A new efficient simulation method for multiple access channel

Yicheng Cui

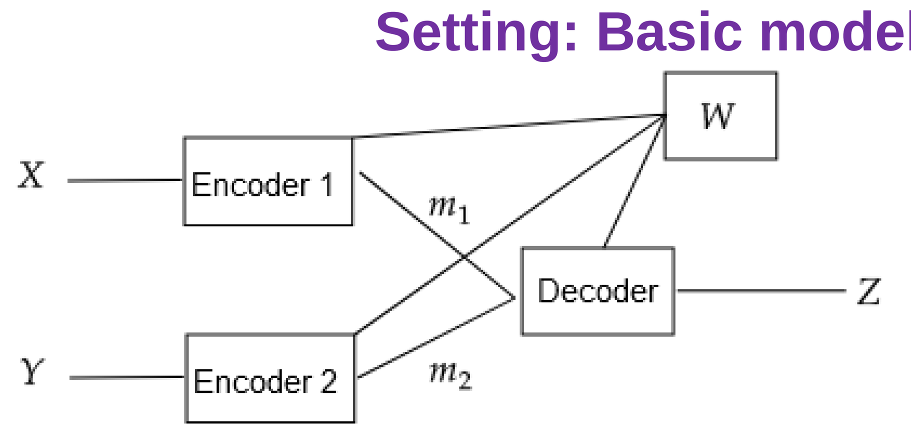
Supervisor: Cheuk Ting Li

Department of Information Engineering, Faculty of Engineering, The Chinese University of Hong Kong, Hong Kong.

Background

- The multiple access channel is widely used in reality. Usually there are one decoder and more than two encoders.
- By simulating multiple access channel, we can solve a lot of problems . For example, the rate region of the quadratic Gaussian CEO problem[2].
- We know point-to-point channel well. However, the channel simulation method for the multiple assess channel is still not well researched.
- This poster serves as an introduction to the channel simulation of the multiple access channel. It aims to solve for some basic results with the experience we have in point-to-point channel.

Setting: Basic model, Requirement and Goal



$$\begin{pmatrix} X \\ Y \end{pmatrix} \sim N\left(0, \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}\right)$$

W: the common randomness

m_1, m_2: descriptions using Huffman

Z: An output determined by all above things. Some linear combination of X, Y plus some Gaussian noise $N \sim N(0, \sigma_N)$.

$$Z = aX + bY + N$$

Requirement:

1. We would like our output $\hat{Z}$ follow the same distribution of Z. $\hat{Z} = \tilde{a}\hat{X} + \tilde{b}\hat{Y}$

2. We want the output satisfy $\delta_{TV}((X, Y, \hat{Z}), (X, Y, Z) \le \epsilon$

Goal:

We want to find the trade off between the average length of m_1, m_2 ($R_1 = H(m_1|\mathrm{W})$ and $R_1 = H(m_1|\mathrm{W})$).

Scheme Design: Basic Idea

1. Choose α such that $Y = K\alpha + B$, $0 \le B < \alpha$
2. Do a channel simulation to allow Encoder 1 and Decoder to simulate an additive noise channel, get $\hat{X}$.
3. 1. Do a channel simulation to allow Encoder 2 and Decoder to simulate an additive noise channel, get $\hat{B}$.
4. Solve σ_{N_X}, σ_{N_B}, $\tilde{a}$, $\tilde{b}$ such that our output $\hat{Z}$ follow the same distribution of Z. $\hat{Z} = \tilde{a}\hat{X} + \tilde{b}\hat{Y}$

$$\hat{X} - X \sim N(0, \sigma_{N_X}^2),\ \hat{X} - X \perp\!\!\!\perp X.$$

$$\hat{B} - B \sim N(0, \sigma_{N_B}^2),\ \hat{B} - B \perp\!\!\!\perp B.$$

$$\hat{Y} = \underset{Y: Y \bmod \alpha = \hat{B}}{\arg\min} |Y - \mathrm{E}[Y|\hat{X}]|.$$

Here, we state this theorem. We are going to prove this theorem in later parts.

Theorem: The requirement and the goal is satisfied if $R_1 \le \frac{1}{2}\log(1 + \frac{1}{\sigma_{N_X}^2})$, $R_2 \le \frac{1}{2}\log(1 + \frac{\alpha^2}{4\sigma_{N_B}^2})$ and $\epsilon \le 2Q(\frac{\alpha}{2\sigma_\theta})$, where $\sigma_\theta^2 = 1 - \frac{\rho^2}{1+\sigma_{N_X}^2} + \sigma_{N_B}^2$. This holds when $\tilde{a}^2\sigma_{N_X}^2 + \tilde{b}^2\sigma_{N_B}^2 = \sigma_N^2$, $\hat{Z}$ follows the same distribution as Z.

Scheme Design: Computation of $E[Y|\hat{X}]$

Gaussian!

Using **linear MMSE** to solve $E[Y|\hat{X}]$

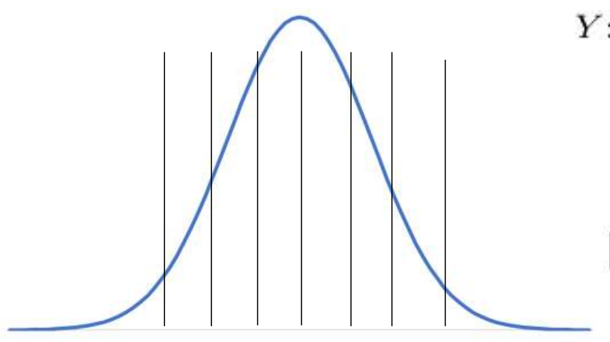
$$Y = \gamma\hat{X} + T, T \perp\!\!\!\perp \hat{X}.$$

$$\mathrm{E}[Y|\hat{X}] = \gamma\hat{X}.$$

$$\mathrm{Cov}(\hat{X}, Y) = \gamma\sigma_{\hat{X}}^2.$$

$$\gamma = \frac{\mathrm{Cov}(\hat{X}, Y)}{\sigma_{\hat{X}}^2} = \frac{\rho}{1 + \sigma_{N_X}^2}.$$

$$\mathrm{Var}(Y - \gamma\hat{X}) = \sigma_Y^2 - \frac{\mathrm{Cov}(\hat{X}, Y)^2}{\sigma_{\hat{X}}^2} = 1 - \frac{\rho^2}{1 + \sigma_{N_X}^2}.$$



$$Y|\hat{X} \sim N(\gamma\hat{X}, 1 - \frac{\rho^2}{1 + \sigma_{N_X}^2}).$$

$$\hat{Y} = \underset{Y: Y \bmod \alpha = \hat{B}}{\arg\min} |Y - \gamma\hat{X}|.$$

$\widetilde{K} = K$ if and only if

$$|Y - \gamma\hat{X} + N_B| \le \frac{\alpha}{2}.$$

$$\mathrm{Var}(Y - \gamma\hat{X} + N_B) = 1 - \frac{\rho^2}{1 + \sigma_{N_X}^2} + \sigma_{N_B}^2.$$

$$Y - \gamma\hat{X} + N_B \sim N(0, 1 - \frac{\rho^2}{1 + \sigma_{N_X}^2} + \sigma_{N_B}^2).$$

Now, we have the mean and the variance of $Y|\hat{X}$

Link Capacity and Total Variation Distance

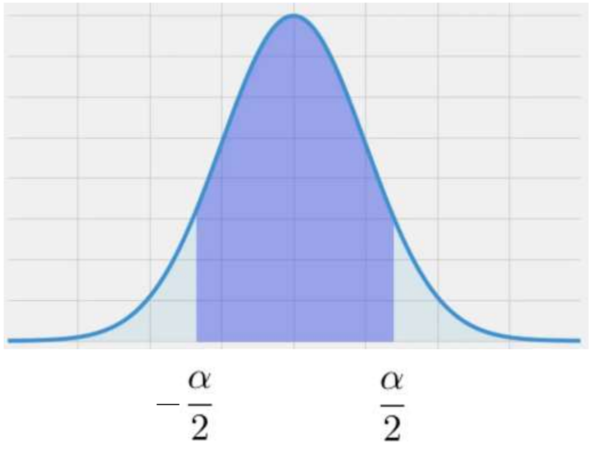
$$\begin{aligned} I(B; \hat{B}) &= h(\hat{B}) - h(\hat{B}|B) \\ &= h(\hat{B}) - h(N_B) \\ &= h(B + N_B) - h(N_B) \\ &\le \frac{1}{2}\log(2\pi e(\frac{\alpha^2}{4} + \sigma_{N_B}^2)) - \frac{1}{2}\log(2\pi e\sigma_{N_B}^2) \\ &\le \frac{1}{2}\log(1 + \frac{\alpha^2}{4\sigma_{N_B}^2}). \end{aligned}$$

$$I(X; \widetilde{X}) \le \frac{1}{2}\log(1 + \frac{1}{\sigma_{N_X}^2}).$$

The decoder estimated $\hat{Y} = \widetilde{K}\alpha + \hat{B}$, the actual output will be: $\hat{Y} = \widetilde{K}\alpha + \hat{B}$.

Suppose the decoder estimated K correctly, i.e., $\widetilde{K} = K$. Then $\hat{Y}' = K\alpha + \hat{B}$.

We want to create $\hat{Z}'$that has the correct distribution, and make $\hat{Z}' = \hat{Z}$ with high probability. Suppose $\hat{Z}' = \tilde{a}\hat{X} + \tilde{b}\hat{Y}'$.



We can estimate the total variation distance between $(X, Y, \hat{Z})$ and (X, Y, Z):

$$\delta_{TV}((X, Y, \hat{Z}), (X, Y, Z)) \le \delta_{TV}((X, Y, \hat{Z}'), (X, Y, Z)) + \delta_{TV}((X, Y, \hat{Z}'), (X, Y, \hat{Z})) \le P(K \ne \widetilde{K}) = 2Q(\frac{\alpha}{2\sigma_\theta}).$$

One-Shot Channel Simulation

This part aims to find the minimum amount of communication needed to simulate this setup. Suppose m_1 and m_2 are prefix-free descriptions of X and Y. The problem is to find the minimum expected description length of m_1 and m_2. Several setting of this problem have been studied. One of them is [3]. With this result, we can get the average length.

$$\begin{aligned} \mathrm{E}[L(m_1)] &\le I(X; \widetilde{X}) + \log(I(X; \widetilde{X}) + 1) + 5 \\ &\le \frac{1}{2}\log(1 + \frac{1}{\sigma_{N_X}^2}) + \log(\frac{1}{2}\log(1 + \frac{1}{\sigma_{N_X}^2}) + 1) + 5. \end{aligned}$$

$$\begin{aligned} \mathrm{E}[L(m_2)] &\le I(B; \hat{B}) + \log(I(B; \hat{B}) + 1) + 5 \\ &\le \frac{1}{2}\log(1 + \frac{\alpha^2}{4\sigma_{N_B}^2}) + \log(\frac{1}{2}\log(1 + \frac{\alpha^2}{4\sigma_{N_B}^2}) + 1) + 5. \end{aligned}$$

Conclusion

With all of the results above, we can prove that the theorem stated above is true. The requirement and the goal is satisfied if $R_1 \le \frac{1}{2}\log(1 + \frac{1}{\sigma_{N_X}^2})$, $R_2 \le \frac{1}{2}\log(1 + \frac{\alpha^2}{4\sigma_{N_B}^2})$ and $\epsilon \le 2Q(\frac{\alpha}{2\sigma_\theta})$, where $\sigma_\theta^2 = 1 - \frac{\rho^2}{1+\sigma_{N_X}^2} + \sigma_{N_B}^2$. This holds when $\tilde{a}^2\sigma_{N_X}^2 + \tilde{b}^2\sigma_{N_B}^2 = \sigma_N^2$, $\hat{Z}$ follows the same distribution as Z.

References

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[3] C. T. Li and A. El Gamal, "Strong Functional Representation Lemma and Applications to Coding Theorems," in IEEE Transactions on Information Theory, vol. 64, no. 11, pp. 6967-6978, Nov. 2018, doi: 10.1109/TIT.2018.2865570.

Contact:

Yicheng Cui

Email: yccui@link.cuhk.edu.hk

Supervisor: Cheuk Ting Li

Email: ctli@ie.cuhk.edu.hk