2020 Spring Numerical School

Finite Difference Problems

1. Solve the following ODE problems

(i) $$y'' = e^x, y(0) = 1, y(1) = e \tag{1}$$

(ii) $$y'' = 0, y(0) = 1, y(1) = 2 \tag{2}$$

Their exact solutions are

(i) $$y = e^x \tag{3}$$

(ii) $$y = x + 1 \tag{4}$$

(a) Derive the theorical truncation error of central difference.

(b) Show the result of each with grid $\Delta x = \frac{1}{25}, \frac{1}{50}, \frac{1}{100}, \frac{1}{200}$ ... (do as thin as you can).

(c) Is the numerical result in (b) match the result in (a)? If not, why?

2. Solve the following PDE problem ($\alpha = 1$) :

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2} \tag{5}$$

With initial condition :

$$u(x, 0) = sin(\pi x) + x + 1 \tag{6}$$

And boundary condition :

$$u(0, t) = 1, u(1, t) = 2 \tag{7}$$

Its exact solution is

$$u(x, t) = e^{-\pi^2 t} \sin(\pi x) + x + 1 \tag{8}$$

(a) Solve it by the following methods :

(i) FTCS : The forward time central space

$$u_i^{n+1} = u_i^n + \frac{\alpha \Delta t}{(\Delta \mathrm{x})^2}(u_{i+1}^n - 2u_i^n + u_{i-1}^n) \tag{9}$$

(ii) BTCS : The backward time central space

$$u_i^{n+1} - \frac{\alpha \Delta t}{(\Delta \mathrm{x})^2}(u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}) = u_i^n \tag{10}$$

(iii) Crank-Nicolson :

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = \frac{\alpha}{2}\left[\frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{(\Delta \mathrm{x})^2} + \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{(\Delta \mathrm{x})^2}\right] \tag{11}$$

(b) Derive the theorical truncation error for each scheme in (a).

(c) Keep decreasing the grid size in space and time suitably to numerically prove (b).

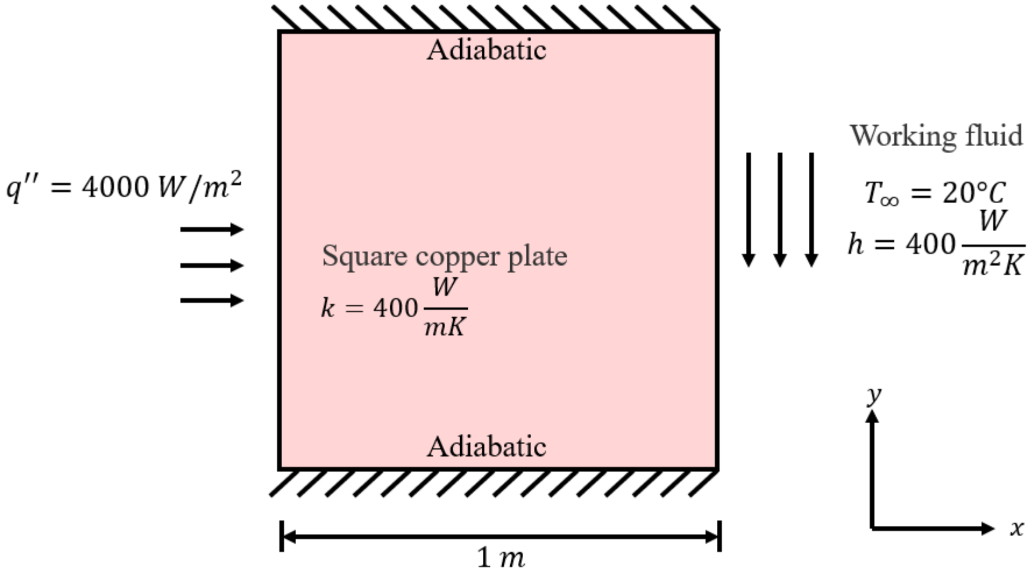
(d) Use Von-Neumann stability analysis to show (i) is conditionally stable with condition :

$$\frac{\alpha \Delta t}{(\Delta \mathrm{x})^2} \leq \frac{1}{2} \tag{12}$$

(e) Use Von-Neumann stability analysis to show (ii) & (iii) is unconditionally stable.

(f) When $t \to \infty$, give a physical explain why the solution is close to the result in 1(ii)?

3. Consider a 2-D, area $1m^2$, square copper plate without heat source/sink. Its thermal conductivity k is 400W/m・K, on the left side the heat flux is given by 4000 W/m^2, and the working fluid flows over the right side with temperature $20°C$ and convection coefficient h = 400 W/m^2K. The top and bottom side is adiabatic. Here we consider in steady operation.



(a) Use thermal resistance diagram to determine the temperature on the left and right side. Moreover, since the steady profile is linear on x direction, please also determine the temperature profile inside.

The following question gives the guideline for simulating this problem. Consider the 2D Poisson equation :

$$\nabla \cdot (k\nabla T) = -\dot{q} \tag{13}$$

(b) Explain why (13) can reduced to

$$\nabla^2 T = T_{xx} + T_{yy} = 0 \tag{14}$$

(c) Write down the boundary condition on each side.

(d) Write down the discretized boundary equation on each side. You can use **Any** of discretization method you like.

(e) Start from $\Delta x = \Delta y = {}^{1}/_{8}$. If $\Delta x = \Delta y = {}^{1}/_{8}$, How many unknowns on the grid ? (This is the size of your matrix.)

(f) Use **Any** of 2D numerical scheme to solve (14) and make sure your result is consistent with (a). Usually the discretization equation of (14) is applied to the interior point and the boundary discretization equation you derived in (d) is applied to the boundary point and the total number of equation of them is equal to the number of unknowns.

4. Base on problem 3, Consider the heat conduction equation which the thermal diffusivity α of copper is given by $10^{-4}\ m^2/s$:

$$\frac{\partial T}{\partial t} = \alpha \nabla^2 T \tag{15}$$

And initial condition :

$$T(x, y, 0) = 20\ (°C) \tag{16}$$

(a) Use **Any** of numerical scheme to solve (15) and make sure your grid is small enough so that the numerical solution is converge.

(b) Suppose the steady solution is $T_s(x, y)$, which is the result of problem 3, and the steady time t_s is defined as the minimal time t to make

$$\max_{(x,y)\in[0,1]\times[0,1]} \frac{|T(x, y, t) - T_s(x, y)|}{|T_s(x, y)|} < 0.01 \tag{17}$$

Or, we can write

$$t_s = \min\left\{t \,\middle|\, \max_{(x,y)\in[0,1]\times[0,1]} \frac{|T(x, y, t) - T_s(x, y)|}{|T_s(x, y)|} < 0.01\right\} \tag{18}$$

What is t_s is this problem? Or in other words, how much time does it take to reach steady state with all place in 1% relative error?

(c) Plot the spatial profile (you can just plot on x direction) before the steady state on each of $t_s/5$ in 1 figure. Note : In your simulation you may not go through the numerical time at $t_s/5$ exactly, in this case you can just pick a closest one to represent as the profile at $t = t_s/5$.