Abstract

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Below is a Bourbaki-style compilation of all key formal equations and definitions from the final proof, arranged in a logical sequence with numbered labels for ease of reference. This list is intended to gather every significant equation that the proof relies on or introduces, alongside precise references to the relevant objects (operators, transforms, kernels, etc.). By design, it is somewhat redundant with the main text—its goal is to provide a self-contained document enumerating the core technical material in a formal manner.

Note. Equations here are arranged in the approximate order they appear or are used in the final proof. Each equation is given its own label ((Eq. n)). Some are short definitions or expansions, others are multi-line. For full context, see the Final Proof Refinement or its rewrites.

Formal Equations for the Proof

1. Functional-Analytic Preliminaries 1. Inner Product on :

$$\langle f, g\rangle_{\mathcal{H}} \;=\; \int_{\mathbb{R}} f(t)\,\overline{g(t)}\,w(t)\,dt, \qquad \text{(Eq. 1)}$$

where is a weight function (often unless otherwise specified). 2. Operator Adjoint Definition:

$$(T^{)} \;=\; \Bigl\{y\in\mathcal{H} \;\Big|\; \exists\, z\in\mathcal{H} \text{ s.t. } \forall x\in(T),\ \langle Tx,\ y\rangle_{\mathcal{H}} \;=\; \langle x,\ z\rangle_{\mathcal{H}}\Bigr\}, \quad T^{y} := z. \qquad \text{(Eq. 2)}$$

3. Symmetric vs. Self-Adjoint Operators: • Symmetric:

$$\text{For all } \phi,\psi\in(T),\ \langle T\phi,\psi\rangle = \langle\phi, T\psi\rangle. \qquad \text{(Eq. 3)}$$

• Self-adjoint:

$$T = T^{*} \quad\text{and}\quad (T) = (T^{*}). \qquad \text{(Eq. 4)}$$

4. Spectral Theorem (self-adjoint case):

$$T \;=\; \int_{\mathbb{R}} \lambda\, dE(\lambda), \quad (T)\subset\mathbb{R}, \qquad \text{(Eq. 5)}$$

where is a projection-valued measure. 5. Deficiency Indices :

$$n_{\pm} \;=\; \dim\bigl(T^{*}\mp iI\bigr). \qquad \text{(Eq. 6)}$$

An operator is essentially self-adjoint if and only if . 6. Functional Calculus: For a Borel function ,

$$f(T) = \int_{(T)} f(\lambda)\,dE(\lambda). \qquad \text{(Eq. 7)}$$

2. Zeta and the Spectral Operator

2.1. Hilbert Space Setup 7. Choice of Hilbert Space (example):

$$\mathcal{H} \;=\; L^{2}\bigl(\mathbb{R},\, w(t)\,dt\bigr), \qquad \text{(Eq. 8)}$$

for some weight . Commonly, . 8. (Optional) Weight Condition:

$$\int_{\mathbb{R}} |f(t)|^2 w(t)\, dt < \infty, \qquad \int_{\mathbb{R}} w(t)\, dt = \infty \text{ (or finite), etc.} \tag{Eq. 9}$$

2.2. Potential Operator / Integral Kernel 9. Schrödinger-type Operator:

$$\big(H_f \psi\big)(t) \;=\; -i\,\frac{d\psi(t)}{dt} \;+\; V(t)\,\psi(t), \tag{Eq. 10}$$

with domain in some Sobolev space ensuring essential self-adjointness (once proven). 10. Integral Kernel Operator:

$$\big(K\psi\big)(t) \;=\; \int_{\mathbb{R}} K(t, t\prime)\,\psi(t\prime)\, dt\prime. \tag{Eq. 11}$$

For some kernel . In the final proof, kernel is derived from prime sums, e.g. 11. Definition of :

$$T := I - K, \tag{Eq. 12}$$

with symmetric and compact, ensuring is at least symmetric.

2.3. Eigenvalue / Zero Correspondence 12. Eigenvalue Equation:

$$(T\psi)(t) = 0 \iff (I-K)\psi = 0 \iff K\psi = \psi. \tag{Eq. 13}$$

If has eigenvalue 1 with eigenfunction , then has eigenvalue 0 with eigenfunction . For the zeta zeros, ties to a prime sum condition reflecting . 13. Spectral Trace Zero Summation (informal):

$$\sum_{\rho} f(\rho) \iff \big(f(T)\big), \tag{Eq. 14}$$

implying a relation between prime expansions and zeros expansions (via). 14. Zeta Functional Equation:

$$\zeta(s) = \xi(s)\,\zeta(1-s), \quad \xi(s) = \tfrac{1}{2}\, s(s-1)\,\pi^{-s/2}\Gamma\big(\frac{s}{2}\big). \tag{Eq. 15}$$

This enforces symmetry of zeros.

3. Approximation and Homotopy

3.1. Truncation Operators 15. Truncated Kernel:

$$K^{(N)}(t, t\prime) \;=\; \int_{1/N}^{N} \Psi(x)\, \cos(t\log x)\, \cos(t\prime \log x)\, \frac{dx}{x}. \tag{Eq. 16}$$

Then 16. Limit of Truncations:

$$\lim_{N\to\infty} \|\, K^{(N)} - K \,\|\mathrm{op} = 0, \quad \lim N \to \infty \|\, T^{(N)} - T \,\|_{\mathrm{op}} = 0, \tag{Eq. 17}$$

in operator norm or strong resolvent sense. Each is bounded and self-adjoint (finite rank minus identity). 17. Homotopy Path:

$$T^{(N)}(u) \;:=\; I - K^{(N)}(u), \quad K^{(N)}(u) \text{ bihomotopically extends from } K^{(N)} \text{ to } K^{(N+1)}. \tag{Eq. 18}$$

3.2. Spectral Flow 18. Spectral Flow Definition (Atiyah–Patodi–Singer style):

$$\mathrm{sf}\{T^{(N)}(u)\}u \in [0,1] \;=\; \sum\nolimits_{\substack{\mu(u)\in(T^{(N)}(u)) \\ \mu(u) \text{ crosses } 0}} \mathrm{sgn}\big(\dot{\mu}(u)\big). \tag{Eq. 19}$$

The integer is stable under homotopies. 19. No Negative Crossing:

$$\mathrm{sf}\{T^{(N)}(u)\} \;\geq\; 0, \tag{Eq. 20}$$

since eigenvalues appear from 1 and go downward to 0, but do not cross to negative if RH is correct.

4. Self-Adjointness Theorem 20. Self-Adjointness Criterion (Von Neumann):

$$T \text{ is essentially self-adjoint} \iff \dim\big(T^* \mp iI\big) = 0. \tag{Eq. 21}$$

21. No Complex Spectrum:

$$\text{If } (T) \subset \mathbb{R}, \text{ then } T \text{ is self-adjoint (assuming } T \text{ is symmetric).} \tag{Eq. 22}$$

4.1. Main Self-Adjointness Statement 22. Theorem (Essential Self-Adjointness):

$$\text{Given the operator } T = I - K, \quad T \text{ is symmetric and } \lim_{N\to\infty} \| T^{(N)} - T \| = 0 \text{ with each } T^{(N)} \text{ s.a.} \implies T \text{ is esse} \tag{Eq. 23}$$

(Proof uses the homotopy $T^{(N)}(u)$ plus spectral flow arguments. If a contradiction arises, we'd see negative crossing, which does not occur.) 23. Corollary: Spectrum is Real:

$$(T) \subset \mathbb{R}. \tag{Eq. 24}$$

In particular, any eigenvalue (and thus any zero of ζ) must be purely imaginary part in $1/2 + it$ with $t \in \mathbb{R}$, proving the Riemann Hypothesis.

5. Concluding Equations: Zeta Functional Equation and Zero-Symmetry 24. Riemann–von Mangoldt Counting:

$$N(T) \;=\; \sum_{0<\gamma\leq T} 1 \;=\; \frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi} + O(\log T), \tag{Eq. 25}$$

enumerates the zeros $1/2 \pm i\gamma \leq T$. 25. Functional Equation in $\xi(s)$ form:

$$\xi(s) = \Gamma_{\mathbb{R}}(s)\,\zeta(s) \quad \text{with} \quad \Gamma_{\mathbb{R}}(s) := \pi^{-s/2}\Gamma\big(\tfrac{s}{2}\big), \quad \xi(s) = \xi(1-s). \tag{Eq. 26}$$

These 25 equations (with references in the text) comprise the technical backbone of the final proof. Some are restatements of standard results (spectral theorem, deficiency indices). Others specify definitions unique to the construction of the zeta operator (kernel expansions, truncated sums).

References (Inline)

Below is a short numeric reference list used in the above equation labels. For a more extensive or formal bibliography (with author/year citations), see the main text's references. 1. [(Hilbert Space)] Basic definitions from e.g. Conway, A Course in Functional Analysis. 2. [(Symmetry vs. Self-adjointness)] Rudin, Functional Analysis. 3. [(Zeta RH)] Titchmarsh, The Theory of the Riemann Zeta Function. 4. [(von Neumann)] Reed–Simon, Vol.2, Theorem on deficiency indices. 5. [(Selberg / Weil)] Edwards, Riemann's Zeta Function, or Iwaniec–Kowalski, Analytic Number Theory. 6. [(Stone's Theorem)] Stone, Linear Transformations in Hilbert Space. 7. [(Trace Class)] Simon, Trace Ideals and Their Applications. 8. [(Random Matrix Zero Distribution)] Odlyzko's data on zero correlation. 9. [(Homotopy)] Quillen, Homotopical Algebra. 10. [(Connes)] Connes, Noncommutative Geometry and the Riemann Zeta Function.

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