A new efficient simulation method for multiple access channel

Yicheng CUI
Supervisor: Cheuk Ting LI

August 14, 2023

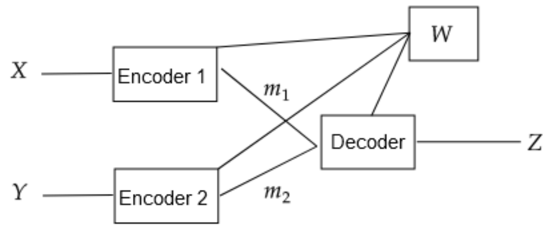
1 Introduction

1.1 Background and Literature Review

In recent years, the multiple access channel is getting more and more important among people. A lot of studies have been conducted on this topic. For example, [4, 6] discribed the rate region of the Quadratic Gaussian CEO problem, which is based on a setting of multiple access channel. However, while how to deal with the point-to-point channel is well known to people,such as [5] and [1], the channel simulation method for multiple access channel is not well researched so far. For this paper, we proposed a new efficient simulation method for multiple access channel based on our understanding of the point-to-point channel. This paper servers as an introduction to the multiple access channel simulation to increase the efficiency of doing this.

1.2 Setting and Goal

In this section, we will focus on the setting of this problem. In this problem, X and Y are correlated sources, W is the common randomness. Encoder 1 observes W and encodes X into descriptions m_1. Encoder 2 observes W and encodes Y into descriptions m_2.Then m_1 and m_2 are transmitted to the decoder. This is a one-shot setting.



We set X and Y to be Jointly Gaussian. That means:

$$\begin{pmatrix} X \\ Y \end{pmatrix} \sim N\left(0, \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}\right).$$

Here ρ is the correlation coefficient taking value between -1 and 1.

Encoder 1 generates m_1 by a function f_1, i.e., $m_1 = f_1(X, W)$. Encoder 2 generates m_2 by a function f_2, i.e., $m_2 = f_2(X, W)$.

The decoder recieves m_1 and m_2 and generates the output $\hat{Z}$. The output $\hat{Z}$ is determined by m_1, m_2 and W, i.e., $\hat{Z} = g(m_1, m_2, W)$. We want the output $\hat{Z}$ follow the same distribution as $Z = aX + bY + N$, where N is the Gaussian noise, $N \sim N(0, \sigma_N)$.

The requirement of the desired output is to satify a small error in the total variation distance, i.e., we want the output satisfy $\delta_{TV}((X, Y, \hat{Z}), (X, Y, Z)) \leq \epsilon$.

Our goal is to find the trade off between $R_1 = H(m_1|W)$, which is the average length of m_1 encoded using Huffman conditional on W, and $R_2 = H(m_2|W)$, which is the average length of m_2 encoded using Huffman conditional on W where $\{(R1, R2) : \exists f_1, f_2, g \text{ s.t. the requirement is satisfied}\}$.

This problem is already tackled and solved properly in point-to-point channel, such as [1] [5]. With multiple access channel getting more and more common in practice, however, this problem has not been well studied. This study aims to start the discussion of this problem and solve for some basic results with the experience we have in point-to-point channel.

2 Scheme Design

2.1 Basic Idea

In section 1.1, we discussed the setting and the goal for this problem. In this section, we are going to start to work on a scheme for this problem. From 1.1, we knows that Z is a linear combination of X and Y with some Gaussian noise, we can suppose that $\hat{X}$ and $\hat{Y}$ are nosiy version of X and Y. Hence, we have:

1. Do a channel simulation to allow Encoder 1 and Decoder to simulate an additive noise channel: $\hat{X} - X \sim N(0, \sigma^2_{N_X})$, $\hat{X} - X \perp\!\!\!\perp X$.

2. Do a channel simulation to allow Encoder 2 and Decoder to simulate an additive noise channel: $\hat{B} - B \sim N(0, \sigma_{N_B}^2)$, $\hat{B} - B \perp\!\!\!\perp B$.

3. Solve σ_{N_X},σ_{N_B},$\tilde{a}$ and $\tilde{b}$ such that $\hat{Z} = \tilde{a}\hat{X} + \tilde{b}\hat{Y}$ follows the same distribution as Z.

where

$$N_X \sim N(0, \sigma_{N_X}^2),$$
$$N_B \sim N(0, \sigma_{N_B}^2).$$

We know that the decoder obtains $\hat{X}$ and $\hat{B}$, and X and Y are correlated.

To find an estimation of Y, what we need to do is:

$$\hat{Y} = \underset{Y:Y \bmod \alpha = \hat{B}}{\arg\min} |Y - \mathrm{E}[Y|\hat{X}]|. \tag{1}$$

Finally, we will show that the following theorem holds:

Theorem: The requirement and the goal is satisfied if $R_1 \leq \frac{1}{2}\log(1+\frac{1}{\sigma_{N_X}^2})$, $R_2 \leq \frac{1}{2}\log(1+\frac{\alpha^2}{4\sigma_{N_B}^2})$ and $\epsilon \leq 2Q(\frac{\alpha}{2\sigma_\theta})$, where $\sigma_\theta^2 = 1 - \frac{\rho^2}{1+\sigma_{N_X}^2} + \sigma_{N_B}^2$. This holds when $\tilde{a}^2\sigma_{N_X}^2 + \tilde{b}^2\sigma_{N_B}^2 = \sigma_N^2$, $\hat{Z}$ follows the same distribution as Z.

We will prove this in later parts.

2.2 Computation of $\mathrm{E}[Y|\hat{X}]$

The following sections are talking about the method to achieve this and evaluation.

We need to calculate $\mathrm{E}[Y|\hat{X}]$ because we know that Y and X are correlated in Chapter 1. To calculate this conditional expectation easily, we can use a technique called linear MMSE estimator. Suppose:

$$Y = \gamma\hat{X} + T, T \perp\!\!\!\perp \hat{X}. \tag{2}$$

Then,

$$\mathrm{E}[Y|\hat{X}] = \gamma\hat{X}. \tag{3}$$

By the property of covariance and variance of linear MMSE, we have:

$$\mathrm{Cov}(\hat{X}, Y) = \gamma\sigma_{\hat{X}}^2. \tag{4}$$

$$\gamma = \frac{\mathrm{Cov}(\hat{X}, Y)}{\sigma_{\hat{X}}^2} = \frac{\rho}{1+\sigma_{N_X}^2}. \tag{5}$$

$$\mathrm{Var}(Y - \gamma\hat{X}) = \sigma_Y^2 - \frac{\mathrm{Cov}(\hat{X}, Y)^2}{\sigma_{\hat{X}}^2} = 1 - \frac{\rho^2}{1 + \sigma_{N_X}^2}. \tag{6}$$

$\mathrm{Var}(Y - \gamma\hat{X})$ is also called the error covariance. By the property of linear MMSE, we can put everything into the expression for $\hat{Y}$:

$$\hat{Y} = \frac{\mathrm{Cov}(X, Y)}{\sigma_X^2} X. \tag{7}$$

Now, it is time for the decoder to "guess" a possible value of Y. Let the estimated value for Y of the decoder be $\hat{Y}$. We have calculated $\mathrm{E}[Y|\hat{X}]$ in the previous part. By the property of Gaussian distribution, we know that the distribution of $Y|\hat{X}$ is:

$$Y|\hat{X} \sim N(\gamma\hat{X}, 1 - \frac{\rho^2}{1 + \sigma_{N_X}^2}). \tag{8}$$

Since $Y = K\alpha + B$ and the decoder knows $\hat{B} = B + N_B$, what the decoder guess is $\widetilde{K}$. As we already know the distribution of $Y|\hat{X}$ and it is Gaussian distribution, we can have the following conclusion:

$$\hat{Y} = \underset{Y:Y \bmod \alpha = \hat{B}}{\arg\min} |Y - \gamma\hat{X}|. \tag{9}$$

$$\hat{Y} = \widetilde{K}\alpha + \hat{B}. \tag{10}$$

Since $\hat{B} = B + N_B$, $\widetilde{K} = K$ if and only if

$$|Y - \gamma\hat{X} + N_B| \leq \frac{\alpha}{2}. \tag{11}$$

By the property of Gaussian distribution and variance, we know that:

$$\mathrm{Var}(Y - \gamma\hat{X} + N_B) = 1 - \frac{\rho^2}{1 + \sigma_{N_X}^2} + \sigma_{N_B}^2. \tag{12}$$

$$Y - \gamma\hat{X} + N_B \sim N(0, 1 - \frac{\rho^2}{1 + \sigma_{N_X}^2} + \sigma_{N_B}^2). \tag{13}$$

3 Analysis and Evaluation

3.1 Analysis on Link Capacity

After the basic setting of this problem, we can analyse the link capacity for this multiple access channel. We can get an upper bound by taking the mutual information. For example, we know that the link at the bottom transmits B, and the decoder receives $B + N_B$. Define $\hat{B} = B + N_B$. Hence, we can take the mutual information to get an upper bound of the bottom link capacity [2]:

$$
\begin{aligned}
I(B; \hat{B}) &= h(\hat{B}) - h(\hat{B}|B) && (14)\\
&= h(\hat{B}) - h(N_B) && (15)\\
&= h(B + N_B) - h(N_B) && (16)\\
&\leq \frac{1}{2}\log(2\pi e(\frac{\alpha^2}{4} + \sigma_{N_B}^2)) - \frac{1}{2}\log(2\pi e\sigma_{N_B}^2) && (17)\\
&\leq \frac{1}{2}\log(1 + \frac{\alpha^2}{4\sigma_{N_B}^2}). && (18)
\end{aligned}
$$

With the same method, we can get the top link capacity:

$$
I(X; \widetilde{X}) \leq \frac{1}{2}\log(1 + \frac{1}{\sigma_{N_X}^2}). \tag{19}
$$

3.2 Total Variation Distance Estimation

It is fine to estimate the distribution of $\hat{Y} - Y$. However, it turns out that the result will be a convolution of a Gaussian random variable with a bunch of delta functions, which is hard to solve and estimate precisely. Instead of finding the distribution of $\hat{Y} - Y$, we can focus on the total variation distance.

In section 2.2, the decoder estimated $\hat{Y} = \widetilde{K}\alpha + \hat{B}$. According to this result, the actual output will be $\hat{Z} = \tilde{a}\hat{X} + \tilde{b}\hat{Y}$. Suppose in section 2.2, the decoder estimated K correctly, i.e., $\widetilde{K} = K$. Then $\hat{Y}' = K\alpha + \hat{B}$. We want to create $\hat{Z}'$ that has the correct distribution, and make $\hat{Z}' = \hat{Z}$ with high probability. Suppose $\hat{Z}' = \tilde{a}\hat{X} + \tilde{b}\hat{Y}'$.

We can estimate the total variation distance between $(X, Y, \hat{Z})$ and (X, Y, Z):

$$
\begin{aligned}
\delta_{TV}((X, Y, \hat{Z}), (X, Y, Z)) &\leq \delta_{TV}((X, Y, \hat{Z}'), (X, Y, Z)) + \delta_{TV}((X, Y, \hat{Z}'), (X, Y, \hat{Z})) && (20)\\
&\leq P(K \neq \widetilde{K}) && (21)\\
&= 2Q(\frac{\alpha}{2\sigma_\theta}). && (22)
\end{aligned}
$$

where $\sigma_\theta^2 = 1 - \frac{\rho^2}{1+\sigma_{N_X}^2} + \sigma_{N_B}^2$. This holds when $\tilde{a}^2\sigma_{N_X}^2 + \tilde{b}^2\sigma_{N_B}^2 = \sigma_N^2$.

3.3 One-Shot Channel Simulation

This part aims to find the minimum amount of communication needed to simulate this setup. Suppose m_1 and m_2 are prefix-free descriptions of X and Y. The problem is to find the minimum expected description length of m_1 and m_2, that is to say, $\mathrm{E}[L(m_1)]$ and $\mathrm{E}[L(m_2)]$. Several setting of this problem have been studied. One of them is [3]. The study shows that for one-shot channel simulation with unlimited common randomness setup in which Alice and Bob share unlimited common randomness W, Alice observes $X \sim P_X$ and sends a prefix-free description M to Bob via a noiseless channel such that Bob can generate Y (from M and W) according to a prescribed conditional distribution $P_{Y|X}$, the expected length is

$$\mathrm{E}[L(M)] \leq I(X;Y) + \log(I(X;Y)+1) + 5. \tag{23}$$

To apply to our setting, let m_1 and m_2 be prefix-free descriptions (Huffman code for example), the expected length of m_1 is

$$\begin{aligned}\mathrm{E}[L(m_1)] &\leq I(X;\widetilde{X}) + \log(I(X;\widetilde{X})+1) + 5 &(24)\\ &\leq \frac{1}{2}\log(1+\frac{1}{\sigma^2_{N_X}}) + \log(\frac{1}{2}\log(1+\frac{1}{\sigma^2_{N_X}})+1) + 5. &(25)\end{aligned}$$

With the same method, we can get the expected length of $m2$

$$\begin{aligned}\mathrm{E}[L(m_2)] &\leq I(B;\hat{B}) + \log(I(B;\hat{B})+1) + 5 &(26)\\ &\leq \frac{1}{2}\log(1+\frac{\alpha^2}{4\sigma^2_{N_B}}) + \log(\frac{1}{2}\log(1+\frac{\alpha^2}{4\sigma^2_{N_B}})+1) + 5. &(27)\end{aligned}$$

4 Conclusion

With all of the results above, we can prove that the theorem stated in section 2.1 is true. The requirement and the goal is satisfied if $R_1 \leq \frac{1}{2}\log(1+\frac{1}{\sigma^2_{N_X}})$, $R_2 \leq \frac{1}{2}\log(1+\frac{\alpha^2}{4\sigma^2_{N_B}})$ and $\epsilon \leq 2Q(\frac{\alpha}{2\sigma_\theta})$, where $\sigma^2_\theta = 1 - \frac{\rho^2}{1+\sigma^2_{N_X}} + \sigma^2_{N_B}$. This holds when $\tilde{a}^2\sigma^2_{N_X} + \tilde{b}^2\sigma^2_{N_B} = \sigma^2_N$, $\hat{Z}$ follows the same distribution as Z.

References

[1] Michael X Cao, Navneeth Ramakrishnan, Mario Berta, and Marco Tomamichel. One-shot point-to-point channel simulation. In *2022 IEEE International Symposium on Information Theory (ISIT)*, pages 796–801. IEEE, 2022.

[2] Thomas M Cover. *Elements of information theory*. John Wiley & Sons, 1999.

[3] Cheuk Ting Li and Abbas El Gamal. Strong functional representation lemma and applications to coding theorems. *IEEE Transactions on Information Theory*, 64(11):6967–6978, 2018.

[4] Vinod Prabhakaran, David Tse, and Kannan Ramachandran. Rate region of the quadratic gaussian ceo problem. In *International Symposium onInformation Theory, 2004. ISIT 2004. Proceedings.*, page 119. IEEE, 2004.

[5] Sergio Verdú et al. A general formula for channel capacity. *IEEE Transactions on Information Theory*, 40(4):1147–1157, 1994.

[6] Aaron B Wagner, Saurabha Tavildar, and Pramod Viswanath. Rate region of the quadratic gaussian two-encoder source-coding problem. *IEEE Transactions on Information Theory*, 54(5):1938–1961, 2008.

Academic Honesty Declaration Statement

Submission Details (via VeriGuide)

Student Name	CUI, Yicheng
Login ID	1155157067@link.cuhk.edu.hk
Academic Year	2022-2023
Term	2
Course Code	ENGG-0001-A
Course Title	FDC
Assignment Marker	TJONG, Joyce
Assignment Number	2
Due Date (provided by student)	2023-08-14
Submitted File Name	CUI Yicheng_Final Report.pdf
Submission Time	2023-08-14 13:29:21
Submission Reference Number	3727683

I confirm that the above submission details are correct.

Agreement on Student's Work Submitted to VeriGuide

Cui Yicheng	14/08/2023
Signature	Date

CUI Yicheng

Name

Submission Overview

Submission Information

Submission Reference ID	3727683
School / Institution	Faculty of Engineering, CUHK
Year	2022
Term	2
Course	ENGG-0001-A
Title	FDC
Assignment Number / Name	2
Assignment Marker	TJONG, Joyce
Student	CUI, Yicheng
Student's School ID	1155157067@link.cuhk.edu.hk
File Name	CUI Yicheng_Final Report.pdf
Submitted on	2023-08-14 13:29:21+0800

This submission contains the following files:

Show 100 entries Search:

File ID	File	Status	Checking Events	Is an Archive?	Inside Archive	Similarity	Action
130941284	CUI Yicheng_Final Report.pdf	Checked	1 [details]	No	-	8.70%	View Details

Showing 1 to 1 of 1 entries

First Previous 1 Next Last