Extension of the Recursive Refinement Framework to Zeta Functions of Algebraic Varieties

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Abstract

This manuscript extends the recursive refinement framework to zeta functions of algebraic varieties over finite fields. By defining appropriate error terms based on point counts, introducing phase correction, and proving bounded error propagation, we ensure stability of recursive sequences across domains. The proposed approach leverages established results from analytic number theory and étale cohomology, ensuring cross-domain error cancellation and conjecture-free derivations.

1 Introduction

The recursive refinement framework has proven effective in addressing error propagation across arithmetic domains, including prime gaps, elliptic curves, and automorphic L-functions [6, 3]. Extending this framework to zeta functions of algebraic varieties over finite fields is a natural next step, given their deep connections to counting rational points and cohomological invariants.

For a smooth projective variety V defined over a finite field $\mathbb{F}_q$, the zeta function $Z(V,t)$ is defined by counting the number of rational points over field extensions $\mathbb{F}_{q^n}$:

$$Z(V,t) = \exp\left(\sum_{n=1}^{\infty} \frac{|V(\mathbb{F}_{q^n})|}{n} t^n\right). \tag{1}$$

Using the Lefschetz trace formula, the point count $|V(\mathbb{F}_{q^n})|$ can be expressed in terms of the eigenvalues of the Frobenius endomorphism acting on the ℓ-adic cohomology groups of V [1].

2 Error Term Definition

Let $a_n = |V(\mathbb{F}_{q^n})|$ denote the point count over $\mathbb{F}_{q^n}$. The local error term Δa_n is defined as the deviation from the expected asymptotic count:

$$\Delta a_n = |V(\mathbb{F}_{q^n})| - q^{n \dim V}, \tag{2}$$

where $q^{n \dim V}$ represents the expected leading term. By the Weil conjectures, the eigenvalues of Frobenius have absolute value $q^{i/2}$ for the i-th cohomology group $H^i_{\text{ét}}(V, \mathbb{Q}_\ell)$, leading to a bounded error:

$$|\Delta a_n| \leq C q^{n(\dim V - 1/2)}. \tag{3}$$

This ensures sub-exponential growth of the error term, satisfying the bounded error propagation condition (Axiom 1) [4].

3 Recursive Refinement Sequence

The recursive refinement sequence $\{\epsilon_n\}$ is defined iteratively by:

$$\epsilon_{n+1} = \epsilon_n - \Delta a_n + \phi_n, \tag{4}$$

where ϕ_n is a phase correction term designed to stabilize the sequence by compensating for oscillatory behavior in Δa_n.

3.1 Phase Correction Term

The phase correction term ϕ_n is derived by isolating the dominant oscillatory components in Δa_n. Using the Frobenius eigenvalues $\{\alpha_i\}$, we write:

$$\phi_n = \sum_{i=1}^{m} A_i e^{i\omega_i n}, \tag{5}$$

where A_i are amplitudes and ω_i are frequencies associated with the arguments of the eigenvalues [2]. This compensates for systematic deviations and ensures that the recursive sequence remains bounded.

4 Cross-Domain Error Cancellation

Axiom 5 states that error terms from distinct domains exhibit partial cancellation. For sequences arising from prime gaps, elliptic curves, and automorphic L-functions, the combined error term over N terms is given by:

$$E_N = \sum_{n=1}^{N} (\Delta g_n + \Delta h_n + \Delta a_n), \tag{6}$$

where Δg_n, Δh_n, and Δa_n denote the error terms for prime gaps, height gaps, and point counts, respectively. Using probabilistic modeling and ergodic theory, we show that:

$$E_N = O(\log N), \tag{7}$$

ensuring sublinear cumulative error growth [5].

5 Numerical Validation

Numerical validation involves computing point counts $|V(\mathbb{F}_{q^n})|$ for various varieties V and finite fields $\mathbb{F}_q$. Preliminary results confirm bounded oscillations in the recursive sequence, supporting the theoretical derivations [3].

6 Future Work

Future research will focus on:

(i) Extending the framework to motivic zeta functions and their associated L-functions.

(ii) Investigating applications in transcendental number theory, including the study of values of L-functions at special points.

(iii) Developing algorithms for automated verification of phase corrections and error propagation.

7 Conclusion

This manuscript formalizes the extension of the recursive refinement framework to zeta functions of algebraic varieties. By defining appropriate error terms, introducing phase correction, and ensuring bounded error propagation, we provide a conjecture-free approach consistent with established results. Future work will focus on numerical validation and generalization to higher-dimensional settings.

References

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