Recursive Refinement Framework for Complex Function Zeros: Addressing Error Propagation through a PDE-Based Approach

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Abstract

This manuscript investigates the recursive refinement framework for computing zeros of complex functions, specifically L-functions and zeta functions, highlighting challenges related to numerical instability and error propagation. Our analysis links the observed erratic error growth to missing prime structure and proposes a new theoretical framework based on partial differential equations (PDEs) to encode prime information explicitly. We introduce regularization strategies, adaptive precision control, and iterative solvers to mitigate instability, culminating in a PDE-driven recursive refinement model that formalizes zero propagation as a dynamical flow.

1 Introduction

Recursive refinement is a powerful method for computing zeros of complex functions. Despite its utility, applying it to high-dimensional L-functions and zeta functions has exposed significant numerical challenges, particularly erratic error growth and divergence due to ill-conditioning. These issues are hypothesized to stem from a fundamental loss of structural information when primes are not explicitly encoded in the refinement process.

This work explores the underlying causes of these instabilities and proposes a PDE-based solution that models zero refinement as a smooth dynamical process. By introducing prime-dependent terms in the recursive update equations, we aim to restore structural information and mitigate error propagation.

2 Recursive Refinement Framework

2.1 General Formulation

Given a complex function $L(s)$ and an initial guess s_0 for a zero, recursive refinement seeks to iteratively update the guess according to:

$$s_{n+1} = s_n - J_L^{-1}(s_n)L(s_n),$$

where J_L is the Jacobian matrix of L with respect to the complex variables.

2.2 Challenges with Error Propagation

Numerical experiments demonstrate that despite using high-precision inputs, error growth becomes erratic, particularly in higher dimensions (e.g., GL(n) for $n \geq 7$). Analysis suggests that the erratic behavior is analogous to working in a transcendental number space without prime structure, leading to unpredictable error propagation. This loss of structure implies that recursive refinement operates in a fundamentally incomplete dimensional space.

3 Theoretical Insights

3.1 Dimensionality and Prime Structure

Traditional dimensionality analysis assumes directionality in a continuous vector space. However, in the context of recursive refinement, dimensionality should be reinterpreted as the ability to generate all relevant numbers from a seed set. The absence of prime information creates a restricted dimensional space, akin to limiting the generated set to even numbers when starting from 2.

This insight parallels results from the Chinese Remainder Theorem, where congruence relations modulo primes define distinct equivalence classes. The absence of prime structure implies missing congruence relations, leading to incomplete error cancellation and erratic refinement behavior.

3.2 PDE-Based Formalization

To address the lack of prime structure, we propose modeling the refinement process as a PDE-driven flow. Let $\psi(s,t)$ represent the position of a zero as a function of a pseudo-time parameter t in the refinement process. The evolution of ψ is governed by:

$$\frac{\partial \psi}{\partial t} = -\nabla J_L^{-1}(\psi)L(\psi) + \sum_p f_p(\psi),$$

where the sum over p represents contributions from prime-dependent correction terms. The function $f_p(\psi)$ encodes the influence of each prime, ensuring that prime structure is explicitly included in the refinement process.

4 Proposed Solutions

4.1 Regularization Techniques

Given the high condition numbers observed in the Jacobian, we propose regularization using Tikhonov damping:

$$J_L^{-1}(s_n) \to (J_L(s_n) + \lambda I)^{-1},$$

where λ is a dynamically chosen parameter to control ill-conditioning.

4.2 Adaptive Precision Control

Adaptive precision control dynamically increases the precision of computations when numerical instability is detected. This approach mitigates the impact of finite precision errors during refinement.

4.3 Iterative Solvers for Jacobian Updates

Replacing direct inversion of the Jacobian with iterative solvers such as GMRES or conjugate gradient methods offers increased robustness in handling ill-conditioned systems.

5 Experimental Results

Preliminary numerical experiments validate the proposed PDE-driven model. Error growth is significantly reduced, and convergence is observed for higher-dimensional L-functions with properly chosen prime correction terms.

6 Conclusion and Future Work

This study introduces a novel PDE-based approach to recursive refinement, explicitly incorporating prime structure to mitigate error propagation. Future work includes extending the model to automorphic L-functions and exploring connections with Frobenius manifolds and integrable systems.