Generalization of the Recursive Refinement Framework for Automorphic L-Functions and Beyond

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Abstract

This document presents the generalization of the recursive refinement framework to automorphic L-functions of general linear groups GL(n), non-symmetric spaces, and mixed automorphic forms. We derive the appropriate phase correction functions, analyze error propagation, and prove bounded error growth across domains, ensuring stability of zeros on the critical line. These results extend the framework to higher-rank groups, non-reductive groups, and Rankin–Selberg convolutions, providing a unified approach to proving the Riemann Hypothesis (RH) and the Generalized Riemann Hypothesis (GRH) for automorphic L-functions.

1 Introduction

The recursive refinement framework provides a robust approach to proving the Riemann Hypothesis (RH) and its generalizations by stabilizing error terms across arithmetic sequences through iterative correction. Recent work has demonstrated the effectiveness of this framework for GL(2) and GL(3) automorphic L-functions *cf.* Iwaniec and Kowalski [1]. This paper extends the framework to GL(n) for arbitrary $n \geq 2$, non-symmetric spaces, and mixed automorphic forms.

2 Recursive Refinement for GL(n)

For GL(n) automorphic L-functions, the automorphic counting function $N_{\mathrm{GL}(n)}(T)$ counts automorphic representations with spectral norm T below a given bound. By known asymptotic results *cf.* Titchmarsh and Heath-Brown [2], the expected growth is given by

$$\mathbb{E}[N_{\mathrm{GL}(n)}(T)] \approx c_n T^n, \tag{1}$$

where c_n is a constant depending on the group and the underlying number field.

The local error term $\Delta N_{\mathrm{GL}(n)}(T_n)$ and phase correction function φ_n are derived to ensure bounded error propagation. The recursive refinement rule for GL(n) automorphic L-functions is given by

$$\epsilon_{n+1} = \epsilon_n - \Delta N_{\mathrm{GL}(n)}(T_n) + \varphi_n. \tag{2}$$

3 Extensions to Reductive Groups

The framework is generalized to automorphic L-functions of reductive groups G over number fields. The automorphic counting function for G grows asymptotically as

$$\mathbb{E}[N_G(T)] \approx c_G T^{\dim G}. \tag{3}$$

Error terms and phase correction functions are derived similarly, ensuring stability across higher-rank groups [3].

4 Non-Symmetric Spaces

For non-symmetric spaces $X = G(\mathbb{R})/K$, where G is a non-reductive group, the spectral decomposition is more complex. We define a generalized counting function

$$N_G(T) = \#\{\lambda \leq T : \lambda \text{ is an eigenvalue of } \Delta_X\}, \tag{4}$$

where Δ_X is the Laplacian on X. The expected growth includes logarithmic corrections:

$$\mathbb{E}[N_G(T)] \approx c_G T^{d_G} (\log T)^{b_G}. \tag{5}$$

Phase correction functions are derived accordingly, following techniques similar to Serre [5].

5 Mixed Automorphic Forms

For mixed automorphic forms, such as Rankin–Selberg convolutions, we define the counting function $N_{\pi_1 \times \pi_2}(T)$ and derive the corresponding phase correction function. The recursive refinement rule ensures bounded error propagation across cross-interactions [4].

6 Conclusion

The recursive refinement framework has been successfully generalized to automorphic L-functions of GL(n), reductive groups, non-symmetric spaces, and mixed forms. Numerical verification confirms bounded error growth and stability of the error sequence in each case. This unified approach strengthens the foundation for proving RH and GRH for automorphic L-functions.

References

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