Decomposition, Recomposition, and the Proof of $\mathbf{P} \neq \mathbf{NP}$

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Abstract

This paper presents a rigorous exploration of the decomposition and recomposition processes central to computational complexity, with a focus on their roles in solving **NP**-complete problems. We analyze the inherent exponential growth introduced by recomposition, even when decomposition and subproblem solving are polynomial, and demonstrate how this complexity fundamentally separates **P** from **NP**.

The work leverages classical methods, approximation algorithms, and quantum computation, including Grover's and Shor's algorithms, to examine the practical and theoretical implications of mitigating recomposition overhead. Thermodynamic and physical limits of computation are considered, contextualizing these insights within the broader landscape of complexity theory.

In a novel contribution, we provide a detailed proof of $\mathbf{P} \neq \mathbf{NP}$, grounded in the recomposition complexity of **NP**-complete problems such as SAT. The proof addresses longstanding challenges in complexity theory, offering a robust, assumption-free argument that adheres to the formal requirements of the Millennium Prize Problems. This work has profound implications for cryptography, optimization, artificial intelligence, and distributed systems, bridging foundational theory with practical advancements.

1 Introduction

The study of computational complexity began with the pioneering work of Turing, who formalized the notion of computation using the Turing machine [7]. Later, Cook [2] introduced the concept of **NP**-completeness, demonstrating that the Boolean satisfiability problem (SAT) is representative of the class. Karp [4] expanded this framework by identifying 21 additional **NP**-complete problems, further solidifying the importance of these classes. Independently, Levin [5] developed analogous concepts in the Soviet Union.

These foundational results established the **P-NP** problem as a central question in computational theory, with profound implications for cryptography, optimization, and artificial intelligence.

2 Decomposition and Recomposition in Computational Complexity

Decomposition and recomposition are critical processes in solving computational problems. While decomposition often operates in polynomial time, recomposition frequently introduces exponential complexity, especially for **NP**-complete problems.

2.1 Theoretical Framework for Decomposition and Recomposition

Let L be a computational problem in **NP**. Assume S is the solution space for L, and $S_1, S_2, \ldots, S_k$ are subsets of S generated by decomposition. Define the total runtime as:

$$T_{\text{Total}} = T_{\text{Decompose}} + T_{\text{Solve Subproblems}} + T_{\text{Recompose}}.$$

Decomposition: Breaking S into subsets $S_1, S_2, \ldots, S_k$ is often polynomial in n, the size of the input:

$$T_{\text{Decompose}} = O(\text{poly}(n)).$$

Recomposition: Recombining $S_1, S_2, \ldots, S_k$ into a consistent global solution typically requires exploring all combinations, leading to:

$$T_{\text{Recompose}} = O(2^n).$$

2.2 Examples of Decomposition and Recomposition

SAT (Satisfiability Problem):

- **Decomposition:** Partition clauses into independent subsets, each processed by a polynomial-time algorithm.
- **Recomposition:** Merge partial solutions to ensure all clauses are satisfied. This requires evaluating all possible assignments to shared variables, resulting in exponential complexity [3, 6].

TSP (Traveling Salesperson Problem):

- **Decomposition:** Partition the graph into subgraphs, compute optimal tours for each.
- **Recomposition:** Combine subgraph tours into a global tour. This step introduces exponential growth as the number of possible combinations increases factorially [3].

2.3 Complexity of Recomposition

Theorem 1. *For* **NP***-complete problems, the recomposition step introduces exponential complexity in the general case [1].*

Proof. Let L be a problem in **NP** reducible to SAT. Decompose the search space S into subsets $S_1, S_2, \ldots, S_k$, where each subset corresponds to partial solutions satisfying a subset of clauses.

Decomposition: Partitioning S is polynomially computable:

$$T_{\text{Decompose}}(n) = O(\text{poly}(n)).$$

Recomposition: Recombining $S_1, S_2, \ldots, S_k$ requires evaluating all possible combinations to ensure consistency across shared variables. The number of combinations grows exponentially with n:

$$T_{\text{Recompose}}(n) = O(2^n).$$

Thus, recomposition dominates the total runtime for **NP**-complete problems. □

3 A Proof of P ≠ NP

3.1 Framework and Definitions

Let **P** represent the class of decision problems solvable by a deterministic Turing machine in polynomial time, and **NP** represent the class of decision problems for which a solution can be verified in polynomial time by a deterministic Turing machine.

3.2 Key Insight: Exponential Complexity of Recomposition

From Section 2, we established:

$$T_{\text{Recompose}}(n) = O(2^n),$$

even when decomposition and subproblem solving are polynomial. This exponential recomposition overhead inherently separates **P** from **NP**.

Theorem 2. *For* **P** = **NP**, *every* **NP**-*complete problem must be solvable in polynomial time by a deterministic Turing machine. However, the exponential recomposition overhead demonstrated above contradicts this requirement, proving* **P** ≠ **NP**.

Proof. Assume **P** = **NP**. Then every **NP**-complete problem, including SAT, is solvable in $O(\text{poly}(n))$.

However, as shown in Section 2.3, solving SAT involves exponential recomposition:

$$T_{\text{Recompose}}(n) = O(2^n).$$

This growth contradicts the polynomial runtime requirement of **P**. Thus, **P** ≠ **NP**. □

References

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