1]$, by

$$s^{\xi}(x) = m_{\sigma_j}, \qquad \text{if} \quad x \in (\xi_{j-1}, \xi_j), \qquad \text{with} \qquad j = 1, ..., n+1, \tag{4}$$

with $\xi \in \Gamma_o$. Define $q \in \{-1, 1\}^n$, by $q_j = \sigma_{j+1} - \sigma_j$, for $j = 1, ..., n$. Hence, q, together with m_{σ_1}, defines a pattern characterized by n transition layers between consecutive minima of W, which can be of the downward $(q_j = -1)$, or upward $(q_j = 1)$ type, for $j = 1, ..., n$. Each component of ξ will give the position of each one of the transition layers. The proof of the following proposition is based on a standard phase plane analysis of the equation $\varepsilon^2 u'' + f(u) = 0$.

PROPOSITION 2.2.1. *Fix σ (or, equivalently, q and m_{σ_1}) as above. Then, there is $\xi^o \in \Gamma$, and an ε-parametrized family of equilibria $\{p_\varepsilon(\cdot)\}_{\varepsilon \in (0, \varepsilon_o]}$ of (2), such that $p_\varepsilon \to s^{\xi^o}$, as $\varepsilon \downarrow 0$, in the $L^2(0, 1)$ norm, with s^{ξ^o} constructed as above, if and only if $\sigma \in \{k, k+1\}^{n+1}$, for some $k \in \{1, ..., N-1\}$.*

In that case, ξ^o is uniquely determined by σ and, if $\{\bar{p}_\varepsilon(\cdot)\}_{\varepsilon \in (0, \varepsilon_o]}$ is another ε-parametrized family of equilibria in the same conditions for the same σ, then, if $\varepsilon_o > 0$ is small enough (depending on σ), then for $\varepsilon \in (0, \varepsilon_o]$, $\bar{p}_\varepsilon = p_\varepsilon$.

For each $\varepsilon > 0$ small, $\xi \in \Gamma_o$, and $j = 1, ..., n$, define u^j by

$$u^j(\varepsilon, \xi, x) = \frac{m_{\sigma_j} + m_{\sigma_{j+1}}}{2} + q_j \Phi\left(\frac{x - \xi_j}{\varepsilon}\right), \tag{5}$$

for all $x \in \mathbf{R}$. Each one of these functions satisfies

$$\varepsilon^2 u^j_{xx} + f(u^j) = 0, \qquad \text{in } \mathbf{R},$$

$$\lim_{x \to -\infty} u^j(x) = m_{\sigma_j}, \qquad \lim_{x \to +\infty} u^j(x) = m_{\sigma_{j+1}},$$

$$u^j(\xi_j) = (m_{\sigma_j} + m_{\sigma_{j+1}})/2.$$

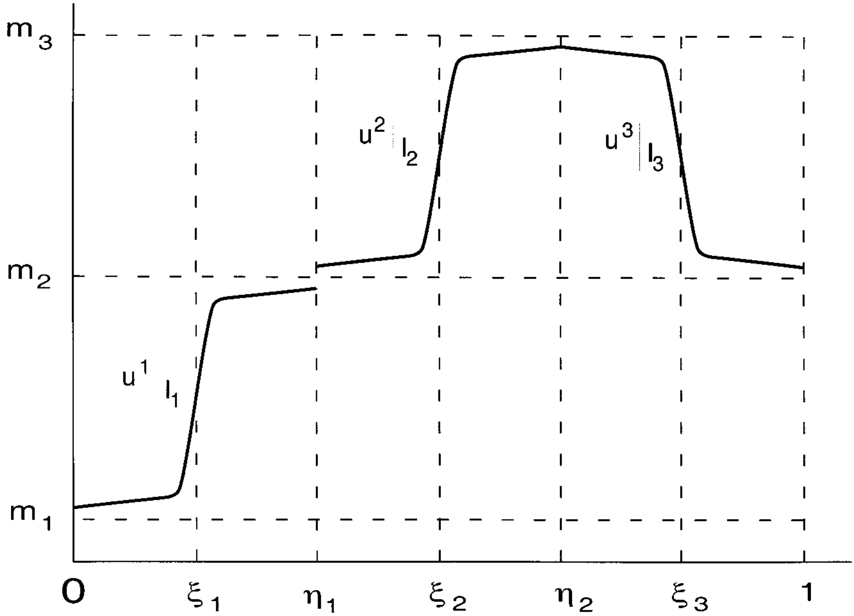

FIG. 1. An example of U.

Notice that $|u_x^j|$ attains its maximum at $x=\xi_j$. For $j=1, ..., n$, consider the subintervals $I_j=(\eta_{j-1}, \eta_j)$, where $\eta_j=(\xi_j+\xi_{j+1})/2$ and define U by

$$U(\varepsilon, \xi, x)=u^j(\varepsilon, \xi, x), \qquad \text{if} \quad x\in I_j,$$

for each $x\in[0, 1]$. Here, we explicitly display the dependence on ε and ξ in the argument. In the next sections, however, this will be omitted, for the sake of notation. Note that at each point $x=\eta_j$, U has a jump discontinuity of an exponential small order in ε, for $\varepsilon>0$ small, or it is continuous but with an exponentially small jump discontinuity in U_x. Therefore, these functions are not in the domain of $\mathscr{L}$. In Fig. 1 we display an example of $U(\varepsilon, \xi, \cdot)$, for a case where $n=3$ and $\sigma=(1, 2, 3, 2)$, and thus, $q=(+1, +1, -1)$.

Now fix $\varepsilon>0$ small. Suppose that, by slightly (in some sense to be defined) perturbing $U(\varepsilon, \xi, x)$, for each $\xi\in\Gamma$, we obtain p such that $\hat{M}:=\{p(\varepsilon, \xi, \cdot)\mid \xi\in\Gamma\}$ is a local invariant manifold for (2), parametrized by ξ. Then, for each $\xi\in\Gamma$, $\mathscr{L}(p(\varepsilon, \xi, \cdot))\in T\hat{M}_\xi$, which is equivalent to saying that there exists a continuous map $c\colon \Gamma\to\mathbf{R}^n$ such that

$$\mathscr{L}(p)=\sum_{k=1}^{n} c^k p_{\xi_k}, \tag{6}$$

for each $\xi \in \Gamma$. The ODE

$$\dot{\xi} = c(\varepsilon, \xi) \tag{7}$$

defines a flow on Γ, which parametrizes the flow on $\hat{M}$; that is, ϕ is a solution of (2) on $\hat{M}$ on some time interval J, if and only if, for all $t \in J$, $\phi(t) = p(\varepsilon, \xi(t), \cdot) \in \hat{M}$, for some solution $\xi(\cdot)$, of (7), defined for all $t \in J$. If the perturbation is small enough, p has the same layer structure as U. Hence, the motion on $\hat{M}$ is essentially given by the motion of the transition layers without changing the layer pattern, with dynamics given by the vector field c. Since, roughly speaking, U fails to give stationary solutions due to exponentially small deviations from the boundary conditions and from C^1 conditions at the points η_j, we can expect that $|c|$ is exponentially small, thus giving an exponentially slow motion of the transition layers.

Therefore, our aim would be to find p close to U in $L^2(0, 1)$ and c such that (6) was satisfied for all $\xi \in \Gamma$. We do not solve (6) directly. Instead, we try to find a manifold $M = \{ p(\xi, \cdot) \mid \xi \in \Gamma \}$, close, in $L^2(0, 1)$, to $\mathscr{U} = \{ U(\varepsilon, \xi, \cdot) \mid \xi \in \Gamma \}$, and c, close to the above "exact" c, satisfying

$$\mathscr{L}(p) = \sum_{k=1}^{n} c^k u^k_{\xi_k}. \tag{8}$$

If p is a small perturbation of U, then M gives a good approximation of $\hat{M}$, being itself an almost invariant manifold, with c giving a good approximation to the flow in it, retaining all the relevant dynamics. Another question to be addressed is, for each ξ, in what fiber F^{ξ} of $\mathscr{U}$ should we look for the perturbation to $U(\varepsilon, \xi, \cdot)$? Note that, since $\mathscr{U}$ is not a differentiable manifold, there is too normal bundle. The choice of F^{ξ} will be made in the next section according to the convenience of the method of resolution and will be given by an almost orthogonality condition. The problem consists then in constructing, for $\varepsilon > 0$ small and for each $\xi \in \Gamma$, a pair $(V, c) \in F^{\xi} \times \mathbf{R}$, such that $U + V \in D(\mathscr{L})$ and

$$\mathscr{L}(U + V) = \sum_{k=1}^{n} c^k u^k_{\xi_k}, \tag{9}$$

obtaining good estimates on V and c and looking at the dynamics defined by (7).

3. A TRANSMISSION PROBLEM

Consider again the bistable nonlinearity, f_o. The following formulation is due to G. Fusco:

LEMMA 3.3.1. *For each $h \in L^2(\mathbf{R})$, such that $\langle h, \Phi' \rangle = 0$, the problem*

$$\text{(i)} \quad \phi'' + f_o'(\Phi(x))\,\phi = h,$$

$$\text{(ii)} \quad \langle \phi, \Phi' \rangle = 0$$

has a unique solution ϕ in $L^2(\mathbf{R})$. Moreover, this problem admits a Green function $G\colon \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ such that the solution of (i)–(ii) *is given by*

$$\phi(x) = \langle G(x, \cdot), h \rangle,$$

for each $x \in \mathbf{R}$ and it satisfies the following estimate, for all $(x, y) \in \mathbf{R}^2$:

$$|G(x, y) - G_\infty(x, y)| \leqslant C e^{-\nu^-(|x| + |y|)}. \tag{10}$$

Here, ν^- is any constant in $(0, \nu)$, and G_∞ is the Green function for the problem $\psi'' - \nu^2\psi = h$, which is given explicitly by

$$G_\infty(x, y) = -\frac{1}{2\nu}\, e^{-\nu\,|x-y|}.$$

Also, G is C^{k_o+1} in $\Omega = \{(x, y) \in \mathbf{R}^2 \mid xy \neq 0 \text{ and } x \neq y\}$, and, in Ω, the partial derivatives with respect to x and y, up to order $k_o + 1$, satisfy estimates that are obtained formally by differentiating $G(x, y) - G_\infty(x, y)$ in (10) *and updating the constant $C > 0$.*

Proof. The problem can be solved explicitly by remarking that equation (i) is equivalent to $(\Phi'\phi' - \Phi''\phi)' = \Phi' h$. Hence, using the condition $\phi(x) \to 0$ as $x \to +\infty$, we can deduce that, if $\phi \in L^2(\mathbf{R})$ is the solution of (i) then there is a real α such that, for all $x \in \mathbf{R}$,

$$\phi(x) = \alpha\Phi'(x) - \Phi'(x) \int_0^x \int_s^{+\infty} \frac{\Phi'(y)}{[\Phi'(s)]^2}\, h(y)\, dy\, ds. \tag{11}$$

Using the orthogonality condition for h, this solution can also be written as

$$\phi(x) = \alpha\Phi'(x) - \Phi'(x) \int_x^0 \int_{-\infty}^s \frac{\Phi'(y)}{[\Phi'(s)]^2}\, h(y)\, dy\, ds. \tag{12}$$

Conversely, each $\alpha \in \mathbf{R}$ gives a $L^2(\mathbf{R})$ solution of (i), ϕ, satisfying (11) and (12). By using (11) if $x \geqslant 0$ and (12) if $x < 0$, and inverting the order of integration, we obtain

$$\phi(x) = \alpha\Phi'(x) + \int_{-\infty}^{+\infty} H(x, y)\, h(y)\, dy,$$

where

$$H(x, y) = -\Phi'(x)\,\Phi'(y)\cdot\begin{cases}\left|\int_0^y \frac{1}{(\Phi')^2}\right|, & \text{if } xy \geqslant 0 \text{ and } |y| \leqslant |x|,\\ \left|\int_0^x \frac{1}{(\Phi')^2}\right|, & \text{if } xy \geqslant 0 \text{ and } |y| > |x|,\\ 0, & \text{if } xy < 0.\end{cases}$$

The constant α is computed so that the orthogonality condition for ϕ is satisfied. We conclude that, for all $x \in \mathbf{R}$,

$$\phi(x) = \int_{-\infty}^{+\infty} G(x, y)\, h(y)\, dy,$$

with

$$G(x, y) = H(x, y) - \frac{\Phi'(x)}{\|\Phi'\|^2} \int_{-\infty}^{\infty} H(\cdot, y)\, \Phi'.$$

The remainder of the proof, which we omit, is based on the analysis of this Green function, using estimate (3). ∎

By a standard procedure, we can prove that this lemma holds true if we take $h \in H^{-1}(\mathbf{R})$ and consider $\langle \cdot, \cdot \rangle$ as the duality product in $H^{-1}(\mathbf{R}) \times H^1(\mathbf{R})$ that generalizes the scalar product in $L^2(\mathbf{R})$.

As a consequence of the previous lemma, for each $j = 1, ..., n$,

$$g^j(x, y) := \varepsilon^{-1} G\left(\frac{x - \xi_j}{\varepsilon}, \frac{y - \xi_j}{\varepsilon}\right),$$

is the Green function for the following problem similar to (i)–(ii):

$$\varepsilon^2 \phi_{xx} + f'_o(u^j)\, \phi = h, \qquad \text{with} \quad \langle h, u^j_x \rangle = 0,$$
$$\langle \phi, u^j_x \rangle = 0.$$

We now consider the following problem: Consider $F = (F^1, ..., F^n) \in [L^2(\mathbf{R})]^n$. Then, find $v = (v_1, ..., v_n) \in [L^2(\mathbf{R})]^n$ and $c = (c^1, ..., c^n) \in \mathbf{R}^n$ such that, for $j = 1, ..., n$, the following equations are satisfied:

$$\begin{aligned}
&\text{(i)} \quad \varepsilon^2 v^j_{xx} + f'_o(u^j)\, v^j = 1_j \left[\sum_{k=1}^{n} c^k u^k_{\xi_k} + F^j \right] + a^j\, \delta_{\eta_{j-1}} + b^j\, \delta_{\eta_j},\\
&\text{(ii)} \quad \langle v^j, u^j_x \rangle = 0,\\
&\text{(iii)} \quad v^k(\eta_k) + u^k(\eta_k) = v^{k+1}(\eta_k) + u^{k+1}(\eta_k), \qquad \text{for} \quad k = j-1, j,\\
&\text{(iv)} \quad v^k_x(\eta_k) + u^k_x(\eta_k) = v^{k+1}_x(\eta_k) + u^{k+1}_x(\eta_k), \qquad \text{for} \quad k = j-1, j,
\end{aligned} \tag{13}$$

where 1_j is the characteristic function of the interval I_j, a^j and b^j are real constants to be determined, δ_η is the delta function with support at the point $\eta \in \mathbf{R}$, and $\langle \cdot, \cdot \rangle$ has the same meaning as before. Also, for all $x \in \mathbf{R}$, we define $v^o(x) := v^1(-x)$ and $v^{n+1}(x) := v^n(2-x)$ and do the same with u^o and u^{n+1}. Assume that this problem has a solution (v, c). Define $U, V \in L^2(0, 1)$ by $V(x) = v^j(x)$, and as before, $U(x) = u^j(x)$, for each $x \in I_j$, $j = 1, ..., n$. Conditions (iii)–(iv), for $k \neq 0, n$, state that $U + V \in C^1(0, 1)$. On the other hand, if $j = 1$ and $k = 0$, since $\eta_o = 0$, equation (iii) is always satisfied and we can ignore it. The same is true if $j = n$ and $k = n$, which corresponds to $\eta_n = 1$. It is also easy to see that, for both $(j, k) = (1, 0)$ and (n, n), equation (iv) reduces to $v^j_x(\eta_k) + u^j_x(\eta_k) = 0$. This implies that $(U_x + V_x)(x) = 0$, at $x = 0, 1$.

Let $p = U + V$. Define f_2: $[L^2(\mathbf{R})]^n \to [L^2(\mathbf{R})]^n$, by considering, for each $v \in [L^2(\mathbf{R})]^n$ and for each $j = 1, ..., n$,

$$f^j_2(v) = f(u^j + v^j) - f(u^j) - f'(u^j)\, v^j.$$

If $F = -f_2(v)$, then (p, c), solves (8). Also, if we had $F^j = \sum_{k=1}^n c^k v^k_{\xi_k} - f^j_2(v)$, for $j = 1, ..., n$, then (p, c) would solve (6). In this second case, we would have found a true invariant manifold with the corresponding flow. Here, we only try to solve the first case which corresponds to the problem described in the end of Section 2.

Also, note that conditions (13ii) correspond to a choice of the fiber F^ξ along which we construct the perturbation to $\mathcal{U}$, as we also mentioned in the end of Section 2.

In the next lemma we compile a number of estimates that will be important in the following. The proof is based on estimates (3) and (10) together with the obvious changes of variables. We denote by $\|\cdot\|$ the $L^2(\mathbf{R})$ norm. Since this causes no confusion, we write $\|\cdot\|_o$ for the sup norm in any of the subintervals I_j as well as in the whole interval $[0, 1]$. Below, as in the rest of the paper C represents a constant independent of (ε, ξ, x) which can be updated from estimate to estimate.

LEMMA 3.3.2. *For $\varepsilon > 0$ small, for each $\xi \in \Gamma_o$ and $j, k = 1, ..., n$,*

$$\left| \int_{I_j} (u^j_x)^2 - \varepsilon^{-1} \|\Phi'\| \right| \leqslant C\varepsilon^{-1}(e^{-\nu(\xi_j - \xi_{j-1})/\varepsilon} + e^{-\nu(\xi_{j+1} - \xi_j)/\varepsilon}), \tag{14}$$

$$\left| \int_{I_j} u^j_x u^k_x \right| \leqslant C\varepsilon^{-2}(e^{-\nu(\xi_j - \xi_{j-1})/\varepsilon} + e^{-\nu(\xi_{j+1} - \xi_j)/\varepsilon}), \qquad \text{if} \quad j \neq k, \tag{15}$$

$$\left|\int_{I_j} g^j(x,\cdot)\, u_x^k\right| \leqslant C\varepsilon^{-1} e^{-\nu\,|x-\xi_j|/\varepsilon}, \tag{16}$$

$$\left|\int_{I_j} (g^j(\cdot,\eta_i))^2\, u_x^k\right| \leqslant C\varepsilon^{-2} e^{-\nu\,|\eta_j-\xi_i|/\varepsilon}, \qquad \textit{for} \quad i=j-1,\, j, \tag{17}$$

Furthermore, if $\phi \in C^o(I_j, \mathbf{R})$ *then*

$$\left|\int_{I_j} u_x^j \phi\right| \leqslant C\,\|\phi\|_o, \tag{18}$$

$$\left|\int_{I_j} g^j(x,\cdot)\,\phi\right| \leqslant C\,\|\phi\|_o. \tag{19}$$

The estimate (17) *is also true if we substitute* η_j *by* η_{j-1}. *Estimates* (16), (17), *and* (19) *are also true if we substitute* g^j *by* $g^j_{x^m}$, $m=1, 2, 3$, *and multiply the right hand side of* (16) *and* (19) *by* ε^{-m} *and* (17) *by* ε^{-2m}.

For the next result, we define $\mathscr{F}_1=(\mathscr{F}_1^1, ..., \mathscr{F}_1^n)^T$, where, for $j=1, ..., n$,

$$\mathscr{F}_1^j=\frac{\varepsilon}{\|\Phi'\|^2} \in t_{I_j} u_x^j F^j$$

and $\mathscr{F}_a=(\mathscr{F}_a^1, ..., \mathscr{F}_a^n)^T$ and $\mathscr{F}_b=(\mathscr{F}_b^1, ..., \mathscr{F}_b^n)^T$, where, for $j=2, ..., n$:

$$\mathscr{F}_a^j=\frac{\varepsilon}{\|\Phi'\|^2}\int_{I_{j-1}} [-\nu g^{j-1}(\eta_{j-1},\cdot)+\varepsilon g_x^{j-1}(\eta_{j-1},\cdot)]\, F^{j-1}$$

$$+\frac{\varepsilon}{\|\Phi'\|^2}\int_{I_j} [\nu g^j(\eta_{j-1},\cdot)-\varepsilon g_x^j(\eta_{j-1},\cdot)]\, F^j,$$

and, for $j=1, ..., n-1$:

$$\mathscr{F}_b^j=\frac{\varepsilon}{\|\Phi'\|^2}\int_{I_j} [\nu g^j(\eta_j,\cdot)+\varepsilon g_x^j(\eta_j,\cdot)]\, F^j$$

$$+\frac{\varepsilon}{\|\Phi'\|^2}\int_{I_{j+1}} [-\nu g^{j+1}(\eta_j,\cdot)-\varepsilon g_x^{j+1}(\eta_j,\cdot)]\, F^{j+1}.$$

Also,

$$\mathscr{F}_a^1=-\frac{2\varepsilon^2}{\|\Phi'\|^2}\int_{I_1} g_x^1(0,\cdot)\, F^1,$$

and

$$\mathscr{F}_b^n = \frac{2\varepsilon^2}{\|\Phi'\|^2} \int_{I_n} g_x^n(1, \cdot)\, F^n.$$

Define also $\mathscr{F}_o = |\mathscr{F}_a| + |\mathscr{F}_b|$. We use the following notation for the jumps at the point η_j:

$$\tau_o^j = u^j(\eta_j) - u^{j+1}(\eta_j),$$
$$\tau_1^j = u_x^j(\eta_j) - u_x^{j+1}(\eta_j).$$

In the example of Fig. 1, $\tau_o^1 < 0$, $\tau_o^2 = 0$, $\tau_1^1 = 0$, and $\tau_1^2 > 0$. Define also $d^\xi = \min_{j=1, \ldots, n+1} (\xi_j - \xi_{j-1})/2$. Still, with respect to notation, we denote by $|\cdot|$ the absolute value in $\mathbf{R}$, the euclidean norm in $\mathbf{R}^n$, or the corresponding matrix norm, depending on the context.

THEOREM 3.3.3. *Problem* (13) *has a unique solution* (v, c), *and, furthermore, it satisfies, for each* $j = 1, \ldots, n$,

$$v^j(x) = \int_{I_j} g^j(x, \cdot) \left[\sum_{k=1}^n c^k u_{\xi_k}^k + F^j \right] + a^j g^j(x, \eta_{j-1}) + b^j g^j(x, \eta_j), \tag{20}$$

$$c^j = \gamma^j + \mathscr{F}_1^j + O(e^{-\nu d^\xi/\varepsilon}(|\gamma| + \mathscr{F}_o + |\mathscr{F}_1|)), \tag{21}$$

where

$$\gamma^j := K_1 \varepsilon (q_{j-1} q_j e^{-\nu(\xi_j - \xi_{j-1})/\varepsilon} - q_j q_{j+1} e^{-\nu(\xi_{j+1} - \xi_j)/\varepsilon}), \tag{22}$$

considering, as before, $q_o = -q_1$, $q_{n+1} = -q_n$, $\xi_o = -\xi_1$, *and* $\xi_{n+1} = 2 - \xi_n$, *and defining*

$$K_1 = \frac{2\nu^2 K^2}{\|\Phi'\|^2}.$$

Moreover, a^j *and* b^j *satisfy the following estimates*:

$$|a^j + \nu\varepsilon\tau_o^{j-1} - \varepsilon^2 \tau_1^{j-1}| \leqslant C(e^{-\nu d^\xi/\varepsilon}(|\gamma| + |\mathscr{F}_1|) + |\mathscr{F}_o|), \tag{23}$$
$$|b^j - \nu\varepsilon\tau_o^j - \varepsilon^2 \tau_1^j| \leqslant C(e^{-\nu d^\xi/\varepsilon}(|\gamma| + |\mathscr{F}_1|) + |\mathscr{F}_o|).$$

Proof. From Lemma 3.3.1 and the remarks following it, if we assume that, for $j = 1, \ldots, n$, the constants c^j, a^j, and b^j are given so that

$$\sum_{k=1}^n c^k \int_{I_j} u_{\xi_k}^k u_x^j = -\int_{I_j} u_x^j F^j - a^j u_x^j(\eta_{j-1}) - b^j u_x^j(\eta_j), \tag{24}$$

then problem (13)(i)–(ii) has a unique solution given by the expression (20). We now prove that, if (13)(iii)–(iv) are also taken in account, then a^j and b^j are uniquely determined. For that matter, we remark that, at each point η_j, $j=1, ..., n-1$ conditions (iii)–(iv), together with (20), result in the pair of conditions

$$\begin{aligned} a^j g^j(\eta_j, \eta_{j-1}) + b^j g^j(\eta_j, \eta_j) \\ - a^{j+1} g^{j+1}(\eta_j, \eta_j) - b^{j+1} g^{j+1}(\eta_j, \eta_{j+1}) = P^j, \end{aligned} \tag{25}$$

$$\begin{aligned} a^j g^j_x(\eta_j, \eta_{j-1}) + b^j g^j_x(\eta_j^-, \eta_j) \\ - a^{j+1} g^{j+1}_x(\eta_j^+, \eta_j) - b^{j+1} g^{j+1}_x(\eta_j, \eta_{j+1}) = Q^j, \end{aligned} \tag{26}$$

where

$$\begin{aligned} P^j = & -\int_{I_j} g^j(\eta_j, \cdot) \left[\sum_{k=1}^n c^k u^k_{\xi_k} + F^j \right] \\ & + \int_{I_{j+1}} g^{j+1}(\eta_j, \cdot) \left[\sum_{k=1}^n c^k u^k_{\xi_k} + F^{j+1} \right] \\ & - u^j(\eta_j) + u^{j+1}(\eta_j), \end{aligned} \tag{27}$$

$$\begin{aligned} Q^j = & -\int_{I_j} g^j_x(\eta_j, \cdot) \left[\sum_{k=1}^n c^k u^k_{\xi_k} + F^j \right] \\ & + \int_{I_{j+1}} g^{j+1}_x(\eta_j, \cdot) \left[\sum_{k=1}^n c^k u^k_{\xi_k} + F^{j+1} \right] \\ & - u^j_x(\eta_j) + u^{j+1}_x(\eta_j). \end{aligned} \tag{28}$$

With respect to the conditions at the boundary points $\eta_o = 0$ and $\eta_n = 1$, we have

$$-a^1 g^1_x(0^+, 0) - b^1 g^1_x(0, \eta_1) = Q^o, \tag{29}$$

$$a^n g^n_x(1, \eta_{n-1}) + b^n g^n_x(1^-, 1) = Q^n, \tag{30}$$

where

$$Q^o = \int_{I_1} g^1_x(0, \cdot) \left[\sum_{k=1}^n c^k u^k_{\xi_k} + F^1 \right] + u^1_x(0) \tag{31}$$

and

$$Q^n = -\int_{I_n} g^n_x(1, \cdot) \left[\sum_{k=1}^n c^k u^k_{\xi_k} + F^n \right] - u^n_x(1). \tag{32}$$

If $\mathbf{a} := (a^1, b^1, ..., a^n, b^n)^T$ and $\mathbf{P} := (Q_o, P_1, Q_1, ..., P_{n-1}, Q_{n-1}, Q_n)^T$, then the $2n \times 2n$ system (25)–(26), (29)–(30), can be written in the form

$$\mathscr{G}\mathbf{a} = \mathbf{P}, \tag{33}$$

with

$$\mathscr{G} = \begin{pmatrix} \mathscr{G}_o & \mathscr{H}_o^+ & & & \\ \mathscr{H}_1^- & \mathscr{G}_1 & \mathscr{H}_1^+ & & \\ & \ddots & \ddots & \ddots & \\ & & \mathscr{H}_{n-1}^- & \mathscr{G}_{n-1} & \mathscr{H}_{n-1}^+ \\ & & & \mathscr{H}_n^- & \mathscr{G}_n \end{pmatrix}, \tag{34}$$

where, for $j = 1, ..., n-1$,

$$\mathscr{G}_j = \begin{pmatrix} g^j(\eta_j, \eta_j) & -g^{j+1}(\eta_j, \eta_j) \\ g_x^j(\eta_j^-, \eta_j) & -g_x^{j+1}(\eta_j^+, \eta_j) \end{pmatrix},$$

$$\mathscr{H}_j^- = \begin{pmatrix} 0 & g^j(\eta_j, \eta_{j-1}) \\ 0 & g_x^j(\eta_j, \eta_{j-1}) \end{pmatrix}, \qquad \mathscr{H}_j^+ = \begin{pmatrix} g^{j+1}(\eta_j, \eta_{j+1}) & 0 \\ g_x^{j+1}(\eta_j, \eta_{j+1}) & 0 \end{pmatrix},$$

and, furthermore

$$\mathscr{G}_o = -g_x^1(0^+, 0), \qquad \mathscr{G}_n = g_x^n(1^-, 1),$$

$$\mathscr{H}_o^+ = (-g_x^1(0, \eta_1) \quad 0), \qquad \mathscr{H}_n^- = (0 \quad g_x^n(1, \eta_{n-1})),$$

$$\mathscr{H}_1^- = (g^1(\eta_1, 0) \quad g_x^1(\eta_1, 0))^T, \qquad \mathscr{H}_{n-1}^+ = (g^n(\eta_{n-1}, 1) \quad g_x^n(\eta_{n-1}, 1))^T.$$

From Lemma 3.3.1 we have, as $\varepsilon \downarrow 0$, for $j = 1, ..., n$,

$$g^j(\eta_k, \eta_k) = -\frac{1}{2\nu\varepsilon} + O(\varepsilon^{-1} e^{-\nu d^\xi/\varepsilon}), \qquad \text{for} \quad k = j-1, j,$$

$$g_x^j(\eta_j^-, \eta_j) = -\frac{1}{2\varepsilon^2} + O(\varepsilon^{-2} e^{-\nu d^\xi/\varepsilon}), \qquad g_x^j(\eta_{j-1}^+, \eta_{j-1}) = \frac{1}{2\varepsilon^2} + O(\varepsilon^{-2} e^{-\nu d^\xi/\varepsilon}),$$

and

$$g^j(\eta_i, \eta_k) = O(\varepsilon^{-1} e^{-\nu d^\xi/\varepsilon}), \qquad g_x^j(\eta_i, \eta_k) = O(\varepsilon^{-2} e^{-\nu d^\xi/\varepsilon}) \qquad \text{if} \quad i \neq k.$$

Therefore, $\mathscr{G} = \hat{\mathscr{G}} + \mathscr{E}$, where $\hat{\mathscr{G}} = \operatorname{diag}(\hat{\mathscr{G}}_o, ..., \hat{\mathscr{G}}_n)$ and $|\mathscr{E}| = O(\varepsilon^{-2} e^{-\nu d^\xi/\varepsilon})$,

$$\hat{\mathscr{G}}_o = \hat{\mathscr{G}}_n = -\frac{1}{2\varepsilon^2}, \qquad \hat{\mathscr{G}}_j = \begin{pmatrix} -1/(2\nu\varepsilon) & 1/(2\nu\varepsilon) \\ -1/(2\varepsilon^2) & -1/(2\varepsilon^2) \end{pmatrix},$$

for $j=1, ..., n-1$. Therefore, for ε small, $\mathscr{G}$ is invertible and, as $\varepsilon \downarrow 0$,

$$\mathscr{G}^{-1} = (I_{2n} + O(|\hat{\mathscr{G}}^{-1}\mathscr{E}|))\, \hat{\mathscr{G}}^{-1},$$

where, in turn, $|\hat{\mathscr{G}}^{-1}\mathscr{E}| = O(\varepsilon^{-1}\, e^{-\nu d^{\xi}/\varepsilon})$, and the unique solution of system (33) satisfies

$$a^1 = -2\varepsilon^2 Q^o + \cdots, \tag{35}$$

$$b^n = -2\varepsilon^2 Q^n + \cdots, \tag{36}$$

and

$$\begin{aligned} b^j &= -\nu\varepsilon P^j - \varepsilon^2 Q^j + \cdots, \\ a^{j+1} &= \nu\varepsilon P^j - \varepsilon^2 Q^j + \cdots, \end{aligned} \tag{37}$$

for $j=1, ..., n-1$. Here, "..." mean terms which are linear on $\mathbf{P}$ and which are $O(e^{-\nu d^{\xi}/\varepsilon}(|P^j| + \varepsilon\, |Q^j|))$, $j=1, ..., n$, as $\varepsilon \downarrow 0$. By substitution in Eqs. (24) we obtain a linear system in the unknowns c^j, $j=1, ..., n$. We now prove that this system is uniquely solvable and estimate the solution.

First, we use the definitions of the terms P^j and Q^j, (27), (28), (31), and (32) in (35)–(37), and then we substitute the expressions that result for a^j and b^j, $j=1, ..., n$, in (24). After collecting in the left hand side of the resulting system all terms in the unknowns c and taking the product of both sides by the factor $-\varepsilon/\|\Phi'\|^2$, we obtain a linear system in the unknown c,

$$[\Pi c]_j = c_o^j + \mathscr{F}_1^j + \mathscr{F}_a^j u_x^j(\eta_{j-1}) + \mathscr{F}_b^j u_x^j(\eta_j). \tag{38}$$

Here, Π is a $n \times n$ matrix that satisfies

$$\Pi = I_n + \mathscr{E}_o,$$

where $\mathscr{E}_o$ is a matrix whose coefficients are $O(e^{-\nu d^{\xi}/\varepsilon})$, as $\varepsilon \downarrow 0$, and the components of c_o are defined by

$$\begin{aligned} c_o^j = \frac{\varepsilon^2}{\|\Phi'\|^2} [&(-\nu\tau_o^{j-1} + \varepsilon\tau_1^{j-1})\, u_x^j(\eta_{j-1}) \\ &+ (\nu\tau_o^j + \varepsilon\tau_1^j)\, u_x^j(\eta_j)], \end{aligned} \tag{39}$$

for $j=1, ..., n$.

Recalling the definition of u^j, (5), the fact that

$$m_{\sigma_{j+1}} - m_{\sigma_{j-1}} = 2\mu(q_j + q_{j+1}),$$

for $j = 2, ..., n$, the oddness of Φ that implies $\Phi((\eta_j - \xi_j)/\varepsilon) = -\Phi((\eta_j - \xi_{j+1})/\varepsilon) = \Phi((\xi_{j+1} - \xi_j)/2\varepsilon)$, and estimate (3), we obtain

$$\begin{aligned}\tau_o^j &= (q_j + q_{j+1}) \left(\Phi \left(\frac{\xi_{j+1} - \xi_j}{2\varepsilon} \right) - \mu \right) \\ &= -(q_j + q_{j+1})\, K e^{-\nu(\xi_{j+1} - \xi_j)/(2\varepsilon)} (1 + \hat{\tau}((\xi_{j+1} - \xi_j)/\varepsilon).\end{aligned} \tag{40}$$

Similarly,

$$\begin{aligned}\tau_1^j &= (q_j - q_{j+1})\, \varepsilon^{-1} \Phi' \left(\frac{\xi_{j+1} - \xi_j}{2\varepsilon} \right) \\ &= (q_j - q_{j+1}) \frac{K\nu}{\varepsilon} e^{-\nu(\xi_{j+1} - \xi_j)/(2\varepsilon)} (1 + \hat{\tau}'((\xi_{j+1} - \xi_j)/\varepsilon).\end{aligned} \tag{41}$$

By using (40), (41) and the asymptotic estimate for u_x^j in (39), we obtain, for $j = 1, ..., n$,

$$c_o^j = \gamma^j + O(e^{-\nu d^\xi/\varepsilon}\, |\gamma|), \tag{42}$$

with γ^j given by (22). By multiplying both sides of (38) by Π^{-1}, we obtain (21). The estimates for a^j and b^j are straightforward consequences of (35)–(37) and of the definitions of P^j and Q^j. ∎

Now, for $0 < \Delta < 1$, define, for each $j = 1, ..., n$,

$$X_\Delta^j := \{\phi \in C^o(I_j) \mid \|\phi\|_o \leqslant \Delta\}$$

and $X_\Delta = X_\Delta^1 \times \cdots \times X_\Delta^n$, with the metric that results from the C^o norm, thus making X_Δ a complete metric space. We denote by $\|\cdot\|_o$ the norms in the spaces $C^o(I_j)$ as well as in the space $C^o(I_1) \times \cdots \times C^o(I_n)$.

LEMMA 3.3.4. *There is a constant, $\Delta > 0$, such that, for $\varepsilon > 0$ small, the nonlinear problem that results from* (13) *by taking $F^j = -f_2^j(v)$ has a unique solution (c_*, v_*) satisfying $v_* \in X_\Delta$. Moreover, the following estimate holds true, for $\varepsilon > 0$ small*:

$$\|v_*\|_o \leqslant C e^{-\nu d^\xi/\varepsilon} \qquad \textit{and} \qquad c_* = \gamma + O(\varepsilon e^{-3\nu d^\xi/\varepsilon}).$$

Proof. The proof is done using Theorem 3.3.3 and the contraction theorem. For a fixed small $\Delta > 0$, consider $v \in X_\Delta$, and let $(\hat{v}, c)$ be the solution of the resulting problem (13). Then,

$$\|F\|_o = \|f_2(v)\|_o \leqslant C\,\|v\|_o^2; \tag{43}$$

$$|\mathscr{F}_o|, |\mathscr{F}_1| \leqslant C\varepsilon\,\|F\|_o \leqslant C\varepsilon\,\|v\|_o^2; \tag{44}$$

therefore, according to (23) and (21), for $j = 1, ..., n$,

$$|a^j|, |b^j| \leqslant C\varepsilon(e^{-\nu d^\xi/\varepsilon} + \|v\|_o^2)$$

and

$$|c| \leqslant C\varepsilon(e^{-2\nu d^\xi/\varepsilon} + \|v\|_o^2).$$

By using the above estimates in (20), we obtain

$$\|\hat{v}\|_o \leqslant C(e^{-\nu d^\xi/\varepsilon} + \|v\|_o^2) \leqslant C(e^{-\nu d^\xi/\varepsilon} + \Delta^2). \tag{45}$$

Therefore, $\Delta > 0$ can be chosen independent of ε, so small that, for $\varepsilon \in (0, \varepsilon_o]$, with $\varepsilon_o > 0$ small, $\|\hat{v}\|_o \leqslant \Delta$, and therefore $\hat{v} \in X_\Delta$. Hence, the correspondence $v \mapsto \hat{v}$ defined above establishes a map from X_Δ into itself.

Now, take $v_o, v_1 \in X_\Delta$. Let

$$F_o := -(f_2^1(v_o), ..., f_2^n(v_o))^T \qquad \text{and} \qquad F_1 := -(f_2^1(v_1), ..., f_2^n(v_1))^T.$$

We consider the problem (13) with $F = F_o$ and $F = F_1$, to which we assign, respectively, the subscripts "o" and "1." Then,

$$\|F_o - F_1\|_o \leqslant C\Delta\,\|v_o - v_1\|_o.$$

Then, going through steps similar to the ones we took to reach estimate (45), we obtain, for $\varepsilon > 0$ small and $j = 1, ..., n$,

$$\varepsilon\,|P_o^j - P_1^j|, \varepsilon^2\,|Q_o^j - Q_1^j| \leqslant C(e^{-\nu d^\xi/\varepsilon}\,|c_o - c_1| + \varepsilon\,\|v_o - v_1\|_o),$$

which immediately gives

$$|a_o^j - a_1^j|, |b_o^j - b_1^j| \leqslant C(e^{-\nu d^\xi/\varepsilon}\,|c_o - c_1| + \varepsilon\,\|v_o - v_1\|_o)$$

and therefore, we can conclude after collecting the terms in $c_o - c_1$ in the left hand side of the equation that results from (24) by subtraction, and proceeding as in the proof of Theorem 3.3.3, we obtain

$$|c_o - c_1| \leqslant C(|\mathscr{F}_{1o} - \mathscr{F}_{11}| + \varepsilon\Delta\,\|v_o - v_1\|_o),$$

and this finally gives, by using (20),

$$\|\hat{v}_o - \hat{v}_1\|_o \leqslant C(\varepsilon^{-1}\,|\mathcal{F}_{1o} - \mathcal{F}_{11}| + \varDelta\,\|v_o - v_1\|_o)$$
$$\leqslant C\varDelta\,\|v_o - v_1\|_o.$$

Hence, we can conclude that if the constant $\varDelta > 0$ is chosen small enough, then the map $v \mapsto \hat{v}$ defines a contraction in $X_\varDelta$, for $\varepsilon > 0$ small. By the contraction theorem, it has a unique fixed point $v_* \in X_\varDelta$. To refine the estimates on v_* and c_*, we start by fixing a $C_o \in (0, 1)$ and choosing $\varDelta > 0$, so that $\varDelta \leqslant (1 - C_o)/C$, where C, in this case, is the constant of the estimate (45). Then, by that estimate,

$$C_o\,\|v_*\|_o \leqslant \|v_*\|_o(1 - C\,\|v_*\|_o) \leqslant Ce^{-\nu d^\xi/\varepsilon},$$

thus proving that $\|v_*\| \leqslant Ce^{-\nu d^\xi/\varepsilon}$, for some $C > 0$. This gives, in turn, $\mathcal{F}_o \leqslant C\varepsilon e^{-2\nu d^\xi/\varepsilon}$ and $|\mathcal{F}_1| \leqslant Ce^{-2\nu d^\xi/\varepsilon}$. However, this last estimate, together with (21), is not sufficient to obtain the estimate on c_*. To obtain that estimate, we use the definition of $\mathcal{F}_1$ to write

$$|\mathcal{F}_1^j| \leqslant C\varepsilon \int_{I_j} |u_x^j|\,|v^j|^2,$$

and we estimate all the terms which result from this inequality when applying (20). Then, using Lemma 3.3.2,

$$\int_{I_j} |u_x^j| \left(\int_{I_j} g^j(x, \cdot) \sum_{k=1}^n c^k u_{\xi_k}^j\right)^2 \leqslant Ce^{-2\nu d^\xi/\varepsilon}\,|c|^2,$$

$$\int_{I_j} |u_x^j| \left(\int_{I_j} g^j(x, \cdot)\,F^j\right)^2 \leqslant C\,\|v^j\|_o^4 \leqslant Ce^{-4\nu d^\xi/\varepsilon},$$

$$\int_{I_j} |u_x^j|\,|a^j g^j(\cdot, \eta_{j-1})|^2 \leqslant C(\|v_*\|_o^2 + e^{-2\nu d^\xi/\varepsilon}\,|c|^2 + e^{-2\nu d^\xi/\varepsilon})\,e^{-\nu d^\xi/\varepsilon}$$
$$\leqslant Ce^{-3\nu d^\xi/\varepsilon},$$

with a similar estimate for the term containing $|b^j g^j(\cdot, \eta_j)|^2$. Therefore $|\mathcal{F}_1| \leqslant C\varepsilon e^{-3\nu d^\xi/\varepsilon}$, which, together with the previous estimate on $\mathcal{F}_o$ and (21), gives $|c_* - \gamma| \leqslant e^{-3\nu d^\xi/\varepsilon}$. ∎

Fix a pattern by giving a $\sigma \in \{1, ..., N\}^n$ or, equivalently, $q \in \{-1, +1\}^n$, together with m_{σ_1}, according to Section 2. Let Γ be the open subset of Γ_o, defined as before.

Define the function $V_*: \Gamma \times [0, 1] \to \mathbf{R}$ by

$$V_*(\xi, x) = v_*^j(x), \qquad \text{for} \quad x \in I_j, \qquad j = 1, ..., n$$

and $p = U + V_*$. If we solve the above problem for each $\xi \in \Gamma$, we obtain a map $(\xi, x) \mapsto p(\xi, x)$ defined in $\Gamma \times [0, 1]$. Define

$$M = \{ p(\xi, \cdot) \mid \xi \in \Gamma \}.$$

Then, we have the following

THEOREM 3.3.5. *For each $\varepsilon > 0$ small, and $\xi \in \Gamma$, if $p = p(\xi, x)$ is as defined above, and $c = c_*(\xi)$, then the following statements hold true:*

(i) *For each $\xi \in \Gamma$, $p(\xi, \cdot) \in C^2(0, 1) \cap C^1([0, 1])$ and*

$$\begin{gathered} \varepsilon^2 p_{xx} + f(p) = \sum_{k=1}^{n} c^k u^k_{\xi_k}, \qquad \text{in} \quad (0, 1) \\ p_x(0) = p_x(1) = 0; \end{gathered} \tag{46}$$

(ii) *For $\xi^o \in \Gamma$, $\gamma(\xi^o) = 0$ if and only if $p(\xi^o, \cdot)$ is an equilibrium of the N-stable reaction–diffusion equation* (2). *In that case, $p_o = p(\xi^o, \cdot)$ will be the only equilibrium in M, and if $\varepsilon > 0$ is small enough, $p_o = p_\varepsilon$, as in Proposition* 2.2.1.

(iii) *The following partial derivatives exist, for $(\xi, x) \in \Gamma \times (0, 1)$, with the corresponding estimate holding true for each integer $\alpha_o \geqslant 0$ and multi-index α, such that $0 \leqslant \alpha_o + |\alpha| \leqslant k_o$:*

$$|\partial_{x^{\alpha_o}} \partial_{\xi^\alpha}(p(\xi, x) - U(\xi, x))| \leqslant C\varepsilon^{-(\alpha_o + |\alpha|)} e^{-\nu d^\xi/\varepsilon},$$

for each $x \in I_j$, $j = 1, ..., n$, and

$$|\partial_{\xi_j}(c(\xi) - \gamma(\xi))| \leqslant Ce^{-3d^\xi \nu/\varepsilon}.$$

(iv) *Consider any (ε, ξ)-independent $C_o \in (0, \rho\nu/3)$. Then there is an open set $\Gamma_1 \subset \Gamma$, independent of ε such that, for $\xi \in \Gamma_1$,*

$$c(\xi) = \gamma(\xi) + O(e^{-C_o/\varepsilon} |\gamma(\xi)|), \tag{47}$$

and, furthermore, if $D^\xi := \max_{j=1, ..., n+1} (\xi_j - \xi_{j-1})/2$, then

$$\hat{\Gamma}_1 := \{ \xi \in \Gamma \mid d^\xi \in (2D^\xi/3 + C_o/3\nu, D^\xi) \} \subset \Gamma_1,$$

and, in particular, Γ_1 *includes the only possible equilibrium of* $\dot{\xi} = c(\xi)$ *in* Γ, *in the case it exists.*

Proof. Part (i) is an immediate consequence of the way we constructed p. We now prove (ii). From (46) it is obvious that, for some $\xi^o \in \Gamma$, $p(\xi^o, \cdot)$ is an equilibrium of (2) if and only if $c(\xi^o) = 0$. Now suppose that $\gamma(\xi^o) = 0$. This is equivalent to $q_{j-1} q_j = -1$, for $j = 1, ..., n+1$, together with $\xi^o = (1/2n, 1/2n + 1/n, ..., 1/2n + (n-1)/n)$. Then, $\eta_j - \xi_j = \xi_j - \eta_{j-1} = 1/(2n)$, for $j = 1, ..., n$. Consider the metric space X_Δ defined before Lemma 3.3.4. Let Y be the set such that $\phi \in Y$ if and only if $\phi \in X_\Delta$ and, furthermore, for each $j = 1, ..., n$,

$$\phi^{j-1}(\eta_j - s) = \phi^j(\eta_j + s), \qquad \phi^j(\xi_j - s) = -\phi^j(\xi_j + s), \tag{48}$$

for all $s \in [0, 1/n]$. Now fix $v \in Y$. Since the functions $u^j - (m_{\sigma_j} + m_{\sigma_{j+1}})/2$ satisfy conditions (48), according to our symmetry hypothesis on f, the functions $x \mapsto F^j(x) \equiv f^j_2(v)(x)$ also satisfy (48). This immediately gives

$$\int_{I_j} u^j_x F^j = 0, \qquad \text{for} \quad j = 1, ..., n,$$

and, thus, $\mathscr{F}_1 = 0$. Also, by the definition of g^j, and since $G(x, y) = G(-x, -y)$, for all $(x, y) \in \mathbf{R}^2$, we obtain, $\mathscr{F}_a + \mathscr{F}_b = 0$. On the other hand, by (42) if $\gamma = 0$ then $c_o = 0$. Since $u^j_x(\eta_{j-1}) = u^j_x(\eta_j)$, for each $j = 1, ..., n$, then (38) gives $c = 0$.

Now consider the image $\hat{v}$ of v under the contraction map defined in the proof of Lemma 3.3.4. Then,

$$\hat{v}^j(x) = \int_{I_j} g^j(x, \cdot)\, F^j + a^j g^j(x, \eta_{j-1}) + b^j g^j(x, \eta_j).$$

Again, it is not difficult to verify that the functions $x \mapsto \int_{I_j} g^j(x, \cdot)\, F^j$ satisfy the properties (48). With respect to the coefficients P^j and Q^j defined in (27), (28), (31), and (32), by the above symmetry properties of G and functions F^j, we obtain $P^j = 0$ and $Q^j = 2(-1)^j Q^o$, for $j = 1, ..., n-1$, and $Q^n = (-1)^n Q^o$. If, for $j = 1, ..., n-1$, we take $a^{j+1} = -a^j$ and $b^j = -a^j$, with $a^1 = -\varepsilon^2 Q^o / (G_{,1}(-l, -l) - G_{,1}(-l, l))$, where $l = 1/(2n\varepsilon)$, then system (33) is satisfied. Since $\mathscr{G}$ is nonsingular, this gives the only solution $\mathbf{a}$ of that system. Noting that, once more, properties (48) are satisfied by the functions $x \mapsto g^j(x, \eta_{j-1}) - g^j(x, \eta_j)$, we finally conclude that $\hat{v} \in Y$. This means that the contraction map $v \mapsto \hat{v}$ maps Y into itself. Since properties (48) obviously extend to functions which are the limits, in C^o norm, of sequences in Y, then Y is closed in X_Δ. We conclude that, if $\xi^o \in \Gamma$ is such that $\gamma(\xi^o)=0$, then the only fixed point v_* of the above contraction map is in Y and therefore $c_*=0$, as we have seen above. Let $p_o=p(\xi^o,\cdot)$. Then, p_o is an equilibrium of the reaction–diffusion equation, and if we make the same computations for the same σ and Γ, for each $\varepsilon>0$ small, we obtain $\lim_{\varepsilon\downarrow 0}\|p_o-s^{\xi^o}\|_{L^2(0,1)}=0$. According to Proposition 2.2.1, if $\varepsilon>0$ is small enough, then $p_o=p_\varepsilon$, and this is the only equilibrium in M.

To prove the converse assertion, consider that $\xi\in\Gamma$ is such that $\gamma(\xi)\neq 0$. Suppose that the problem we are considering corresponds to a pattern such that $q_{j-1}q_j=-1$, for $j=1, ..., n+1$. Then $\xi\in\Gamma\backslash\{\xi^o\}$, and therefore, since $p(\xi,\cdot)\to s^{\xi}$ and $p(\xi^o,\cdot)\to s^{\xi^o}$, as $\varepsilon\downarrow 0$, then $p(\xi,\cdot)\neq p(\xi^o,\cdot)$, for $\varepsilon>0$ sufficiently small. This means that $p(\xi,\cdot)$ is not an equilibrium of the reaction–diffusion equation, as we have seen above. If we are considering a case where there is at least a $j\in\{1, ..., n+1\}$ such that $q_jq_{j-1}=+1$, then, since there is no equilibria close to s^{ξ}, for any $\xi\in\Gamma$, we conclude that, for $\varepsilon>0$ small, $p(\xi,\cdot)$ is not an equilibrium.

With respect to (iii), for $\alpha_o+|\alpha|=0$, this is already stated in Lemma 3.3.4. For the other cases, we only give here a sketch of the proof. First, for F sufficiently smooth, we study the derivatives of the explicit formulas that lead us to v and c. By using estimates similar to the ones in Lemma 3.3.2 but for the higher derivatives, which contain extra negative powers of ε, we obtain estimates which are similar to the ones obtained in Theorem 3.3.3. By proceeding as in Lemma 3.3.4, we obtain candidates to the derivatives of v_* as fixed points of the corresponding maps. By standard uniform contraction arguments we can conclude that they are, in fact, the derivatives of v_* and we can obtain estimates for the derivatives of v_* and c_* similar to the ones obtained in Lemma 3.3.4 but with the extra negative powers of ε.

To prove (iv), we first suppose that $q_jq_{j-1}=-1$, for $j=1, ..., n+1$. Since $c(\xi^o)=\gamma(\xi^o)=0$, then, if ξ is in an r-ball centered in ξ^o, $B_r(\xi^o)$, contained in Γ, we have

$$c(\xi)-\gamma(\xi)=\bar{M}(\xi)(\xi-\xi^o),$$

where the entries of the $n\times n$ matrix $\bar{M}(\xi)$ are, for $i, j=1, ..., n$,

$$\bar{m}_{ij}(\xi)=\partial_{\xi_j}(c^i(\bar{\xi}^i)-\gamma^i(\bar{\xi}^i)),$$

for a ξ-dependent set $\{\bar{\xi}^1, ..., \bar{\xi}^n\}\subset B_r(\xi^o)$. On the other hand,

$$\gamma(\xi)=\hat{M}(\xi)(\xi-\xi^o),$$

where the entries of the $n\times n$ matrix $\hat{M}(\xi)$ are, for i, $j=1, ..., n$,

$$\hat{m}_{ij}(\xi)=\partial_{\xi_j}\gamma^i(\hat{\xi}^i),$$

for another ξ-dependent set $\{\hat{\xi}^1, ..., \hat{\xi}^n\}\subset B_r(\xi^o)$. The matrix $\hat{M}(\xi)$ has structure

$$\hat{M}(\xi)=\begin{pmatrix} 2\alpha_1+\beta_1 & -\beta_1 & & & \\ -\alpha_2 & \alpha_2+\beta_2 & -\beta_2 & & \\ & \ddots & \ddots & \ddots & \\ & & -\alpha_{n-1} & \alpha_{n-1}+\beta_{n-1} & -\beta_{n-1} \\ & & & -\alpha_n & \alpha_n+2\beta_n \end{pmatrix},$$

with $\alpha_j=K_1\nu e^{-\nu(\hat{\xi}^j_j-\hat{\xi}^j_{j-1})/\varepsilon}$ and $\beta_j=K_1\nu e^{-\nu(\hat{\xi}^j_{j+1}-\hat{\xi}^j_j)/\varepsilon}$. Since any matrix of the above type with strictly positive α_j and β_j coefficients is invertible, then $\hat{M}(\xi)$ is invertible and

$$c(\xi)-\gamma(\xi)=\bar{M}(\xi)\,\hat{M}(\xi)^{-1}\,\gamma(\xi). \tag{49}$$

On the other hand, det $\hat{M}$ is a sum of terms, each one being a product of n coefficients α_j, β_j times a positive constant. Using this, together with the fact that if $\xi\in B_r(\xi^o)$ then $|\xi_j-\xi_{j-1}-1/n|\leqslant 2r$, we obtain, for some $C>0$,

$$\det\hat{M}(\xi)\geqslant Ce^{-\nu(1+2rn)/\varepsilon}.$$

For similar reasons we obtain the following estimate for the cofactors:

$$|\hat{M}_{ij}|\leqslant Ce^{-\nu(n-1)(1/n-2r)/\varepsilon}.$$

Also, by (iii)

$$|\bar{M}(\xi)|\leqslant Ce^{-3/2\,\nu(1/n-2r)/\varepsilon}.$$

Using the last three estimates we finally obtain

$$|\bar{M}(\xi)\,\hat{M}(\xi)^{-1}|\leqslant Ce^{-\nu(1/2n-r(1+4n))/\varepsilon}.$$

If we define $C_1=\max(C_o, \nu/4n)$ and choose $0<r\leqslant(1/2n-C_1\nu^{-1})/(1+4n)$, we obtain that, for $\xi\in B_r(\xi^o)$,

$$|\bar{M}(\xi)\,\hat{M}(\xi)^{-1}|\leqslant Ce^{-C_1/\varepsilon}\leqslant Ce^{-C_o/\varepsilon}.$$

This, together with (49), proves the estimate (47) for all ξ in $B_r(\xi^o)$. Suppose now that $\xi\in\hat{\Gamma}_1\backslash B_r(\xi^o)$. Then, if we call $d_j=(\xi_j-\xi_{j-1})/2$, for

$j=1, ..., n+1$, then, at least for one such j, say, j^*, $|d_{j^*}-d_{j^*+1}|>r/n^2$. It is enough to consider $d_{j^*}>d_{j^*+1}$, the other case being similar. Then,

$$|\gamma^{j^*}(\xi)| \geqslant K_1 \varepsilon e^{-2\nu/\varepsilon\, d^*_{j+1}} \, |1-e^{-2\nu/\varepsilon(d^*_j - d^*_{j+1})}|$$

$$\geqslant C\varepsilon e^{-2\nu/\varepsilon\, d^*_{j+1}} \geqslant C\varepsilon e^{-2\nu D^{\xi}/\varepsilon},$$

where, as usual, $C>0$ is independent of (ε, ξ). Since $\xi \in \hat{\Gamma}_1$, we have

$$|\gamma(\xi)| \geqslant |\gamma^{j^*}(\xi)| \geqslant C\varepsilon e^{C_o/\varepsilon} e^{-3\nu d^{\xi}/\varepsilon}$$

and again the estimate (47) is satisfied.

Now consider the case when, there are two contiguous layers, j and $j-1$, of the same type; that is, $q_j q_{j-1}=1$. In this case, there is j^* such that $q_{j^*} q_{j^*-1} = -q_{j^*+1} q_{j^*}$, and in that case,

$$|\gamma(\xi)| \geqslant |\gamma^{j^*}(\xi)| = K_1 \varepsilon \, |e^{-2\nu/\varepsilon\, d^*_j} + e^{-2\nu/\varepsilon\, d^*_{j+1}}|.$$

The arguments used in the last part of the previous case can be used here for all $\xi \in \hat{\Gamma}_1$. ∎

Remark 3.1. Although we do not give here a proof of that, the map $\xi \mapsto p^{\xi}$ from Γ into $C^o([0, 1])$ defines a one-to-one parametrization of M. The proof can be done as an application of the implicit function theorem in the same way as in [8], slightly modified since the fibrations are different.

The set M is then a manifold whose elements are functions exponentially close to the functions U^{ξ}, for ε small, therefore exhibiting the same transition layer structure. Our results suggest that M is an approximation to a true invariant manifold M, whose elements still have the same layers structure. According to (47), the motion in that invariant manifold should be characterized by the slow motion of transition layers with exponentially small speed, as $\varepsilon \downarrow 0$.

From the analysis of the vector field, we can draw some conclusions. Like the well-known case of the singularly perturbed bistable equation, while the transition layers are uniformly far apart, they move with a speed of order $O(e^{-C/\varepsilon})$, the motion of each layer being essentially determined by its distance to the two layers which are immediately before and after it. For the two extreme layers, their interaction with the boundary of the interval can also be reduced to the interaction with virtual layers which are their mirror image with respect to the two boundary points. However, a significant difference arises here when generalizing the way the layers interact with each other. In fact, a j layer and a $j+1$ layer tend to attract each other if $q_j q_{j+1} = -1$, and they tend to repel each other if $q_j q_{j+1} = +1$. The resulting motion is the sum of these interactions. In the bistable case only the first case can occur. These last facts have dramatic consequences for the dynamics on the slow motion manifold corresponding to each particular transition layer pattern. As a matter of fact, the condition $\gamma = 0$ is satisfied in a point of Γ only in the case where $q_j q_{j+1} = -1$, for $j = 0, 1, ..., n$, and, by Theorem 3.3.5, this condition is a necessary and sufficient one for the existence of an equilibrium on M. If this condition is satisfied, then, like in the bistable case, this manifold can be taken as the unstable manifold of an equilibrium, less a small subset close to the boundary.

4. A PERTURBATION

Consider one of the patterns for which there are, at least, two consecutive layers both of the increasing or the decreasing type. Therefore, the slow motion manifold considered above is not associated to any equilibrium. It is the main goal of this section to show how, in particular cases, the potential that defined the nonlinearity of (2) can be perturbed by a small term, for $\varepsilon > 0$ small, in such a way that the new equation has an equilibrium in a slow motion manifold corresponding to the same transition layers pattern.

Here, we consider the specific case of a pattern with two transition layers, both of the increasing type; that is, $n = 2$ and $q = (+1, +1)$. We also consider that the potential W is of the simplest type which, in the sense of Section 2, is compatible with this configuration, which is the tristable type. Therefore $\sigma = (1, 2, 3)$. We consider the three minimizers as being $m_1 = -1$, $m_2 = 0$, and $m_3 = 1$.

We now construct the perturbation of the potential. Let Ψ: $\mathbf{R} \to \mathbf{R}$ be a smooth even function such that Ψ is decreasing in $[0, +\infty]$ and, for a small $r > 0$,

$$\Psi(u) = \begin{cases} 0, & \text{if} \quad |u| > 2r, \\ 1, & \text{if} \quad |u| < r. \end{cases}$$

Consider $\psi = -\Psi'$, and for some ε-dependent parameter $\alpha(\varepsilon) \in \mathbf{R}$, consider the new equation

$$u_t = \varepsilon^2 u_{xx} + f(u) + \alpha(\varepsilon)\, \psi(u), \tag{50}$$

with Neumann b.c.'s. We consider the associated problem (13) with $F^j = -\alpha(\varepsilon)\, \psi(u^j + v^j) - f_2^j(v)$, for $j = 1, 2$, similar to Section 3. If we start by neglecting v, we obtain $F^j(x) = -\alpha(\varepsilon)\, \psi(u^j(x))$, for $x \in I_j$, $j = 1, 2$. Then,

$$\mathscr{F}_1^1 = -\frac{\varepsilon\alpha(\varepsilon)}{\|\Phi'\|^2}\int_0^{\eta_1} \psi(u^1)\, u_x^1\, dx = \frac{\varepsilon\alpha(\varepsilon)}{\|\Phi'\|^2}\left[\Psi(u^1(\eta_1)) - \Psi(u^1(0))\right] = \frac{\varepsilon\alpha(\varepsilon)}{\|\Phi'\|^2},$$

$$\mathscr{F}_1^2 = -\frac{\varepsilon\alpha(\varepsilon)}{\|\Phi'\|^2}\int_{\eta_1}^{1} \psi(u^2)\, u_x^2\, dx = \frac{\varepsilon\alpha(\varepsilon)}{\|\Phi'\|^2}\left[\Psi(u^2(1)) - \Psi(u^2(\eta_1))\right] = -\frac{\varepsilon\alpha(\varepsilon)}{\|\Phi'\|^2},$$

since $|u^1(0)|, |u^2(1)| > 2r$ and $|u^1(\eta_1)|, |u^2(\eta_1)| < r$, for $\varepsilon > 0$ sufficiently small. On the other hand, for $\varepsilon > 0$ small,

$$|\mathscr{F}_a^j|, |\mathscr{F}_b^j| \leqslant C\varepsilon\, |\alpha(\varepsilon)|, \qquad \text{for} \quad j = 1, 2.$$

Then, by (38), the ODE $\dot{\xi} = c(\xi)$ has an equilibrium at some $\xi^* \in \Gamma$ if and only if the following system is satisfied,

$$\begin{aligned} &c_o^1(\xi^*) + \frac{\varepsilon\alpha(\varepsilon)}{\|\Phi'\|^2} + \mathscr{F}_a^1 u_x^1(0) + \mathscr{F}_b^1 u_x^1(\eta^*) = 0, \\ &c_o^2(\xi^*) - \frac{\varepsilon\alpha(\varepsilon)}{\|\Phi'\|^2} + \mathscr{F}_a^2 u_x^2(\eta^*) + \mathscr{F}_b^2 u_x^2(1) = 0, \end{aligned} \tag{51}$$

where $\eta^* = (\xi_1^* + \xi_2^*)/2$. Now, we take $\eta^* = 1/2$. In the case,

$$c_o^1(\xi^*) = \frac{\varepsilon^2}{\|\Phi'\|^2}\left[-2\varepsilon u_x^1(0)\, u_x^1(0) + 2\nu u^1(1/2)\, u_x^1(1/2)\right],$$

$$c_o^2(\xi^*) = \frac{\varepsilon^2}{\|\Phi'\|^2}\left[-2\nu u^2(1/2)\, u_x^2(1/2) + 2\varepsilon u_x^2(1)\, u_x^2(1)\right].$$

Since $\xi_1^* = 1 - \xi_2^*$ and $1/2 - \xi_1^* = \xi_2^* - 1/2$, then $u_x^1(0) = u_x^2(1)$, $u^1(1/2) = -u^2(1/2)$, and $u_x^1(1/2) = u_x^2(1/2)$. Hence

$$c_o^1(\xi^*) = -c_o^2(\xi^*). \tag{52}$$

By the definition of $g^j(\cdot, \cdot)$, $j = 1, 2$, and the fact that $G(x, y) = G(-x, -y)$, for all $(x, y) \in \mathbf{R}^2$, we obtain, for all $y \in [0, 1/2]$, $g_x^1(0, y) = -g_x^2(1, 1-y)$ and $g^1(1/2, y) = g^2(1/2, 1-y)$. Then, taking also into account that

$$\psi(u^1(x)) = -\psi(u^2(1-x)), \qquad \text{for} \quad x \in [0, 1/2],$$

we obtain $\mathscr{F}_a^1 = -\mathscr{F}_b^2$ and $\mathscr{F}_a^2 = -\mathscr{F}_b^1$. Therefore,

$$\mathscr{F}_a^1 u_x^1(0) + \mathscr{F}_b^1 u_x^1(1/2) = -\mathscr{F}_a^2 u_x^2(1/2) - \mathscr{F}_b^2 u_x^2(1). \tag{53}$$

Then, by (52) and (53), both Eqs. (51) are satisfied if and only if only the first one is. On the other hand, since

$$\mathscr{F}_a^1 u_x^1(0) + \mathscr{F}_b^1 u_x^1(1/2) = O(e^{-2\nu\xi_1^*/\varepsilon} + e^{-\nu(\xi_2^* - \xi_1^*)/\varepsilon})\, \varepsilon\, |\alpha(\varepsilon)|,$$

as $\varepsilon \downarrow 0$, then, for $\varepsilon > 0$ small, Eqs. (51) defined uniquely $\alpha(\varepsilon)$ and, moreover,

$$\alpha(\varepsilon) = -\|\Phi'\|^2\, \varepsilon^{-1} c_o^1(\xi^*)(1 + O(e^{-\nu d^{\xi^*}/\varepsilon})).$$

In particular, $|\alpha(\varepsilon)| = O(e^{-2\nu d^{\xi^*}/\varepsilon})$, as $\varepsilon \downarrow 0$. Define

$$\hat{c}_o^1(\xi) = c_o^1(\xi) + \frac{\varepsilon\alpha(\varepsilon)}{\|\Phi'\|^2} + \mathscr{F}_a^1 u_x^1(0) + \mathscr{F}_b^1 u_x^1(\eta_1),$$

$$\hat{c}_o^2(\xi) = c_o^2(\xi) - \frac{\varepsilon\alpha(\varepsilon)}{\|\Phi'\|^2} + \mathscr{F}_a^2 u_x^2(\eta_1) + \mathscr{F}_b^2 u_x^2(1),$$

for $\alpha(\varepsilon)$, $\mathscr{F}_a$, and $\mathscr{F}_b$ as above. Note that

$$\hat{c}_o(\xi) = \gamma(\xi) - \gamma(\xi^*) + O(e^{-3\nu d^{\xi}/\varepsilon} + e^{-3\nu d^{\xi^*}/\varepsilon}). \tag{54}$$

Then, we have the following:

THEOREM 4.4.1. *For $\varepsilon > 0$ small, there are functions $\hat{p}\colon \Gamma \times [0, 1] \to \mathbf{R}$ and $\hat{c}\colon \Gamma \to \mathbf{R}^2$ such that the following statements are true:*

(i) *For each $\xi \in \Gamma$, $\hat{p}(\xi, \cdot) \in C^2(0, 1) \cap C^1([0, 1])$, and*

$$\begin{aligned} \varepsilon^2 \hat{p}_{xx} + f(\hat{p}) - \alpha(\varepsilon)\, \psi(\hat{p}) &= \hat{c}^1 u_{\xi_1}^1 + \hat{c}^2 u_{\xi_2}^2, \qquad \text{in} \quad (0, 1) \\ \hat{p}_x(0) = \hat{p}_x(1) &= 0. \end{aligned} \tag{55}$$

(ii) *The following partial derivatives exist for all $(\xi, x) \in \Gamma \times (0, 1)$, with the corresponding estimates holding true for each integer $\alpha_o \geqslant 0$ and multiindex α such that $0 \leqslant \alpha_o + |\alpha| \leqslant k$,*

$$|\partial_{x^{\alpha_o}} \partial_{\xi^\alpha} (\hat{p}(x, \xi) - U(\xi, x))| \leqslant C\varepsilon^{-(\alpha_o + |\alpha|)}\, e^{-\nu d^{\xi}/\varepsilon},$$

for each $x \in I_j$, $j = 1, ..., n$, and

$$|\hat{c}(\xi)|, \varepsilon\, |\hat{c}_{\xi_j}(\xi)| \leqslant C\varepsilon e^{-2d^{\xi}\nu/\varepsilon}.$$

(iii) *For an $r > 0$, ε-independent, there is $\xi^o \in \Gamma$ such that $\xi_j^o \in (\xi_j^* - r\varepsilon, \xi_j^* + r\varepsilon)$, $j = 1, 2,$*

$$\hat{c}(\xi^o) = 0,$$

and hence $\hat{p}(\xi^o, \cdot)$ *is an equilibrium of* (50) *in*

$$\hat{M} := \{ \hat{p}(\xi, \cdot) \mid \xi \in \Gamma \}.$$

Proof. Consider the space X_Δ, for $\Delta > 0$ small, defined in Section 3, and pick $\hat{v} \in X_\Delta$. By the mean value theorem, there are functions $w^j\colon I_j \to \mathbf{R}$, $j = 1, 2$, such that, for each $x \in I_j$, $w^j(x) \in [u^j(x) - |\hat{v}^j|, u^j(x) + |\hat{v}^j|]$, and

$$\psi(u^j + v^j) = \psi(u^j) + \psi'(w^j)\, v^j, \qquad \text{in } I_j, \qquad j = 1, 2.$$

Consider the problem (13) with

$$F^j = -\alpha(\varepsilon)\, \psi(u^j) - \alpha(\varepsilon)\, \psi'(w^j)\, \hat{v}^j - f_2^j(\hat{v}),$$

and define, $\mathscr{F}_1$, $\mathscr{F}_a$, and $\mathscr{F}_b$ relative to this F. Let $(\hat{c}, v)$ be the solution of that problem. Define also, for $j = 1, 2$,

$$\hat{F}^j = F^j + \alpha(\varepsilon)\, \psi(u^j).$$

Define, accordingly, $\hat{\mathscr{F}}_1$, $\hat{\mathscr{F}}_a$, and $\hat{\mathscr{F}}_b$. Then (38) can be written, for $j = 1, 2$, as

$$\begin{aligned}[\Pi\hat{c}]_j &= c_o^j + \mathscr{F}_1^j + \mathscr{F}_a^j u_x^j(\eta_{j-1}) + \mathscr{F}_1^b u_x^j(\eta_j)\\ &= \hat{c}_o^j + \mathscr{F}_1^j + \mathscr{F}_a^j u_x^j(\eta_{j-1}) + \mathscr{F}_1^b u_x^j(\eta_j).\end{aligned}$$

Since

$$\|\hat{F}^j\|_o \leqslant C(\alpha(\varepsilon) + \|\bar{v}\|_o)\, \|\bar{v}\|_o,$$

we obtain

$$|\hat{\mathscr{F}}_1|, |\hat{\mathscr{F}}_a|, |\hat{\mathscr{F}}_b| \leqslant C\varepsilon(\alpha(\varepsilon) + \|\bar{v}\|_o)\, \|\bar{v}\|_o,$$

and we can proceed as in the proof of 3.3.4 to prove that there is $v^* \in X_\Delta$ such that the solution of (13) when we consider $\bar{v} = v^*$ is (c^*, v^*) and

$$\|v^*\|_o \leqslant Ce^{-\nu d^\xi/\varepsilon} \qquad \text{and} \qquad c^* = \hat{c}_o + O(\varepsilon e^{-3\nu d^\xi/\varepsilon}), \tag{56}$$

as $\varepsilon \downarrow 0$. Again referring to c^* as $\hat{c}$ and defining $\hat{p}$ the way we did just before Theorem 3.3.5, we prove (i) and the part of (ii) corresponding to the zero order derivatives. Here, again, we omit the proof of the remainder of (ii), which can be done as in Theorem 3.3.5.

To prove (iii), from (56) and (54), we write

$$\hat{c}(\xi) = \gamma(\xi) - \gamma(\xi^*) + O(e^{-3\nu d^\xi/\varepsilon} + e^{-3\nu d^{\xi^*}/\varepsilon}). \tag{57}$$

Fixing $\eta_1 = 1/2$, we can define the scalar function, $\bar{\gamma}(\xi_1) := \gamma^1(\xi_1, 1-\xi_1)$. Obviously, $\gamma^2(\xi_1, 1-\xi_1) = -\bar{\gamma}(\xi_1)$. Similarly define $\bar{c}(\xi_1) := \hat{c}^1(\xi_1, 1-\xi_1)$. Then,

$$\bar{\gamma}'(\xi_1) = C(e^{-2\nu\xi_1/\varepsilon} - e^{-2\nu(1/2-\xi_1)/\varepsilon}).$$

Since $\xi^* \in \Gamma$ was chosen arbitrarily, subject only to the condition $\xi_1^* + \xi_2^* = 1$, we assume that this was chosen so that, for some $r > 0$ small, $\xi_1^* < 1/2 - \xi_1^* - 3r\varepsilon$. If $|\xi_1 - \xi_1^*| < r\varepsilon$, then $\xi_1 < 1/2 - \xi_1 - r\varepsilon$. In this case $d^\xi = \xi_1$ and $1/2 - \xi_1 > d^\xi + r\varepsilon$. Therefore,

$$\bar{\gamma}'(\xi_1) > C(1 - e^{-2\nu r})\, e^{-2\nu d^\xi/\varepsilon},$$

and, for all $\xi_1 \in [\xi_1^*, \xi_1^* + r\varepsilon)$,

$$\bar{\gamma}'(\xi_1) > C(1 - e^{-2\nu r})\, e^{-2\nu d^{\xi^*}/\varepsilon}.$$

Hence, redefining the constant $C > 0$ we have, by the mean value theorem,

$$\bar{\gamma}(\xi_1^* + r\varepsilon) - \bar{\gamma}(\xi_1^*) > C\varepsilon e^{-2\nu d^{\xi^*}/\varepsilon}.$$

Similarly, we can prove that, if $\xi \in (\xi_1^* - r\varepsilon, \xi_1^*]$ then

$$\bar{\gamma}(\xi_1^* - r\varepsilon) - \bar{\gamma}(\xi_1^*) < -C\varepsilon e^{-2\nu d^{\xi^*}/\varepsilon}.$$

On the other hand, if $|\xi_1 - \xi_1^*| = r\varepsilon$, then $e^{-3\nu d^\xi/\varepsilon} \leqslant Ce^{-3\nu d^{\xi^*}/\varepsilon}$, and by (57), if $\varepsilon > 0$ is sufficiently small then

$$\bar{c}(\xi_1^* - r\varepsilon) < 0, \qquad \text{and} \qquad \bar{c}(\xi_1^* + r\varepsilon) > 0,$$

and by the continuity of $\bar{c}$ we can conclude the existence of $\xi_1^o \in (\xi_1^* - r\varepsilon, \xi_1^* + r\varepsilon)$ such that

$$\hat{c}^1(\xi_1^o, 1 - \xi_1^o) = \bar{c}(\xi_1^o) = 0.$$

Since, by analyzing the terms $\hat{\mathscr{F}}_1$, $\hat{\mathscr{F}}_a$, and $\hat{\mathscr{F}}_b$ in a way similar to what we have done to arrive at (53), we obtain $c^2(\xi_1^o, 1-\xi_1^o) = -c^1(\xi_1^o, 1-\xi_1^o)$, then (iii) is proved. ∎

5. FINAL COMMENTS

In the Introduction, we wrote that, in this study, we would not give a proof that the manifold M we constructed is in fact very close to a true invariant manifold of the scalar reaction–diffusion equation considered and that c provides a good approximation to the motion on that invariant manifold. We can prove that there is a slow channel around M, in the spirit of [8] and [4]. For that proof we need the ε-estimates that we have obtained in our theorems, combined with spectral information about the linearization of $\mathscr{L}$ about M; that is, on

$$L^{\xi}u = \varepsilon^2 u_{xx} + f'(p^{\xi})\, u,$$

with domain $D = \{u \in H^2(0, 1) \mid u_x = 0, \text{ at } x = 0, 1\}$. In fact, based on the fact that this linear operator is the same as the one associated to the bistable equation with the transition layers in the same positions, the spectral analysis carried out in [8] can be entirely applied here. Hence, we conclude that the eigenvalues separate into two sets: the first n eigenvalues, which are small eigenvalues satisfying $|\lambda_j(\varepsilon)| = O(e^{-C/\varepsilon})$, and the others, which satisfy $\lambda_j(\varepsilon) < -C$, for ε-independent C. After proving that the space spanned by the eigenvectors corresponding to the small eigenvalues is close to TM_{ξ}, for each $\xi \in \Gamma$, with the estimates being uniform on ξ, we can make the same analysis as in [4] to prove the slow channel result.

However, with the same results we can still prove the invariant manifold theorem, as stated in the Introduction. Note that here we cannot rely on the existence of a hyperbolic equilibrium on the invariant manifold and, furthermore, there is no uniqueness of the invariant manifold. A proof generalizing the results on [11] to higher manifold dimensions will be done in some other place.

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