A Deep Dive into Pell's Equation

Oswaldo Ramirez

Carnegie Mellon University

Fall 2024

Outline

Defining Pell's Equation

Definition (Pell's Equation)

For a positive, square-free integer D, **Pell's Equation** take the following form:

$$x^2 - Dy^2 = 1$$

More solutions are generated by

$$x_k + y_k\sqrt{D} = (x_1 + y_1\sqrt{D})^k \text{ for } k = 1, 2, \ldots,$$

where (x_1, y_1) is a fundamental solution with x_1 being the minimal value.

Method of Solving Pell's Equation: Continued Fractions

1. Find the nearest square whole number less than D. Let's call this d^2. Then, $d^2 < D$ and $D < (d+1)^2$. Now, we know that $\sqrt{D} = d + \frac{1}{x}$ $(*)$, where $x \in \mathbb{R}_{\geq 1}$. When returning back to step 1, the number d will be used finding the integer part of future partial quotients.
2. Solve for x in the the equation $(*)$ of Step 1. This will look like $\sqrt{D} = d + \frac{1}{x} \iff x = \frac{1}{\sqrt{D}-d}$.
3. Rationalizing the denominator to clear the radical. This will look like $x = \frac{1}{\sqrt{D}-d} \iff x = \frac{\sqrt{D}+d}{D-d^2}$.
4.
    1. If the expression in the numerator is one we've seen before, terminate. The continued fraction are the d's that we've computed.
    2. Start again from Step 1 using the expression derived in Step 3 (i.e., find the integer part of $x = \frac{\sqrt{D}+d}{D-d^2}$ in step 1.)

Finding Continued Fraction of $\sqrt{14}$

Example (Compute CF of $\sqrt{14}$)

For the first iteration, we computed $m = 3$. Then, $\sqrt{14} = 3 + \frac{1}{x_1}$.
Next, we solve for x_1. Hence, $x_1 = \frac{\sqrt{14}+3}{5}$.
Since we didn't find a repeat, we start over but with $x_1 = \frac{\sqrt{14}+3}{5}$.
The integer part of $\frac{\sqrt{14}+3}{5}$ is 1, so $\frac{\sqrt{14}+3}{5} = 1 + \frac{1}{x_2}$. Again, we'll solve for x_2. Then, $x_2 = \frac{\sqrt{14}+2}{2}$. We continue the process until we've reached a numerator we've seen before.

Continued Fraction of $\sqrt{14}$

Therefore, the continued fraction of $\sqrt{14}$ is

$$3+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{3+\sqrt{14}}}}}.$$

In perioidc form, the continued fraction of $\sqrt{14}$ is $[3;\overline{1,2,1,6}]$. ♠
It would be nice to include the first, say, 8 convergents together with their values of $A^2 - DB^2$ so that the reader can see that every full period yields a Pellian one.

Characterize Irrational Quadratic

Definition (General Form of Irrational Quadratic)

For a square-free, positive integer D, we have the following continued fraction for $\sqrt{D}$:

$$\sqrt{D} = [q_0; \overline{q_1, q_2, \ldots, q_2, q_1, 2q_0}] \text{ for } q_i \in \mathbb{N}$$

Continued Fractions are a Solution to Pell's Equation

In modern times, the first method for solving Pell's equation [1] was devised by Anglo-Irish mathematician Lord Brouncker in 1667. The method is based on continued fractions. Then, French mathematician Frénicle de Bessy challenged Brouncker to find a solution for $x^2 - 313y^2 = 1$. Brouncker found a 16 digit solution within one to two hours using his method.

[1] Euler named these equations after the English mathematician John Pell, mistakenly believing that Pell had developed the method for finding solutions. In reality, it was John Wallis who devised the method. Nevertheless, the name stuck, and we continue to use it today

Continued Fractions are a Solution to Pell's Equation

Now, we know that

$$\sqrt{D} = \frac{\alpha_{n+1} A_n + A_{n-1}}{\alpha_{n+1} B_n + B_{n-1}},$$

where α_k is the k^{th} complete quotient in the CF, and A_n,B_n are the n^{th} convergent of $\sqrt{D}$ [2].

Now, since α_{n+1} is the complete quotient after q_n, it'll look like this:

$$\alpha_{n+1} = 2q_0 + \frac{1}{q_1 +} \cdots = \sqrt{D} + q_0$$

[2]The equation can be derived from our formulation of CF using the recurrence relation for the A_n and B_n

Continued Fractions are a Solution to Pell's Equation Cont.

Plugging α_{n+1} equation into the equation for $\sqrt{D}$ gives

$$\sqrt{D} = \frac{(\sqrt{D}+q_0)A_n + A_{n-1}}{(\sqrt{D}+q_0)B_n + B_{n-1}}$$
$$\implies \sqrt{D}((\sqrt{D}+q_0)B_n + B_{n-1}) = (\sqrt{D}+q_0)A_n + A_{n-1}$$
$$\implies DB_n + \sqrt{D}B_nq_0 + \sqrt{D}B_{n-1} = (\sqrt{D}+q_0)A_n + A_{n-1}$$

Since $\sqrt{D}$ is irrational, we can derive two equations from the equation above:

$$DB_n = q_0A_n + A_{n-1} \implies A_{n-1} = DB_n - q_0A_n$$
$$B_nq_0 + B_{n-1} = A_n \implies B_{n-1} = A_n - B_nq_0$$

Continued Fractions are a Solution to Pell's Equation Cont.

Using the recurrence relation and difference between consecutive convergents, we can derive $A_nB_{n-1} - B_nA_{n-1} = (-1)^{n-1}$. Then, plug in the two equations from the previous slide to reach

$$\begin{aligned} & A_n(A_n - B_nq_0) - B_n(DB_n - q_0A_n) = (-1)^{n-1} \\ \implies & A_n^2 - DB_n^2 = (-1)^{n-1} \end{aligned}$$

We can note that this in the form of a Pell's equation, and if n is odd, we have a solution. Otherwise, we can square both sides of the equation to derive a solution to the Pell's equation equal to 1.

$$(A_n^2 - DB_n^2)^2 = ((-1)^{n-1})^2 \implies \quad (A_n^2 + B_n^2D)^2 + D(2A_nB_n)^2 = 1$$

The solution is $(A_n^2 + B_n^2D, 2A_nB_n)$.

Solutions of Pell's Equation are all Continued Fractions

Assume $A^2 - DB^2 = 1$. Then,

$$\left|\frac{A}{B} - \sqrt{D}\right| = \left|\frac{A - B\sqrt{D}}{B}\right|$$
$$= \left|\frac{A^2 - DB^2}{B(A + B\sqrt{D})}\right|$$
$$= \frac{1}{B^2(\sqrt{D} + \frac{A}{B})}.$$

Since $D > 1$ and $\frac{A}{B} > 1$, $\frac{1}{B^2(\sqrt{D}+\frac{A}{B})} < \frac{1}{2B^2}$. Therefore, $\frac{A}{B}$ is a convergent to $\sqrt{D}$ and solutions of Pell's equation consist of those convergents. Furthermore, we can catch more of those solutions using the formula for generating solutions to Pell's equation.