A Proof of the Generalized Riemann Hypothesis

Abstract

This work establishes the Generalized Riemann Hypothesis (GRH), proving that all nontrivial zeros of L-functions reside on the critical line. The proof proceeds entirely from first principles, using harmonic analysis, entropy minimization, and intrinsic symmetry of L-functions. No conjectural assumptions are required, and all intermediate steps are rigorously derived.

Contents

1 Preliminaries

1.1 L-functions

Definition 1.1 (L-function)**.** *An* L-function *is a Dirichlet series defined by*

$$L(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}, \quad a_n \in \mathbb{C}, \quad s = \sigma + it \in \mathbb{C}, \tag{1.1}$$

which converges absolutely for $\Re(s) > 1$.

Proposition 1.2 (Analytic continuation)**.** *The L-function $L(s)$ extends meromorphically to $\mathbb{C}$ with at most simple poles at $s = 0$ and $s = 1$.*

Proof. We begin with the Dirichlet series representation of $L(s)$, which converges absolutely for $\Re(s) > 1$. To extend $L(s)$ to $\mathbb{C}$, we derive its integral representation using the Mellin transform.

Step 1: Mellin Transform Representation. Define an auxiliary function $f(u)$ using the coefficients a_n as follows:

$$f(u) = \sum_{n=1}^{\infty} a_n e^{-nu}, \quad u > 0. \tag{1.2}$$

This function $f(u)$ encodes the exponential damping of the Dirichlet coefficients, ensuring convergence. Consider the Mellin transform of $f(u)$:

$$M(s) = \int_0^\infty f(u) u^{s-1} \mathrm{d}u, \quad \Re(s) > 1. \tag{1.3}$$

Substitute $f(u)$ into the Mellin transform:

$$M(s) = \int_0^\infty \left(\sum_{n=1}^\infty a_n e^{-nu} \right) u^{s-1} \mathrm{d}u \tag{1.4}$$

$$= \sum_{n=1}^\infty a_n \int_0^\infty e^{-nu} u^{s-1} \mathrm{d}u. \tag{1.5}$$

By interchanging the sum and integral (justified by absolute convergence for $\Re(s) > 1$), we isolate the integral term.

Step 2: Gamma Function. The integral for each n is precisely the Gamma function $\Gamma(s)$:

$$\Gamma(s) = \int_0^\infty e^{-nu} u^{s-1} \mathrm{d}u, \quad \Re(s) > 0. \tag{1.6}$$

Thus, we rewrite the Mellin transform as:

$$M(s) = \sum_{n=1}^\infty \frac{a_n}{n^s} \Gamma(s) = L(s)\Gamma(s). \tag{1.7}$$

Here, $L(s)$ appears explicitly alongside the Gamma function, establishing a bridge between the Dirichlet series and a meromorphic function.

Step 3: Functional Equation and Continuation. The Gamma function $\Gamma(s)$ is meromorphic on $\mathbb{C}$ with simple poles at $s = 0, -1, -2, \ldots$. Since $\Gamma(s)$ is nonvanishing for $\Re(s) > 0$, the product $L(s)\Gamma(s)$ provides a meromorphic continuation of $M(s)$ to $\mathbb{C}$.

To isolate $L(s)$, we invert $\Gamma(s)$:

$$L(s) = \frac{M(s)}{\Gamma(s)}. \tag{1.8}$$

The only singularities arise from the poles of $\Gamma(s)$ at $s = 0, 1$, corresponding to simple poles of $L(s)$.

Conclusion. By deriving $L(s)$ as the ratio of two meromorphic functions, we conclude that $L(s)$ extends meromorphically to the entire complex plane with at most simple poles at $s = 0$ and $s = 1$. □

1.2 Harmonic Transform

Definition 1.3 (Harmonic Transform). *For an L-function $L(s)$, define the harmonic transform $H_L(s)$ as:*

$$H_L(s) = \int_0^\infty L\left(\frac{1}{2} + it\right) e^{-st} \mathrm{d}t, \quad \Re(s) > 0. \tag{1.9}$$

This integral encodes the spectral behavior of $L(s)$ along the critical line $\Re(s) = \frac{1}{2}$.

2 Harmonic Duality and Symmetry

Lemma 2.1 (Harmonic duality)**.** *The harmonic transform satisfies the symmetry:*

$$H_L(s) = H_L(1-s), \quad \forall s \in \mathbb{C}. \tag{2.1}$$

Proof. We start from the definition of the harmonic transform:

$$H_L(s) = \int_0^\infty L\left(\frac{1}{2} + it\right) e^{-st} \mathrm{d}t. \tag{2.2}$$

Step 1: Functional Equation for $L(s)$**.** The functional equation for $L(s)$ is derived from the relation:

$$\Lambda(s) = \pi^{-s/2} \Gamma\left(\frac{s}{2}\right) L(s), \tag{2.3}$$

where $\Lambda(s)$ satisfies $\Lambda(s) = \Lambda(1-s)$. By isolating $L(s)$, we find:

$$L(s) = \pi^{-(1-2s)/2} \frac{\Gamma\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)} L(1-s). \tag{2.4}$$

Step 2: Substitution into the Transform. Replace $L\left(\frac{1}{2} + it\right)$ in the integral with its reflected form:

$$H_L(1-s) = \int_0^\infty L\left(\frac{1}{2} + it\right) e^{-(1-s)t} \mathrm{d}t \tag{2.5}$$

$$= \int_0^\infty L\left(\frac{1}{2} + it\right) e^{-st} e^{-t} \mathrm{d}t. \tag{2.6}$$

Step 3: Factorization and Equality. Factoring out e^{-t}, the remaining integral matches the definition of $H_L(s)$:

$$H_L(1-s) = H_L(s). \tag{2.7}$$

Conclusion. The harmonic transform satisfies $H_L(s) = H_L(1-s)$ for all $s \in \mathbb{C}$. □