CS-67720 Metric Embedding Theory and Its Algorithmic Applications

Lecture 2

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2.1 Embedding into Normed Spaces

We continue the study with the classic results of embedding into Euclidean normed space.

2.1.1 Embedding into ℓ_2

As we have seen in the previous lecture there are metric spaces that do not admit isometric embedding into ℓ_2. In addition, we have noted that the identity embedding of C_4 into the plane has distortion $\sqrt{2}$.

Claim 2.1. *Any embedding of C_4 into ℓ_2 has distortion at least $\sqrt{2}$.*

Recall the following basic inequalities of ℓ_2 space (the first is true in any inner product space).
Parallelogram Law: $\forall x, y \in \ell_2$ it holds that: $\|x+y\|_2^2 + \|x-y\|_2^2 = 2\left(\|x\|_2^2 + \|y\|_2^2\right)$.
Quadrilateral Inequality: $\forall x, y, z, t \in \ell_2$ it holds that:

$$\|x-z\|_2^2 + \|y-t\|_2^2 \leq \|x-y\|_2^2 + \|y-z\|_2^2 + \|z-t\|_2^2 + \|t-x\|_2^2 .$$

To prove the inequality, it is enough to prove it for the points on the line.

Proof of Claim 2.1. Let $f\colon C_4 \to \ell_2$ be any embedding with distortion α. Assume w.l.o.g. that f is non-contractive (since distortion is invariant under scaling). Let $\{A, B, C, D\}$ denote the points of C_4, and let $f(A) = A', f(B) = B', f(C) = C', f(D) = D'$. By the quadrilateral inequality and by the definition of distortion we obtain:

$$\|A'-C'\|_2^2 + \|B'-D'\|_2^2 \leq \|A'-B'\|_2^2 + \|B'-C'\|_2^2 + \|C'-D'\|_2^2 + \|D'-A'\|_2^2 \leq$$

$$\leq \alpha^2 \left(d(A,B)^2 + d(B,C)^2 + d(C,D)^2 + d(D,A)^2\right) = 4\alpha^2.$$

Also, ${\|A'-C'\|_2}^2 + {\|B'-D'\|_2}^2 \geq d(A,C)^2 + d(B,D)^2 = 8$. Thus, $4\alpha^2 \geq 8 \Rightarrow \alpha \geq \sqrt{2}$. $\square$

As we mentioned in the previous lecture, Bourgain has shown that any metric space embeds into ℓ_2 with distortion $O(\log n)$. Moreover, in [6], the bound on distortion was shown to be tight:

Theorem 2.2 ([6])**.** *Every embedding of an n-vertex constant-degree expander into an ℓ_2 space, of any dimension, has distortion $\Omega(\log n)$.*

We will show a slightly weaker lower bound. Particularly, we will show that there exists an n-point metric space, any embedding of which into ℓ_2 requires distortion of $\Omega(\sqrt{\log n})$.

Definition 2.1. *For an integer $d \geq 1$, the set $\{0,1\}^d$ is called a hypercube of dimension d. Each string in the set is called a vertex of the hypercube. The edges of a hypercube are the pairs of vertices that differ at exactly one coordinate.*

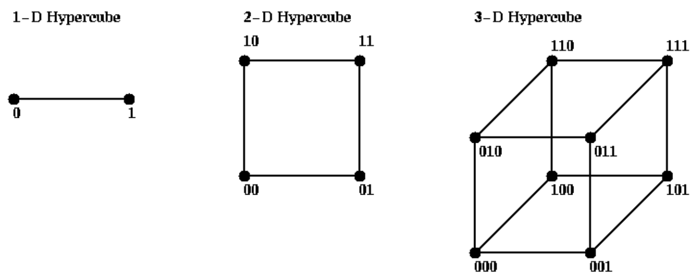

Figure 2.1: Hypercubes.

We can think of a hypercubes as unweighted graphs. Thus, we can consider metric spaces defined by this graphs. Let H_d denote the metric space defined over the vertices of a d-dimensional hypercube, imposed by its graph structure. Note that this metric is exactly the ℓ_1-distance, moreover, when the points are binary strings, the ℓ_1-distance is exactly the Hamming distance.

We have shown already that any embedding of H_2 ($= C_4$) into ℓ_2 requires distortion of at least $\sqrt{2}$. We generalize this result.

Theorem 2.3. *Any embedding of H_d (for $d \geq 2$) into ℓ_2, incurs distortion of at least $\sqrt{d}$.*

Denoting $n = 2^d$, we get distortion at least $\sqrt{d} = \sqrt{\log n}$. Note that the distortion of the identity embedding of H_d into ℓ_2 is $\sqrt{d}$.

Proof. (by Enflo [2]) Recall that for any $u \neq v \in H^d$, (u,v) is an edge iff $\|u-v\|_1 = 1$. Let E denote the set of all edges of H_d. Note that $|E| = d2^{d-1}$, since for each vertex u there are exactly d edges that contain u, and thus all the vertices together cover all the edges twice. Let $F = \{(u,\bar{u}) | u \in H_d\}$ denote the set of all the longest diagonals of H_d, where $\bar{u}$ denotes the negation of u. Then $|F| = 2^{d-1}$.

We prove the following generalization of the quadrilateral inequality.

[**Hyper-Quadrilateral Inequality**]: For any $f : \{0,1\}^d \to \ell_2$ it holds that

$$\sum_{(u,\bar{u})\in F} \|f(u) - f(\bar{u})\|_2^2 \leq \sum_{(u,v)\in E} \|f(u) - f(v)\|_2^2 .$$

Assume the inequality is correct, and let $f \colon H_d \to \ell_2$ be a (w.l.o.g.) non-contractive embedding with distortion α. Then, we have

$$2^{d-1} \cdot d^2 \leq \sum_{(u,\bar{u})\in F} \|f(u) - f(\bar{u})\|_2^2 \leq \sum_{(u,v)\in E} \|f(u) - f(v)\|_2^2 \leq \alpha^2 \cdot d \cdot 2^{d-1},$$

implying $\alpha \geq \sqrt{d}$ as required. Thus it remains to prove the inequality.

The proof is by induction on d. The claim is true for $d = 2$ (the quadrilateral inequality). Assume the claim is true for any $d' < d$. Denote by X the set of 2^d vertices of the cube $\{0,1\}^d$. Let X_0 and X_1 be the sets of all points of X with last coordinate being 0 and 1, respectively. Then $|X_0| = |X_1| = 2^{d-1}$. Let E_0 and E_1 be the sets of edges of X_0 and X_1, respectively, dnd let F_0 and F_1 be the sets of longest diagonals of X_0 and X_1, respectively. Note that E_0, E_1 are also edges of X, but F_0, F_1 are not the longest diagonals of X. By the induction hypothesis X_0 and on X_1, we obtain that:

$$\sum_{(u0,\bar{u}0)\in F_0} \|f(u0) - f(\bar{u}0)\|_2^2 \leq \sum_{(u0,v0)\in E_0} \|f(u0) - f(v0)\|_2^2,$$

and

$$\sum_{(u1,\bar{u}1)\in F_1} \|f(u1) - f(\bar{u}1)\|_2^2 \leq \sum_{(u1,v1)\in E_1} \|f(u1) - f(v1)\|_2^2,$$

where for a string $u \in \{0,1\}^{d-1}$, and a bit $b \in \{0,1\}$, we denote by ub the concatenation of u with b (i.e. $ub \in \{0,1\}^d$).

For each vertex $u0 \in X_0$, consider the vertices $\bar{u}1 \in X_1$, $\bar{u}0 \in X_0$, $u1 \in X_1$. Note that $(u0, \bar{u}1)$ and $(\bar{u}0, u1)$ form two longest diagonals of X. Moreover, by the quadrilateral inequality we have

$$\|f(u0) - f(\bar{u}1)\|_2^2 + \|f(\bar{u}0) - f(u1)\|_2^2 \leq$$
$$\leq \|f(u0) - f(\bar{u}0)\|_2^2 + \|f(u1) - f(\bar{u}1)\|_2^2 + \|f(u0) - f(u1)\|_2^2 + \|f(\bar{u}0) - f(\bar{u}1)\|_2^2.$$

The first two arguments are the longest diagonals of X_0 and X_1 respectively, and the last two arguments are the edges of X, connecting the subcubes X_0 and X_1. Therefore, if we apply this argument to all the points of X_0 (essentially, to the half of the points, since we don't want duplications) we will "cover" all the longest diagonals of X, by means of all the longest diagonals of X_0 and X_1, and all the edges of X going between X_0 and X_1:

$$\sum_{(v,\bar{v})\in F} \|f(v) - f(\bar{v})\|_2^2 \leq \sum_{(u0,\bar{u}0)\in F_0} \|f(u0) - f(\bar{u}0)\|_2^2 + \sum_{(u1,\bar{u}1)\in F_1} \|f(u1) - f(\bar{u}1)\|_2^2 + \sum_{(w,s)\in E_{0,1}} \|f(w) - f(s)\|_2^2,$$

where $E_{0,1} = \{(w,s) \mid w \in X_0, s \in X_1, \|w - s\|_1 = 1\}$. Applying the induction's hypothesis ,we conclude the lemma.

□

Open Problem 2.1. *What is the lowest distortion $\alpha(n)$ such that all n-point metric spaces in ℓ_1 are $\alpha(n)$ embeddable into ℓ_2? The conjecture is that distortion is $O(\sqrt{\log n})$. In [1] the authors proved that $\alpha(n) = O(\sqrt{\log n}\log(\log n))$.*

2.2 Embedding into ℓ_∞ With Low Distortion

In the last lecture we have seen the Frechet's embedding [4]: Every n-point metric isometrically embeds into ℓ_∞^{n-1}. In fact, Frechet proved a stronger statement: Every *separable* metric space isometrically embeds into ℓ_∞ (of infinite dimension). The construction of the embedding is similar to the finite case we have seen in class, where the "location" points are the points of the countable dense set of the metric space. Thus, we can use a finite ϵ-net as the location points, loosing in precision for the sake of dimension.

Theorem 2.4. *(Farago [3]) Let $(X, \|\cdot\|)$ be a normed space of dimension d. Let $S \subset X$ be a bounded subset. Then, for any $0 < \epsilon < \frac{1}{2}$, there is an embedding $f : S \to \ell_\infty^{O(\frac{1}{\epsilon^d})}$, with distortion $1 + \epsilon$.*

Before the proof of the theorem, we shall define several concepts and prove some lemmas.

Definition 2.2. *Let (X, d) be a metric space and $N \subseteq X$.*
N is ϵ-dense in X, if $\forall x \in X$ there exists $y \in N$ such that $d(x, y) \leq \epsilon$.
N is ϵ-separated, if $\forall x, y \in N$ it holds that $d(x, y) > \epsilon$.
N is ϵ-net of X, if N is ϵ-dense and is ϵ-separated.

The idea is to show that there exists an ϵ-net of S of size $k = O\left(\frac{1}{\epsilon}\right)^d$, and that the Frechet embedding to the points of the net has distortion $1 + \epsilon$.

Namely, we will show that there is $N = \{z_1, z_2, \ldots, z_k\} \subseteq S$, such that N is ϵ-net of S, of size $k = O\left(\frac{1}{\epsilon}\right)^d$, and that for the embedding f defined by $\forall\, x \in S, \quad f(x) = (\|x - z_1\|, \|x - z_2\|, \ldots \|x - z_k\|)$, it holds that

$$\forall\, x \neq y \in S, \quad \frac{1}{1+\epsilon} \|x - y\| \leq \|f(x) - f(y)\|_\infty \leq \|x - y\|.$$

Lemma 2.5 (A bounded, separated set in a normed space is not too large)**.** *Let $R > 0$, $0 < \epsilon \leq R/2$. Let $(X, \|\|)$ be a d dimensional normed space, $B_R = \{x \in X \mid \|x\| \leq R\}$. If $A \subset B_R$ is ϵ- separated, then $|A| = O\left(\frac{R}{\epsilon}\right)^d$.*

Proof. We use the standard volume argument to bound the size of A. Denote $k = |A|$. The closed balls of radius $\frac{\epsilon}{2}$ around the points in A are disjoint, and their union is contained in a closed ball of radius $(R + \frac{\epsilon}{2})$ around 0.

There are various definitions for the volume of the set in some normed space. We can use a common one – the Jordan measure, which can be defined for all normed spaces over $\mathbb{R}$. Recall that not any set is Jordan measurable, but it is known that a bounded set is Jordan measurable if and only if its boundary has Jordan measure zero. In addition, it can be shown that the boundary of a convex set has Jordan measure zero. Therefore, since the closed ball of radius r of any normed space in $\mathbb{R}$ is a convex set, it is Jordan measurable.

We can assume w.l.o.g. that $V = \mathbb{R}^d$. In addition, denote by $B_{\frac{\epsilon}{2}}$ and by $B_{(R+\frac{\epsilon}{2})}$ the closed balls of the appropriate radii under the $\|\|_X$. Note that $B_{(R+\frac{\epsilon}{2})} = \left(\frac{R+\frac{\epsilon}{2}}{\frac{\epsilon}{2}}\right) \cdot B_{\frac{\epsilon}{2}}$. Therefore,

$$Vol\left(B_{(R+\frac{\epsilon}{2})}\right) = \left(\frac{R + \frac{\epsilon}{2}}{\frac{\epsilon}{2}}\right)^d Vol\left(B_{\frac{\epsilon}{2}}\right).$$

Therefore,

$$k \leq \left(\frac{R + \epsilon/2}{\epsilon/2}\right)^d = O\left(\frac{R}{\epsilon}\right)^d.$$

$\square$

Claim 2.6. *Let $R > 0$, $0 < \epsilon \leq R/2$. In every normed space X of dimension d, there is an ϵ-net $N \subset B_R$, of size $O\left(\frac{R}{\epsilon}\right)^d$.*

Proof. We build N greedily. At the beginning $N = \emptyset$. Choose an arbitrary point in B_R and add it to N. Let B be the union of closed balls of radius ϵ around the points in N. As long as there is a point in $B_R \setminus B$, add it to N. When the algorithm stops, the resulting set N is clearly ϵ-dense and ϵ-separated. Therefore, N is an ϵ-net and by lemma 2.5, $|N| = O\left(\frac{R}{\epsilon}\right)^d$. $\square$

In fact, one can prove that for any $\epsilon > 0$, for any metric space X there exists an ϵ-net of X.

It is not clear how to find a net efficiently in a general normed space. However, if the norm is ℓ_p-norm then we can take the grid (that covers the set S) with side length of $\epsilon/d^{\frac{1}{p}}$, and sieve the points that are too close to the points that have been taken to the net. This can be done in running time of $O(kk'd)$ where k is the number of points in the grid and k' is the number of points that survived in the net.

Note that we can assume w.l.o.g. that $S \subset B_1$, so it is enough to prove the following theorem.

Theorem 2.7. *Let $0 < \epsilon \leq 1/2$, and let X be a normed space of dimension d. Then B_1 can be embedded into l_∞^k, with distortion $1 + \epsilon$ and $k = O\left(\frac{1}{\epsilon}\right)^d$.*

Proof. By Claim 2.6 we can find an $(\epsilon/8)$- net N of B_2 of size $k = O\left(\frac{1}{\epsilon}\right)^d$. Let $z_1 \ldots z_k$ be the points of N. We define the embedding $f : B_1 \to l_\infty^k$ by setting

$$f(x) = (\|x - z_1\|, \ldots, \|x - z_k\|), \quad \forall x \in B_1.$$

We show that for all $x \neq y \in B_1$, it holds that $\frac{1}{1+\epsilon} \leq \frac{\|f(x)-f(y)\|_\infty}{\|x-y\|} \leq 1$.

From the triangle inequality, for all $1 \leq i \leq k$, $|\,\|x - z_i\| - \|y - z_i\|\,| \leq \|x - y\|$, so the right hand side of the inequality is correct. To prove the left hand side of the inequality we divide into two cases: $\|x - y\| \geq 1/2$ and $\|x - y\| < 1/2$. (Note that if $x = z_i$ or $y = z_j$ then trivially $\frac{\|f(x)-f(y)\|_\infty}{\|x-y\|} \geq 1$.)

We show that there exists $1 \leq i \leq k$ such that $\frac{|\|y-z_i\|-\|x-z_i\||}{\|y-x\|} \geq \frac{1}{1+\epsilon}$. (Namely, not all the coordinates are very close to each other. From this follows that the embedding is one to one.)

Case A: $\|x - y\| \geq 1/2$**.**

Let z_i be the closest point of N to x. (Note that $\|y - z_i\| \neq \|x - z_i\|$, as otherwise, by the triangle inequality $1/2 \leq \|x - y\| \leq \|x - z_i\| + \|y - z_i\| \leq \epsilon/4 \leq 1/8$). Therefore,

$$\frac{|\,\|y - z_i\| - \|x - z_i\|\,|}{\|y - x\|} \geq \frac{\|y - x\| - 2\,\|x - z_i\|}{\|y - x\|} = 1 - 2\frac{\|x - z_i\|}{\|y - x\|} \geq 1 - 2\frac{\epsilon/8}{1/2} = 1 - \epsilon/2 \geq \frac{1}{1 + \epsilon}.$$

Case B: $\|x - y\| < 1/2$.

We want to choose a point x', such that x' is on the line passing through x, y, and $\|x' - y\| = 1$. Denote $\lambda = \|x - y\|$. Let $x' = 1/\lambda(x - (1 - \lambda)y)$. Then:

- $x = \lambda x' + (1 - \lambda)y$,

- $\|x' - y\| = 1$,
- $x' \in B_2$, since by the triangle inequality, $\|x'\| \leq \|x' - y\| + \|y\| \leq 2$.

Therefore, by the case A, if z_i is the closest point of N to x', then (recall $\|y - x'\| = 1$)

$$| \, \|y - z_i\| - \|x' - z_i\| \, | \geq 1/(1+\epsilon).$$

By the triangle inequality (or convexity of the norm) we have

$$\|x - z_i\| \leq \lambda \|x' - z_i\| + (1-\lambda) \|y - z_i\| .$$

Therefore,

$$\left| \frac{\|y - z_i\| - \|x - z_i\|}{\lambda} \right| \geq \left| \frac{\|y - z_i\| - (\lambda \|x' - z_i\| + (1-\lambda) \|y - z_i\|)}{\lambda} \right| = \Big| \, \|y - z_i\| - \|x' - z_i\| \, \Big| \geq 1/(1+\epsilon),$$

which completes the proof.

□

Remark 2.8. *Essentially, making a more complicated arguments (from Banach space theory) can prove a stronger result: every d-dimensional normed space X can be embedded into $\ell_\infty^{O(1/\epsilon^d)}$, with distortion $1+\epsilon$. See [5] for an outline of such proof.*

References

[1] Sanjeev Arora, James R. Lee, and Assaf Naor. Euclidean distortion and the sparsest cut. In *In Proceedings of the 37th Annual ACM Symposium on Theory of Computing*, pages 553–562. ACM Press, 2005.

[2] P. Enflo. On the nonexistence of uniform homeomorphisms between l_p-spaces. *Ark. Mat.*, 8(4):103 – 105, 1969.

[3] András Faragó. Low distortion metric embedding into constant dimension. In *Theory and Applications of Models of Computation - 8th Annual Conference, TAMC 2011, Tokyo, Japan, May 23-25, 2011. Proceedings*, pages 114–123, 2011.

[4] M. Frchet. Les dimensions d'un ensemble abstrait. *Mathematische Annalen*, 68:145–168, 1910.

[5] P. Indyk. Algorithmic applications of low-distortion geometric embeddings. In *Proceedings of the 42Nd IEEE Symposium on Foundations of Computer Science*, FOCS '01, Washington, DC, USA, 2001. IEEE Computer Society.

[6] Nathan Linial, Eran London, and Yuri Rabinovich. The geometry of graphs and some of its algorithmic applications. *Combinatorica*, 15(2):215–245, 1995.