Proving Hopf's Umlaufsatz using Algebraic Topology

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1 Introduction

In one of the earlier lectures, we examined the proof of Hopf's Umlaufsatz. In doing so, we used—without proof—the fact that homotopic maps to $\mathbb{S}^1$ have the same degree; that is, the degree of a map is invariant under homotopy. The goal of this paper is to review and present a proof of this important result.

The proof relies on several fundamental concepts in algebraic topology: covering spaces, homotopy, and the homotopy lifting property. We begin by introducing covering spaces, described via covering maps $\pi : \widetilde{B} \to B$ that are continuous and surjective, with the additional local homeomorphism condition that distinguishes coverings. We then briefly review homotopy, which captures the idea of continuously deforming one map into another. Finally, we introduce the homotopy lifting property, which allows us to "lift" homotopies to covering spaces in a controlled way—an essential ingredient in our proof.

2 Covering Spaces

Definition 1 (Covering Space)**.** *Let $\widetilde{B}$ and B be subsets of $\mathbb{R}^3$. We say that $\pi : \widetilde{B} \to B$ is a covering map if*

1. *π is continuous and $\pi(\widetilde{B}) = B$.*
2. *Each point $p \in B$ has a neighborhood U in B (to be called a* ***distinguished neighborhood*** *of p) such that*
$$\pi^{-1}(U) = \bigcup_{\alpha} V_\alpha,$$
where the V_α's are pairwise disjoint open sets such that the restriction of π to V_α is a homeomorphism of V_α onto U. $\widetilde{B}$ is then called a ***covering space*** *of B.*

Example 1. *Let $P \subset \mathbb{R}^3$ be a plane. By fixing a point $q_0 \in P$ and two orthogonal unit vectors $\vec{e_1}, \vec{e_2}$, with origin at q_0, every $q \in P$ is characterized by coordinates $(u, v) = q$ given by*

$$q - q_0 = ue_1 + ve_2.$$

Essentially, we shifted q so that it sits at q_0, which is our origin. Now, let $S = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = 1\}$ be the right circular cylinder whose axis is the z-axis and define the map $\pi : P \to S$ as the following:
$$(u, v) \mapsto (\cos(u), \sin(u), v).$$
For some geometric intuition, the map wraps the plane P around the cylinder S (infinitely many times). Next, we must prove π is covering map by showing it satisfies the two conditions.

Condition 1***:*** *Let $(u_0, v_0) \in P$. Then, π restricted to the band $R = \{(u, v) \in P : u_0 - \pi \leq u \leq u_0 + \pi$ covers S entirely. Since trigonometric functions are continuous, π must be continuous. Furthermore, $\pi(P) = S$. Therefore, π satisfies the first condition.*

Condition 2***:*** *Let $p \in S$ and define $U = S - r$, where r is the generator opposite to the generator passing through p. Now, we must show that U is a distinguished neighborhood of p. Well, let $(u_0, v_0) \in P$ such that $\pi(u_0, v_0) = p$ and choose V_n as the band given by*

$$V_n = \{(u, v) \in P : u_0 + (2n - 1)\pi < u < u_0 + (2n + 1)\pi\} \quad \textit{for } n = 0, \pm 1, \pm 2, \ldots$$

By construction, we know for $i \neq j$, $V_i \cap V_j = \emptyset$ since the intervals are different. Furthermore, $\bigcup_n V_n = \pi^{-1}(U)$ as we can think of unrolling sections of the cylinder. Lastly, we must show that π

is a homeomorphism when restricted to V_n for $n \in \mathbb{N} \cup \{0\}$. Well, it's clear that $\pi|_{V_n}$ is bijective, continuous and inverse is continuous. Finally, we can conclude that π is a covering.

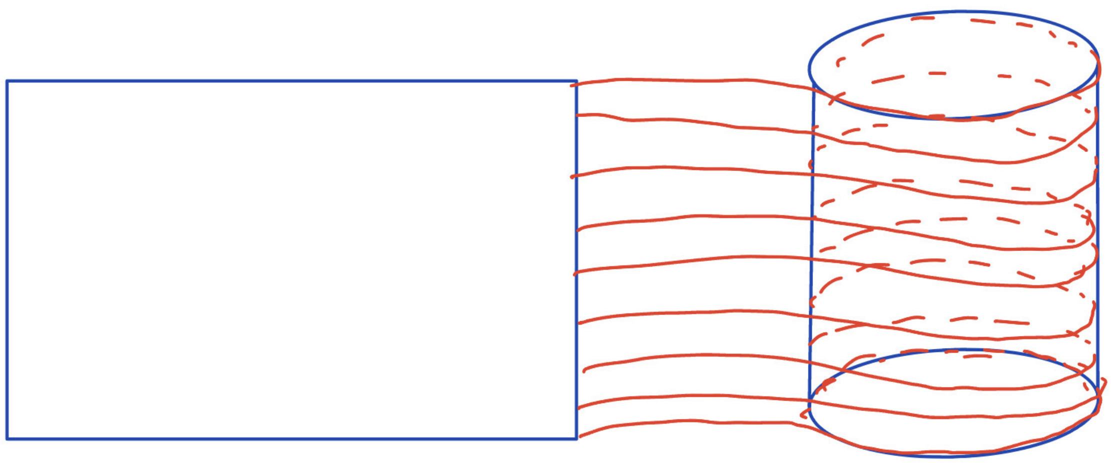

Figure 1: Covering of Cylinder using Plane

Example 2. *For our second example, we shall look at the helix*

$$H = \{(x, y, z) \in \mathbb{R}^3 : x = \cos(t), y = \sin(t), z = bt, t \in \mathbb{R}\}$$

and let

$$\mathbb{S}^1 = \{(x, y, 0) \in \mathbb{R}^3 : x^2 + y^2 = 1\}$$

be the unit circle. Now, we'll define the covering $\pi : H \to \mathbb{S}^1$ as the following:

$$(x, y, z) \mapsto (x, y, 0).$$

***Condition** 1: It is immediate that π is continuous and $\pi(H) = \mathbb{S}^1$.*

***Condition** 2: Let $p \in \mathbb{S}^1$. We shall prove $U = \mathbb{S}^1 - \{q\}$, where $q \in \mathbb{S}^1$ is the point symmetric to p. Let $t_0 \in \mathbb{R}$ such that*

$$\pi(\cos(t_0), \sin(t_0), bt_0) = p.$$

For V_n, we'll take a slice of the arc of the helix corresponding to the interval

$$(t_0 + (2n-1)\pi, t_0 + (2n+1)\pi) \subset \mathbb{R} \quad \text{for } n = 0, \pm 1, \pm 2, \ldots$$

Again, the V_n are pairwise disjoint because the V_n's correspond to different arcs of slices of the arc of the helix. Furthermore, $\pi^{-1}(U) = \bigcup_n V_n$ can be proven using double containment. For the forward direction, let $x \in \pi^{-1}(U)$. Then, x lies in the slice of some arc in the helix. In particular, it lies in some V_n. Thus, $x \in \bigcup_n V_n$. For the other direction, we use similar logic. Lastly, π restricted to V_n yields a homeomorphism onto U.

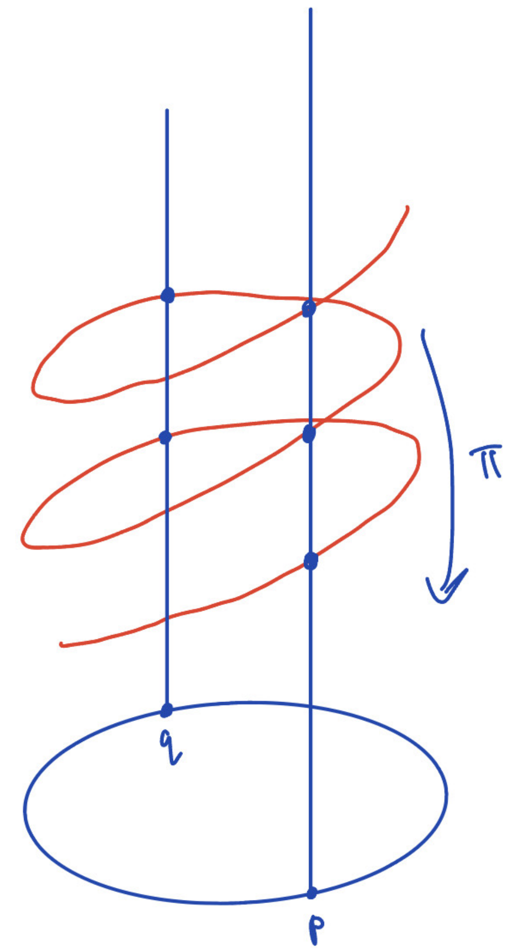

Figure 2: Covering of Unit Circle using Helix

3 Arc Lifting Property

Next, we'll need introduce a proposition that will be useful when proving homotopic liftings:

Proposition 1. *Let $\pi : \widetilde{B} \to B$ be a covering map, $\alpha : [0,l] \to B$ an arc in B, and $\widetilde{p}_0 \in \widetilde{B}$ a point of $\widetilde{B}$ such that $\pi(\widetilde{p}_0) = \alpha(0) = p_0$. Then there exists a unique lifting $\widetilde{\alpha} : [0,l] \to \widetilde{B}$ of α with origin at $\widetilde{p}_0$, that is, with $\widetilde{\alpha}(0) = \widetilde{p}_0$.*

Proof. We break the proof into a proof of uniqueness and existence.

Uniqueness: Let $\widetilde{\alpha}, \widetilde{\beta} : [0,l] \to \widetilde{B}$ be two liftings of α with origin at $\widetilde{p}_0$. Let $A \subset [0,l]$ be the set of points $t \in [0,l]$ such that $\widetilde{\alpha}(t) = \widetilde{\beta}(t)$. A is non-empty and closed in $[0,l]$. Now, we want to show that A is open in $[0,l]$ to conclude $A=[0,l]$. Consider $t_0 \in [0,l]$ such that $\widetilde{\alpha}(t_0)=\widetilde{\beta}(t_0)=\widetilde{p}$. Consider the neighborhood V of $\widetilde{p}$ in which π is a homeomorphism. Since $\widetilde{\alpha},\widetilde{\beta}$ are continuous, there exists $I_{t_0} \subset [0,l]$ containing V such that $\widetilde{\alpha}(I_{t_0}) \subset V$ and $\widetilde{\beta}(I_{t_0}) \subset V$. Since $\pi \circ \widetilde{\alpha} = \pi \circ \widetilde{\beta}$ that implies $\widetilde{\alpha}=\widetilde{\beta}$ in I_{t_0}. Therefore, A is open, $A=[0,l]$ and the two liftings have the same values for all $t \in [0,l] \implies \widetilde{\alpha}=\widetilde{\beta}$.

Existence: Since α is continuous, $\forall \alpha(t) \in B$ there exists $I_t \subset [0,l]$ containing t such that $\alpha(I_t)$ is contained in a distinguished neighborhood of $\alpha(t)$. The family I_t, $t \in [0,l]$, is an open covering of $[0,l]$; however, by compactness, it admits a finite subcovering namely $I_0, I_1, \ldots, I_n$. Now, we will begin with the construction. Assume $0 \in I_0$ (otherwise, enumerate again). Since $\alpha(I_0)$ is contained in a distinguished U_0 of p, there exists V_0 of $\widetilde{p}$ such that the restriction π_0 of π to V_0 is a homeomorphism on U_0. For $t \in I_0$, we define

$$\widetilde{\alpha}(t) = \pi_0^{-1} \circ \alpha(t).$$

By construction, $\widetilde{\alpha}(0) = \pi_0^{-1}(\alpha(0)) = \widetilde{p}_0$ and $\pi \circ \widetilde{\alpha}(t) = \pi(\pi_0^{-1}(\alpha(t))) = \alpha(t)$. Suppose now that $I_1 \cap I_0 \neq \emptyset$ (otherwise, reorder intervals). Let $t_1 \in I_1 \cap I_0$. Again, since $\alpha(I_1)$ is contained in a distinguished neighborhood U_1 of $\alpha(t_1)$. We may define a lifting of α in I_1 with origin at $\widetilde{\alpha}(t_1)$. By uniqueness, this arc agrees with $\widetilde{\alpha}$ in $I_1 \cap I_0$ and is thus an extension of α on $I_1 \cap I_0$. Continuing until n, we get an arc $\widetilde{\alpha} : [0,l] \to \widetilde{B}$ such that $\widetilde{\alpha}(0) = \widetilde{p}_0$ and $\pi \circ \widetilde{\alpha}(t) = \alpha(t) \quad \forall t \in [0,l]$ ☐

4 Homotopy

Definition 2 (Homotopy)**.** *Let $B \subset \mathbb{R}^3$ and $\alpha_0 : [0,l] \to B$, $\alpha_1 : [0,l] \to B$ be two arcs of B, joining the points*

$$p = \alpha_0(0) = \alpha_1(0) \quad \textit{and} \quad q = \alpha_0(l) = \alpha_1(l).$$

*We say that α_0 and α_1 are **homotopoic** if there exists a continuous map $H : [0,l] \times [0,1] \to B$ such that*

1. *$H(s,0) = \alpha_0(s), H(s,1) = \alpha_1(s), \quad s \in [0,l]$.*
2. *$H(0,t) = p, H(l,t) = q, \quad t \in [0,1]$.*

*The map H is called a **homotopy** between α_0 and α_1.*

For every $t \in [0,1]$, the arc $\alpha_t : [0,l] \to B$ given by $\alpha_t(s) = H(s,t)$ is called an **arc of the homotopy** H.

Example 3. *From lecture, we saw the example when $B = \mathbb{R}^2$ and α_0, α_1 are any simple curve with the same endpoints p and q. Then,*

$$H(s,t) = \alpha_0(1-t) + \alpha_1(s)t.$$

Evidently, all of the conditions for a homotopy are satisfied:

- *H is continuous,*
- *$H(0,t) = p$,*
- *$H(1,t) = q$,*
- *$H(s,0) = \alpha_0(s)$,*
- *$H(s,1) = \alpha_1(s)$.*

Therefore, H is a homotopy.

5 Homotopy Lifting Property

Proposition 2. *Let* $\pi : \widetilde{B} \to B$ *be a local homeomorphism with the property of lifting arcs. Let* α_0,$\alpha_1 : [0, l] \to B$ *be two arcs of* B *joining the points of* p *and* q*, let*

$$H : [0, l] \times [0, 1] \to B$$

be a homotopy between α_0 *and* α_1*, and let* $\widetilde{p} \in \widetilde{B}$ *be a point of* $\widetilde{B}$ *such that* $\pi(\widetilde{p}) = p$*. Then there exists a lifting* $\widetilde{H}$ *of* H *with origin at* $\widetilde{p}$*.*

Proof. Similar to the proof of the arc lifting property, we'll break this proof into the proof of uniqueness and existence.

Uniqueness: Let $\widetilde{H}_1, \widetilde{H}_2$ be two liftings of H with $\widetilde{H}_1(0, 0) = \widetilde{H}_2(0, 0) = \widetilde{p}$. Let A be the set of points $(s, t) \in [0, l] \times [0, 1] = Q$ such that $\widetilde{H}_1(s, t) = \widetilde{H}_2(s, t)$ is nonempty and closed in Q. Since $\widetilde{H}_1, \widetilde{H}_2$ are continuous and π is a local homeomorphism, we get A is open in Q. Lastly, by the connectedness of Q, we have $A = Q$. Since $\widetilde{H}_1$ and $\widetilde{H}_2$ agree on all points, we have $\widetilde{H}_1 = \widetilde{H}_2$.
Existence: Let $\alpha_t(s) = H(s, t)$ be the arc of the homotopy H. Define $\widetilde{H}$ as the following:

$$\widetilde{H}(s, t) = \widetilde{\alpha}_t(s), \quad s \in [0, l], t \in [0, 1],$$

where $\widetilde{\alpha}_t$ is the lifting of α_t with origin at $\widetilde{p}$. All the necessary properties of a homotopy are satisfied. However, we still need to prove $\widetilde{H}$ is continuous. We leave this as a reading exercise (pp. 380-381 of Ddo Carmo, M. *Differential Geometry of Curves and Surfaces*). $\square$

Example 4. *Let's revisit example* 1*. We'll take a look at the homotopy lifting of homotopic curves in the cylinder. Below is a figure of the homotopy lifting:*

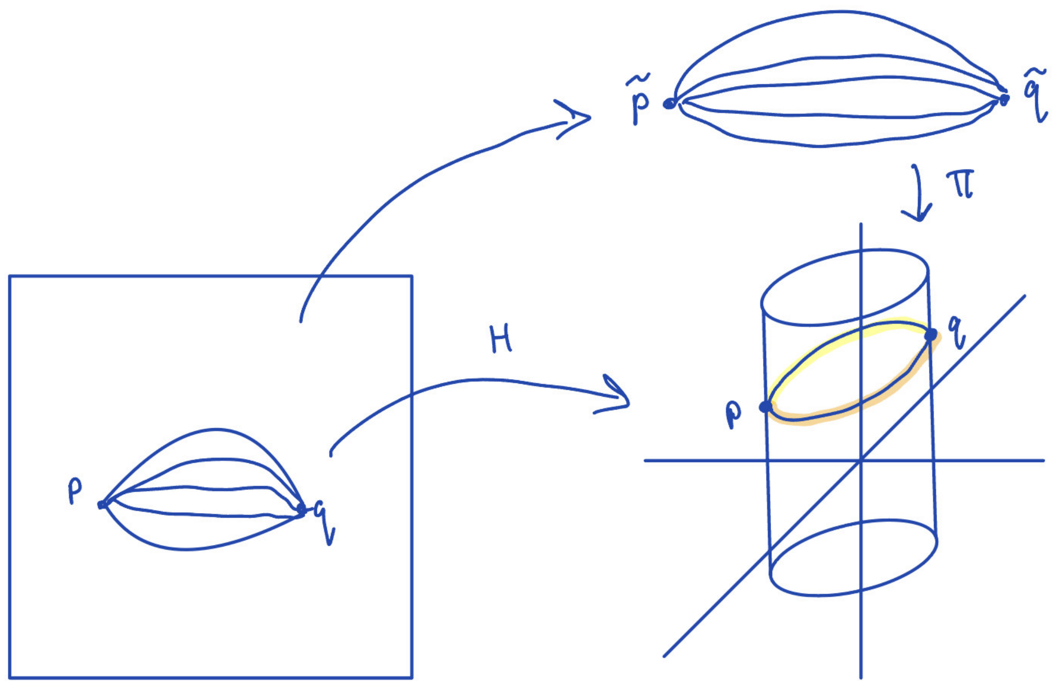

Figure 3: Homotopy Lifting of Plane to Cylinder Example

Finally, we use this powerful proposition to generalize the homotopy lifting property, which will be useful for proving degree is invariant of homotopy. We state the necessary proposition:

Proposition 3. *Let $\pi : \widetilde{B} \to B$ be a local homeomorphism with the property of lifting arcs. Let $\alpha_0, \alpha_1 : [0, l] \to B$ be two arcs of B joining the points p and q and $\widetilde{p} \in \widetilde{B}$ such that $\pi(\widetilde{p}) = p$. If α_0 and α_1 are homotopic, then the liftings $\widetilde{\alpha}_0$ and $\widetilde{\alpha}_1$ of α_0 and α_1, respectively, with origin $\widetilde{p}$, are homotopic.*

Proof. Let H be the homotopy between α_0 and α_1, and let $\widetilde{H}$ be its lifting, with origin at $\widetilde{p}$. All there is to show is $\widetilde{H}$ is a homotopy of the arc liftings $\widetilde{a}_0$ and $\widetilde{a}_1$. By the uniqueness of arc liftings,

$$\widetilde{H}(s, 0) = \widetilde{\alpha}_0(s), \quad \widetilde{H}(s, 1) = \widetilde{\alpha}_1(s), \quad s \in [0, l].$$

Therefore, the first condition of homotopy is satisfied. Next, we lift the constant arc $H(0, t) = p$ to $\widetilde{H}(0, t)$ with origin $\widetilde{p}$. Using uniqueness, we see

$$\widetilde{H}(0, t) = \widetilde{p}, \quad t \in [0, 1].$$

By similar argument, we get

$$\widetilde{H}(l,t) = \widetilde{q}, \quad t \in [0,1].$$

That satisfies the second condition of homotopy, so $\widetilde{H}$ is a homotopy between $\widetilde{\alpha}_0$ and $\widetilde{\alpha}_1$. □

6 Degree Theory using Covering Spaces

We first must develop some degree theory using coverings. Let's look at the unit circle $\mathbb{S}^1 = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ and let $\pi : \mathbb{R} \to \mathbb{S}^1$ be the covering of the unit circle by the real line given by

$$\pi(x) = (\cos(x), \sin(x)), \quad x \in \mathbb{R}.$$

Let $\varphi : \mathbb{S}^1 \to \mathbb{S}^1$ be a continuous map. First, we think of the domain of φ as a closed interval $[0,l]$ with its end points 0 and l identified. Thus, φ can be thought of as a continuous map with $\varphi(0) = \varphi(l) = p \in \mathbb{S}^1$. Thus, φ is a closed arc at p in the unit circle, which we know can be lifted to a unique arc $\widetilde{\varphi} : [0,l] \to \mathbb{R}$, starting at a point $x \in \mathbb{R}$ with $\pi(x) = p$. Since $\pi(\widetilde{\varphi}(0)) = \pi(\widetilde{\varphi}(l))$, the difference $\widetilde{\varphi}(l) - \widetilde{\varphi}(0)$ is an integral multiple of 2π. The integer $\deg(\varphi)$ given by

$$\widetilde{\varphi}(l) - \widetilde{\varphi}(0) = (\deg(\varphi))2\pi$$

Intuitively, we can think of the degree as the number of times that φ can wrap $[0,l]$ around the unit circle. Next, we should show that the definition of degree is independent of the choices of p and x. To see this, we encourage the reader to see **page** 399 from Ddo Carmo, M. *Differential Geometry of Curves and Surfaces.*

Now, let's look at an example.

Example 5. *Let's again consider the unit circle* $\mathbb{S}^1 = \{(x,y) \in \mathbb{R}^2 : x = \cos(t), y = \sin(t), t \in \mathbb{R}\}$. *Consider*

$$\varphi(\cos(t), \sin(t) = (\cos(nt), \sin(nt)).$$

Next, we have the lifting of $\widetilde{\varphi} : [0, 2\pi] \to \mathbb{R}$ *defined as* $t \mapsto nt$. *Therefore,*

$$\widetilde{\varphi}(2\pi) - \widetilde{\varphi}(0) = (\deg(\varphi))2\pi \implies 2\pi n - 0 = (\deg(\varphi))2\pi \implies \deg(\varphi) = n.$$

This matches our geometric interpretation of "wrapping" because φ *wraps* $[0, 2\pi]$ *around the unit circle* n *times at* $t = 0, \frac{2\pi}{n}.\frac{4\pi}{n}, \ldots, \frac{2\pi n}{n} = 2\pi$.

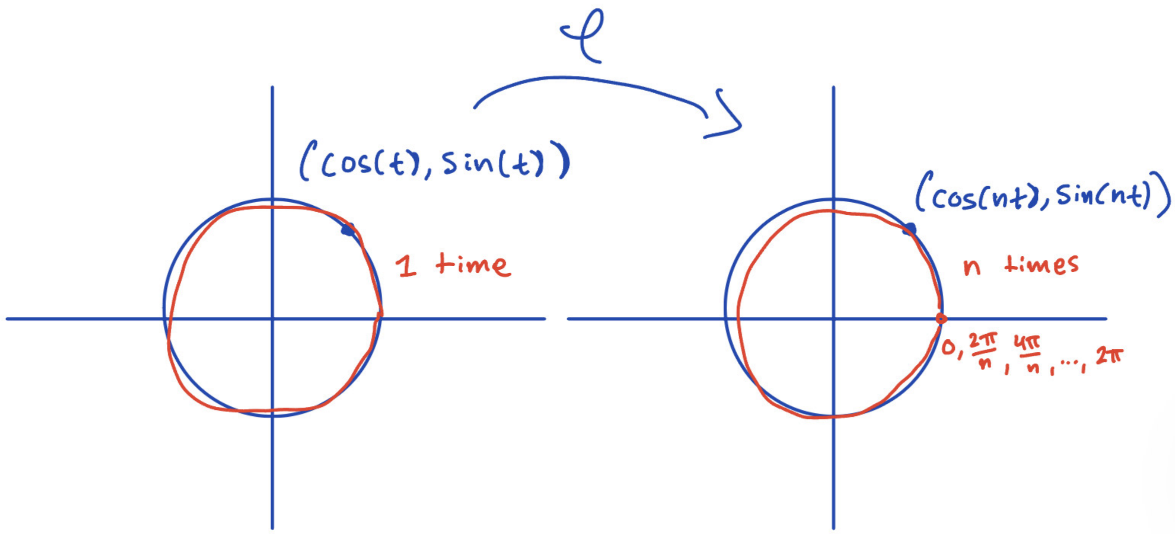

Figure 4: Showing the Degree of φ is n on the Unit Circle.

6.1 Degree is invariant under homotopy

Let $\varphi_1, \varphi_2 : \mathbb{S}^1 \to \mathbb{S}^1$ be continuous maps. Fix a point $p \in \mathbb{S}^1$, thus obtaining two closed arcs at $p, \varphi_1, \varphi_2 : [0, l] \to \mathbb{S}^1, \varphi_1(0) = \varphi_2(0) = p$. If φ_1 and φ_2 are homotopic, then $\deg \varphi_1 = \deg \varphi_2$. This follows from **proposition 3 from section 5**, which says the liftings of φ_1, φ_2 start from a fixed point $x \in \mathbb{R}$ are homotopic, and hence have the same endpoints.

7 Review Proof of Hopf's Umlaufsatz Theorem

First, let's recall the definition of **rotation index**. Let $\alpha : [0, l] \to \mathbb{R}^2$ be a regular plane closed curve, and let $\varphi : [0, l] \to \mathbb{S}^1$ be given by

$$\varphi(t) = \frac{\alpha'(t)}{|\alpha'(t)|}, \quad t \in [0, l].$$

φ is the tangent map of α and the degree of φ is the **rotation index** of α. Now, we state the Hopf's Umlaufsatz theorem:

Theorem 1 (Hopf's Umlaufsatz)**.** *Let $\beta : [0, l] \to \mathbb{R}^2$ be a plane, regular, simple, closed curve. Then, the rotation index of β is ± 1 (depending on the orientation of β).*

We won't revisit the proof of the theorem, but we'll cover the strategy from lecture. The key idea was to make use of the fact that the degree is invariant of homotopy. Then, by applying the homotopy lifting property, we can deform β into a curve for which the rotation index is visibly ± 1.

References

[1] Manfredo P. do Carmo. *Differential Geometry of Curves and Surfaces.* Prentice-Hall, Englewood Cliffs, NJ, 1976.