Gauss Elimination and LU Decomposition Example

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System of equations

$$2x_1 - 4x_2 + 2x_3 = 6 \quad (1)$$
$$4x_1 - 9x_2 + 7x_3 = 20 \quad (2)$$
$$2x_1 + x_2 + 3x_3 = 14 \quad (3)$$

Associated matrices

$$\underbrace{\begin{bmatrix} 2 & -4 & 2 \\ 4 & -9 & 7 \\ 2 & 1 & 3 \end{bmatrix}}_{A}$$

(Step 1)

(2′) = (2) - (1)× 2; (1′)=(1); (3′)=(3)

$$2x_1 - 4x_2 + 2x_3 = 6 \quad (1')$$
$$-x_2 + 3x_3 = 8 \quad (2')$$
$$2x_1 + x_2 + 3x_3 = 14 \quad (3')$$

(2) = 2×(1) + 1×(2′); (1)=(1′); (3)=(3′)

$$\underbrace{\begin{bmatrix} 2 & -4 & 2 \\ 4 & -9 & 7 \\ 2 & 1 & 3 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{L_1} \underbrace{\begin{bmatrix} 2 & -4 & 2 \\ 0 & -1 & 3 \\ 2 & 1 & 3 \end{bmatrix}}_{A'}$$

(Step 2)

(3″) = (3′) - (1′)× 1; (1″)=(1′); (2″)=(2′)

$$2x_1 - 4x_2 + 2x_3 = 6 \quad (1'')$$
$$-x_2 + 3x_3 = 8 \quad (2'')$$
$$5x_2 + x_3 = 8 \quad (3'')$$

(3′) = 1×(1′) + 1×(3″); (1′)=(1″); (2′)=(2″)

$$\underbrace{\begin{bmatrix} 2 & -4 & 2 \\ 0 & -1 & 3 \\ 2 & 1 & 3 \end{bmatrix}}_{A'} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \underbrace{\begin{bmatrix} 2 & -4 & 2 \\ 0 & -1 & 3 \\ 0 & 5 & 1 \end{bmatrix}}_{A''}$$

$$= \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}}_{L_2} \underbrace{\begin{bmatrix} 2 & -4 & 2 \\ 0 & -1 & 3 \\ 0 & 5 & 1 \end{bmatrix}}_{A''}$$

(Step 3)

(3‴) = (3″) - (2″)× (-5); (1‴)=(1″); (2‴)=(2″)

$$
\begin{aligned}
2x_1 - 4x_2 + 2x_3 &= 6 && (1''') \\
-x_2 + 3x_3 &= 8 && (2''') \\
16x_3 &= 48 && (3''')
\end{aligned}
$$

(3″) = -5×(2″) + ×(3‴); (1″)=(1‴); (2″)=(2‴)

$$
\underbrace{\begin{bmatrix} 2 & -4 & 2 \\ 0 & -1 & 3 \\ 0 & 5 & 1 \end{bmatrix}}_{A''} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -5 & 1 \end{bmatrix}}_{L_3} \underbrace{\begin{bmatrix} 2 & -4 & 2 \\ 0 & -1 & 3 \\ 0 & 0 & 16 \end{bmatrix}}_{A'''}
$$

Thus

$$
\begin{aligned}
A &= L_1 A' \\
&= L_1 L_2 A'' \\
&= L_1 L_2 L_3 A''' \\
&= \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -5 & 1 \end{bmatrix} \begin{bmatrix} 2 & -4 & 2 \\ 0 & -1 & 3 \\ 0 & 0 & 16 \end{bmatrix} \quad = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & -5 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 2 & -4 & 2 \\ 0 & -1 & 3 \\ 0 & 0 & 16 \end{bmatrix}}_{U}.
\end{aligned}
$$