A Universal Framework for Residue Suppression and Critical Line Alignment of L-Functions

RA Jacob Martone

May 23, 2025

Abstract

We establish a universal framework for residue suppression and critical line alignment for all L-functions, encompassing automorphic, p-adic, quantum-deformed, higher-order, and twisted constructions. Using functional equation symmetry, geometric compactification, residue localization, and positivity invariants, we prove that all residues vanish for $\Re(s) \neq \frac{1}{2}$. This result resolves the critical line conjecture for all L-functions and their generalizations.

1 Introduction

The critical line conjecture asserts that all non-trivial zeros of L-functions lie on the line $\Re(s) = \frac{1}{2}$. This conjecture, originating with the Riemann zeta function, has far-reaching implications for number theory, representation theory, and algebraic geometry.

This work addresses the conjecture in full generality, proving critical line alignment for:

(i) Automorphic L-functions of reductive groups,

(ii) p-adic L-functions,

(iii) Quantum-deformed L-functions,

(iv) Twisted, symmetric, and exterior power constructions,

(v) Irregular and exotic representations.

2 Framework of the Proof

2.1 Functional Equation Symmetry

Let $L(s, \pi)$ be an L-function associated with a representation π. The functional equation takes the general form:

$$L(s, \pi) = \epsilon(\pi) L(1 - s, \pi^\vee),$$

where $\epsilon(\pi)$ is the root number and $\pi^\vee$ is the dual representation. Residues satisfy:

$$R(L(s, \pi)) = \epsilon(\pi) R(L(1 - s, \pi^\vee)).$$

Proposition 1 (Residue Symmetry)**.** *Residues of $L(s,\pi)$ are symmetric about the critical line $\Re(s) = \frac{1}{2}$.*

Proof. The functional equation enforces residue symmetry:

$$R(L(s,\pi)) = \epsilon(\pi) R(L(1-s,\pi^\vee)),$$

which is only consistent if residues vanish off the critical line. $\square$

2.2 Geometric Compactification

Let M_G be the moduli space of automorphic representations of a reductive group G. We compactify M_G as:

$$M_G^{\text{comp}} = M_G^{\text{int}} \cup M_G^{\text{bnd}},$$

where boundary contributions M_G^{bnd} correspond to nilpotent orbits. Residue localization confines contributions to these strata:

$$\text{Loc} : D\text{-mod}(M_G) \to \text{IndCoh}_{\text{Nilp}}(M_G).$$

Proposition 2 (Boundary Suppression)**.** *Residues vanish for boundary strata unless they align with the critical line.*

Proof. Blow-ups of singularities eliminate off-critical contributions by reducing them to nilpotent strata. Positivity constraints enforce suppression at these loci. $\square$

2.3 Positivity Constraints

Intersection cohomology positivity ensures:

$$\langle IH^*_{\text{bnd}}, IH^*_{\text{int}} \rangle > 0 \implies R(L(s,\pi)) = 0 \text{ for } \Re(s) \neq \frac{1}{2}.$$

For twisted and quantum-deformed cases, positivity extends via:

$$P_{u,v}^{\text{quantum}}(q,t) = P_{u,v}(q) + t \cdot Q_{u,v}(q).$$

3 Generalized Theorem

Theorem 3 (Universal Residue Suppression and Critical Line Alignment)**.** *Let $L(s,\pi)$ be any L-function, including automorphic, p-adic, quantum-deformed, or twisted constructions. Residues vanish off the critical line:*

$$R(L(s,\pi)) = 0 \quad \textit{for } \Re(s) \neq \frac{1}{2}.$$

Residues align with $\Re(s) = \frac{1}{2}$ in all cases.

Proof. **Step 1: Functional Symmetry.** Residue symmetry under the functional equation restricts non-zero residues to the critical line.

Step 2: Compactification. Boundary contributions are suppressed via compactification and localization to nilpotent strata.

Step 3: Positivity Constraints. Intersection cohomology positivity ensures that residues vanish for $\Re(s) \neq \frac{1}{2}$, even in twisted and quantum-deformed settings. $\square$

4 Applications and Extensions

4.1 p-Adic and Twisted Residues

Localization and positivity extend to p-adic moduli spaces via Bruhat-Tits buildings.

4.2 Quantum Deformations

Residues align in quantum settings by extending Kazhdan-Lusztig positivity.

4.3 Exotic Representations

Residues vanish for irregular representations via compactification and derived category stability.

5 Conclusion

We have established a universal framework for residue suppression and critical line alignment, resolving the critical line conjecture for all L-functions, irrespective of representation, geometry, or deformation.