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Lyapunov Functionals and Stability for FitzHugh–Nagumo Systems

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1. INTRODUCTION

Consider the following system of partial differential equations, known as the FitzHugh–Nagumo equations,

$$\begin{cases} u_t = d_1\, \Delta u + f(u) - v \\ v_t = d_2\, \Delta v + \delta u - \gamma v \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x) \end{cases} \qquad x \in \Omega. \tag{1}$$

Here $\Omega \subset \mathbb{R}^n$ is a bounded connected set with smooth boundary and we will consider either Dirichlet or Neumann homogeneous boundary conditions. The numbers d_1, δ and γ are positive parameters, while d_2 is taken to be nonnegative. The function f is such that the system will have a global attractor—in some cases, such as the stability results in Sections 4 and 5, it is sufficient to assume f to be smooth enough. A typical function f which will ensure the existence of a global attractor is the cubic function $f(u) = -\mu_0(u - r_0)(u - r_1)(u - r_2)$, where μ_0 is a positive number.

This system is a simplified version of the Hodgkin–Huxley set of equations which describes the behaviour of electrical impulses in the axon of the squid, and several different variations of it have been considered in the mathematics literature. These range from introducing several simplifications to the model (letting $d_2 = 0$, for instance [CS, FR]), to the study of the existence of travelling and standing waves when $\Omega = \mathbb{R}$, [J, KT, S].

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It is well known that under general conditions on the nonlinearity f (Lipschitz continuity and a sign condition ensuring the existence of invariant rectagles in the (u, v)-plane is sufficient) problem (1) generates a semigroup in $X = L^2 \times L^2$ with a compact global attractor [T, Ma]. Here we choose the setting of the fractional power spaces for this problem (see [Ha, He] for example). Taking X as the underlying space and denoting by $A\colon D(A) \subset X \to X$ the linear operator $A\varphi = -\Delta\varphi$ defined on $D(A) = \{\varphi \in H^2(\Omega, \mathbb{R}^2)\colon \varphi$ satisfies the boundary conditions on $\partial\Omega\}$ we have that problem (1) generates a nonlinear semigroup on $X^\alpha = D(A^\alpha)$. Moreover, if α is taken larger than $n/4$, for $n \leqslant 3$, we have that $X^\alpha \subset L^\infty(\Omega, \mathbb{R}^2)$ and the semigroup has a compact global attractor in X^α, [C].

The existence of a Lyapunov functional for (1) has been established for appropriate nonlinearities and under some parameter regimes. Its importance stems from the abundance of results on gradient-like systems, ranging from the asymptotic behaviour of solutions to the Morse structure of the global attractor (see [Ha] for a general reference and [Mi, Kal] for specific results). For example, taking the cubic nonlinearity $f(u) = \mu(u - u^3)$, if $d_2 = 0$ the system generated by (1) is gradient-like in the region $\delta \leqslant \gamma^2$, [CS]. Amazingly enough, the same result holds for a shadow system obtained from (1) by a limiting process as $d_2 \to \infty$, [N1]. This follows from the Lyapunov function in [Kal] after some rescaling, (see Section 5). In fact, the Lyapunov functions exhibited in both cases differ only in the adaptation forced by the limiting process $d_2 \to \infty$. However, this does not seem to be easily adapted to the general case $(0 < d_1, d_2 < \infty)$ without imposing some restrictions on the parameters. Using a different approach for a particular parameter case, Rothe and Mottoni have exhibited in [RM] a Lyapunov function for (1) under an awkward condition on the nonlinearity's first derivative. In this paper, we present in Section 2 a different Lyapunov functional for (1) holding for all positive values of the parameters d_1 and d_2, and only being restricted by the relative values of γ and of the supremum of the derivative of the function f. In Section 3 we draw some immediate consequences for the structure of the attractor. Finally, in Sections 4 and 5 we consider the limiting cases $d_2 = 0$ and $d_2 \to \infty$ for which we are also able to provide Lyapunov functionals valid under the restriction mentioned above. In these two cases, we also give a complete description of the type of stationary solutions that may be stable.

2. EXISTENCE OF A LYAPUNOV FUNCTIONAL

In the following we will assume f to be a $\mathscr{C}^r$-function with $r \geqslant 1$. This implies the $\mathscr{C}^r$ regularity of the semigroup generated by (1), [He]. Our main result is the following

PROPOSITION 2.1. *Let*

$$\mu = \sup_{u \in \mathbb{R}} (f'(u)),$$

let C_P be Poincaré's constant for the domain Ω, and consider the functional

$$\mathscr{L}(u, v) = \int_\Omega \frac{u_t^2}{2} + \frac{v_t^2}{2\delta} + \gamma d_1 \frac{|\nabla u|^2}{2} - \gamma d_2 \frac{|\nabla v|^2}{2\delta} - \gamma F(u) + \gamma uv - \gamma^2 \frac{v^2}{2\delta}\, dx,$$

where F denotes a primitive of f. Then, in the case of homogeneous Neumann boundary conditions, if $\gamma > \mu$ the functional $\mathscr{L}$ is nonincreasing along trajectories of (1), and it will be strictly decreasing except at stationary points. In the case of homogeneous Dirichlet boundary conditions this can be improved to $d_1 C_P + \gamma \geqslant \mu$.

Proof. This functional is defined and continuous on $D(A)$ and by the smoothing action of the semigroup, the solutions of (1) are in $D(A)$ for all $t > 0$, ([He, S]). Therefore we compute

$$\dot{\mathscr{L}}(u, v) = \int_\Omega u_t u_{tt} + \frac{v_t v_{tt}}{\delta} + \gamma\, d_1\, \nabla u.\, \nabla u_t - \frac{\gamma\, d_2}{\delta} \nabla v.\, \nabla v_t - \gamma f(u)\, u_t$$

$$+ \gamma(u_t v + u v_t) - \frac{\gamma^2}{\delta}\, v v_t\, dx$$

$$= \int_\Omega u_t[d_1\, \Delta u_t + f'(u)\, u_t - v_t] + \frac{v_t}{\delta}[d_2\, \Delta v_t + \delta u_t - \gamma v_t]$$

$$- \gamma\, d_1 u_t\, \Delta u + \frac{\gamma\, d_2}{\delta} \Delta v v_t - \gamma f(u)\, u_t + \gamma(u_t v + u v_t) - \frac{\gamma^2}{\delta}\, v v_t\, dx$$

$$= \int_\Omega - d_1\, |\nabla u_t|^2 + f'(u)\, u_t^2 - \frac{d_2}{\delta} |\nabla v_t|^2 - \frac{\gamma}{\delta} v_t^2$$

$$- \gamma[d_1\, \Delta u + f(u) - v]\, u_t + \frac{\gamma}{\delta}[d_2\, \Delta v + \delta u - \gamma v]\, v_t\, dx$$

$$= - \int_\Omega d_1 |\nabla u_t|^2 + \frac{d_2}{\delta} |\nabla v_t|^2 + [\gamma - f'(u)]\, u_t^2\, dx.$$

This is negative except when $|\nabla u_t|^2 = |\nabla v_t|^2 = u_t^2 = 0$, which gives that u_t has to be zero at a point t_1 where $\dot{\mathscr{L}}$ vanishes. From the first equation in (1) it now follows that $v_t = u_{tt}$ at such a point. Since $\mathscr{L}$ cannot increase, it will either decrease for t in a positive neighbourhood of t_1, in which case the result follows, or it will remain constant. In this second case, u_t must remain equal to zero, from which it follows that u_{tt} also vanishes and thus the same happens to v_t. This completes the proof in the case of Neumann boundary conditions. For Dirichlet boundary conditions it only remains to apply Poincaré's inequality to the first term inside the integral to obtain that

$$\dot{\mathscr{L}}(u, v) \leqslant -\int_\Omega [C_P\, d_1 + \gamma - f'(u)]\, u_t^2 + \frac{d_2}{\delta} |\nabla v_t|^2\, dx. \qquad \blacksquare$$

Due to the presence of the term $-\gamma d_2 |\nabla v|^2/2\delta$, one might suspect that $\mathscr{L}$ is not bounded from below on its domain. However, it is possible to show that $\mathscr{L}$ is bounded from below along trajectories of (1). In the following, we assume that f satisfies conditions ensuring the existence of positively invariant rectangles for (1). Then, the nonlinear semigroup generated by (1) has a global attractor which is bounded in $H^1(\Omega, \mathbb{R}^2)$, (see [T]). In this case, we obtain a uniform lower bound for $\mathscr{L}$.

PROPOSITION 2.2. *The continuous functional* $\mathscr{L}: D(A) \to \mathbb{R}$ *is lower bounded.*

Proof. Let M denote a $H^1(\Omega, \mathbb{R}^2)$ bound on the compact global attractor $\mathscr{A}$ of the semigroup, and let c_0 denote an upper bound on F. Using standard estimates we obtain the following lower bound on $\mathscr{L} \mid \mathscr{A}$:

$$\begin{aligned}
\mathscr{L}(u, v) &\geqslant \int_\Omega \gamma uv - \gamma\, d_2 \frac{|\nabla v|^2}{2\delta} - \gamma F(u) - \gamma^2 \frac{v^2}{2\delta}\, dx \\
&\geqslant -\frac{\gamma}{2} \|u\|_{L^2}^2 - \frac{\gamma}{2} \|v\|_{L^2}^2 - \gamma\, d_2 \frac{\|v\|_{H^1}^2}{2\delta} - \gamma c_0 |\Omega| - \gamma^2 \frac{\|v\|_{L^2}^2}{2\delta} \\
&\geqslant -\left(\gamma + \frac{\gamma^2}{2\delta} + \frac{\gamma\, d_2}{2\delta}\right) M - \gamma c_0\, |\Omega| \stackrel{\text{def}}{=} c_1.
\end{aligned}$$

Since $\mathscr{L}$ is nonincreasing along trajectories of (1) we conclude that the Lyapunov functional $\mathscr{L}$ is lower bounded on its domain, $\mathscr{L}(u, v) \geqslant c_1$ for every $(u, v) \in D(A)$. $\blacksquare$

Although we shall not be using the fact explicitly, it is interesting to view equation (1) as a scalar wave equation with damping. This can be achieved by differentiating the first equation with respect to time to obtain

$$u_{tt} = d_1\, \Delta u_t + f'(u)\, u_t - d_2\, \Delta v - \delta u + \gamma v.$$

Substituting v and Δv by the expression obtained from the first equation in (1) we obtain, after rearranging terms,

$$u_{tt}+[\gamma-f'(u)]\,u_t=-d_1 d_2\,\Delta^2 u+(d_1+d_2)\,\Delta u_t \\ +[\gamma\,d_1-d_2\,f'(u)]\,\Delta u-d_2|\nabla u|^2\,f''(u)+\gamma f(u)-\delta u, \tag{2}$$

with boundary conditions $u=0$ and $\Delta u=-f(0)$ in the case of Dirichlet boundary conditions for (1). From this it is clear why one should expect the sign of the term $\gamma-f'(u)$ to play a role in the existence of the Lyapunov functional.

3. STRUCTURE OF THE ATTRACTOR

In this section we consider system (1) in the case of Neumann boundary conditions

$$\partial u/\partial v|_{\partial\Omega}=\partial v/\partial v|_{\partial\Omega}=0. \tag{3}$$

We describe the attractor for a restricted set of parameters, collecting and adapting some of the results scattered throughout the literature.

In the following we let f satisfy the assumptions:

(i) $f(0)=0$;

(ii) $f(u)/u>f'(u)$ for $u\neq 0$;

(iii) $f(u)/u<-\delta/\gamma$ for all large $|u|$.

Condition (i) ensures that system (1) has the trivial solution $(u, v)\equiv(0, 0)$. In fact, any stationary homogeneous solution of (1), (3) can be used as reference and translated to the origin. In this case, (i) is always satisfied up to a change of variables.

Condition (ii) is a soft-spring type condition on the nonlinearity, and it is slightly weaker than the simpler form

$$\operatorname{sign} f''(u)=\operatorname{sign} u.$$

It appears, for example, in the study of the scalar one-dimensional reaction-diffusion equation

$$u_t=u_{xx}+f(u), \qquad 0<x<1 \tag{4}$$

(known as the Chafee–Infante problem, [CI]). In the following we will make a comparison between both problems concerning the structure of the attractor. Under condition (ii) we have that $\mu = f'(0)$, and for our purposes it does not make any difference if we consider

$$f(u) = \mu(u - u^3). \tag{5}$$

The bifurcation diagram of the stationary solutions of (4), (5) under Neumann boundary conditions is well known. Besides the 3 homogeneous solutions, $u = 0$ and $u = \pm 1$, it consists of a sequence of supercritical bifurcations of the trivial solution occurring at the values $\mu = \lambda_k = k^2\pi^2$, $k = 1, 2, ...$ and generating pairs of solutions globally defined for all $\mu > \lambda_k$, (see Fig. 1). The fact that the bifurcations are supercritical and the bifurcating solution branches are global (without any secondary bifurcations) is a consequence of condition (ii).

Finally, (iii) is a growth condition ensuring the dissipativeness of the dynamical system generated by (1). In fact, (iii) implies the existence of an invariant rectangle for (1) (see [S] for a reference). This condition is used here to ensure a $\mathscr{C}^0$ bound on the stationary solutions of (1) by an application of the maximum principle.

The stationary solutions of this problem satisfy a nonlinear scalar equation of the form $Lu = f(u)$, where the (linear) operator L is defined by (see for example [R])

$$Lu = -d_1\, \Delta u + \delta(-d_2\, \Delta + \gamma)^{-1}\, u$$

together with Neumann boundary conditions. Note that L is well defined for all positive γ, that it is self-adjoint and that its spectrum consists only of eigenvalues which we will denote by λ_k, $k = 0, 1, ...$ We then have that

$$\lambda_k = d_1\, \gamma_k + \delta(d_2\, \gamma_k + \gamma)^{-1}, \qquad k = 0, 1, ...,$$

where the γ_k's denote the eigenvalues of $-\Delta$, satisfying

$$0 = \gamma_0 < \gamma_1 \leqslant \gamma_2 \leqslant \cdots \leqslant \gamma_k \to +\infty.$$

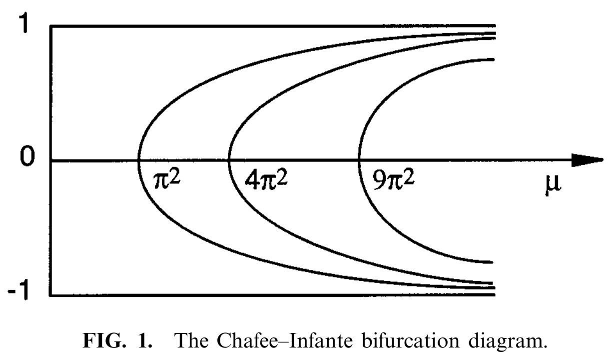

FIG. 1. The Chafee–Infante bifurcation diagram.

Therefore, $\lambda_0 = \delta/\gamma$ and $\lambda_k > 0$ for all $k \geqslant 0$. However, the ordering of these eigenvalues strongly depends on the parameters. In the following we let $\alpha = \inf\{\lambda_k : k \geqslant 0\}$ denote the first eigenvalue of L. We also let $\beta = \inf\{\lambda_k : k \geqslant 0, \lambda_k \neq \alpha\}$ denote the second eigenvalue of L if α is simple, and define $\beta = \alpha$ otherwise.

Regarding the homogeneous solutions, the situation is quite simple. If $\mu \leqslant \delta/\gamma$ the only stationary homogeneous solution is the trivial solution, and if $\mu > \delta/\gamma$ there are exactly 3 homogeneous solutions. The stability of these solutions (or any other stationary solution) is easily established from the linearization of the system (1). It turns out that an equilibrium solution (u_0, v_0) is hyperbolic if and only if the operator $L - f'(u_0)$, subject to the Neumann boundary conditions, is nonsingular. It follows that the trivial equilibrium is non-hyperbolic if, and only if, $\mu = \lambda_k$ for $k = 0, 1, \ldots$. A similar statement holds for the other two homogeneous equilibria (for $\mu > \delta/\gamma = \lambda_0$). Moreover, they are always hyperbolic and stable if in their case $f'(u_0) \leqslant 0$. The origin is stable if $\mu \leqslant \alpha$ and unstable if $\mu > \alpha$.

The condition of non-hyperbolicity of the origin, $\mu = \lambda_k$, is the same appearing in the Chafee–Infante problem. Recall however that the index k does not, in general, reflect the ordering of the eigenvalues. For $k = 0$ we simply have $\mu = \delta/\gamma$. For $k \geqslant 1$ we obtain the following primary bifurcation curves in the diffusion parameter space $(d_1, d_2) \in \mathbb{R}^2_+$:

$$d_2 = \Gamma_k(d_1; \mu, \delta, \gamma) \stackrel{\text{def}}{=} \left(\frac{\delta}{\mu - d_1\,\gamma_k} - \gamma\right) \Big/ \gamma_k, \qquad k = 1, 2, \ldots \tag{6}$$

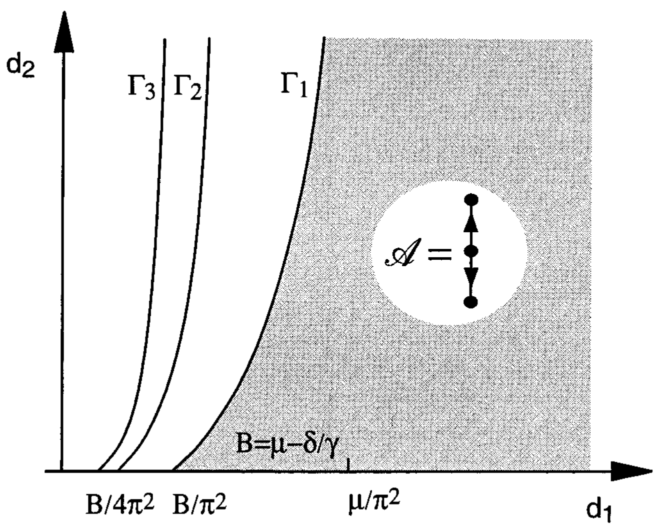

FIG. 2. Primary bifurcation curves and global attractors for $n = 1$, $\Omega = (0, 1)$. Case $\mu > \delta/\gamma$.

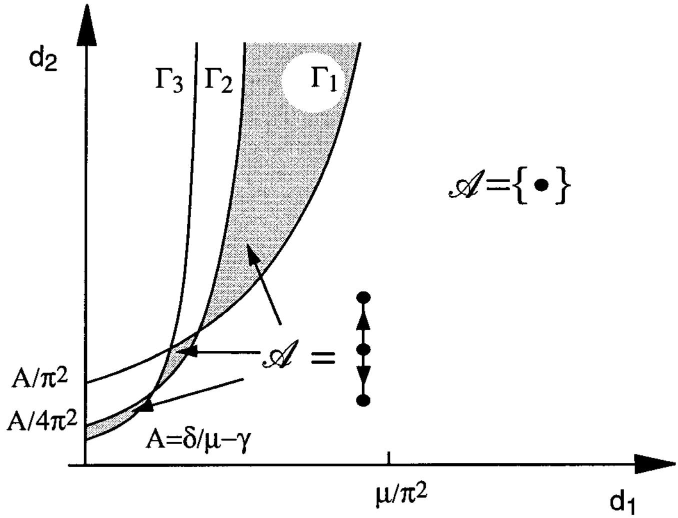

FIG. 3. Primary bifurcation curves and global attractors for $n=1$, $\Omega=(0,1)$. Case $\mu<\delta/\gamma$.

In the one-dimensional case and in the case of some particular domain geometries, it is possible to compute the eigenvalues γ_k explicitly, completing the information on the primary bifurcation curves represented by the graphs of the functions Γ_k (see for example [N1]). Of course, for $n=1$, $\Omega=(0,1)$, we have that $\gamma_k=k^2\pi^2$. The cases $\mu<\delta/\gamma$ and $\mu>\delta/\gamma$ are qualitatively different as shown in Figs. 2 and 3.

As in the case of the Chafee–Infante problem, it can be shown (see [R]) that condition (ii) implies that $\mu=\lambda_k$ correspond to supercritical bifurcation points for

$$Lu=f(u), \qquad \partial u/\partial v|_{\partial\Omega}=0. \tag{7}$$

Going back to our system (1), we can study the bifurcation diagram of its stationary solutions by varying d_1 from $+\infty$ to 0 and taking $d_2=$ constant in the parameter space (d_1, d_2). It follows that, at least locally around the trivial solution, this bifurcation diagram is qualitatively like the one of Figure 1 (called a bistable bifurcation diagram by Mischaikow [Mi]).

As already pointed out, the existence of a Lyapunov functional for system (1) provides much information on the structure of its attractor. Let $\mathscr{A}$ denote the global attractor of the semigroup generated by (1) and let $\mathscr{E}$

denote the set of its equilibrium points. When the condition of Proposition 2.1 ($\mu < \gamma$) is fulfilled and all the equilibria are hyperbolic, the attractor $\mathscr{A}$ is the (finite) union of all the unstable manifolds of the equilibria,

$$\mathscr{A} = \bigcup_{e \in \mathscr{E}} W^u(e).$$

The finer structure of the geometry of the attractor involves the study of these manifolds which are composed essentially of heteroclinic orbits between equilibria. In the absence of transversality results for this study, one usually turns to topological methods. Using Conley index, Mischaikow [Mi] has shown that the flow of a gradient-like system with a bistable equilibria bifurcation diagram (i.e., one like Figure 1) and the flow of the Chafee–Infante problem (4), (5), up to a topological semiconjugacy are the same. Therefore, this is how we expect the attractor $\mathscr{A}$ to behave (when $\mu < \gamma$) as long as the bifurcation diagram for (1) is qualitatively bistable, that is, exhibiting only primary bifurcations, all at the origin and supercritical. Unfortunately, in view of the results of [Kal] (see also [N1]) we expect the appearance of secondary bifurcations in regions of the parameter space (d_1, d_2) corresponding to small values of d_1 and large values of d_2. Nevertheless, secondary bifurcations are prevented in other regions. In fact, it follows from a result of Lazer and McKenna [LM], (see also Rothe [R]), that in a certain region of the parameter space no other bifurcations can occur globally. Therefore, in that region the attractor $\mathscr{A}$ remains very simple. In terms of the bifurcation parameter, the secondary bifurcations cannot appear before the occurrence of the second primary bifurcation of the origin. The full statement is contained in the next theorem which is a slight modification of Theorem 1 of [LM] conveniently adapted.

THEOREM 3.1. *Let f be a $\mathscr{C}^2$ function satisfying the conditions* (i–iii). *Then, if $\mu = f'(0) \leqslant \alpha$, system* (1) *has only one stationary solution which is the trivial solution, while if $\alpha < \mu < \beta$ then it has exactly* 3 *stationary solutions* (*in* $\mathscr{C}^{2+\nu}$), *one of which is the trivial one. Furthermore, the origin is stable in the first case and is unstable in the second while the other two solutions are stable.*

For completeness we sketch the main steps of the proof in the Appendix and refer to [LM] for details.

It follows from this result that the attractor is a singleton when $\mu \leqslant \alpha$ and is composed of three equilibria connected by two heteroclinic orbits when $\alpha < \mu < \beta$ (in this case the unstable manifold of the origin is one-dimensional). In the diffusion parameter space the corresponding regions are easily determined (as depicted in Figs. 2 and 3).

To finish, we remark that a very similar result is obtained if instead of Neumann one considers Dirichlet boundary conditions.

4. THE CASE OF $d_2 = 0$

In this section we consider the system that is obtained from (1) by letting d_2 vanish, that is

$$\begin{cases} u_t = d_1 \, \Delta u + f(u) - v \\ v_t = \delta u - \gamma v \\ u(x, 0) = u_0(x), \quad v(x, 0) = v_0(x) \end{cases} \tag{8}$$

plus Neumann or Dirichlet homogeneous boundary conditions. Note that if we also let d_2 vanish in the functional used in Section 1 we still obtain a Lyapunov functional for the above system, under the same conditions on the parameters, that is, assuming that γ is larger than the maximum of the derivative of f in the case of Neumann boundary conditions, and larger than this value minus $d_1 C_P$ in the case of Dirichlet boundary conditions. This gives a region (in the parameter space) where the system has a Lyapunov functional which is different from that of Conley and Smoller [CS]. In that paper they consider equation (8) on the interval $(-l, l)$ and with $d_1 = 1$. From their result it follows that

$$\mathscr{L}_1(u, v) = \int_\Omega d_1 \frac{|\nabla u|^2}{2} - F(u) + \frac{\delta}{\gamma} u^2 - uv + \frac{\gamma}{2\delta} v^2 \, dx$$

is a Lyapunov functional provided that $\gamma^2 \geqslant \delta$.

The functionals $\mathscr{L}_1$ and $\mathscr{L}$ lead to different regions in the parameter space $(\gamma, \delta) \in \mathbb{R}^2_+$ where the system (8) is gradient-like. For a comparison, we depict these regions in Fig. 4.

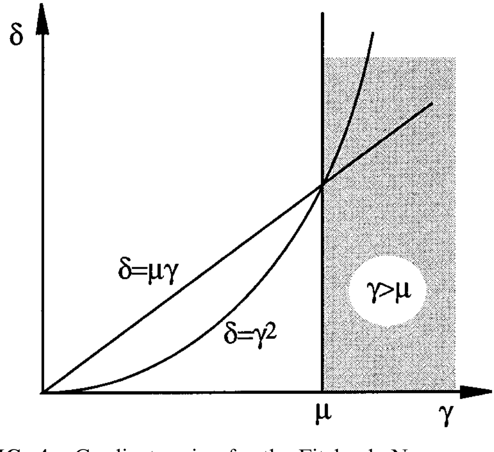

FIG. 4. Gradient region for the Fitzhugh–Nagumo system.

4.1. *Stability of Stationary Solutions*

By letting d_2 vanish in equation (2) we see that equation (8) can be written as a scalar strongly damped wave equation of the form

$$u_{tt} + [\gamma - f'(u)]\, u_t = d_1\, \Delta u_t + \gamma\, d_1\, \Delta u + \gamma f(u) - \delta u.$$

Thus, in order to study the type of stationary solutions of (8) that may be stable, we can use the comparison results established for that type of equations in [F3]. In this particular case, these extend a result given in [S], in one dimension, regarding the type of solutions that may be stable.

As the diffusion coefficient in the second equation is now taken to be zero we have that the elliptic system giving the stationary solutions of (8) can be reduced to a scalar equation of the form

$$d_1\, \Delta u + f(u) - \theta u = 0, \qquad \theta = \delta/\gamma$$

with the second component being given by $v = \theta u$. We shall thus use the corresponding parabolic equation, that is,

$$u_t = d_1\, \Delta u + f(u) - \theta u, \tag{9}$$

to obtain the following

THEOREM 4.1. *Let the pair* $(u, v) = (\bar{u}, \bar{v})$ *be a stationary solution of* (8). *Then the dimension of the unstable manifold of this solution is greater than or equal to the dimension of the unstable manifold of* $\bar{u}$ *when considered as a stationary solution of the parabolic equation* (9).

Proof. It is a direct consequence of the instability results given in [F3] applied to the strongly damped wave equation associated with (8). ∎

As a direct corollary we have

COROLLARY 4.2. *In the case of homogeneous Neumann boundary conditions where either the domain* Ω *is convex or the function* $f(u)$ *is convex, the only stationary solutions that may be stable are constant.*

In the case of homogeneous Dirichlet boundary conditions and Ω *being a ball, the only stationary solutions that may be stable do not change sign in* Ω.

5. THE SHADOW SYSTEM ($d_2 \to \infty$)

In this section we analyze the behaviour of the shadow system to the set of equations (1), that is, the system that is obtained by letting one of the diffusion parameters go to infinity, in the case of homogeneous Neumann boundary conditions [HS, Kal]. In this instance, it is of interest to consider what happens as $d_2 \to \infty$, in which case we obtain the following system of nonlocal equations

$$\begin{cases} u_t = d_1\ \Delta u + f(u) - v, & x \in \Omega \\ v_t = \dfrac{\delta}{|\Omega|} \displaystyle\int_\Omega u\, dx - \gamma v, & \\ u_\nu(x,t) = 0, & x \in \partial\Omega \\ u(x,0) = u_0(x), \qquad v(0) = v_0 & \end{cases} \tag{10}$$

Scalar reaction-diffusion equations including nonlocal terms of this type have received some attention in the literature recently. Here we have a system of two equations where the nonlocal term corresponds to the average of one of the variables. This will make it possible to use some of the theory developed for the scalar case to study the stability of stationary solutions of (10). Of particular relevance here will be the results presented in [F1, F2, F3, FV].

As remarked already in the introduction, this shadow system also exhibits a gradient-like behavior for a certain range of the parameters. System (10) was considered in [Kal] in the one-dimensional case $\Omega = (0, 1)$ (with $\gamma = 1$ but with a constant added to the second equation). This system was shown to possess a Lyapunov functional in a certain parameter region. After a rescaling, their result leads to the functional

$$\mathscr{L}_2(u,v) = \frac{1}{|\Omega|} \int_\Omega \gamma\, d_1 \frac{|\nabla u|^2}{2} - \gamma F(u)\, dx + \frac{\delta}{2} \bar{u}^2 + \frac{1}{2\delta} (\delta \bar{u} - \gamma v)^2,$$

where $\bar{u} = \int_\Omega u\, dx / |\Omega|$. The same computation as in [Kal] leads to

$$\dot{\mathscr{L}}_2(u,v) = -\frac{1}{|\Omega|} \int_\Omega \gamma \left(d_1\ \Delta u + f(u) - \frac{\delta}{\gamma} \bar{u} \right)^2 dx - \frac{\gamma^2 - \delta}{\delta \gamma} (\delta \bar{u} - \gamma v)^2.$$

Therefore, $\mathscr{L}_2$ is a Lyapunov functional for (10) provided that $\gamma^2 \geqslant \delta$. This is entirely analogous to the case $d_2 = 0$ considered before.

As in the previous section, here we also obtain a different Lyapunov functional for the shadow system valid in the region $\gamma > \mu$ (see again these regions depicted in Figure 4).

5.1. *Existence of a Lyapunov Functional*

As before, we begin by showing the existence of a Lyapunov functional for certain values of the parameter γ. Following Section 4, it is useful to begin by rewriting (10) as a (nonlocal) strongly damped wave equation which in this case takes the form

$$u_{tt}+[\gamma-f'(u)]\,u_t=d_1\,\gamma\,\Delta u+d_1\,\Delta u_t+\gamma f(u)-\frac{\delta}{|\Omega|}\int_\Omega u\,dx. \qquad (11)$$

From this we again see that the sign of the term $\gamma-f'(u)$ will be important in determining whether all solutions must converge to an equilibrium or a more complex behaviour can be expected. Multiplying both sides of (11) by u_t and integrating over Ω gives, after rearranging terms and integration by parts, that

$$\frac{d}{dt}\left[\int_\Omega \frac{u_t^2}{2}+d_1\,\gamma\,\frac{|\nabla u|^2}{2}-\gamma F(u)\,dx+\frac{\delta}{2|\Omega|}\left(\int_\Omega u\,dx\right)^2\right]$$

$$=-\int_\Omega [\gamma-f'(u)]\,u_t^2+d_1\,|\nabla u_t|^2\,dx.$$

It now follows that if $\gamma>f'(u)$ for all u, then the term on the right-hand side will be negative, except at points where u_t vanishes identically. As in the proof of Proposition 2.1 this implies that v_t is also zero. We thus have

PROPOSITION 5.1. *Let μ be as in Theorem 2.1. Then if $\gamma>\mu$ the functional*

$$\mathcal{L}(u)=\int_\Omega \frac{u_t^2}{2}+d_1\,\gamma\,\frac{|\nabla u|^2}{2}-\gamma F(u)\,dx+\frac{\delta}{2|\Omega|}\left(\int_\Omega u\,dx\right)^2$$

is nonincreasing along trajectories of (10), *and it will be strictly decreasing except at stationary points.*

In this case, it will be enough to assume that F is bounded above, for example assuming that $uf(u)<0$ for all large $|u|$, to ensure that $\mathcal{L}$ is bounded from below.

5.2. *Stability of Stationary Solutions*

In this section we shall study the type of stationary solutions of system (10) that may be stable. In contrast with what may happen in the general case of system (1), here we will see that for a stationary solution to be stable it will have to satisfy certain restrictions. We shall begin by determining some conditions that stable solutions must satisfy, and then show by means of an example that, in some sense, these are optimal. Our main result is the following

THEOREM 5.2. *Let the pair* $(u, v) = (\bar{u}, \bar{v})$ *be a stationary solution of* (10). *Then it can only be stable if there exists a scalar (local) parabolic equation of the form* $u_t = d\,\Delta u + g(u)$ *which has* $\bar{u}$ *as a stationary solution and such that the linear operator obtained by linearizing this equation around this solution has at most one positive eigenvalue.*

Proof. The idea of the proof is basically an adaptation of the techniques used in [F3] for the case of the strongly damped wave equation, the main difference now being the presence of the nonlocal term.

Let $(\bar{u}, \bar{v})$ be a stationary solution of (10). Then the linearized eigenvalue problem associated with (10) around this solution is given by

$$\begin{cases} d_1\,\Delta u + f'(\bar{u})\,u - v = \lambda u \\ \dfrac{\delta}{|\Omega|}\displaystyle\int_\Omega u\,dx - \gamma v = \lambda v. \end{cases} \tag{12}$$

For $\lambda \neq -\gamma$, this can be reduced to a single equation of the form

$$d_1\,\Delta u + a(x)\,u - \frac{\delta}{(\lambda+\gamma)\,|\Omega|}\int_\Omega u\,dx = \lambda u, \tag{13}$$

which in turn can be written as

$$\begin{cases} d_1\,\Delta u + a(x)\,u + p\displaystyle\int_\Omega u\,dx = \lambda u \\ p = -\dfrac{\delta}{(\lambda+\gamma)\,|\Omega|}. \end{cases} \tag{14}$$

If we now consider the first of these equations separately, we obtain λ as a function of the parameter p, and the eigenvalues of the original problem will correspond to the solutions of the equation $\lambda(p) = -\gamma - \delta/(|\Omega|\,p)$. It thus follows that for real roots of this equation with $p \in (-\delta/(\gamma\,|\Omega|), 0)$ there correspond real positive eigenvalues of (12). For real values of p this eigenvalue problem is self-adjoint, and thus the eigencurves $\lambda(p)$ consist of continuous curves in the plane $p-\lambda$ which are defined for all $p \in \mathbb{R}$.

Assume now that when $p = 0$ there are at least two real positive eigenvalues of (14). From the results in [FV]—see also [F1]—the two corresponding eigencurves must have disjoint images, except in the case where one of them is constant for all p. Thus, they cannot intersect each other except when one of them is constant and so at least one of these curves must remain above the p-axis. We then have at least one intersection between the graphs of these eigencurves and the function $-\gamma-\delta/(|\Omega| p)$ for p in the interval $(-\delta/(\gamma\,|\Omega|), 0)$, to which there corresponds at least one positive eigenvalue.

Finally, note that the first component of the stationary solution $\bar{u}$ must satisfy

$$d_1\, \Delta\bar{u}+f(\bar{u})-\frac{\delta}{\gamma\,|\Omega|}\int_\Omega \bar{u}\,dx=0,$$

that is, $\bar{u}$ satisfies a local elliptic equation of the form

$$d_1\, \Delta u+f(u)-K=0,$$

where

$$K=\frac{\delta}{\gamma\,|\Omega|}\int_\Omega \bar{u}\,dx.$$

As the linearized eigenvalue problem for the corresponding parabolic equation is

$$d_1\, \Delta u+a(x)\,u=\lambda u,$$

the result follows. ▮

From the proof it is also clear that if the dimension of the unstable manifold of $\bar{u}$ when considered as a stationary solution of the auxiliary local parabolic equation is $m>0$, then the dimension of the unstable manifold of $(\bar{u}, \bar{v})$ is at least $m-1$. Thus, as a straightforward corollary we obtain

COROLLARY 5.3. *In the one-dimensional case if the first component of a stationary solution of* (10) *has m isolated extremum points in the interior of* Ω, *then the dimension of the unstable manifold will be at least m. In particular, the only stationary solutions that may be stable are those that have a monotone first component.*

For a shadow system with different conditions on the reaction terms a similar result was obtained previously by Nishiura [N2].

In [Kal] the authors studied the dynamics on the invariant subspace formed by monotone solutions (in space). The above result implies that, provided we are in a parameter region where there exists a Lyapunov functional, then this invariant subspace will contain all the stable dynamics.

Using an argument similar to that in the proof of Theorem (5.2), we see that there are some situations for which instability of solutions of the associated local parabolic problem implies instability of solutions of (10).

PROPOSITION 5.4. *Let the pair* $(u, v) = (\bar{u}, \bar{v})$ *be a stationary solution of* (10). *If the first eigenvalue* σ_0 *of the linearization of the associated scalar parabolic problem around the corresponding stationary solution* $\bar{u}$ *satisfies*

$$\sigma_0 > \delta/(\gamma\,|\Omega|),$$

then the stationary solution $(\bar{u}, \bar{v})$ *is unstable.*

Note that in the above condition the eigenvalue σ_0 is actually a function of the three variables δ, γ and $|\Omega|$. This means that it is not possible to conclude that having $\delta/(\gamma\,|\Omega|)$ sufficiently small automatically implies the instability of the stationary solution.

APPENDIX

Here we sketch the proof of Theorem 3.1 following closely [LM]. We want to find the solutions of

$$Lu = f(u), \qquad \partial u/\partial \nu|_{\partial\Omega} = 0. \tag{15}$$

From the hypothesis on f we have that this equation has a trivial solution, and the linearization around it is given by $Lw = \mu w$. Then we have

LEMMA 1 [LM, Lemma 2]. *If* u *is a nontrivial solution of* (15), *then for any nonzero* $w \in H^1$ *we have*

$$\int_\Omega wLw - f'(u)\, w^2\, dx > 0.$$

Proof. Let us denote by $\tilde{\lambda}_k$ the eigenvalues of the operator $L - \mu I$, conveniently ordered in k. These are of the form $\lambda_j - \mu$, and in particular we have $\tilde{\lambda}_1 = \alpha - \mu$ and $\tilde{\lambda}_2 = \beta - \mu$. Since $v = u$ is a nontrivial solution of

$$Lv = \frac{f(u)}{u}\, v,$$

it follows that there exists a zero eigenvalue for the nonhomogeneous eigenvalue problem

$$Lv - \frac{f(u)}{u}\, v = \hat{\lambda} v, \qquad \partial v/\partial \nu|_{\partial\Omega} = 0,$$

that is, there exists some k for which $\lambda_k = 0$. The soft–spring condition (ii) gives that

$$f'(0) - f(u)/u = -\int_0^u (f(s)/s)'\, ds = \int_0^u (f(s)/s - f'(s))/s\, ds > 0$$

for $u \neq 0$, and thus $f(u)/u < f'(0)$. Therefore, the comparison principle resulting from the Courant–Fisher min-max characterization of eigenvalues implies that $\hat{\lambda}_k \geqslant \tilde{\lambda}_k$. On the other hand, the condition $\alpha < \mu < \beta$ implies that $\hat{\lambda}_1 = 0$ and thus $\hat{\lambda}_k > 0$ for $k \geqslant 2$. As a result, we have that

$$\int_\Omega wLw - \frac{f(u)}{u} w^2\, dx \geqslant 0$$

which leads to the conclusion. Moreover, the equality holds only if

$$\int_\Omega [f(u)/u - f'(u)]\, w^2\, dx$$

is zero, i.e. $uw = 0$ a.e., hence the strict inequality for nontrivial u and w. We remark that for $\mu \leqslant \alpha$ there are no nontrivial solutions of (15). ∎

From the maximum principle and condition (iii) on f it is possible to obtain the following $\mathscr{C}^0$ estimate on the solutions of (15):

LEMMA 2 [LM, Lemma 3]. *If there exists a positive number u_0 such that f satisfies* (iii) *for $|u| \geqslant u_0$, then any solution of* (15) *satisfies the a priori estimate $|u(x)| \leqslant u_0$ for all $x \in \Omega$.*

Proof. Suppose that $\max_{x \in \Omega} |u(x)| = u_1 > u_0$. Then, $v = Bu$ satisfies $\gamma\, |v(x)| \leqslant u_1$ for all $x \in \bar{\Omega}$ (this is the maximum principle applied to $-d_2\, \Delta v + \gamma v = u$, see [PW]). Now, if $x_1 \in \Omega$ and $u(x_1) = u_1$, then $\Delta u(x_1) \leqslant 0$ and (from (iii)) $u(x_1) < -\frac{\gamma}{\delta} f(u(x_1)) = \frac{\gamma}{\delta}\, d_1\, \Delta u(x_1) - \gamma v(x_1) \leqslant u_1$, a contradiction. The same holds if we take $u(x_1) = -u_1$. ∎

The operator L in (15) is a linear self-adjoint positive operator defined on H^1. Using Rellich's theorem we define a compact linear self-adjoint operator L^{-1}: $L^2 \to L^2$ rewriting (15) in the following abstract form:

$$u = N(u) \tag{16}$$

where N is a nonlinear continuous compact operator given by $N(u) = L^{-1}\tilde{f}(u)$, and $\tilde{f}$ is a convenient function (with a bounded derivative) satisfying $f(u) = \tilde{f}(u)$ for $|u| \leqslant u_0$. By regularity results, solutions of (16) are $\mathscr{C}^2$ functions satisfying (15). Conversely, solutions of (15) satisfy (16).

The essential part of the proof of Theorem 3.1 is then an application of the Leray–Schauder degree theory. One shows that the degree of $u - N(u)$ with respect to 0 and a large ball is $+1$, the zeros of (16) are isolated and the index of any nontrivial solution is $+1$ while the index of the trivial solution is ± 1 according to the value of μ. Then, the proof of Theorem 3.1 follows by a simple counting argument.

LEMMA 3 [LM, Lemma 4] *There exists a positive number R such that any solution of* (16) *satisfies* $\|u\|_{L^2} < R$. *Furthermore, if B_R denotes the ball of radius R in L^2, then $d(u - Nu, B_R, 0)$, the Leray–Schauder degree of $u - Nu$ with respect to B_R and* 0, *satisfies*

$$d(u - Nu, B_R, 0) = 1.$$

Proof. Clearly there is a constant c such that $uf(u) < c$ for all real u. Let $\tilde{f}$ be such that also $u\tilde{f}(u) < c$ for all real u. Take $t \in [0, 1]$ and let u denote a solution of $u = tNu$. By the regularity theory, u is a $\mathscr{C}^2$ solution of $Lu = t\tilde{f}(u)$, $\partial u/\partial v|_{\partial\Omega} = 0$, and we have

$$\int_\Omega uLu\, dx = t\int_\Omega u\tilde{f}(u)\, dx.$$

Therefore we conclude that (for some convenient θ)

$$\theta \|u\|_{L^2} \leqslant (u, Lu) = t\int_\Omega u\tilde{f}(u)\, dx \leqslant c\,|\Omega|,$$

which gives the existence of the constant R depending only on θ, c and Ω such that $\|u\|_{L^2} < R$. Since $u - tNu \neq 0$ for $\|u\|_{L^2} = R$ and $0 \leqslant t \leqslant 1$, using the homotopy invariance of the Leray–Schauder degree we obtain $d(u - Nu, B_R, 0) = d(u, B_R, 0) = 1$. ∎

LEMMA 4 [LM, Lemma 5]. *If w is a nontrivial solution of* (16), *then w is an isolated zero of $u - Nu$ and has Leray–Schauder index*

$$i(u - Nu, w) = 1.$$

Proof. Since N is compact we can use the Leray–Schauder theorem. Let w satisfy $w - Nw = 0$ and consider the linearized operator around w, $T = I - DN(w)$:

$$Tu = u - L^{-1}f'(w)\, u.$$

We now show that if w is nontrivial, then the operator $\tau I - DN(w)$ is nonsingular for every $\tau \geqslant 1$. It then follows that w is isolated. The Leray–Schauder theorem [Nir] then implies that $i(u - Nu, w) = (-1)^\sigma = 1$, since σ, the number of eigenvalues of $DN(w)$ (counting multiplicities) larger than 1, is zero.

Let now $z \in L^2$ be a solution of $\tau z - L^{-1}f'(w)\, z = 0$. By regularity z is actually $\mathscr{C}^2$ and satisfies

$$\tau Lz - f'(w)\, z = 0, \qquad \partial w/\partial v|_{\partial\Omega} = 0.$$

Therefore,

$$\int_\Omega zLz - f'(w)\, z^2\, dx \leqslant \int_\Omega \tau zLz - f'(w)\, z^2\, dx = 0$$

and Lemma 1 now implies that $z = 0$. ∎

LEMMA 6 [LM, Lemma 6] *The trivial solution of $u - Nu = 0$ is isolated and has Leray–Schauder index*

$$i(u - Nu, 0) = \begin{cases} 1, & \mu < \alpha \\ -1, & \alpha < \mu < \beta. \end{cases}$$

Proof. Again we have that if $z \in L^2$ satisfies $z - \mu L^{-1} z = 0$ it follows that $z \in \mathscr{C}^2$ and

$$Lz - \mu z = 0, \qquad \partial z/\partial v|_{\partial\Omega} = 0.$$

Hence, for $\mu \neq \lambda_k$, the operator $I - DN(0)$ is nonsingular and $u = 0$ is an isolated zero of $u - Nu$. As before

$$i(u - Nu, 0) = (-1)^\sigma,$$

where σ is the number of eigenvalues (counting multiplicities) of $DN(0)$ greater than 1. To compute σ we take the usual counting argument [Nir] considering the spectrum of $\tau I - DN(0)$ as τ decreases from $+\infty$ to 1. Then, if $z \neq 0$ satisfies $\tau z - DN(0)\, z = 0$ we have that

$$\tau Lz - \mu z = 0, \qquad \partial z/\partial v|_{\partial\Omega} = 0.$$

which occurs only if $\tau = \mu/\lambda_k$ for some k. Therefore σ is zero for $\mu < \alpha$ and $\sigma = 1$ for $\alpha < \mu < \beta$, concluding the proof. ∎

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