[Inverse matrix]

(Def)

For $n \times n$ matrix A, if there exist a $n \times n$ matrix B such that $AB = BA = I_n$

$AB = I$ $\quad n \times n$

$p \times q$, $\quad p = n$

$r \times s$, $\quad s = n$

$q = r$

$BA = I$ $\quad n \times n$

$r \times s$, $\quad r = n$

$p \times q$, $\quad q = n$

$q = r = n$.

⟨Determinant⟩ ; $n \times n$ matrix

<Inverse Matrix>

(Def)

Let A be a $n \times n$ matrix

Then, if $^{\exists}B$ s.t. $n \times n$ matrix and $AB = BA = I_n$,

then B is a inverse of A denoted by A^{-1}.

<Determinant>

Def. (order 2)

Let $A_{2\times 2} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $\det(A) = |A| = ad-bc$

(ex) $A = \begin{pmatrix} 1 & 2 \\ 3 & 7 \end{pmatrix}$ $\det(A) = 1\cdot 7 - 2\cdot 3 = 1$

<Theorem>

Let A be a 2×2 matrix.

$\det(A) \neq 0 \overset{}{\underset{\text{iff}}{\Longleftrightarrow}} A$ is <u>invertable</u>

A has inverse matrix.

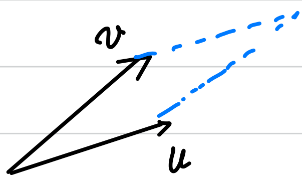
Assume that $\det(A) \neq 0$.

Then, there exist 2×2 matrix B

such that $AB = BA = I$. $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$B = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Fact.



<Def>

For $A_{n\times n}$, $\det(A) = \sum_{j=1}^{n} (-1)^{i+j} A_{ij} \cdot \det(\tilde{A}_{ij})$

(ex) $C = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 3 & -1 \\ -4 & 5 & 0 \end{pmatrix}$ $\det(C) = ?$

5

$\cancel{1\ 0\ 2}$
$0\ 3\ -1$
$-4\ 5\ 0$

$\sum_{j=1}^{3} (-1)^{1+j} A_{1j} \cdot \det(\tilde{A}_{1j})$

$(-1)^2 \cdot A_{11} \cdot \det(\tilde{A}_{11}) = 1 \cdot (5)$

$(-1)^3 \cdot A_{12} \cdot \det(\tilde{A}_{12}) = 0$

$(-1)^4 \cdot A_{13} \cdot \det(\tilde{A}_{13}) = 1 \cdot 2 \cdot (12)$

$\therefore \det(C) = 29$

Elementary row operation & determinants

1. Exchange two rows.

2. multiple scalar for a row

3. multiple ~~of~~ scalar for one row and add another row.

def

Example.

-2

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 2 & -5 \\ 6 & -4 & 3 \end{pmatrix} \xrightarrow{R_3 \leftrightarrow R_2} A_1 = \begin{pmatrix} 0 & 1 & 1 \\ 6 & -4 & 3 \\ 1 & 2 & -5 \end{pmatrix}$$

$$A \xrightarrow{R_2 \to 3R_2} A_2 = \begin{pmatrix} 0 & 1 & 1 \\ 3 & 6 & -15 \\ 6 & -4 & 3 \end{pmatrix}$$

$$A \xrightarrow{R_2 \to R_2 - 2R_1} A_3 = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & -7 \\ 6 & -4 & 3 \end{pmatrix}$$

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 2 & -5 \\ 6 & -4 & 3 \end{pmatrix}$$

$(-1)^{1+j} \cdot A_{1j} \cdot \det(\tilde{A}_{1j})$

$0 + (-1)^3 \cdot 1 \cdot (3+30) + (-1)^4 \cdot 1 \cdot (-4-12)$

$= -33 - 16 = -49$

$$A_1 = \begin{pmatrix} 0 & 1 & 1 \\ 6 & -4 & 3 \\ 1 & 2 & -5 \end{pmatrix}$$

$\det(A_1) = -\det(A)$

$$A_2 = \begin{pmatrix} 0 & 1 & 1 \\ 3 & 6 & -15 \\ 6 & -4 & 3 \end{pmatrix}$$

$\det(A_2) = 3\det(A)$

$$A_3 = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & -7 \\ 6 & -4 & 3 \end{pmatrix}$$

$\det(A_3) = \det(A)$

$\det(I_n) = 1$ $\quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ $\quad e_1$

Elementary matrix → exchange two rows

e_2

e_3

Properties

1. $\det(AB) = \det(A)\det(B)$

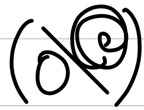
2. if $\det(A) \neq 0$, $\det(A^{-1}) = \dfrac{1}{\det(A)}$

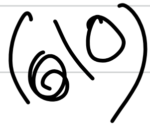
3. $\det(A) = \det(A^T)$

Properties of the Determinant

1. If B is a matrix obtained by interchanging any two rows or interchanging any two columns of an $n \times n$ matrix A, then $\det(B) = -\det(A)$.
2. If B is a matrix obtained by multiplying each entry of some row or column of an $n \times n$ matrix A by a scalar k, then $\det(B) = k \cdot \det(A)$.
3. If B is a matrix obtained from an $n \times n$ matrix A by adding a multiple of row i to row j or a multiple of column i to column j for $i \neq j$, then $\det(B) = \det(A)$.

4. The determinant of an upper triangular matrix is the product of its diagonal entries. In particular, $\det(I) = 1$.
5. If two rows (or columns) of a matrix are identical, then the determinant of the matrix is zero.

Triangular matrix — upper triangular 

Triangular matrix — lower triangular 

$\Downarrow$

A

$\det(A) = a_{11} a_{22} \cdots a_{nn}$

$A = \begin{pmatrix} 1 & 2 & 0 \\ 0 & -1 & 2 \\ 0 & 0 & 3 \end{pmatrix}$ $\quad \det(A) = 1 \cdot (-1) \cdot 3$

$= -3.$

diagonal matrix

ex) $\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -3 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

$6 \qquad -6 \qquad 0$

$A = \begin{pmatrix} & \end{pmatrix}$, $\det(A) = ?$

$A \xrightarrow{\text{elementary row operation}} \boxed{B} \to$ triangular / diagonal.

(d) $\begin{pmatrix} 1 & -2 & 3 \\ -1 & 2 & -5 \\ 3 & -1 & 2 \end{pmatrix} = A$ $\times(-3)$

det(A).

$-3 \quad 6 \quad -9$

$\begin{pmatrix} 1 & -2 & 3 \\ 0 & 0 & -2 \\ 0 & 5 & -7 \end{pmatrix} \to B$

$R_2 \to R_3$

$\begin{pmatrix} 1 & -2 & 3 \\ 0 & 5 & 7 \\ 0 & 0 & -2 \end{pmatrix} \to C$

$\det(A) = \det(B)$

$\det(A) = 10$

$\det(C) = -\det(B)$

$\det(C) = -10$, $\det(B) = 10$

(g) $$\begin{pmatrix} 1 & 0 & -2 & 3 \\ -3 & 1 & 1 & 2 \\ 0 & 4 & -1 & 1 \\ 2 & 3 & 0 & 1 \end{pmatrix}$$

$0 \quad 4 \quad -1$

$5x/19 \quad 95$

$$\begin{pmatrix} 1 & 0 & -2 & 3 \\ 0 & 1 & -5 & 11 \\ 0 & 4 & -1 & 1 \\ 2 & 3 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & -2 & 3 \\ 0 & 1 & -5 & 11 \\ 0 & 0 & 19 & -43 \\ 2 & 0 & 15 & -32 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & -2 & 3 \\ 0 & 1 & -5 & 11 \\ 0 & 0 & 19 & -43 \\ 0 & 0 & 19 & -38 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & -2 & 3 \\ 0 & 1 & -5 & 11 \\ 0 & 0 & 19 & -43 \\ 0 & 0 & 0 & 5 \end{pmatrix}$$

$1 \cdot 1 \cdot 19 \cdot 5 = 95$