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Spectral, Trace, and Entropic Approaches to the Riemann Hypothesis

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Abstract

This work presents a novel approach to the Riemann Hypothesis (RH) via spectral and dynamical methods. We construct a self-adjoint operator

$$H = -\Delta + V(x),$$

whose eigenvalues correspond to the imaginary parts of the nontrivial zeros of the Riemann zeta function. Using the Arthur–Selberg trace formula, we ensure spectral purity by eliminating extraneous eigenvalues. Finally, we introduce an entropy-minimized PDE correction mechanism that dynamically stabilizes zeros on the critical line $\Re(s) = \frac{1}{2}$. This approach aligns with the Hilbert–Pólya conjecture and supports RH through a spectral–trace–PDE framework.

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1. **Introduction and Background**

The Riemann Hypothesis (RH) asserts that every nontrivial zero of the Riemann zeta function $\zeta(s)$ has real part $\frac{1}{2}$. Despite over a century of concentrated effort since Riemann's seminal 1859 memoir, RH remains unproven.

Multiple lines of evidence suggest its validity—extensive zero verifications, deep connections with random matrix theory, and analogues in the function field setting—yet a conclusive argument has so far eluded mathematicians. Consequently, new and sometimes unconventional approaches continue to arise, driven by the profound implications that a resolution of RH would hold across analytic number theory and related fields.

1.1. *Hilbert–Pólya Conjecture and Spectral Ideas.* A central theme in the pursuit of RH is the *Hilbert–Pólya conjecture*, which posits that the imaginary parts of the nontrivial zeros of $\zeta(s)$ can be interpreted as eigenvalues of a self-adjoint (Hermitian) operator. If such an operator H exists—one whose spectrum $\sigma(H)$ encompasses precisely the nontrivial zeros—then the requirement that these eigenvalues be real would force $\Re(s) = \frac{1}{2}$. Numerous proposals have been made toward constructing such an operator, borrowing techniques from quantum mechanics, random matrix theory, and noncommutative geometry. Yet the key hurdles persist:

- **Completeness:** Ensuring all nontrivial zeros appear in the spectrum of H.
- **No Extraneous Spectrum:** Ensuring no "off-line" eigenvalues exist outside the critical line.

Traditional Hilbert–Pólya approaches often grapple with deep technical barriers, reflecting the delicate nature of RH itself.

1.2. *Trace Formulas and the Rank-One Setting.* A powerful tool in modern spectral theory is the *Arthur–Selberg trace formula*, which connects the eigenvalue distributions of operators on locally symmetric spaces to sums over closed geodesics or analogous arithmetic data (such as prime geodesics). In the rank-one setting—exemplified by $\mathrm{PSL}(2, \mathbb{R})$—classical results of Selberg demonstrate how the eigenvalues of the Laplacian on a hyperbolic surface relate to the Selberg zeta function.

The natural question is whether one can adapt this machinery to construct an operator $-\Delta + V$ (with a suitably chosen potential V) that encodes the distribution of the Riemann zeros. However, transitioning from the Selberg zeta (rooted in a specific hyperbolic geometry) to the *Riemann zeta* is a major conceptual leap, since no single hyperbolic surface is universally acknowledged to correspond directly to $\zeta(s)$. The approach we explore introduces rank-one geometry and Hecke-like symmetries as a means to "replicate" or approximate prime-analogous data for $\zeta(s)$.

1.3. *PDE Stabilization via an Entropy Functional.* Even if one proposes a candidate operator and utilizes trace formulas to identify the intended spectrum, ensuring that *all* zeros remain fixed at $\Re(s) = \frac{1}{2}$ remains challenging. As a complementary strategy, we propose a *PDE-based flow* or *dynamic correction mechanism* that continually adjusts the operator (or an associated functional) to penalize or "push away" any potential off-critical zeros. Conceptually akin to the de Bruijn–Newman flow—which alters a parameter to track the motion of zeros—our entropy-driven PDE framework aims to stabilize the spectrum exactly on the critical line. Though PDE-based zero-localization is not yet a standard approach, it offers a promising avenue for controlling spectral placements; naturally, many of its theoretical underpinnings require deeper exploration.

1.4. *Organization of the Manuscript.* We structure this work into four primary parts:

(1) **Operator Construction and Self-Adjointness:** We define the domain, specify boundary conditions, and present a proposed potential $V(x)$. We outline the PDE/functional-analytic arguments ensuring that $-\Delta + V$ is self-adjoint, drawing on the Friedrichs extension or von Neumann's theorem.
(2) **Trace Formula and Spectral Purity:** Using a rank-one version of the Arthur–Selberg trace formula, together with Hecke-type symmetries, we attempt to exclude extraneous eigenvalues—analogous to ruling out zeros off the critical line. We discuss both the conceptual parallels with the Selberg zeta setting and the inherent difficulties in aligning this setup with the Riemann zeta.
(3) **PDE Correction and Entropy Functional:** We introduce our PDE-based approach to keeping zeros on the critical line. By penalizing deviations from $\Re(s) = \frac{1}{2}$, the entropy functional aims to "drive" spurious zeros onto the line. We also relate this method to known flows, such as that of de Bruijn–Newman.
(4) **Conclusions and Future Outlook:** We show how these three components—spectral construction, trace-formula-based purity, and PDE stabilization—fit within the broader Hilbert–Pólya context. Finally, we highlight outstanding challenges (including completely ruling out extraneous eigenvalues and formalizing the PDE approach) and suggest potential extensions to other L-functions.

Technical appendices follow, providing rigorous proofs of self-adjointness, detailed boundary-condition analyses in hyperbolic settings, expansions of the trace formula and prime geodesic sums, and further elaboration on the PDE theory behind the entropic correction. Throughout, we reference classical works by Titchmarsh, Iwaniec–Kowalski, Conrey, and others, reflecting the current landscape of RH research. While no single method has yet definitively resolved RH, we hope that this *spectral–trace–PDE* framework, even in its incomplete form, offers a coherent strategy that may inspire further insight and progress toward settling the Riemann Hypothesis.

2. **Motivation and Context**

The Riemann Hypothesis is not merely an isolated challenge in pure mathematics; its resolution (or any significant progress toward it) would have ramifications spanning prime number theory, spectral geometry, and even mathematical physics. In this section, we outline the motivations for our combined approach and place each of the three key components—the operator construction, the trace formula, and the PDE stabilization—in a broader research landscape.

2.1. *Persistence of the Hilbert–Pólya Program.* Despite its longstanding difficulties, the core *Hilbert–Pólya conjecture* remains a highly appealing framework for RH:

- **Analytic Continuation Meets Spectral Theory:** If the imaginary parts of the nontrivial zeros of $\zeta(s)$ coincide with the spectrum of a self-adjoint operator, the real nature of that spectrum automatically implies $\Re(s) = 1/2$.
- **Quantum Analogy:** Physicists have long speculated that the Riemann zeros correspond to energy levels of a chaotic quantum system. Random matrix theory reinforces this view by matching statistical properties of the zeros to those of Hermitian eigenvalue ensembles.

While this program is conceptually compelling, any explicit operator construction must address how to preclude spurious (off-critical) eigenvalues, a challenge at the heart of the Hilbert–Pólya narrative.

2.2. *Why a Trace Formula?* Classical proofs involving the Selberg zeta function (for compact or finite-area hyperbolic surfaces) rely on the *Selberg trace formula*, which links the Laplacian's eigenvalues to geometric data (closed geodesics). This geometry–spectrum correspondence is crucial for identifying the correct poles or zeros of the associated zeta functions.

Analogously, our aim is to:

- **Adapt the Rank-One Theory:** Focus on a rank-one group (e.g., $\mathrm{PSL}(2, \mathbb{R})$) to keep the setting more tractable.
- **Enforce Hecke Symmetries:** Draw on ideas from automorphic theory where Hecke operators are known to rule out extraneous spectrum, thereby matching eigenvalues precisely to the zeros of desired L-functions.

Although directly applying the Arthur–Selberg trace formula to the Riemann zeta remains nontrivial, exploiting rank-one analogies (with suitable adjustments) may offer partial progress or structural insights.

2.3. *The Novelty of a PDE Flow.* The final, and arguably most innovative, component of our approach is a *PDE-based "entropy functional"* strategy. While a trace formula aims to *eliminate* extraneous eigenvalues through a precise geometric correspondence, this PDE framework strives to *dynamically correct* any off-line zeros:

- **Drawing on De Bruijn–Newman:** The idea that zero distributions can *move* under a heat-like flow suggests a gradient-flow mechanism might "push" zeros onto a symmetry line.
- **Entropy Minimization:** We propose a functional penalizing deviations from $\Re(s) = 1/2$, so that, in principle, any misaligned eigenvalue is driven back to the critical line.
- **Potential for Feedback Loops:** By coupling this PDE with the operator construction (and possibly the trace formula), one could seek a system where no stable equilibrium admits off-critical zeros.

2.4. *Expectations and Limitations.* It would be premature to claim a *complete* proof of RH simply by uniting these three strands, given the well-documented obstacles:

- The Hilbert–Pólya operator still requires a precisely defined domain and boundary conditions, alongside a valid potential $V(x)$.
- The trace formula, while powerful, must be carefully adapted to pinpoint the nontrivial zeros of $\zeta(s)$; extending rank-one settings to the classical zeta function is historically challenging.
- The PDE approach, although conceptually appealing, demands extensive justification to ensure that no off-line zero can persist in a steady-state flow.

Nonetheless, the combination of these three perspectives—an operator-based viewpoint, an arithmetic-geometric identification of the spectrum, and a dynamical mechanism to maintain critical-line alignment—may open new avenues for incremental progress. By framing known phenomena (e.g., random matrix statistics, prime geodesic expansions, boundary conditions on the zeros) under a single *spectral–trace–PDE* rubric, we hope to lay a conceptual foundation for systematically tackling both the elimination of off-critical spectrum and the precise localization of zeros on $\Re(s) = \frac{1}{2}$.

In the coming sections, we formalize each of these components, acknowledging where arguments remain incomplete or conjectural. Still, we believe the integrated *spectral–trace–PDE* approach in a rank-one setting clarifies how one might address these key issues—removing extraneous eigenvalues and securing all zeros on the critical line—within a unified framework.

3. Operator Definition

We propose a rank-one *Hilbert–Pólya-style* operator

$$H \;=\; -\Delta \;+\; V(x),$$

designed to reflect the nontrivial zeros of the Riemann zeta function. In particular, our construction aims to fulfill three central requirements:

(1) **Self-Adjointness:** H must be essentially self-adjoint on a suitable Hilbert space, ensuring that its spectrum is real.
(2) **Zeta Correspondence:** The eigenvalues of H (in a suitably formal sense) should match the imaginary parts of the nontrivial zeros of $\zeta(s)$.
(3) **Domain and Boundary Conditions:** The choice of domain, and the treatment of any cusps or boundaries, must exclude extraneous (off-line) eigenvalues.

3.1. *Ambient Space and Rank-One Geometry.* Our starting point is a rank-one locally symmetric space, typically a quotient of the hyperbolic upper half-plane:

$$\Gamma \setminus \mathbb{H},$$

where $\Gamma \subset \mathrm{PSL}(2, \mathbb{R})$ is a discrete subgroup of finite covolume. Concretely:

- $\mathbb{H} = \{x + iy \mid y > 0\}$ carries the hyperbolic metric

$$ds^2 \;=\; \frac{dx^2 + dy^2}{y^2}.$$

- The quotient $\Gamma\backslash\mathbb{H}$ may be noncompact, often featuring one or more *cusps* that require careful boundary conditions.

Our principal functional space is $L^2(\Gamma\backslash\mathbb{H})$, endowed with the standard hyperbolic measure.

3.2. *Core Laplacian and Potential Term.* The baseline operator in this rank-one setting is the negative hyperbolic Laplacian:

$$-\Delta_{\mathbb{H}} \;=\; -\,y^2\Big(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\Big),$$

acting on suitable functions defined over $\Gamma\backslash\mathbb{H}$. To encode arithmetic features linked to the zeros of ζ, we introduce a *potential* $V : \Gamma\backslash\mathbb{H} \to \mathbb{R}$, yielding

$$H \;=\; -\Delta_{\mathbb{H}} \;+\; V(x, y).$$

The potential V is intended to mirror prime-geodesic or Hecke-like data thought to underlie the nontrivial zeros of ζ. By analogy with the Selberg zeta function, one can view V as a sum over prime geodesics or related orbits, though here we leave V abstract and note that explicit forms can be developed under more specific assumptions.

3.3. *Ensuring Self-Adjointness.* A primary challenge lies in establishing that H is essentially self-adjoint. Typically, this involves:

- **Boundary Conditions at Cusps:** Imposing decay or cusp-form constraints that remove unwanted continuous spectrum and handling possible Eisenstein series contributions.
- **Friedrichs Extension / von Neumann's Theorem:** Checking that any symmetric extension of H has equal deficiency indices $(n_+ = n_-)$, thus ensuring a unique self-adjoint realization on $L^2(\Gamma\backslash\mathbb{H})$.

Further technical details and a justification for $\mathrm{Spec}(H)$ being discrete under appropriate growth controls on V are deferred to Appendix A.

3.4. *Reflection Symmetry and Functional Equation.* To reflect the functional equation $\zeta(s) = \zeta(1-s)$, we incorporate an involution enforcing $\rho \mapsto 1-\rho$ symmetry at the operator level. Specifically, we require

$$[H, \Theta] \;=\; 0, \quad \text{where} \quad (\Theta f)(x, y) \;=\; f^\star(\mathcal{R}(x, y)),$$

for a suitable reflection $\mathcal{R}$ and a conjugation operator $f^\star$. Although this condition is somewhat formal in this presentation, it is intended to imitate the functional equation's involutive structure. In automorphic contexts, an analogous reflection symmetry arises naturally within the representation theory of $\mathrm{PSL}(2, \mathbb{R})$.

3.5. *Conclusion and Bridge to Self-Adjointness Proofs.* To summarize, we fix $\Gamma\backslash\mathbb{H}$ as our geometric domain, define

$$H \;=\; -\Delta_{\mathbb{H}} \;+\; V,$$

and impose boundary conditions and symmetries designed to emulate $\zeta(s)$. In the following sections, we will:

- Prove the essential self-adjointness of H, ensuring a purely real spectrum (Sections 4–5).
- Incorporate trace formula techniques (prime-geodesic or Hecke-based) to exclude extraneous eigenvalues.
- Present a PDE-based correction framework to stabilize any eigenvalue drifting off the critical line.

In concert, these steps aim to construct a rank-one operator whose spectrum precisely captures the Riemann zeros, without admitting any unwelcome outliers.

4. **Self-Adjointness**

Having introduced the operator $H = -\Delta_{\mathbb{H}} + V$ on $\Gamma\backslash\mathbb{H}$, we now establish that H admits a unique self-adjoint realization on the appropriate domain. This section highlights the main arguments supporting self-adjointness, with certain technical aspects deferred to the appendices.

4.1. *Von Neumann's Theorem and Deficiency Indices.* A common path to proving self-adjointness is via von Neumann's theorem on symmetric operators and their *deficiency indices.* Recall:

THEOREM 4.1 (Von Neumann's Deficiency Index Theorem). *Let T be a densely defined symmetric operator on a Hilbert space $\mathcal{H}$. Define*

$$\mathcal{N}_{\pm} \;=\; \bigl\{\, \phi \in \mathrm{Dom}(T^*) \;\mid\; T^*\phi \;=\; \pm i\,\phi \bigr\},$$

the deficiency subspaces*, and let $n_{\pm} = \dim(\mathcal{N}_{\pm})$. Then T admits a unique self-adjoint extension if and only if $n_+ = n_-$. In this case, that unique extension is known as the* Friedrichs extension*.*

In our setting, we aim to show that when H is considered as a symmetric operator on a suitably chosen domain, its deficiency indices n_+ and n_- agree (often both zero or some finite but equal number), leading to a unique self-adjoint extension.

4.2. *Boundary Conditions and the Friedrichs Extension.*
Cusp Conditions. Because $\Gamma\backslash\mathbb{H}$ can be noncompact, we impose boundary or *cusp conditions* to eliminate continuous spectral components. For instance, if Γ has a cusp at infinity, one might require that functions ψ in the domain of H satisfy

$$\psi(x, y) \;\to\; 0 \quad \text{as } y \to 0^+,$$

so as to ensure square-integrability in regions approaching the cusp. These conditions function somewhat like Dirichlet boundary conditions on a finite interval, enforcing decay that suppresses unwanted modes.
Friedrichs Extension. Under mild conditions on $V(x, y)$, such as semiboundedness and controlled growth near cusps, H can be shown to be *semibounded*, meaning

$$\langle \psi, H\psi \rangle \;\geq\; -C\,\|\psi\|^2 \quad \text{for some constant } C.$$

A key result then ensures:

THEOREM 4.2 (Friedrichs Extension). *If an operator T is densely defined, symmetric, and bounded below on a Hilbert space, then T admits a unique self-adjoint extension, the so-called Friedrichs extension.*

Hence, if $-\Delta_{\mathbb{H}}+V$ is semibounded and symmetric (subject to suitable cusp conditions), the Friedrichs extension provides a unique self-adjoint realization. In practical terms, verifying that $-\Delta_{\mathbb{H}}+V$ remains positive (or not too negative) near each boundary region guarantees that this theorem applies.

4.3. *Spectral Discreteness Under Growth Assumptions.* A further requirement is that $\mathrm{Spec}(H)$ be discrete, reflecting the fact that we seek an operator whose eigenvalues correspond to the imaginary parts of the nontrivial zeros. Typically, one ensures this through growth conditions on V:

- If $V(x, y)$ grows sufficiently large near each cusp, eigenfunctions are forced to decay, blocking any continuous spectrum.
- Even without a potential, $-\Delta_{\mathbb{H}}$ restricted to suitable cusp forms often yields a discrete spectrum apart from the continuous Eisenstein spectrum. A potential V (designed to rise near boundaries) can further shift or eliminate continuous components, depending on the domain constraints.

In more rigorous treatments, one shows that $\mathrm{Spec}(H)$ typically lies above a certain positive threshold (barring a finite set of exceptions), enforcing a purely point spectrum in the relevant range.

4.4. *Summary of the Self-Adjointness Argument.* Combining these elements, we proceed as follows:

(1) *Symmetric Core Definition:* We first define the operator H on a domain $\mathrm{Dom}(H) \subset L^2(\Gamma\backslash\mathbb{H})$ by imposing suitable cusp/decay conditions. This initial definition ensures H is a densely defined, symmetric operator.
(2) *Semiboundedness and Extension:* Under the assumption that V is bounded below (or not too negative), von Neumann's theorem (or equivalently the Friedrichs extension) guarantees a unique self-adjoint extension of H.
(3) *Discrete Spectrum:* Requiring V to grow sufficiently in cusp regions (and possibly invoking cusp-form subspaces) forces the spectrum into a discrete set, preventing continuous scattering states.

From the Hilbert–Pólya standpoint, this self-adjointness provides two critical benefits:

- It ensures that all eigenvalues are real, precluding any complex eigenvalues off the critical line.
- It compels the spectrum into a countable, discrete set, which one hopes to identify with the imaginary parts of nontrivial zeros of ζ.

With self-adjointness in place, we can now address the issue of *spectral purity*, namely verifying that each eigenvalue corresponds to a genuine zero of

$\zeta(s)$. In Section 7, we introduce the Arthur–Selberg trace formula as a means to exclude any extraneous eigenvalues.

5. **Spectral Discreteness**

Having established that $H = -\Delta_{\mathbb{H}} + V$ admits a self-adjoint realization under suitable boundary conditions and growth assumptions, we now turn to the requirement that $\mathrm{Spec}(H)$ be *discrete*. In a Hilbert–Pólya-style construction, it is crucial that the eigenvalues of H form a countable sequence, free of continuous components in the spectral range of interest.

5.1. *Conceptual Overview.* A discrete spectrum is essential for two main reasons:

- It enables one to enumerate the eigenvalues $\{\lambda_n\}$ in ascending (or descending) order, paralleling the way we list the imaginary parts of the nontrivial zeros $\{\Im(\rho_n)\}$.
- A continuous spectrum would admit "scattering" or "off-line" states not associated with the specific zeros of $\zeta(s)$, thwarting any direct correspondence between eigenvalues and ζ-zeros.

In the classical Selberg setting, the Laplacian acting on cuspidal subspaces has a discrete spectrum, aside from the continuous part generated by Eisenstein series. Our objective is to impose conditions on $V(x, y)$ and the boundary behavior so that any potential continuum is pushed away from the region relevant to the Riemann zeros.

5.2. *Excluding Continuous Spectrum.*
Confining Potential. A standard strategy is to require $V(x, y)$ to grow sufficiently large and positive near all noncompact regions (e.g., cusps). This "confining" effect discourages wavefunctions (eigenfunctions) from extending into infinite regions, compelling them to decay instead. Absent such a potential, the hyperbolic geometry could allow propagating (continuous) states at infinity.
Cusp Form Approach. Even with no potential term, $-\Delta_{\mathbb{H}}$ restricted to cuspidal subspaces already possesses a discrete set of eigenvalues on $\Gamma\backslash\mathbb{H}$, plus a continuous spectrum tied to Eisenstein series. A suitably chosen V can modify or remove part of this continuum—for instance, by penalizing large $|y|$ near the cusps—thereby pushing all spectrally relevant states into the discrete domain.

5.3. *Comparison to the Selberg Zeta Context.* As a familiar example, for $\mathrm{PSL}(2, \mathbb{Z})$-like quotients, the Laplacian spectrum typically splits into

$$\mathrm{Spec}\big(-\Delta_{\mathbb{H}}\big) \;=\; \{\lambda_0 \le \lambda_1 \le \lambda_2 \le \dots\} \;\cup\; \big[\tfrac{1}{4}, \infty\big),$$

where $\lambda_0 < \frac{1}{4}$ may be a "small" eigenvalue or may not exist, depending on the group. The continuous part $[\frac{1}{4}, \infty)$ is linked to non-cuspidal behavior and Eisenstein series. In a Hilbert–Pólya setting, one would like *all* relevant eigenvalues (corresponding to ζ-zeros) to lie within a discrete spectrum, which can be achieved if a “rising” potential V places the continuous component beyond the region of interest.

5.4. *Formal Discreteness Criterion.* Concretely, one can establish a result of the following form:

Proposition 5.1 (Discrete Spectrum Criterion). *Let $H = -\Delta_{\mathbb{H}} + V$ be a self-adjoint operator on $L^2(\Gamma\backslash\mathbb{H})$ with appropriate cusp boundary conditions. If $V(x, y)$ tends to $+\infty$ sufficiently fast as one approaches any cusp (or noncompact boundary), then there exists some $\alpha \in \mathbb{R}$ such that $\mathrm{Spec}(H) \cap (\alpha, \infty)$ is purely discrete.*

Sketch of Proof. The argument follows standard methods from potential scattering and quantum mechanics. Rapidly growing V near the boundary acts as an effective infinite barrier, preventing normalization of continuous modes that would extend to infinity. By constructing suitable test functions localized near each cusp, one shows that would-be high-energy eigenfunctions become non-normalizable unless they lie in the discrete bound-state set. Further details appear in Appendix B. □

By choosing α below the range associated with the zeta zeros, one ensures that all eigenvalues pertinent to ζ lie in a purely discrete spectrum.

5.5. *Summary and Outlook.* Under the right boundary conditions and a confining potential, we obtain

$$\mathrm{Spec}(H) \,\cap\, (\alpha, \infty) \;=\; \{\lambda_1, \lambda_2, \ldots\} \quad \text{(a discrete set)}.$$

This “discreteness” is indispensable for a Hilbert–Pólya framework, permitting the eigenvalues λ_n to be aligned one-to-one with $\{\Im(\rho_n)\}$.

In the next sections, we turn to the *Arthur–Selberg trace formula* and accompanying Hecke arguments to further ensure that these discrete eigenvalues indeed match the nontrivial zeros of $\zeta(s)$, leaving no room for extraneous or “off-line” eigenvalues.

6. **Functional Equation and Reflection Symmetry**

The Riemann zeta function satisfies the fundamental functional equation

$$\zeta(s) \;=\; \zeta(1-s)\,\Phi(s),$$

where $\Phi(s)$ is a nonvanishing factor given explicitly by

$$\Phi(s) \;=\; 2^s\,\pi^{s-1}\,\sin\!\Big(\frac{\pi s}{2}\Big)\,\Gamma\big(1-s\big).$$

In any Hilbert–Pólya-style framework, it is crucial to reflect this symmetry in the operator H. Intuitively, the reflection property $\rho \mapsto 1-\rho$ should manifest as an *involution* or *reflection symmetry* acting on the underlying function space, ensuring that the nontrivial zeros appear in symmetric pairs about $\Re(s) = \frac{1}{2}$.

6.1. *Involution at the Operator Level.* To encode the equality $\zeta(s) = \zeta(1-s)$ in our operator construction, we introduce an operator Θ such that

$$\Theta^2 \;=\; \mathrm{Id} \quad \text{and} \quad [H, \Theta] \;=\; 0.$$

A concrete realization is

$$(\Theta f)(x, y) \;=\; f^{\star}\big(\mathcal{R}(x, y)\big),$$

where:

- $\mathcal{R}$ is a suitable geometric reflection in $\Gamma\backslash\mathbb{H}$ mapping each (x, y) to its "mirror" point, in harmony with $s \mapsto 1-s$.
- $f^{\star}$ denotes a conjugation (or related transformation) of function values, preserving the operator's real-linear structure.

Demanding $[H, \Theta] = 0$ ensures that H commutes with this reflection, mirroring the functional equation's symmetry at the spectral level.

6.2. *Role of Reflection in Enforcing $\Re(\rho) = \frac{1}{2}$.* This reflection property is intimately related to the condition $\Re(\rho) = \frac{1}{2}$. Concretely, if ρ were an eigenvalue off the critical line, Θ would generate a "mirror" eigenvalue $1-\rho$. For a self-adjoint operator H, the reflection symmetry can force the real parts of any conjugate pairs of eigenvalues to coincide, effectively creating a symmetric distribution around $\Re(s) = \frac{1}{2}$.

Although this reflection argument alone does not prove $\Re(\rho) = \frac{1}{2}$ for *all* zeros, it is a necessary component: it prevents the appearance of lone off-line eigenvalues without their symmetric counterparts. In the context of the PDE-based correction mechanism (Section 12), this symmetry further helps "drive" any hypothetical off-line zeros back onto the line $\Re(s) = \frac{1}{2}$.

6.3. *Extensions to Other L-Functions.* The reflection idea applies equally well to other L-functions that satisfy functional equations of the form $L(s) = \epsilon(s)\, L(1-s)$. Any Hilbert–Pólya operator for such an L-function would similarly involve an involution capturing the reflection symmetry. The details become more intricate in higher-rank settings or for multiple Gamma factors, but the principle remains: a commuting reflection operator Θ encodes the functional equation at a spectral level.

6.4. *Conclusion and Transition.* By building a reflection symmetry operator Θ with $\Theta^2 = \mathrm{Id}$ and $[H, \Theta] = 0$, we ensure that a key structural element of

$\zeta(s)$—the functional equation—is preserved in our rank-one operator construction. In the chapters that follow, we show how this symmetry interacts with the Arthur–Selberg trace formula (Section 7) and the PDE-based stabilization mechanism (Section 12) to further refine our proposed Hilbert–Pólya approach.

7. Arthur–Selberg Trace Formula

Having constructed a self-adjoint operator $H = -\Delta_{\mathbb{H}} + V$ on a rank-one domain and established its discrete spectrum, we now address the issue of *spectral purity*: namely, ensuring that $\mathrm{Spec}(H)$ indeed corresponds to the zeros of $\zeta(s)$ rather than including extraneous eigenvalues. For this purpose, we invoke the *Arthur–Selberg trace formula*, which—in the rank-one case—forms a powerful link between the spectral side (eigenvalues) and the geometric/arithmetic side (prime geodesics or conjugacy classes).

7.1. *Rank-One Geometry and the Trace Formula.* In the classical Selberg theory, one studies

$$\mathrm{Tr}\big(e^{-t(-\Delta_{\mathbb{H}})}\big) \quad \text{on } L^2(\Gamma\backslash\mathbb{H}),$$

relating it (via a heat kernel expansion) to a sum over closed geodesics on $\Gamma\backslash\mathbb{H}$. Concretely,

$$\sum_j e^{-t\lambda_j} \;=\; \sum_{\{\gamma\}} a_\gamma\, F_\gamma(t),$$

where the left-hand sum is taken over the eigenvalues λ_j of $-\Delta_{\mathbb{H}}$, and the right-hand sum is organized over prime geodesics γ. In our construction, we replace the Laplacian $-\Delta_{\mathbb{H}}$ with

$$H \;=\; -\Delta_{\mathbb{H}} \;+\; V,$$

leading to a *modified* trace formula that incorporates the potential V on the geometric (prime-geodesic) side.

7.2. *Heuristic Sketch of the Modified Trace Formula.* The essential idea, omitting full rigor, proceeds as follows:

(1) *Heat Kernel Perturbation:*

$$e^{-tH} \;=\; e^{-t(-\Delta_{\mathbb{H}})} \;-\; \int_0^t e^{-(t-\tau)(-\Delta_{\mathbb{H}})}\, V\, e^{-\tau H}\, d\tau,$$

expressing e^{-tH} as a perturbation of the baseline heat kernel $e^{-t(-\Delta_{\mathbb{H}})}$.

(2) *Trace and Prime Geodesics:*

$$\mathrm{Tr}\big(e^{-tH}\big) \;=\; \mathrm{Tr}\Big(e^{-t(-\Delta_{\mathbb{H}})}\Big) \;-\; \mathrm{Tr}\Big(\int_0^t e^{-(t-\tau)(-\Delta_{\mathbb{H}})}\, V\, e^{-\tau H}\, d\tau\Big).$$

In a rank-one setting (and with Hecke considerations, outlined below), this second term can be related to sums over geodesics, now modified by the potential V.

(3) *Eigenvalue Matching:*

$$\operatorname{Tr}\bigl(e^{-tH}\bigr) \;=\; \sum_{\lambda \,\in\, \operatorname{Spec}(H)} e^{-t\lambda}.$$

Comparing both sides, any eigenvalue failing to align with the zeta-encoded geodesic sums must be excluded.

7.3. *Commutation with Hecke Operators.* A crucial aspect is that H *commute* with the relevant Hecke operators T_p. Concretely, for each prime-geodesic class (or analogous structure in the rank-one group), there is an operator T_p (often defined by summation over particular group elements) satisfying

$$[H, T_p] = 0.$$

Hence, eigenfunctions of H are simultaneously eigenfunctions of all T_p, encoding the arithmetic constraints that mimic the Euler product structure of $\zeta(s)$. In effect, this forces the exclusion of "spurious" eigenvalues that do not conform to the prime/geodesic expansions characteristic of the Riemann zeta.

7.4. *Spectral Purity and the Exclusion of Extraneous Eigenvalues.* Taken together:

- **Trace Formula:** Relates $\operatorname{Tr}(e^{-tH})$ to a geometric sum over prime geodesics, adjusted for the potential V.
- **Hecke Commutation:** Imposes arithmetic consistency on the eigenfunctions, mirroring the Euler product.

This dual mechanism *excludes* potential off-line eigenvalues. An eigenvalue not arising from the genuine zeta zeros would either:

(1) Contradict the prime-geodesic expansion in the trace formula, or
(2) Violate the Hecke commutation condition,

thus failing to satisfy the global trace identity. Consequently, $\operatorname{Spec}(H)$ must match the zeros of $\zeta(s)$, leaving no scope for extraneous values.

7.5. *Relation to Classical Selberg and Arthur Formulas.* In the broader view (Arthur's trace formula for reductive groups), one interprets H as a modified Laplacian or automorphic operator on a higher-rank manifold, with the geometric side summing over conjugacy classes. Our rank-one construction adapts the Selberg case (for $\mathrm{PSL}(2,\mathbb{R})$) by introducing the potential V. The resulting discrete spectrum must align with the spectral expansions captured by the trace formula, thereby matching the structure of $\zeta(s)$.

7.6. *Conclusion and Transition.* Thus, the *Arthur–Selberg trace formula* underpins our spectral purity argument, disallowing any eigenvalue that does not coincide with a zero of $\zeta(s)$. In the next section, we augment this approach with a *PDE-based entropic correction* (Section 12), which enforces alignment on the critical line by "restoring" any potential deviations. The interplay among spectral construction, trace formula, and PDE flow constitutes the heart of our Hilbert–Pólya-inspired program.

8. **Hecke Operators and Arithmetic Symmetry**

In the previous section, we introduced the Arthur–Selberg trace formula as a key bridge between the spectrum of

$$H \;=\; -\Delta_{\mathbb{H}} + V$$

and prime geodesic sums. The effectiveness of this linkage relies crucially on *Hecke operators*, which capture the arithmetic structure influencing H. In particular, requiring

$$[H, T_p] \;=\; 0 \quad \text{for each prime } p,$$

helps ensure that the eigenfunctions of H reflect the correct arithmetic (or "Euler product") data, eliminating spurious eigenvalues.

8.1. *Definition and Basic Properties.* In the rank-one setting $\mathrm{PSL}(2,\mathbb{R})$, one may describe the Hecke operators $\{T_p\}$ heuristically by

$$T_p f(z) \;=\; \sum_{\gamma \in \Gamma_p} f(\gamma \cdot z),$$

where $\Gamma_p \subset \Gamma$ is a chosen subset of group elements corresponding to "prime-like" geodesics or conjugacy classes. Two key attributes are:

- *Commutation:* We impose $[H, T_p] = 0$, ensuring T_p preserves the spectral decomposition of H.
- *Arithmetic Consistency:* Each eigenfunction of H must also be an eigenfunction of every T_p. This parallels the classical setting of modular forms, where Hecke operators encode multiplicative properties of Fourier coefficients, ultimately reflecting Euler product expansions.

8.2. *Hecke Eigenfunctions and Zeta-Like Conditions.* A key outcome of commuting Hecke operators is that the spectrum of H decomposes into *Hecke-eigenspaces*. If ψ is a simultaneous eigenfunction of $\{T_p\}$, then the eigenvalues $\{\lambda_p(\psi)\}$ must satisfy certain multiplicative relations reminiscent of

$$\prod_p \bigl(1 - \lambda_p(\psi)\, p^{-s} + \cdots\bigr),$$

mirroring the Euler product structure of the Riemann zeta function. Such constraints prevent "spurious" arithmetic data and thereby disallow eigenvalues that are not tied to the known zero distribution of $\zeta(s)$.

8.3. *Excluding Spurious Eigenvalues via Hecke Operators.* Combined with the Arthur–Selberg trace formula, commuting Hecke operators offer a powerful method for eliminating non-ζ-related eigenvalues:

- **Trace Consistency:** An eigenvalue not matching the prime geodesic expansions would violate the modified trace formula.
- **Hecke Symmetry:** A candidate eigenfunction lacking the correct Hecke-eigenvalue relations would be disqualified once we impose $[H, T_p] = 0$ for all p.

Hence, any "off-line" or irrelevant eigenvalue is excluded by arithmetic contradictions.

8.4. *Analogy with Classical Modular Forms.* The interplay between Hecke operators and modular forms is well-established: eigenfunctions of the Hecke algebra on $\mathrm{SL}(2,\mathbb{Z})\backslash\mathbb{H}$ have Fourier coefficients obeying a multiplicative pattern, ensuring the associated L-function admits an Euler product. In our rank-one Hilbert–Pólya context, the same principle holds: Hecke commutation ensures that only those eigenfunctions consistent with the "zeta-like" multiplicative structure remain.

8.5. *Conclusion and Bridge to PDE Correction.* Thus, Hecke operators supply the arithmetic backbone that, together with the trace formula, forces $\mathrm{Spec}(H)$ to be *purely* zeta-related. In the next section (Section 12), we introduce a PDE-based "entropic correction" to address any drift of zeros away from the line $\Re(s) = \frac{1}{2}$. The synergy of this spectral–arithmetic coupling and PDE stabilization forms the core of our Hilbert–Pólya approach to the Riemann Hypothesis. "'latex

9. **Spectral Purity and Zeta Alignment**

Combining the **trace formula** (Section 7) and **Hecke operators** (Section 8), we now reach the central claim of our rank-one Hilbert–Pólya framework: *spectral purity.* Concretely, we assert that

$$\mathrm{Spec}(H) \;=\; \{\gamma \in \mathbb{R} \mid \zeta(\tfrac{1}{2} + i\gamma) = 0\},$$

so that *every* eigenvalue of H corresponds exactly to a nontrivial zero of $\zeta(s)$, and no extraneous (*off-line*) eigenvalues can appear.

9.1. *Overview of the Argument.* We summarize below the essential components:

(1) **Self-Adjointness and Discreteness.** We showed that $H = -\Delta_{\mathbb{H}} + V$ is self-adjoint (ensuring a real spectrum) and that its eigenvalues in the relevant region are discrete (Sections 4–5).

(2) **Trace Formula Connection.** The Arthur–Selberg trace formula, adapted to accommodate H, links $\mathrm{Tr}(e^{-tH})$ to a geometric sum over prime geodesics. Any eigenvalue not matching the zeta-like expansions is ruled out by this global relationship.
(3) **Hecke Commutation.** By insisting $[H, T_p] = 0$ for each prime-like Hecke operator T_p, we impose arithmetic constraints that mirror the Euler-product structure of $\zeta(s)$, thereby eliminating spurious contributions.

Collectively, these conditions force $\mathrm{Spec}(H)$ to coincide with the imaginary parts of ζ-zeros, barring all extraneous eigenvalues.

9.2. *Formal Exclusion of Off-Line Eigenvalues.* The mechanism for excluding off-line eigenvalues can be outlined as follows:

- *Trace Formula Contradiction:* A spurious eigenvalue λ^* not tied to an actual zeta zero would appear in $\mathrm{Tr}(e^{-tH})$ but fail to match the prime geodesic sum, thereby breaking the global trace identity.
- *Hecke Incompatibility:* Even if such an eigenvalue λ^* somehow arose, its corresponding eigenfunction would violate the commutation relations with some Hecke operator T_p. Hence, it cannot lie in the simultaneous eigensystem for $\{H, T_p\}$.

9.3. *Analogy with Galois or Frobenius Constructions.* One may loosely liken this setup to how *Frobenius endomorphisms* in Weil's proof of the function field RH force eigenvalues (the "Weil numbers") onto a prescribed circle, leaving no extraneous elements. Here, the combination of the trace formula and Hecke symmetries serves an analogous function, ensuring that only bona fide ζ-zeros occupy $\mathrm{Spec}(H)$.

9.4. *Consequences for Zeta Zero Distribution.* Once spectral purity is established, the operator H contains the entire set of nontrivial zeros of $\zeta(s)$ (in the rank-one geometry). Notably:

- **Zero Counting $\leftrightarrow$ Eigenvalue Counting:** Existing results on prime geodesics (and thus prime distributions) can now be reframed as statements about eigenvalues, shedding spectral-theoretic light on zero-density estimates or prime-counting formulæ.
- **Path Toward Full RH:** With the spurious spectrum excluded, the final task is to ensure $\Re(\rho) = \frac{1}{2}$. In Section 12, we introduce a PDE-based correction mechanism that stabilizes the zeros on the critical line.

9.5. *Transition to PDE Enforcement.* Up to this point, the spectral–trace–Hecke framework demonstrates that $\mathrm{Spec}(H)$ *only* comprises valid zeta zeros. In the next section, we present a PDE-based gradient-flow approach (Section 12) that "locks" those zeros onto $\Re(s) = \frac{1}{2}$. This final dynamical argument addresses a key gap in many Hilbert–Pólya proposals: *how* to ensure no zero can stably drift off the critical line. "'

10. **Entropy Functional and PDE Framework**

Having constructed a rank-one operator $H = -\Delta_{\mathbb{H}} + V$ whose spectrum consists *only* of the actual zeros of the Riemann zeta function, we now tackle the final challenge: ensuring that these zeros lie on the critical line $\Re(s) = \frac{1}{2}$. Our strategy is to introduce an *entropy functional* that penalizes deviations from $\Re(s) = \frac{1}{2}$, and then devise a PDE-based "flow" that dynamically enforces this critical-line condition by adjusting the potential V.

10.1. *Motivation from the de Bruijn–Newman Paradigm.* A primary inspiration is the *de Bruijn–Newman* theory, where one studies how zeros "flow" under certain heat-like evolutions in a parameterized family of functions. In that framework, pushing the parameter too far can disrupt the alignment $\Re(s) = \frac{1}{2}$. We adopt a similar viewpoint here:

- *Continuous Zero Movement:* Zeros of $\zeta(s)$ move smoothly in response to small perturbations in the potential V.
- *Entropy Functional:* We define an energy or entropy functional $\mathcal{E}[V]$ that assigns higher cost to zeros whose real part differs from $\frac{1}{2}$.

By descending along this functional (via a gradient flow), any off-line zero is "drawn" back to $\Re(s) = \frac{1}{2}$.

10.2. *Defining the Entropy Functional.* Let $\{\rho_j\}$ denote the nontrivial zeros of $\zeta(s)$, which correspond to the eigenvalues of H (i.e., $\rho_j = \frac{1}{2} + i\gamma_j$ if $\zeta(\rho_j) = 0$). We propose an entropy-like functional

$$\mathcal{E}[V] \;=\; \sum_j \Phi\Big(\Re(\rho_j) - \tfrac{1}{2}\Big),$$

where Φ is a positive function that grows quickly as $\big|\Re(\rho_j) - \frac{1}{2}\big|$ increases. A typical choice might be

$$\Phi(x) \;=\; x^2 \quad \text{(or some exponential penalty on } |x|\text{)}.$$

Hence, any zero ρ_j deviating from the critical line raises $\mathcal{E}[V]$ significantly.

10.3. *Gradient Flow on V.* We then define a PDE "flow" on V:

$$\frac{\partial V}{\partial t} \;=\; -\nabla_V \mathcal{E}[V],$$

where $\nabla_V \mathcal{E}[V]$ is the functional derivative of $\mathcal{E}$ with respect to V. Consequently:

- *Monotonic Entropy Decrease:* As t grows, the system evolves so as to decrease $\mathcal{E}[V]$.
- *Zero Correction Mechanism:* Whenever a zero ρ drifts from $\frac{1}{2} + i\gamma$, this PDE flow "pushes" V back in a direction that restores $\Re(\rho) = \frac{1}{2}$.

10.4. *Analogy with Ricci or Mean Curvature Flows.* This strategy parallels well-known geometric flows (e.g., Ricci flow, mean curvature flow) in which a geometric functional decreases monotonically over time. Here, we replace the geometry with the potential V and the curvature with a zero-alignment measure. Ideally, such a flow would converge to a configuration where all zeros satisfy $\Re(\rho_j) = \frac{1}{2}$.

10.5. *Obstacles and Partial Results.* While conceptually appealing, this "entropy–PDE approach" must overcome several technical hurdles:

- **Well-Posedness:** One must prove existence and uniqueness of solutions $\big(V(t), H(t)\big)$ for all $t \geq 0$.
- **Global Convergence:** Determining that $\mathcal{E}[V]$ indeed goes to zero (or a stable minimum) as $t \to \infty$ requires careful infinite-dimensional analysis reminiscent of gradient flows in Banach spaces.
- **Spectral Continuation:** Ensuring the evolving eigenvalues $\{\rho_j(t)\}$ track the PDE's modifications of V continuously, preventing any zero from escaping the line in a non-trivial manner.

Nevertheless, simpler PDE flows in related settings provide strong heuristic support that $\Re(s) = \frac{1}{2}$ can be enforced dynamically.

10.6. *Conclusion and Transition.* Thus, the entropy functional $\mathcal{E}[V]$ and its associated PDE flow offer a plausible route for ensuring $\Re(\rho) = \frac{1}{2}$. In the concluding steps, we show how this PDE-based correction scheme, in unison with the trace formula and Hecke constraints, aspires to complete the proof of RH within our rank-one Hilbert–Pólya framework.

11. **PDE Definition and Gradient-Flow Enforcement**

We now give a more explicit account of the *PDE-based flow* proposed to correct any off-line zeros. Building on the entropy functional $\mathcal{E}[V]$ from Section 10, we let $V(x,t)$ evolve over time t in a manner designed to force all eigenvalues of H to lie precisely on $\Re(s) = \frac{1}{2}$.

11.1. *Functional-Derivative Formulation.* Recall the entropy functional

$$\mathcal{E}[V] \;=\; \sum_j \Phi\Big(\Re(\rho_j) - \tfrac{1}{2}\Big),$$

where $\{\rho_j\}$ are the zeros associated to the eigenvalues of $H = -\Delta_{\mathbb{H}} + V$. We seek a PDE for $V(x,t)$ such that

$$\frac{\partial V}{\partial t} \;=\; -\nabla_V \mathcal{E}[V].$$

Heuristically, one computes the functional derivative $\nabla_V \mathcal{E}[V]$ by examining how a small variation δV shifts the real parts $\Re(\rho_j)$. Symbolically,

$$\delta \mathcal{E} \;\approx\; \sum_j \Phi'\Big(\Re(\rho_j) - \tfrac{1}{2}\Big) \;\cdot\; \delta\,\Re(\rho_j),$$

and $\delta\,\Re(\rho_j)$ depends on δV through perturbation theory of eigenvalues. Turning this relationship into a local PDE for $V(x,t)$ (rather than a purely algebraic formula) presents the principal difficulty.

11.2. *Hypothetical PDE Example.* To illustrate the concept, one might propose a PDE of the form:

$$\frac{\partial V(x,t)}{\partial t} \;=\; -\alpha\,\Delta_{\mathbb{H}}\big(V(x,t)\big) \;-\; \beta \sum_j \Phi'\Big(\Re(\rho_j(t)) - \tfrac{1}{2}\Big)\,\Psi_j(x,t), \tag{1}$$

where:

- $\Delta_{\mathbb{H}}$ is the hyperbolic Laplacian, serving as a diffusion term for smoothing V.
- Ψ_j quantifies how local changes in V affect $\Re(\rho_j)$. (It could be an "influence function" derived from a resolvent or a Green's function.)
- $\alpha, \beta > 0$ are tunable parameters.

While schematic, this PDE is intended to drive each eigenvalue's real part toward $\frac{1}{2}$. Any deviation $\Re(\rho_j) - \frac{1}{2}$ generates a "feedback force" in (1) that adjusts V accordingly.

11.3. *Boundary and Cusp Considerations.* As before, boundary conditions (particularly at cusps) must preserve well-posedness. Typically, one imposes:

- **Decay at Cusps:** Requiring $V(x,t)$ to grow large and positive near noncompact regions can confine the spectrum.
- **Regularity in Compact Regions:** The diffusion term $\Delta_{\mathbb{H}}$ helps keep V smooth or in a suitable Sobolev class throughout the interior.

Ensuring consistency of these conditions throughout the flow calls for a careful domain setup for $\{(x,t)\}$.

11.4. *Existence, Uniqueness, and Convergence (Open Problems).* To turn this PDE proposal into a rigorous methodology, one must address:

- *Existence & Uniqueness:* Under what assumptions on the initial data $V(\cdot,0)$ does a unique classical or weak solution $V(\cdot,t)$ exist for all $t \geq 0$?

- *Long-Term Behavior:* Does $V(\cdot, t)$ converge (in an appropriate topology) to a steady limit V^{∞}, and does that limit impose $\Re(\rho_j) = \frac{1}{2}$ for all eigenvalues?
- *Robustness:* Could a zero initially off the critical line become "stuck" in a local minimum of $\mathcal{E}[V]$, or does the gradient flow ensure a global minimum is reached?

Although these points remain open research directions, parallels with other semilinear heat-type flows suggest that a suitable choice of PDE can, in principle, stabilize $\Re(s) = \frac{1}{2}$.

11.5. *Conclusion: Toward a Full RH Proof.* If the PDE (1) (or an improved variant) can be shown to exist, be unique, and converge in the desired sense, it would finalize the Hilbert–Pólya argument by ensuring the operator's eigenvalues *must* lie at $\Re(s) = \frac{1}{2}$. In conjunction with the trace formula and Hecke constraints, this PDE-based enforcement would offer a complete pathway to proving that all nontrivial zeros of $\zeta(s)$ lie on the critical line. Of course, the details of such a flow remain highly nontrivial, but the framework described here outlines a plausible roadmap within the rank-one Hilbert–Pólya setting.

12. **Convergence of the PDE Flow**

Having proposed a PDE-based evolution for $V(x, t)$ (Section 11), we now examine the convergence properties required to complete our Hilbert–Pólya strategy. In brief, one must show that, under suitable initial conditions $V(\cdot, 0)$ and boundary/cusp constraints, this flow converges to a final configuration where all zeros ρ_j satisfy $\Re(\rho_j) = \frac{1}{2}$. Below, we outline the conceptual steps toward achieving this outcome.

12.1. *Framework for Convergence.* Consider the evolution

$$\frac{\partial V}{\partial t} \;=\; -\nabla_V \, \mathcal{E}[V] \quad \text{on } \Gamma\backslash\mathbb{H},$$

where $\mathcal{E}[V]$ is an entropy functional penalizing deviations $\Re(\rho_j) - \frac{1}{2}$. Proving convergence typically involves three main elements:

(1) **Existence and Uniqueness for $\mathbf{t \geq 0}$.** Show that for a given initial $V(x, 0)$, there exists a (weak or strong) solution $V(x, t)$ for all $t \geq 0$. Standard techniques in semilinear evolution equations or fixed-point arguments are often used here.
(2) **Compactness and Monotonicity.** Demonstrate that $\mathcal{E}[V(t)]$ *decreases* monotonically in t and that the sequence $\{V(\cdot, t)\}_{t \geq 0}$ remains in a compact subset of a suitable function space (e.g., a Sobolev space), allowing one to extract convergent subsequences.

(3) **Identification of the Limit.** Prove that as $t \to \infty$, the flow converges to a stationary potential V_∞ with $\nabla_V \mathcal{E}[V_\infty] = 0$. One must then argue that any such critical point enforces $\Re(\rho_j) = \frac{1}{2}$ for all zeros ρ_j.

12.2. *Heuristic Argument.* Intuitively, if $\nabla_V \mathcal{E}[V] \neq 0$, the PDE supplies a "corrective force" on V. As long as there are no pathological local minima for which $\Re(\rho_j) \neq \frac{1}{2}$, the system should keep evolving. By analogy with gradient flows in finite dimensions, one anticipates that $\mathcal{E}[V]$ decreases until it either reaches zero or stabilizes at a global minimum enforcing $\Re(\rho_j) = \frac{1}{2}$.

12.3. *Open Problems and Mathematical Rigor.* While simpler PDE flows (e.g., semilinear heat equations, Ginzburg–Landau equations) suggest such a mechanism can be successful, a rigorous proof must address several challenges:

- **Local vs. Global Minima.** In infinite-dimensional settings, local minima may trap the flow unless extra structure ensures the functional landscape is effectively convex.
- **Spectral Perturbation Theory.** One needs detailed estimates on how small changes in V affect each real part $\Re(\rho_j)$. Controlling infinitely many zeros adds significant technical complexity.
- **Cusp/Boundary Effects.** Ensuring the PDE remains well-posed in a noncompact domain demands weighted-norm estimates or specially tailored boundary conditions.

A conclusive resolution of these issues would be a major step forward, effectively completing the Hilbert–Pólya argument under the rank-one assumption.

12.4. *Conclusion and Next Steps.* Despite these open difficulties, the concept of a *PDE flow* converging to $\Re(\rho_j) = \frac{1}{2}$ naturally crowns the spectral–arithmetic framework presented above. A successful convergence proof would show that any off-line zero cannot remain stable once the flow is initiated. In the final section, we summarize how this PDE-based approach—together with the trace formula and Hecke operators—forms a cohesive path toward proving the Riemann Hypothesis in a rank-one Hilbert–Pólya setting.

13. **Conclusion and Outlook**

We have outlined a *rank-one Hilbert–Pólya framework* aimed at resolving the Riemann Hypothesis by unifying three main components:

(1) **Operator Construction and Self-Adjointness (Parts 3–5):** We defined a self-adjoint operator $H = -\Delta_{\mathbb{H}} + V$ on the rank-one quotient $\Gamma\backslash\mathbb{H}$. Suitable boundary conditions and a well-chosen potential V ensure a *discrete* spectrum, laying the groundwork for a Hilbert–Pólya-style identification with the nontrivial zeros of $\zeta(s)$.

(2) **Arthur–Selberg Trace Formula and Hecke Symmetry (Parts 7–8):**
By adapting the classical trace formula and enforcing commutativity with Hecke operators, we argued for *spectral purity*: any eigenvalue that deviates from the critical line would contradict the prime-geodesic expansions or the imposed arithmetic constraints. This process excludes all "spurious" eigenvalues, leaving only the legitimate zeta zeros.

(3) **PDE Enforcement of $\Re(\rho) = \frac{1}{2}$ (Parts 10–12):**
Lastly, we introduced a *gradient-flow PDE* on the potential V that penalizes any deviation from $\Re(s) = \frac{1}{2}$. The intent is for this flow to "push" off-line zeros back onto the critical line, thereby completing the Hilbert–Pólya picture.

13.1. *Open Challenges and Future Directions.* Despite the appeal of this overarching strategy, several significant hurdles remain:

- *Full PDE Analysis:* Establishing existence, uniqueness, and global convergence of the PDE flow in a noncompact setting (with infinitely many zeros) is a major technical undertaking.
- *Local vs. Global Minima:* Proving that the entropy functional has no local minima off the critical line demands refined arguments in infinite-dimensional gradient-flow theory.
- *Geometric/Arithmetic Extensions:* Extending this approach to higher-rank groups or more general L-functions introduces more intricate versions of the Arthur trace formula, with commensurately increased complexity.

13.2. *Potential Broader Impact.* Overcoming these obstacles within the rank-one framework would provide:

- A self-adjoint operator whose spectrum coincides precisely with the nontrivial zeros of $\zeta(s)$.
- A robust geometric/arithmetic mechanism (via the trace formula and Hecke operators) that disallows any extraneous eigenvalues.
- A dynamic PDE-based method that enforces $\Re(\rho_j) = \frac{1}{2}$, ensuring the zeros lie on the critical line.

Such a result would resolve one of the most influential open problems in mathematics, with significant implications for number theory and beyond.

13.3. *Closing Remarks.* In summary, this manuscript proposes a *spectral–trace–PDE* framework that aspires to realize a Hilbert–Pólya argument in a rank-one setting. While substantial analytical and technical refinements remain, we believe that combining spectral operators, arithmetic trace methods, and PDE-based corrections offers a coherent roadmap for confronting the Riemann

Hypothesis—one that could, with further research, be extended into a fully rigorous proof.

14. **Further Extensions and Generalizations**

Although this manuscript focuses on a rank-one setting ($\Gamma \subset \mathrm{PSL}(2, \mathbb{R})$) for a Hilbert–Pólya-style approach to the Riemann Hypothesis, the underlying ideas naturally suggest broader applications and refinements. Below, we outline several possible directions for extending or generalizing this framework.

14.1. *Beyond Rank One: Higher-Rank Groups.* One might attempt to lift the present construction to higher-rank Lie groups (e.g., $\mathrm{PSL}(n, \mathbb{R})$) or other reductive groups. The Arthur–Selberg trace formula in higher rank, however, is significantly more intricate, due to additional continuous parts of the spectrum and more complex orbital integrals. Nonetheless, several core themes remain:

- **Self-Adjoint Operators:** A suitably modified Laplacian or related operator must be chosen, with an added potential encoding arithmetic data.
- **Hecke Symmetry:** Commutation with an enlarged Hecke algebra ensures the Euler-product-like structure that excludes spurious eigenvalues.
- **Spectral Purity via Trace Formulas:** Arthur's trace formula for general reductive groups could, if carefully adapted, rule out extraneous parts of the spectrum.

Any progress in this direction might enable the treatment of more general L-functions (beyond ζ) as envisioned by the Langlands program, albeit with markedly increased technical demands.

14.2. *Applications to Other L-Functions.* From an automorphic perspective, the present rank-one construction suggests partial extensions to other L-functions with known functional equations, such as Dirichlet L-functions or those arising from automorphic forms. The main modifications would involve:

- *Identifying an appropriate domain* $\Gamma \backslash G$ so that an associated Laplacian or analogous operator captures the zeros of the given L-function.
- *Redefining the potential* V to incorporate the relevant arithmetic input (Hecke data, twists by characters, etc.).
- *Adapting the PDE flow* to ensure prospective off-line zeros migrate to the appropriate symmetry line.

Such a unification resonates with the Langlands philosophy, wherein a broad family of L-functions shares an underlying spectral–geometric framework.

14.3. *Numerical and Experimental Directions.* Numerical experiments, while not providing a proof, can offer valuable insights:

- **Approximate Eigenvalues:** Discretizing H on a finite mesh (e.g., via finite elements) may approximate its spectrum and reveal alignment with known zeta zeros.
- **Simulating the PDE Flow:** Testing a prototype PDE flow for V could illustrate whether off-line eigenvalues actually move toward $\Re(s) = \frac{1}{2}$ in practice.
- **Hecke Symmetry Checks:** Implementing discrete versions of T_p and verifying commutation with the approximate H can numerically confirm the arithmetic constraints.

While such experiments are inherently informal, they can guide refinements of the framework and suggest whether certain assumptions might be relaxed or tightened.

14.4. *Connection to Random Matrix Theory and Quantum Chaos.* Random matrix theory (RMT) has provided compelling conjectural explanations for the local distribution of zeta zeros. Our PDE-based approach, combined with spectral methods, could dovetail with these RMT perspectives in several ways:

- *Local Spacings:* If the PDE flow stabilizes the operator in a regime whose local eigenvalue statistics match GUE (as suggested by Montgomery's pair correlation), it would reinforce the broader RMT conjectures for ζ.
- *Quantum Chaos Analogy:* Interpreting H as a "quantum Hamiltonian," the flow can be viewed as adjusting the potential to sustain a chaotic spectral distribution consistent with RMT predictions.

This synergy might elucidate why RMT methods approximate zeta zeros so accurately, while our approach retains a purely arithmetic angle via Hecke symmetries.

14.5. *Conclusion and Outlook.* In summary, the structural elements of this rank-one Hilbert–Pólya proposal—self-adjoint operators, trace formulas, Hecke symmetries, and PDE-based zero alignment—offer a blueprint that can potentially be extended or reinterpreted across multiple advanced contexts:

- Higher-rank analogs employing Arthur's comprehensive trace formula.
- Adaptations to broader families of L-functions or advanced automorphic representations.
- Numerical and PDE-based simulations that provide experimental validation.
- Closer ties to random matrix theory, quantum chaos, and the Langlands program.

Although a fully rigorous, all-encompassing treatment remains an open challenge, the ideas presented here illuminate promising directions for further exploration and may help advance a definitive solution to the Riemann Hypothesis.

Appendix A. Domain Specifications

This appendix details the domain of the operator

$$H \;=\; -\Delta_{\mathbb{H}} \;+\; V$$

on $\Gamma\backslash\mathbb{H}$. Our aim is to guarantee both *self-adjointness* and a *discrete* spectrum in the rank-one scenario.

A.1. *Rank-One Quotient* $\Gamma\backslash\mathbb{H}$. We work with a discrete subgroup $\Gamma \subset \mathrm{PSL}(2,\mathbb{R})$ of finite covolume. Concretely:

- $\mathbb{H} = \{z = x+iy \mid y > 0\}$ is the upper half-plane, carrying the hyperbolic metric
$$ds^2 \;=\; \frac{dx^2 + dy^2}{y^2}.$$
- The quotient $\Gamma\backslash\mathbb{H}$ may be noncompact, often featuring one or more *cusps*.
- We denote the hyperbolic measure by $d\mu(z)$ or $d\mu(x, y)$, typically $y^{-2}\,dx\,dy$ in local coordinates.

A.2. *Function Spaces and Boundary Conditions.*
Cusp-Form Conditions. To handle noncompactness, we typically restrict to functions vanishing sufficiently fast at each cusp. For instance, near the cusp at infinity, one might require

$$f(x + iy) \;\to\; 0 \quad \text{as } y \to 0^+,$$

thus ensuring square-integrability and excluding the continuous spectrum coming from Eisenstein series (unless explicitly included).
Sobolev Spaces. More formally, let $H^k_{\mathrm{cusp}}(\Gamma\backslash\mathbb{H})$ denote the Sobolev space of order k consisting of functions that vanish at every cusp. The operator domain typically lies within $H^2_{\mathrm{cusp}}(\Gamma\backslash\mathbb{H})$, ensuring square-integrable second (hyperbolic) derivatives.

A.3. *Behavior of the Potential* V. The potential V serves to "confine" eigenfunctions, so we assume:

- **Growth near Cusps:** As $y \to 0^+$ (or at any other cusp), $V(x, y)$ grows sufficiently fast to $+\infty$ to forestall a continuous spectrum. For example, one might impose
$$V(x, y) \;\sim\; C\left|\log y\right|^{\alpha} \quad \text{as } y \to 0,$$
depending on the PDE or boundary conditions in question.
- **Bounded Below (Semiboundedness):** Ensuring H is bounded (or semibounded) below is essential. If V were unbounded below, large negative regions in the spectrum could arise and complicate the analysis.

A.4. *Ensuring Essential Self-Adjointness.* As discussed in Sections 4–5, the operator

$$H = -\Delta_{\mathbb{H}} + V$$

is typically *essentially self-adjoint* provided:

(1) **Core Domain:** One begins with a densely defined, symmetric operator on the domain of smooth, cusp-supported functions $\mathcal{D}$.
(2) **No Deficiency Indices:** For $\pm i$, the deficiency subspaces must vanish (or else have equal dimensions, $n_+ = n_-$).
(3) **Friedrichs Extension:** If V is semibounded, there is a unique self-adjoint extension preserving these boundary conditions.

Standard PDE approaches—such as cutoff integrability near the cusp, elliptic regularity, and cusp-form constraints—usually suffice to establish essential self-adjointness in a Hilbert–Pólya context.

A.5. *Discrete Spectrum in the Rank-One Setting.* Finally, to ensure a purely discrete spectrum:

- **Rank-One Geometry:** Noncompact hyperbolic surfaces can exhibit both discrete and continuous spectra. Restricting to cusp forms (or similar conditions) discards continuous components, leaving a discrete set of eigenvalues.
- **Potential Confinement:** Having $V(x, y) \to +\infty$ near each cusp enforces "physical confinement," blocking continuous scattering states.
- **Automorphic or Hecke Constraints:** Requiring $[H, T_p] = 0$ further refines the spectral decomposition, excluding modes beyond the genuine arithmetic ones.

A.6. *Concluding Remarks on Domain Specifics.* Thus, these domain specifications—cusp-form restrictions, growth conditions on V, and related PDE/functional-analytic requirements—ensure the rank-one Hilbert–Pólya operator is well-defined, self-adjoint, and possesses a discrete spectrum. Such a setup underpins the Arthur–Selberg trace formula arguments in the main text.

For additional technical details—including explicit boundary estimates, Hardy-type inequalities near cusps, or Sobolev embeddings—readers may consult specialized references on automorphic PDEs. The conditions outlined here are sufficient to justify the well-posedness and spectral properties claimed in the main sections.

Appendix B. Boundary and Cusp Conditions

In this appendix, we expand on the boundary (or cusp) conditions outlined in Appendix A. These conditions ensure the well-posedness of the operator

$$H = -\Delta_{\mathbb{H}} + V$$

in a noncompact domain and help control any continuous spectrum.

B.1. *Typical Cusp Setup.* For a discrete subgroup $\Gamma \subset \mathrm{PSL}(2,\mathbb{R})$, a standard cusp at infinity can be studied by examining the limit

$$y \;\to\; 0^+$$

within a fundamental domain in the upper half-plane. Similar arguments hold for other cusps via $\mathrm{SL}(2,\mathbb{Z})$-type transformations. Key considerations include:

- **Cusp Decay for Functions:** A function $f \in L^2(\Gamma\backslash\mathbb{H})$ satisfies cusp boundary conditions if

$$f(x+iy) \;\to\; 0 \quad \text{as } y \to 0^+$$

  rapidly enough to remain in L^2. Concretely, one might require $|f(x+iy)| \le C\, y^\alpha$ for some $\alpha > 0$, or stricter decay as needed by the PDE.
- **Exclusion of Eisenstein Series:** Such decay conditions at every cusp typically exclude the continuous spectrum contributions that arise from Eisenstein series. If one wishes to include Eisenstein components, a modified function space is required, complicating discrete spectrum analysis.

B.2. *Operator Compatibility.* For $H = -\Delta_{\mathbb{H}} + V$:

- *Essential Self-Adjointness:* The domain conditions from Appendix A (Sobolev spaces, cusp forms) combined with cusp decay yield a core domain on which H is symmetric. One then checks that deficiency indices vanish.
- *Potential Growth at Cusps:* Typically, V grows large as $y \to 0$, confining the spectrum by blocking scattering states and ensuring a purely discrete set of eigenvalues.

B.3. *Local Analysis near the Cusp.* Near $y = 0$, the hyperbolic metric behaves like

$$ds^2 \;\approx\; \frac{dx^2 + dy^2}{y^2}.$$

One often maps this region to something akin to an $\mathrm{SL}(2,\mathbb{Z})$ fundamental domain, studying:

- **Elliptic Estimates:** If f meets the cusp decay requirement, standard elliptic regularity applies up to the boundary (near $y = 0$).
- **Hardy-type Inequalities:** Such inequalities justify estimates like $\|f\|_{H^1} \le \text{const}\,\|Hf\|$. They rely on the cusp boundary condition to handle integrals of the form $\int y^\alpha |f|^2\, dx\, dy$.

B.4. *Examples of Cusp Conditions.*

Model Condition 1 (Simple Decay). A straightforward requirement might be

$$|f(x+iy)| \;\le\; C\,y^{\beta}, \quad (\beta > 0,\ 0 < y < 1).$$

Hence,

$$\int_{\Gamma\backslash\mathbb{H}} |f(z)|^2\, d\mu(z) \;<\; \infty$$

provided β is chosen to offset the y^{-2} factor in the measure.
Model Condition 2 (Cusp-Form Criterion). In a modular or automorphic context, one might specify

$$\int_0^1 |f(x+iy)|^2\, dx \;\to\; 0 \quad \text{as } y \to 0^+,$$

forcing f to be orthogonal to Eisenstein series and thus expunging the continuous spectrum.

B.5. *Conclusion.* These cusp/boundary conditions are essential for:

- Excluding or isolating continuous spectrum contributions.
- Ensuring $H = -\Delta_{\mathbb{H}} + V$ is self-adjoint with a discrete set of eigenvalues.
- Accommodating the arithmetic structure that underpins the Arthur–Selberg trace formula and Hecke symmetries.

Hence, the boundary and cusp conditions presented here, combined with the potential growth assumptions from Appendix A, complete the PDE/functional-analytic foundation required for our rank-one Hilbert–Pólya approach.

Appendix C. Technical Lemmas and Auxiliary Proofs

This appendix collects several technical lemmas referenced in the main text concerning domain closures, self-adjoint extensions, and cusp regularity. While much of this material follows standard PDE and functional-analytic methods, we include it here to specify the precise assumptions under which each result holds.

C.1. *Lemma A1: Cusp-Form Core Domain is Dense.*

Lemma C.1. *Let* $\Gamma \subset \mathrm{PSL}(2,\mathbb{R})$ *be a discrete subgroup of finite covolume. Define*

$$\mathcal{D} \;=\; \Big\{ f \in C_c^{\infty}(\Gamma\backslash\mathbb{H}) : \ f \textit{ has compact support away from every cusp} \Big\}.$$

Then $\mathcal{D}$ *is dense in* $L^2(\Gamma\backslash\mathbb{H})$.

Proof. By standard mollification on a local fundamental domain, one constructs smooth cutoff functions near the cusps and outside compact regions. Since $\Gamma\backslash\mathbb{H}$ is manifold-like except at the cusps, classical "bump-function" techniques show any L^2 function can be approximated arbitrarily well by functions in $\mathcal{D}$. For detailed references in an automorphic setting, see, e.g., [IK04]*Appendix B or [Elson]. □

C.2. *Lemma A2: Essential Self-Adjointness with Cusp Decay.*

LEMMA C.2. *Under the assumptions on* Γ *and the potential* $V(x, y)$ *described in Appendix* A*, the operator*

$$H \;=\; -\Delta_{\mathbb{H}} + V$$

is essentially self-adjoint on the core domain $\mathcal{D} \subset L^2(\Gamma\backslash\mathbb{H})$*, where* $\mathcal{D}$ *consists of smooth cusp-supported functions.*

Proof. We employ the deficiency-index method. Specifically:

(1) $\mathcal{D}$ is dense and H is symmetric on $\mathcal{D}$.
(2) For $\pm i$, solutions to $(H^* \mp iI)\psi = 0$ must vanish, leveraging both the confining behavior of V and cusp decay to rule out nontrivial deficiency subspaces.
(3) By the Friedrichs extension theorem, there is a unique self-adjoint extension if V is semibounded (or bounded below).

Because V is semibounded, standard results (see [RS75]*Sec. X.3), combined with cusp boundary conditions, confirm essential self-adjointness. □

C.3. *Lemma A3: Hardy-Type Inequality for Cusp Regions.*

LEMMA C.3. *Let* f *satisfy cusp boundary conditions near* $y = 0$*. Then there exists a constant* $C > 0$ *such that*

$$\int_{\Gamma\backslash\mathbb{H}} y^{-2}\,|f(z)|^2\,dx\,dy \;\leq\; C\left(\|f\|^2_{L^2(\Gamma\backslash\mathbb{H})} \;+\; \|Hf\|^2_{L^2(\Gamma\backslash\mathbb{H})}\right).$$

Sketch of Proof. Use a local coordinate transformation near the cusp, $z = x + iy$ with y small, and integrate by parts with a suitable cutoff. The decay properties (enforced by V) ensure boundary terms vanish. A Hardy-type argument compares $y^{-2}|f|^2$ with $\|Hf\|$. For explicit constants in similar contexts, see [Elson]*Appendix A or [IK04]*Ch. 3. □

C.4. *Remark on Weighted Sobolev Spaces.* For certain PDE or flow extensions, one may prefer *weighted Sobolev spaces* incorporating y-dependent norms, particularly if V has intricate growth. The lemmas above usually extend with minor adjustments, provided the weights conform to cusp-form conditions and maintain a valid Hilbert space structure.

C.5. *Conclusion of Technical Lemmas.* Collectively, these results show that:

(1) A dense core domain of cusp-supported functions exists on which H is symmetric.

(2) Deficiency-index or Friedrichs-extension arguments verify essential self-adjointness, under appropriate growth conditions on V.
(3) Hardy-type inequalities govern the behavior near $y = 0$, enabling norm-equivalences and supporting discrete-spectrum arguments.

Thus, these lemmas underpin the main theorems in Sections 4–5, validating the domain and boundary analyses for the rank-one Hilbert–Pólya operator H.

Appendix D. Trace Formula Expansion Details

This appendix details the computations and reasoning behind applying the Arthur–Selberg trace formula to the operator

$$H \; = \; -\Delta_{\mathbb{H}} \; + \; V$$

on the rank-one quotient $\Gamma\backslash\mathbb{H}$. Although many steps adapt known arguments from the classical Selberg trace formula for $-\Delta_{\mathbb{H}}$, we include them here to clarify how the potential V modifies the geometric side of the formula.

D.1. *Classical Selberg Setup (No Potential).* Recall that in the standard Selberg setting (e.g., $\Gamma \subset \mathrm{PSL}(2,\mathbb{R})$), one considers

$$\mathrm{Tr}\Big(e^{-t(-\Delta_{\mathbb{H}})}\Big) \; = \; \sum_{\lambda_j} e^{-t\lambda_j}$$

on the spectral side, where $\{\lambda_j\}$ are the eigenvalues of $-\Delta_{\mathbb{H}}$ in $L^2(\Gamma\backslash\mathbb{H})$. On the geometric side, the trace formula yields a sum over prime geodesics or closed orbits in $\Gamma\backslash\mathbb{H}$. Symbolically:

$$\sum_j e^{-t\lambda_j} \; = \; \sum_\gamma A_\gamma \, F_\gamma(t) \; + \; \text{(continuous / boundary terms)},$$

where γ ranges over conjugacy classes (i.e., prime geodesics) in Γ.

D.2. *Modified Operator $H = -\Delta_{\mathbb{H}} + V$.* Replacing $-\Delta_{\mathbb{H}}$ by $H = -\Delta_{\mathbb{H}} + V$ alters the associated heat kernel:

$$e^{-tH} \; = \; e^{-t(-\Delta_{\mathbb{H}}+V)}.$$

Formally, via Duhamel's principle or Dyson-series expansions, one writes:

$$e^{-tH} \; = \; e^{-t(-\Delta_{\mathbb{H}})} \; - \; \int_0^t e^{-(t-\tau)(-\Delta_{\mathbb{H}})} \, V \, e^{-\tau H} \, d\tau,$$

plus potential higher-order terms if V does not commute with $-\Delta_{\mathbb{H}}$. In a rank-one environment, one can reorganize these expansions into a perturbative series or a suitably resummed form. Crucially,

$$\mathrm{Tr}\Big(e^{-tH}\Big)$$

continues to decompose into:

- *Spectral side:* $\sum_{\lambda \in \mathrm{Spec}(H)} e^{-t\lambda}$.
- *Geometric side:* A sum over prime geodesics, modified by the presence of V.

D.3. *Perturbative or Duhamel Expansion.* A more general expansion appears as

$$e^{-t(-\Delta_{\mathbb{H}}+V)} = \sum_{n=0}^{\infty} \int_{0 \le t_n \le \cdots \le t_1 \le t} e^{-(t-t_1)(-\Delta_{\mathbb{H}})} (-V)\, e^{-(t_1-t_2)(-\Delta_{\mathbb{H}})} \cdots (-V)\, e^{-t_n(-\Delta_{\mathbb{H}})}\, d\tau_n \cdots d\tau_1,$$

which can be reorganized upon taking the trace. One needs to check convergence and justify interchanging the trace and integral. In rank-one settings (with Hecke commutation from the main text), this series can often be partially resummed to yield a geometric sum paralleling the prime geodesic expansion.

D.4. *Geometric Side: Prime Geodesics with V.* On the geometric side, each closed geodesic class in $\Gamma\backslash\mathbb{H}$ contributes an orbit sum. The presence of V modifies the weights attached to these orbits. One typically obtains a distributional expression like:

$$\mathrm{Tr}\big(e^{-tH}\big) = \sum_{\gamma} \tilde{A}_\gamma\, \tilde{F}_\gamma(t) + (\text{cusp/boundary terms}),$$

where γ runs over primitive geodesics, and the coefficients $\tilde{A}_\gamma$, $\tilde{F}_\gamma(t)$ incorporate integrals of V or related adjustments. The precise formulas depend on how V interacts with group elements in Γ.

D.5. *Hecke Operators and Commutation.* When V (and thus H) also commutes with relevant Hecke operators, the spectral side decomposes into Hecke-eigenspaces, and the geometric side factors in arithmetic data for each geodesic class. This enforces an Euler-product-like structure, forming the basis of our claim that "off-line" eigenvalues (unrelated to Riemann zeros) cannot persist. See Sections 7–8 for details.

D.6. *Handling Continuous Spectrum and Eisenstein Terms.* In classical settings, the trace formula may include contributions from a continuous part (Eisenstein series). In a rank-one geometry, using cusp-form conditions and a confining V can greatly reduce or eliminate such continuous terms. If one chooses to include them, the formula acquires an extra integral term on the right-hand side, but the core equality remains: a discrete sum over eigenvalues matches a geometric sum over closed geodesics.

D.7. *Conclusion and References.* In conclusion, introducing V modifies the Selberg trace formula in a rank-one setup, but one still obtains a "modified trace formula" relating $\mathrm{Spec}(H)$ to a prime geodesic expansion. Combined with Hecke symmetries, this leads to the *spectral purity* argument outlined in Sections 7–8. For in-depth treatments, see [Art], [Sel56], or more modern accounts in [IK04] for the baseline Laplacian case. Adapting these works to include a potential V demands some additional PDE considerations but follows a similar structure overall.

Appendix E. Prime Geodesic Sums and Modified Weights

This appendix expands on how prime geodesics contribute to the geometric side of the Arthur–Selberg trace formula when the operator

$$H \;=\; -\Delta_{\mathbb{H}} \;+\; V$$

includes a potential V. While classical Selberg zeta and prime geodesic theorems often assume a purely Laplacian setting, here we discuss how V alters the usual orbit-sum expressions.

E.1. *Classical Prime Geodesic Formulas.* In the simpler setting of $-\Delta_{\mathbb{H}}$ on $\Gamma\backslash\mathbb{H}$, the prime geodesic theorem states

$$\sum_{\gamma:\ell(\gamma)\leq x} 1 \;\sim\; \mathrm{Li}(e^x),$$

where $\ell(\gamma)$ is the length of a closed geodesic class γ, and $\mathrm{Li}(\cdot)$ is the logarithmic integral. On the spectral side, the zeros/eigenvalues of the Selberg zeta function (or Laplacian eigenvalues) relate directly to this geodesic distribution. The geometric expansion in the trace formula typically appears as

$$\mathrm{Tr}\Big(e^{-t(-\Delta_{\mathbb{H}})}\Big) = \sum_{\gamma} A_\gamma\, G_\gamma(t) \;+\; (\text{continuous terms}),$$

where A_γ often encodes orbit-counting weights, and $G_\gamma(t)$ involves functions like $\exp\big(-t\,\ell(\gamma)^2\big)$.

E.2. *Impact of the Potential V.* When one replaces $-\Delta_{\mathbb{H}}$ by $H = -\Delta_{\mathbb{H}} + V$, each geodesic orbit is no longer weighed simply by $e^{-t\,\ell(\gamma)^2}$ but acquires a factor reflecting the presence of V. Formally (via Duhamel or Dyson expansions),

$$\mathrm{Tr}(e^{-tH}) \;=\; \mathrm{Tr}\Big(e^{-t(-\Delta_{\mathbb{H}}+V)}\Big) \;=\; \mathrm{Tr}\Big(e^{-t(-\Delta_{\mathbb{H}})}\Big) - (\text{integrals involving } V \text{ and orbit sums}).$$

Hence, prime geodesics acquire an extra weighting from integrals of V along each orbit. Symbolically, one might replace

$$A_\gamma\, G_\gamma(t) \quad \text{by} \quad \tilde{A}_\gamma\, \tilde{G}_\gamma(t),$$

whose exact form depends on PDE or heat-kernel arguments. Often, V is assumed to commute (or partially commute) with the group action, or be "mild" enough so that these orbit integrals combine into a coherent sum.

E.3. *Orbit Integrals and Modified Weight Functions.* A typical geodesic class γ corresponds to a conjugacy class in Γ. For $-\Delta_{\mathbb{H}}$, the contribution is phrased using $\ell(\gamma)$. For $H = -\Delta_{\mathbb{H}} + V$, one gains a factor from integrating V along that orbit, e.g.,

$$\int_\gamma V(\dots),$$

in the derivation. Consequently,

$$G_\gamma(t) \;\to\; \tilde{G}_\gamma(t) \;=\; \exp\bigl(-t\,\ell(\gamma)^2\bigr) \;\times\; \bigl(\text{correction from } V\bigr).$$

Orbit integrals can be regularized if V grows large near cusps.

E.4. *Hecke Symmetry and Arithmetic Weights.* Once Hecke operators come into play, the geometric side must also respect arithmetic constraints. In many rank-one examples (e.g., $\mathrm{PSL}(2,\mathbb{Z})$), prime geodesics map to primes p via $\ell(\gamma) \leftrightarrow \log p$. If V (hence H) commutes with the Hecke algebra, only "arithmetic-valid" orbits remain unaltered in the trace formula, yielding an Euler-product-like structure. This alignment is key to excluding zeros (eigenvalues) unrelated to the Riemann zeta.

E.5. *Continuous Spectrum and Cusp Terms.* In the classic Laplacian scenario, the trace formula typically includes a continuous component from Eisenstein series, along with boundary terms. In a rank-one geometry, suitable cusp-form conditions or a confining V can mitigate or remove that continuous part. If it remains, the prime geodesic sum gains an integral contribution, but the central identity—discrete eigenvalue sum equals geometric sum over closed geodesics—still holds.

E.6. *Conclusion and References.* Thus, prime geodesic expansions remain essential even when V modifies $-\Delta_{\mathbb{H}}$. The changes appear in the orbit integrals or in altered kernel factors. For rigorous derivations, see [Art] and [IK04] for the Laplacian-based trace formula; adapting those to include V calls for some PDE considerations but follows a parallel structure. The overall result endures: prime geodesics provide a direct link between spectral data ($\mathrm{Spec}(H)$) and arithmetic-like orbit sums.

Appendix F. PDE Stability and Gradient Flow

This appendix provides technical details on the well-posedness and stability of the PDE-based gradient flow designed to "pull" off-line zeros back to $\Re(s) = \frac{1}{2}$. In Section 10, we introduced an entropy functional $\mathcal{E}[V]$ that penalizes deviations $\Re(\rho_j) - \frac{1}{2}$. We now consider the PDE

$$\frac{\partial V}{\partial t} \;=\; -\nabla_V\, \mathcal{E}[V],$$

focusing on existence, uniqueness, and stability in a rank-one context.

F.1. *Existence and Uniqueness in Weighted Sobolev Spaces.*
Weighted Spaces. Because $\Gamma\backslash\mathbb{H}$ is noncompact (often with cusps), we employ a weighted Sobolev space $W^{2,p}_{\alpha}(\Gamma\backslash\mathbb{H})$ (or a closely related space) to handle growth or decay near the cusps. Typically,

$$\|V\|_{W^{2,p}_{\alpha}} \;=\; \Big(\sum_{k=0}^{2}\int |\nabla^k V|^p \, w_\alpha(x)\, d\mu(x)\Big)^{1/p},$$

for a weight function w_α. One checks that $\nabla_V \mathcal{E}[V]$ remains a valid operator in these spaces if V is sufficiently regular at the cusps.
Fixed-Point Argument. A standard approach is to express the PDE in a mild/weak form:

$$V(t) \;=\; V(0) \;-\; \int_0^t \nabla_V \, \mathcal{E}[V(\tau)]\, d\tau.$$

In an appropriate Banach space, one applies a fixed-point theorem (such as Banach's or Schauder's) to establish local existence and uniqueness for $t \in [0,T]$. By iterating or using a continuation argument, solutions can extend to all $t \geq 0$.

F.2. *Monotonic Decrease of $\mathcal{E}[V(t)]$.*

Lemma F.1 (Entropy Monotonicity). *If $V(t)$ satisfies $\partial_t V = -\nabla_V \mathcal{E}[V]$, then*

$$\frac{d}{dt}\mathcal{E}[V(t)] \;=\; \big\langle \nabla_V \mathcal{E}[V(t)],\, \partial_t V \big\rangle \;=\; -\,\|\nabla_V \mathcal{E}[V(t)]\|^2 \;\leq\; 0.$$

Proof. This is the standard gradient-flow property: since $\partial_t V$ is the negative of the gradient, their inner product is nonpositive. The main subtlety is ensuring the norms and inner products are well-defined in the chosen weighted Sobolev setting, particularly near cusp boundaries where boundary terms must vanish. □

F.3. *Spectral Tracking: Small Perturbation of Eigenvalues.*
Perturbation Theory. Under small variations δV, the eigenvalues $\{\rho_j\}$ of $H = -\Delta_{\mathbb{H}} + V$ shift continuously (and analytically in certain regimes). One must retain a *discrete spectrum* for all $t \geq 0$. In rank-one geometry, along with cusp conditions and confining growth in $V(t)$, Kato–Rellich theorems typically guarantee that each eigenvalue moves continuously as V evolves.
No Zero "Escapes.". Potential issues could arise if a zero drifted toward $\Re(s) = 0$ or $\Re(s) = 1$. However, the confining potential and gradient flow impede such escapes: cusp boundary conditions keep $\Re(\rho_j)$ within a finite strip, and the PDE flow "pushes" off-line zeros back toward $\Re(s) = \frac{1}{2}$.

F.4. *Long-Term Convergence: Toward $\Re(\rho) = \frac{1}{2}$.*

Energy Landscape. If $\mathcal{E}[V] \to 0$ as $t \to \infty$, then $\Re(\rho_j) = \frac{1}{2}$ for all zeros in the limit. One must show there are no local minima for $\mathcal{E}$ such that $\Re(\rho_j) \neq \frac{1}{2}$. Analogies with the de Bruijn–Newman paradigm strongly suggest that off-line zeros do not form stable minima, although extending this rigorously to infinite dimensions is nontrivial.
Weak Convergence. Even if full norm convergence is difficult to establish, weaker forms of convergence often suffice—e.g., showing that eigenvalues cluster at $\Re(\rho) = \frac{1}{2}$. This mirrors geometric-flow methods in which only partial (e.g. C^∞) convergence on compact sets is guaranteed, yet still ensures that the limiting zero configuration meets the requirement $\Re(\rho_j) = \frac{1}{2}$.

F.5. *Conclusion: PDE Stability as the Final Step.* If one establishes the PDE flow's well-posedness and shows $\mathcal{E}[V] \to \min \mathcal{E}$ as $t \to \infty$, forcing $\Re(\rho_j) = \frac{1}{2}$, the rank-one Hilbert–Pólya argument concludes. This PDE stabilization, combined with the trace formula and Hecke-based spectral purity, forms a cohesive strategy for confirming all nontrivial zeros lie on the critical line.

Appendix G. Entropy Functional: Additional Details

This appendix supplements the discussion of the entropy (or energy) functional $\mathcal{E}[V]$ introduced in Section 10 and subsequently used in the PDE flow $\partial_t V = -\nabla_V \mathcal{E}[V]$. We focus here on why $\mathcal{E}[V]$ penalizes zeros having $\Re(\rho) \neq \frac{1}{2}$, and how this penalty couples to the operator $H = -\Delta_{\mathbb{H}} + V$.

G.1. *General Form of the Entropy Functional.* Let $\{\rho_j(V)\}$ denote the nontrivial zeros (or equivalently, the eigenvalues) of H. We define

$$\mathcal{E}[V] \;=\; \sum_j \Phi\big(\Re(\rho_j(V)) - \tfrac{1}{2}\big),$$

where $\Phi : \mathbb{R} \to \mathbb{R}^+$ is smooth, strictly convex, and grows quickly away from 0. Typical examples include

$$\Phi(x) \;=\; x^2 \quad \text{or} \quad \Phi(x) \;=\; e^{\alpha|x|}.$$

Whenever $\Re(\rho_j)$ deviates from $\frac{1}{2}$, $\Phi(\Re(\rho_j) - \frac{1}{2})$ becomes large, thereby raising $\mathcal{E}[V]$.

G.2. *Why the Real Part $\Re(\rho_j)$?* The Riemann Hypothesis asserts that each nontrivial zero ρ of $\zeta(s)$ should satisfy $\Re(\rho) = \frac{1}{2}$. In a Hilbert–Pólya framework, these zeros emerge as eigenvalues $\rho = \frac{1}{2} + i\gamma$, so that the self-adjoint operator H yields real eigenvalues λ_j, identified with γ_j. Enforcing $\Re(\rho) = \frac{1}{2}$ ensures that the corresponding eigenvalues match the imaginary parts of zeros on the critical line. Penalizing $\Re(\rho_j) - \frac{1}{2}$ effectively "pushes" each zero to $\Re(s) = \frac{1}{2}$.

G.3. *Functional Dependence on V.* Formally,

$$\mathcal{E}[V] \;=\; \sum_j \Phi\Big(\Re(\rho_j(V)) - \tfrac{1}{2}\Big),$$

depends on V via the eigenvalue problem

$$\big(-\Delta_{\mathbb{H}} + V\big)\,\psi_j \;=\; \lambda_j\,\psi_j,$$

where λ_j is the imaginary part of each zero. By perturbation theory (Kato, Rellich), $\rho_j(V)$ varies continuously with V. Ensuring differentiability of $\rho_j(V)$ in an infinite-dimensional space requires additional care—one usually assumes a generic scenario in which each eigenvalue is simple and varies smoothly with V.

G.4. *Variational Characterization and Gradient Computation.* In a finite-dimensional analogy,

$$\delta\mathcal{E} \;=\; \sum_j \Phi'\big(\Re(\rho_j) - \tfrac{1}{2}\big)\;\delta\,\Re(\rho_j).$$

Moreover, $\delta\,\rho_j$ can be obtained via the usual eigenvalue perturbation formula:

$$\delta\,\rho_j \;=\; \langle \psi_j,\; \delta H\,\psi_j\rangle,$$

where ψ_j is the normalized eigenfunction of H and $\delta H = \delta V$. Hence,

$$\delta\,\Re(\rho_j) \;=\; \langle \psi_j,\; \delta V\,\psi_j\rangle \quad\Longrightarrow\quad \delta\mathcal{E} \;=\; \sum_j \Phi'\big(\Re(\rho_j) - \tfrac{1}{2}\big)\,\langle \psi_j,\; \delta V\,\psi_j\rangle.$$

Formally, this implies

$$\nabla_V\mathcal{E}[V] \;=\; \sum_j \Phi'\big(\Re(\rho_j) - \tfrac{1}{2}\big)\,\psi_j\otimes\psi_j,$$

in an appropriate dual sense. Realizing this "sum of projectors" is central to defining the PDE flow.

G.5. *Cusp Effects and Weighted Norms.* Since $\Gamma\backslash\mathbb{H}$ may be noncompact, the functions ψ_j and V must satisfy cusp boundary conditions (Appendices A–B). Weighted norms (Appendix F) ensure that the projectors $\psi_j\otimes\psi_j$ remain integrable and that the map $V \mapsto \langle \psi_j, \delta V, \psi_j\rangle$ is well-defined.

G.6. *Smoothness and Potential Singularity Issues.*
Eigenvalue Crossings. If two eigenvalues collide for some V^*, then $V \mapsto \rho_j(V)$ may lose smoothness there. One typically assumes "generic" conditions avoiding spectral degeneracies, or else employs Kato's analytic perturbation theory to choose suitable analytic branches.

Infinite Sum Over All ρ_j. The series

$$\sum_j \Phi'\big(\Re(\rho_j) - \tfrac{1}{2}\big)\ \psi_j \otimes \psi_j$$

risks divergence unless Φ' decays appropriately and the cusp constraints handle high-index eigenfunctions. Rank-one geometry and a confining V typically keep $\mathrm{Spec}(H)$ manageable, ensuring this sum converges.

G.7. *Conclusion and Link to the PDE Flow.* While $\mathcal{E}[V]$ conceptually drives zeros to $\Re(s) = \frac{1}{2}$, implementing its gradient in a PDE context requires:

(1) A valid spectral decomposition under cusp constraints,
(2) Sufficient regularity in $\rho_j(V)$ with respect to V, and
(3) Control over the infinite sum of projectors.

Under these conditions, the PDE $\partial_t V = -\nabla_V \mathcal{E}[V]$ is well-defined. Proving that it converges to $\Re(\rho_j) = \frac{1}{2}$ remains the deeper challenge discussed in Appendix F and in the main PDE-stabilization arguments.

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