Basis (기저)

V: vector space

$v_1, v_2, \cdots, v_n \in V$

$\beta = \{v_1, v_2, \cdots, v_n\}$

$\mathrm{Span}(\beta) = \{v \mid a_1v_1 + a_2v_2 + \cdots + a_nv_n = v\}$

1. $\underline{\mathrm{Span}(\beta)} = V$

2. $v_1, v_2, \cdots, v_n$; <u>linearly independent</u>

   $a_1v_1 + a_2v_2 + \cdots + a_nv_n = 0$

   를 만족하는 $a_1, \cdots, a_n$이

   " $a_1 = a_2 = \cdots = a_n = 0$ " 뿐일 때.

* "V" vector space ; basis 개수: ∞

dimension = basis를 이루는 벡터 개수

Ordered basis

⇨ 순서가 정해진 basis

< Coordinate Vector >

<Def>

$\beta = \{v_1, v_2, \dots, v_n\}$

Let V be a vector space and $\boxed{\beta}$ be a ordered basis of V.

For any $v \in V$, there exist $a_1, a_2, \dots, a_n$ s.t.

$v = a_1 v_1 + a_2 v_2 + \cdots + a_n v_n$.

Then $[v]_\beta = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$ is a coordinate vector of v with β.

ex) $\beta_1 = \{\overset{v_1}{\overset{\shortparallel}{(1,2)}}, \overset{v_2}{\overset{\shortparallel}{(-2,1)}}\}$, $\beta_2 = \{(3,4), (1,-1)\}$

Note that β_1, β_2 are ordered basis of $\mathbb{R}^2$

$[(-3,4)]_{\beta_1} = ?$

How to compute $[(-3,4)]_{\beta_1}$?

$\Rightarrow$

step 1. $(-3,4) = a_1(1,2) + a_2(-2,1) \Rightarrow a_1 = 1,\ a_2 = 2$

$= \boxed{1 \cdot} v_1 + \boxed{2 \cdot} v_2$

step 2. $[(-3,4)]_{\beta_1} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$[(13,8)]_{\beta_2} = ?$, $[(6,0)]_{\beta_1} = ?$

$\beta_1 = \{(1,2), (-2,1)\}$, $\beta_2 = \{(3,4), (1,-1)\}$

Note that β_1, β_2 are ordered basis of $\mathbb{R}^2$

$[v]_{\beta_2} = \begin{bmatrix} 2 \\ 5 \end{bmatrix} = 2\cdot(3,4) + 5\cdot(1,-1) = (11,3)$

1) $a_1(3,4) + a_2(1,-1) = (13,8)$

$$\begin{cases} 3a_1 + a_2 = 13 \\ 4a_1 - a_2 = 8 \end{cases}$$

$a_1 = 3$, $a_2 = 4$

$= \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

$4a_1 + 2a_2 = 0$

$5a_1 = 6$

2) $a_1(1,2) + a_2(-2,1) = (6,0)$

$$\begin{cases} a_1 - 2a_2 = 6 \\ 2a_1 + a_2 = 0 \end{cases}$$

$a_1 = 6 + 2a_2$

$2(6+2a_2) + a_2 = 0$

$12 + 5a_2 = 0$

$a_2 = -\frac{12}{5}$

$a_1 = \frac{6}{5}$

$= \begin{bmatrix} \frac{6}{5} \\ -\frac{12}{5} \end{bmatrix}$

$(1, 2, -1), (1, 0, 2), (2, 1, 1)$

$a_1(1, 2, -1) + a_2(1, 0, 2) + a_3(2, 1, 1) = 0$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 2 & 0 & 2 & 0 \\ -1 & 2 & 1 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 2 & 0 & 2 & 0 \\ 0 & 3 & 3 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & -2 & -2 & 0 \\ 0 & 3 & 3 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 3 & 3 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] //$$

2개

3개

(c) $\{(1,2,-1),(1,0,2),(2,1,1)\} = \beta$

1. $\mathrm{span}(\beta) = \mathbb{R}^3$

dependent..

2. linearly independent $\Rightarrow$ $b_1(1,2,-1) + b_2(1,0,2) + b_3(2,1,1)$
$= (0,0,0)$

$\mathbb{R}^3$ (x,y,z)

$a_1(1,2,-1) + a_2(1,0,2) + a_3(2,1,1) = (x,y,z)$

$$\begin{aligned} a_1 + a_2 + 2a_3 &= x \\ 2a_1 + a_3 &= y \\ -a_1 + 2a_2 + a_3 &= z \end{aligned}$$

$a_3 = y - 2a_1$

⟨Linear transformation⟩

a scalar

$\mathbb{R}$ $\mathbb{C}$ → a+bi

Definition. *Let* $\boxed{\mathsf{V}}$ *and* W *be vector spaces (over* $\cancel{F}$*). We call a function* $\mathsf{T}\colon \mathsf{V} \to \mathsf{W}$ *a* ***linear transformation from*** V ***to*** W *if, for all* $x, y \in \mathsf{V}$ *and* $c \in F$*, we have*

$$\boxed{\begin{array}{ll} \text{(a)} & \mathsf{T}(x+y) = \mathsf{T}(x) + \mathsf{T}(y) \\ \text{(b)} & \mathsf{T}(cx) = c\mathsf{T}(x). \end{array}}$$ and

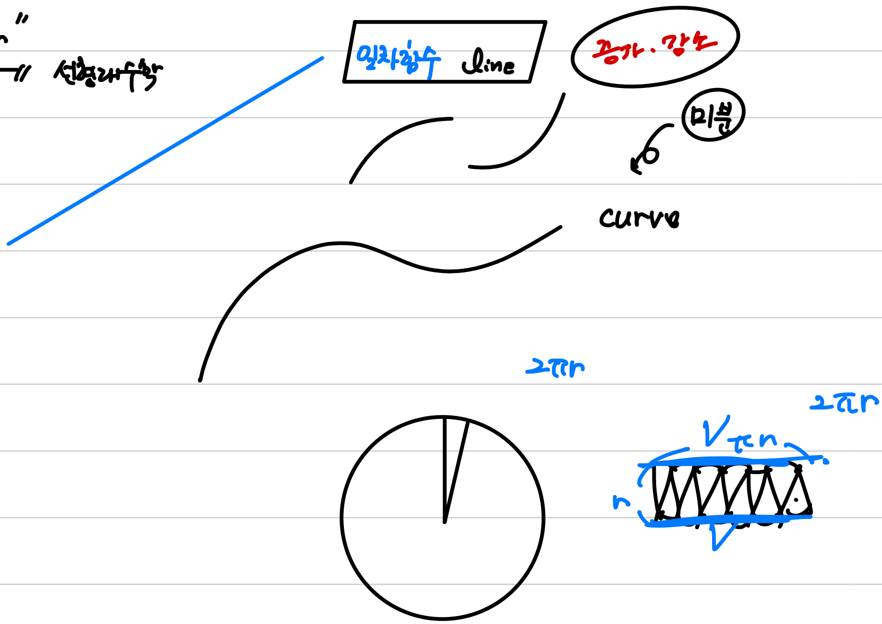
linear

Domain.

$f : V \longrightarrow W$

$$\left(\begin{array}{l} T(x+y) = T(x) + T(y) \\ T(cx) = c\,T(x) \end{array} \right)$$

"linear" // 선형대수학



$$\begin{cases} T(x+y) = T(x) + T(y) \\ T(cx) = cT(x) \end{cases}$$

1. If T is linear, then $\mathsf{T}(\mathit{0}) = \mathit{0}$.
2. T is linear if and only if $\boxed{\mathsf{T}(cx + y) = c\mathsf{T}(x) + \mathsf{T}(y)}$ for all $x, y \in \mathsf{V}$ and $c \in F$.

   필요충분 ⇔

3. If T is linear, then $\mathsf{T}(x - y) = \mathsf{T}(x) - \mathsf{T}(y)$ for all $x, y \in \mathsf{V}$.
4. T is linear if and only if, for $x_1, x_2, \ldots, x_n \in \mathsf{V}$ and $a_1, a_2, \ldots, a_n \in F$, we have

1. $T(0_v) = 0_w$

By 2, $c = 1$ 2. $y = 0_v$

$T(cx+y) = cT(x) + T(y)$

$T(0_v) = T(0_v + 0_v) = T(0_v) + T(0_v)$

$T(0_v) = T(0_v) + \underbrace{T(0_v)}_{= 0_w}$

$T(x-y) = T(x) - T(y)$

$x-y = x+(-y)$

$T(x) + T(-y)$

$-T(y)$

y의 덧셈에 대한 역원

$y + 0 = y$

항등원

$y + \square = 0$

$T(y)$

$\underbrace{T(-y)}_{V} = \underbrace{-T(y)}_{W}$

V (y)

$T(x_1+x_2+x_3) = T(x_1) + T(x_2) + T(x_3)$

$T(ax_1+bx_2+cx_3) = aT(x_1) + bT(x_2) + cT(x_3)$

$c \in \mathbb{R}$

$x, y \in \mathbb{R}$ $\quad x = (x_1, x_2, x_3)$, $y = (y_1, y_2, y_3)$

2. $\mathsf{T}\colon \mathsf{R}^3 \to \mathsf{R}^2$ defined by $\mathsf{T}(a_1, a_2, a_3) = (a_1 - a_2, 2a_3)$.

linear transformation?

$T(cx+y) = cT(x) + T(y)$.

1. $T(x+y) \overset{?}{=} T(x) + T(y)$

$T((x_1+y_1, x_2+y_2, x_3+y_3)) = (x_1+y_1-x_2-y_2,\ 2x_3+2y_3)$

$T(x) = (x_1 - x_2, 2x_3)$

$T(y) = (y_1 - y_2, 2y_3)$

$(x_1 - x_2 + y_1 - y_2,\ 2x_3 + 2y_3)$ -

2. $T(cx) = c\,T(cx)$ $\quad (x_1 - x_2,\ 2x_3)$.

$T((cx_1, cx_2, cx_3)) = (cx_1 - cx_2, 2cx_3)$

3. $\mathsf{T}\colon \mathsf{R}^2 \to \mathsf{R}^3$ defined by $\mathsf{T}(a_1, a_2) = (a_1 + a_2, 0, 2a_1 - a_2)$.

$x = (x_1, x_2)$, $y = (y_1, y_2)$

1. $T(x+y) = T(x) + T(y)$

$T((x_1+y_1, x_2+y_2)) = (x_1+y_1+x_2+y_2, 0, 2x_1+2y_1-x_2-y_2)$

$T(x)+T(y) = (x_1+x_2, 0, 2x_1-x_2) + (y_1+y_2, 0, 2y_1-y_2)$

$= (x_1+x_2+y_1+y_2, 0, 2x_1+2y_1-x_2-y_2)$

2. $T(cx) = cT(x)$

$T(cx) = (c \cdot x_1 + c \cdot x_2, 0, 2c \cdot x_1 - c \cdot x_2)$

$c \cdot T(x) = c \cdot (x_1+x_2, 0, 2x_1-x_2) = (cx_1+cx_2, 0, 2 \cdot cx_1 - cx_2)$

Definitions. *Let* V *and* W *be vector spaces, and let* $\mathsf{T}\colon \mathsf{V} \to \mathsf{W}$ *be linear. We define the **null space** (or **kernel**)* $\mathsf{N}(\mathsf{T})$ *of* T *to be the set of all vectors* x *in* V *such that* $\mathsf{T}(x) = 0$*; that is,* $\mathsf{N}(\mathsf{T}) = \{x \in \mathsf{V} \colon \mathsf{T}(x) = 0\}$.

*We define the **range** (or **image**)* $\mathsf{R}(\mathsf{T})$ *of* T *to be the subset of* W *consisting of all images (under* T*) of vectors in* V*; that is,* $\mathsf{R}(\mathsf{T}) = \{\mathsf{T}(x) \colon x \in \mathsf{V}\}$.

<Def>

Let V, W be Vector space and $T: V \to W$ be a linear transformation.

$N(T) = \{v \mid v \in V \text{ s.t. } T(v) = 0_W\}$ is a null space of T

$T(v) = 0$

P_n : n차이하의 다항식

이차이하의 다항식

5. $\mathsf{T}\colon \mathsf{P}_2(R) \to \mathsf{P}_3(R)$ defined by $\mathsf{T}(f(x)) = xf(x) + f'(x)$.

3차이하의 다항식

$N(T) = \{0\}$

a, b, c

$ax^2 + bx + c = f(x)$

$f'(x) = 2ax + b$

$x(ax^2 + bx + c) + 2ax + b$

$\{ax^3 + bx^2 + (c+2a)x + b\}$: $R(T)$

$a = 0 \quad b = 0, \quad c + 2a = 0, \quad b = 0$

$2a + c = 0$

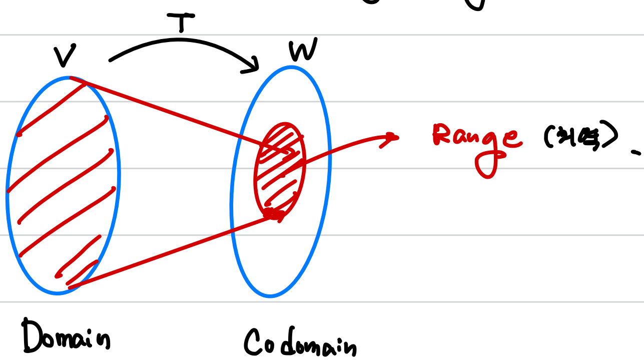
$c = 0$

이차항의 계수 = 상수항

$R(T) = \{g(x) \mid g(x) = ax^3 + bx^2 + cx + b\}$

Let V, W be Vector space and $T: V \to W$ be a linear transformation.

$R(T) = \{T(v) \mid v \in V\}$ is a range (image) of T.



Theorem 2.2. *Let* V *and* W *be vector spaces, and let* $\mathsf{T}\colon \mathsf{V} \to \mathsf{W}$ *be linear. If* $\beta = \{v_1, v_2, \ldots, v_n\}$ *is a basis for* V, *then*

$$\mathsf{R}(\mathsf{T}) = \operatorname{span}(\mathsf{T}(\beta)) = \operatorname{span}(\{\mathsf{T}(v_1), \mathsf{T}(v_2), \ldots, \mathsf{T}(v_n)\}).$$

$T: V \to W$

$\beta =$ basis of V $\Rightarrow$ $\{T(v_1), T(v_2), \cdots, T(v_n)\}$; basis of $R(T)$.

$= \underbrace{\{v_1, \ldots, v_n\}}$ //

$v \in V$ $\quad v = a_1v_1 + a_2v_2 + a_3v_3 + \cdots + a_nv_n$ 이므로,

$T(v) = T(a_1v_1 + a_2v_2 + a_3v_3 + \cdots + a_nv_n)$

$= a_1T(v_1) + a_2T(v_2) + \cdots + a_nT(v_n)$; $T(v_i)$의 linear combination

$T(v)$는 $T(v_1), T(v_2), \ldots, T(v_n)$의 linear combination으로 표현가능.

$\operatorname{span}\{T(v_1), T(v_2), \cdots, T(v_n)\} = R(T)$

↳ $R(T)$의 basis

linearly independent? ✓

$v_1 \sim v_n$; linearly independent

$a_1T(v_1) + a_2T(v_2) + \cdots + a_nT(v_n) = 0$

$T(a_1v_1) + T(a_2v_2) + \cdots + T(a_nv_n) = 0$

$T(\underbrace{a_1v_1 + \cdots + a_nv_n}) = 0$ $\quad$ T가 linear

$\quad\quad \shortparallel$

$\quad\quad 0$ $\quad\quad\quad\quad$ $T(0) = 0$

$a_1 = a_2 = \cdots = 0$

Definitions. *Let* V *and* W *be vector spaces, and let* $\mathsf{T}\colon \mathsf{V} \to \mathsf{W}$ *be linear. If* $\mathsf{N}(\mathsf{T})$ *and* $\mathsf{R}(\mathsf{T})$ *are finite-dimensional, then we define the* ***nullity*** *of* T*, denoted* $\text{nullity}(\mathsf{T})$*, and the* ***rank*** *of* T*, denoted* $\text{rank}(\mathsf{T})$*, to be the dimensions of* $\mathsf{N}(\mathsf{T})$ *and* $\mathsf{R}(\mathsf{T})$*, respectively.*

<Def>

$\dim(N(T)) = \text{nullity}$

$\dim(R(T)) = \text{rank}$

Theorem 2.3 (Dimension Theorem). *Let* V *and* W *be vector spaces, and let* $\mathsf{T}\colon \mathsf{V} \to \mathsf{W}$ *be linear. If* V *is finite-dimensional, then*

$$\operatorname{nullity}(\mathsf{T}) + \operatorname{rank}(\mathsf{T}) = \dim(\mathsf{V}).$$