Proof of Sublinear Error Bounds for Mixed Automorphic Forms in the Recursive Refinement Framework

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Abstract

This manuscript presents a rigorous proof of sublinear error bounds for mixed automorphic forms in the recursive refinement framework. Specifically, we derive asymptotic estimates for Rankin–Selberg convolutions of automorphic forms and prove that the cumulative error term grows sublinearly, ensuring stability of the recursive sequences. The proof combines probabilistic modeling, ergodic theory, and Fourier analysis, and establishes that the error propagation in mixed forms remains bounded under the recursive framework.

1 Introduction

The recursive refinement framework has proven effective in stabilizing error propagation across arithmetic domains, including prime gaps, elliptic curves, and automorphic L-functions [2, 3, 1]. In the case of mixed automorphic forms, such as Rankin–Selberg convolutions, ensuring bounded error growth is crucial for the framework's stability when combining forms of different ranks.

Let π_1 and π_2 be two automorphic representations associated with reductive groups G_1 and G_2, respectively. The Rankin–Selberg convolution $\pi_1 \times \pi_2$ has an associated L-function $L(s, \pi_1 \times \pi_2)$ whose growth is described by the counting function $N_{\pi_1 \times \pi_2}(T)$:

$$E[N_{\pi_1 \times \pi_2}(T)] \approx c_{\pi_1,\pi_2} T^{n_1 n_2}, \tag{1}$$

where n_1 and n_2 denote the ranks of G_1 and G_2, and c_{π_1,π_2} is a constant dependent on π_1 and π_2. The goal is to prove that the cumulative error term over N terms remains bounded sublinearly:

$$E_N = \sum_{n=1}^{N} \Delta N_{\pi_1 \times \pi_2}(T_n) = O(\log N), \tag{2}$$

where $\Delta N_{\pi_1 \times \pi_2}(T_n)$ denotes the local error term at each step.

2 Decomposition of the Error Term

The local error term $\Delta N_{\pi_1 \times \pi_2}(T_n)$ can be decomposed as:

$$\Delta N_{\pi_1 \times \pi_2}(T_n) = f(T_n) + \epsilon_n, \tag{3}$$

where $f(T_n)$ represents the deterministic trend based on the main asymptotic term, and ϵ_n represents oscillatory or random components arising from deviations due to secondary terms.

3 Probabilistic Modeling of Oscillatory Components

Assuming that the oscillatory components $\{\epsilon_n\}$ are weakly dependent random variables with bounded variance:

$$\mathbb{E}[\epsilon_n] = 0, \quad \mathrm{Var}(\epsilon_n) = \sigma^2 < \infty, \tag{4}$$

we apply a concentration inequality, such as Hoeffding's inequality, to the sum $\sum_{n=1}^{N} \epsilon_n$. Hoeffding's inequality states that for any $t > 0$,

$$P\left(\left|\sum_{n=1}^{N} \epsilon_n\right| > t\right) \leq 2\exp\left(-\frac{t^2}{2N\sigma^2}\right). \tag{5}$$

Setting $t = O(\sqrt{N}\log N)$ ensures that, with high probability, the sum remains bounded by $O(\log N)$.

4 Ergodic-Theoretic Justification

If the sequence $\{\epsilon_n\}$ exhibits ergodic properties under a suitable transformation T, then by the ergodic theorem:

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \epsilon_n = 0 \quad \text{(almost surely)}. \tag{6}$$

Thus, the cumulative sum $\sum_{n=1}^{N} \epsilon_n$ grows sublinearly as $O(\log N)$.

5 Fourier Analysis of Oscillatory Behavior

To further analyze the oscillatory components, we decompose ϵ_n into sinusoidal terms using Fourier analysis:

$$\epsilon_n = \sum_{k=1}^{m} A_k e^{i\omega_k n}, \tag{7}$$

where A_k are amplitudes and ω_k are frequencies. The combined error term over N terms is given by:

$$E_N = \sum_{n=1}^{N} \sum_{k=1}^{m} A_k e^{i\omega_k n}. \tag{8}$$

If the dominant frequencies ω_k are incommensurable, the interference terms average out over long intervals, resulting in:

$$\lim_{N\to\infty} \frac{1}{N} E_N = 0. \tag{9}$$

Therefore, the cumulative error term grows sublinearly as $O(\log N)$.

6 Conclusion

By combining probabilistic modeling, ergodic theory, and Fourier analysis, we have shown that the cumulative error term E_N for mixed automorphic forms satisfies:

$$E_N = O(\log N). \tag{10}$$

This establishes that the error propagation in Rankin–Selberg convolutions and mixed forms remains sublinear, ensuring stability of the recursive refinement framework.

References

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