3. Let $\mathsf{T}\colon \mathsf{R}^2 \to \mathsf{R}^3$ be defined by $\mathsf{T}(a_1, a_2) = (a_1 - a_2, a_1, 2a_1 + a_2)$. Let β be the standard ordered basis for R^2 and $\gamma = \{(1,1,0), (0,1,1), (2,2,3)\}$. Compute $[\mathsf{T}]_\beta^\gamma$. If $\alpha = \{(1,2), (2,3)\}$, compute $[\mathsf{T}]_\alpha^\gamma$.

$$\beta = \{ \underset{V_1}{(1,0)}, \underset{V_2}{(0,1)} \}$$

$$T(V_1) = (1,1,2)$$

$$T(V_2) = (-1,0,1)$$

$$(1,1,2) = a_1(1,1,0) + a_2(0,1,1) + a_3(2,2,3)$$

$$a_1 + \quad + 2a_3 = 1 \rightarrow a_1 = 1 - 2a_3$$

$$a_1 + a_2 + 2a_3 = 1$$

$$+ a_2 + 3a_3 = 2 \rightarrow a_2 = 2 - 3a_3$$

$$\Rightarrow 1 \cancel{- 2a_3} + 2 - 3a_3 \cancel{+ 2a_3} = 1$$

$$a_3 = \frac{2}{3}, \quad a_1 = -\frac{1}{3}, \quad a_2 = 0$$

$$\Rightarrow [T(V_1)]_\beta^\gamma = \begin{bmatrix} -\frac{1}{3} \\ 0 \\ \frac{2}{3} \end{bmatrix}$$

$$(-1,0,1) = a_1(1,1,0) + a_2(0,1,1) + a_3(2,2,3)$$

$$a_1 + \quad + 2a_3 = -1 \rightarrow a_1 = -1-2a_3$$

$$a_1 + a_2 + 2a_3 = 0$$

$$+ a_2 + 3a_3 = 1 \rightarrow a_2 = 1-3a_3$$

$$\Rightarrow -1 \cancel{-2a_3} + 1 - 3a_3 \cancel{+2a_3} = 0$$

$$a_3 = 0, \quad a_1 = -1, \quad a_2 = 1$$

$$\Rightarrow [T(v_2)]_\beta^\gamma = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

$$\therefore [T]_\beta^\gamma = \begin{bmatrix} -\frac{1}{3} & -1 \\ 0 & 1 \\ \frac{2}{3} & 0 \end{bmatrix}$$

$[T]_\beta^\gamma$

1. $\beta = \{v_1, v_2, \cdots, v_n\}$

2. $T(v_1)$, $T(v_2)$ $\cdots$ $T(v_n)$

3. $[T(v_1)]_\gamma$, $[T(v_2)]_\gamma$ $\cdots$

4. $\begin{bmatrix} [T(v_1)]_\gamma & [T(v_2)]_\gamma & \cdots & [T(v_n)]_\gamma \end{bmatrix}$

$T : V \rightarrow W$

β $\{v_1, \cdots, v_n\}$

γ $\{u_1, \cdots, u_m\}$

$[w]_\gamma$

$\boxed{T(v) = w}$

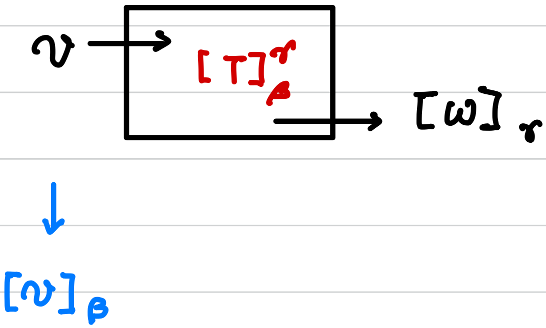
$\underline{T(a_1v_1 + a_2v_2 + \cdots + a_nv_n)}$

$a_1T(v_1) + a_2T(v_2) + \cdots + a_nT(v_n)$

$T(v_1) = b_1w_1 + b_2w_2 + \cdots + b_nw_m$

$T(v_2) = c_1w_1 + \cdots + c_nw_n$

$T(v) = [w]_\gamma$ $T(v) = [w]_\gamma$



$T(v) = w$ $\quad [w]_\gamma \longrightarrow \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$

$w = \overset{\times}{b_1} w_1 + b_2 w_2 + \cdots + b_m w_m$

$T(v) = T(a_1 v_1 + a_2 v_2 + \cdots + a_n v_n)$

$= a_1 T(v_1) + a_2 T(v_2) + a_3 \cdots + a_n T(v_n)$

$\Downarrow$

$\boxed{c_1} w_1 + c_2 w_2 + \cdots + c_m w_m$

w_1의 계수

$d_1 w_1 + d_2 w_2 + \cdots + d_m w_m$

e_1

$a_1 c_1 + a_2 d_1 + a_3 e_1 + \cdots + a_n \square = w_1$ 의 계수

$[\, a_1 \; a_2 \; a_3 \; \cdots \; a_n \,] \begin{bmatrix} c_1 \\ d_1 \\ \vdots \\ \; \end{bmatrix}$

$\Downarrow$

$[v]_\beta$

$\beta = \{v_1, v_2, \dots, v_n\}$: ordered basis of V

$T.\ V \longrightarrow W$

$\gamma = \{w_1, w_2, \dots, w_m\}$: ordered basis of W

Claim. $[T(v)]_\gamma = [T]_\beta^\gamma [v]_\beta$

Let $v = a_1 v_1 + a_2 v_2 + \cdots + a_n v_n$

$\quad // \ w$

and $T(v) = b_1 w_1 + b_2 w_2 + \cdots + b_m w_m$.

$$T(v_i) = \sum_{j=1}^{m} C_{ij} w_j = C_{i1} w_1 + C_{i2} w_2 + \cdots + C_{im} w_m$$

$[T(v_2)]$

$\boxed{C_{21}}$

$$\begin{aligned} T(v) &= T(a_1 v_1 + a_2 v_2 + \cdots + a_n v_n) \\ &= a_1 T(v_1) + a_2 T(v_2) + \cdots + a_n T(v_n) \\ &= a_1 (C_{11} w_1 + C_{12} w_2 + \cdots +) + a_2 (\underline{\boxed{C_{21}}} w_1 + \cdots \quad) \end{aligned}$$

$$w_1 \dashrightarrow a_1 C_{11} + a_2 C_{21} + a_3 C_{31} + \cdots + a_n C_{n1}$$

$[v]_\beta$

$$\underbrace{(a_1 \ a_2 \ a_3 \ \cdots \ a_n)} \begin{pmatrix} \boxed{C_{11}} \\ \boxed{C_{21}} \\ C_{31} \\ \vdots \\ C_{n1} \end{pmatrix} = \boxed{b_1}$$

$\to [T(v_1)]_\gamma$ 첫번째.

$\to [T(v)]_\gamma$의 첫번째

$\to [T(v_i)]_\gamma$의 첫번째

$$(a_1 \ a_2 \ a_3 \ \cdots \ a_n) \begin{pmatrix} C_{12} \\ C_{22} \\ C_{32} \\ \vdots \\ C_{n2} \end{pmatrix} = b_2$$

$\to [T(v_i)]_\gamma$의 두번째.

$$\begin{pmatrix} c_{11}\\ c_{21}\\ c_{31}\\ \vdots\\ c_{m1}\end{pmatrix}\begin{pmatrix} a_1 & a_2 & a_3 & \cdots & a_n\end{pmatrix}$$

$$\begin{pmatrix} c_{11} & c_{21} & c_{31} & \cdots & c_{m1}\end{pmatrix}\begin{pmatrix} a_1\\ a_2\\ a_3\\ \vdots\\ a_n\end{pmatrix}$$

$$\begin{pmatrix} c_{11} & c_{21} & c_{31} & c_{41} & \cdots & c_{m1}\\ \vdots & & & & & \\ c_{1n} & c_{2n} & c_{3n} & \cdots & & c_{mn}\end{pmatrix}\begin{pmatrix} a_1\\ \vdots\\ a_n\end{pmatrix} = \begin{pmatrix} b_1\\ b_2\\ \vdots\\ b_m\end{pmatrix}$$

$\boxed{[T]^{\gamma}_{\beta}}$

$[T(v_i)]_{\gamma}$

$[v]_{\beta}$

$$[T]_\beta^\gamma [v]_\beta = [T(v)]_\gamma$$

$$T: V \to \overset{\gamma}{W} \qquad L: \overset{\alpha}{U} \to \overset{\beta}{V}$$

$$(T \circ L)$$

$$\begin{tikzcd} U \arrow[r] \arrow[d, Rightarrow] & V \\ u \arrow[ur, bend right] & \end{tikzcd}$$

$$\underset{\rightsquigarrow //}{T(L(u))}$$

$$[T]_\beta^\gamma \underbrace{[L]_\alpha^\beta [u]_\alpha}$$

$$[T]_\beta^\gamma \; [Lu]_\beta$$

$$[TLu]_\gamma$$

$T : \overset{\alpha}{V} \rightarrow \overset{\beta}{W}$: $[T]^{\beta}_{\alpha}$; $m \times n$ Matrix

$\dim V = n$, $\dim V = m$ $[T]^{\beta}_{\alpha} [v]_{\alpha} = [T(v)]_{\beta}$

Matrix multiplication $\Rightarrow$ Composition of two linear transformation

Compute A, $A B$, BC, CB, and CA.

Let $g(x) = 3 + x$. Let $\mathsf{T}\colon \mathsf{P}_2(R) \to \mathsf{P}_2(R)$ and $\mathsf{U}\colon \mathsf{P}_2(R) \to \mathsf{R}^3$ be the linear transformations respectively defined by

$$\mathsf{T}(f(x)) = f'(x)g(x) + 2f(x) \quad \text{and} \quad U(a + \overset{0}{b}x + \overset{1}{c}x^2) = (a + b, c, a - b)$$

Let β and γ be the standard ordered bases of $\mathsf{P}_2(R)$ and R^3, respectively.

$\{(1,0,0),(0,1,0),(0,0,1)\}$

(a) Compute $[\mathsf{U}]_\beta^\gamma$, $[\mathsf{T}]_\beta$, and $[\mathsf{UT}]_\beta^\gamma$ directly. Then use Theorem 2.11 to verify your result.

(b) Let $h(x) = 3 - 2x + x^2$. Compute $[h(x)]_\beta$ and $[\mathsf{U}(h(x))]_\gamma$. Then use $[\mathsf{U}]_\beta^\gamma$ from (a) and Theorem 2.14 to verify your result.

$\{1, x, x^2\}$.

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}$$

$$U(1) = (1,0,1) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} ,,$$

$$U(x) = (1,0,-1) = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$U(x^2) = (0,1,0) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

. Let $g(x) = 3 + x$. Let $\mathsf{T}\colon \mathsf{P}_2(R) \to \mathsf{P}_2(R)$ and $\mathsf{U}\colon \mathsf{P}_2(R) \to \mathsf{R}^3$ be the linear transformations respectively defined by

$$\mathsf{T}(f(x)) = f'(x)g(x) + 2f(x) \quad \text{and} \quad U(a + bx + cx^2) = (a + b, c, a - b).$$

Let β and γ be the standard ordered bases of $\mathsf{P}_2(R)$ and R^3, respectively.

(a) Compute $[\mathsf{U}]_\beta^\gamma$, $[\mathsf{T}]_\beta$, and $[\mathsf{UT}]_\beta^\gamma$ directly. Then use Theorem 2.11 to verify your result.

(b) Let $h(x) = \boxed{3 - 2x + x^2.}$ Compute $[h(x)]_\beta$ and $[\mathsf{U}(h(x))]_\gamma$. Then use $[\mathsf{U}]_\beta^\gamma$ from (a) and Theorem 2.14 to verify your result.

$$[h(x)]_{\beta} = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$$

$$3 - 2x + x^2 = \underline{\textcircled{3}\,1} + (-2)\cdot \underline{x} + 1\cdot \underline{x^2}$$

Note that $\beta = \{\overset{v_1}{1}, \underset{v_2}{x}, \underset{v_3}{x^2}\}$

$$T(x^2) = 2x\cdot(3+x) + 2\cdot x^2$$
$$= 6x + 2x^2 + 2x^2$$
$$= 4x^2 + 6x$$

$$T(1) = 0\cdot(3+x) + 2\cdot 1 = 2$$
$$= 2\cdot 1 + 0\cdot x + 0\cdot x^2$$

$$T(x) = 1\cdot(3+x) + 2\cdot x = \underline{3x+3}$$

$$[T(1)]_\beta = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} \qquad [T(x^2)]_\beta = \begin{bmatrix} 0 \\ 6 \\ 4 \end{bmatrix}$$

$$[T(x)]_\beta = \begin{bmatrix} 3 \\ 3 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} 2 & 3 & 0 \\ 0 & 3 & 6 \\ 0 & 0 & 4 \end{bmatrix}$$

$\beta = \{1, x, x^2\}$ $\quad h(x) = 3 - 2x + x^2$

$h(1) = 3 - 2 + 1 = 2$ $\quad [h(1)]_\beta = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$

$h(x) = 3 - 2x + x^2$ $\quad [h(x)]_\beta = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$ $\quad \Rightarrow [h(x)]_\beta =$

$h(x^2) = 3 - 2x^2 + x^4$ $\quad [h(x^2)]_\beta =$

$U(3 - 2x + x^2) = (a+b, c, a-b)$ $\quad U(v_1) =$

$\gamma = \{(1,0,0), (0,1,0), (0,0,1)\}$

Compute A^t, A^tB, BC^t, CB, and CA.

Let $g(x) = 3 + x$. Let $\mathsf{T}\colon \mathsf{P}_2(R) \to \mathsf{P}_2(R)$ and $\mathsf{U}\colon \mathsf{P}_2(R) \to \mathsf{R}^3$ be the linear transformations respectively defined by

$$\mathsf{T}(f(x)) = f'(x)g(x) + 2f(x) \quad \text{and} \quad U(a + bx + cx^2) = (a + b, c, a - b).$$

Let β and γ be the standard ordered bases of $\mathsf{P}_2(R)$ and R^3, respectively.

(a) Compute $[\mathsf{U}]_\beta^\gamma, [\mathsf{T}]_\beta$, and $[\mathsf{UT}]_\beta^\gamma$ directly. Then use Theorem 2.11 to verify your result.

(b) Let $h(x) = 3 - 2x + x^2$. Compute $[h(x)]_\beta$ and $[\mathsf{U}(h(x))]_\gamma$. Then use $[\mathsf{U}]_\beta^\gamma$ from (a) and Theorem 2.14 to verify your result.

$U(h(x)) = U(3 - 2x + x^2) = (1, 1, 5) \Rightarrow \begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix}$

$\textcircled{\beta}$ ✓γ γ ✓γ

$a = 3, \ b = -2, \ c = 1$

$\square \Rightarrow$

$T(1) = 1 + 0 + 0 = 1$

$T(x) = 0 + 2x + 1 + 0 = 2x+1$

$T(x^2) = 0 + 0 + 4x^2 + 4x + 1$

120. *Polynomial Transformation.* Let $T : \mathcal{P}_2 \to \mathcal{P}_2$ be the linear transformation defined by $T(p(x)) = p(2x+1)$, that is,

$$T(a_0 + a_1x + a_2x^2) = a_0 + a_1(2x+1) + a_2(2x+1)^2.$$

$[T]_B^\gamma$

Find $[T]_B$ with respect to the basis $B = \{1, x, x^2\}$. , $\gamma = \{1-x, 2x-1, x^2\}$.

$1 = a_1(1-x) + a_2(2x-1) + a_3(x^2)$

$1 = a_1 - a_1x + 2a_2 \cdot x - a_2 + a_3x^2$

$= a_3x^2 - a_1x + 2a_2x + a_1 - a_2$

$a_3 = 0$, $-a_1 + 2a_2 = 0$, $a_1 - a_2 = 1$

$a_2 = 1$

$a_1 = 2$ $\Rightarrow [T(1)]_\gamma = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$

$2x+1 = a_3x^2 - a_1x + 2a_2x + a_1 - a_2$

$a_3 = 0$, $-a_1 + 2a_2 = 2$, $a_1 - a_2 = 1$

$a_2 = 3$

$a_1 = 4$ $\Rightarrow [T(x)]_\gamma = \begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix}$

$4x^2 + 4x + 1 = a_3x^2 - a_1x + 2a_2x + a_1 - a_2$

$a_3 = 4$, $-a_1 + 2a_2 = 4$, $a_1 - a_2 = 1$

$a_2 = 5$

$a_1 = 6$ $\Rightarrow [T(x^2)]_\gamma = \begin{bmatrix} 6 \\ 5 \\ 4 \end{bmatrix}$ $\therefore \begin{bmatrix} 2 & 4 & 6 \\ 1 & 2 & 5 \\ 0 & 0 & 4 \end{bmatrix}$

$T: \mathbb{R}^2 \to \mathbb{R}^3.$

$T(x,y) = (x+y,\ 2x-y,\ 3x+5y)$

$$\beta = \{\underset{v_1}{(1,1)},\ \underset{v_2}{(0,-1)}\}$$

$\gamma = \{(1,1,1),\ (1,0,1),\ (0,0,1)\}$

$[T]_\beta^\gamma$

$T(v_1) = (2,1,8)$ $\quad a_1(1,1,1)+a_2(1,0,1)+a_3(0,0,1)$

$a_1+a_2 = 2$

$a_1 = 1$

$a_1+a_2+a_3 = 8$

$a_1 = 1,\ a_2 = 1,\ a_3 = 6$

$\Rightarrow [T(v_1)]_\gamma = \begin{bmatrix} 1 \\ 1 \\ 6 \end{bmatrix}$

$T(v_2) = (-1,1,-5)$ $\quad a_1(1,1,0)+a_2(1,0,1)+a_3(0,0,1)$

$a_1+a_2 = -1$

$a_1 = 1$

$a_1+a_2+a_3 = -5$

$a_1 = 1,\ a_2 = -2,\ a_3 = -4$

$\Rightarrow [T(v_2)]_\gamma = \begin{bmatrix} 1 \\ -2 \\ -4 \end{bmatrix}$ $\quad \therefore \begin{bmatrix} 1 & 1 \\ 1 & -2 \\ 6 & -4 \end{bmatrix}$

<Dimension theorem>

<Theorem>

0

- T is a one-to-one if and only if (iff) Nullity = 0

1. $T(\underline{0}) = 0$

Null space $= \{ v \mid T(v) = 0 \}$

$= \{ 0 \}$; zero vector space ; dimension = 0 .

<Theorem>

$\underset{m \times n}{[T]_\beta^\gamma} \, \boxed{[v]_\beta}^{\,n \times 1} = [T(v)]_\gamma$ $\qquad v = a_1 v_1 + a_n$

$T : V \to W$

$\quad\;\; \Downarrow \quad\; \Downarrow$

$\quad\;\; \beta \quad\;\; \gamma$

$A = [T]_\beta^\gamma$, $\dim V = n$, $\dim W = m$.

Then, note that A is a $m \times n$ matrix .

$$\text{Rank} = \dim(\underbrace{\text{col}(A)}_{\Downarrow})$$

span{columns of A}

행렬 A의 열 열들로 생성

ex) $A = \begin{pmatrix} 2 & 1 & 3 \\ 1 & 2 & 5 \end{pmatrix}$