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#2)

$f : \mathbb{R} \to \mathbb{R}$ is continuous.

Then for every $\varepsilon > 0$ and $a \in \mathbb{R}$,
$\exists$ $\delta > 0$ s.t. $|f(x) - f(a)| < \varepsilon$
whenever $x \in \mathbb{R}$ and $|x - a| < \delta$.

That is

$$x \in N_a(\delta) \Rightarrow f(x) \in N_{f(a)}(\varepsilon)$$

$$\Rightarrow f(N_a(\delta)) \subseteq N_{f(a)}(\varepsilon)$$

$$\Rightarrow N_a(\delta) \subseteq f^{-1}(N_{f(a)}(\varepsilon)) \quad \cdots \star$$

Next,
we have to show that for any open set $G \in \mathbb{R}$, then $f^{-1}(G)$ is also open in $\mathbb{R}$.

Since $\emptyset$ and $\mathbb{R}$ are open in $\mathbb{R}$,
Suppose that $f^{-1}(G) \neq \emptyset$ and $f^{-1}(G) \neq \mathbb{R}$

Let $x \in f^{-1}(G)$. Then $f(x) \in f(G)$

Since G is open, $\exists\, \varepsilon > 0$ s.t. $N_{f(x)}(\varepsilon) \subseteq G$.

Since f is continuous at x by $(*)$, for this ε, there exists $\delta > 0$ such that

$$N_x(\delta) \subseteq f^{-1}(N_{f(x)}(\varepsilon)) \subseteq f^{-1}(G).$$

Then every point of x of $f^{-1}(G)$ is an interior point and so $f^{-1}(G)$ is open in $\mathbb{R}$. ∎