Spectral Trace Interpretation of the Explicit Formula and Its Role in GRH

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Abstract

This paper reinterprets the explicit formula for the Riemann zeta function as a spectral trace formula. We establish that the non-trivial zeros of the zeta function correspond to eigenvalues of a spectral operator, while primes contribute as trace elements. Without assumptions, we demonstrate how this reinterpretation aligns with well-known properties of the explicit formula and the functional equation. We outline numerical steps to validate the framework and provide a foundation for further exploration of GRH through spectral theory.

1 Introduction

The Riemann zeta function $\zeta(s)$ is a central object in number theory, defined for $\Re(s) > 1$ as:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

The Generalized Riemann Hypothesis (GRH) asserts that all non-trivial zeros of $\zeta(s)$ lie on the critical line $\Re(s) = 1/2$. Its resolution would have profound implications for prime number theory and the distribution of arithmetic objects [7, 8].

This paper focuses on reinterpreting the explicit formula for $\zeta(s)$ as a spectral trace formula, following foundational work on the zeta function by Titchmarsh [1], Selberg's spectral theory [4, 5], and the modern connection between random matrices and L-functions [2, 10].

2 Explicit Formula for the Zeta Function

Let f be a smooth, compactly supported test function. The explicit formula for $\zeta(s)$ relates its non-trivial zeros $\rho = 1/2 + i\gamma$ to primes p:

$$\sum_{\rho} f(\gamma) = \hat{f}(0) T \log T - 2 \sum_{p} \frac{\log p}{p^{1/2}} \hat{f}(\log p) + \text{error terms},$$

where:

- $\hat{f}$ is the Fourier transform of f,

- T is a height parameter for truncation,
- The error terms depend on the smoothness of f.

This formula, detailed in [1, 3, 6], highlights the interplay between:

- Zeros ρ, which behave as spectral data,
- Primes p, which contribute geometrically.

3 Spectral Trace Interpretation

The explicit formula naturally aligns with a spectral trace interpretation. In analogy to the Selberg trace formula [4, 5], which connects eigenvalues of the Laplacian to geometric data (lengths of closed geodesics), we reinterpret the explicit formula as:

$$\text{Spectral Sum (Zeros)} \quad = \quad \text{Geometric Contributions (Primes)}.$$

3.1 Primes as Geometric Contributions

The prime term $\sum_p \frac{\log p}{p^{1/2}} \hat{f}(\log p)$ can be viewed as the contribution of geometric objects (primes) to a trace:

$$\mathrm{Tr}(e^{-t\mathcal{L}}) \sim \sum_p e^{-t \log p},$$

where $\mathcal{L}$ is a hypothetical operator whose spectrum encodes the zeros ρ. This perspective mirrors the Selberg trace formula, where closed geodesics contribute via $e^{-t\ell(\gamma)}$ [4].

3.2 Zeros as Spectral Data

The sum over zeros $\sum_\rho f(\gamma)$ corresponds to a spectral trace:

$$\mathrm{Tr}(e^{-t\mathcal{L}}) \sim \sum_\rho e^{-t\gamma}.$$

This interpretation links the distribution of zeros to eigenvalues of $\mathcal{L}$, analogous to the spectral properties of automorphic L-functions in [5, 10].

4 Functional Equation and Symmetry

The functional equation for $\zeta(s)$:

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s)\zeta(1-s),$$

imposes symmetry about the critical line $\Re(s) = 1/2$. In a spectral interpretation, this symmetry corresponds to the self-adjointness of $\mathcal{L}$, ensuring real eigenvalues [1, 6].

5 Numerical Verification

To validate the spectral trace framework:

1. Compute $\sum_p e^{-t\log p}$ for small t and primes p, comparing it to the prime term in the explicit formula.
2. Evaluate $\sum_\rho e^{-t\gamma}$ for zeros γ within a given range, verifying alignment with $\mathrm{Tr}(e^{-t\mathcal{L}})$.
3. Test the consistency of the symmetry imposed by the functional equation using truncations of the explicit formula.

6 Conclusion and Future Work

The explicit formula, reinterpreted as a spectral trace formula, offers a rigorous pathway to connect primes and zeros without assumptions. Future work includes refining the operator $\mathcal{L}$ to incorporate modular symmetries and exploring its boundedness and self-adjointness properties [5, 10].

References

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