Vector space

$\langle V, +, \cdot \rangle$

1. $x+y = y+x$
2. $x+(y+z) = (x+y)+z$
3. $x+0 = x$ (additive identity)
4. $x + x^{-1} = 0$ (" inverse)
5. $1 \cdot x = x$ (scalar multi- identity)
6. $c(x+y) = cx + cy$
7. $(c+d)x = cx + dx$
8. $(ab)x = a(bx)$.

$\mathbb{R}^n$, P_n n차 이하인 다항식 $M_{m \times n}(\mathbb{R})$, . .

Vector space (벡터 공간) — finite ⇒ $\{0\}$: zero vector space.
— infinite

Subspace (부분공간) : 1. subset of Vector space
W — subset; V — Vector space

2. vector space

$\forall x, y \in W,\ c \in F$

① $cx + y \in W$

② $0 \in W$

linear combination (선형결합)

" vector " $v_1, v_2, \cdots, v_n$

linear combination of $v_1, v_2, \cdots, v_n$

$= a_1 v_1 + a_2 v_2 + \cdots + a_n v_n$ where $a_i \in F$.

- **linearly (in) dependent**

$a_1 v_1 + a_2 v_2 + a_3 v_3 + \cdots + a_n v_n = 0$ 라고 할때,

$a_1 = a_2 = \cdots = a_n = 0 \Rightarrow$ linearly independent.

" 가 아닌 경우 $\Rightarrow$ " " dependent.

Q1. $(2,3,1)$, $(-1,5,2)$, $(0,1,1)$, $(1,2,0)$

linearly independent / dependent

$\Rightarrow$

$$a_1(2,3,1) + a_2(-1,5,2) + a_3(0,1,1) + a_4(1,2,0) = 0$$

$2a_1 - a_2 + \quad + a_4 = 0$

$3a_1 + 5a_2 + a_3 + 2a_4 = 0$

$a_1 + 2a_2 + a_3 \quad = 0$

$a_3 = -a_1 - 2a_2$ $\qquad$ $3a_1 + 5a_2 - a_1 - 2a_2 - 4a_1 + 2a_2 = 0$

$a_4 = -2a_1 + a_2$ $\qquad$ $-2a_1 + 5a_2 = 0$

$a_2 = \frac{2}{5}a_1$

a_1, $a_2 = \frac{2}{5}a_1$, $a_3 = -\frac{9}{5}a_1$, $a_4 = \frac{1}{5}a_1$

$\beta = \{v_1, v_2, \cdots, v_n\}$

Span(β)

$= \{v \mid v = a_1 v_1 + a_2 v_2 + \cdots + a_n v_n$ for some $a_i \in F\}$.

$\mathbb{R}^2$

· Basis of V ✓

1. span $\beta = V$

2. β linearly independent.

✓ $\{(1,2), (-1,0)\}$

✓ $\{(1,0), (0,1)\}$

Dimension

Fact.

1. $v_i \in V \Rightarrow \text{span}(\{v_1, v_2, \cdots, v_n\}) \subseteq V$

2. dimension = basis를 이루는 vector 개수

Q1. $\underset{v_1}{(2,3,1)}$, $\underset{v_2}{(-1,5,2)}$, $\underset{v_3}{(0,1,1)}$, $\underset{v_4}{(1,2,0)}$

linearly independent / dependent

$v_i \in \mathbb{R}^3 \Rightarrow$ 3

3개

Vector space V → $\dim(V) = \boxed{n}$

$v_i \in V$ for $i = 1, 2, \dots, n$ 에 대하여,

$v_1, \dots, v_n$ 이 linearly independent ⇒ $\{v_1, \dots, v_n\}$ is a basis of V

ex) $(2, 1, 0)$, $(-1, 1, 3)$ (x)

linear system (연립일차방정식)

$$\begin{cases} x_1 + 2x_2 - x_3 = -1 \\ 2x_1 + 2x_2 - x_3 = 1 \\ 3x_1 + 5x_2 - 2x_3 = -1 \end{cases} \longrightarrow \left[\begin{array}{ccc|c} 1 & 2 & -1 & -1 \\ 2 & 2 & -1 & 1 \\ 3 & 5 & -2 & -1 \end{array}\right]$$

Augmented matrix

Elementary row operation.

RREF

Elementary Row operation

1. 서로 다른 두 행의 위치를 바꾼다. ☆
2. 한 행에 스칼라곱을 해준다 ✳
3. 한 행에 스칼라곱을 한 후 다른 행에 더해준다. ★

$$\begin{cases} x_1 + 2x_2 - x_3 = -1 \\ 2x_1 + 2x_2 - x_3 = 1 \\ 3x_1 + 5x_2 - 2x_3 = -1 \end{cases} \qquad \left[\begin{array}{ccc|c} 1 & 2 & -1 & -1 \\ 2 & 2 & -1 & 1 \\ 3 & 5 & -2 & -1 \end{array}\right]$$

$$\Big\downarrow \; R_2 \Leftrightarrow R_3$$

식순서만 바뀜.

$$\begin{cases} x_1 + 2x_2 - x_3 = -1 \\ 3x_1 + 5x_2 - 2x_3 = -1 \\ 2x_1 + 2x_2 - x_3 = 1 \end{cases} \qquad \left[\begin{array}{ccc|c} 1 & 2 & -1 & -1 \\ 3 & 5 & -2 & -1 \\ 2 & 2 & -1 & 1 \end{array}\right]$$

$$\Big\downarrow \; R_1 \to 2R_1$$

$$\begin{cases} 2x_1 + 4x_2 - 2x_3 = -2 \\ 3x_1 + 5x_2 - 2x_3 = -1 \\ 2x_1 + 2x_2 - x_3 = 1 \end{cases} \qquad \left[\begin{array}{ccc|c} 2 & 4 & -2 & -2 \\ 3 & 5 & -2 & -1 \\ 2 & 2 & -1 & 1 \end{array}\right]$$

$$\begin{cases} 2x_1 + 4x_2 - 2x_3 = -2 \\ 3x_1 + 5x_2 - 2x_3 = -1 \\ 2x_1 + 2x_2 - x_3 = 1 \end{cases}$$

$$\left[\begin{array}{ccc|c} 2 & 4 & -2 & -2 \\ 3 & 5 & -2 & -1 \\ 2 & 2 & -1 & 1 \end{array}\right]$$

$$\downarrow \; R_3 \to -R_1 + R_3$$

$$\begin{array}{c} \begin{array}{cccc} -2 & -4 & 2 & 2 \end{array} \\ \left[\begin{array}{ccc|c} 2 & 4 & -2 & -2 \\ 3 & 5 & -2 & -1 \\ 0 & -2 & 1 & 3 \end{array}\right] \end{array} \checkmark$$

(기약)행 사다리꼴.

(Reduced) Row Echelon Form ⇒ (R)REF

REF

1. 모든 성분이 0인 행은 가장 아래에 있다
2. 각 행에서 처음으로 나오는 0이 아닌 수는 1이고, 이 1을 leading 1 (선도 1)이라고 한다.
3. 윗행의 leading 1은 아래행의 leading 1보다 왼쪽에 있다.
4. leading 1이 속한 column의 나머지 성분은 0이다.

RREF

(ex) leading one

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & -1 \\ 0 & 2 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \times$$

$$\left(\begin{array}{ccccc|c} 1 & 0 & 0 & 2 & 0 & 3 \\ 0 & 1 & 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & -1 & 4 \end{array}\right)$$

1 REF / 2 RREF / 3 None

$$\left(\begin{array}{ccccc|c} 1 & 0 & 0 & 1 & 2 & 5 \\ 0 & 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 & 1 & 3 \end{array}\right)$$

오전

목 9시

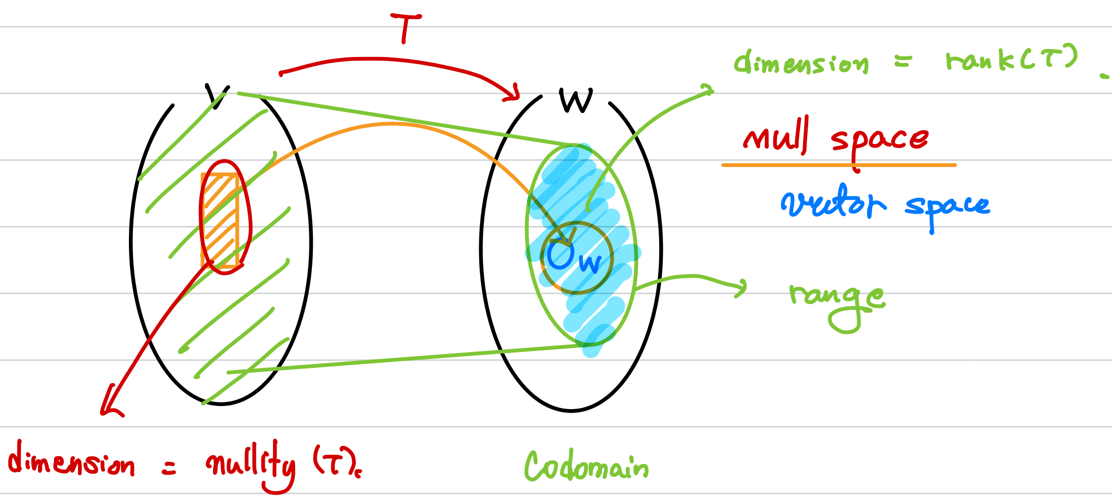
Linear Transformation

$T : V \longrightarrow W$

For any $v_1, v_2 \in V$, $\left(\begin{array}{l} T(v_1+v_2) = T(v_1)+T(v_2) \\ T(av_1) = aT(v_1) \end{array} \right)$ linearity

$T(a_1v_1 + a_2v_2 + \cdots + a_nv_n)$

$= a_1T(v_1) + a_2T(v_2) + \cdots + a_nT(v_n)$.



ordered

Vector space V with basis $\beta = \{v_1, \dots, v_n\}$

$v \in V \Rightarrow a_1 v_1 + a_2 v_2 + \cdots + a_n v_n = v$

$$[v]_\beta = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_n \end{bmatrix}$$

exercise 1.

$\mathbb{R}^3$ with ordered basis $\beta = \{\overset{v_1}{(1,0,1)}, \overset{v_2}{(1,1,0)}, \overset{v_3}{(0,1,1)}\}$

$$[(5,7,6)]_\beta = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$$

$a_1 + a_2 = 5$

$a_2 + a_3 = 7 \qquad a_3 = 7 - a_2$

$a_1 + a_3 = 6$

$a_1 + a_2 = 5$

$a_1 - a_2 = -1 \qquad a_1 = 2, \; a_2 = 3, \; a_3 = 4$

Definition. *Using the notation above, we call the $m \times n$ matrix A defined by $A_{ij} = a_{ij}$ the* ***matrix representation of*** T ***in the ordered bases*** β **and** γ *and write $A = [\mathsf{T}]_\beta^\gamma$. If $\mathsf{V} = \mathsf{W}$ and $\beta = \gamma$, then we write $A = [\mathsf{T}]_\beta$.*

$\wedge$

$T: V^{n} \longrightarrow W^{m}$

$T(v) = w \in W$

$T(v_1) = k_1 w_1 + k_2 w_2 + \cdots + k_m v_m$

$$\begin{pmatrix} k_1 \\ \vdots \\ k_m \end{pmatrix} = A의\ 1열$$

$$\mathsf{T}(v_j) = \sum_{i=1}^{m} a_{ij} w_i \quad \text{for } 1 \leq j \leq n.$$

$n \leftarrow V \to W \to m$

$$[T]_\beta^\gamma = A_{m \times n}$$

$$T(v_1) = k_1 w_1 + k_2 w_2 + \cdots + k_m w_m$$

$\begin{bmatrix} k_1 \\ \vdots \\ k_m \end{bmatrix}$ A의 1열

A의 j열

$[T(v_j)]_\gamma$

$$T(v_2) = s_1 w_1 + s_2 w_2 + \cdots + s_m w_n$$

$\begin{bmatrix} s_1 \\ \vdots \\ s_m \end{bmatrix}$ A의 2열,

$\vdots$

1×3

dimension $= 3$ $\quad$ 1 $\quad\quad$ $\dim(P_n) = n+1$

$T : \boxed{P_2(\mathbb{R})} \to \mathbb{R}$ by $\boxed{T(f(x)) = f(2)}$ $\quad\quad$ $P_n \cong \mathbb{R}^{n+1}$

$\Downarrow$

$1, x, x^2, \cdots, x^{n-1}, x^n$

$\underline{\beta} = \{ \overset{v_1}{1}, \overset{v_2}{x}, \overset{v_3}{x^2} \}$: ordered basis of $P_2(\mathbb{R})$

$\underline{\gamma = \{1\}}$ $\quad$ " $\quad$ " $\mathbb{R}$

Compute $[T]_\beta^\gamma$ $\quad\quad$ $f(x) = 1$

Step 1.

$T(v_1) = T(1) = 1 = 1 \cdot 1$

$T(v_2) = T(x) = 2 = 2 \cdot 1$

$T(v_3) = T(x^2) = 4 = 4 \cdot \underset{v_\gamma}{1}$

$[T]_\beta^\gamma = ([T(v_1)]_\gamma \quad [T(v_2)]_\gamma \quad [T(v_3)]_\gamma)$

$= (1 \quad 2 \quad 4)$

Define

$$\underset{3}{\mathsf{T}\colon \mathsf{P}_2(R)} \to \underset{4}{\mathsf{M}_{2\times 2}(R)} \quad \text{by} \quad \mathsf{T}(f(x)) = \begin{pmatrix} f'(0) & 2f(1) \\ 0 & f''(3) \end{pmatrix},$$

where $'$ denotes differentiation. Compute $[\mathsf{T}]_\beta^\alpha$.

4×3

$f(x) = x^2$

$f(x) = 1.$

$$\alpha = \left\{ \underset{w_1}{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}, \underset{w_2}{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}, \underset{w_3}{\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}}, \underset{w_4}{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}} \right\},$$; $M_{2\times2}(\mathbb{R})$ 의 ordered basis

$\beta = \{ \overset{v_1}{1}, x, x^2 \}$

$$[T]_\beta^\alpha = \begin{pmatrix} \boxed{0} & \boxed{1} & \boxed{0} \\ \boxed{2} & \boxed{2} & \boxed{2} \\ \boxed{0} & \boxed{0} & \boxed{0} \\ \boxed{0} & \boxed{0} & \boxed{2} \end{pmatrix}$$

$$[T(v_1)]_\alpha = [T(1)]_\alpha = \left[\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}\right]_\alpha$$

$$= [0 \cdot w_1 + 2 \cdot w_2 + 0 \cdot w_3 + 0 \cdot w_4]_\alpha$$

$$= \begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \end{bmatrix}$$

$$T(v_2) = T(x) = \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}$$

$$= 1 \cdot w_1 + 2 \cdot w_2 + 0 \cdot w_3 + 0 \cdot w_4$$

$$[T(v_2)]_\alpha = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}$$

$$T(v_3) = T(x^2) = \begin{pmatrix} 0 & 2 \\ 0 & 2 \end{pmatrix}$$

$$= 0 \cdot w_1 + 2 \cdot w_2 + 0 \cdot w_3 + 2 \cdot w_4$$

$$[T(v_3)]_\alpha = \begin{bmatrix} 0 \\ 2 \\ 0 \\ 2 \end{bmatrix}$$

(a) Define $\mathsf{T}\colon \mathsf{M}_{2\times 2}(F) \to \mathsf{M}_{2\times 2}(F)$ by $\mathsf{T}(A) = A^t$. Compute $[\mathsf{T}]_\alpha$.

$$\alpha = \left\{ \underset{V_1}{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}, \underset{V_2}{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}, \underset{V_3}{\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}}, \underset{V_4}{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}} \right\}$$

$$T(V_1) = T\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = 1\cdot V_1 + 0\cdot V_2 + 0\cdot V_3 + 0\cdot V_4 \Rightarrow [T(V_1)]_\alpha = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$T(V_2) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = 0\cdot V_1 + 0\cdot V_2 + 1\cdot V_3 + 0\cdot V_4 \Rightarrow [T(V_2)]_\alpha = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$T(V_3) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = 0\cdot V_1 + 1\cdot V_2 + 0\cdot V_3 + 0\cdot V_4 \Rightarrow [T(V_3)]_\alpha = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$T(V_4) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = 0\cdot V_1 + 0\cdot V_3 + 0\cdot V_2 + 1\cdot V_4 \Rightarrow [T(V_4)]_\alpha = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$\therefore [T]_\alpha = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$A^t = A^T = A'$$

$$a_{ij} \Rightarrow a_{ji}$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \end{pmatrix}^T = \begin{pmatrix} 1 & 0 \\ 2 & 1 \\ 3 & 1 \end{pmatrix}$$

$T: V \to \boxed{W}$

$\operatorname{rank}(T) = \operatorname{rank}([T]_\beta^\gamma)$

linear transformation의 rank

range $\Rightarrow$ dimension

Theorem 3.5. *The rank of any matrix equals the maximum number of its linearly independent columns; that is, the rank of a matrix is the dimension of the subspace generated by its columns.*

Matrix $\underset{\text{X}}{\overset{\text{O}}{\rightleftarrows}}$ linear transformation

Theorem 3.5. *The rank of any matrix equals the maximum number of its linearly independent columns; that is, the rank of a matrix is the dimension of the subspace generated by its columns.*

rank(A) $V \to W$

matrix linear transformation.

의 range의 dim.

ex) $A = \begin{pmatrix} 2 & 3 & 0 \\ 3 & 2 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & 0 \end{pmatrix}$ linear transformation.

rank (A) = 3

rank (A) = 2.

rank(A) = 1.

rank (A) = linearly independent 한 개수

Elementary row operation.

RREF. leading 1의 개수.

E $\rightarrow$ RREF

rank

$$A = \begin{pmatrix} 2 & 1 & 0 & 1 \\ -1 & 2 & 3 & 4 \end{pmatrix}$$

$$\begin{pmatrix} 2 \\ -1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} \begin{pmatrix} 0 \\ 3 \end{pmatrix} \begin{pmatrix} 1 \\ 4 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -2 & -3 & -4 \\ 2 & 1 & 0 & 1 \end{pmatrix} \qquad -2 \quad 4 \quad 6 \quad 8$$

$$\begin{pmatrix} 1 & -2 & -3 & -4 \\ 0 & 5 & 6 & 9 \end{pmatrix}$$

$$\begin{pmatrix} \boxed{1} & -2 & -3 & -4 \\ 0 & \boxed{1} & \frac{6}{5} & \frac{9}{5} \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 & 1 & 1 \\ 1 & 4 & 0 & 1 & 2 \\ 0 & 2 & -3 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 & 3 & 1 & 1 \\ 0 & 2 & -3 & 0 & 1 \\ 0 & 2 & -3 & 0 & 1 \\ 0 & -2 & -3 & -1 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 & 3 & 1 & 1 \\ 0 & 1 & -\frac{3}{2} & 0 & \frac{1}{2} \\ 0 & 2 & -3 & 0 & 1 \\ 0 & -2 & -3 & -1 & -1 \end{pmatrix}$$

$$\downarrow$$

$$\begin{pmatrix} 1 & 2 & 3 & 1 & 1 \\ 0 & 1 & -\frac{3}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{6} & 0 \\ 0 & 0 & -6 & -1 & 0 \end{pmatrix} \leftarrow \begin{pmatrix} 1 & 2 & 3 & 1 & 1 \\ 0 & 1 & -\frac{3}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{6} & 0 \\ 0 & -2 & -3 & -1 & -1 \end{pmatrix} \leftarrow \begin{pmatrix} 1 & 2 & 3 & 1 & 1 \\ 0 & 1 & -\frac{3}{2} & 0 & \frac{1}{2} \\ 0 & 0 & -6 & -1 & 0 \\ 0 & -2 & -3 & -1 & -1 \end{pmatrix}$$

$$\downarrow$$

$$\begin{pmatrix} 1 & 2 & 3 & 1 & 1 \\ 0 & 1 & -\frac{3}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{6} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow \text{rank}(A) = 3$$

$A_{n \times n}$ $\quad rank(A) = n \Rightarrow$ inverse 존재

$\Downarrow$

$\Rightarrow$ 구할 수도 있음.

(ex)

$\begin{pmatrix} 2 & -5 \\ -1 & 3 \end{pmatrix}$ inverse?

RREF

$\left(\begin{array}{cc|cc} 2 & -5 & 1 & 0 \\ -1 & 3 & 0 & 1 \end{array}\right)$ $-2 \quad 6 \quad 0 \quad 2$

$\left(\begin{array}{cc|cc} -1 & 3 & 0 & 1 \\ 2 & -5 & 1 & 0 \end{array}\right) \rightarrow \left(\begin{array}{cc|cc} 1 & -3 & 0 & -1 \\ 2 & -5 & 1 & 0 \end{array}\right)$

$\left(\begin{array}{cc|cc} 1 & 0 & 3 & 5 \\ 0 & 1 & 1 & 2 \end{array}\right) \Leftarrow \left(\begin{array}{cc|cc} 1 & -3 & 0 & -1 \\ 0 & 1 & 1 & 2 \end{array}\right)$

inverse

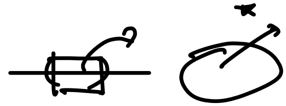
$0 \quad 3 \quad 3 \quad 6$

(ex) $\begin{pmatrix} 2 & 3 & 1 \\ 0 & 1 & -1 \\ -1 & -1 & -1 \end{pmatrix}$ inverse

$$\left(\begin{array}{ccc|ccc} 2 & 3 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ -1 & -1 & -1 & 0 & 0 & 1 \end{array}\right)$$

$-2 \quad -2 \quad -2 \quad 0 \quad 0 \quad 2$

$$\left(\begin{array}{ccc|ccc} 0 & 1 & -1 & 1 & 0 & 2 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & -1 \end{array}\right)$$



$$\left(\begin{array}{ccc|ccc} 0 & 1 & -1 & 1 & 0 & 2 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & -1 \end{array}\right)$$