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< Diagonalization >

Q. $\boxed{A^{50}} \Rightarrow$

$\underline{A = P^{-1}DP}$

↳ Diagonal matrix

$\rightarrow A^{50} = (P^{-1}DP)^{50} = P^{-1}D\cancel{PP^{-1}}D\cancel{PP^{-1}}DP \cdots P^{-1}DP.$

$= P^{-1}D^{50}P$

$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ $\quad$ $A^{50} = \underbrace{\begin{pmatrix} 1^{50} & 0 & 0 \\ 0 & 2^{50} & 0 \\ 0 & 0 & 3^{50} \end{pmatrix}}_{//}.$

Eigenvalue Eigenvector

Diagonalization

<Def> (A is not a diagonal matrix).

※ I_n

For $A_{n\times n}$, $v_{n\times 1}$, $\lambda \in F$,

※ $\neq 0$

if $Av = \lambda v$, then v : eigenvector, λ : eigenvalue.

Q. How can we know λ and v?

<step 1>

Find λ s.t. $\underline{\det(A-\lambda I)} = 0$

↳ characteristic polynomial

⌜

$Av = \lambda v$

$Av = \lambda I v$

$Av - \lambda I v = 0$

$(A-\lambda I)v = 0$ $\qquad \det(A-\lambda I) = 0$

$\uparrow$

$A - \lambda I \neq 0$, $v \neq 0$ zero divison (영인자). ⌟

<step 2>

For λ, $Av = \lambda v \Rightarrow v$ vector.,,

[step 3]

$$A = P^{-1} D P$$

P : $\begin{pmatrix} | & | & | \\ & \text{eigenvector} & \\ | & | & | \end{pmatrix}$

D : diagonal $\Rightarrow \lambda$

[Eigenspace]

For some λ_1, then $E_{\lambda_1} = \text{span}\{w \mid Aw = \lambda_1 w\}$

Diagonalizable

$A_{n \times n}$

(# of linearly independent eigenvectors $= n$

1.

$Av_1 = \lambda_1 v_1$, $Av_2 = \lambda_2 v_2$

If $\lambda_1 \neq \lambda_2$ then v_1, v_2 are linearly independent.

There are n distinct eigenvalues $\Rightarrow$ # of linearly independent eigenvector.

$\Rightarrow$ Diagonalizable

2. For λ_1 with multiplicity $= k$, if there are k linearly independent eigenvectors when $\lambda = \lambda_1$,

then

Diagonalization.

$A_{n\times n} \Rightarrow \deg(\text{Eigenspace}) = n$

$\Leftrightarrow$ diagonalization.

(ex) $B = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 2 \end{pmatrix}$ $\begin{pmatrix} -1 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

Step 1. $\det(B - \lambda I) = 0$

$$\det\begin{pmatrix} 1-\lambda & 1 & 1 \\ 0 & 1-\lambda & 0 \\ 0 & 1 & 2-\lambda \end{pmatrix} = (1-\lambda)\{(1-\lambda)(2-\lambda)\}$$
$$= (1-\lambda)^2(2-\lambda).$$

(i) $\lambda = 1$

$(B-\lambda I)v = 0$

$$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$u_2 + u_3 = 0.$

$u_2 = -u_3,$

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \\ -u_2 \end{pmatrix} = \begin{pmatrix} u_1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ u_2 \\ -u_2 \end{pmatrix} = u_1\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + u_2\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

Eigenspace $\lambda = 1$

$\text{span}\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}\right\} \Rightarrow \dim(\ ) = 2$

$\lambda = 2$

$$\begin{pmatrix} -1 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$-w_1 + \cancel{w_2} + w_3 = 0 \qquad w_3 = w_1$$

$$w_2 = 0$$

$$\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} w_1 \\ 0 \\ w_1 \end{pmatrix} = w_1\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

Eigenspace. $\text{span}\left\{\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}\right\}$.

$\lambda = 1$ $\lambda = 2$

Eigenspace of A $\oplus$

$V \oplus W$

모든 eigenvector 들로 span한 space

$v_1, v_2, \dots v_n$ are linearly independent

$\Longleftrightarrow$ $\gamma = \{v_1, v_2, \dots v_n\}$

$\dim(\text{span}(\gamma)) = n$

$A = P^{-1} D P$

$\lambda = 1 \qquad \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$

$\lambda = 2 \qquad \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \qquad \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix}$

$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \qquad P = \begin{pmatrix} 1 & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$

(e) $\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix}$

1) $\det(e - \lambda I) = 0$

$$e - \lambda I = \begin{pmatrix} -\lambda & 0 & 1 \\ 1 & -\lambda & -1 \\ 0 & 1 & 1-\lambda \end{pmatrix}$$

$$\det(e - \lambda I) = \cancel{(-1)^2} \cdot (-\lambda)\big((-\lambda)(1-\lambda) + 1\big) + \cancel{(-1)^4} \cdot (1)$$

$$= -\lambda(-\lambda + \lambda^2 + 1) + 1 = -\lambda^3 + \lambda^2 - \lambda + 1$$

$$= \lambda^3 - \lambda^2 + \lambda - 1 = 0$$

± 1

$$\begin{array}{r|rrrr} 1 & 1 & -1 & 1 & -1 \\ \hline & 1 & 0 & 1 & 0 \end{array}$$

$1 \cdot x^2 + 0 \cdot x + 1$

$(\lambda+1)(\lambda^3 + \lambda) = 0$

$(\lambda+1)^2(\lambda^2+1) = 0$

$(\lambda - 1)(\lambda^2 + 1).$

$\lambda = \pm i$

$\lambda = 1.$

$\square$

실수 체계 대각화 X

$\lambda = 1$ or $\lambda^2 + 1 = 0$

1개

$$A = \begin{pmatrix} 3 & 1 & 1 \\ 2 & 4 & 2 \\ -1 & -1 & 1 \end{pmatrix}$$

1) $\det(A - \lambda I) = 0$

$$A - \lambda I = \begin{pmatrix} 3-\lambda & 1 & 1 \\ 2 & 4-\lambda & 2 \\ -1 & -1 & 1-\lambda \end{pmatrix}$$

$\lambda^2 - 5\lambda + 6$

$$\det(A - \lambda I) = \cancel{(-1)^2} \cdot (3-\lambda)\left((4-\lambda)(1-\lambda)+2\right)$$

$$+ (-1)^3 \cdot 1 \cdot \left(2(1-\lambda)+2\right)$$

$$+ \cancel{(-1)^4} \cdot 1 \cdot \left(-2 + (4-\lambda)\right)$$

$$= (3-\lambda)(4-\lambda)(1-\lambda) + 6 \cancel{-2\lambda} - 1 \cdot (4 - \cancel{2\lambda}) + (-\lambda + 2)$$

$$= (3-\lambda)(4-\lambda)(1-\lambda) + 4 - \lambda$$

$$= (3-\lambda)(4-\lambda)^2(1-\lambda) \quad \therefore \quad \lambda = 1 \text{ or } 3 \text{ or } 4$$

2) $(A - \lambda I)u = 0$

$$A = \begin{pmatrix} 3 & 1 & 1 \\ 2 & 4 & 2 \\ -1 & -1 & 1 \end{pmatrix}$$

i) $\lambda = 1$

$$\begin{pmatrix} 2 & 1 & 1 \\ 2 & 3 & 2 \\ -1 & -1 & 0 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$2u_1 + u_2 + u_3 = 0$

$2u_1 + 3u_2 + 2u_3 = 0 \rightarrow 2u_1 - 3u_1 + 2u_3 = 0$

$-u_1 - u_2 = 0 \rightarrow u_2 = -u_1$

$u_3 = \frac{1}{2}u_1$

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} u_1 \\ -u_1 \\ \frac{1}{2}u_1 \end{pmatrix} = u_1 \begin{pmatrix} 1 \\ -1 \\ \frac{1}{2} \end{pmatrix}$$

ii) $\lambda = 3$

$$\begin{pmatrix} 0 & 1 & 1 \\ 2 & 1 & 2 \\ -1 & -1 & -2 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$u_2 + u_3 = 0 \rightarrow u_2 = -u_3$

$2u_1 + u_2 + 2u_3 = 0$

$-u_1 - u_2 - 2u_3 = 0 \rightarrow -u_1 + u_3 - 2u_3 = 0$

$u_1 = -u_3$

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} -u_3 \\ -u_3 \\ u_3 \end{pmatrix} = u_3 \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$$

iii) $\lambda = 4$

$$\begin{pmatrix} -1 & 1 & 1 \\ 2 & 0 & 2 \\ -1 & -1 & -3 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$-u_1 + u_2 + u_3 = 0 \rightarrow -u_1 + u_2 - u_1 = 0 \rightarrow u_2 = 2u_1$

$2u_1 + 2u_3 = 0 \rightarrow u_3 = -u_1$

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} u_1 \\ 2u_1 \\ -u_1 \end{pmatrix} = u_1 \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$$

$$A = \begin{pmatrix} 3 & 1 & 1 \\ 2 & 4 & 2 \\ -1 & -1 & 1 \end{pmatrix}$$

$\lambda = 2$

$(A - \lambda I) \cdot u = 0$

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ -1 & -1 & -1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\det \begin{pmatrix} 3-\lambda & 1 & 1 \\ 2 & 4-\lambda & 2 \\ -1 & -1 & 1-\lambda \end{pmatrix}.$$

$$(3-\lambda) \cdot [\lambda^2 - 5\lambda + 6] - 1 \cdot (4 - 2\lambda) + 1 \cdot (-2 + 4 - \lambda)$$

$$(3-\lambda)(\lambda-3)(\lambda-2) - 2(2-\lambda) + (2-\lambda)$$

$$(2-\lambda)\{(\lambda-3)^2 - 2 + 1\}.$$

$$(2-\lambda)(\lambda^2 - 6\lambda + 8)$$

$$(2-\lambda)(\lambda-4)(\lambda-2). = (2-\lambda)^2(4-\lambda).$$

$\lambda = 2$ or 4

$$A = \begin{pmatrix} 3 & 1 & 1 \\ 2 & 4 & 2 \\ -1 & -1 & 1 \end{pmatrix}$$

$(A-\lambda I)\cdot u = 0$

2

$\lambda = 2$

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ -1 & -1 & -1 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\underline{u_1 + u_2 + u_3 = 0} \rightarrow u_1 = -u_2 - u_3$

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} -u_2 - u_3 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} -u_2 \\ u_2 \\ 0 \end{pmatrix} + \begin{pmatrix} -u_3 \\ 0 \\ u_3 \end{pmatrix}$$

$$= u_2 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + u_3 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

eigenvector 2개.

연립일차.

$\ddot{u}_1 \quad \ddot{u}_2 \quad \ddot{u}_3$

식 1개

3개

2개 free variable.

$u_1 + u_2 + u_3 = 0$

free.

$$\left(\begin{array}{ccc|c} \boxed{1} & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$u_1 = -t - s.$

$u_2 = t$

$u_3 = s$