Conjectures and Proofs for Galois-Theoretic Extensions of Automorphic L-Functions

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Introduction

This paper presents conjectures and proofs for Galois-theoretic extensions of automorphic L-functions, focusing on non-critical values. Motivic Galois groups are introduced to describe the symmetries and period relations of non-critical values. The paper also explores the independence of conjugates and proposes a Galois closure for transcendental values.

Conjectures

Conjecture 1: Motivic Galois Groups Acting on Non-Critical Values

Let $\mathcal{L}(s, \pi)$ be an automorphic L-function associated with a reductive group representation π, and let $s_0 \notin \{s \mid \Re(s) = \frac{1}{2}\}$ be a non-critical point. There exists a motivic Galois group $G_{\mathcal{M}}$ acting on the category of motives associated with π, such that the orbit of the non-critical value $\mathcal{L}(s_0, \pi)$ under $G_{\mathcal{M}}$ generates all possible linear combinations of periods corresponding to the motive $\mathcal{M}(\pi)$.

Conjecture 2: Independence of Conjugates of Non-Critical Values

Let $\mathcal{L}(s_0, \pi)$ be a non-critical value of an automorphic L-function. Define the conjugates of $\mathcal{L}(s_0, \pi)$ as the set:

$$\mathrm{Conj}(\mathcal{L}(s_0, \pi)) = \{\mathcal{L}(s_0, \pi') \mid \pi' \sim \pi \text{ (motivic equivalence)}\}.$$

If $\mathcal{L}(s_0, \pi)$ is transcendental, then the set of conjugates is linearly independent over the rationals, assuming the motivic periods are algebraically independent.

Conjecture 3: Galois Closure for Transcendental Values

The Galois closure of a non-critical value $\mathcal{L}(s_0, \pi)$ is the smallest field extension containing all possible conjugates under the motivic Galois group action. If $\mathcal{L}(s_0, \pi)$ is transcendental, then its Galois closure contains infinitely many algebraically independent periods.

Proof Sketch

Proof of Conjecture 1: Motivic Galois Group Action

Let $\mathcal{C}$ denote the category of motives associated with automorphic forms corresponding to representations π. By Grothendieck's theory of motives, the periods of $\mathcal{M}(\pi)$ can be expressed as integrals of algebraic differential forms over algebraic cycles. Define the motivic Galois group $G_{\mathcal{M}}$ as the group of automorphisms of $\mathcal{C}$ that preserve period relations. Applying $G_{\mathcal{M}}$ generates an orbit corresponding to all linear combinations of periods.

Proof of Conjecture 2: Independence of Conjugates

Suppose π and π' are two motivic equivalent representations. By Baker's theorem on linear forms in logarithms, logarithms of algebraically independent numbers are linearly independent over the rationals. Since non-critical values are sums involving logarithms of primes, the conjecture follows by assuming motivic periods are algebraically independent.

Proof of Conjecture 3: Galois Closure

Let K be the field generated by the non-critical value $\mathcal{L}(s_0, \pi)$. The Galois closure $\overline{K}$ is the smallest extension of K containing all conjugates under the action of the motivic Galois group. If $\mathcal{L}(s_0, \pi)$ is transcendental, then the action of $G_{\mathcal{M}}$ generates infinitely many conjugates, corresponding to different combinations of motivic periods. Thus, $\overline{K}$ contains infinitely many algebraically independent elements.

Conclusion

This paper proposes conjectures regarding the action of motivic Galois groups on non-critical values of automorphic L-functions. Proof sketches were provided for the transitive action of Galois groups, the independence of conjugates, and the definition of Galois closures. Future work will involve numerical validation of these conjectures and further exploration of transcendence properties.