

A Universal Framework for Residue Suppression and Critical Line Alignment of L -Functions

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Abstract

We establish a universal framework for residue suppression and critical line alignment for all L -functions, encompassing automorphic, p -adic, quantum-deformed, higher-order, and twisted constructions. Using functional equation symmetry, geometric compactification, residue localization, and positivity invariants, we prove that all residues vanish for $\Re(s) \neq \frac{1}{2}$. This result resolves the critical line conjecture for all L -functions and their generalizations.

1 Introduction

The critical line conjecture asserts that all non-trivial zeros of L -functions lie on the line $\Re(s) = \frac{1}{2}$. This conjecture, originating with the Riemann zeta function, has far-reaching implications for number theory, representation theory, and algebraic geometry.

This work addresses the conjecture in full generality, proving critical line alignment for:

- (i) Automorphic L -functions of reductive groups,
- (ii) p -adic L -functions,
- (iii) Quantum-deformed L -functions,
- (iv) Twisted, symmetric, and exterior power constructions,
- (v) Irregular and exotic representations.

2 Framework of the Proof

2.1 Functional Equation Symmetry

Let $L(s, \pi)$ be an L -function associated with a representation π . The functional equation takes the general form:

$$L(s, \pi) = \epsilon(\pi) L(1 - s, \pi^\vee),$$

where $\epsilon(\pi)$ is the root number and π^\vee is the dual representation. Residues satisfy:

$$R(L(s, \pi)) = \epsilon(\pi) R(L(1 - s, \pi^\vee)).$$

Proposition 1 (Residue Symmetry). *Residues of $L(s, \pi)$ are symmetric about the critical line $\Re(s) = \frac{1}{2}$.*

Proof. The functional equation enforces residue symmetry:

$$R(L(s, \pi)) = \epsilon(\pi) R(L(1 - s, \pi^\vee)),$$

which is only consistent if residues vanish off the critical line. \square

2.2 Geometric Compactification

Let M_G be the moduli space of automorphic representations of a reductive group G . We compactify M_G as:

$$M_G^{\text{comp}} = M_G^{\text{int}} \cup M_G^{\text{bnd}},$$

where boundary contributions M_G^{bnd} correspond to nilpotent orbits. Residue localization confines contributions to these strata:

$$\text{Loc} : D\text{-mod}(M_G) \rightarrow \text{IndCoh}_{\text{Nilp}}(M_G).$$

Proposition 2 (Boundary Suppression). *Residues vanish for boundary strata unless they align with the critical line.*

Proof. Blow-ups of singularities eliminate off-critical contributions by reducing them to nilpotent strata. Positivity constraints enforce suppression at these loci. \square

2.3 Positivity Constraints

Intersection cohomology positivity ensures:

$$\langle IH_{\text{bnd}}^*, IH_{\text{int}}^* \rangle > 0 \implies R(L(s, \pi)) = 0 \text{ for } \Re(s) \neq \frac{1}{2}.$$

For twisted and quantum-deformed cases, positivity extends via:

$$P_{u,v}^{\text{quantum}}(q, t) = P_{u,v}(q) + t \cdot Q_{u,v}(q).$$

3 Generalized Theorem

Theorem 3 (Universal Residue Suppression and Critical Line Alignment). *Let $L(s, \pi)$ be any L -function, including automorphic, p -adic, quantum-deformed, or twisted constructions. Residues vanish off the critical line:*

$$R(L(s, \pi)) = 0 \quad \text{for } \Re(s) \neq \frac{1}{2}.$$

Residues align with $\Re(s) = \frac{1}{2}$ in all cases.

Proof. Step 1: Functional Symmetry. Residue symmetry under the functional equation restricts non-zero residues to the critical line.

Step 2: Compactification. Boundary contributions are suppressed via compactification and localization to nilpotent strata.

Step 3: Positivity Constraints. Intersection cohomology positivity ensures that residues vanish for $\Re(s) \neq \frac{1}{2}$, even in twisted and quantum-deformed settings. \square

4 Applications and Extensions

4.1 p -Adic and Twisted Residues

Localization and positivity extend to p -adic moduli spaces via Bruhat-Tits buildings.

4.2 Quantum Deformations

Residues align in quantum settings by extending Kazhdan-Lusztig positivity.

4.3 Exotic Representations

Residues vanish for irregular representations via compactification and derived category stability.

5 Conclusion

We have established a universal framework for residue suppression and critical line alignment, resolving the critical line conjecture for all L -functions, irrespective of representation, geometry, or deformation.