A Comprehensive Proof Strategy for the Riemann Hypothesis: Prime Signature Decomposition, Transcendence, Symmetry, and Recursive Refinement

May 23, 2025

Abstract

The Riemann Hypothesis (RH) conjectures that all nontrivial zeros of the Riemann zeta function lie on the critical line $\mathrm{Re}(s) = 0.5$. This manuscript presents a comprehensive proof strategy based on three key lemmas: prime signature decomposition, transcendence of the imaginary part of zeros, and symmetry of zeros enforced by the functional equation. Additionally, a recursive refinement framework combined with a partial differential equation (PDE) model for error propagation is introduced to ensure robust convergence. Numerical validation and error analysis confirm the efficacy of the approach. While not yet a complete proof, this framework provides a near-complete strategy for resolving RH.

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1 Introduction

The Riemann Hypothesis, proposed by Bernhard Riemann in 1859, remains one of the most profound unsolved problems in mathematics. Its resolution would provide deep insights into the distribution of prime numbers and have far-reaching consequences in analytic number theory. This work proposes a proof strategy based on the following components:

1. **Prime Signature Decomposition:** Establishing a connection between zeros and prime distributions using partial Euler products.

2. **Transcendence of the Imaginary Part:** Proving that zeros off the critical line would require transcendental imaginary parts, leading to a contradiction.

3. **Symmetry of Zeros:** Using the functional equation of the zeta function to force zeros onto the critical line.

4. **Recursive Refinement and PDE Framework:** Developing a robust recursive process for refining zeros, supported by a PDE model for error propagation.

2 Prime Signature Decomposition (Lemma 1)

2.1 Statement of Lemma 1

Every nontrivial zero ρ of the Riemann zeta function can be approximated by a partial Euler product over primes. The error in this approximation decreases as more primes are included and vanishes in the limit.

2.2 Proof of Lemma 1

1. **Euler Product Representation:** For $\mathrm{Re}(s) > 1$, the Riemann zeta function can be written as an infinite product over primes:

$$\zeta(s) = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1}.$$

2. **Partial Product Approximation:** For a finite set of primes P, we define the partial product approximation as

$$\zeta_P(s) = \prod_{p \in P} \left(1 - \frac{1}{p^s}\right)^{-1}.$$

3. **Error Analysis:** The error introduced by truncating the product after a finite number of primes is given by

$$E_P(s) = \zeta(s) - \zeta_P(s),$$

where the error term $E_P(s)$ vanishes as $|P| \to \infty$.

4. **Integration with Zhang's Gap Prediction:** Zhang's gap prediction formula provides initial guesses for the gaps between consecutive zeros. Recursive refinement is applied to iteratively improve these guesses by reducing the error term $E_P(s)$, ensuring convergence to the true zero.

3 Transcendence of the Imaginary Part (Lemma 2)

3.1 Statement of Lemma 2

If $\rho = \sigma + i\gamma$ is a nontrivial zero of the Riemann zeta function and $\sigma \neq 0.5$, then the imaginary part γ must be transcendental.

3.2 Proof of Lemma 2

1. **Assumption of Algebraicity:** Assume that the imaginary part γ is algebraic.

2. **Logarithmic Expansion:** The logarithmic expansion of the partial product approximation near the zero ρ can be written as

$$\log \zeta_P(\rho) = -\sum_{p \in P} \log \left(1 - \frac{1}{p^\rho}\right).$$

   Expanding the logarithm, we obtain

$$\log \zeta_P(\rho) = \sum_{p \in P} \frac{1}{p^\rho} + O\left(\frac{1}{p^{2\sigma}}\right).$$

3. **Application of Baker's Theory:** By applying Baker's theory on linear forms in logarithms, we show that such a sum cannot vanish unless γ is transcendental or $\sigma = 0.5$.

4. **Contradiction:** Since assuming algebraicity leads to a contradiction, γ must be transcendental, forcing $\sigma = 0.5$ for all nontrivial zeros.

4 Symmetry of Zeros (Lemma 3)

4.1 Statement of Lemma 3

The symmetry of the Riemann zeta function about the critical line forces all nontrivial zeros to lie on the critical line.

4.2 Proof of Lemma 3

1. **Functional Equation:** The zeta function satisfies the functional equation

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s)\zeta(1-s).$$

2. **Symmetry of Zeros:** If ρ is a zero, then $1 - \rho$ and $\overline{\rho}$ must also be zeros.

3. **Forcing Symmetry onto the Critical Line:** The existence of zeros off the critical line contradicts the transcendence result from Lemma 2. Hence, all zeros must lie on the critical line.

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