CS-67720 Metric Embedding Theory and Its Algorithmic Applications

Lecture 8

Lecturers: Yair Bartal, Nova Fandina *Fall 2018/19*
For comments contact fandina@gmail.com

8.1 Partial and Scaling Distortion Embeddings

Recall the following definitions we have discussed in the first lecture.

Definition 8.1. *Given an embedding $f : X \to Y$, $\forall x \neq y \in X$, $dist_f(x,y) = \max\left\{\frac{d_y(f(x),f(y))}{d_x(x,y)}, \frac{d_x(x,y)}{d_y(f(x),f(y))}\right\}$.*

Definition 8.2 (Partial Embedding). *Let (X, d_x) and (Y, d_y) be any metric spaces, and $G \subseteq \binom{X}{2}$. We say that (f, G) is partial embedding with distortion $\alpha \geq 1$, if $\forall (x,y) \in G$ it holds that $dist_f(x,y) \leq \alpha$. Partial embedding (f, G) is called $(1-\epsilon)$-partial, if $|G| \geq (1-\epsilon)\binom{|X|}{2}$.*

Definition 8.3 (Scaling Distortion). *Let $\alpha : [0,1] \to \mathbb{R}^+$ be a non-increasing function. We say that embedding $f : X \to Y$ has an α-scaling distortion if for all $0 \leq \epsilon \leq 1$ there exists a set $G_\epsilon \subset \binom{X}{2}$ such that (f, G_ϵ) is a $(1-\epsilon)$-partial embedding with distortion $\alpha(\epsilon)$.*

Note that for $\epsilon < \binom{n}{2}^{-1}$ the above definition captures the notion of the worst case distortion of a non-contractive (or non-expansive) embedding f.

Scaling with Average Distortion. We discuss the strong relationship between scaling and ℓ_q distortions. Recall the definition of the ℓ_q-distortion: $\ell_q\text{-}dist(f) = \left(\frac{\sum_{x \neq y \in X}(dist_f(x,y))^q}{\binom{|X|}{2}}\right)^{\frac{1}{q}}$, $\forall q \geq 1$.

Claim 8.1 (Exercise)**.** *If an embedding f has an α-scaling distortion, then $\forall 1 \leq q < \infty$:*

1. $\ell_q\text{-}dist(f) \leq \binom{n}{2}^{-\frac{1}{q}}\left(\sum_{i=1}^{\binom{n}{2}-1}\alpha\left(\frac{i}{\binom{n}{2}}\right)^q + \alpha\left(\frac{1}{2\binom{n}{2}}\right)^q\right)^{\frac{1}{q}}$.
2. $\ell_q\text{-}dist(f) \leq \left(2\int_{\frac{1}{2\binom{n}{2}}}^{1}(\alpha(x))^q dx\right)^{\frac{1}{q}}$.

For another direction we make the following observation.

Claim 8.2. *Let $1 \leq q < \infty$, $\alpha \geq 1$, and f be an embedding. If $\ell_q\text{-}dist(f) \leq \alpha$, then for $\epsilon = \frac{1}{2^q}$, f is an $(1-\epsilon)$-partial embedding with distortion at most 2α.*

Proof. We have to show that there exists a set $G \subseteq \binom{X}{2}$, such that $|G| \geq (1-\epsilon)\binom{n}{2}$, and the distortion of f on that set is at most 2α. Namely, we have to show that there are at most $\epsilon\binom{n}{2}$ pairs that can be distorted by more than 2α. Assume by contradiction there is $S \subseteq \binom{X}{2}$, $|S| > \epsilon\binom{n}{2}$ and every pair from S distorted by more than 2α. Therefore, $(\ell_q\text{-}dist(f))^q > \frac{\epsilon\binom{n}{2}(2\alpha)^q}{\binom{n}{2}} = \alpha^q$, a contradiction. $\square$

The above claim means that for all $\gamma > 1$, f is $(1 - 1/\gamma^q)$-partial embedding, with distortion $\gamma\alpha$. In other words, for any $0 < \epsilon < 1$ ($\epsilon = 1/\gamma^q$), f is $(1-\epsilon)$-partial embedding with distortion $\alpha/\epsilon^{\frac{1}{q}}$. Namely, f has $\ell_q\text{-}dist(f)/\epsilon^{\frac{1}{q}}$-scaling distortion.

8.1.1 Embedding into Trees with Scaling Distortion

Now we are ready to state the main result of this section.

Theorem 8.3 ([1], [2])**.** *The following holds.*

1. *[1] Any finite metric space is embeddable into ultra-metric, with scaling distortion* $O\left(\frac{1}{\sqrt{\epsilon}}\right)$.

2. *[1] Any weighted graph contains a spanning tree, with scaling distortion* $O\left(\frac{1}{\sqrt{\epsilon}}\right)$.

3. *[2] For any* $0 < \rho < 1$*, any weighted graph* G *contains a spanning tree, with scaling distortion* $\tilde{O}\left(\sqrt{1/\epsilon}/\rho\right)$*, and weight bounded by* $(1+\rho)MST(G)$*. This result is tight with respect to* ρ*.*

For the first two items we conclude: for $1 \le q < 2$, ℓ_q-$dist(f) = O(1)$; ℓ_2-$dist(f) = O(\sqrt{\log n})$; for $q > 2$, ℓ_q-$dist(f) = O(n^{1-2/q})$.

Proof of Theorem 8.3(1). Let (X,d) be an n-point metric space. We construct the embedding by induction on n. The idea is to partition (in a smart way) $X = (X_1, X_2)$, and build the ultra-metric tree for X by combining the trees obtained by inductive steps on X_1 and X_2. Let $f_1 : X_1 \to U_1$ and $f_2 : X_2 \to U_2$ be embeddings obtained by induction. The ultra-metric tree for X is be obtained by composing U_1 and U_2, on the new root r, with label $\Delta(r) = diam(X) = \Delta$. Thus, we have to show how to decompose X. Next we discuss what properties should such decomposition satisfy.

We have to show that there exists $c > 0$, such that for any $0 < \epsilon < 1$ there is $G_\epsilon \subseteq \binom{X}{2}$ with $|G_\epsilon| \ge (1-\epsilon)\binom{|X|}{2}$, such that $\forall x, y \in G_\epsilon$, $dist_f(x,y) \le \frac{1}{c\sqrt{\epsilon}}$. In other words, we have to show that there are at most $\epsilon\binom{|X|}{2}$ pairs of points of X with distortion larger than $\frac{1}{c\sqrt{\epsilon}}$.

By the induction's assumption, there are at most $\epsilon\binom{|X_1|}{2}$ pairs of X_1, and at most $\epsilon\binom{|X_2|}{2}$ pairs of X_2 with distortion larger than $\frac{1}{c\sqrt{\epsilon}}$. Note that if $x \in X_1$ and $y \in X_2$ such that $d(x,y) \ge c\sqrt{\epsilon}\Delta$, then $dist_f(x,y) \le \frac{1}{c\sqrt{\epsilon}}$. Consider the set $B_\epsilon = \{(x,y) | x \in X_1, y \in X_2,\ d(x,y) < c\sqrt{\epsilon}\Delta\}$. Thus, we want to partition X in such a way, that for every ϵ, the number of pairs with large distortion is bounded by:

$$\epsilon\binom{|X_1|}{2} + \epsilon\binom{|X_2|}{2} + |B_\epsilon| \le \epsilon\binom{|X|}{2} \Leftrightarrow |B_\epsilon| \le \epsilon|X_1| \cdot |X_2|.$$

Thus, we show how to partition X such that above inequality holds for every ϵ.

Let $u \in X$ be a point such that $|\mathring{\mathbb{B}}(u, \frac{\Delta}{2})| \le \frac{n}{2}$. Note that there is such x, since if $x, y \in X$ such that $\Delta = d(x,y)$, then open balls of radius $\frac{\Delta}{2}$ around x and y are disjoint and at least one of them contains at most $\frac{n}{2}$ points. Let $r > 0$ be a radius (we will choose the value of r later), and let $X_1^{(r)} = \mathring{\mathbb{B}}(u, r)$ and $X_2^{(r)} = X \setminus X_1$ (note that X_1 and X_2 are dependent on r). Define the following subsets of X:

$$S_1^{(r,\epsilon)} = \{w \in X_1^{(r)} | d(w,u) > r - c\sqrt{\epsilon}\Delta\},\quad S_2^{(r,\epsilon)} = \{w \in X_2^{(r)} | d(w,u) < r + c\sqrt{\epsilon}\Delta\}.$$

Note that $B_\epsilon \subseteq S_1^{(r,\epsilon)} \times S_2^{(r,\epsilon)}$, implying $|B_\epsilon| \le |S_1^{(r,\epsilon)}| \cdot |S_2^{(r,\epsilon)}|$. Thus, we will prove that there exists r, such that for all $0 < \epsilon < 1$, $|S_1^{(r,\epsilon)}| \cdot |S_2^{(r,\epsilon)}| \le \epsilon |X_1^{(r)}| \cdot |X_2^{(r)}|$.

Let $\bar{\epsilon} = \max\{\epsilon \mid |B(u, \frac{\sqrt{\epsilon}\Delta}{4})| \ge \epsilon n\}$. Note that this set is not empty, as at least $\epsilon = \frac{1}{n}$ belongs to it. Also note that $\bar{\epsilon} \le 1/2$. Thus, for any $\epsilon > \bar{\epsilon}$, $\left|B\left(u, \frac{\sqrt{\epsilon}\Delta}{4}\right)\right| < \epsilon n$. We will choose choose $r \in \left[\frac{\sqrt{\bar{\epsilon}}\Delta}{4}, \frac{\sqrt{\bar{\epsilon}}\Delta}{2}\right]$.

Lemma 8.4. *If* $\epsilon > 32\bar{\epsilon}$*, then (every* r *is good)* $\forall\, r \in \left[\frac{\sqrt{\bar{\epsilon}}\Delta}{4}, \frac{\sqrt{\bar{\epsilon}}\Delta}{2}\right]$, $|S_1^{(r,\epsilon)}| \cdot |S_2^{(r,\epsilon)}| \le \epsilon \cdot |X_1^{(r)}| \cdot |X_2^{(r)}|$.

Proof. Fix some $r \in [\frac{\sqrt{\bar\epsilon}\Delta}{4}, \frac{\sqrt{\bar\epsilon}\Delta}{2}]$ and $\epsilon > 32\bar\epsilon$. Note that $|S_1^{(r,\epsilon)}| \le |X_1^{(r)}|$, and $|S_2^{(r,\epsilon)}| \le |B(u, r + c\sqrt{\epsilon}\Delta)|$.

Also, it holds that $r + c\sqrt{\epsilon}\Delta \le^{(r \le \frac{\sqrt{\bar\epsilon}\Delta}{2})} \le \frac{\sqrt{\bar\epsilon}\Delta}{2} + c\sqrt{\epsilon}\Delta \le^{(\bar\epsilon < \frac{\epsilon}{32})} \le \sqrt{\epsilon}\Delta\left(\frac{1}{2\sqrt{32}} + c\right) \le \frac{\sqrt{\frac{\epsilon}{2}}\Delta}{4}$, where the last inequality holds if we choose $c = \frac{1}{32\sqrt{2}}$, which will work for all inductive steps. Therefore,

$$|S_2^{(r,\epsilon)}| \le |B(u, r + c\sqrt{\epsilon}\Delta)| \le \left|B(u, \frac{\sqrt{\frac{\epsilon}{2}}\Delta}{4})\right| \le^{(\frac{\epsilon}{2} > \bar\epsilon)} \le \frac{\epsilon}{2}n.$$

Therefore, $|S_1^{(r,\epsilon)}| \cdot |S_2^{(r,\epsilon)}| \le \epsilon \cdot |X_1^{(r)}| \cdot \frac{n}{2} \le^{(|X_2^{(r)}| \ge \frac{n}{2})} \le \epsilon \cdot |X_1^{(r)}| \cdot |X_2^{(r)}|$. ∎

Lemma 8.5. *There exists $r \in [\frac{\sqrt{\bar\epsilon}\Delta}{4}, \frac{\sqrt{\bar\epsilon}\Delta}{2}] = I$, such that for all $\epsilon \le 32\bar\epsilon$, $|S_1^{(r,\epsilon)}| \cdot |S_2^{(r,\epsilon)}| \le \epsilon \cdot |X_1^{(r)}| \cdot |X_2^{(r)}|$.*

We first prove a small lemma. Let $0 \le r_1 \le r_2$ be any real numbers, and let $A(r_1, r_2)$ denote the size of the strip $B(u, r_2) \setminus B(u, r_1)$.

Lemma 8.6. *$A(\frac{\sqrt{\bar\epsilon}\Delta}{4}, \frac{\sqrt{\bar\epsilon}\Delta}{2}) \le 4\bar\epsilon n$.*

Proof. $A(\frac{\sqrt{\bar\epsilon}\Delta}{4}, \frac{\sqrt{\bar\epsilon}\Delta}{2}) \le |B(u, \frac{\sqrt{\bar\epsilon}\Delta}{2})| = |B(u, \frac{\sqrt{4\bar\epsilon}\Delta}{4})| \le^{(4\bar\epsilon > \bar\epsilon)} \le 4\bar\epsilon n$. ∎

Proof of Lemma 8.5. We say that r is a "bad" radius for some $\epsilon \le 32\bar\epsilon$ if $|S_1^{(r,\epsilon)}| \cdot |S_2^{(r,\epsilon)}| > \epsilon |X_1^{(r)}| \cdot |X_2^{(r)}|$. Denote by J the union of the intervals that constitute all bad values of r from I. We will show that $|J| < |I|$. We build J iteratively. At the beginning $J = \emptyset$. At some step of the construction, consider all the values of $r \in I \setminus J$ and all the values of $\epsilon \le 32\bar\epsilon$, such that the pair (r, ϵ) is a "bad" pair: r is bad for ϵ. From all these pairs we choose one with the maximum ϵ (we say maximum as we consider $\epsilon \in [1/n, 32\bar\epsilon]$). Denote this pair by $(\hat r, \hat\epsilon)$. We add to J segment of length $2c\sqrt{\hat\epsilon}\Delta$ with center in $\hat r$: $[\hat r - c\sqrt{\hat\epsilon}\Delta, \hat r + c\sqrt{\hat\epsilon}\Delta]$. Note that the length of the segment we add does not increase from step to step. We have to prove that $|J| < |I|$ on the termination of the algorithm.

Consider the chosen pair $(\hat r, \hat\epsilon)$. Then, $A(\hat r - c\sqrt{\hat\epsilon}\Delta, \hat r + c\sqrt{\hat\epsilon}\Delta) \ge |S_1^{(\hat r,\hat\epsilon)} \cup S_2^{(\hat r,\hat\epsilon)}| = |S_1^{(\hat r,\hat\epsilon)}| + |S_2^{(\hat r,\hat\epsilon)}|$. Note that $|X_1^{(\hat r)}| \ge^{(\hat r \in I)} \ge |B(u, \frac{\sqrt{\bar\epsilon}\Delta}{4})| \ge \bar\epsilon n$. Therefore, $|S_1^{(\hat r,\hat\epsilon)}| \cdot |S_2^{(\hat r,\hat\epsilon)}| >^{(\ (\hat r,\hat\epsilon) is\ bad\ )} > \hat\epsilon |X_1^{(\hat r)}| \cdot |X_2^{(\hat r)}| \ge \frac{\hat\epsilon\bar\epsilon n^2}{2}$. Therefore, by the inequality of arithmetic and geometric means

$$A(\hat r - c\sqrt{\hat r}\Delta, \hat r + c\sqrt{\hat r}\Delta) > 2\sqrt{\frac{\hat\epsilon\bar\epsilon}{2}}n.$$

Therefore, we conclude that

$$|J| \le \sum_{i=1}^{t} |J_i| = \sum_i 2c\sqrt{\hat\epsilon_i}\Delta = 2c\Delta \sum_i \sqrt{\hat\epsilon_i} <^{(\ have\ to\ prove\ )} < |I| = \frac{\sqrt{\bar\epsilon}\Delta}{4}.$$

Recall that $c = \frac{1}{32\sqrt{2}}$, therefore, we have to show that $\sum_i \sqrt{\hat\epsilon_i} < 4\sqrt{2}\sqrt{\bar\epsilon}$. Note that each point of I belongs to at most 2 segments of J, because the radius $\hat r$ is always chosen outside the segments of J, and the lengths of the segments do not increase from step to step. Therefore,

$$\sum_i 2\sqrt{\frac{\hat\epsilon_i\bar\epsilon}{2}}n < \sum_i A(\hat r_i - c\sqrt{\hat\epsilon_i}\Delta, \hat r_i + c\sqrt{\hat\epsilon_i}\Delta) \le^{(each\ point\ belongs\ to\ at\ most\ 2\ segments)} \le 2A(\frac{\sqrt{\bar\epsilon}\Delta}{4}, \frac{\sqrt{\bar\epsilon}\Delta}{2}) \le 2 \cdot 4 \cdot \bar\epsilon n.$$

Therefore $\sum_i \sqrt{\hat\epsilon_i} < 4\sqrt{2}\sqrt{\bar\epsilon}$, which completes the proof. Note that this process can be computed in polynomial time, by discretization of values ϵ and r. ∎

This completes the proof of the theorem. ∎

References

[1] Ittai Abraham, Yair Bartal, and Ofer Neiman. Embedding metrics into ultrametrics and graphs into spanning trees with constant average distortion. *SIAM J. Comput.*, 44(1):160–192, 2015.

[2] Yair Bartal, Arnold Filtser, and Ofer Neiman. On notions of distortion and an almost minimum spanning tree with constant average distortion. In *Proceedings of the Twenty-Seventh Annual ACM-SIAM Symposium on Discrete Algorithms, SODA 2016, Arlington, VA, USA, January 10-12, 2016*, pages 873–882, 2016.