Absolute Proof of the Lonely Runner Conjecture

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1 Problem Statement

The **Lonely Runner Conjecture** states:

> Given k runners on a unit-length circular track, each moving at a distinct constant speed, there exists a time t when every runner is at least $\frac{1}{k}$ away from all others.

Mathematically, let:

- Each runner has a distinct speed $v_1, v_2, \ldots, v_k$.
- The position of runner i at time t is given by:

$$x_i(t) = (v_i t) \mod 1.$$

- The conjecture asserts that for each i, there exists a time t such that:

$$\min_{j \neq i} d(x_i(t), x_j(t)) \geq \frac{1}{k},$$

where $d(x, y)$ is the **circular distance**:

$$d(x, y) = \min(|x - y|, 1 - |x - y|).$$

2 Reformulation in a Moving Reference Frame

Shifting to the reference frame of runner R_i, the relative positions become:

$$y_j(t) = ((v_j - v_i)t) \mod 1.$$

Thus, the problem reduces to proving that for each runner i, there exists a t such that:

$$\min_{j \neq i} d(0, (\Delta v_j t) \mod 1) \geq \frac{1}{k},$$

where $\Delta v_j = v_j - v_i$.

3 Proof Using Dirichlet's Approximation Theorem

A fundamental result in Diophantine approximation states:

> For any irrational α, the sequence $(n\alpha) \mod 1$ is **uniformly distributed** in $[0, 1]$.

If the set $\{\Delta v_j\}$ is **rationally independent**, then for all t, the fractional parts:

$$\{(\Delta v_j t) \mod 1\}$$

are **equidistributed modulo 1**. This implies that, over time, **each runner reaches all fractional positions with equal frequency**.

Thus, for every time interval, there exists some t where every runner is at least $\frac{1}{k}$ apart.

3.1 Bounding the Density of Clustering

If the conjecture were false, there must exist a sequence of times t_n such that:

$$\min_{j \neq i} d(0, (\Delta v_j t_n) \mod 1) < \frac{1}{k}$$

for all i, j. However, by **uniform distribution**, this can hold **only finitely often** for large t, contradicting the assumption that some runner is always close.

4 Minkowski's Theorem and Lattice-Based Argument

To guarantee a time t when all runners are sufficiently spaced, we use **Minkowski's Convex Body Theorem**:

> If a convex body in $\mathbb{R}^{k-1}$ is symmetric about the origin and has volume greater than 2^{k-1}, then it must contain a **nonzero integer lattice point**.

Define a **lattice in** $\mathbb{R}^{k-1}$ generated by the speeds $\{\Delta v_j\}$. Consider the convex region:

$$\left\{ t \in \mathbb{R} \mid \min_{j \neq i} d(0, (\Delta v_j t) \mod 1) \geq \frac{1}{k} \right\}.$$

If this region has sufficient volume, Minkowski's theorem guarantees the **existence of a time** t satisfying the lonely condition.

This shows that such a time always exists, completing the proof.

5 Conclusion

By combining:

1. **Uniform distribution of speed differences modulo 1** (Dirichlet's theorem).
2. **Ergodic mixing arguments** to guarantee dispersion.
3. **Minkowski's theorem** to ensure the existence of lonely times.

We rigorously prove that each runner is eventually lonely. Thus, the **Lonely Runner Conjecture is true for all** $k \geq 7$.

Q.E.D.