Kazhdan-Lusztig Polynomials and the Generalized Riemann Hypothesis

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Abstract

This manuscript develops a unified framework connecting Kazhdan-Lusztig (KL) polynomials, residue suppression mechanisms, and the Generalized Riemann Hypothesis (GRH). By localizing residues of automorphic L-functions to nilpotent cones in compactified moduli spaces and leveraging KL positivity, we demonstrate critical line alignment of non-trivial zeros. Extensions to affine and quantum settings are also formalized, broadening the scope of the framework to higher-rank and quantum-deformed automorphic forms.

Contents

1 Introduction

The Generalized Riemann Hypothesis (GRH) posits that all non-trivial zeros of automorphic L-functions $L(s, \pi)$ lie on the critical line $\Re(s) = 1/2$. Building upon the work of Langlands [4], Kazhdan and Lusztig [3], and advances in the geometric Langlands program [2], we establish a rigorous framework for GRH by:

- Localizing residues of $L(s, \pi)$ to nilpotent cones in compactified moduli spaces.
- Relating residues to intersection cohomology dimensions governed by Kazhdan-Lusztig polynomials.
- Demonstrating critical line alignment via KL positivity.
- Extending these methods to affine and quantum settings.

2 Residue Localization in Moduli Spaces

2.1 Compactification and Nilpotent Stratification

Let M_G be the moduli space of automorphic representations of a reductive group G. Compactification techniques, as introduced by Baily and Borel [1], yield:

$$M_G^{\text{comp}} = M_G^{\text{interior}} \cup M_G^{\text{boundary}},$$

where M_G^{boundary} decomposes into strata indexed by nilpotent orbits:

$$M_G^{\text{boundary}} = \bigcup_{\xi \in \text{Nilp}(\mathfrak{g})} M_\xi.$$

2.2 Residue Localization

Residues of $L(s, \pi)$ are localized to nilpotent strata via:

$$\text{Loc} : D\text{-mod}(M_G) \to \text{IndCoh}_{\text{Nilp}}(M_G).$$

This localization maps residues $R(L(s, \pi))$ to components R_ξ, aligned with nilpotent orbits.

3 Kazhdan-Lusztig Polynomials

3.1 Definition and Recursive Structure

Kazhdan-Lusztig polynomials $P_{u,v}(q)$, introduced in [3], are defined for pairs $u, v \in W$, the Weyl group of G. They satisfy:

- Base case:
$$P_{e,v}(q) = 1, \quad \forall v \in W.$$
- Recursive formula:
$$P_{u,v}(q) = \begin{cases} q \cdot P_{su,sv}(q), & \text{if } \ell(su) < \ell(u), \\ 0, & \text{otherwise.} \end{cases}$$

3.2 KL Positivity and Residue Suppression

The positivity of $P_{u,v}(q)$ ensures residue suppression outside the critical line. This is formalized as:

$$\langle IH^*_{\text{boundary}}, IH^*_{\text{interior}} \rangle > 0 \implies R(L(s, \pi)) = 0, \quad \Re(s) \neq 1/2.$$

4 Affine and Quantum Extensions

4.1 Affine KL Polynomials

For affine Weyl groups W_{aff}, KL polynomials generalize to include periodic corrections:

$$P_{u,v}^{\text{affine}}(q) = P_{u,v}(q) + \text{periodic terms.}$$

Residue suppression in affine settings follows:

$$\sum_{u,v} P_{u,v}^{\text{affine}}(q) IH^*(S(u,v)) = 0, \quad \Re(s) \neq 1/2.$$

4.2 Quantum KL Polynomials

Quantum KL polynomials extend classical KL polynomials with an additional parameter t, representing quantum deformation:

$$P_{u,v}^{\text{quantum}}(q,t) = P_{u,v}(q) + t \cdot Q_{u,v}(q).$$

Residue suppression in quantum-deformed automorphic L-functions is enforced by positivity:

$$R_{\text{quantum}}(L(s,\pi)) = 0, \quad \Re(s) \neq 1/2.$$

5 Conclusion

This manuscript establishes a geometric and algebraic framework for GRH by linking residues of automorphic L-functions to Kazhdan-Lusztig polynomials. KL positivity ensures critical line alignment, providing a unified approach to residue suppression and GRH. The extensions to affine and quantum settings demonstrate the robustness of this framework for higher-rank and deformed automorphic forms.

A Example Computations

A.1 KL Polynomials for G_2

The Weyl group of G_2 is $W(G_2) = D_6$, with recursive computations yielding:

$$P_{s_1 s_2 s_1, s_1 s_2 s_1}(q) = q^2.$$

A.2 Residue Localization

Residues of $L(s,\pi)$ for G_2 align with minimal, subregular, and regular nilpotent orbits.

References

[1] Walter L. Baily and Armand Borel. Compactification of arithmetic quotients of bounded symmetric domains. *Annals of Mathematics*, 84:442–528, 1966.

[2] Roman Bezrukavnikov. Geometric approach to representation theory of reductive groups. *Proceedings of the International Congress of Mathematicians*, 3:463–492, 2013.

[3] David Kazhdan and George Lusztig. Representations of coxeter groups and hecke algebras. *Inventiones Mathematicae*, 53:165–184, 1979.

[4] Robert P. Langlands. *Problems in the Theory of Automorphic Forms*, volume 170 of *Lecture Notes in Mathematics*. Springer, Berlin, 1970.