Diffusion neutral and diffusion dependent patterning

Ovidiu Radulescu[1], Sergei Vakulenko[2],

[1]IRMAR, UMR CNRS 6625, Université de Rennes 1, France
[2]Institute for Mech. Engineering Problems Sanct Petersburg, Russia

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Abstract

In this paper a connection between reaction-diffusion systems with small diffusion coefficients and the corresponding shorted systems (where diffusion is removed) is investigated. We obtain new sufficient conditions showing when solutions are completely defined by the corresponding shorted system and when, in contrast to this, pattern formation depends on diffusion. We call the two situations, diffusion neutral and diffusion dependent patterning. We also discuss the consequences of being in one or in the other of the two situations and present examples from developmental biology and from fluid mechanics.

1 Introduction

1.1 Statement of the problem

We study initial value boundary problems for reaction-diffusion systems with spatially inhomogeneous reaction terms:

$$u_t = \epsilon^2 D \nabla^2 u + f(u, x, \epsilon t), \tag{1.1}$$

where $u = u(x,t) \in \mathbf{R}^n$ is an unknown vector function, $x \in \Omega \subset \mathbf{R}^q$, where Ω is a compact domain with a regular boundary $\partial\Omega$, $t \geq 0$. In equation (1.1), D is a diagonal matrix with positive entries, i.e., $D = diag\{d_1, d_2, ..., d_n\}$, ϵ is a parameter for diffusion intensity, f is a reaction term depending, in general, on x and also slowly on time. Initial data has the following form

$$u(x, 0) = u_0(x). \tag{1.2}$$

In applications, boundary conditions are usually of the Neumann type. This means that there is no flux across the boundary:

$$\nabla u(x) \cdot n(x) = 0, \quad x \in \partial\Omega, \tag{1.3}$$

where $n(x)$ is an unit normal vector to the boundary at x. To avoid additional difficulties connected with boundary layers, we sometimes consider a simplified problem, when Ω is a box in $\mathbf{R}^q$, with periodical boundary conditions in x.

Two distinct situations are considered here. In the first situation time evolution of solutions ("patterning") is defined essentially by the reaction term and does not suffer qualitative changes in the limit of vanishing diffusion. The corresponding patterning mechanism will be called diffusion neutral. In the second situation patterning depends essentially on diffusion and the patterning mechanism will be called diffusion dependent. The patterns that we are studying contain interfaces, defined as regions where one or several components of the function u change abruptly. The two types of patterning correspond to two different types of interfaces.

Diffusion neutral patterning appears when dynamics of the "shorted system" (described by Eq.(1.1) with diffusion removed, $d_i = 0$) has, as a global attractor, an unique stable rest point (that can depend on x). In this case we prove theorems on a connection between solutions of the shorted system and the original reaction diffusion system. The existence of interfaces in the pattern results from the steep spatial dependence of the reaction term on x.

Examples of diffusion dependent patterning are rather extensive and will shall not treat them exhaustively here (we shall not discuss the important example of the Turing instability [37, 24]). Here we deal with the case when two local attractors of the shorted system associated to (1.1) coexist at some positions x. In this case, patterning is diffusion dependent and it is incorrect to remove the diffusion in the equations. Here an interface appears as a localized jump in space from one attractor to another and represents a traveling wave solution of the singularly perturbed reaction-diffusion system. Stationary patterns are formed by interfaces corresponding to zero velocity traveling waves. There exists a rather well developed mathematical theory describing this situation in the one-component case (see [2, 9, 5, 20, 23, 4] among many others). Traveling and stationary interfaces occur in many systems in biology (see [21]) and in the phase transition theory [22, 29, 27, 28, 26]. Their existence is well known for reaction-diffusion equations with bistable nonlinearities and for gradient or monotone reaction-diffusion systems [41, 42, 43]. For the sake of completeness and in view of applications, we slightly extend some classical results on the interface structure and motion in the diffusion dependent case.

1.2 Examples

Notice that mathematical models in developmental biology [45, 19, 30, 24] and in the theory of phase transitions [29, 27] lead to such reaction-diffusion equations with a spatial inhomogeneous reaction term. Two different approaches are fundamental in the theory of pattern formation.

The first approach, pioneered by the work of Turing [37], identifies diffusion-driven instability as a patterning mechanism. In Turing systems [18, 21, 24] a spatial dependence of the reaction term $f = f(u, x)$ is not necessary. However, the instability is possible only in very special regions of the parameter space [15].

The second approach is inspired from the ideas of developmental biologists such as Driesch and later Wolpert, who postulated the need for a pre-pattern that frames the so-called "positional information" and guides the patterning [45, 44]. In Wolpert systems, the translation symmetry is broken from the very beginning and the spatial dependence of the reaction term is essential. Recently, it has been shown that the chemical support of the pre-pattern is a maternal protein (morphogen) gradient developed in eggs soon after fecundation [30]. Biologists seem to reject the role of diffusion in Wolpert systems. This belief is backed up by the fact that without diffusion monotone morphogen profiles can induce arbitrarily complex layered patterns consisting of many narrow interfaces [45, 39, 38]. Other examples of Wolpert systems can be found in fluid mechanics. Flow-induced phase transitions can trigger the formation of bands separated by narrow interfaces [22, 29, 27, 28].

Both for Turing and for Wolpert systems, interfaces may play an important role in patterning. An interface can be defined as a region of strong inhomogeneity. Fick's law implies that diffusion is important at interfaces, but it says nothing on the role of diffusion for the control of the interface position and dynamics. In particular, how do the patterns behave in the limit of vanishing diffusion? In this paper, we discuss these issue for the less studied Wolpert systems.

Our main example is the gene circuit model proposed to describe early stages of Drosophila (fruit-fly) morphogenesis [19, 30]:

$$(u_i)_t = \epsilon^2 d_i \nabla^2 u_i + R_i \sigma_\alpha(\sum_{j=1}^{n} K_{ij} u_j + \sum_{k=1}^{p} J_{ik} m_k(x) - h_i) - \lambda_i u_i, \tag{1.4}$$

where u_j are zygotic gene concentrations, K is a matrix describing pair gene interaction between zygotic genes, J is a matrix describing pair interaction between zygotic genes and maternal morphogens, h_i are thresholds, m_i are functions of x which define maternal morphogens, $R_i > 0$ and $\lambda_i > 0$ are production and degradation rates. Here $\sigma_\alpha(h) = \sigma(\alpha h)$, σ is a monotone and smooth (at least twice differentiable) "sigmoidal" function such that

$$\sigma(-\infty) = 0, \quad \sigma(+\infty) = 1. \tag{1.5}$$

The function σ_α becomes a step-like function as its sharpness α tends to ∞. Typical examples can be given by

$$\sigma(h) = \frac{1}{1 + \exp(-h)}, \quad \sigma(h) = \frac{1}{2}\left(\frac{h}{\sqrt{1+h^2}} + 1\right). \tag{1.6}$$

Slow dependence of K, J, h, λ on time can be also considered.

Our second example is a generalization of the Allen-Cahn model of phase transitions. The spatial homogeneous version of this model has been discussed in connection to equilibrium first order phase transitions [1], population dynamics [10], metastability phenomena [2, 25]. It has been used as a toy model to describe shear banding of complex fluids [29]. The spatial inhomogeneous model can be described by the following reaction-diffusion equation with Neumann no flux boundary conditions on some compact set $\Omega \subset \mathbf{R}^q$:

$$u_t = \epsilon^2 u_{xx} - A^2(x,\tau)[u - u_0(x,\tau)][u - u_2(x,\tau)][u - u_1(x,\tau)], \tag{1.7}$$

where $\tau = \epsilon t$, $u_0(x,\tau) < u_2(x,\tau) < u_1(x,\tau)$ and $u_i(x,\tau), A(x,\tau) > 0$ are smooth (at least C^2), real functions.

1.3 Definitions and main results

We are interested in the influence of diffusion on patterning. We shall call a pattern formation mechanism "diffusion neutral", if the solution $u^\epsilon(x,t)$ of system (1.1) converges uniformly to a pattern $v(x,t)$ that can be found in the absence of diffusion, as the diffusion coefficients converge to zero. In this case the diffusion term is a regular perturbation. The pattern formation is "diffusion dependent", if this does not hold.

More precisely, let us formulate the following definition.

Let $u^\epsilon(x,t)$ denote the solution of reaction-diffusion system (1.1) with Neumann conditions (1.2) or with periodical boundary conditions in x on $\Omega = [0,1]^q$ and initial data $u^\epsilon(x,0) = u_0(x)$.

Let $v(x,t)$ be the solution of the following system of ordinary differential equations (we shall refer to it as the "shorted" system):

$$v_t = f(v,x,t), \quad v(x,0) = u_0(x). \tag{1.8}$$

Definition. *Patterning defined by problem (1.1), (1.2) with initial data $u_0(x)$ is diffusion neutral, if the estimate*

$$|u^\epsilon(x,t) - v(x,t)| < r_\epsilon$$

holds for any $x \in \Omega, t > 0$, where $r_\epsilon \to 0$ as $\epsilon \to 0$. The number r_ϵ is independent of x,t but can depend on initial data. Otherwise, patterning is diffusion dependent.

Stationary patterns and interfaces

An important situation in pattern formation is the case, where system (1.1) has steady state solutions. In order to study stationary patterns, we consider the autonomous case of system (1.1), where f does not depend on t.

There occur two essentially different cases that are illustrated in Fig. 1:

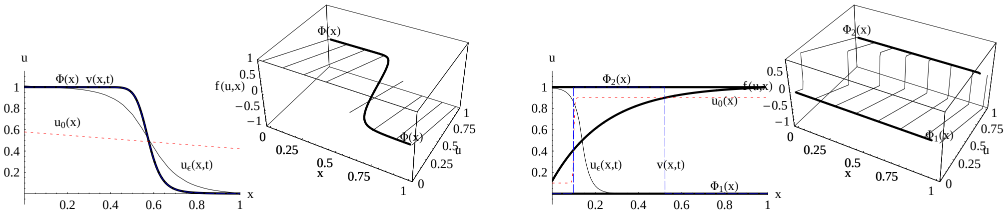
Case I. For any x, the shorted equation has an unique point attractor $v = \phi(x)$, where ϕ is a solution of the equation $f(\phi(x), x) = 0$. This attractor, for each x, attracts globally all trajectories of the shorted system.

For small ϵ, the solution $u^\epsilon(x,t)$ is close to the solution $v(x,t)$ of the shorted equation. This function v tends to $\phi(x)$ for large times. In this case a narrow interface can occur only if the function $f(\phi, x)$ is, in a sense, "sharp" in x. We shall refer to such a region as type 1 interface or "transition layer", because the variations of one or several components of the function $\phi(x)$ are strong across it. It is the case, for example, if, in the Eq. (1.4), the parameter α is large. For $n = p = q = 1$, $\sigma(h) = 1/[1 + exp(-h)]$, $J_{11}m_1(x) = kx$, $K_{11} = 0$, the solution ϕ has the following form

$$\phi(x) = \frac{A}{1 + exp[-\alpha(kx - h_1)]}, \quad A > 0. \tag{1.9}$$

This solution describes an interface at $x = h_1/k$ whose width scales with $1/(k\alpha)$ and is controlled by the nonlinearity of the sigmoidal function σ.

Case II. There exist domains in x, where the shorted equation $f(u,x) = 0$ has several different attractors, for example $\phi_1(x), \phi_2(x), ..., \phi_k(x)$. The interfaces correspond to a localized jump from an attractor to another one, the functions $\phi_k(x)$ having otherwise slow variation. In this case, the interface width scales with ϵ and is diffusion controlled. We shall refer to these as type 2 interfaces or



a) b)

Figure 1: Gene circuit model for $n = 1, p = 1, q = 1, \sigma_\alpha(x) = 1/(1 + exp(-\alpha x))$, $m(x) = exp(-x)$. a) Diffusion neutral patterning: $\alpha = 80.0$, $K_{11} = 0$, $J_{11} = 1.0$, $h_1 = 0.1$, $\epsilon^2 = 0.1$. b) Diffusion dependent patterning: $\alpha = 400.0$, $K_{11} = 1.7$, $J_{11} = 1.5$, $h_1 = 1.71$, $\epsilon^2 = 0.001$; although $u^\epsilon(x,t)$ does not depend on initial data $u_0(x)$ and has only one kink, the solution $v(x,t)$ of the shorted system depends on initial data and can have many kinks.

"diffusion layers". Diffusion layers can be avoided if the initial data $u_0(x)$ lies in an attraction basin of only one attractor, for instance $\phi_1(x)$. In this situation, patterning is again diffusion neutral.

Our main result for diffusion neutral patterning are Theorems 2.1 and 2.2 that we summarize here.

Let us suppose that in the autonomous case the shorted system (1.8) has only point attractors $\phi_k(x)$, with disjoint open attraction basins $B_k(x)$ for all x. The case of bifurcations, where several curves $\phi_k(x)$ touch one another [36] needs a special treatment and will not be considered in this paper. Furthermore, let us assume that the initial data $u_0(x)$ is contained in $B_1(x)$ for any x. Then the solution $u^\epsilon(x,t)$ of system (1.1) with periodic boundary conditions stays, for all times t, in a small neighborhood (whose size shrinks to zero uniformly when $\epsilon \to 0$) of the solution $v(x,t)$ of the shorted system : patterning is diffusion neutral.

If, for some x, the shorted system possesses stable limit cycles or chaotic attractors, we can prove only a weak variant of Theorem 2.1, where u^ϵ stays close to v on a large but bounded time interval $O(-\log \epsilon)$. It is difficult to say something beyond this interval. If the attractor of the shorted system is a stable limit cycle, rigorous results are unknown, however, a formal asymptotic solution can be obtained by the Kuramoto method [17]. This solution shows that, for generic initial data, patterning is diffusion dependent. Nothing is known on the structure of the solution u^ϵ in the case of chaotic attractors of the shorted system. However, taking into account that such attractors contain many periodic trajectories, one can suppose that in this case patterning is diffusion dependent as well.

Some properties of diffusion dependent patterning are given by Theorems 3.1 and 4.2. Theorem 3.1 stipulates that patterning with diffusion layers is necessarily diffusion dependent. Theorem 4.2 extends previous results of [5] by allowing a slow time dependence of the reaction term on timescales slower than the diffusion time. For the one-component, one-dimensional patterning problem we reduce the dynamics to a differential equation for the position of the diffusion layer.

2 Diffusion neutral patterning

Let us formulate some assumptions. Suppose that the reaction term f is a C^2 regular function of x, u and the initial data u_0 have at least C^2-regularity on the domain Ω. Let us consider the case when Ω is a compact box in $\mathbf{R}^m$ and let us set periodical boundary conditions along all its edges.

Our next assumption is concerned with the dynamics of the shorted system (1.8). Notice that (1.8) is a system of ordinary differential equations in $\mathbf{R}^n$ whose right hand sides and initial data involve x as a parameter.

Let us remind definitions of invariant domains and invariant rectangles [34]. Let U be a domain in $\mathbf{R}^n$ having a C^1- smooth boundary ∂U and let $n(u)$, $u \in \partial U$ be the outward pointing unit vector field.

We say that U is an *invariant domain* for the system (1.8) at fixed x, if there holds

$$n(u) \cdot f(u, x) < 0 \quad (u \in \partial U). \tag{2.1}$$

The following condition guarantees that, for each fixed x, (1.8) defines a global semiflow in $U(x)$:

Dissipativity condition for shorted system: *For any $x \in \Omega$ there exists an invariant domain $U(x)$. Moreover, the initial data satisfy $u_0(x) \in U(x)$.*

Let us turn to the reaction-diffusion system (1.1). If all the diffusion coefficients are equal: $d_i = d$ we define invariant domains as above, but we set the additional condition that the invariant domain Π should be *convex* [34]. For example, it can be a ball. Notice that Π is independent of x. If the diffusion coefficients are different, instead of convex invariant domains, we use *invariant rectangles U*. In this case, a stronger assumption guarantees global existence and uniqueness of solutions of the problem (1.1),(1.2):

Uniform Dissipativity condition: *There exists an invariant rectangle $\Pi \subset \mathbf{R}^n$ (a convex invariant domain, if $d_i \equiv d$) such that for any x*

$$n(u) \cdot f(u, x) < 0 \quad (u \in \partial \Pi), \tag{2.2}$$

and

$$u_0(x) \in \Pi \quad (x \in \Omega).$$

This condition is always satisfied by the genetic circuit model (1.4), and by the Allen-Cahn model (1.7) when the functions u_0, u_1 are uniformly bounded.

Let us recall the definition of local attractor and of their attraction basin for dissipative systems of ordinary differential equations. Let us assume that the dissipativity condition holds. For a fixed x, the set $\mathcal{A}_x$ is a local attractor for the shorted system, if this set has an attraction basin $B(\mathcal{A}_x)$ containing an open neighborhood V_x of $\mathcal{A}_x$. The attraction basin $B(\mathcal{A}_x)$ consists of all $u_0(x)$ such that the trajectory $u(t, x, u_0(x))$ starting at $u_0(x)$ approaches to $\mathcal{A}_x$ for large times:

$$\mathrm{dist}[u(t, x, u_0(x)), \mathcal{A}_x] \to 0 \quad (t \to \infty).$$

Attraction basin condition: *Suppose that for each x $\mathcal{A}_x = \{\phi(x)\}$ is a local point attractor of the system* (1.8)*. Moreover, assume that the initial data $u_0(x)$ lie, for all x, in the corresponding basin $B(\mathcal{A}_x)$:*

$$u_0(x) \in B(\mathcal{A}_x) = B(\{\phi(x)\}). \tag{2.3}$$

Let us formulate a condition for the stability of $\mathcal{A}_x$. Let $M(x)$ be the derivative of f at $\phi(x)$:

$$f(\phi + w, x) - f(\phi, x) = M(x)w + h(w, x), \quad |h| < C|w|^2 \tag{2.4}$$

for small w and some constant C. Suppose that there holds the following

Strong linear stability assumption: *For any $x \in \Omega$ the corresponding matrix $M(x)$ at $\phi(x)$ satisfies the following condition:*

$$\sum_{j \neq i} |M_{ij}(x)| + M_{ii}(x) \leq -b < 0. \tag{2.5}$$

This condition ensures that the spectrum of the matrix $M(x)$ lies in the negative half-plane being separated by an uniform gap from the imaginary axis. Hence, $\{\phi(x)\}$ has an open attraction basin B_x for all x.

We can state now

Theorem 2.1. *Let $f = f(u, x)$ and u_0 are sufficiently smooth as formulated above.*

Under the dissipativity condition, the strong linear stability and the attraction basin conditions for sufficiently small ϵ the solution u^ϵ of the time autonomous version of problem (1.1), (1.2) exists for all $t > 0$ and stays close to the solution of the shorted system (1.8), *i.e.,*

$$u^\epsilon(x, t) = v(x, t) + \tilde{v}^\epsilon(x, t), \tag{2.6}$$

where the correction $\tilde{v}^\epsilon$ satisfies the estimate

$$|\tilde{v}^\epsilon(x, t)| < c\epsilon^s, \quad t \geq 0, \tag{2.7}$$

where $c > 0$ is a constant independent of ϵ, $s > 0$.

Furthermore, for sufficiently large times $t > t_\epsilon$ the following result holds:

$$u^\epsilon(x, t) = \phi(x) + \tilde{u}^\epsilon(x, t), \tag{2.8}$$

where $\tilde{u}^\epsilon(x, t)$ satisfies the following estimate uniformly in x:

$$|\tilde{u}^\epsilon(x, t)| < c_1\epsilon^2. \tag{2.9}$$

Remark 1. Although the uniform dissipativity condition is needed for an unique global solution of (1.1), (1.2), a priori estimates ensuring the global existence of solutions hold *for small diffusion coefficients* under the essentially weaker dissipativity condition for the shorted system. The proof of the Theorem 2.1 is given in the Appendix.

Remark 2. The estimates (2.6), (2.7) can be extended to the non-autonomous case, where f depends slowly on time but we omit these details.

If $d_1 = d_2 = ... = d_n = d$, this result can be improved. In this case the strong linear stability condition on $M(x)$ can be weakened. It is sufficient to suppose that the spectrum of $M(x)$ lies in the left half-plane for each x. More precisely, let us suppose that there holds the following

Weak linear stability assumption: *For each fixed x the solution of the linear evolution equation*

$$v_t = M(x)v,$$

satisfies the estimate

$$|v(t)| \le |v(0)| \exp(-\sigma t), \quad \sigma > 0 \tag{2.10}$$

where σ is independent of x.

Then, we have

Theorem 2.2. *Suppose that the conditions of Theorem 2.1, where the strong stability condition is replaced by the weak stability assumption, hold. Moreover, let the diffusion coefficients be equal*

$$d_1 = ... = d_n = d\epsilon^2 > 0. \tag{2.11}$$

Then, for sufficiently small ϵ, the solution u^ϵ of problem (1.1), (1.2) exists for all $t > 0$, stays close to a solution of shorted system (1.8) *and satisfies the estimates (2.7), (2.9).*

The proof is given in the Appendix.

3 Diffusion dependent patterning

If condition (2.3) of Theorem 2.1 is invalid, i.e., if initial data $u_0(x)$ lie in attraction basins of different attractors of the shorted system for different x, then the estimates of the previous section hold only on a time interval of order $O(-\log \epsilon)$. The main reason for that is the following. In this case the derivatives of the solution $v(x,t)$ of the shorted equations with respect to x increase unboundedly at some fixed positions $x = q_0$ as $t \to \infty$. In the presence of diffusion, one can expect that there exist diffusion layers connecting different attractors of the shorted system. These layers are mobile and they eventually reach equilibrium positions which are generally different from the initial positions q_0.

To understand this situation, let us remind some basic definitions from the theory of finite dimensional dynamical systems [31]. Let us consider the time autonomous shorted system (1.8) ($f = f(v,x)$) for a fixed $x \in \Omega$ and let us suppose that the uniform dissipativity condition holds in a ball $B(R_0)$ in $\mathbf{R}^n$. Let us recall that $u_0(x) \in B(R_0)$ is wandering if there exists a neighborhood $U(u_0)$ of u_0 and a time $T_0 > 0$ such that for all $t > T_0$ the trajectory $v(t,x,u_0(x))$ starting from $u_0(x)$ does not intersect $U(u_0)$: $v(t,x,u_0(x)) \notin U(u_0), \ t > T_0$. In the theory of finite dimensional dynamical systems, the set of nonwandering points play a key role [31]. It is a closed invariant set which contains the ω-limit sets of all trajectories ($w \in \mathbf{R}^n$ is the ω- limit set of the trajectory $v(t,x,u_0(x))$ if and only if there is a sequence $t_k \to \infty$ such that $v(t_k,x,u_0(x)) \to w$ as $k \to \infty$). In our case these sets depend on x, since x plays the role of a parameter. Typically, the nonwandering set Θ_x consists of some connected components ${A^i}_x$. Some components are local attractors (i.e., A^i_x attracts all trajectories starting in an open neighborhood of A^i_x), others are saddle points or repellers. Intuitively, initial interface position can be associated with x such that the corresponding trajectory $v(x,t,u_0(x))$ goes to a saddle point or hangs in a fixed repeller as $t \to \infty$ [20].

However, a rigorous mathematical analysis is nontrivial even in the simplest cases. To illustrate, let us consider two simple examples. In these examples we set periodic boundary conditions in $x \in \Omega$, supposing that Ω is a box.

Example 1. Let us consider a system of Allen-Cahn's type:

$$u_t = \epsilon^2 \Delta u + a(x)(u - u^3), \quad u \in \mathbf{R}, \tag{3.1}$$

where $a(x) > 0, x \in \Omega \subset \mathbf{R}^q$. The nonwandering set of the shorted system is $\{-1, 0, 1\}$, where $1, -1$ are local attractors and 0 is a saddle point. If $u_0(x) > 0$ for all $x \in \Omega$ or if $u_0(x) < 0$ for all $x \in \Omega$, one can apply Theorems 2.1 or 2.2 and $u(t, x) \to \pm 1 + \mathcal{O}(\epsilon^s)$ as $t \to \infty$, respectively.

If there are points, where $u_0(x) = 0$, we can expect a more complicated behavior. The analysis [20] shows that the gradient of $u(x, t)$ increases to ∞ at the set $S = \{x : \ u_0(x) = 0\}$. Fife [7] described in detail the development of the inner layer in the one-dimensional case ($q = 1$) and when the set S contains a single point.

For arbitrary $u_0(x)$, the set S can be a rather complicated, fractal set. If $u_0 \in C^\infty$ and if the rank of the derivative $\mathrm{d}u_0(x)$ is $r, 0 \le r \le q$ then by the implicit function theorem S is a smooth sub-manifold of Ω of co-dimension r. Like in [20], we use the concept of "generic situation", which is standard in differential topology [12]. If the initial data u_0 are smooth and generic, $\mathrm{d}u_0(x)$ has full rank $r = 1$ (which is the dimension of the u-phase space), hence $codim S = 1$. This means that, within times t of order $-\log \epsilon$, one obtains an interface appearance at some hypersurfaces in Ω. These hypersurfaces are initial positions for interface evolution for $t >> O(-\log \epsilon)$.

Example 2. Let us consider a system of Ginzburg-Landau's type:

$$u_t = \epsilon^2 \Delta u + a(x)(u - |u|^2 u), \tag{3.2}$$

where u is an unknown complex valued function, $a(x) > 0$ in Ω.

The non wandering set of the shorted system is the union of the attracting limit cycle $|u| = 1$ and the repeller $\{0\}$. The set S is now, for generic smooth $u_0(x)$, a submanifold of codimension 2. This means that if $dim\ \Omega = 1$, generically, we have no singularity growth, if $dim\ \Omega = 2$ one obtains vortices localized at some points and if $dim\ \Omega = 3$ we have vortex lines [20].

Let us now formulate a result identifying the diffusion dependent patterning situation considered in the above examples.

Let us denote by $v(t, x, u_0(x))$ the solution of the time autonomous shorted system (1.8) with the initial data $u_0(x)$.

Let us formulate the following conditions on the dynamics of (1.8):

C1 *For each x the nonwandering set Θ_x of* (1.8) *is a union of the two disjoint subsets, $\Theta_x = A_x \cup A'_x$, where A_x is an attracting set for an open neighborhood V_x, $A_x \subset V_x$. This means that the trajectories $v(t, x, v_0)$, starting at the points $v_0 \in V_x$, satisfy the condition:*

$$dist\{v(t, x, v_0), A_x\} \to 0$$

as $t \to \infty$. We also assume that V_x continuously depends on x, i.e.,

$$dist\{V_x, V_y\} \to 0$$

as $x \to y$.

C2 *The sets A_x are separated from the other components of the nonwandering sets Θ_y, i.e.,*

$$\inf_{x,y} dist\{A_x, A'_y\} > 0.$$

These conditions are fulfilled in the examples (3.1), (3.2). They also hold if $f(v, x)$ and has some continuous non-intersecting branches of roots $v_k(x)$, $k = 1, 2, ..., M$, where $f_v(v_k(x), x) < 0$, serving as point attractors. It is the case of the Allen-Cahn model. It is more difficult to check these conditions for the genetic circuit model with small degradation rates λ_i. In this case, bifurcations of the attractor sets are possible.

Theorem 3.1. *Consider the time autonomous case $f = f(u, x)$, $f \in C^2$, where Ω is a box and we set the periodic boundary conditions. Assume that our reaction-diffusion system satisfies the uniform dissipativity condition (2.2).*

Suppose the shorted system satisfies conditions **C1, C2**. *Suppose, moreover, that there exist two distinct points $x_0, x_1 \in \Omega$ such that the solutions of the shorted system satisfy*

$$v(t, x_0, u_0(x_0)) \to A_{x_0},$$

and the ω -limit set of the trajectory $v(t, x_1, u_0(x_1))$ lies in the complement A'_{x_1} of A_{x_1}.

Then patterning with the initial data $u_0(x)$ is diffusion dependent.

Proof. Let us suppose that patterning is diffusion neutral. Let us consider the set Ω_V consisting of $x \in \Omega$ such that the trajectory $v(t, x, u_0(x))$ enters V_x for some $t = t_0(x)$: $v(t_0, x, u_0(x)) \in V_x$. It is clear Ω_V is an open set since V_x depends continuously on x and trajectories $v(x, t, u_0(x))$ also depend continuously on x within time bounded intervals. If $x \in \Omega_V$, the corresponding trajectory $v(t, x, u_0(x))$ of shorted system (1.8) converges to A_x. Indeed, this trajectory enters V_x, and since V_x is contained in the basin attraction of A_x, this trajectory tends to A_x as $t \to \infty$.

Let $\overline{\Omega}_V$ be the closure of Ω_V in Ω. As Ω_V is open in the connected set Ω, there exists a point x_* such that $x_* \in \overline{\Omega}_V$ and $x_* \notin \Omega_V$. Thus, for any $\delta > 0$ there exists $y_\delta \in \Omega_V$ such that $dist(y_\delta, x_*) < \delta$. Consider now the trajectories $v(t, y_\delta, u_0(y_\delta))$ and the trajectory $v(t, x_*, u_0(x_*))$. The first ones converge to the corresponding sets $A_{y_\delta} \subset V_{y_\delta}$. The trajectory $v(t, x_*, u_0(x_*))$ approaches to a ω -limit set C_* as $t \to \infty$ which does not intersect A_{x_*}. Thus this ω -limit set C_* lies in the complement A'_{x_*}. Therefore, this set C_* does not intersect any A_y for any y (due to assumption **C2**). Thanks to these limit properties of the trajectories $v(t, x_*, u_0(x_*))$ and $v(t, y_\delta, u_0(y_\delta))$ one concludes that, for sufficiently small $\kappa > 0, \delta > 0$ there is a time moment $T(\kappa, \delta)$ such that

$$|v(t, y_\delta, u_0(y_\delta)) - v(t, x_*, u_0(x_*))| > \kappa, \quad t = T(\kappa, \delta).$$

Let us choose a small ϵ such that $r_\epsilon < \kappa/2$, where r_ϵ is a constant from the definition of diffusion neutral patterning, see subsection 1.2.

Let us fix such κ, ϵ. Then the previous estimate and the definition of diffusion neutral patterning yield

$$|u(t, y_\delta) - u(t, x_*)| > \frac{\kappa}{2}, \quad t = T(\kappa, \delta),$$

where $dist(y_\delta, x_*) < \delta$. This holds for any δ. Letting $\delta \to 0$ one notices that

$$|\nabla u(x_*, T(\kappa, \delta))| \to \infty \quad (\delta \to 0).$$

However, solutions of our problem (1.1), (1.2) are a priori bounded due to the uniform dissipativity. Therefore, the Schauder a priori estimate is fulfilled: $|u_x| < c\epsilon^{-1}$ [16]. This fact gives us a contradiction with the last limit relation. The theorem is proved.

4 Properties of diffusion layers

In the case of diffusion dependent patterning, interfaces are diffusion layers. Let us characterize here, their structural and mobility properties. Unfortunately, there is not a general theory on such layers for $n > 1$. The diffusion layers can be described in the one-component case $n = 1$, and for $n > 1$ only in the case of gradient systems, or in the case of monotone systems [41, 42, 43]. Let us remind that gene circuits generate a monotone dynamics if the interaction matrix K_{ij} satisfies the cooperativity condition $K_{ij} \geq 0,\ i \neq j$.

It is convenient to analyze the motion of diffusion layers in two steps. We shall only deal with the one-component case $n = 1$.

I. Internal (diffusion) layer problem in infinite homogeneous media

Let us ignore temporarily the dependence of f on the space variable x and on the slow time variable $\tau = \epsilon t$ and let us suppose that f has the shape in Fig. 1 b). This means that $f(u, x, \tau)$ satisfies the following assumptions (bistability): for each x, τ, the function $f(u, x, \tau)$ has only three roots $w_0(x, \tau), w_1(x, \tau)$ and $w_2(x, \tau))$ such that

$$f_u(w_0, x, \tau) < 0, \quad f_u(w_1, x, \tau) < 0, \quad f_u(w_2, x, \tau) > 0, \quad w_0 < w_2 < w_1, \tag{4.1}$$

for any x, τ.

Notice that this assumption is satisfied for the Allen-Cahn model. It is also satisfied for gene circuits under some conditions such as in the situation illustrated in Fig. 1 b).

Furthermore, let us consider that the pattern is infinite, i.e., $x \in (-\infty, +\infty)$. This leads to the following problem involving the parameters q, τ:

$$u_t = u_{xx} + f(u, q, \tau), \tag{4.2}$$

$$\lim_{x \to -\infty} u(x) = w_0(q, \tau), \quad \lim_{x \to +\infty} u(x) = w_1(q, \tau). \tag{4.3}$$

Denote by $\Phi(u, x, \tau)$ the primitive $\Phi(u, x, \tau) = \int_{w_0}^{u} f(s, x, \tau) ds$.

The well known result for this problem can be formulated as follows.

Proposition 4.1 *The internal (diffusion) layer problem (4.2), (4.3) has traveling wave solutions $u = \psi(x - V(q, \tau)t, q, \tau)$, where the velocity V is a functional of f (and thus depends on the parameters q, τ). The velocity $V(q, \tau) = 0$ if $\Phi(w_0, q, \tau) = \Phi(w_1, q, \tau)$. The function ψ is increasing in x for any fixed q, τ and satisfies the following exponential estimates*

$$|\psi(z) - w_0| < C \exp(b_0 z), \quad z < 0, \tag{4.4}$$

$$|\psi(z) - w_1| < C \exp(-b_1 z), \quad z > 0, \tag{4.5}$$

and

$$|\psi_z| < C\exp(-b|z|), \quad b = \min\{b_0, b_1\}, \tag{4.6}$$

where C, b_i, b are positive constants depending on q, τ.

Proof Below we sometimes omit dependence on q, τ in notation. The function $\psi(\xi, q, \tau)$ satisfies the equation

$$\psi_{\xi\xi} + V\psi_\xi + f(\psi, q, \tau) = 0. \tag{4.7}$$

The existence of a solution of equation (4.7) satisfying exponential estimates is well known, for example, [6, 42]. Multiplying equation (4.7) by ψ_ξ, replacing $f = \Phi_\psi$ and integrating one obtains :

$$\Phi(w_0) - \Phi(w_1) = \frac{V}{2}\int_{-\infty}^{\infty} (\psi_\xi)^2 \, d\xi. \tag{4.8}$$

This equation shows that the velocity $V = 0$ and thus the traveling wave is a stationary interface if $\Phi(w_0) = \Phi(w_1)$.

Remark 1. If $V = 0$, equation (4.7) has a first integral $\frac{1}{2}\psi_\xi^2 + \Phi(\psi) = \Phi(w_0)$. This fact allows us to calculate the integral

$$\int_{-\infty}^{\infty} (\psi_\xi)^2 \, d\xi = \sqrt{2}\int_{w_0}^{w_1} \sqrt{\Phi(w_0) - \Phi(\psi)} d\psi.$$

Thus, one obtains an approximate formula valid for small interface velocities:

$$V \approx \frac{\sqrt{2}[\Phi(w_0) - \Phi(w_1)]}{\int_{w_0}^{w_1} \sqrt{\Phi(w_0) - \Phi(\psi)} d\psi} \tag{4.9}$$

Remark 2. This result can be generalized for monotone systems, see [41, 42, 43]. However, even for weakly non-monotone systems this result is not correct: there are waves with a very complex structure, whose wave front profiles change with position and time [40]. For these examples the wave front cannot be presented as a function of $x - Vt$ and Proposition 4.1 does not hold; we shall not consider these questions here.

Example 1.

Let us consider the interface properties for a genetic circuit with a single zygotic gene and a single morphogen ($n = 1, K_{11} = 1, m(q) = J_{11}m_1(q) > 0$, $\lambda = \lambda_1$, $h = h_1$). Supposing that α (sharpness) is a large parameter, then for positions such that $h - 1/\lambda + \mathcal{O}(\alpha^{-1}) < m(q) < h + \mathcal{O}(\alpha^{-1})$ we obtain two stable stationary solutions ($u_- = \mathcal{O}(\alpha^{-1})$, $u_+ = \lambda^{-1} + \mathcal{O}(\alpha^{-1})$ and one unstable u_s defined by the relations $u_s = h - m + \alpha^{-1}v$, $\sigma(v) = \lambda(h - m + \alpha^{-1}v)$, $|v| < C$.

In this case one has $\Phi(u) = (u + m - h)\mathrm{H}(u + m - h) - \lambda u^2/2 + \mathcal{O}(\alpha^{-1})$, where H is the Heaviside step function. By this relation and (4.8) one finds

$$V(q) = \rho(q)[\frac{1}{2\lambda} - h + m(q)] + \mathcal{O}(\alpha^{-1}), \tag{4.10}$$

where $\rho > 0$ for all q. Using (4.9) one obtains the approximation $\rho(q) \approx 2\lambda\sqrt{2\lambda}$ valid for small velocities. We have a stationary interface if $m(q) = h - \frac{1}{2\lambda} + \mathcal{O}(\alpha^{-1})$.

Example 2.

The shorted equation of the Allen-Cahn model has two attractor nodes u_0, u_1 and a saddle u_2. The values u_0, u_1 are the two maxima and u_2 is the minimum of the fourth order polynomial potential $\Phi(u)$. In this case the traveling wave velocity is

$$V(q,\tau) = \sqrt{2A(q,\tau)}[u_0(q,\tau) + u_1(q,\tau) - 2u_2(q,\tau)]. \tag{4.11}$$

There exist thus stationary interfaces, where the saddle is precisely at half distance between the two nodes:

$$u_2(q,\tau) = [u_0(q,\tau) + u_1(q,\tau)]/2. \tag{4.12}$$

II. Internal (diffusion) layer problem in heterogeneous media

Due to Proposition 4.1 we can get a formal asymptotic solution of the interface motion problem. Let us set

$$u_{as} = \psi(\theta(x,t,\epsilon), q, \tau) + O(\epsilon), \tag{4.13}$$

where

$$\theta = \epsilon^{-1}(x - q(t)) \tag{4.14}$$

and where a unknown function $q(t)$ determines the localization of the narrow interface. By substituting the expression for u_{as} in equation (1.1) one observes that the terms of the principal order $O(1)$ vanish under the condition

$$\frac{dq}{dt} = \epsilon V(q, \epsilon t). \tag{4.15}$$

One can therefore expect that this equation describes the interface (diffusion layer) propagation. If f depends only on x, there exist stationary solutions of equation (4.15). Indeed, if $q_1, q_2, ..., q_l$ are roots of the function $V(q)$ then the diffusion layer is immobile in any one of the positions q_j. These equilibrium positions can be stable or unstable. It is easy to see that if the derivative $V_q^j > 0$, then the corresponding position is unstable, and if $V_q^j < 0$, the position is stable. In this situation there are stable stationary solutions with diffusion layers.

After this heuristic reasoning let us formulate the main result of this section.

Let us assume that the function f satisfies *uniform dissipativity* condition, namely, there exists positive constants M_1, M_2 uniform in x, τ such that

$$f(u,x,\tau) < 0, \quad u > M_1, \tag{4.16}$$

$$f(u,x,\tau) > 0, \quad u < -M_2. \tag{4.17}$$

This assumption ensures the existence of an unique smooth solution of (1.1) (see [11]). Notice that the uniform dissipativity condition always holds for one-component genetic circuits. It also holds for the Allen-Cahn model if the functions $u_0(q,\tau), u_1(q,\tau)$ are uniformly bounded.

Theorem 4.2 (on the interface motion). *Let us consider the reaction-diffusion equation*

$$u_t = \epsilon^2 u_{xx} + f(u,x,\epsilon t), \quad u(x,0) = u_0(x), \tag{4.18}$$

where $x \in [0,1]$, under the zero Neumann boundary conditions

$$u_x(0,t) = u_x(1,t) = 0.$$

Assume $f \in C^2$. Let us suppose that the assumptions of Proposition 4.1 hold and

$$w_1(x,\tau) - w_0(x,\tau) > \delta_0 > 0,$$
$$\sup_{x,\tau} f_u(w_i, x, \tau) = -\mu_i < 0, \quad i = 0, 1, \tag{4.19}$$

where $x \in [0,1]$, $\tau > 0$. Moreover, assume that

$$(w_0)_x(0,\tau) < 0, \quad (w_1)_x(1,\tau) > 0. \tag{4.20}$$

I. *Let $\psi(\xi, q, \tau)$ be an interface solution of (4.7) with asymptotic boundary conditions $\psi(-\infty, q, \tau) = w_0(q,\tau)$, $\psi(\infty, q, \tau) = w_1(q,\tau)$ (see Proposition 4.1). Suppose that the initial data are sufficiently close to this interface. More precisely,*

$$|u_0(x) - \psi(\epsilon^{-1}(x - q_0), q_0, 0)| < \bar{c}_0 \epsilon^s, \quad s \in (0,1). \tag{4.21}$$

Then, if ϵ is small enough, the solution of the problem has the interface form

$$u = \psi(\epsilon^{-1}(x - q(t)), q(t), \epsilon t) + v, \tag{4.22}$$

where the correction v satisfies the estimate

$$|v| < C\epsilon^s, \quad C > \bar{c}_0 > 0 \tag{4.23}$$

and the time evolution of the interface position q is defined by the differential equation

$$\frac{dq}{dt} = \epsilon(V(q, \epsilon t) + R(q, \epsilon, t)), \quad q(0) = q_0, \tag{4.24}$$

where

$$|R| < c' \epsilon^{s_1}, \quad s_1 > 0.$$

Equation (4.24) holds while $q > c\epsilon^{s_2}$ and $q < 1 - c\epsilon^{s_2}$, where $s_2 \in (0,1)$.

II. *Consider the time-independent case: $f = f(u,x)$. If there is a point q_* such that*

$$V(q_*) = 0, \quad V'(q_*) < 0 \tag{4.25}$$

then there is a stationary, well localized at $x = q_*$ interface solution tending to a step-like function as $\epsilon \to 0$.

The proof of the theorem is given in the Appendix.

Remarks . The assertion **II** is a consequence of results obtained by Fife [5], who also considered the time evolution of interfaces in the case $f = f(u,x)$ [7]. Condition (4.20) is technical and simplify estimates. Condition (4.19) yields that the functions $b_i, i = 0, 1$ defined by Eqs. 5.5, 5.6 satisfy $b_i(x,\tau) > \tilde{b}_i > 0$, $i = 0, 1$, for all $x \in [0,1]$, $\tau > 0$.

5 Applications

5.1 Is the segmentation of the fruit fly diffusion neutral?

Diffusion neutral patterning is believed to be the main mechanism in the segmentation of the fruit fly [45]. The purpose of this section is to discuss the correctness of this hypothesis.

The one-dimensional patterning dynamics of gap genes of the fruit fly has been studied in [30], where the gene circuit model is used (1.4).

This model satisfies our uniform dissipativity condition with Π a large enough rectangle. The weak linear stability follows trivially from the fact that $\phi(x)$ is a non-degenerate stable steady state of the shorted equation and from the regularity of the function σ. In order to prove that patterning is diffusion neutral one has to check only one more condition : the attraction basin condition. Next, we discuss this condition in some particular cases.

The steady states of the shorted system of (1.4) are solutions of:

$$u_i = \frac{R_i}{\lambda_i}\sigma_\alpha(\sum_{j=1}^{n} K_{ij}u_j + \sum_{k=1}^{p} J_{ik}m_k(x) - h_i) = G_i(u). \tag{5.1}$$

For large λ_i (or small R_i) the application $u \to G(u)$ is a contraction map and thus an unique stable rest point u_0 exists. In this case patterning is diffusion neutral. For smaller λ_i it is possible to have coexistence of several point attractors. A rather standard necessary criterium for multi-stationarity is the existence of a positive loop in the interaction directed graph defined by the matrix K [35]. For competitive gene circuits ($K_{ij} \leq 0$ for all $i \neq j$) such as the gap gene circuit of the fruit fly a positive loop means a loop made of an even number of interactions. As pointed out by Smale [32] competitive systems in dimension n can have any dynamics that is possible in $n-1$ dimensions. Chaotic attractors could be expected for $n \geq 4$ and limit cycles for $n \geq 3$. Competitive gene circuits for $n = 2$ have only point attractors and for $n = 3$ all attractors are either limit circles or points [13, 14]. Cooperative gene circuits ($K_{ij} \geq 0$ for all $i \neq j$) are monotone and monotone systems have particularly simple dynamics: almost all trajectories converge to point equilibria [33].

Although there are no general methods for finding attractors of the gene circuit's shorted system, in some special cases there are algorithms allowing the exhaustive determination of point attractors (solutions of (5.1)). In the following we discuss such a special situation.

Let us consider the case $\alpha >> 1$ and let us restrict for simplicity to the case of a single morphogen. Then the point attractors of the shorted equations have the following form as $\alpha \to \infty$:

$$u_i^a = R_i\lambda_i^{-1}s_i(x) + O(\alpha^{-1}), \quad s_i(x) \in \{0, 1\}. \tag{5.2}$$

Locally in x the point attractors of the shorted equation are discrete and are indexed by the set $\{0, 1\}^n$. They are the solutions of the following binary programming problem:

$$\sum_{j=1}^{n} \tilde{K}_{ij}s_j(x) + m(x) > \tilde{h}_i, \quad if \ s_i(x) = 1, \tag{5.3}$$

$$\sum_{j=1}^{n} \tilde{K}_{ij} s_j(x) + m(x) < \tilde{h}_i, \quad if \ s_i(x) = 0, \tag{5.4}$$

where $\tilde{K}_{ij} = \frac{K_{ij} R_j}{J_{i1}\lambda_j}$, $\tilde{h}_i = \frac{h_i}{J_{i1}}$, $m(x) = m_1(x)$.

The shorted system has other steady state solutions (for instance saddle points) that can not be obtained by this method.

In the fruit fly embryo $m(x)$ is a monotonous function of x ($q = 1, x \in \mathbb{R}$). The solution of the programming problem can be given by specifying for each possible steady state $\mathcal{S}^{(k)} = (s_1^{(k)}, s_2^{(k)}, \ldots, s_n^{(k)}) \in \{0,1\}^n$ the domain $\mathcal{I}^{(k)}$ in x, where this state exists. If $m(x)$ is monotonous, then all $\mathcal{I}^{(k)}$ are intervals:

$$\mathcal{I}^{(k)} = \{x | \max_{s_i^{(k)}=1} (\tilde{h}_i - \sum_j \tilde{K}_{ij} s_j^{(k)}) < m(x) < \min_{s_i^{(k)}=0} (\tilde{h}_i - \sum_j \tilde{K}_{ij} s_j^{(k)})\}.$$

The pattern consists of segments corresponding to different states $\mathcal{S}^{(k)}$. If the intervals $\mathcal{I}^{(k)}$ have non-overlapping interiors, patterning is diffusion neutral and the interfaces separating segments are transition layers. For overlapping interval interiors, depending on initial data, one can have diffusion layers and diffusion dependent patterning.

Example 1 : $n = 1$, one component.

In the gene circuit model, the one-component case describes the situation of a zygotic gene u_1 which is not regulated by other zygotic genes $u_j, j \neq 1$, i.e. $K_{1j} = 0$ for any $j \neq 1$.

In this case there are two possible point attractors, $\mathcal{S}^{(1)} = (1)$,$\mathcal{S}^{(2)} = (0)$, coding for $u_1^a = R(\lambda)^{-1} + O(\alpha^{-1})$, $u_2^a = O(\alpha^{-1})$, and existing in the intervals $I^{(1)} = \{x | m(x) > \tilde{h}_1 - \tilde{K}_{11}\}$, $I^{(2)} = \{x | m(x) < \tilde{h}_1\}$, respectively. A saddle steady state $u_3^s(x)$ also exists (this should be calculated by a different method). If $K_{11} = 0$ (gene 1 has no regulation effect on itself) the two interval interiors do not overlap. The patterning is diffusion neutral. The interface between the two states is a transition layer and for $\epsilon << \alpha^{-1}$ its width is of order $\mathcal{O}(\alpha^{-1})$ (see Fig. 1 a). This one-component case is the one usually used by biologists to illustrate Wolpert's positional information mechanisms [45] and the diffusion neutrality hypothesis is entirely justified here.

The case $K_{11} < 0$ (gene 1 inhibits itself) needs special treatment. Indeed, in this case there is a gap between the intervals $I^{(1)}$ and $I^{(2)}$ on which the boolean programming problem has no solution at all. This phenomenon is due to the singular character of the limit $\alpha \to \infty$ and disappears for finite α. Let us consider the function $f_\alpha(u, m) = \sigma_\alpha(K_{11}u + J_{11}m - h_1) - \lambda_1 u$. The function f is monotonic in m. The steady state of the shorted equation satisfies $f_\alpha(u, m) = 0$. If $m \in (\tilde{h}_1, \tilde{h}_1 - \tilde{K}_{11})$, then $f_\alpha(0, m) > 0$, $f_\alpha(1/\lambda_1, m) < 0$ and, therefore, there is an unique $u_0(m)$, $0 < u_0(m) < 1/\lambda_1$ such that $f_\alpha(u_0(m), m) = 0$. The function $\phi(x) = u_0(m(x))$ describes a smooth transition layer connecting the states $u_1 = \mathcal{O}(\alpha^{-1})$ and $u_2 = 1/\lambda_1 + \mathcal{O}(\alpha^{-1})$. Patterning is diffusion neutral because the branch of attractors $\phi(x)$ is unique. The pattern contains a transition layer whose width does not reduce to zero as $\alpha \to \infty$.

If $K_{11} > 0$ (gene 1 activates itself) there is an overlap of the intervals $I^{(1)}$,$I^{(2)}$ and depending on the initial data it is possible to have type 2 interfaces (diffusion layers) and diffusion dependent patterning. The diffusion layer forms at a position q_0 satisfying $u_0(q_0) = u_3^s(q_0)$ (u_3^s gives the position of the saddle in the phase space). According to Theorem 4.2 the width of the fully formed diffusion layer scales like ϵ and vanishes in the limit of zero diffusion Fig. 1 b). Furthermore, equation (4.10) giving the kink velocity allows to obtain the following approximate equation for the position of the diffusion layer:

$$\frac{dq}{dt} = 2\lambda\epsilon(2\lambda)^{1/2}[(2\lambda)^{-1} - h + m(q)]. \tag{5.5}$$

The layer is at rest at a position q_* such that $m(q_*) = h - (2\lambda)^{-1}$.

Example 2: $n = 2$, two components.

Let us suppose that $\tilde{K}_{ii} = 0, i = 1, 2, \tilde{K}_{12} \leq 0, \tilde{K}_{21} \leq 0$. In this case there are four possible steady states, $\mathcal{S}^{(1)} = (1, 1)$,$\mathcal{S}^{(2)} = (0, 0)$,$\mathcal{S}^{(3)} = (1, 0)$,$\mathcal{S}^{(4)} = (0, 1)$. The corresponding existence intervals are $I^{(1)} = \{x|m(x) > \max(\tilde{h}_1 - \tilde{K}_{12}, \tilde{h}_2 - \tilde{K}_{21})\}$, $I^{(2)} = \{x|m(x) < \min(\tilde{h}_1, \tilde{h}_2)\}$, $I^{(3)} = \{x|\tilde{h}_1 < m(x) < \tilde{h}_2 - \tilde{K}_{21}\}$, $I^{(4)} = \{x|\tilde{h}_2 < m(x) < \tilde{h}_1 - \tilde{K}_{12}\}$. Notice that $I^{(3)}$ and $I^{(4)}$ can overlap if the following condition is satisfied:

$$\tilde{h}_1 \leq \tilde{h}_2 \leq \tilde{h}_1 - \tilde{K}_{12}, \quad \text{or} \quad \tilde{h}_2 \leq \tilde{h}_1 \leq \tilde{h}_2 - \tilde{K}_{21}. \tag{5.6}$$

In general, existence of a diffusion layer connecting $\mathcal{S}^{(3)}$ and $\mathcal{S}^{(4)}$ depends on the initial data and on condition (5.6).

A diffusion layer always exists if an extremity of $I^{(3)}$ belongs to the interior of $I^{(4)}$, or reciprocally if an extremity of $I^{(4)}$ belongs to the interior of $I^{(3)}$. This condition corresponds to strict inequalities in (5.6).

The interiors of $I^{(1)}$, and of $I^{(2)}$ do not overlap on the interiors of other intervals, therefore the interfaces separating the states $\mathcal{S}^{(1)}$ or $\mathcal{S}^{(2)}$ from any other states are transition layers. Contrary to the preceding example, there are no longer gaps between intervals meaning that for small diffusion coefficients the widths of all the transition layers are of the order $\mathcal{O}(\alpha^{-1})$.

To conclude, in this case diffusion neutrality is justified for interfaces involving $\mathcal{S}^{(1)}$ or $\mathcal{S}^{(2)}$ but it is not always justified for interfaces separating $\mathcal{S}^{(3)}$ from $\mathcal{S}^{(4)}$.

A discussion of real data will be presented in a separate publication.

5.2 Shear banding in pipe flow, a diffusion dependent patterning

Shear banding is an instability of complex fluids observed in solution of wormlike micelles. In [29] a toy model consisting of one reaction-diffusion equation has been used to mimic the basic properties of the phenomenon. This equation has similar properties with respect to the existence and the propagation of kinks as the system yielded by a more realistic Johnson-Segalman model with stress diffusion [22, 28, 29]. In the Poiseuille (pipe flow) flow geometry the equation reads:

$$S_t = \epsilon^2 S_{xx} - S\left[1 + \left(\frac{\sigma(x,\tau) - S}{\eta}\right)^2\right] + \frac{\sigma(x,\tau) - S}{\eta}, \quad \tau = \epsilon t, \tag{5.7}$$

where S is the part of the total shear stress carried by the micelles, ϵ^2 is the stress diffusion coefficient, $0 < \eta < 1/8$ is the retardation parameter, i.e., the ratio between the viscosity of the solvent and the viscosity of the micelles. σ is the total shear stress, which in the Poiseuille geometry depends linearly on the distance x from the pipe axis $\sigma(x,\tau) = g(\tau)x$, the slow time function $g(\tau)$ is the pressure gradient that sustains the flow.

The shorted equation for this system is

$$S_t = f(S,\eta,\sigma) - S\left[1 + \left(\frac{\sigma - S}{\eta}\right)^2\right] + \frac{\sigma - S}{\eta}. \tag{5.8}$$

If $0 < \eta < 1/8$, the third order polynomial $f(S,\eta,\sigma)$ has one real root which is a stable attractor for (I) $\sigma < \sigma_1(\eta)$, or (II) $\sigma > \sigma_2(\eta)$, and three real roots among which two stable attractors $S_0(\sigma), S_1(\sigma)$ and a saddle $S_2(\sigma)$ for (III) $\sigma_1(\eta) < \sigma < \sigma_2(\eta)$. One can notice that in the case (III) the model is equivalent to the Allen-Cahn model with the parameters $A = \frac{1}{\eta}$, $u_i = S_i(\sigma), i = 1, 3$.

Starting from the pipe axis σ increases with x and one passes from the situation (I) close to the axis to the situation (III) and eventually, if the pressure gradient is large enough to the situation (II) close to the pipe walls. Hence if $g > \sigma_1(\eta)/L$ where L is the half width of the pipe, an interface can form parallel to the walls, separating bands of high (close to the pipe axis) and low viscosity (close to the walls). In this example, patterning is diffusion dependent. In the presence of diffusion, the interface is mobile and its position q propagates according to the following (approximate) equation:

$$\frac{dq}{dt} = \sqrt{\frac{2D}{\eta}}[S_0(g(\tau)q) + S_1(g(\tau)q) - 2S_2(g(\tau)q)]. \tag{5.9}$$

If g is not time dependent the interface will be at rest in the position q_* satisfying $S_0(gq_*) + S_1(gq_*) - 2S_2(gq_*) = 0$. If such a position does not exist, steady flow will not be banded. If such a rest point exists, then the asymptotic relaxation of the interface position is exponential $q - q_* = C_1 \exp(-t/\tau)$, with $\tau^{-1} = \sqrt{\frac{2D}{\eta}} g[S_0'(gq_*) + S_1'(gq_*) - 2S_2'(gq_*)]$. This relation has been used in [28] to estimate D from rheological measurements.

Appendix: proofs

Proof of Theorem 2.1. The proof uses the following Lemma, which is a slightly modified version of a comparison theorem for systems of reaction-diffusion equations [34]. To formulate this Lemma, we introduce two sets depending on two vectors w^-, w^+. We denote

$$E_+^i(w^+, w^-) = \{\xi | \xi_i = w^+, w^- \leq \xi_j \leq w^+, \forall j \neq i\},$$

$$E_-^i(w^+, w^-) = \{\xi | \xi_i = w^-, w^- \leq \xi_j \leq w^+, \forall j \neq i\}.$$

Lemma A.1 *Let $w(x,t) \in \mathbb{R}^n$ be the solution of problem (1.1), (1.2) with zero Neumann or periodic boundary conditions and initial data $w(x,0) = w_0(x) \in \mathbb{R}^q$. Let the time dependent functions $w^+(t), w^-(t)$ satisfy*

$$w_t^+ \geq \max_i \sup_{\xi \in E_+^i(w^+,w^-)} f_i(\xi, x),$$

$$w_t^- \le \min_i \inf_{\xi\in E_-^i(w^+,w^-)} f_i(\xi,x),\ \textit{for any}\ \ x\in\Omega,$$

where $1 \le i \le n$.

Moreover, let us suppose

$$w^-(0) \le w_i(x,0) \le w^+(0),\ \textit{for any}\ \ x\in\Omega.$$

Then

$$w^-(t) \le w_i(x,t) \le w^+(t),\ \textit{for any}\ t>0,\ \ x\in\Omega.$$

To simplify the proof, we proceed with it in two parts, I and II. In the first part we show that $u^\epsilon(x,t)$ stays within distance $o(\epsilon^s)$ from the solution $v(x,t)$ of the shorted system for times $t < t_\epsilon = -a\log(\epsilon)$. We use the exponential decay of $v(x,t)$ to $\phi(x)$ in order to show that at $t = t_\epsilon$, the solution $u^\epsilon(x,t)$ is within distance $o(\epsilon^s)$ from $\phi(x)$. In the second part we apply again the lemma and find that $u^\epsilon(x,t)$ remains within distance $o(\epsilon^s)$ from $v(x,t)$ for all $t \ge t_\epsilon$.

Part I. Let $w(x,t) = u^\epsilon(x,t) - v(x,t)$, where $v(x,t)$ is the solution of the shorted system with the same initial data as $u^\epsilon(x,t)$, $v(x,0) = u_0(x)$.

Notice that the function w satisfies the equation

$$w_t = \epsilon^2 D\Delta w + f(v+w,x) - f(v,x) + \epsilon^2 g. \tag{5.10}$$

One has $|f(v+w,x) - f(v,x)| < C_2|w|$ and $g = \Delta v$. Moreover, C^2 regularity of f and the initial data $u_0(x)$ imply that $|g| < C_1$.

Let us apply the comparison lemma to $w_t^+ = C_2 w^+ + C_1\epsilon^2$, $w^+(0) = 0$, $w^-(t) = -w^+(t)$. From the relation $w^+(t) = C_1 {C_2}^{-1}\epsilon^2(\exp(C_2 t) - 1)$ it follows that $|w(x,t)| \le C_3\epsilon^{2-C_2 a}$ for $t \le t_\epsilon = -a\log\epsilon$ for positive a such that $C_2 a < 2$.

Now, using the spectral properties of the matrix $M(x)$ and the attractive nature of $\phi(x)$, we have

$$|v(x,t) - \phi(x)| < B\exp(-bt), t > T_0 \tag{5.11}$$

that holds uniformly for any $x\in\Omega$.

At $t = t_\epsilon$, one has $|v(x,t_\epsilon) - \phi(x)| \le \delta\epsilon^{ba}$, hence $|u^\epsilon(t_\epsilon,x) - \phi(x)| \le C_4\epsilon^s$, where $\min(2 - C_2 a, ba) > s > 0$.

Part II. Let us define $\bar w(x,t)$ by $u^\epsilon(x,t) = \bar w(x,t) + \phi(x)$. Then $\bar w$ satisfies the equation:

$$\bar w_t = \epsilon^2 D\Delta\bar w + M(x)\bar w + \epsilon^2\Delta\phi + h(\bar w,x,t), \tag{5.12}$$

where $|h| \le C_5|\bar w|^2$.

From the C^2 regularity of f one obtains $|\Delta\phi| < C_6$. Indeed, ϕ satisfies $f(\phi(x),x) = 0$. Under the linear stability condition one can use the implicit function theorem that gives $|\nabla\phi| < \bar C$ and $|\Delta\phi| < C_6$.

Let us choose w^+ to satisfy the equation

$$w_t^+ = -bw^+ + C_5(w^+)^2 + C_6\epsilon^2,$$

$w^+(t_\epsilon) = |u^\epsilon(x,t_\epsilon) - \phi(x)|$, and $w^-(t) = -w^+(t)$. Using the strong stability assumption on M, we find that $\max_i \sup_{\xi\in E_+^i(w^+,w^-)} \sum_j M_{ij}\xi_j \le -bw^+$. This fact ensures that the function w^+ satisfies Lemma A.1. The function w^+ is less than $c\epsilon^s$ at $t = t_\epsilon$ and $|w^+| < c\epsilon^s$ for all $t \geq t_\epsilon$. Thus $|u^\epsilon(x,t) - \phi(x)| < c\epsilon^s, t \geq t_\epsilon$. Together with estimate (5.11) this proves the first inequality (2.7). To prove (2.9), let us observe that estimate (2.7) can be improved as follows. Since $w^+ < c\epsilon^s$, we have

$$w_t^+ < -bw^+ + C_7\epsilon^{s'}w^+ + C_6\epsilon^2 < -\frac{b}{2}w^+ + C_6\epsilon^2.$$

This implies that

$$w_+ < \bar{w}(t_\epsilon)\exp(-\frac{b}{2})(t-t_\epsilon) + 2C_1b^{-1}\epsilon^2(1-\exp(-\frac{b}{2})(t-t_\epsilon)).$$

This completes the proof.

Proof of Theorem 2.2. The first part of the proof repeats, without any changes, the proof of Theorem 2.1 (in fact, this part uses no properties of $M(x)$).

The second part must be modified and uses now the weak stability assumption and condition (2.11).

Let us consider again the equation

$$w_t = \epsilon^2 d\Delta w + M(x)w + \epsilon^2 g + h(w,x,t), \tag{5.13}$$

where $g = \Delta\phi$, $|h| < c|w|^2$. To estimate the solutions w of this equation, we introduce the matrix $W(x)$ of size $n \times n$ defined as follows [3]

$$W(x) = \int_0^\infty \exp(M(x)^\dagger t)\exp(M(x)t)dt.$$

This matrix is correctly defined since estimate (2.10) holds. The matrix W is symmetric and positively defined. Let (u,v) denote the inner scalar product in $\mathbf{R}^n$ and $|u|$ is the norm. Then,

$$(Wu,u) = \int_0^\infty |\exp(M(x)t)u|^2 dt \leq \rho|u|^2, \quad \rho = \frac{1}{2\sigma} > 0$$

and the norms $|u|$ and $(Wu,u)^{1/2}$ are equivalents.

Moreover, let us notice that the definition of W entails [3]

$$WM + M^\dagger W = -I.$$

Let us define now a scalar function $R(x,t)$ by $R^2 = (Ww,w)$ and let us calculate the time derivative of this function for solutions w of equation (5.13). We obtain

$$\frac{1}{2}(R^2)_t = d\epsilon^2[(W\Delta w,w) + (Ww,\Delta w)] + (W(h+\epsilon^2 g),w) + (w,W(h+\epsilon^2 g)) +$$

$$+(WMw,w) + (Ww,Mw).$$

Notice that

$$(WMw,w) + (Ww,Mw) = -|w|^2.$$

Furthermore,

$$|(W(h+\epsilon^2 g),w) + (w,W(h+\epsilon^2 g)| \leq c_0R^3 + c_1\epsilon^2 R.$$

Let us consider the term $Y = (W\Delta w, w) + (Ww, \Delta w)$. This term can be represented as

$$Y = \Delta R^2 - 2(W\nabla w, \nabla w) - 2(\nabla W \nabla w, w) - 2(\nabla W w, \nabla w).$$

Let us notice that

$$|(\nabla W \nabla w, w) + (\nabla W w, \nabla w)| \leq c_2(\mu |\nabla w|^2 + \mu^{-1}|w|^2).$$

Let us choose μ such that $c_2\mu < \rho$. Then we find that

$$Y \leq \Delta R^2 + c_2\mu^{-1}|w|^2 \leq \Delta R^2 + c_3 R^2.$$

As a result, one obtains the differential inequality

$$\frac{1}{2}(R^2)_t \leq \epsilon^2 d\Delta R^2 + c_4\epsilon^2 R + c_5 R^3 - R^2\rho^{-1} + \epsilon^2 c_6 R^2, \quad t \geq t_\epsilon,$$

where $R(t_\epsilon) = c_7\epsilon^s,\ s > 0$. By the standard scalar comparison principle [34] this differential inequality implies that

$$R < c_8\epsilon^{s'}, \quad s' > 0.$$

The theorem is proved.

Proof of Theorem 4.2.

Proof of part I. Below some key estimates can be simplified if w_0, w_1 are independent of x, τ. We reduce the general situation to this case by introducing a new variable $\tilde{u}$:

$$u = \beta(x,\tau)\tilde{u} + w_0(x,\tau), \quad \beta = w_1(x,\tau) - w_0(x,\tau), \tag{5.14}$$

where $\tau = \epsilon t$.

For the new unknown function $\tilde{u}$ the corresponding values $\tilde{w}_i$ are $0, 1$, respectively. Moreover, the boundary conditions take the following form

$$\beta\tilde{u}_x(0,t) = -w_{0x}(0,\tau) - \beta_x(0,\tau)\tilde{u}(0,t), \tag{5.15}$$

$$\beta\tilde{u}_x(1,t) = -w_{0x}(1,\tau) - \beta_x(1,\tau)\tilde{u}(1,t). \tag{5.16}$$

Making the change $u \to \tilde{u}$ one obtains the equation

$$\tilde{u}_t = \epsilon^2\tilde{u}_{xx} + \tilde{f}(\tilde{u}, x, \epsilon t) + \epsilon g_1(u, u_x, x, \epsilon t), \tag{5.17}$$

where

$$\begin{aligned} g_1(x,t,\epsilon) &= \beta^{-1}(\beta_\tau\tilde{u} + \epsilon\beta_{xx}\tilde{u} + 2\epsilon\beta_x\tilde{u}_x + \epsilon w_{0xx} - w_{0\tau}) \\ \tilde{f}(\tilde{u}, x, \tau) &= f(\beta\tilde{u} + w_0, x, \tau) \end{aligned} \tag{5.18}$$

We notice that $|g_1| < c$, with a constant c uniform in x, t, ϵ. Below, to simplify formulas, we omit the symbol tilde in notation.

To prove assertion **I**, we use the comparison principle [8]. Our construction of a supersolution follows from [6], in a modified form, since here, with respect two [6], there are two additional difficulties: smallness of ϵ and the nonlinearity

f depending on x, τ. Since $w_1 > w_0$, the function ψ defined by Prop. 4.1, is increasing in $\theta = \epsilon^{-1}(x - q)$ for all fixed τ, q and thus $\psi_\theta > 0$.

As a supersolution u^+ we take

$$u^+ = \psi(\theta, q, \tau) + \delta, \tag{5.19}$$

where δ is a small number, ψ is defined by Prop. 4.1 and satisfies

$$-V(q, \tau)\psi_\theta = \psi_{\theta\theta} + f(\psi, q, \tau). \tag{5.20}$$

The function q is defined by

$$\frac{dq}{dt} = \epsilon(V(q, \tau) - \delta_1), \quad q(0) = q_0, \tag{5.21}$$

where the "unperturbed" speed $V(q, \tau)$ is defined by Prop. 4.1. Both constants δ, δ_1 depend on ϵ.

Let us check that u^+ satisfies the comparison principle [8]. We must check three main inequalities [8]. The first, according to Theorem 17 (Chapter II) from [8], at the boundaries $x = 0, 1$ our supersolution must satisfy the inequalities

$$\beta u_x^+(0, t) + w_{0x}(0, \tau) + \beta_x(0, \tau)u^+(0, t) < 0, \tag{5.22}$$

$$\beta u_x^+(1, t) + w_{0x}(1, \tau) + \beta_x(1, \tau)u^+(1, t) > 0. \tag{5.23}$$

One can check that, under the conditions $q > c\epsilon^{s_2}$ and $q < 1 - c\epsilon^{s_2}$, $s_2 \in (0, 1)$, both inequalities hold due to (4.20) and our explicit formula for u^+. In fact, up to exponentially small corrections, the left hand side of inequality (5.22) is $(w_0)_x(0, \tau)$ and the left hand side of (5.23) is $(w_1)_x(1, \tau)$.

The second, at $t = 0$ our supersolution must majorize the initial data:

$$u^+(x, 0) > u_0(x).$$

This estimate holds due to relations (5.19), (5.21) and (4.21) if $\delta = C\epsilon^s$ and $C > \bar{c}_0$.

The third, it is necessary to check the following inequality:

$$0 > (V - \delta_1)\psi_\theta + \psi_{\theta\theta} + f(\psi + \delta, x, \tau) - \epsilon\psi_q(V - \delta_1) - \epsilon\psi_\tau + \epsilon g_1(\psi + \delta, \psi_x, x, \epsilon t). \tag{5.24}$$

Let us notice that

$$|\psi_\tau|, |\psi_q| < c.$$

Furthermore, by the Poincaré inequality we have $a\delta_0^2 < \int_{-\infty}^{\infty} \psi_\theta^2 d\theta, a > 0$ and by equation (4.8), one obtains $|V(q, \tau)| < b/\delta_0^2$.

Using the definition of ψ and the last estimates, let us replace inequality (5.24) by a stronger inequality

$$0 > -\delta_1\psi_\theta + f(\psi + \delta, x, \tau) - f(\psi, q, \tau) + C_1\epsilon. \tag{5.25}$$

This inequality can be rewritten as

$$0 > -\delta_1\psi_\theta + F_1 + F_2 + C_1\epsilon, \tag{5.26}$$

where

$$F_1 = f(\psi + \delta, x, \tau) - f(\psi, x, \tau),$$

$$F_2 = f(\psi, x, \tau) - f(\psi, q, \tau).$$

Let us introduce two domains Ω_1 and Ω_2 depending on t. The domain Ω_1 is an union of the two intervals

$$\Omega_1^+ = \{q < x < q - a_1\epsilon \log \epsilon\}, \tag{5.27}$$

$$\Omega_1^- = \{q + a_0\epsilon \log \epsilon < x < q\}, \tag{5.28}$$

and Ω_2 is a complementary domain

$$\Omega_2 = \Omega - \Omega_1. \tag{5.29}$$

Here the constants $a_i > 0$ are uniform in ϵ and will be chosen below together with δ, δ_1.

We choose these parameters as follows:

$$\delta = C\epsilon^s, \quad \delta_1 = C_1\epsilon^{s_1}, \quad s, s_1 > 0,$$

where $C > \bar{c}_0$, the numbers s_1, s and a_i satisfy

$$s_1 < s < s_1 + a_1\tilde{b}_1 < 1, \tag{5.30}$$

$$s < s_1 + a_0\tilde{b}_0 < 1. \tag{5.31}$$

It is clear that, given positive $\tilde{b}_i$, such a choice of a_i, s_1, s is always possible. Below all constants C, C_k, c, c_k are uniform in ϵ as $\epsilon \to 0$.

Let us estimate F_2. Suppose $x > q$. One has

$$|F_2| \leq |f_\xi(\psi, \xi, \tau)||x - q|,$$

where $\xi \in [q, x]$. Since $f(w_1, q, \tau) = 0$ for any q and w_1 is independent of q (due to our assumption in beginning of the demonstration) one has $f_q(w_1, q, \tau) = 0$.

Thus the last estimate gives

$$|F_2| \leq c\epsilon|\psi(\theta, q, \tau) - w_1||x - q|.$$

Using the exponential estimates $|\psi - w_1| < C\exp(-\tilde{b}_1\theta)$ for the interfaces (see Prop. 4.1) and the fact that $\theta\exp(-\tilde{b}_1\theta)$ is a bounded function on $(0, \infty)$, one has finally

$$|F_2| \leq c_1\epsilon, \quad x > q.$$

The same inequality can be derived for $x < q$. Using this estimate, we replace main inequality (5.26) to a stronger inequality:

$$0 > -\delta_1\psi_\theta + F_1 + C_2\epsilon, \tag{5.32}$$

where C_2 is uniform in ϵ as $\epsilon \to 0$.

Let us turn into F_1. Let $x > q, x \in \Omega_2$. We use now the following remark (see also [6]). If ρ is sufficiently small, for example, $|\rho| < r_1$, where r_1 is independent of ϵ, then

$$f_u(w_1 + \rho + \rho_1, x, \tau) < -\mu_1/2$$

for any $\rho_1 \in [0, \delta)$. Therefore, one obtains (due to our conditions on $f_u(w_1, x, \tau)$)

$$F_1 < -c_3\delta, \quad x \in \Omega_2, \ x > q.$$

First let us consider (5.32) in the domain Ω_2 for $x > q$ (the case $x < q$ can be considered in a similar way). By replacing F_1 by its upper estimate one derives a stronger inequality

$$0 > C_2\epsilon - c_3\delta - \delta_1\psi_\theta.$$

This holds due to our choice of the parameter s and the interface monotonicity ($\psi_\theta > 0$), since for small ϵ one has $\delta = \epsilon^s$ and $\epsilon = o(\delta)$ as $\epsilon \to 0$.

Let us check inequality (5.32) in Ω_1^+.

Thanks to the exponential asymptotics

$$\psi_{\theta\theta} = -\kappa(q,\tau)\exp(-b_1(q,\tau)\theta) + O(\exp(-2b_1\theta)), \quad \theta \to \infty,$$

where $\kappa > 0$, the function ψ is convex for sufficiently large θ, i.e.,

$$\psi_{\theta\theta}(\theta, q, \tau) < 0, \quad \theta > \theta_0(q,\tau).$$

Moreover due to (4.19) and the asymptotics $\psi(+\infty, q, \tau) = w_1$ of the interface ψ, there is a number θ_1 such that $F_1(\psi, x, \tau) < 0$ for $x > q + \epsilon\theta_1$. Let us set $\theta_2 = \max\{\theta_0, \theta_1\}$. Consider the subdomain W_+ of Ω_1^+, where $x > q + \epsilon\theta_2$. In this domain $F_1 < 0$ and, due to the convexity $\psi_{\theta\theta} < 0$ in W, the function ψ_θ takes the minimal value at $x = q - a_1\epsilon\log\epsilon$.

Let us check now inequality (5.32) for the points $x \in W$. Due to this property of ψ, and our choice of s, s_1, a_1, in W one has

$$C_2\epsilon < c\epsilon^{s_1 + a_1\tilde{b}_1} \le \delta_1\psi_\theta$$

for small ϵ. Since in this domain W one has $F_1 < 0$, main inequality (5.32) holds in W.

If $x \in \Omega_1^+$ and $x \notin W$, then $\psi_\theta > \kappa_0$, where $\kappa_0 > 0$ is independent of ϵ. Therefore, main inequality (5.26) again holds for sufficiently small ϵ. In fact, $|F_1| < c\delta$ and $F_1 + C_2\epsilon = o(\delta_1)$ as $\epsilon \to 0$ under conditions (5.30) and (5.31) on the choice s_1, s.

We have proved that the function u^+ is actually a supersolution. In a similar way, we can construct an analogous subsolution. Thus the theorem is proved.

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