$A = \{a, b, \dots\}$

$a \cdot b = 0$

$a = 0$ or $b = 0$) a: real number

$AB = 0$

$A = 0$ or $B = 0$ × zero divison

Abstract Algebra (현대 대수학).

Group (군)

closed

binary operation (이항연산자)

$\langle G, * \rangle$

$a, b \in G \quad a*b \in G$

(i) $(a*b)*c = a*(b*c)$; Associative.

(ii) $\exists\, e \in G$ s.t. $a*e = e*a = a$

identity

(iii) $\forall a \in G, \ \exists\, b \in G$ s.t. $a*b = b*a = e$

inverse.

Ring / Field.

$\langle R, +, \cdot \rangle$

Commutative,

$\langle R, + \rangle$; abelian group,

$\langle R, \cdot \rangle$; semi-group ; associative.

Homomorphism

$a_1, a_2 \cdots \quad b_1, b_2 \ldots$

$\quad\ \ \pi \qquad\quad \pi$

1. Group homo - ; $\varphi : G_1 \to G_2$

$$\varphi(a_1 * a_2) = \underbrace{\varphi(a_1)}_{G_2} * \underbrace{\varphi(a_2)}_{G_2}$$

2. Ring homo -

$\varphi : R_1 \to R_2$

$$\varphi(a \underset{R_1}{\cdot} b) = \varphi(a) \underset{R_2}{\cdot} \varphi(b)$$

Isomorphism

동형이 .

φ : homomorphism .

bijective

bijection

$\mathbb{Z}_n = \{0, 1, 2, \cdots, n-1\}$

$\mathbb{Z}_6 = \{0, 1, 2, 3, 4, 5\}$ ✓

$$3 \neq 9$$
$$\downarrow \qquad \downarrow$$
$$3 = 3$$

$2 \cdot 3 = 6 = 0$

zero divison

$a \cdot b = 0$

$a \neq 0,\ b \neq 0$

zero divison

zero divison.

§2.4 Invertible & Isomorphism.

Rings and Fields

(Def) Group, Ring, Field,

Homomorphism, Isomorphism

φ: ring homomorphism, bijective

Unity ; multiplication identity

uniqueness?

Unit ; For a ring with unity, if $a \in R$ has mult. inverse

$\Rightarrow$ unit

Division Ring : every non-zero elements is a unit

Field : Commutative division ring

multiplicate.

Subring

<example> $Ra = \{ ra \mid r \in R \}$; a subring generated by a.

Nonempty subset of R is subring iff $a-b \in S$, $ab \in S$

Zero-divisor ;

$\underbrace{a\,b}_{\text{zero divisor}} = 0$ but $a \neq 0$ and $b \neq 0$

In the ring $\mathbb{Z}_n$,
$a \in \mathbb{Z}$ is a zero divisor iff $\gcd(a,n) \neq 1$) $\Rightarrow$ $\mathbb{Z}_p$ with p: prime has no zero divisor

Cancellation law iff no zero divisors.

Integral Domain

Commutative ring with unity and there is no zero divisor.

Every field is a integral domain.

Inverse

$T: A \to B$

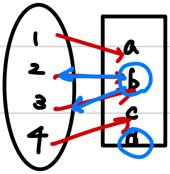
$T^{-1}: B \to A$ } invertiable.

$(T \circ T^{-1})(x) = x$

Inverse

* inverse function.

$A \to B$ bijective



Inverse of linear transformation

* linear transformation *

$[T]_\beta^\gamma$ ↲ Matrix

$T: \overset{n}{V} \to \overset{n}{W}$; linear transformation.

If $L: W \to V$ s.t. inverse of T exist,

then T: isomorphism.

(c) $\beta = \{2x^2 - x, 3x^2 + 1, x^2\}$ and $\beta' = \{1, x, x^2\}$

(d) $\beta = \{x^2 - x + 1, x + 1, x^2 + 1\}$ and

$\beta' = \{x^2 + x + 4, 4x^2 - 3x + 2, 2x^2 + 3\}$

$P_2(\mathbb{R})$ = 실수계수를 갖는 이차이하의 다항식

β, β' be an ordered basis of $P_2(\mathbb{R})$

$p(x) = a_0 + a_1 x + a_2 x^2 \in P_2(\mathbb{R})$

(i) β에 대해 표현

$b_1(x^2 - x + 1) + b_2(x+1) + b_3(x^2+1) = p(x)$

$[p(x)]_\beta = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

$[p(x)]_\beta = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \Rightarrow 1 \cdot (x^2 - x + 1) + 2 \cdot (x+1) + 3 \cdot (x^2+1)$

(ii) β'에 대해 표현

$p(x) = c_1(x^2 + x + 4) + c_2(4x^2 - 3x + 2) + c_3(2x^2 + 3)$

$[p(x)]_{\beta'} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$

$[p(x)]_\beta = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ 일 때 $\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$

$[T]_{\beta}^{\gamma}$ 1. $T(\beta)$를 γ에 대해서.

(c) $\beta = \{2x^2 - x, 3x^2 + 1, x^2\}$ and $\beta' = \{1, x, x^2\}$ (v_1, v_2, v_3)

(d) $\beta = \{x^2 - x + 1, x + 1, x^2 + 1\}$ and
$\beta' = \{x^2 + x + 4, 4x^2 - 3x + 2, 2x^2 + 3\}$ (w_1, w_2, w_3)

Note that both of β and β' are ordered basis of $P_2(\mathbb{R})$

$[I]_\beta^{\beta'} = ?$

Step 1. $[v_1]_{\beta'}$, $[v_2]_{\beta'}$, $[v_3]_{\beta'}$

= change of coordinate matrix.

$x^2 - x + 1 = a_1(x^2+x+4) + a_2(\quad) + a_3(\quad)$.

$V = \mathbb{R}^2$.

(a) $\beta = \{e_1, e_2\}$ and $\beta' = \{(a_1, a_2), (b_1, b_2)\}$

(b) $\beta = \{(-1, 3), (2, -1)\}$ and $\beta' = \{(0, 10), (5, 0)\}$

(c) $\beta = \{(2, 5), (-1, -3)\}$ and $\beta' = \{e_1, e_2\}$

$V_1 \quad V_2$

$[V_1]_{\beta'}$ $(-1,3) = a_1(0,10) + a_2(5,0)$
$= (5a_2, 10a_1)$ $\Rightarrow [V_1]_{\beta'} = \begin{bmatrix} \frac{3}{10} \\ -\frac{1}{5} \end{bmatrix}$

$[V_2]_{\beta'}$ $(2,-1) = (5a_2, 10a_1)$

$\Rightarrow [V_2]_{\beta'} = \begin{bmatrix} -\frac{1}{10} \\ \frac{2}{5} \end{bmatrix}$

$\therefore [I]_\beta^{\beta'} = \begin{bmatrix} \frac{3}{10} & -\frac{1}{10} \\ -\frac{1}{5} & \frac{2}{5} \end{bmatrix}$

$$\beta = \{ \underset{v_1}{x^2 - x},\ \underset{v_2}{x^2+1},\ \underset{v_3}{x-1} \}$$

$$\beta' = \{ 5x^2-2x-3,\ -2x^2+5x+5,\ 2x^2-x-3 \}$$

$[v_1]_{\beta'}$

$$x^2 - x = a_1(5x^2-2x-3) + a_2(-2x^2+5x+5) + a_3(2x^2-x-3)$$

$$= (5a_1 - 2a_2 + 2a_3)x^2 + (-2a_1 + 5a_2 - a_3)x + (-3a_1 + 5a_2 - 3a_3)$$

$$5a_1 - 2a_2 + 2a_3 = 1$$
$$-2a_1 + 5a_2 - a_3 = -1 \quad \to \quad a_3 = -2a_1 + 5a_2 + 1$$
$$-3a_1 + 5a_2 - 3a_3 = 0$$

$$5a_1 - 2a_2 + 2(-2a_1 + 5a_2 + 1) = 1$$
$$a_1 + 8a_2 = -1$$
$$a_1 = -1 - 8a_2$$
$$= -\frac{41}{17}$$

$$-3a_1 + 5a_2 - 3(-2a_1 + 5a_2 + 1) = 0$$
$$3a_1 - 10a_2 = 3$$
$$-\ \cancel{3a_1 + 24a_2 = -3}$$
$$-34a_2 = 6$$
$$a_2 = \frac{6}{34} = \frac{3}{17}$$

$$a_3 = \frac{82}{17} + \frac{15}{17} + \frac{17}{17}$$
$$= \frac{114}{17}$$

$$[v_1]_{\beta'} = \begin{bmatrix} -\frac{41}{17} \\ \frac{3}{17} \\ \frac{114}{17} \end{bmatrix}$$

$[V_2]_{\beta'}$

$$x^2+1 = a_1(5x^2-2x-3)+a_2(-2x^2+5x+5)+a_3(2x^2-x-3)$$
$$= (5a_1-2a_2+2a_3)x^2+(-2a_1+5a_2-a_3)x + (-3a_1+5a_2-3a_3)$$

$$5a_1-2a_2+2a_3 = 1$$
$$-2a_1+5a_2-a_3 = 0 \quad \rightarrow \quad a_3 = -2a_1+5a_2$$
$$-3a_1+5a_2-3a_3 = 1$$

$$5a_1-2a_2+2(-2a_1+5a_2) = 1$$
$$a_1+8a_2 = 1$$
$$a_1 = 1+\frac{8}{7} = \frac{15}{7}$$
$$a_3 = -\frac{30}{7}-\frac{5}{7} = -\frac{35}{7} = -5$$

$$-3a_1+5a_2-3(-2a_1+5a_2) = 1$$
$$3a_1-10a_2 = 1$$
$$3a_1-24a_2 = 3$$
$$14a_2 = -2$$
$$a_2 = -\frac{1}{7}$$

$$[V_2]_{\beta'} = \begin{bmatrix} \frac{15}{7} \\ -\frac{1}{7} \\ -5 \end{bmatrix}$$

$[V_3]_{\beta'}$

$$x-1 = a_1(5x^2-2x-3)+a_2(-2x^2+5x+5)+a_3(2x^2-x-3)$$

$$= (5a_1-2a_2+2a_3)x^2+(-2a_1+5a_2-a_3)x + (-3a_1+5a_2-3a_3)$$

$$5a_1-2a_2+2a_3 = 0$$

$$-2a_1+5a_2-a_3 = 1 \quad \rightarrow \quad a_3 = -2a_1+5a_2-1$$

$$3a_1+5a_2-3a_3 = -1$$

$$5a_1-2a_2+2(-2a_1+5a_2-1)=0 \qquad 3a_1+5a_2-3(-2a_1+5a_2-1)=-1$$

$$a_1+8a_2 = 2 \qquad 9a_1-10a_2 = -4$$

$$a_1 = 2-\frac{88}{31} \qquad -\ \underline{9a_1-72a_2 = 18}$$

$$= -\frac{26}{31} \qquad 62a_2 = -22$$

$$a_2 = -\frac{11}{31}$$

$$a_3 = \frac{52}{31}+\left(-\frac{55}{31}\right)-1$$

$$= \frac{52-55-31}{31} = \frac{-34}{31}$$

$$[V_2]_{\beta'} = \begin{bmatrix} -\frac{26}{31} \\ -\frac{11}{31} \\ -\frac{34}{31} \end{bmatrix}$$

$$[I]_{\beta}^{\beta'} = \begin{bmatrix} -\frac{41}{17} & \frac{15}{7} & -\frac{26}{21} \\ \frac{3}{17} & -\frac{1}{7} & -\frac{11}{31} \\ \frac{14}{17} & -5 & -\frac{36}{21} \end{bmatrix}$$

[Determinant] $n \times n$

⟨Def⟩

For a 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\det(A) = |A| = ad - bc$

(ex) $B = \begin{pmatrix} 3 & 2 \\ -1 & 0 \end{pmatrix}$ $\det(B) = 3\cdot 0 - (-1)\cdot 2 = 2$.

⟨Theorem⟩

$M_{2\times 2} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $\det(M) \neq 0 \overset{\text{iff (if and only if)}}{\longleftrightarrow}$ inverse matrix

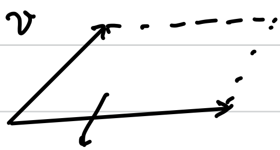
$$\Downarrow$$

$$\frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$\det(M_{2\times 2}) = ?$

$\mathbb{R}^2$ $v = \begin{bmatrix} a \\ c \end{bmatrix}$

$u = \begin{bmatrix} b \\ d \end{bmatrix}$



$A = \det\begin{pmatrix} v & u \end{pmatrix}$

Determinants of $M_{n\times n}$ $(n \geq 3)$

$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j}\, a_{ij} \cdot \det(\tilde{A}_{ij})$$

1행 1열

$$\begin{pmatrix} \boxed{0 \quad 1 \quad 2} \\ -1 \quad 0 \quad -3 \\ 2 \quad 3 \quad 0 \end{pmatrix}$$

$$(-1)^{1+1} \cdot 0 \cdot \det\begin{pmatrix} 0 & -3 \\ 3 & 0 \end{pmatrix} + (-1)^{1+2} \cdot 1 \cdot \det\begin{pmatrix} -1 & -3 \\ 2 & 0 \end{pmatrix}$$

$$+ (-1)^{1+3} \cdot 2 \cdot \det\begin{pmatrix} -1 & 0 \\ 2 & 3 \end{pmatrix}$$

$$= (-1) \times 1 \times 6 + 1 \times 2 \cdot (-3) = -12 \neq 0.$$

$\det(I_n) = 1$

$(-1)^2 \times 1 \times x$

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow 1$$

$\to I \; (n-1)\times(n-1)$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$)$

Elementary Row Operation

1. 서로 다른 두 행의 위치를 바꾼다 B $\det(A) = -\det(B)$

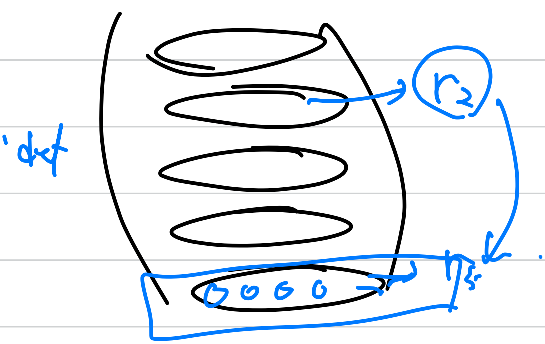
2. 한 행에 스칼라배 해준다 A의 한 행에 k배. = B $k\det(A) = \det(B)$

3. 한 행에 스칼라배를 한 후 다른 행에 더해준다. det가 같음.

ex)

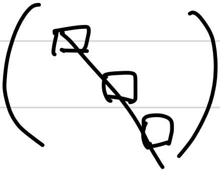
$$\det\left(\begin{pmatrix} 2 & 3 & 1 \\ 4 & 8 & 2 \\ 1 & 5 & 7 \end{pmatrix}\right) = \det\left(\begin{pmatrix} 2 & 3 & 1 \\ 0 & 2 & 0 \\ 1 & 5 & 7 \end{pmatrix}\right)$$

A가 같은 행을 두개 $\det(A) = 0$



$r_2 = r_5$

대각행렬



det

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -3 \end{pmatrix} \qquad 1 \cdot 2 \cdot (-3),$$

[Inverse matrix]

(Def)

For $n\times n$ matrix A, if there exist a $n\times n$ matrix B such that $AB=BA=I_n$

$AB = I$ $p\times q$, $r\times s$, $n\times n$ $p = n$, $s = n$ $q=r$

$BA = I$ $r\times s$, $p\times q$, $n\times n$ $r = n$, $q = n$ $q=r=n$.

⟨Determinant⟩ ; $n\times n$ matrix

<Determinant>