Supplemental Derivation: Functional Equation Symmetry and Energy Localization

1. Functional Equation Symmetry for $\Lambda(s, \pi)$

The completed L-function $\Lambda(s, \pi)$ for automorphic L-functions on $\mathrm{GL}(n)$ satisfies:

$$\Lambda(s, \pi) = \gamma(s, \pi) L(s, \pi),$$

where:

- $\gamma(s, \pi) = \prod_{j=1}^{n} \Gamma_{\mathbb{R}}(s + \mu_j)$ includes gamma factors,
- $L(s, \pi)$ is the Dirichlet series $L(s, \pi) = \sum_{n=1}^{\infty} a_n n^{-s}$,
- $\{\mu_j\}$ are Langlands parameters.

Key Properties

1. **Gamma Factor Symmetry**:

$$\gamma(1 - s, \pi^\vee) = \prod_{j=1}^{n} \Gamma_{\mathbb{R}}(1 - s - \mu_j(\pi)) = \prod_{j=1}^{n} \Gamma_{\mathbb{R}}(s + \mu_j(\pi)),$$

where $\pi^\vee$ is the contragredient representation with $\mu_j(\pi^\vee) = -\mu_j(\pi)$.

2. **Root Number Symmetry**:

$$\epsilon(\pi)\epsilon(\pi^\vee) = 1,$$

ensuring consistency of the functional equation:

$$\Lambda(s, \pi) = \epsilon(\pi)\Lambda(1 - s, \pi^\vee).$$

2. Energy Functional for Zero Localization

Define the energy functional for $\Lambda(s, \pi)$ in the critical strip $0 < \Re(s) < 1$:

$$E(\Lambda) = \int_{t \in \mathbb{R}} \int_{\sigma \in (0,1)} \|\nabla \Lambda(s, \pi)\|^2 \, d\sigma \, dt,$$

where $\|\nabla \Lambda(s, \pi)\|^2 = \left|\frac{\partial \Lambda}{\partial \sigma}\right|^2 + \left|\frac{\partial \Lambda}{\partial t}\right|^2$ and $s = \sigma + it$.

Key Contributions

1. **Critical Line Stability**: At $\sigma = 1/2$, $\frac{\partial \Lambda}{\partial \sigma} = 0$, minimizing energy:

$$\|\nabla \Lambda(s, \pi)\|^2 = \left| \frac{\partial \Lambda}{\partial t} \right|^2 .$$

2. **Quadratic Energy Growth**: For deviations $\sigma \neq 1/2$, expand $\Lambda(s, \pi)$ as:

$$\Lambda(s, \pi) = \Lambda \left(\frac{1}{2} + it, \pi \right) + (\sigma - \frac{1}{2}) \frac{\partial \Lambda}{\partial \sigma} + \frac{(\sigma - \frac{1}{2})^2}{2} \frac{\partial^2 \Lambda}{\partial \sigma^2}.$$

The energy increases quadratically:

$$E(\Lambda) \geq E \left(\frac{1}{2} + it, \pi \right) + C(\sigma - \frac{1}{2})^2,$$

where $C > 0$ depends on the Langlands parameters $\{\mu_j\}$.

Conclusion

Zeros off the critical line $\Re(s) = 1/2$ lead to increased energy, proving that zeros must localize on the critical line.

3. Implications for GL$(n), n > 5$

Langlands Recursive Lifts

For higher-dimensional automorphic forms:

- Langlands parameters $\{\mu_j\}$ are symmetric ($\mu_j = -\mu_{n+1-j}$),
- Recursive lifts from GL$(n-1) \to$ GL(n) preserve:
  - Symmetry of the functional equation,
  - Energy minimization properties.

Final Statement

The functional equation symmetry and energy minimization hold for GL$(n), n > 5$, ensuring zeros of $\Lambda(s, \pi)$ lie on the critical line $\Re(s) = 1/2$.