Diagonalization [Eigen —]

Similar Matrices.

$$A = P^{-1}\cancel{B}P$$

$$D$$

Dia

$$A^n = P^{-1}D^nP$$

$$A^n = \begin{pmatrix} 2 & 1 \\ 4 & 3 \end{pmatrix}^{100}$$

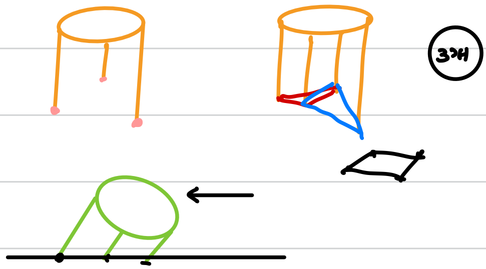
Eigenvalue

$Av = \lambda v$

eigenvector.

1. eigenvalue

2. ______ "



3개

$v \neq 0$ $A \neq \lambda I$

$Av = \lambda v,$

$Av = \lambda I v$

$(A - \lambda I)v = 0$ zero - divison

$\det(A - \lambda I) = 0$

<Characteristic polynomial>

λ

$(2-\lambda)^{1}(3-\lambda)^{2}(1-\lambda)^{1}$

$\lambda = 1, 2, 3$

$\Downarrow \quad \Downarrow \quad \Downarrow$

$1 \quad 1 \quad 2$

multiplicity

For $n \times n$ matrix A, assume that
$Av = \lambda v$ for scalar λ and vector v.

Then, λ : eigenvalue

For $\lambda = \lambda_1$, $Av_1 = \lambda_1 v_1 \Rightarrow v_1$: eigenvector of $\lambda = \lambda_1$.

Eigenspace ...?

Note that multiplicity of $\lambda_1 = k$, then there are at most k eigenvectors for $\lambda = \lambda_1$.
linearly independent.

Suppose that

$$B = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 2 \end{pmatrix}.$$

$$\begin{pmatrix} 1-\lambda & 1 & 1 \\ 0 & 1-\lambda & 0 \\ 0 & 1 & 2-\lambda \end{pmatrix}$$

$$\det(B-\lambda I) = (1-\lambda)\cdot(1-\lambda)(2-\lambda) = (1-\lambda)^2(2-\lambda).$$

$\lambda_1 = 1$

$$Bv = \lambda v \Rightarrow \underbrace{(B-\lambda I)}_{\Downarrow} v = 0$$

$$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} u_2+u_3 \\ 0 \\ u_2+u_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

$$u_2 = -u_3$$

$$\begin{pmatrix} u_1 \\ u_2 \\ -u_2 \end{pmatrix} = \begin{pmatrix} u_1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ u_2 \\ -u_2 \end{pmatrix}$$

$$= u_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + u_2 \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

$$\underbrace{\text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \right\}}_{} = \text{Eigenspace for } \lambda = 1.$$ ②

$\lambda = 2$

$$\boxed{} \quad \boxed{}$$

$$\begin{pmatrix} 3-\lambda & 1 & 0 \\ 0 & 3-\lambda & 0 \\ 0 & 0 & 4-\lambda \end{pmatrix}$$

$\det(A-\lambda I)$

$(3-\lambda)^2(4-\lambda)=0$

X

$\lambda=3$ ②

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} w_2 \\ 0 \\ w_3 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad w_2=w_3=0.$$

(1) $\rightarrow$ 4

$$\begin{pmatrix} w_1 \\ 0 \\ 0 \end{pmatrix}=w_1\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

Diagonalizable

[Theorem 1]

Let the multiplicity of $\lambda_1 = k_i$, then, there are at most k_i linearly independent eigenvectors.

[Theorem]

Let v_1 is an eigenvector of λ_1

(v_2 〃 of λ_2)

If $\lambda_1 \neq \lambda_2$, then v_1 and v_2 linearly independent

$A_{n\times n}$. There are n ~~distinct~~ linearly independent eigenvectors.

$\Leftrightarrow$ A is diagonalizable.

For $A_{n \times n}$,

Case 1. There are n distinct eigenvalues.

n of linear independent eigenvectors.

$\Rightarrow$ Diagonalizable

Case 2. There are $k < n$ distinct eigenvalues.

multiplicity $\Rightarrow$ For $\lambda = \lambda_1$ with multiplicity $= p$

For $\lambda = \lambda_1$, there are p linearly independent eigenvectors.

ex) $(1-\lambda)(2-\lambda)(3-\lambda)$ ok

✓

$(1-\lambda)^{2}(2-\lambda)^{1}$ $\quad$ $3>4$ $\quad$ X

①

$E_\lambda = \text{span}\{v \mid Av = \lambda_1 v\}$.

$\lambda = \lambda_1$

2개의 v

eigen space $\Rightarrow E_\lambda$

$$A\begin{pmatrix}7 & -4 & 0\\ 8 & -5 & 0\\ 6 & -6 & 3\end{pmatrix}$$

$$A-\lambda I = \begin{pmatrix}7-\lambda & -4 & 0\\ 8 & -5-\lambda & 0\\ 6 & -6 & 3-\lambda\end{pmatrix}$$

$$\det(A-\lambda I) = (-1)^2\cdot(7-\lambda)\left((-5-\lambda)(3-\lambda)\right)$$
$$+(-1)^3\cdot(-4)\left(8(3-\lambda)\right)$$
$$=(7-\lambda)(-5-\lambda)(3-\lambda)+32(3-\lambda)$$
$$=(3-\lambda)\left(-(7-\lambda)(+5+\lambda)+32\right)$$
$$=(3-\lambda)\left(-(-\lambda^2+2\lambda+35)+32\right)$$
$$=(3-\lambda)(\lambda^2-2\lambda-3)$$
$$=(3-\lambda)(\lambda-3)(\lambda+1)$$
$$=(3-\lambda)^2(1-\lambda)$$

i) $\lambda = 3$

$(A-\lambda I)\cdot v = 0$

$$\begin{pmatrix}4 & -4 & 0\\ 8 & -8 & 0\\ 6 & -6 & 0\end{pmatrix}\begin{pmatrix}v_1\\ v_2\\ v_3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

$v_1 - v_2 = 0$

$v_1 = v_2$

$$\therefore \begin{pmatrix}v_1\\ v_1\\ v_3\end{pmatrix} = v_1\begin{pmatrix}1\\1\\v_3\end{pmatrix}$$

$$\begin{pmatrix}u_1\\ u_1\\ 0\end{pmatrix}+\begin{pmatrix}0\\0\\u_3\end{pmatrix} = u_1\begin{pmatrix}1\\1\\0\end{pmatrix} + u_3\begin{pmatrix}0\\0\\1\end{pmatrix}.$$

Diagonalize

$$\begin{pmatrix} 3 & 1 & 1 \\ 2 & 4 & 2 \\ -1 & -1 & 1 \end{pmatrix} = A$$

$\det(A - \lambda I) = 0$

$$A - \lambda I = \begin{pmatrix} 3-\lambda & 1 & 1 \\ 2 & 4-\lambda & 2 \\ -1 & -1 & 1-\lambda \end{pmatrix}$$

$$\det(A-\lambda I) = (-1)^2 \cdot (3-\lambda)\left((4-\lambda)(1-\lambda)+2\right) + (-1)^3 \cdot \left(2(1-\lambda)+2\right) + (-1)^4 \cdot \left(-2(4-\lambda)\right)$$

$$= (3-\lambda)(4-\lambda)(1-\lambda) + 6 - 2\lambda + (-4+2\lambda) + (2\lambda - 8)$$

$$= (3-\lambda)(4-\lambda(1-\lambda)) + 6 \cancel{-2\lambda} - 4 \cancel{+2\lambda} + 2\lambda - 8$$

$$2\lambda - 6$$

$$2(\lambda-3)$$

$$= -2(3-\lambda)^2(4-\lambda)(1-\lambda)$$

$\therefore \lambda = 1$ or 3 or 4

$(A - \lambda I)\cdot U = 0$

$$A = \begin{pmatrix} 3 & 1 & 1 \\ 2 & 4 & 2 \\ -1 & -1 & 1 \end{pmatrix}$$

i) $\lambda = 1$

$$\begin{pmatrix} 2 & 1 & 1 \\ 2 & 3 & 2 \\ -1 & -1 & 0 \end{pmatrix}\begin{pmatrix} U_1 \\ U_2 \\ U_3 \end{pmatrix} = 0$$

$2U_1 + U_2 + U_3 = 0$

$2U_1 + 3U_2 + 2U_3 = 0$

$-U_1 - U_2 = 0 \to U_2 = -U_1$

$2U_1 - 3U_1 + 2U_3 = 0 \to U_3 = \frac{1}{2}U_1$

$\Rightarrow U_1 \begin{pmatrix} 1 \\ -1 \\ \frac{1}{2} \end{pmatrix}$

$(A - \lambda I) \cdot u = 0$ $\quad A = \begin{pmatrix} 3 & 1 & 1 \\ 2 & 4 & 2 \\ -1 & -1 & 1 \end{pmatrix}$

i) $\lambda = 3$

$$\begin{pmatrix} 0 & 1 & 1 \\ 2 & 1 & 2 \\ -1 & -1 & -2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = 0$$

$u_2 + u_3 = 0 \quad u_2 = -u_3$

$2u_1 + u_2 + 2u_3 = 0$

$-u_1 - u_2 - 2u_3 = 0$ $\quad \Rightarrow u_3 \begin{pmatrix} -\frac{1}{2} \\ -1 \\ 1 \end{pmatrix}$

$2u_1 - u_3 + 2u_3 = 0 \rightarrow u_1 = -\frac{1}{2}u_3$

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{pmatrix}$$

$$\det(A - \lambda I) = 0$$

$$A - \lambda I = \begin{pmatrix} 1-\lambda & 1 & 0 \\ 0 & 1-\lambda & 2 \\ 0 & 0 & 3-\lambda \end{pmatrix}$$

$$\det(A - \lambda I) = (-1)^6 \cdot (3-\lambda)(1-\lambda)(1-\lambda) = 0$$

$$\lambda = 1 \text{ or } 3$$

$$(A - \lambda I)v = 0$$

(i) $\lambda = 1$

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} u_2 = 0 \\ u_3 = 0 \end{cases}$$

$$\begin{pmatrix} u_1 \\ 0 \\ 0 \end{pmatrix} \Rightarrow u_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + u_2 \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

ii) $\lambda = 3$

$$\begin{pmatrix} -2 & 1 & 0 \\ 0 & -2 & 2 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$-2u_1 + u_2 = 0 \rightarrow u_1 = \tfrac{1}{2}u_2$$

$$-u_2 + u_3 = 0 \rightarrow u_3 = u_2$$

$$\Rightarrow u_2 \begin{pmatrix} \frac{1}{2} \\ 1 \\ 1 \end{pmatrix}$$