SIAM J. APPL. MATH.
Vol. 60, No. 3, pp. 778–802

A METASTABLE SPIKE SOLUTION FOR A NONLOCAL REACTION-DIFFUSION MODEL*

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Abstract. An asymptotic reduction of the Gierer–Meinhardt activator-inhibitor system in the limit of large inhibitor diffusivity leads to a singularly perturbed nonlocal reaction diffusion equation for the activator concentration. In the limit of small activator diffusivity, a one-spike solution to this nonlocal model is constructed. The spectrum of the eigenvalue problem associated with the linearization of the nonlocal model around such an isolated spike solution is studied in both a one-dimensional and a multidimensional context. It is shown that the principal eigenvalues in the spectrum are exponentially small in the limit of small activator diffusivity. The nonlocal term in the eigenvalue problem is essential for ensuring the existence of such exponentially small principal eigenvalues. These eigenvalues are responsible for the occurrence of an exponentially slow, or metastable, spike-layer motion for the time-dependent problem. Explicit metastable spike dynamics are derived by using a projection method, which enforces a limiting solvability condition on the solution to the linearized problem.

Key words. spike, nonlocal eigenvalue problem, metastability, exponentially small eigenvalue

AMS subject classifications. 35B25, 35C20, 35P15

PII. S0036139998338340

1. Introduction.

In 1957 Turing [14] proposed a mathematical model for morphogenesis which describes the development of complex organisms from a single cell. He speculated that localized peaks in the concentration of a chemical substance, known as an inducer or morphogen, could be responsible for a group of cells developing differently from the surrounding cells. He then demonstrated, with linear analysis, how a nonlinear reaction diffusion system could possibly generate such isolated peaks [14]. Later, Gierer and Meinhardt [4] demonstrated the existence of such solutions numerically for the following dimensionless reaction diffusion system:

$$A_t = \epsilon^2 \Delta A - A + \frac{A^p}{H^q}\,, \quad \mathbf{x} \in \Omega\,, \quad t > 0\,, \tag{1a}$$

$$\tau H_t = D_h \Delta H - \mu H + \frac{A^m}{H^s} \quad \mathbf{x} \in \Omega\,, \quad t > 0\,, \tag{1b}$$

$$\partial_n A = 0\,, \quad \partial_n H = 0\,, \quad \mathbf{x} \in \partial\Omega\,, \tag{1c}$$

where A, H, ϵ, D_h, μ, and τ represent the scaled activator concentration, inhibitor concentration, activator diffusivity, inhibitor diffusivity, inhibitor decay rate, and reaction time constant for this reaction. Here Ω is a closed bounded domain in $\mathbb{R}^N$, ∂_n indicates the outward normal derivative, and the exponents (p, q, m, s) satisfy

$$p > 1, \qquad q > 0, \qquad m > 0, \qquad s \geq 0, \qquad 0 < \frac{p-1}{q} < \frac{m}{s+1}\,. \tag{2}$$

This system was then used in [4] to model head formation in the *hydra*.

*Received by the editors May 6, 1998; accepted for publication (in revised form) February 17, 1999; published electronically February 10, 2000. This research was supported by NSERC grant 5-81541.

http://www.siam.org/journals/siap/60-3/33834.html

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The numerical studies of [4] and, more recently, those of [6] have revealed that in the limit $\epsilon \to 0$, the activator concentration A for (1) can have localized regions of spatial extent $O(\epsilon)$, where its concentration is elevated from a constant background concentration. Such solutions are called spike-type solutions. Some properties of spike-type equilibrium solutions have been obtained in [7], [10], and [13]. In this paper we will examine the dynamical behavior of a one-spike solution to (1) in the limit $D_h \to \infty$ and $\epsilon \to 0$. In this limit, (1) reduces to a nonlocal reaction diffusion model. Before we give an explicit outline of the paper we introduce an appropriate scaling of (1) for spike-type solutions and we derive the nonlocal reaction diffusion model corresponding to the limit $D_h \to \infty$.

The amplitude of a spike solution to (1) will tend to infinity as $\epsilon \to 0$. Therefore, we introduce new variables for which the spike solution has an $O(1)$ amplitude as $\epsilon \to 0$. To this end, we introduce a and h by

$$A = \epsilon^{-\nu_a} a\,, \qquad H = \epsilon^{-\nu_h} h, \tag{3}$$

where ν_a and ν_h are to be found. To balance the terms in (1a), we require

$$-\nu_a = -\nu_a p + q\nu_h. \tag{4}$$

We will construct a solution in which the spike has its support in an $O(\epsilon)$ region near some point in Ω. Therefore, to obtain an additional equation relating ν_a and ν_h, we consider an average balancing of (1b). Specifically, we integrate (1b) over the domain to get

$$\tau \int_\Omega H_t \, d\mathbf{x} = -\mu \int_\Omega H \, d\mathbf{x} + \int_\Omega \frac{A^m}{H^s} \, d\mathbf{x}\,. \tag{5}$$

Since A will be localized to an $O(\epsilon)$ region about the spike center $\mathbf{x}_0$, we scale $\mathbf{x}$ in the last term by $\mathbf{y} = \epsilon^{-1}(\mathbf{x} - \mathbf{x}_0)$. Balancing the terms in this equation we get

$$-\nu_h = -\nu_a m + \nu_h s + N\,. \tag{6}$$

The solution of (4) and (6) yields,

$$\nu_a = \frac{Nq}{(1-p)(1+s)+mq}, \qquad \nu_h = \frac{N(p-1)}{(1-p)(1+s)+mq}. \tag{7}$$

In terms of these new variables, (1) becomes

$$a_t = \epsilon^2 \Delta a - a + \frac{a^p}{h^q}, \qquad \text{in} \quad \Omega, \quad t > 0\,, \tag{8a}$$

$$\tau h_t = D_h \Delta h - \mu h + \epsilon^{-N} \frac{a^m}{h^s}, \qquad \text{in} \quad \Omega, \quad t > 0\,, \tag{8b}$$

$$\partial_n a = 0\,, \quad \partial_n h = 0 \quad \text{on} \quad \partial\Omega\,. \tag{8c}$$

Next we examine (8) in the limit $D_h \to \infty$. In this limit, h will be spatially independent to leading order and we will obtain a nonlocal equation for a.

We begin by writing h as a power series in D_h^{-1},

$$h = h_0 + D_h^{-1} h_1 + \cdots\,. \tag{9}$$

Substituting (9) into (8), and collecting powers of D_h^{-1} results in

$$\Delta h_0 = 0, \quad \text{in} \quad \Omega\,; \qquad \partial_n h_0 = 0 \quad \text{on} \quad \partial\Omega\,, \tag{10a}$$

$$\Delta h_1 = \tau h_{0t} + \mu h_0 - \epsilon^{-N}\frac{a^m}{h_0^s}, \quad \text{in} \quad \Omega\,; \qquad \partial_n h_1 = 0 \quad \text{on} \quad \partial\Omega\,. \tag{10b}$$

Equation (10a) implies that $h_0 = h_0(t)$. Then, by imposing a solvability condition on (10b), we obtain an equation for $\dot{h}_0 \equiv dh_0/dt$:

$$\tau \dot{h}_0 + \mu h_0 - \epsilon^{-N}\frac{1}{|\Omega|}\int_\Omega \frac{a^m}{h_0^s}\,d\mathbf{x} = 0. \tag{11}$$

Here $|\Omega|$ is the volume of Ω. Typically τ is small and thus the dynamics of h are much faster than that of a. So we set $\dot{h}_0 = 0$ in (11) to obtain

$$h_0 = \left(\frac{\epsilon^{-N}}{\mu|\Omega|}\int_\Omega a^m\,d\mathbf{x}\right)^{\frac{1}{s+1}}. \tag{12}$$

In the analysis below, we will consider only the leading order term of h. Thus, we label h_0 by h. Substituting the value of h into (8) results in the following scalar nonlocal reaction diffusion equation:

$$a_t = \epsilon^2\Delta a - a + \frac{a^p}{h^q}, \quad \text{in} \quad \Omega\,, \quad t > 0\,, \tag{13a}$$

$$h = \left(\frac{\epsilon^{-N}}{\mu|\Omega|}\int_\Omega a^m\,d\mathbf{x}\right)^{\frac{1}{s+1}}, \tag{13b}$$

$$\partial_n a = 0 \quad \text{on} \quad \partial\Omega\,. \tag{13c}$$

This system, referred to as the *shadow system* [11], is studied below. It will also be studied in a one-dimensional domain $x \in [-1, 1]$. In this case, (13) reduces to

$$a_t = \epsilon^2 a_{xx} - a + \frac{a^p}{h^q}, \qquad -1 < x < 1\,, \quad t > 0\,, \tag{14a}$$

$$h = \left(\frac{\epsilon^{-1}}{2\mu}\int_{-1}^{1} a^m\,dx\right)^{\frac{1}{s+1}}, \tag{14b}$$

$$a_x(\pm 1, t) = 0\,. \tag{14c}$$

The outline of the paper is as follows. In section 2 we consider the one-dimensional problem (14). In this case, we construct a one-spike quasi-equilibrium solution. We examine the stability and dynamics of this solution by analyzing the spectrum of the linear operator resulting from a linearization of (14) about this nonconstant solution. This eigenvalue problem is a nonlocal Sturm–Liouville problem of the type considered in [3]. A combination of analytical and numerical techniques will be used to demonstrate that the principal eigenvalue of this operator is exponentially small. The nonlocal term is essential for this conclusion. The exponentially small eigenvalue will be estimated asymptotically. A differential equation characterizing the motion of the center of the spike will be derived in the limit $\epsilon \to 0$ by using a limiting solvability condition, which requires that the solution to the quasi-steady linearized problem has no component in the eigenspace associated with the exponentially small eigenvalue. This procedure is known as the projection method and has been used in other contexts (see [15], [16], and [17]). The resulting ODE for the motion of the center of the spike shows that the spike drifts exponentially slowly towards the point on the boundary closest to the initial location of the spike. This metastable behavior is verified by calculating full numerical solutions to (14). In section 3 we give a similar analysis of metastable spike-layer motion for the multidimensional problem (13). Finally, in section 4 we close with a few remarks and possible extensions of this work.

2. A spike in a one-dimensional domain. We first construct a one-spike quasi-equilibrium solution a_E for (14) in the form

$$a = a_E(x;x_0) \equiv h^\gamma u_c(y)\,, \qquad y = \epsilon^{-1}(x-x_0)\,, \qquad \gamma = q/(p-1)\,. \tag{15}$$

Here x_0, with $|x_0| < 1$, is the center of the spike. The function $u_c(y)$, called the canonical spike solution, satisfies

$$u_c^{''} - u_c + u_c^p = 0, \qquad 0 < y < \infty\,, \tag{16a}$$

$$u_c^{'}(0) = 0\,; \qquad u_c(y) \sim \alpha e^{-y}\,, \qquad \text{as} \quad y \to \infty\,. \tag{16b}$$

It is easily seen from phase plane considerations that such a solution exists. In terms of this solution, $h = h_E$, where

$$h_E = \left(\frac{1}{2\epsilon\mu} \int_{-1}^{1} u_c^m \, dx \right)^{\frac{p-1}{(s+1)(p-1)-qm}}\,. \tag{17}$$

Since u_c is localized near x_0, we estimate

$$h_E \sim \left(\frac{\beta}{\mu} \right)^{\frac{p-1}{(s+1)(p-1)-qm}}\,, \qquad \beta \equiv \int_0^\infty \left[u_c(y)\right]^m \, dy\,. \tag{18}$$

To determine numerical values for certain asymptotic quantities below we must compute $u_c(y)$, β, and α numerically. The constant α is obtained by integrating (16):

$$\log(\alpha) = \frac{\log\left(\frac{p+1}{2}\right)}{p-1} + \int_0^{\left(\frac{p+1}{2}\right)^{\frac{1}{p-1}}} \left[\frac{-1}{\sqrt{\eta^2 - \frac{2}{p+1}\eta^{p+1}}} - \frac{1}{\eta} \right] d\eta. \tag{19}$$

To compute u_c numerically, we use the asymptotic boundary condition $u_c^{'} + u_c = 0$ at $y = y_L$, where y_L is a large positive constant. To compute solutions for various values of p, we use a continuation procedure starting from the special analytical solution,

$$u_c(y) = \frac{3}{2}\text{sech}^2\left(\frac{y}{2}\right)\,, \tag{20}$$

which holds when $p = 2$. Thus, when $p = 2$ we get $\alpha = 6$. The boundary value solver COLNEW (see [1]) is then used to solve the resulting boundary value problem. In Figure 1, we plot the numerically computed $u_c(y)$ when $p = 2, 3, 4$.

We note that, for any x_0 with $|x_0| < 1$, the solution $a_E(x;x_0)$ will satisfy the steady-state problem corresponding to (14a), but will fail to satisfy the boundary conditions in (14c) by only exponentially small terms as $\epsilon \to 0$. Thus, we expect that the spectrum of the eigenvalue problem associated with the linearization about a_E will contain an exponentially small eigenvalue.

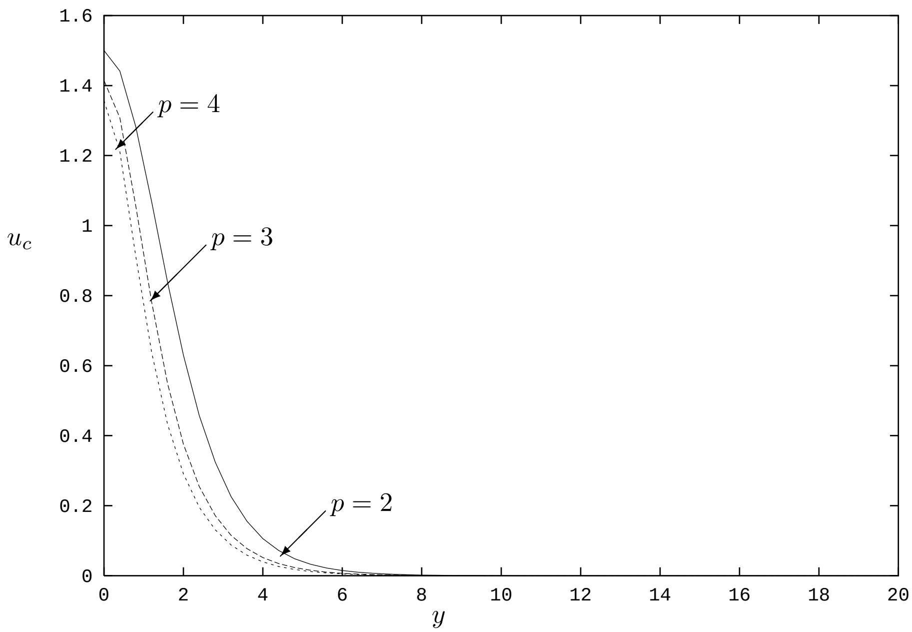

FIG. 1. *Numerical solution for $u_c(y)$ when $p = 2, 3, 4$.*

2.1. The nonlocal eigenvalue problem. Let $x_0 \in (-1, 1)$ be fixed and linearize (14) around a_E, h_E. We obtain the eigenvalue problem for the linearization by introducing ϕ and η by

$$a(x,t) = a_E(x; x_0) + e^{\lambda t}\phi(x)\,, \tag{21a}$$

$$h(t) = h_E + e^{\lambda t}\eta\,, \tag{21b}$$

where $\phi \ll 1$ and $\eta \ll 1$. Substituting (21) into (14) we obtain the following nonlocal eigenvalue problem of Sturm–Liouville type on $[-1, 1]$:

$$L_\epsilon \phi \equiv \epsilon^2 \phi_{xx} + (-1 + p u_c^{p-1})\phi - \frac{mq\epsilon^{-1}u_c^p}{2\beta(s+1)} \int_{-1}^{1} u_c^{m-1}\phi\, dx = \lambda\phi, \tag{22a}$$

$$\phi_x(\pm 1) = 0. \tag{22b}$$

The nonlocal integral term in (22) will drastically change the nature of the eigenvalue problem.

In (22), $u_c = u_c\left[\epsilon^{-1}(x - x_0)\right]$. Therefore, we will seek only eigenfunctions that are localized near $x = x_0$. These eigenfunctions are of the form

$$\tilde{\phi}(y) = \phi(x_0 + \epsilon y)\,, \qquad y = \epsilon^{-1}(x - x_0)\,. \tag{23}$$

Therefore, we can replace the finite interval by an infinite interval in the integral in (22) and impose a decay condition for $\tilde{\phi}(y)$ as $y \to \pm\infty$. This gives us the nonlocal eigenvalue problem for the infinite domain $-\infty < y < \infty$:

$$\tilde{L}_\epsilon \tilde{\phi} \equiv \tilde{\phi}_{yy} + (-1 + p u_c^{p-1})\tilde{\phi} - \frac{mq u_c^p}{2\beta(s+1)} \int_{-\infty}^{\infty} u_c^{m-1} \tilde{\phi}\, dy = \lambda \tilde{\phi}, \tag{24a}$$

$$\tilde{\phi} \to 0 \quad \text{as} \quad y \to \pm\infty\,. \tag{24b}$$

To treat the nonlocal eigenvalue problem, we split the operator L_ϵ into two parts,

$$\mathbb{A}\phi \equiv \epsilon^2 \phi_{xx} + (-1 + p u_c^{p-1})\phi, \qquad \mathbb{B}\phi \equiv \frac{mq\epsilon^{-1} u_c^p}{2\beta(s+1)} \int_{-1}^{1} u_c^{m-1} \phi\, dx\,. \tag{25}$$

We define a new operator L_δ by $L_\delta \phi \equiv \mathbb{A}\phi - \delta \mathbb{B}\phi$, where δ, with $0 < \delta < 1$, is a continuation parameter. When $\delta = 0$ we have a simple Sturm–Liouville problem. At $\delta = 1$ we have our full nonlocal eigenvalue problem (22). We define $\tilde{L}_\delta$, $\tilde{\mathbb{A}}$, and $\tilde{\mathbb{B}}$ in a similar fashion, but on the extended domain $-\infty < y < \infty$ with the appropriate boundary conditions at $\pm\infty$.

The operator $\tilde{L}_\epsilon$ has a zero eigenvalue with eigenfunction u_c', which decays exponentially as $|y| \to \infty$. To see this, we differentiate (16a) with respect to y, to show that $\tilde{\mathbb{A}} u_c' = 0$. This is translation invariance. In addition, due to the symmetry of $u_c(y)$, we also have $\tilde{\mathbb{B}} u_c' = 0$. For the finite domain problem (22), the function $u_c'\left[\epsilon^{-1}(x - x_0)\right]$ fails to satisfy the equation and boundary conditions in (22) by only exponentially small terms as $\epsilon \to 0$. Therefore, as estimated carefully below, the presence of the finite domain will perturb the zero eigenvalue and corresponding eigenfunction of the extended problem (24) by only an exponentially small amount. Thus, L_ϵ has an exponentially small eigenvalue.

The function $u_c(y)$ has a unique maximum at $y = 0$ and thus the eigenfunction $u_c'(y)$ has exactly one zero at $y = 0$. This implies that $u_c'(y)$ corresponds to the second eigenfunction of $\tilde{\mathbb{A}}$. The principal eigenvalue of $\tilde{\mathbb{A}}$ is simple, real and positive, and independent of ϵ. By the term principal eigenvalue we mean the eigenvalue with the largest real part. The principal eigenvalue of $\mathbb{A}$ is exponentially close to the principal eigenvalue of $\tilde{\mathbb{A}}$. Hence, in the absence of the nonlocal term, the operator L_ϵ has an $O(1)$ positive eigenvalue and no metastable spike motion can occur.

Since $\tilde{L}_\delta$ has a positive eigenvalue when $\delta = 0$, we must consider what happens to this eigenvalue as δ ranges from 0 to 1. If this eigenvalue remains positive then, since the eigenvalues of L_δ and $\tilde{L}_\delta$ will differ by only exponentially small amounts as $\epsilon \to 0$, we can conclude that the one-spike quasi-equilibrium solution is unstable. Alternatively, if this eigenvalue crosses through zero at some finite value of $\delta < 1$ and remains in the left half-plane $\mathrm{Re}(\lambda) < 0$ up to $\delta = 1$, then the principal eigenvalue of L_δ when $\delta = 1$ (which corresponds to our eigenvalue problem (22)) will be exponentially small. Hence, if this occurs, the one-spike solution is anticipated to be metastable.

The calculation of the eigenvalues of $\tilde{L}_\delta$ will require some numerical analysis. Thus, we will work with specific parameter sets. We first consider the set $(p, q, m, s) = (2, 1, 2, 0)$, which is commonly used in simulations. For this parameter set, we begin by reviewing some exact results for the spectrum of the local eigenvalue problem

$$\tilde{\mathbb{A}}\tilde{\phi} \equiv \tilde{\phi}_{yy} + (-1 + p u_c^{p-1})\tilde{\phi} = \lambda \tilde{\phi}\,, \qquad -\infty < y < \infty\,, \tag{26a}$$

$$\tilde{\phi} \to 0 \quad \text{as} \quad y \to \pm\infty\,. \tag{26b}$$

This problem has three isolated eigenvalues and a continuous spectrum. When $p = 2$,

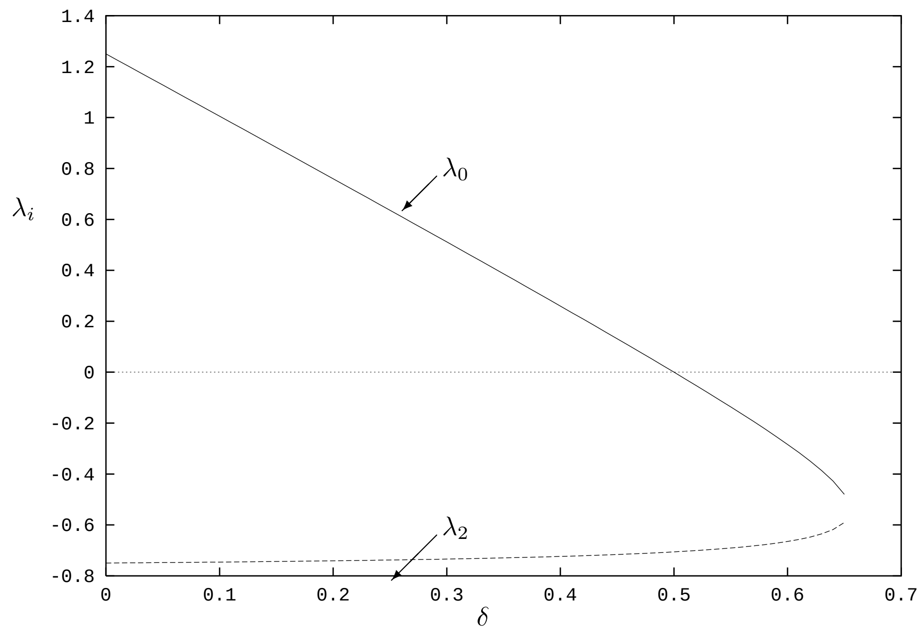

FIG. 2. λ_0 *and* λ_2 *versus* δ *for the parameter set* $(p, q, m, s) = (2, 1, 2, 0)$.

these three isolated eigenpairs are (see [8])

$$\lambda_0 = 5/4\,, \qquad \tilde{\phi}_0 = \operatorname{sech}^3(y/2)\,, \tag{27}$$

$$\lambda_1 = 0\,, \qquad \tilde{\phi}_1 = \tanh(y/2)\operatorname{sech}^2(y/2)\,, \tag{28}$$

$$\lambda_2 = -3/4\,, \qquad \tilde{\phi}_2 = 5\operatorname{sech}^3(y/2) - 4\operatorname{sech}(y/2)\,. \tag{29}$$

Since these eigenfunctions, written in terms of $y = \epsilon^{-1}(x - x_0)$, will fail to satisfy the boundary conditions in (22) by only exponentially small terms as $\epsilon \to 0$, we expect that the eigenvalues of $\mathbb{A}$ will be only slightly perturbed from those of $\tilde{\mathbb{A}}$. As we have previously noted, the zero eigenvalue of (26) will persist for $\tilde{L}_\delta$ as δ ranges from zero to one. Hence, there is an eigenvalue of (22) that is exponentially small as $\epsilon \to 0$.

To numerically compute the eigenvalue branches $\lambda_0(\delta)$ and $\lambda_2(\delta)$ of $\tilde{L}_\delta$ for which $\lambda_0(\delta) \to 5/4$ and $\lambda_2(\delta) \to -3/4$, as $\delta \to 0$, we use the initial guesses provided above for $\delta = 0$ and a continuation procedure to compute these eigenvalues as δ increases. The computations are done using COLNEW. The analysis of [3] showed that these eigenvalue branches are smooth functions of δ, which cannot terminate suddenly at some value of δ. Hence, δ is a natural homotopy parameter. In Figure 2 we plot the numerically computed branches $\lambda_0(\delta)$ and $\lambda_2(\delta)$ versus δ. As can be seen from this graph, $\lambda_0 \approx 0$ for $\delta = 1/2$. As δ increases past $1/2$, λ_0 becomes negative and then complex. At this point, COLNEW is no longer able to track the eigenvalue. As δ increases from 0 to 1, λ_0 is decreasing and λ_2 is increasing. At a value of $\delta \approx 0.65$ the two eigenvalues collide and split into complex conjugates eigenvalues with negative real parts. To track the eigenvalues beyond $\delta \approx 0.65$ one must employ a different numerical technique. We accomplish this by discretizing the finite domain problem (22), which has eigenvalues exponentially close to those of $\tilde{L}_\delta$. This is done using a centered difference approximation applied to the second derivative and Simpson's rule applied to the integral. Thus, the operator L_δ is approximated by a discrete linear operator $\mathcal{L}_\delta$. The eigenvalues of the continuous problem may then be approximated by the eigenvalues of this matrix. Numerical calculations of the eigenvalue λ_0 of $\mathcal{L}_\delta$ are shown in Table 1. Since the real part of λ_0 remains negative as $\delta \to 1$, we conclude that the one-spike quasi-equilibrium solution is stable for this parameter set. Similar computations, with similar conclusions, have been performed for other values of p, q, m, and s. For a wide range of these parameters a recent analytical result of [18], which we summarize in section 4, gives a rigorous proof that $\mathrm{Re}(\lambda) < 0$ when $\delta = 1$.

TABLE 1
δ and λ_0 for the case $(p, q, m, s) = (2, 1, 2, 0)$.

δ	λ_0
0.0	1.2518
0.1	1.0073
0.2	0.76149
0.3	0.51345
0.4	0.26158
0.5	0.0052548
0.6	-0.28247
0.7	$-.59237 + 0.15315i$
0.8	$-.71522 + 0.23035i$
0.9	$-.84093 + 0.23008i$
1.0	$-.98551 + 0.14507i$

2.2. An exponentially small eigenvalue. In the previous section, we showed that the principal eigenvalue of L_ϵ is exponentially small. The nonlocal term in (22) was found to be essential to this conclusion. We denote the eigenpair corresponding to the exponentially small eigenvalue by λ_1, ϕ_1. To predict the dynamics of the quasi-equilibrium solution, we must obtain a very accurate estimate of λ_1. We expect that $\phi_1 \sim C_1 u_c'\left(\epsilon^{-1}(x - x_0)\right)$ in the outer region away from $O(\epsilon)$ boundary layers near $x = \pm 1$. The behavior of ϕ_1 in these regions will be analyzed using a boundary layer analysis.

The eigenfunction ϕ_1 has the boundary layer form

$$\phi_1(x) = C_1 \left(u_c'\left[\epsilon^{-1}(x - x_0)\right] + \phi_l\left[\epsilon^{-1}(x + 1)\right] + \phi_r\left[\epsilon^{-1}(1 - x)\right] \right) . \tag{30}$$

Here $\phi_l(\eta)$ and $\phi_r(\eta)$ are boundary layer correction terms and C_1 is a normalization constant given by

$$C_1 = \left(\epsilon \hat{\beta}\right)^{-1/2} , \qquad \text{where} \quad \hat{\beta} = \int_{-\infty}^{\infty} \left[u_c'(y)\right]^2 \, dy \, . \tag{31}$$

In the boundary layer region near $x = -1$, $u_c'\left[\epsilon^{-1}(x - x_0)\right]$ is exponentially small as $\epsilon \to 0$. Thus, as $\epsilon \to 0$, $\phi_l(\eta)$ satisfies

$$\phi_l'' - \phi_l = 0, \qquad 0 \le \eta < \infty \, , \tag{32a}$$

$$\phi_l'(0) \sim -\alpha e^{-\epsilon^{-1}(1 + x_0)} \, . \tag{32b}$$

A similar equation determines $\phi_r(\eta)$. Solving the boundary layer equations we get

$$\phi_l(\eta) = \alpha e^{-\epsilon^{-1}(1 + x_0)} e^{-\eta}, \tag{33a}$$

$$\phi_r(\eta) = -\alpha e^{-\epsilon^{-1}(1 - x_0)} e^{-\eta} \, . \tag{33b}$$

To estimate λ_1 we first derive Lagrange's identity for $(u, L_\epsilon v)$, where $(u,v) \equiv \int_{-1}^{1} uv\,dx$. Using integration by parts we derive

$$(v, L_\epsilon u) = \epsilon^2 \left(u_x v - v_x u\right)\big|_{-1}^{1} + (u, L_\epsilon^* v)\,, \tag{34}$$

where

$$L_\epsilon^* v \equiv \epsilon^2 v_{xx} + (-1 + p u_c^{p-1}) v - \frac{mq\epsilon^{-1} u_c^{m-1}}{2\beta(s+1)} \int_{-1}^{1} u_c^p v\,dx\,. \tag{35}$$

We now apply this identity to the functions u_c' and $\phi_1(x)$ to get

$$\lambda_1 (u_c', \phi_1) = -\epsilon \phi_1 u_c''\big|_{-1}^{1} + (\phi_1, L_\epsilon^* u_c')\,. \tag{36}$$

We examine each of the terms in (36). We begin with (u_c', ϕ_1). The dominant contribution to this integral arises from the region near $x = x_0$, where $\phi_1 \sim C_1 u_c'$. Therefore, the inner product can be estimated as

$$(u_c', \phi_1) = C_1 (u_c', u_c') \sim C_1 \epsilon \hat{\beta}\,. \tag{37}$$

Next, to estimate $-\epsilon \phi_1 u_c''|_{-1}^{1}$, we use our asymptotic estimates of u_c and ϕ_1. Since $u_c(z) \sim \alpha e^{-|z|}$ as $z \to \pm\infty$ we have that $u_c'' \sim \alpha e^{-\epsilon^{-1}(1 \mp x_0)}$ at the endpoints $x = \pm 1$. In addition, using the previous boundary layer results for ϕ_1 we get the following estimate for $\phi_1(\pm 1)$:

$$\phi_1(\pm 1) \sim \mp 2 C_1 \alpha e^{-\epsilon^{-1}(1 \mp x_0)}\,. \tag{38}$$

Using these results, we get

$$-\epsilon \phi_1 u_c''\big|_{-1}^{1} \sim 2\epsilon C_1 \alpha^2 \left(e^{-2\epsilon^{-1}(1+x_0)} + e^{-2\epsilon^{-1}(1-x_0)} \right)\,. \tag{39}$$

The only term left to examine is $(\phi_1, L_\epsilon^* u_c')$. Since u_c' is a solution to the local operator, we have

$$\begin{aligned} L_\epsilon^* u_c' &= -\frac{mq\epsilon^{-1} u_c^{m-1}}{2\beta(s+1)} \int_{-1}^{1} u_c^p u_c'\,dx, \\ &\sim -\frac{mq\epsilon^{-1}\alpha^{p+1} u_c^{m-1}}{2\beta(s+1)(p+1)} \left(E_+^{p+1} - E_-^{p+1} \right)\,, \end{aligned} \tag{40}$$

where

$$E_\pm^p \equiv e^{-p\epsilon^{-1}(1 \pm x_0)}\,. \tag{41}$$

In a similar way, the term $(\phi_1, L_\epsilon^* u_c')$ is approximated by

$$\begin{aligned} (\phi_1, L_\epsilon^* u_c') &\sim -\frac{mq\epsilon^{-1}\alpha^{p+1} C_1}{2\beta(s+1)(p+1)} \left(E_+^{p+1} - E_-^{p+1} \right) \int_{-1}^{1} u_c^{m-1} u_c'\,dx, \\ &\sim -\frac{mq\epsilon^{-1}\alpha^{p+m+1} C_1}{2\beta(s+1)(p+1)m} \left(E_+^{p+1} - E_-^{p+1} \right) \left(E_+^{m} - E_-^{-m} \right)\,. \end{aligned} \tag{42}$$

Since $p > 1$ and $m > 0$, upon comparing the terms in (42) and (39), it is clear that the second term on the right side of (36) is asymptotically negligible compared to the first

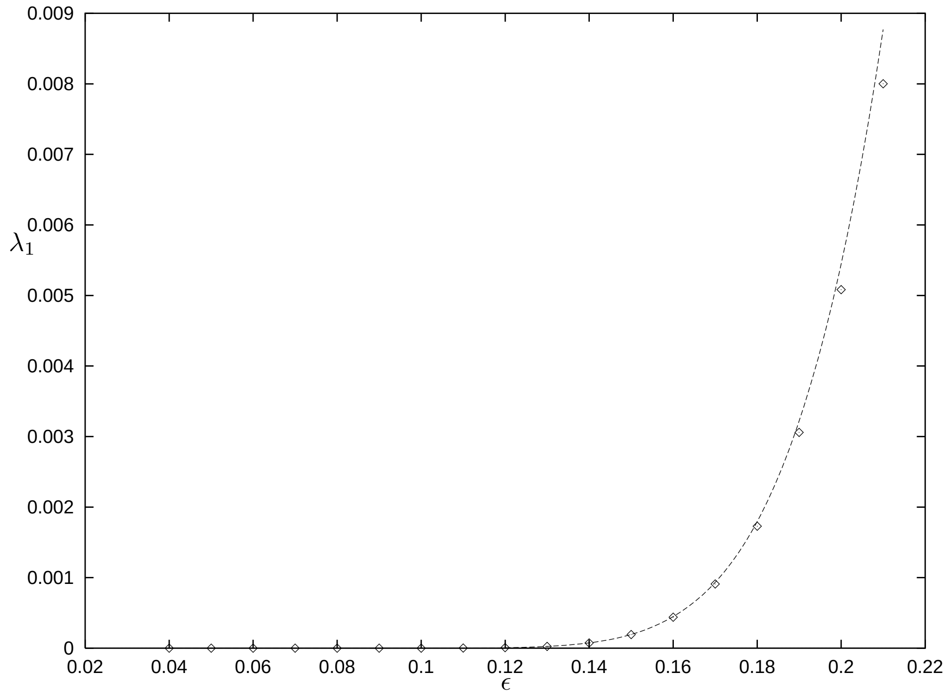

FIG. 3. λ_1 *versus* ϵ *for the parameter set* $(p, q, m, s) = (2, 1, 2, 0)$.

term. Finally, substituting (37) and (39) into (36), we get the following asymptotic estimate for the exponentially small eigenvalue λ_1 as $\epsilon \to 0$:

$$\lambda_1 \sim 2\alpha^2 \hat{\beta}^{-1} \left(e^{-2\epsilon^{-1}(1+x_0)} + e^{-2\epsilon^{-1}(1-x_0)} \right) . \tag{43}$$

In (43), α and $\hat{\beta}$ are defined in (19) and (31), respectively. The estimate (43) holds for p, q, m, and s satisfying (2). Since $\lambda_1 > 0$ the equilibrium spike will be unstable, but it will persist for exponentially long times.

To verify the estimate for λ_1, we also numerically estimate λ_1 by solving (22) using COLNEW. In Figure 3, we compare the numerically computed values of λ_1 (dots) with the asymptotic estimate (43) (dashed curve) for various values of ϵ for the parameter set $(p, q, m, s) = (2, 1, 2, 0)$. Similar favorable comparisons can be made for other parameter sets.

We end this section with a few remarks. First, we recall that λ_1 and $\phi_1 \sim C_1 u_c'\left(\epsilon^{-1}(x - x_0)\right)$ are an eigenpair of L_δ when $\delta = 0$. To within negligible exponentially small terms, this eigenpair remains an eigenpair of L_δ as δ ranges from 0 to 1. To see this, we note that the only difference between the calculations of the eigenvalue for the local problem and for the nonlocal problem is that the term $(L_\epsilon^* u_c', \phi_1)$ in (36) would be replaced by $(\mathbb{A}\phi_1, \phi_1) = 0$, since $\mathbb{A}$ is self-adjoint. In the final calculation of λ_1, the term $(L_\epsilon^* u_c', \phi_1)$ was ignored since it is asymptotically exponentially smaller than the other terms in (36). Second, we note that λ_1, ϕ_1 is exponentially close to an eigenpair λ_1^*, ϕ_1^* of the adjoint operator, L_ϵ^*. For the same reasoning as above, ϕ_1^* would have the same interior behavior near $x = x_0$ as ϕ_1 and the same boundary layer correction terms near $x = \pm 1$. Repeating the calculation to find λ_1^*, we would arrive at the same estimate as in (43).

2.3. The slow motion of the spike. The quasi-equilibrium solution fails to satisfy the steady-state problem corresponding to (14) by only exponentially small terms for any value of x_0 in $|x_0| < 1$. Moreover, the linearization about this solution admits a principal eigenvalue that is exponentially small. Therefore, we expect that the one-spike quasi-equilibrium solution evolves on an exponentially slow time-scale. We will now derive an equation of motion for the center of the spike corresponding to the quasi-equilibrium solution. To do so we first linearize (14) about $a(x,t) = h^{\gamma} u_c\left[\epsilon^{-1}(x - x_0(t))\right]$, where the spike location $x_0 = x_0(t)$ is to be determined. For a fixed x_0 we have shown that the linearization around this solution has an exponentially small principal eigenvalue as $\epsilon \to 0$. By eliminating the projection of the solution on the eigenfunction corresponding to this eigenvalue, we will derive an equation of motion for $x_0(t)$.

We begin by linearizing around a moving spike solution by writing

$$a(x,t) = a_E(x; x_0(t)) + w(x,t)\,, \tag{44}$$

where a_E is defined in (15) and $x_0(t)$ is the trajectory of the spike. Since (22) does not have an $O(1)$ positive eigenvalue, we may assume that $w \ll a_E$ and $w_t \ll \partial_t a_E$. Substituting (44) into (14), we get

$$L_\epsilon w = \partial_t a_E\,, \qquad -1 < x < 1\,, \quad t \geq 0\,, \tag{45a}$$

$$w_x(\pm 1, t) = -\partial_x a_E(\pm 1; x_0)\,. \tag{45b}$$

Next, we expand w in terms of the eigenfunctions ϕ_i of L_ϵ as

$$w = \sum_{i=0}^{\infty} G_i(t)\phi_i\,. \tag{46}$$

The solvability condition for w is that w is orthogonal to the eigenspace of L_ϵ^* associated with the exponentially small eigenvalue. Let ϕ_i^* be the ith eigenfunction of L_ϵ^*. Then, since $(\phi_i, \phi_j^*) = \delta_{ij}$, we integrate by parts to show that

$$G_i(t) = (w, \phi_i^*) = \frac{1}{\lambda_i^*}\left[(L_\epsilon w, \phi_i^*) - \epsilon^2 w_x \phi_i^*|_{-1}^{1}\right]\,, \tag{47}$$

where $L_\epsilon^* \phi_i^* = \lambda_i^* \phi_i^*$. Using (45), we have

$$G_i(t) = \frac{1}{\lambda_i^*}\left[(\partial_t a_E, \phi_i^*) + \epsilon^2 \phi_i^* \partial_x a_E|_{-1}^{1}\right]\,. \tag{48}$$

As discussed previously, when $\epsilon \ll 1$, the nonlocal term in the eigenvalue problem (22) is insignificant in asymptotic estimation of the eigenspace associated with the exponentially small eigenvalue of L_ϵ. Therefore, we can replace ϕ_1^* and λ_1^* by ϕ_1 and λ_1 in (48), where ϕ_1 and λ_1 are given in (30) and (43), respectively.

Since $\lambda_1 \to 0$ exponentially as $\epsilon \to 0$, we must impose the limiting solvability condition that $G_1 = 0$. This projection step yields the following implicit differential equation for $x_0(t)$:

$$(\partial_t a_E, \phi_1) = -\epsilon^2 \phi_1 \partial_x a_E|_{-1}^{1}\,. \tag{49}$$

The dominant contribution to the left side of (49) arises from the region near x_0. For $\epsilon \to 0$, we calculate

$$(\partial_t a_E, \phi_1) \sim -C_1 h_E^{\gamma} \dot{x}_0 \hat{\beta}, \tag{50}$$

where $\dot{x}_0 \equiv dx_0/dt$. Finally, we can evaluate the right side of (49) using our estimates for $\phi_1(\pm 1)$ in (38) and for $u_c(z)$ as $z \to \infty$. This yields our main result of this section.

PROPOSITION 1 (metastability). *A metastable spike solution for* (14) *is represented by* $a(x,t) = a_E(x; x_0(t))$*, where* a_E *is defined in* (15) *and* $x_0(t)$ *satisfies*

$$\dot{x}_0(t) \sim \frac{2\alpha^2 \epsilon}{\hat{\beta}} \left[e^{-2(1-x_0)/\epsilon} - e^{-2(1+x_0)/\epsilon} \right]. \tag{51}$$

Here α *and* $\hat{\beta}$ *are defined in* (19) *and* (31)*, respectively.*

For a given initial condition $x_0(0) \in (-1,1)$, this ODE shows that the spike drifts towards the endpoint that is closest to the initial location $x_0(0)$. As a consistency check on our solvability condition $G_1 = 0$, we note from (31), (43), (50), and (51) that if the solvability condition were not imposed, then $G_1 \phi_1 = O(\epsilon^{-1/2})$. This would violate our linearization assumption.

To verify the asymptotic result (51) we computed numerical solutions to (14) for various values of ϵ for the parameter set $(p,q,m,s) = (2,1,2,0)$. The computations were done by using a variable coefficient variable time step backward-differentiation (BDF) scheme to integrate in time (see [12] where a similar scheme is used). COLNEW was then used to solve the boundary value problem at each time step. At each time step the solution is calculated using a third and fourth order BDF scheme to estimate the error and determine the next maximum allowable time step. Comparisons of these results (solid curve) with a numerical integration of the asymptotic ODE (51) (dots) may be found in Figures 4–6.

In computing numerical solutions to (14), the initial condition $a_E(x; x_0(0))$, defined in (15), was used with $x_0(0) = -0.4$.

3. A spike in a multidimensional domain. We now construct a quasi-equilibrium solution a_E for (13). This is done in a similar manner as in the one-dimensional case, except that the quasi-equilibrium solution will be radially symmetric about the center of the spike. Thus, we look for a steady-state solution to (13) in all of $\mathbb{R}^N$ of the form

$$a = a_E(\mathbf{x}; \mathbf{x_0}) \equiv h^{\gamma} u_c(\rho)\,, \qquad \rho = \epsilon^{-1}|\mathbf{x} - \mathbf{x_0}|\,, \qquad \gamma = q/(p-1)\,, \tag{52}$$

where $\mathbf{x_0}$ is contained inside Ω (i.e., $\text{dist}(x_0, \partial\Omega) \gg O(\epsilon)$). The function $u_c(\rho)$, called the canonical spike solution, is a nonnegative radially symmetric function, which decays exponentially as $\rho \to \infty$. It satisfies

$$u_c'' + \frac{N-1}{\rho} u_c' - u_c + u_c^p = 0\,, \qquad \rho > 0\,, \tag{53a}$$

$$u_c'(0) = 0\,; \qquad u_c(\rho) \sim \alpha \rho^{(1-N)/2} e^{-\rho}\,, \quad \text{as} \quad \rho \to \infty\,, \tag{53b}$$

where $\alpha > 0$ is some constant. In dimension $N > 2$, we require that $p < p_c$, where p_c is the critical Sobolev exponent for dimension N. In terms of this solution, $h = h_E$, where

$$h_E = \left(\frac{\epsilon^{-N}}{\mu|\Omega|} \int_{\Omega} u_c^m \, d\mathbf{x} \right)^{\frac{p-1}{(s+1)(p-1)-qm}}\,. \tag{54}$$

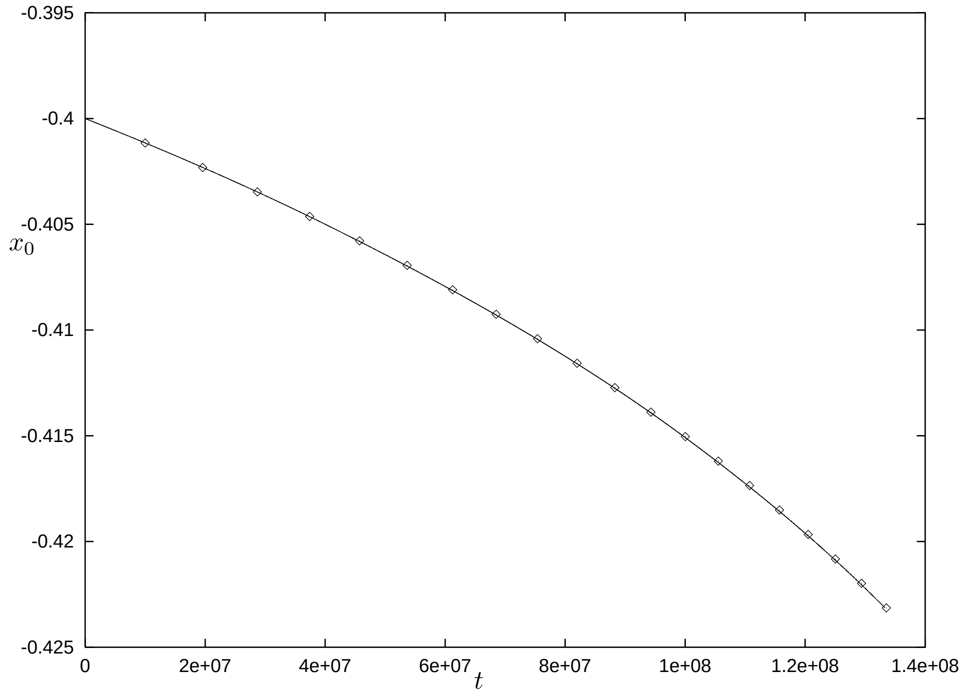

FIG. 4. x_0 *versus* t *for* $\epsilon = .05$.

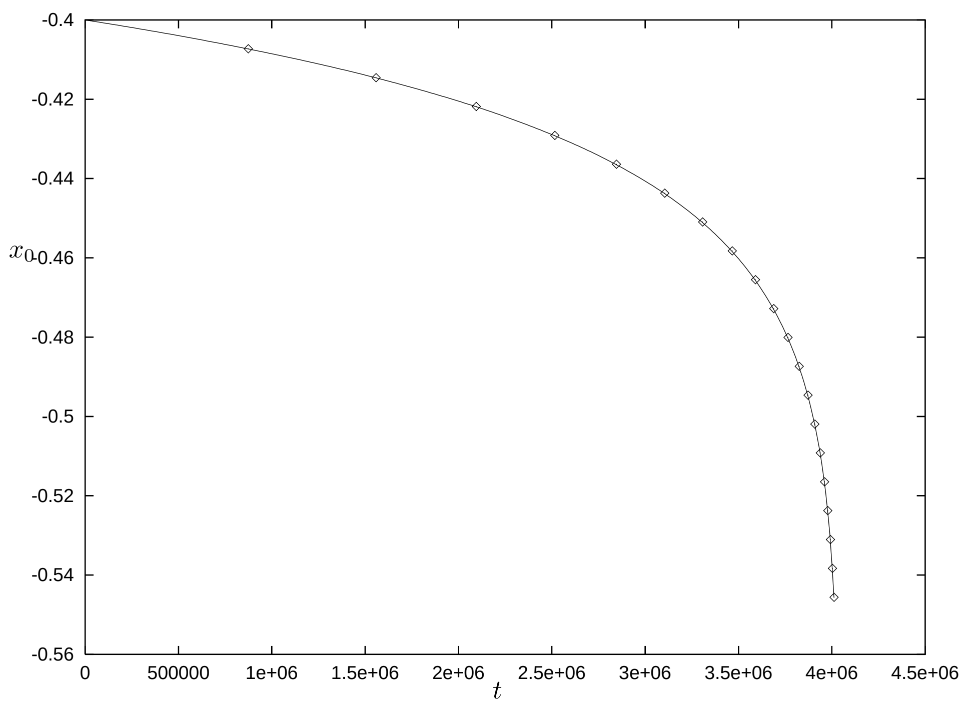

FIG. 5. x_0 *versus* t *for* $\epsilon = .06$.

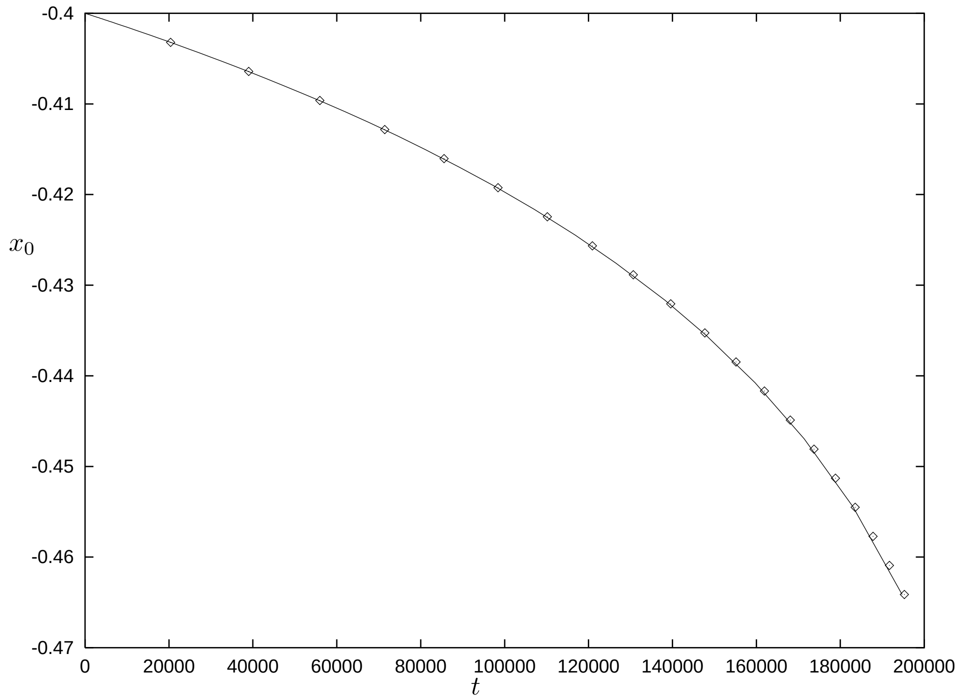

FIG. 6. x_0 *versus* t *for* $\epsilon = .07$.

Here $|\Omega|$ is the volume of Ω. Since u_c is localized near $\mathbf{x_0}$, for $\epsilon \to 0$ we get

$$h_E \sim \left(\frac{\Omega_N}{\mu|\Omega|} \int_0^\infty u_c^m \, \rho^{N-1} d\rho \right)^{\frac{p-1}{(s+1)(p-1)-qm}}, \tag{55}$$

where Ω_N is the surface area of the unit N-dimensional sphere.

Recall that in the one-dimensional case and with $p = 2$ we have the exact solution $u_c(\rho) = \frac{3}{2}\text{sech}^2(\frac{\rho}{2})$, and hence $\alpha = 6$. To find numerical solutions for $u_c(\rho)$ and for α in other dimensions, we will treat N as a real parameter and use N (and p for $p \neq 2$) as continuation parameters. We can use the far-field asymptotic behavior (53b) to obtain the boundary condition $u_c' = \frac{(1-N)}{2\rho} u_c$, which we impose at some large value $\rho = \rho_L$ in our numerical computations of (53). The computations are done using COLNEW. In Figure 7 we plot the numerically computed solution $u_c(\rho)$ for $N = 1, 2, 3$ when $p = 2$.

Since $\text{dist}(x_0, \partial\Omega) \gg O(\epsilon)$, we again note that a_E will satisfy the steady-state problem for (13a), but will fail to satisfy the no-flux boundary condition (13c) by only exponentially small terms for any value of $\mathbf{x}_0$ in the interior of Ω. Thus, we expect that the spectrum of the eigenvalue problem associated with the linearization about a_E contains exponentially small eigenvalues.

3.1. The nonlocal eigenvalue problem. Let $\mathbf{x_0} \in \Omega$ be fixed, and linearize (13) about a_E, h_E. We obtain the eigenvalue problem for this linearization by intro-

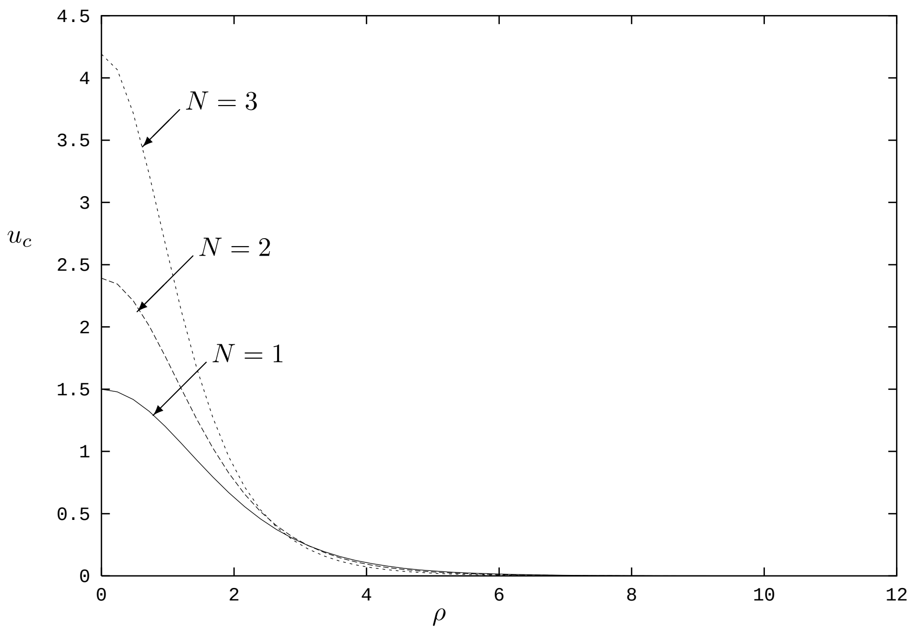

FIG. 7. *Numerical solution for* $u_c(\rho)$ *when* $N = 1, 2, 3$ *and* $p = 2$.

ducing ϕ and η defined by

$$a(\mathbf{x}, t) = a_E(\mathbf{x}; \mathbf{x_0}) + e^{\lambda t}\phi(\mathbf{x}), \tag{56a}$$

$$h(t) = h_E + e^{\lambda t}\eta\,, \tag{56b}$$

where $\phi \ll a_E$ and $\eta \ll h$. Substituting (56) into (13) we obtain, after a lengthy calculation, the following nonlocal eigenvalue problem:

$$L_\epsilon\phi \equiv \epsilon^2\Delta\phi + (-1 + pu_c^{p-1})\phi - \frac{mq\epsilon^{-N}u_c^p}{\beta_N\Omega_N(s+1)}\int_\Omega u_c^{m-1}\phi\, d\mathbf{x} = \lambda\phi, \quad \text{in} \quad \Omega, \tag{57a}$$

$$\partial_n\phi = 0 \quad \text{on} \quad \partial\Omega\,. \tag{57b}$$

Here $u_c = u_c(\rho)$, and β_N is defined by

$$\beta_N = \int_0^\infty u_c^{m-1}\rho^{N-1}\, d\rho\,. \tag{58}$$

Since u_c is localized near $\rho = 0$, which corresponds to $\mathbf{x} = \mathbf{x}_0$, we will seek only eigenfunctions that are localized near $\mathbf{x} = \mathbf{x}_0$. These eigenfunctions are of the form

$$\tilde{\phi}(\mathbf{y}) = \phi(\mathbf{x}_0 + \epsilon\mathbf{y})\,, \qquad \mathbf{y} = \epsilon^{-\mathbf{1}}(\mathbf{x} - \mathbf{x_0})\,. \tag{59}$$

Therefore, we can replace Ω by $\mathbb{R}^n$ in (57a) and impose a decay condition for $\tilde{\phi}$ as $|\mathbf{y}| \to \infty$. This gives us the nonlocal eigenvalue problem for the infinite domain

(60a)
$$\hat{L}_\epsilon \tilde{\phi} \equiv \Delta_y \tilde{\phi} + (-1 + p u_c^{p-1})\tilde{\phi} - \frac{mq u_c^p}{\beta_N \Omega_N (s+1)} \int_{\mathbb{R}^N} u_c^{m-1} \tilde{\phi}\, d\mathbf{y} = \tilde{\lambda}\tilde{\phi}, \quad \text{in} \quad \mathbb{R}^N,$$

(60b)
$$\tilde{\phi} \to 0 \qquad \text{as} \quad |\mathbf{y}| \to \infty\,.$$

In this problem $u_c = u_c(|\mathbf{y}|)$. If, in addition, we consider an eigenfunction that is radially symmetric (i.e., $\tilde{\phi} = \tilde{\phi}(\rho)$, where $\rho = |\mathbf{y}|$), then (60) reduces to

(61a)
$$\tilde{L}_\epsilon \tilde{\phi} \equiv \Delta_\rho \tilde{\phi} + (-1 + p u_c^{p-1})\tilde{\phi} - \frac{mq u_c^p}{\beta_N (s+1)} \int_0^\infty u_c^{m-1} \tilde{\phi} \rho^{N-1}\, d\rho = \tilde{\lambda}\tilde{\phi}, \quad \rho > 0,$$

(61b)
$$\tilde{\phi} \to 0 \quad \text{as} \quad \rho \to \infty\,,$$

where $\Delta_\rho \tilde{\phi} \equiv \tilde{\phi}'' + (N-1)\rho^{-1}\tilde{\phi}'$.

We now analyze the spectrum of these eigenvalue problems. We first note that, for each $i = 1, \ldots, N$, the function $\tilde{\phi}_i = \partial_{y_i} u_c(|\mathbf{y}|)$ satisfies (60). Here y_i is the ith coordinate of $\mathbf{y}$. This follows from the combined effects of translation invariance and the vanishing of the integral in (60) by symmetry considerations. Thus, (60) has a zero eigenvalue of multiplicity N with corresponding eigenfunctions $\tilde{\phi}_i = \partial_{y_i} u_c(|\mathbf{y}|)$ for $i = 1, \ldots, N$. Each of these eigenfunctions has one nodal line. These eigenpairs will be perturbed by only exponentially small terms as a result of the finite domain. Hence, there are N eigenvalues of (57) that are exponentially small, and they are estimated below. The goal is to determine whether these are the principal eigenvalues of (57).

We claim that these are not the principal eigenvalues for (57) when the nonlocal term in (57) is absent. To see this, suppose that the nonlocal term in (57), (60), and (61) is absent. The corresponding eigenvalue problems are then local and self-adjoint, and several key properties follow. In particular, since $\tilde{\phi}_i = \partial_{y_i} u_c(|\mathbf{y}|)$ is an eigenfunction of (60) with a zero eigenvalue and has one nodal line, the local eigenvalue problem must have a simple, positive eigenvalue, which is independent of ϵ. The corresponding positive, radially symmetric eigenfunction satisfies the local version of (61). The effect of the finite domain in (57) is to perturb this eigenvalue by only exponentially small terms. Thus, when the nonlocal term is absent no metastable behavior can occur.

The effect of the nonlocal term will be to ensure that the exponentially small eigenvalues are the principal eigenvalues for the nonlocal eigenvalue problem (57). Thus, the quasi-equilibrium solution will be metastable if we can show that the principal eigenvalue of (61) has a negative real part. To do so, we compute the eigenvalues and eigenfunctions of the radially symmetric problem (61), where a continuation parameter δ, with $0 \le \delta \le 1$, multiplies the nonlocal term

(62a)
$$L_\delta \tilde{\phi} \equiv \Delta_\rho \tilde{\phi} + (-1 + p u_c^{p-1})\tilde{\phi} - \delta \frac{mq u_c^p}{\beta_N (s+1)} \int_0^\infty u_c^{m-1} \tilde{\phi} \rho^{N-1}\, d\rho = \lambda\tilde{\phi}, \quad \rho > 0,$$

(62b)
$$\tilde{\phi} \to 0 \quad \text{as} \quad \rho \to \infty\,.$$

We compute the eigenvalues of this problem as a function of δ and in particular track the first eigenvalue $\lambda_0(\delta)$. We will show that the positive principal eigenvalue $\lambda_0(0)$, which occurs when the nonlocal term is absent, will cross through zero into the left half-plane as δ increases. Thus, we must show that the first eigenvalue $\lambda_0(\delta)$ has a negative real part when $\delta = 1$.

For the parameter set $(p = 2, q = 1, m = 2, s = 0)$, in Figures 8 and 9 we plot the first two eigenvalues $\lambda_0(\delta)$ and $\lambda_{N+1}(\delta)$ of (62) as a function of δ for $N = 2$ and $N = 3$, respectively. Here λ_{N+1} is the first eigenvalue in the sequence for (60) following the zero λ_i, $i = 1, \ldots, N$. These computations were done using COLNEW. These plots clearly indicate that $\lambda_0(\delta)$ crosses through 0 before $\delta = 1$. At some value of δ, λ_0 and λ_{N+1} collide and become complex. To track the eigenvalues past the point where they become complex, we use the same technique as in the one-dimensional case. The differential operator is approximated by a matrix and the eigenvalues of the matrix are then approximations of the eigenvalues of the differential operator. Using this numerical procedure, we give numerical values for the real and imaginary part of $\lambda_0(\delta)$ in Table 2. This table shows that the real part of λ_0 is negative when $\delta = 1$. Similar computations, with similar conclusions, can be performed for other values of p, q, m, and s. The recent result of [18] for the spectrum of (60), which we summarize in section 4, confirms this result.

3.2. An exponentially small eigenvalue. We will now use a boundary layer analysis to construct a composite approximation to the eigenfunctions corresponding to the exponentially small eigenvalues of (57). The corresponding eigenfunctions are well approximated by $\partial_{x_i} u_c$, for $i = 1, \ldots, N$ in the interior of the domain and each of these eigenfunctions has a boundary layer correction term near $\partial\Omega$ in order to satisfy the no-flux boundary condition on $\partial\Omega$. In order to resolve the boundary layer we

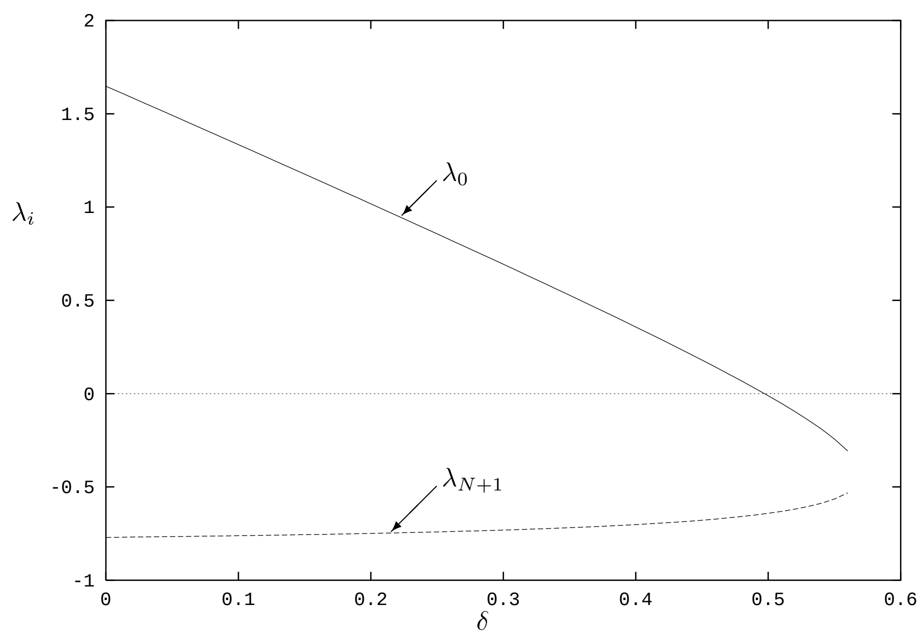

FIG. 8. *$\lambda_0(\delta)$ and $\lambda_{N+1}(\delta)$ versus δ in $\mathbb{R}^2$ for the parameter set $(p = 2, q = 1, m = 2, s = 0)$.*

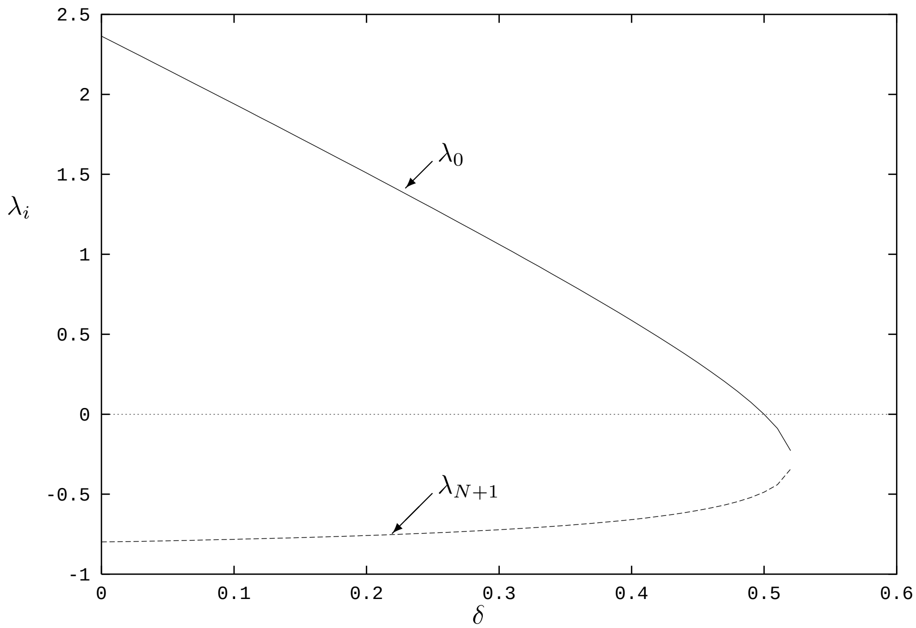

FIG. 9. $\lambda_0(\delta)$ *and* $\lambda_{N+1}(\delta)$ *versus* δ *in* $\mathbb{R}^3$ *for the parameter set* $(p = 2, q = 1, m = 2, s = 0)$.

TABLE 2
δ *and* λ_0 *in* $\mathbb{R}^2$ *and* $\mathbb{R}^3$ *for the case of* $(p = 2, q = 1, m = 2, s = 0)$.

δ	λ_0 in $\mathbb{R}^2$	λ_0 in $\mathbb{R}^3$
0.00000	1.6388	2.3703
0.05000	1.4814	2.1588
0.10000	1.3231	1.9456
0.15000	1.1638	1.7304
0.20000	1.0030	1.5125
0.25000	0.84032	1.2910
0.30000	0.67516	1.0646
0.35000	0.50641	0.83098
0.40000	0.33218	0.58554
0.45000	0.14857	0.31741
0.50000	-.055026	-.019898
0.55000	-.37526	$-.33843 + 0.29744i$
0.60000	$-.48239 + 0.24569i$	$-.44368 + 0.45028i$
0.65000	$-.56115 + 0.33165i$	$-.54978 + 0.54508i$
0.70000	$-.64059 + 0.38475i$	$-.65696 + 0.60964i$
0.75000	$-.72097 + 0.41770i$	$-.76550 + 0.65310i$
0.80000	$-.80268 + 0.43510i$	$-.87584 + 0.67970i$
0.85000	$-.88640 + 0.43886i$	$-.98857 + 0.69170i$
0.90000	$-.97333 + 0.42959i$	$-1.1045 + 0.69037i$
0.95000	$-.10657 + 0.40726i$	$-1.2249 + 0.67652i$
0.10000	$-.11678 + 0.37248i$	$-1.3513 + 0.65089i$

must define a local coordinate system. Let $\hat{\eta}$ represent the distance from a point in Ω to $\partial\Omega$, where $\hat{\eta} < 0$ corresponds to the interior of Ω. Let ζ correspond to the other $N-1$ orthogonal coordinates. To localize the region near $\partial\Omega$, we let $\eta = \epsilon^{-1}\hat{\eta}$. The eigenfunction on the finite domain can then be approximated by

$$\phi_i = C_i \left(\partial_{x_i} u_c(\rho) + \hat{\phi}_i \right), \tag{63}$$

where C_i is a normalization constant and $\hat{\phi}_i$ is a boundary layer correction term. Using the fact that u_c is exponentially small near $\partial\Omega$ we get the following boundary layer problem:

$$\partial_{\eta\eta}\hat{\phi}_i - \hat{\phi}_i = 0, \quad \eta < 0, \tag{64}$$

$$\partial_\eta \hat{\phi}_i = -\epsilon \partial_{\hat{\eta}} \left(\partial_{x_i} u_c \right)|_{\eta=0}, \quad \text{on} \quad \eta = 0, \tag{65}$$

$$\hat{\phi}_i \to 0 \text{ as } \eta \to -\infty. \tag{66}$$

We require that $\hat{\phi}_i \to 0$ as $\eta \to -\infty$ to match to the outer solution. The solution is

$$\hat{\phi}_i = \epsilon g_i(\zeta) e^{\eta}, \quad \text{where} \quad g_i(\zeta) \equiv -\partial_{\hat{\eta}}(\partial_{x_i} u_c)|_{\eta=0}. \tag{67}$$

Thus, the composite asymptotic solution for the eigenfunction is

$$\phi_i = C_i \left[\partial_{x_i} u_c + \epsilon g_i(\zeta) e^{\hat{\eta}/\epsilon} \right], \quad i = 1, \dots, N. \tag{68}$$

Below we need an estimate for ϕ_i on $\partial\Omega$. To do so we need to calculate g_i. Let x_{0i} represent the ith coordinate of $\mathbf{x}_0$. So, setting $r = |\mathbf{x} - \mathbf{x}_0|$, we apply the chain rule, which gives

$$g_i \sim -\frac{(x_i - x_{0i})}{\epsilon^2 r} \left[u_c''(r/\epsilon) \mathbf{r} \cdot \mathbf{n} \right], \tag{69}$$

where $\mathbf{n}$ is the outward unit normal to Ω. Since $u_c(\rho) \sim \alpha \rho^{(1-N)/2} e^{-\rho}$ as $\rho \to \infty$, we get that

$$g_i \sim -\alpha \epsilon^{(N-5)/2} (x_i - x_{0i}) r^{-(1+N)/2} e^{-r/\epsilon} \mathbf{r} \cdot \mathbf{n} \quad \text{on} \quad \partial\Omega. \tag{70}$$

Combining (68) with (70), we get an asymptotic approximation for ϕ_i on $\partial\Omega$,

$$\phi_i \sim -C_i \alpha^2 \epsilon^{(N-3)/2} (x_i - x_{0i}) r^{-(1+N)/2} e^{-r/\epsilon} (1 + \mathbf{r} \cdot \mathbf{n}) \quad \text{on} \quad \partial\Omega. \tag{71}$$

In order to complete our asymptotic estimate of the exponentially small eigenvalues, we apply Green's identity to ϕ_i and $\partial_{x_i} u_c$ to get the following relationship:

$$\lambda_i (\partial_{x_i} u_c, \phi_i) = -\epsilon^2 \int_{\partial\Omega} \phi_i \partial_n (\partial_{x_i} u_c) dS + \left(L_\epsilon^* \left[\partial_{x_i} u_c \right], \phi_i \right). \tag{72}$$

Here L_ϵ^* is the adjoint of L_ϵ,

$$L_\epsilon^* v \equiv \epsilon^2 \Delta v - v + u_c^{p-1} v - \frac{mq\epsilon^{-N} u_c^{m-1}}{\beta_N \Omega_N (s+1)} \int_\Omega u_c^p v \, d\mathbf{x}. \tag{73}$$

We will now estimate each term in (72). Since $\partial_{x_i} u_c$ is an exact solution to the local problem, we have that

$$L_\epsilon^* \left(\partial_{x_i} u_c \right) = -\frac{mq\epsilon^{-N} u_c^{m-1}}{\beta_N \Omega_N (s+1)} \int_\Omega u_c^p \partial_{x_i} u_c \, d\mathbf{x}. \tag{74}$$

Next, since u_c is radially symmetric and localized to a small region in the interior of Ω, it is clear that $\int_\Omega u_c^p \partial_{x_i} u_c \, d\mathbf{x} \sim \int_\Omega u_c^p \partial_{x_j} u_c \, d\mathbf{x}$, as $\epsilon \to 0 \ \forall \, i, j = 1, \ldots, N$. Thus, we may write the above expression as

$$L_\epsilon^* \left(\partial_{x_i} u_c\right) \sim -\frac{mq\epsilon^{-N} u_c^{m-1}}{N\beta_N \Omega_N (s+1)} \int_\Omega \sum_{i=1}^N u_c^p \partial_{x_i} u_c \, d\mathbf{x} \, . \tag{75}$$

Then, an application of the divergence theorem results in

$$L_\epsilon^* \left(\partial_{x_i} u_c\right) \sim -\frac{mq\epsilon^{-N} u_c^{m-1}}{N\beta_N \Omega_N (s+1)} \int_{\partial\Omega} \left(\frac{u_c^{p+1}}{p+1}\right) dS. \tag{76}$$

On the boundary of Ω, $u_c \sim \alpha \epsilon^{(N-1)/2} r^{(1-N)/2} e^{-\epsilon^{-1}|\mathbf{x}-\mathbf{x}_0|}$. Therefore, the integral in (76) will be exponentially small. We then estimate the integral in (76) to get the following bound:

$$\left|L_\epsilon^* \left(\partial_{x_i} u_c\right)\right| < J\epsilon^{-N} |\partial\Omega| u_c^{m-1} \epsilon^{(N-1)(p+1)/2} \rho_0^{(1-N)(p+1)/2} e^{-\epsilon^{-1}(p+1)\rho_0} \, , \tag{77}$$

where

$$J = \frac{mq\alpha^{p+1}}{N\beta_N \Omega_N (s+1)(p+1)} \, . \tag{78}$$

Here $\rho_0 = \mathrm{dist}(\mathbf{x}_0, \partial\Omega)$. Therefore, with $\phi_i \sim C_i \partial_{x_i} u_c$, we have

$$\begin{aligned} \left|\left(L_\epsilon^* \left(\partial_{x_i} u_c\right), \phi_i\right)\right| < {} & JC_i \epsilon^{-N} |\partial\Omega| \epsilon^{(N-1)(p+1)/2} \\ & \times \rho_0^{(1-N)(p+1)/2} e^{-\epsilon^{-1}(p+1)\rho_0} \int_\Omega u_c^{m-1} \partial_{x_i} u_c \, d\mathbf{x} \, . \end{aligned} \tag{79}$$

A similar procedure shows that

$$\begin{aligned} \left|\left(L_\epsilon^* \left(\partial_{x_i} u_c\right), \phi_i\right)\right| < {} & \frac{\alpha^m |\partial\Omega|^2 JC_i \epsilon^{-N}}{mN} \\ & \times \epsilon^{(N-1)(p+m+1)/2} \rho_0^{-(N-1)(p+m+1)/2} e^{-\epsilon^{-1}\rho_0(p+m+1)} \, . \end{aligned} \tag{80}$$

Therefore, we conclude that

$$\left|\left(L_\epsilon^* \left(\partial_{x_i} u_c\right), \phi_i\right)\right| = 0 \left(\epsilon^b e^{-\epsilon^{-1}\rho_0(p+m+1)}\right) \tag{81}$$

for some b. We will show that this term is exponentially smaller than the first term on the right side of (72), and therefore, we can ignore it.

Now we estimate the left-hand side of (72). Since ϕ_i and $\partial_{x_i} u_c$ are exponentially small outside of a neighborhood of $\mathbf{x} = \mathbf{x}_0$, this inner product is dominated by the contribution from $\mathbf{x} = \mathbf{x}_0$. Using a Laplace-type approximation, we can approximate the inner product to get

(82)

$$\left(\partial_{x_i} u_c, \phi_i\right) \sim \frac{C_i}{\epsilon^2} \int_\Omega \left[u_c'(r/\epsilon)\right]^2 \left(\frac{x_i - x_{0i}}{r}\right)^2 d\mathbf{x} \sim \frac{C_i \epsilon^{N-2}}{N} \int_{\mathbb{R}^N} \left[u_c'(\rho)\right]^2 \rho^{N-1} \, d\rho \, d\theta \, ,$$

where θ represents the $N-1$ angular coordinates. Since the integrand is independent of θ,

$$(83) \qquad (\partial_{x_i} u_c, \phi_i) \sim C_i \epsilon^{N-2} \Omega_N \hat{\beta}_N / N, \quad \text{where} \quad \hat{\beta}_N \equiv \int_0^\infty \left[u_c'(\rho)\right]^2 \rho^{N-1}\, d\rho\,.$$

Here Ω_N is the surface area of the N-dimensional unit sphere. Then we determine C_i by using the normalization relation $\int_\Omega \phi_i^2\, d\mathbf{x} = 1$ to obtain

$$(84) \qquad C_i = \left(\frac{N}{\hat{\beta}_N \Omega_N}\right)^{1/2} \epsilon^{(2-N)/2}.$$

Finally, we get our asymptotic estimate of λ_i by substituting (83) and (71) into (72), and using the estimate $\partial_n(\partial_{x_i} u_c) \sim \alpha \epsilon^{(N-5)/2} r^{-(N+1)/2} e^{-r/\epsilon}$, on $\partial\Omega$. In this way, we get

$$(85) \qquad \lambda_i \sim \frac{\alpha^2 N}{\hat{\beta}_N \Omega_N} \int_{\partial\Omega} (x_i - x_{0i})^2 r^{-(1+N)} e^{-2r/\epsilon} (\mathbf{r}\cdot\mathbf{n})(1+\mathbf{r}\cdot\mathbf{n})\, dS\,,$$

where $\mathbf{r} = (\mathbf{x}-\mathbf{x}_0)/r$ and $r = |\mathbf{x}-\mathbf{x}_0|$, with $\mathbf{x} \in \partial\Omega$. As a consistency check we observe, by comparing the asymptotic orders of the two terms on the right side of (72), that the second term is asymptotically negligible compared to the first term, since the exponents satisfy $p+m+1 > 1$.

The surface integral in (85) can be evaluated asymptotically by using a multi-dimensional Laplace technique. Assume that there exists a unique point $\mathbf{x_m} \in \partial\Omega$, where $r_m = \text{dist}(\mathbf{x}_0, \partial\Omega)$ is minimized. If we parameterize the boundary near ζ_m (where $\mathbf{x}(\zeta_m) = \mathbf{x}_m$) such that each coordinate ζ_i of ζ corresponds to arclength along one of the principal directions through ζ_m, then for any smooth $F(r)$, we have (see [15])

$$(86) \qquad \int_{\partial\Omega} r^{1-N} F(r) e^{-2r/\epsilon}\, dS = \left(\frac{\pi\epsilon}{r_m}\right)^{(N-1)/2} F(r_m) K(r_m) e^{-2r_m/\epsilon}\,,$$

where

$$(87) \qquad K(r_m) \equiv (1 - r_m/R_1)^{-1/2}(1-r_m/R_2)^{-1/2}\cdots(1-r_m/R_{N-1})^{-1/2}\,.$$

Here $R_j > 0$, for $j = 1, \ldots, N-1$ are the principal radii of curvature of $\partial\Omega$ at $\mathbf{x_m}$. This result assumes that the nondegeneracy condition $R_j > r_m$, $j = 1, \ldots, N-1$, holds. In this way, we obtain the following explicit asymptotic estimate for the exponentially small eigenvalue:

$$(88) \qquad \lambda_i \sim \frac{2\alpha^2 N}{\hat{\beta}_N \Omega_N} \left(\frac{\pi\epsilon}{r_m}\right)^{(N-1)/2} \left(\frac{\mathbf{r}_m \cdot \mathbf{e}_i}{r_m}\right)^2 K(r_m) e^{-2r_m/\epsilon}\,,$$

where $\mathbf{e}_i$ is the standard unit basis vector in the ith direction and $\mathbf{r}_m \equiv (\mathbf{x}_m - \mathbf{x}_0)/r_m$.

3.3. The slow motion of the spike. Now we derive an ODE characterizing the metastable spike dynamics. We first linearize (13) about a moving spike by writing

$$(89) \qquad a(\mathbf{x}, t) = a_E(\mathbf{x}; \mathbf{x}_0(t)) + w(\mathbf{x}, t)\,,$$

where $a_E(\mathbf{x};\mathbf{x}_0)$ is defined in (52). Since (57) does not have an $O(1)$ positive eigenvalue, we may assume that $w \ll a_E$, $w_t \ll \partial_t a_E$ uniformly in time. Substituting (89) into (13), we obtain

$$L_\epsilon w = \partial_t a_E, \quad \text{in} \quad \Omega, \tag{90a}$$

$$\partial_n w = -\partial_n a_E, \quad \text{on} \quad \partial\Omega. \tag{90b}$$

Next, we expand w in terms of the eigenfunctions ϕ_i of L_ϵ as

$$w = \sum_{i=0}^{\infty} G_i(t)\phi_i. \tag{91}$$

We assume that the eigenfunctions form a complete set. However, this is not required for the construction of the solvability condition, as the key requirement is that w is orthogonal to the eigenspace of L_ϵ^* associated with the exponentially small eigenvalues. Let ϕ_i^* be the ith eigenfunction of L_ϵ^*. Then, since $(\phi_i, \phi_j^*) = \delta_{ij}$, we integrate by parts to show that

$$G_i(t) = (w, \phi_i^*) = \frac{1}{\lambda_i^*}\left[(L_\epsilon w, \phi_i^*) - \epsilon^2 \int_{\partial\Omega} w_n \phi_i^* \, dS\right], \tag{92}$$

where $L_\epsilon^*\phi_i^* = \lambda_i^*\phi_i^*$. Using (90), we have

$$G_i(t) = \frac{1}{\lambda_i^*}\left[(\partial_t a_E, \phi_i^*) + \epsilon^2 \int_{\partial\Omega} \partial_n a_E \phi_i^* \, dS\right]. \tag{93}$$

As seen in (72)–(81), the nonlocal term in the eigenvalue problem $L_\epsilon\phi = \lambda\phi$ is insignificant when $\epsilon \ll 1$ in the asymptotic estimation of the eigenspace associated with the exponentially small eigenvalues of L_ϵ. Therefore, for $i = 1, \ldots, N$, we can replace ϕ_i^* and λ_i^* by ϕ_i and λ_i in (93), where ϕ_i and λ_i are given in (68) and (88), respectively.

Since $\lambda_i \to 0$ exponentially as $\epsilon \to 0$, for $i = 1, \ldots, N$, we must impose the limiting solvability conditions that $G_i = 0$, for $i = 1, \ldots, N$. This projection step yields the following implicit differential equation for $\mathbf{x}_0(t)$:

$$(\partial_t a_E, \phi_i) = -\epsilon^2 \int_{\partial\Omega} \partial_n a_E \phi_i \, dS. \tag{94}$$

The dominant contribution to the left side of (94) arises from the region near $\mathbf{x}_0$, and we calculate

$$(\partial_t a_E, \phi_i) \sim -\frac{C_i h_E^\gamma}{N} \dot{x}_{0i} \Omega_N \hat{\beta}_N \epsilon^{N-2}. \tag{95}$$

Finally, we can evaluate the right side of (94) using our estimates for ϕ_i on $\partial\Omega$ in (71) and for $u_c(\rho)$ as $\rho \to \infty$. This yields the main result of this section.

Proposition 2 (metastability). *A metastable spike solution for* (13) *is represented by* $a(\mathbf{x},t) = a_E(\mathbf{x};\mathbf{x}_0(t))$, *where* a_E *is defined in* (52) *and* $\mathbf{x}_0(t)$ *satisfies*

$$\dot{\mathbf{x}}_0 \sim \frac{\epsilon N \alpha^2}{\hat{\beta}_N \Omega_N} \int_{\partial\Omega} \hat{\mathbf{r}}\, r^{1-N} e^{-2r/\epsilon} (1 + \hat{\mathbf{r}} \cdot \mathbf{n}) \hat{\mathbf{r}} \cdot \mathbf{n} \, d\mathbf{S}. \tag{96}$$

Here $\hat{\mathbf{r}} = (\mathbf{x} - \mathbf{x}_0) r^{-1}$, $r = |\mathbf{x} - \mathbf{x}_0|$, $\mathbf{x} \in \partial\Omega$, *and* $\mathbf{n}$ *is the unit outward normal to* $\partial\Omega$. *In addition,* α *and* $\hat{\beta}_N$ *are defined in* (53b) *and* (83), *respectively.*

There are a few corollaries that follow from this result.

Corollary 1 (equilibrium). *An equilibrium solution for* (13) *is represented by* $a(\mathbf{x},t) = a_E(\mathbf{x};\mathbf{x}_{0e})$, *where* $\mathbf{x}_{0e}$ *is a root of* $I(\mathbf{x}_0)$, *where*

$$I(\mathbf{x}_0) \equiv \int_{\partial\Omega} \hat{\mathbf{r}}\, r^{1-N} e^{-2r/\epsilon}(1+\hat{\mathbf{r}}\cdot\mathbf{n})\hat{\mathbf{r}}\cdot\mathbf{n}\, d\mathbf{S}\,. \tag{97}$$

Using formal asymptotic analysis, it was shown in [15] that for a strictly convex domain $\mathbf{x}_{0e}$ is unique and is centered at an $O(\epsilon)$ distance from the center of the uniquely determined largest inscribed sphere for Ω. A similar result has recently been proved in [19].

Assuming that there is a unique point $\mathbf{x}_m \in \partial\Omega$ closest to the initial center $\mathbf{x}_0(0)$ of the spike, we can evaluate the surface integral in (96) using Laplace's method to get the following explicit result.

Corollary 2 (explicit motion). *Let* $\mathbf{x}_m$ *be the point on* $\partial\Omega$ *closest to* $\mathbf{x}_0(0)$. *Then, for* $t > 0$, *the spike moves in the direction of* $\mathbf{x}_m$ *and the distance* $r_m(t) = |\mathbf{x}_m - \mathbf{x}_0(t)|$ *satisfies the first order nonlinear differential equation*

$$\dot{r}_m = -\xi r_m \left(\frac{\epsilon}{r_m}\right)^{(N+1)/2} K(r_m)e^{-2r_m/\epsilon}\,, \tag{98}$$

where

$$\xi \equiv \frac{2N\alpha^2}{\Omega_N\hat{\beta}_N}\pi^{(N-1)/2}\,. \tag{99}$$

Here $\hat{\beta}_N$ *is defined in* (83) *and* $K(r_m)$ *is determined in terms of the principal radii of curvature of* $\partial\Omega$ *in* (87).

This result is valid up until the time when the spike approaches to within an $O(\epsilon)$ distance of $\mathbf{x}_m$. If the initial condition for (98) is $r_m(0) = r_0$, then the time T needed for $r_m(T) = 0$ is readily found to be

$$T \sim \frac{\epsilon^{(1-N)/2} r_0^{(N-1)/2}}{2K(r_0)\xi} e^{2r_0/\epsilon}\,. \tag{100}$$

Once the spike reaches the boundary, it moves in the direction of increasing mean curvature until it reaches an equilibrium point where the mean curvature of the boundary has a local maximum (see [2]). The existence of such equilibrium solutions, where the spike is located at these special points on the boundary, is demonstrated rigorously in [5] and [9].

4. Conclusions. Although this analysis was carried out on the Gierer–Meinhardt system, the results may be generalized to a much wider class of nonlocal reaction diffusion equations that have localized spike solutions. The key feature of the analysis lies in determining if the exponentially small eigenvalue associated with the linearization around the spike solution is in fact the principal eigenvalue of this linearization. Consider a scalar reaction diffusion model with spatially homogeneous coefficients, and with no nonlocal effects, that is capable of supporting interior spike-type solutions having exponential decay. For such a problem, it has been demonstrated in [15] that there will always be one eigenvalue bounded above from zero as well as an exponentially small eigenvalue. This positive principal eigenvalue eliminates the possibility of metastability. However, for a corresponding nonlocal problem, we have suggested using a homotopy argument that the nonlocal term may push this positive eigenvalue into the left half-plane without affecting the exponentially small eigenvalue. Thus, the principal eigenvalue of the nonlocal problem can be exponentially small. In the case of the limiting form of the Gierer–Meinhardt equations analyzed above, this has been confirmed using numerical techniques that employ a homotopy between the local and nonlocal operators. A limitation of this analysis is that it usually involves extensive computations. However, the recent rigorous analysis of [18] allows us to avoid these computations for specific ranges of the parameters. More specifically, it is proved in [18] that the eigenvalues of the problem (60) and its one-dimensional counterpart (24) satisfy $\mathrm{Re}(\lambda) \leq 0$ when either $m = 2$ and $1 < p \leq 1 + 4/N$ or when $m = p + 1$ and $1 < p < p_c$. Here p_c is the critical Sobolev exponent for dimension N.

We are currently investigating phenomena associated with the full Gierer–Meinhardt model in one spatial dimension when D_h is finite. Numerical studies show that the dynamics in the case of a large, but finite, D_h are very different in character from the case of infinite D_h. As D_h is decreased from infinity, a single spike centered at $x_0 = 0$ will become stable. As D_h is decreased further the system appears to admit stable multiple spike solutions. In particular, there appears to be a sequence of critical values α_n of D_h, with $\alpha_n > \alpha_{n+1}$ for which an n-spike equilibrium solution is stable when $D_h < \alpha_n$ and is unstable when $D_h > \alpha_n$. Work on verifying this conjecture analytically is in progress.

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