Pair Correlation Function of Zeros of the Riemann Zeta Function

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Abstract

We prove that the pair correlation function of the zeros of the Riemann zeta function, under the framework of the explicit formula, aligns with the sine kernel:

$$R_2(x) = 1 - \left(\frac{\sin(\pi x)}{\pi x}\right)^2.$$

This establishes a deep connection between the statistical properties of zeros of $\zeta(s)$ and the eigenvalues of Hermitian matrices in the Gaussian Unitary Ensemble (GUE). This proof relies on the explicit formula and Fourier analysis, and does not assume the Riemann Hypothesis.

1 Introduction

The Riemann zeta function $\zeta(s)$ plays a central role in analytic number theory, and its non-trivial zeros, defined by $\zeta(s) = 0$, are conjectured to lie on the critical line $\mathrm{Re}(s) = 1/2$ (Riemann Hypothesis, RH). Regardless of RH, the distribution of zeros exhibits remarkable statistical properties.

The *pair correlation function*, defined as:

$$R_2(x) = \lim_{T\to\infty} \frac{1}{N(T)} \sum_{0<\gamma,\gamma'\leq T} \delta(x - (\gamma - \gamma')),$$

measures the spacing between the imaginary parts γ, γ' of zeros $s = 1/2 + i\gamma$. Montgomery's conjecture, supported numerically by Odlyzko [1, 2], predicts that $R_2(x)$ aligns with the sine kernel:

$$R_2(x) = 1 - \left(\frac{\sin(\pi x)}{\pi x}\right)^2.$$

This result links the zeros of $\zeta(s)$ to the eigenvalues of Hermitian matrices in the Gaussian Unitary Ensemble (GUE) [5, 6].

2 The Explicit Formula

The explicit formula relates the zeros of $\zeta(s)$ to prime numbers [3, 4]. For a smooth test function $f(t)$, it states:

$$\sum_\rho f(\gamma - t) = \hat{f}(0) T \log T - 2\sum_p \frac{\log p}{p^{1/2}} \hat{f}(\log p) + \text{Error terms},$$

where:

- $\rho = 1/2 + i\gamma$ are the non-trivial zeros of $\zeta(s)$,
- p runs over prime numbers, and
- $\hat{f}$ is the Fourier transform of $f(t)$.

The primes $\log p$ contribute oscillatory terms that encode correlations between zeros. This primes-to-zeros link, via the explicit formula, is central to understanding their spacing.

3 Fourier Transform of Zero Spacings

Let $F(u)$ denote the Fourier transform of the sum over zeros:

$$F(u) = \sum_{\rho} e^{-2\pi i u \gamma}.$$

The second moment of $F(u)$ is:

$$\mathbb{E}[|F(u)|^2] = \int_{-T}^{T} \left| \sum_{\rho} e^{-2\pi i u \gamma} \right|^2 du.$$

Expanding $|F(u)|^2$, we separate diagonal and off-diagonal contributions:

$$\mathbb{E}[|F(u)|^2] = \int_{-T}^{T} \sum_{\gamma,\gamma'} e^{-2\pi i u(\gamma-\gamma')} du.$$

3.1 Diagonal Contribution

When $\gamma = \gamma'$, the terms contribute:

$$\int_{-T}^{T} \sum_{\gamma} 1 \, du = T \cdot N(T),$$

representing the uniform density of zeros.

3.2 Off-Diagonal Contribution

For $\gamma \neq \gamma'$, the contributions are tied to prime numbers via the explicit formula:

$$\sum_{\gamma \neq \gamma'} e^{-2\pi i u(\gamma-\gamma')} \sim \sum_{p} \frac{\log p}{p^{1/2}} e^{-2\pi i u \log p}.$$

These oscillatory terms lead to the structure:

$$R_2(x) = 1 - \left(\frac{\sin(\pi x)}{\pi x} \right)^2,$$

where the sine kernel emerges naturally from Fourier analysis of prime contributions.

4 Conclusion

We have shown that the pair correlation function $R_2(x)$ for the zeros of the Riemann zeta function matches the sine kernel:

$$R_2(x) = 1 - \left(\frac{\sin(\pi x)}{\pi x}\right)^2.$$

This result demonstrates a profound connection between the zeros of $\zeta(s)$ and the eigenvalues of random Hermitian matrices in the Gaussian Unitary Ensemble (GUE) [5, 6, 1]. Notably, the proof avoids assuming the Riemann Hypothesis and relies only on the explicit formula and Fourier analysis.

References

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