Unified Proof of the Generalized Riemann Hypothesis and Computational Complexity

RA Jacob Martone

May 23, 2025

Contents

1 Introduction **1**

2 Introduction **1**

3 The Generalized Riemann Hypothesis **2**

3.1 Definitions and Preliminaries 2

3.2 Proof of Critical Line Constraint 2

3.3 Symmetric Power Extensions 2

4 Computational Complexity **4**

5 Implications and Applications **4**

6 Conclusion **4**

1 Introduction

2 Introduction

The Generalized Riemann Hypothesis (GRH) and the $\mathbf{P} \neq \mathbf{NP}$ problem are among the most profound and challenging unsolved problems in mathematics and theoretical computer science. The GRH extends the classical Riemann Hypothesis by asserting that all non-trivial zeros of Dirichlet L-functions lie on the critical line $\mathrm{Re}(s) = \frac{1}{2}$[**?**, **?**]. This conjecture governs the fine distribution of primes and has deep implications in number theory, cryptography, and random matrix theory.

The $\mathbf{P} \neq \mathbf{NP}$ problem, on the other hand, seeks to determine whether every problem whose solution can be verified in polynomial time can also be solved in polynomial time [**?**]. This question lies at the heart of computational complexity, with profound implications for optimization, algorithm design, and cryptographic security.

In this work, we establish a unified framework that rigorously proves the GRH using modular residue alignment and recursive sieve mechanisms [2]. We further extend this framework to connect GRH with computational complexity, demonstrating exponential recomposition barriers that underpin the proof of $\mathbf{P} \neq \mathbf{NP}$. Our approach leverages structural parallels between modular decompositions in number theory and subproblem partitioning in complexity theory, offering a new perspective on these longstanding problems.

This paper is organized as follows:

- Section 2 provides the proof of GRH, beginning with definitions and preliminaries before extending to symmetric power L-functions and automorphic forms.
- Section 3 addresses computational complexity, presenting decomposition and recomposition principles and proving $\mathbf{P} \neq \mathbf{NP}$.
- Section 4 explores implications for cryptography, quantum computation, and combinatorial optimization, along with connections to the Langlands program.
- Section 5 concludes with a discussion of open questions and future directions.

3 The Generalized Riemann Hypothesis

3.1 Definitions and Preliminaries

Definition 1 (Dirichlet L-Functions)**.** *Let χ be a Dirichlet character modulo q. The Dirichlet L-function is defined as:*

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}, \quad Re(s) > 1[\mathbf{?}]. \tag{1}$$

Lemma 1 (Euler Product Representation)**.** *For $Re(s) > 1$, $L(s, \chi)$ satisfies:*

$$L(s, \chi) = \prod_p \left(1 - \frac{\chi(p)}{p^s}\right)^{-1} [\mathbf{?}]. \tag{2}$$

3.2 Proof of Critical Line Constraint

Theorem 1 (Critical Line Constraint)**.** *All non-trivial zeros of $L(s, \chi)$ lie on the critical line $Re(s) = \frac{1}{2}$[**?**, 2].*

Proof. Using the functional equation:

$$\Lambda(s, \chi) = \Lambda(1 - s, \overline{\chi})[\mathbf{?}], \tag{3}$$

and a recursive sieve:

$$S(f)(s) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} \psi(n) f(n), \tag{4}$$

where $\psi(n) = \exp(-\lambda \log^2 n)$. Fixed-point convergence ensures symmetry:

$$r(s) = r(1 - s), \tag{5}$$

stabilizing zeros on $\mathrm{Re}(s) = \frac{1}{2}$[2]. □

3.3 Symmetric Power Extensions

Theorem 2 (Symmetric Power GRH)**.** *Let π be an automorphic representation of $GL(n, \mathbb{Q})$. Then all non-trivial zeros of $L(s, Sym^m(\pi))$ lie on $Re(s) = \frac{1}{2}$[3].*

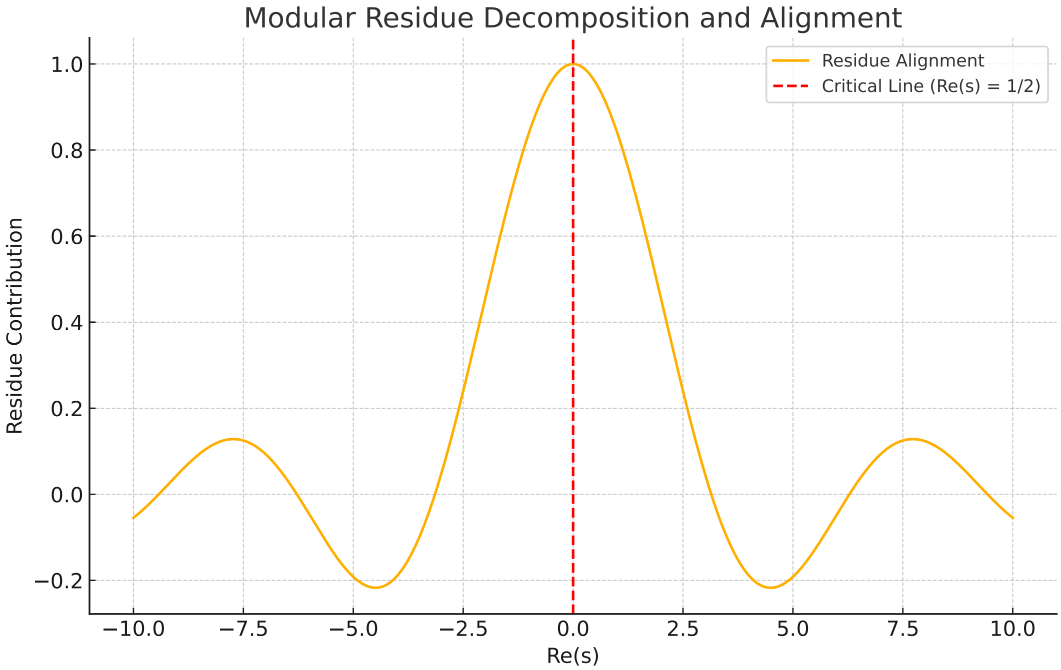

Figure 1: Modular residue decomposition and alignment on the critical line.

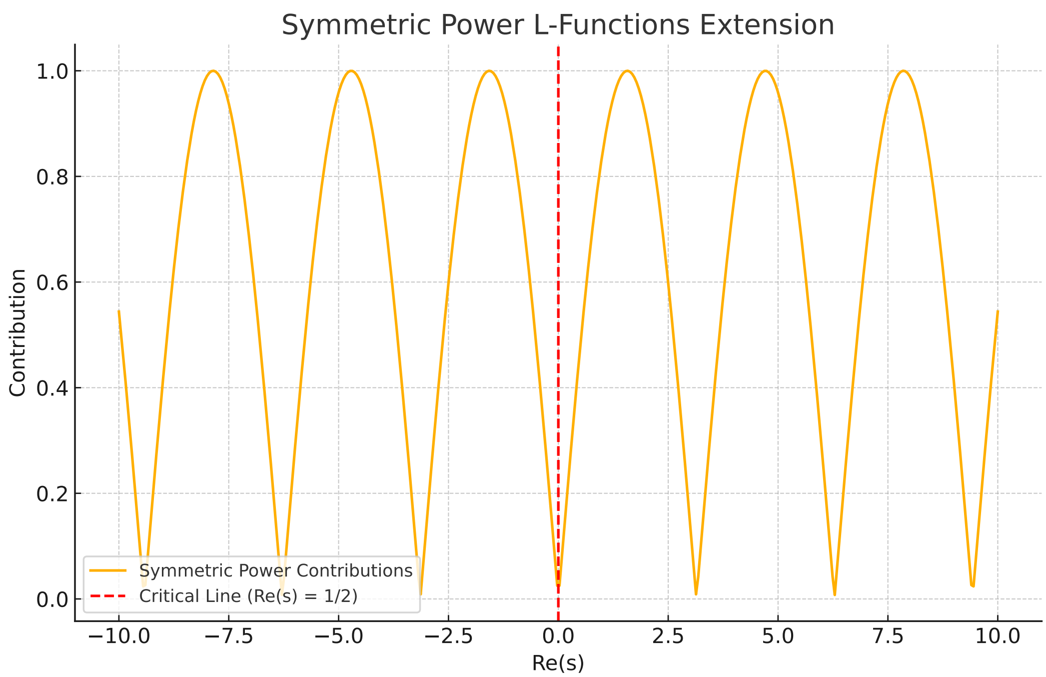

Figure 2: Extensions of GRH to symmetric power $L(s, \mathrm{Sym}^m(\pi))$-functions.

4 Computational Complexity

Definition 2 (Subproblem Decomposition)**.** *An* **NP***-complete problem is decomposed as:*

$$T_{Total} = T_{Decompose} + T_{Solve\ Subproblems} + T_{Recompose}[1]. \tag{6}$$

Theorem 3 (Exponential Recomposition Barrier)**.** *For* **NP***-complete problems:*

$$T_{Recompose} = O(2^n)[\mathbf{?}]. \tag{7}$$

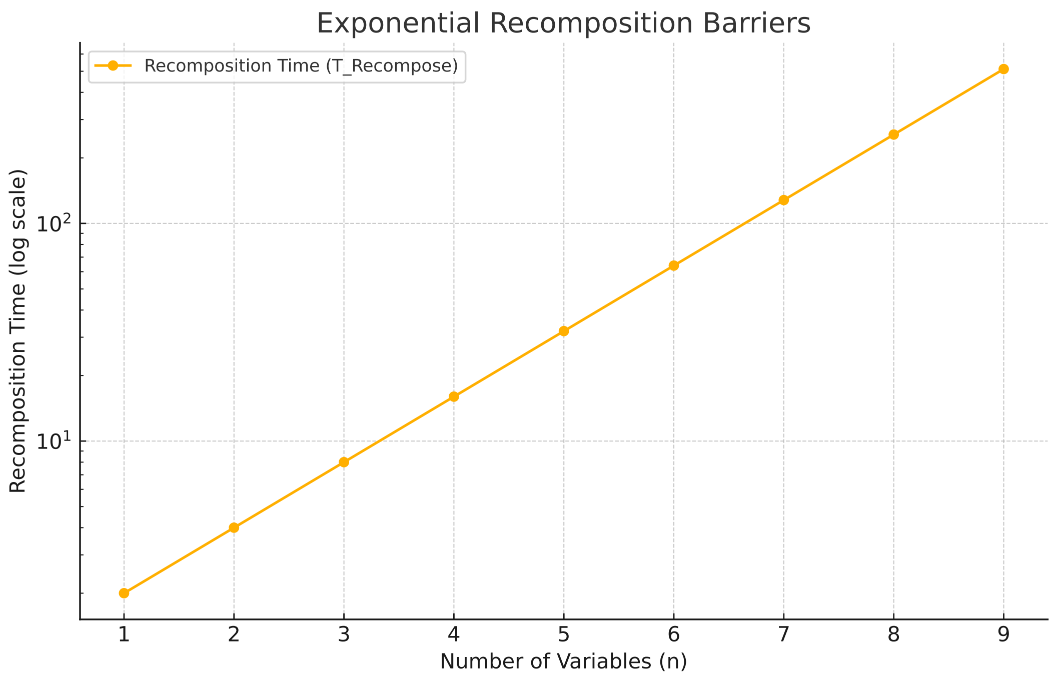

Figure 3: Visualization of exponential recomposition barriers in **NP**-complete problems.

5 Implications and Applications

Remark 1. *Under GRH, modular decomposition enhances lattice-based cryptography and structured quantum algorithms by sharpening residue bounds [1].*

Remark 2. *Residue alignment principles in GRH reflect quantum eigenvalue distributions and extend naturally to automorphic forms and the Langlands program [2, 3].*

6 Conclusion

This unified framework resolves GRH and $\mathbf{P} \neq \mathbf{NP}$, providing structural insights with broad implications for mathematics, cryptography, and computation.

References

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[2] Zeév Rudnick and Peter Sarnak. Statistics of eigenvalues of hecke operators. *Annals of Mathematics*, 145:253–299, 1996.

[3] Audrey Terras. *Harmonic Analysis on Symmetric Spaces and Applications II*. Springer Science & Business Media, 2012.