< linear transformation >

☆ $T: V \to W$, $\left(\begin{array}{l} T(v_1+v_2) = T(v_1) + T(v_2) \\ T(cv) = cT(v) \end{array}\right) \Leftrightarrow T(v_1+cv_2) = T(v_1)+cT(v_2)$

(ex) Rotation, reflection, projection

$N(T) := \{ v \in V \mid T(v) = 0_W \}$ Null space

$R(T) := \{ w \in W \mid \exists v \text{ s.t. } T(v) = w \}$ Range

$= \{ T(v) \in W \mid \forall v \in V \}$

$$\left(\begin{array}{l} \text{Nullity} = \dim(N(T)) \\ \text{Rank} = \dim(R(T)) \end{array}\right)$$

$N(T)$: subspace of V, $R(T)$: subspace of W

1. subset

2. vector space

$T: V \to W$, β: basis of V, $= \{v_1, v_2, \ldots, v_n\}$.

Then, $\{T(v_1), T(v_2), \ldots, T(v_n)\}$ is a basis of $R(T)$

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Then, $\{T(v_1), T(v_2), \cdots, T(v_n)\}$ is a basis of $R(T)$

$\Downarrow$

V의 basis $\beta = \{v_1, v_2, \cdots, v_n\}$에 대하여

$T: V \to W$ 일때, // $R(T)$의 기저 $= \{T(v_1), T(v_2), \cdots, T(v_n)\}$이다.

linear transformation

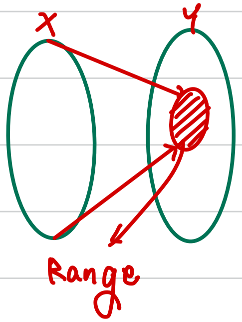
$v \in V$ 일때 $T(v)$

1. 기저...?

생성 (span)

1) $T(v_1), T(v_2), \cdots, T(v_n)$의 선형결합으로 $R(T)$의 벡터 표현 가능...

2) 선형독립.



$R(T)$의 원소 $= w = T(v)$

$T(v)$가 $T(v_1), T(v_2), \cdots, T(v_n)$의 ... ?

$v_1, \cdots v_n$ V

$$v = a_1v_1 + a_2v_2 + \cdots + a_nv_n$$

$$T(v) = T(a_1v_1 + a_2v_2 + \cdots + a_nv_n)$$

$$= a_1T(v_1) + a_2T(v_2) + \cdots + a_nT(v_n)$$

$T(0) = 0$

$T(0) = 0$

선형독립 $a_1v_1 + \cdots + a_nv_n = 0$

$v_1 \sim v_n$

$$a_1 T(v_1) + a_2T(v_2) + \cdots + a_nT(v_n) = 0$$

$$T(\boxed{a_1v_1 + a_2v_2 + \cdots + a_nv_n}) = 0$$

$= 0$

$$\boxed{a_1 = a_2 = \cdots = a_n = 0}$$

Dimension Theorem

$\boxed{\text{Nullity} + \text{Rank} = \dim(V)}$

$T: V \to W$

실수계수를 갖는

$P_n(\mathbb{R})$: n차 이하의 다항식

$a_1(1+x) + a_2(x^2-1) + a_3(x^2-4x)$

$a_2 x^2 + a_3 x^2 + a_1 x - 4a_3 x + a_1 - a_2$

$a_2 + a_3 = -7 \quad a_1 - a_2 = 4$

$a_1 - 4a_3 = 6$

$a_3 = -7 - a_2$

$a_1 = 4 + a_2$

$a_2 + 4 - 4(-7 - a_2) = 6$

$5a_2 = -26$

$$= \begin{bmatrix} -\frac{6}{5} \\ -\frac{26}{5} \\ -\frac{9}{5} \end{bmatrix}$$

$\therefore a_2 = -\frac{26}{5} \quad a_3 = -7 + \frac{26}{5} = \frac{-9}{5} \quad a_1 = 4 + \left(-\frac{26}{5}\right) = -\frac{6}{5}$

Example.

$T : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by

$T(a_1, a_2, a_3) = (a_1+a_2, a_2+a_3, a_3+a_1)$.

Let β be standard ordered basis for $\mathbb{R}^3$

and $\gamma = \{(1,1,0), (0,1,1), (2,2,3)\}$

Compute $[T]_\beta^\gamma = ?$

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Compute $[T]_\beta^\gamma = ?$

step 1.

$$\beta = \{\underset{v_1}{(1,0,0)}, \underset{v_2}{(0,1,0)}, \underset{v_3}{(0,0,1)}\}$$

$(0,1,1)$ -

$T(v_1) = (1,0,1)$, $T(v_2) = (1,1,0)$, $T(v_3)$

step 2.

$T(v_1) = (1,0,1) \Rightarrow a_1(1,1,0) + a_2(0,1,1)$

$+ a_3(2,2,3)$

$T: \mathbb{R} \to \mathbb{R}^3$ be defined by

$T(a_1, a_2, a_3) = (a_1 + a_2,\ a_2 + a_3,\ a_3 + a_1)$.

Let $\gamma = \{ \overset{v_1}{(1,1,0)}, \underset{v_2}{(0,1,1)}, \underset{v_3}{(2,2,3)} \}$ be a basis of $\mathbb{R}^3$

Compute $[T]_\gamma$

Step 1. $T(v_i)$ 구하기

$T(v_1) = T(1,1,0) = (2,1,1)$

$T(v_2) = (1,2,1)$

$T(v_3) = (4,5,5)$

(Step 2) $[T(v_i)]_\gamma$

$T(v_1) = (2,1,1) = a_1(1,1,0) + a_2(0,1,1) + a_3(2,2,3)$

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

$a_1 + 2a_3 = 2$

$a_1 + a_2 + 2a_3 = 1$

$a_2 + 3a_3 = 1$.

$T: \mathbb{R} \to \mathbb{R}^3$ be defined by

$T(a_1, a_2, a_3) = (a_1+a_2,\ a_2+a_3,\ a_3+a_1)$.

Let $\gamma = \{(1,0,0), (0,1,0), (0,0,1)\}$, be a basis of $\mathbb{R}^3$

Compute $[T]_\gamma$

$$[T]_\gamma = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

step1.

$T(V_1) = (1, 0, 1)$

$T(V_2) = (1, 1, 0)$

$T(V_3) = (0, 1, 1)$

step2.

$T(V_1) = (1,0,1) = a_1(1,0,0) + a_2(0,1,0) + a_3(0,0,1)$

$a_1 = 1,\ a_2 = 0,\ a_3 = 1$

$$[T(V_1)]_r = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$T(V_2) = (1,1,0) = a_1(1,0,0) + a_2(0,1,0) + a_3(0,0,1)$

$a_1 = 1,\ a_2 = 1,\ a_3 = 0$

$$[T(V_2)]_r = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$T(V_3) = (0,1,1) = a_1(1,0,0) + a_2(0,1,0) + a_3(0,0,1)$

$a_1 = 0,\ a_2 = 1,\ a_3 = 1$

$$[T(V_3)]_r = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

step 2.

$T(v_1) = (1,0,1) \Rightarrow a_1(1,1,0) + a_2(0,1,1)$

$+ a_3(2,2,3)$

$a_1 + 2a_3 = 1 \rightarrow a_1 = 1-2a_3$

$a_1 + a_2 + 2a_3 = 0$

$a_2 + 3a_3 = 1 \rightarrow a_2 = 1-3a_3$

$1-2a_3 + 1-3a_3 + 2a_3 = 0$

$-3a_3 = -2 \quad \therefore a_3 = \frac{2}{3}$

$a_1 = -\frac{1}{3} \Rightarrow [T(V_1)]^{\gamma}_{\beta} = \begin{bmatrix} -\frac{1}{3} \\ -1 \\ \frac{2}{3} \end{bmatrix}$

$a_2 = -1$

$T(V_2) = (1,1,0) = a_1(1,1,0) + a_2(0,1,1) + a_3(2,2,3)$

$a_1 + 2a_3 = 1 \rightarrow a_1 = 1-2a_3$

$a_1 + a_2 + 2a_3 = 1$

$a_2 + 3a_3 = 0 \rightarrow a_2 = -3a_3$

$1-2a_3 - 3a_3 + 2a_3 = 1$

$a_3 = 0, \quad a_2 = 0, \quad a_1 = 1$

$\Rightarrow [T(V_2)]^{\gamma}_{\beta} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$$T(v_3) = (0,1,1) = a_1(1,1,0) + a_2(0,1,1) + a_3(2,2,3)$$

$$a_1 + 2a_3 = 0 \rightarrow a_1 = -2a_3$$

$$a_1 + a_2 + 2a_3 = 1$$

$$a_2 + 3a_3 = 1 \rightarrow a_2 = 1 - 3a_3$$

$$-2a_3 + 1 - 3a_3 + 2a_3 = 1$$

$$a_3 = 0, \quad a_1 = 0, \quad a_2 = 1$$

$$\Rightarrow [T(v_3)]_\beta^\gamma = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$[T]_\beta^\gamma$$

$$= \begin{bmatrix} & 1 & 0 \\ & 0 & 1 \\ & 0 & 0 \end{bmatrix}$$

ordered basis

$$T : \underset{\beta}{V} \longrightarrow \underset{\gamma}{W} \quad , \quad [T]_\beta^\gamma = ?$$

$\beta = \{v_1, v_2, \cdots, v_n\}$

$\gamma = \{w_1, w_2, \cdots, w_n\}$

step 1.

$\beta = \{v_1, \cdots, v_n\}$ 에 대해 $T(v_i)$를 구한다.

step 2. $[T(v_i)]_\gamma$

$T(v_i)$를 γ에 대해 표현

$$T(v_i) = a_1 w_1 + a_2 w_2 + \cdots + a_m w_m$$

$$\Downarrow$$

$$\begin{bmatrix} a_1 \\ \vdots \\ a_m \end{bmatrix} = [T(v_i)]_\gamma$$

$$[T]_\beta^\gamma = \begin{bmatrix} [T(v_1)]_\gamma & \cdots & [T(v_n)]_\gamma \end{bmatrix}$$

$m \times n$ matrix.

$\mathbb{R}^2$

$T(a_1, a_2) = (a_1 - a_2, a_1, 2a_1 + a_2)$

β = standard ordered basis, $\gamma = \{(1,1,0), (0,1,1), (2,2,3)\}$

$\{\underset{v_1}{(1,0,0)}, \underset{v_2}{(0,1,0)}, \underset{v_3}{(0,0,1)}\}$

step1.

$T(v_1) = (0, 1, 2)$

$T(v_2) = (-1, 0, 1)$

$T(v_3) = (0, 0, 1)$

step2.

$T(v_1) = (0,1,2) = a_1(1,1,0) + a_2(0,1,1) + a_3(2,2,3)$

$a_1 + \quad + 2a_3 = 0 \qquad a_1 = -2a_3$

$a_1 + a_2 + 2a_3 = 1$

$a_2 + 3a_3 = 2 \qquad a_2 = 2 - 3a_3$

$\to -2a_3 + 2 - 3a_3 + 2a_3 = 1$

$a_3 = \frac{1}{3}, \; a_1 = -\frac{2}{3}, \; a_2 = 1 \Rightarrow \begin{bmatrix} -\frac{2}{3} \\ 1 \\ \frac{1}{3} \end{bmatrix}$

$T(v_2) = (-1,0,1) = a_1(1,1,0) + a_2(0,1,1) + a_3(2,2,3)$

$a_1 + \quad + 2a_3 = -1 \to a_1 = -1 - 2a_3$

$a_1 + a_2 + 2a_3 = 0$

$a_2 + 3a_3 = 1 \to a_2 = 1 - 3a_3$

$\to -1 - 2a_3 + 1 - 3a_3 + 2a_3 = 0$

$a_3 = 0, \; a_1 = -1, \; a_2 = 1 \Rightarrow \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$

$T(a_1, a_2) = (a_1 - a_2, a_1, 2a_1 + a_2)$

β = standard ordered basis, $\gamma = \{(1,1,0), (0,1,1), (2,2,3)\}$

$\parallel$

$(1,0),\ (0,1)$

$T(1,0) = (1,1,2) = a_1(1,1,0) + a_2 \cdots \quad a_3$

$T(0,1) = (-1,0,1) = b_1 \quad b_2 \quad b_3$

$$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{bmatrix}$$

$$T(v_3') = (0,0,1) = a_1(1,1,0) + a_2(0,1,1) + a_3(2,2,3)$$

$$a_1 + \quad + 2a_3 = 0 \rightarrow a_1 = -2a_3$$

$$a_1 + a_2 + 2a_3 = 0$$

$$a_2 + 3a_3 = 1 \quad a_2 = 1 - 3a_3$$

$$\rightarrow -2a_3 + 1 - 3a_3 + 2a_3 = 0$$

$$a_3 = \frac{1}{3}, \quad a_1 = -\frac{2}{3}, \quad a_2 = 0 \Rightarrow \begin{bmatrix} -\frac{2}{3} \\ 0 \\ \frac{1}{3} \end{bmatrix}$$

$$\therefore [T]_\beta^\gamma = \begin{bmatrix} -\frac{2}{3} & -1 & -\frac{2}{3} \\ 1 & 1 & 0 \\ \frac{1}{3} & 0 & \frac{1}{3} \end{bmatrix}$$