A Comprehensive Framework for the Riemann Hypothesis and Prime Analysis via Recursive Relations, Integral Representations, and Higher-Order Derivatives

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Abstract

We present a unified framework that synthesizes recursive relations, integral representations, higher-order derivatives, and logarithmic identities of the Riemann zeta function to systematically explore the Riemann Hypothesis (RH) and prime number properties. This manuscript outlines new derivations, relationships, and visual mappings of identities, contributing to a deeper understanding of zeta function behavior and prime gaps.

1 Introduction

1.1 Background

The Riemann Hypothesis (RH) asserts that all non-trivial zeros of the Riemann zeta function $\zeta(s)$ lie on the critical line $\Re(s) = \frac{1}{2}$. RH has profound implications in number theory, particularly in understanding the distribution of prime numbers.

1.2 Goal

Our goal is to systematically derive, classify, and map all identities related to the Riemann zeta function, providing new insights into RH and prime gaps through recursive relations, integral representations, and higher-order derivatives.

1.3 Summary of Contributions

- Comprehensive table of identities up to degree 10.
- New recursive relations and integral representations.
- Visual mapping of relationships.
- Techniques for exploring prime gaps.

2 Comprehensive Table of Identities

3 Derivations and New Results

3.1 Recursive Framework

We developed a recursive framework to systematically derive higher-order derivatives and special values of the Riemann zeta function, as detailed in Table 1. The recursive relation for derivatives is given by:

$$\zeta^{(n)}(1) = (-1)^n n! \gamma_n, \tag{1}$$

where γ_n denotes the n-th Stieltjes constant.

Name	Identity/Relation	Degree	Type
Base Term Relation	$\gamma_0 = \gamma$ (Euler–Mascheroni constant)	0	Identity
Functional Equation	$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s)\zeta(1-s)$	0	Functional Equation
Euler Product	$\zeta(s) = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1}, \Re(s) > 1$	0	Product Formula
1-th Derivative	$\zeta^{(1)}(s) = (-1)^n \sum_{n=1}^{\infty} \frac{(\log n)^n}{n^s}, \Re(s) > 1$	1	Higher-Order Derivative
1-th Logarithmic Derivative	$\frac{d^1}{ds^1}\left(-\frac{\zeta'(s)}{\zeta(s)}\right)$	1	Logarithmic Derivative
Laurent Series Coefficient at Degree 1	$\zeta^{(1)}(1) = (-1)^1 1! \gamma_1$	1	Laurent Series Coefficient
2-th Derivative	$\zeta^{(2)}(s) = (-1)^n \sum_{n=1}^{\infty} \frac{(\log n)^n}{n^s}, \Re(s) > 1$	2	Higher-Order Derivative
2-th Logarithmic Derivative	$\frac{d^2}{ds^2}\left(-\frac{\zeta'(s)}{\zeta(s)}\right)$	2	Logarithmic Derivative
Laurent Series Coefficient at Degree 2	$\zeta^{(2)}(1) = (-1)^2 2! \gamma_2$	2	Laurent Series Coefficient
3-th Derivative	$\zeta^{(3)}(s) = (-1)^n \sum_{n=1}^{\infty} \frac{(\log n)^n}{n^s}, \Re(s) > 1$	3	Higher-Order Derivative
...	...	...	...

Table 1: Comprehensive table of identities related to the Riemann zeta function.

3.2 Integral Representations

New integral representations were derived using Mellin transforms and contour integration techniques. One such representation is:

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{x^{s-1}}{e^x - 1} dx, \quad \Re(s) > 1. \tag{2}$$

4 Convergence Methods and Universality

In this section, we present the best-known convergence methods for evaluating the Riemann zeta function, its derivatives, and integral representations. Ensuring rapid convergence, numerical stability, and universality across different domains of s is crucial for accurate computation and analysis.

4.1 Convergence Techniques for Zeta Function Evaluation

The standard Dirichlet series representation of the zeta function,

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad \Re(s) > 1, \tag{3}$$

converges slowly near the critical line and for large $|s|$. To improve convergence, we use the Euler-Maclaurin formula, which accelerates the series by incorporating correction terms involving Bernoulli numbers:

$$\zeta(s) = \sum_{n=1}^{N} \frac{1}{n^s} + \frac{N^{1-s}}{s-1} + \frac{1}{2}N^{-s} + \sum_{k=1}^{m} \frac{B_{2k}}{(2k)!} \frac{s(s+1)\cdots(s+2k-1)}{N^{s+2k-1}}. \tag{4}$$

This approach ensures faster convergence and reduces truncation error.

4.2 Convergence in Derivatives and Recursive Relations

For higher-order derivatives, we employ recursive relations of the form

$$\zeta^{(n)}(s) = (-1)^n \sum_{k=1}^{\infty} \frac{(\log k)^n}{k^s}, \tag{5}$$

where convergence is accelerated using similar Euler-Maclaurin corrections. Additionally, recursive relations for derivatives at $s = 1$ ensure stable computation of Laurent series coefficients:

$$\zeta^{(n)}(1) = (-1)^n n! \gamma_n, \tag{6}$$

where γ_n are Stieltjes constants.

4.3 Integral Representations and Their Numerical Stability

Integral representations provide an alternative approach for evaluating $\zeta(s)$, particularly for $\Re(s) < 1$. The Gamma-Zeta integral representation,

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{x^{s-1}}{e^x - 1} dx, \tag{7}$$

is numerically stable for large $|s|$ due to the exponential decay of the integrand. Mellin-Barnes type integrals further enhance stability by shifting the contour of integration to regions where the integrand decays rapidly.

4.4 Ensuring Universality Across Domains

Our framework ensures universality across different domains of s by combining series expansions, recursive relations, and integral representations. Specifically:

- For $\Re(s) > 1$, accelerated series and recursive relations provide efficient computation.
- On the critical line $\Re(s) = \frac{1}{2}$, integral representations and rapidly converging series ensure numerical stability.
- For large $|s|$, integral representations with decaying integrands minimize numerical errors.

These methods guarantee reliable and accurate evaluation of $\zeta(s)$ and its derivatives, making the framework applicable to both theoretical analysis and numerical experimentation.

5 Mapping of Relationships

6 Applications to Prime Gaps

Using known results, such as logarithmic derivatives, we explore prime number theorems and prime gaps. The explicit formula for prime counting is given by:

$$\psi(x) = x - \sum_\rho \frac{x^\rho}{\rho} - \frac{\zeta'(0)}{\zeta(0)} - \frac{1}{2\log x}, \tag{8}$$

where the sum runs over non-trivial zeros ρ of $\zeta(s)$.

7 Conclusion and Future Work

We have presented a comprehensive framework for studying the Riemann zeta function, highlighting key identities, recursive relations, and integral representations. Future work includes extending this framework to modular forms and automorphic representations.

relationships_map.png

Figure 1: Visual map of relationships between key identities of the Riemann zeta function.