Contents

0.1 Goal in Course

To be able to learn math outside of a classroom environment with the help of only the internet (and possibly a mentor)

Test with Adarsh

Test 2 with Adarsh

Test 3

0.2 Misc.

Abbott Recommends to memorize all the definitions

Each definition is followed by a proof and a few examples

Good Youtube Playlists: `https://www.youtube.com/watch?v=L-XLcmHwohO&list=PL22w63XsKjqxqaF-Q7MSyeSG1W1_xaQoS`

Neat Latex Tool: `http://detexify.kirelabs.org/classify.html`

0.3 The Completeness Axiom

Big Idea: The reals has no ”gaps”

4.1 There exists a maximum and minimum for any nonempty subset of R

4.2 IOW: real numbers can upper and lower bound a set

(a) S has an upper bound if there exists a $M \geq s$ for all $s \in S$

(b) S has a lower bound if there exists an $M \leq s$ for all $s \in S$

(c) S is bounded if there exists an upper and lower bound

4.3 sup is least upper bound, and inf is greatest upper bound

4.4 Completeness Axiom: Every nonempty subset S of $\mathbb{R}$ that is bounded above has a least upper bound. In other words, sup S exists and it is a real number.

4.5 Corollary: Every nonempty subset S of $\mathbb{R}$ that is bounded below has a greatest lower bound inf S.

4.4 & 4.5 IOW: A upper and lower bound exist.

4.6 Archimedean Property: If $a > 0$ and $b > 0$, then for some positive integer n, we have $na > b$.

4.7 Denseness of $\mathbb{Q}$: If a,b $\in \mathbb{R}$ and a $<$ b, there is a rational $r \in \mathbb{Q}$ such that a $<$ r $<$ b.

0.4 Limits of Sequences

Definition: A function with domain $\{n \in \mathbb{Z} : n \geq m\}$ where m is 1 or 0

Notation: $(s_n)_{n=m}^{\infty}$ or $(s_n)_n \in \mathbb{N}$ and sequence noted by parantheses while set is defined by curly brackets

Defintion: values need to close to limit for all large n

Definition 7.1: For each $\epsilon > 0$ there exists a number N s.t. n > N implies $|s_n - s| < \epsilon$

0.5 8. A Discussion about Proofs

0.5.1 Example 1

Prove $lim \frac{1}{n^2} = 0$.

1. $n > N$ s.t. $|s_n - s| < \epsilon$, find N.
2. $|\frac{1}{n^2} - 0| < \epsilon$
3. $\frac{1}{\epsilon} < n^2$
4. $\frac{1}{\sqrt{\epsilon} < n}$
5. Since $n > N$, $N = \frac{1}{\sqrt{\epsilon}}$.
6. This implies somehow that the proof converges.

0.5.2 Example 2

0.6 10. Monotone Sequences and Cauchy Sequences

10.1 Defintion: A monotonic or monotone sequence is one that is either increasing or decreasing

10.2 Theorem: All bounded monotone sequences converge.

10.6 Defintion: Let (s_n) be a sequence in $\mathbb{R}$. We define $limsups_n = lim_{N \to \infty} inf\{s_n : n > N\}$ and $liminfs_n = lim_{N \to \infty} inf\{s_n : n > N\}$.

Note: This definition does not restrict (s_n) to be bounded. If (s_n) is not bounded above, $sup\{s_n : n > N\} = +\infty$ for all N and we decree $limsups_n = +\infty$. Likewise, if (s_n) i not bounded below, $inf\{s_n : n > N\} = -\infty$ for ll N and we decree $liminfs_n = -\infty$.

We say $limsups_n < limsup\{n \in \}$.

Useful Link: `https://math.stackexchange.com/questions/2322902/how-to-deal-with-lim-sup-and-`

10.7 Theorem: (i) If $lims_n$ is defined, then $liminfs_n = lims_n = limsups_n$.
(ii) If $liminfs_n = limsups_n$, then $lims_n$ is defined and $lims_n = liminfs_n = limsups_n$.

Proof: We use the notation $u_n = inf\{s_n : n > N\}$, $v_n = sup\{s_n : n > N\}$, $u = limu_N = liminfs_n$ and $v = limv_N = limsups_n$. (i) Suppose $lims_n = +\infty$. Let M be a positive real number. Then there is a positive integer N so that $n > N$ implies $s_n > M$. Then $u_N = infs_n : n > N \geq M$. It follows that $m > N$ implies $u_m \geq M$. In other words, the sequence (u_N) satisfied the condition defining $limu_N = +\infty$, i.e., $liminfs_n = +\infty$. Likewise $limsups_n = +\infty$.

The case $lims_n = -\infty$ is handled in a similar manner. Now suppose lim $s_n = -\infty$ is handled in a similar manner. Now suppose $lims_n = s$ where s is a real numbersber. Consider $\epsilon > 0$. There exist a positive integer N such that $|s_n - s| < \epsilon$ for $n > N$. Thus $s_n < s + \epsilon$ for $n > N$, so $v_N = sups_n : n > N \leq s + \epsilon$. Also, $m > N$ implies $v_m \leq s + \epsilon$ for all $\epsilon > 0$, no matter how small, we conclude $limsups_n a \leq s = lims_n$. A similar agrument shows $lims_n \leq liminfs_n$. Since $liminfs_n \leq limsups_n$, we infer all three numbers equal: $liminfs_n = lims_n = limsups_n$.

(ii)

10.8 Definition: (s_n) of real numbers is a Cauchy sequence if: for each $\epsilon > 0$, there exists a number N, such that m,n > N implies $|s_n - s_m| < \epsilon$.

0.7 Useful Sites

https://math.berkeley.edu/ ianagol/113.F10/proofs.pdf