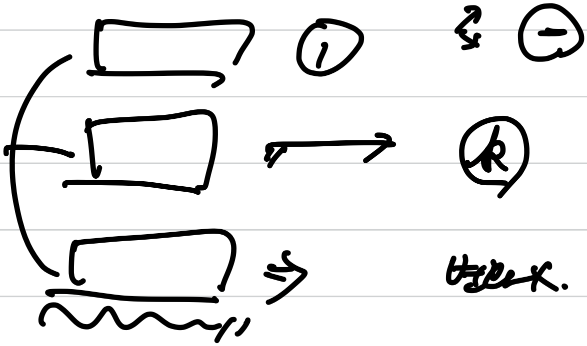
Determinant



Diagonalize (Eigen decomposition).

Diagonalization

<Def>

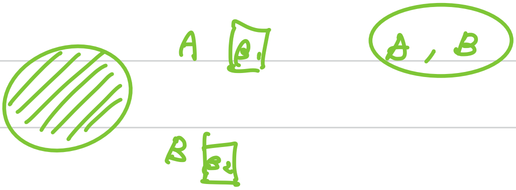
For $A_{n\times n}$ and $B_{n\times n}$, if there is a $P_{n\times n}$ s.t. $A = P^{-1}BP$,

then A is similar to B.

기저를 바꿈.

닮음행렬 : $A = P^{-1}BP$ 의 관계에서 A, B

A와 B는 같은 선형변환을 나타냄.



$$A = P^{-1}DP$$

↳ Diagonal Matrix ex) $\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ $\begin{pmatrix} 3 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ ...

For given A, find P & D

$$A = P^{-1}DP$$

7. 다음 행렬 $A=\begin{pmatrix}5 & 2\\ 2 & 2\end{pmatrix}$가 주어질 때, $A^{2021}=\begin{pmatrix}a & b\\ c & d\end{pmatrix}$에서 $5d$를 고르시오.

① $6^{2021}+4$ ② $2\cdot 6^{2021}-2$ ③ $4\cdot 6^{2021}+1$ ④ $4\cdot 6^{2021}+4$ ⑤ 보기 중 답 없음

A^{2021}

$A=P^{-1}DP$

$A^{2021}=(P^{-1}DP)^{2021}$

$=(P^{-1}D\cancel{P})(\cancel{P^{-1}}D\cancel{P})(\cancel{P^{-1}}DP)\cdots(P^{-1}DP)$

I I

$=P^{-1}D^{2021}P$

Diagonal matrix

$$B=\begin{pmatrix}2 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -3\end{pmatrix} \qquad B^{2000}=\begin{pmatrix}2^{2000} & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & (-3)^{2000}\end{pmatrix}$$

1. 항상 decomposition 가능?

2. 어떻게 하지 ... ?

Eigen Decomposition

(Def)

For $n \times n$ matrix $A_{n\times n}$,

vector $n\times 1$

if there is a λ and v s.t. $Av = \lambda v$, then λ : eigenvalue

scalar

v: eigenvector $\neq 0$

ex) $A = \begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix}$

eigenvector

Note that $\begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix}\begin{pmatrix} 3 \\ 4 \end{pmatrix} = 5\begin{pmatrix} 3 \\ 4 \end{pmatrix}$

eigenvalue (λ)

Eigenvalue

$Av = \lambda v$

$Av = \lambda I v$

$Av - \lambda I v = 0$

$\underbrace{(A - \lambda I)}\, v = \underline{\underline{0}}$

↳ nonzero vector

$\det(A - \lambda I) = 0$;

Characteristic polynomial

* $AB = 0$

~~$A = 0$ or $B = 0$~~

$A \neq 0,\ B \neq 0 \Rightarrow AB = 0$

$\det(A) = 0,\ \det(B) = 0$

(ex) $\begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix} = A$

$A - \lambda I = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$

$= \begin{pmatrix} 1-\lambda & 2 \\ 3 & 2-\lambda \end{pmatrix}$

$\det(A - \lambda I)$

$\det\begin{pmatrix} 1-\lambda & 2 \\ 3 & 2-\lambda \end{pmatrix} = (1-\lambda)(2-\lambda) - 6$

$= \lambda^2 - 3\lambda + 2 - 6 = \lambda^2 - 3\lambda - 4 = 0$

$(\lambda - 4)(\lambda + 1) = 0$.

$\lambda = 4$ or -1

$$\boxed{\textbf{(b)}\quad A = \begin{pmatrix} 0 & -2 & -3 \\ -1 & 1 & -1 \\ 2 & 2 & 5 \end{pmatrix}} \quad \text{fo}$$

(c) $A = \begin{pmatrix} i & 1 \\ \end{pmatrix}$ for $F = C$

$\det(A - \lambda I) = 0$

$$A - \lambda I = \begin{pmatrix} 0 & -2 & -3 \\ -1 & 1 & -1 \\ 2 & 2 & 5 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}$$

$$= \begin{pmatrix} \cancel{-\lambda} & \cancel{-2} & \cancel{-3} \\ -1 & 1-\lambda & -1 \\ 2 & 2 & 5-\lambda \end{pmatrix}$$

$$(-1)^{1+i} \cdot A_{1i} \cdot \det(\tilde{A}_{1i})$$

$$= (-1)^2 \cdot (-\lambda) \cdot \big((1-\lambda)(5-\lambda)+2\big)$$

$$+ (-1)^3 \cdot (-2) \cdot \big(-(5-\lambda)+2\big)$$

$$+ (-1)^4 \cdot (-3) \cdot \big(-2-2\cdot(1-\lambda)\big) = 0$$

$$-\lambda(\lambda^2-6\lambda+7) + 2(\lambda-3) - 3(2\lambda-4) = 0$$

$$-\lambda^3 + 6\lambda^2 - 7\lambda + 2\lambda - 6 - 6\lambda + 12 = 0$$

$$-\lambda^3 + 6\lambda^2 - 11\lambda + 6 = 0$$

$$\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$$

$$(\lambda-1)(\lambda^2-5\lambda+6)$$

$$0 = (\lambda-1)(\lambda-3)(\lambda-2).$$

$\lambda = 1, 2, 3$

$x-1$

$\boxed{1}x^3-6x^2+11x\boxed{-6}=0$

6의 약수

$\pm 1 \quad 2 \quad 3 \quad 6$

0

$x-1=0$

$1\,)\;x^3-6x^2+11x-6$

1	1	-6	11	-6
		1	-5	6
	1	-5	6	0

x^3-4x^2+5x-2 $\qquad$ $\pm 1, 2.$

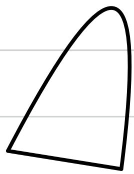
$$\begin{array}{r|rrrr} 1 & 1 & -4 & 5 & -2 \\ & & 1 & -3 & 2 \\ \hline & 1 & -3 & 2 & \boxed{0} \end{array}$$

$(x-1)(x^2-3x+2)$

$(x-1)(x-2)(x-1)$

$\Downarrow$

x^2-3x+2



$\det(A-\lambda I)=0$

$A=\begin{pmatrix}1&3\\4&2\end{pmatrix}-5I$

$Av=\lambda v$

$(A-\lambda I)v=0$

$v=\begin{pmatrix}v_1\\v_2\end{pmatrix}$

$\det(A-\lambda I)=0$

$\begin{pmatrix}3&3\\4&4\end{pmatrix}$

$\det\begin{pmatrix}1-\lambda&3\\4&2-\lambda\end{pmatrix}=(\lambda-1)(\lambda-2)-12=\lambda^2-3\lambda-10=0$

$(\lambda-5)(\lambda+2)=0.$

$\lambda=5$ or -2.

$v_2=\frac{4}{3}v_1$

(i) $\boxed{\lambda=5}$

$3v_2=4v_1$

$\begin{pmatrix}-4&3\\4&-3\end{pmatrix}\begin{pmatrix}v_1\\v_2\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$

$-4v_1+3v_2=0$

$\cancel{4v_1-3v_2=0}$

Eigenspace

$\begin{pmatrix}v_1\\v_2\end{pmatrix}=\begin{pmatrix}v_1\\\frac{4}{3}v_1\end{pmatrix}=v_1\begin{pmatrix}1\\\frac{4}{3}\end{pmatrix}=\frac{v_1}{3}\begin{pmatrix}3\\4\end{pmatrix}$

$u_1+u_2=0$

(ii) $\lambda=-2$

$\begin{pmatrix}3&3\\4&4\end{pmatrix}\begin{pmatrix}u_1\\u_2\end{pmatrix}=\begin{pmatrix}3u_1+3u_2\\4u_1+4u_2\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$ $\quad\begin{pmatrix}u_1\\-u_1\end{pmatrix}=u_1\begin{pmatrix}1\\-1\end{pmatrix}$

$$A=\begin{pmatrix}0&-2&-3\\-1&1&-1\\2&2&5\end{pmatrix},\quad \lambda=1,2,3$$

$$(A-\lambda I)v=0$$

(i) $\lambda=1$

$$\begin{pmatrix}-1&-2&-3\\-1&0&-1\\2&2&4\end{pmatrix}\begin{pmatrix}v_1\\v_2\\v_3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

$v_1+2v_2+3v_3=0$

$-v_1 \quad -v_3=0 \qquad v_3=-v_1$

$v_1+v_2+v_3=0 \qquad v_2=v_1$

$$\begin{pmatrix}v_1\\v_1\\-v_1\end{pmatrix}=v_1\begin{pmatrix}1\\1\\-1\end{pmatrix}$$

(ii) $\lambda=2$

$$\begin{pmatrix}-2&-2&-3\\-1&-1&-1\\2&2&3\end{pmatrix}\begin{pmatrix}v_1\\v_2\\v_3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

$-2v_1-2v_2-3v_3=0$

$v_1+v_2+v_3=0 \qquad v_2=-v_1-v_3$

$2v_1+2v_2+3v_3=0$

$2v_1-2v_1-2v_3+3v_3=0$

$v_2=-v_1,\quad v_3=0$

$$\Rightarrow\begin{pmatrix}v_1\\-v_1\\0\end{pmatrix}=v_1\begin{pmatrix}1\\-1\\0\end{pmatrix}$$

(iii) $\lambda = 3$

$$\begin{pmatrix} -3 & -2 & -3 \\ -1 & -2 & -1 \\ 2 & 2 & 2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$3v_1 + 2v_2 + 3v_3 = 0$

$v_1 + 2v_2 + v_3 = 0$ $\quad$ $v_1 - 2v_1 - 2v_3 + v_3 = 0 \to -v_1 - v_3 = 0$ $\quad$ $v_3 = -v_1$

$v_1 + v_2 + v_3 = 0$ $\quad$ $v_2 = -v_1 - v_3$ $\quad$ $v_2 = 0$

$$\therefore \begin{pmatrix} v_1 \\ 0 \\ -v_1 \end{pmatrix} = v_1 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

$$\det(A - \lambda I) = 0$$

$$A - \lambda I = \begin{pmatrix} 3-\lambda & 1 & 0 \\ 0 & 3-\lambda & 0 \\ 0 & 0 & 4-\lambda \end{pmatrix}$$

$$(-1)^{3+i} \cdot A_{3i} \cdot \det(A_{3i})$$

$$\det(A - \lambda I) = (-1)^6 \cdot (4-\lambda) \cdot \left((3-\lambda)^2\right)$$

$$= (4-\lambda)(3-\lambda)(3-\lambda) = 0$$

$$\lambda = 4, 3$$

$$(A - \lambda I)v = 0$$

i) $\lambda = 3$

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$v_2 = 0$

$v_3 = 0$

ii)

$$\begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$-v_1 + v_2 = 0 \rightarrow v_1 = -v_2$

$v_2 = 0$

3개

$$\begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

2개 1개

$$\det\begin{pmatrix} 3-\lambda & 1 & 0 \\ 0 & 3-\lambda & 0 \\ 0 & 0 & 4-\lambda \end{pmatrix} = (3-\lambda)^2(4-\lambda).$$

$\lambda = 3, 4$ $\quad A-\lambda I$

$\lambda = 3$

X

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \quad v_2 = 0,\ v_3 = 0,\ v_1 =$$

$$\begin{pmatrix} v_1 \\ 0 \\ 0 \end{pmatrix} = v_1 \boxed{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}$$

1개

$\lambda = 4$

$$\begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$v_1 = 0$

$v_2 = 0$

$v_3 = ?$

$$\begin{pmatrix} 0 \\ 0 \\ v_3 \end{pmatrix} = v_3 \boxed{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}$$

대각화 $A = P^{-1} D P$ 언제?

Fact.

① For $A_{n\times n}$, degree of characteristic poly. ($\det(A-\lambda I)$) $= n$

② For given λ, there are at most k linearly independent eigenvectors when the multiplicity $= k$.

(ex) ↳ $\det(A-\lambda I) = (3-\lambda)^2(4-\lambda)$

$\lambda = 3$ ···> at most 2 eigenvectors

linearly independent

$\lambda = 4$ →

③ Eigenspace for $\lambda = \lambda_1$ $=$ span{ eigenvectors for $\lambda = \lambda_1$ }
(E_{λ_1})

④ Let $Av_1 = \lambda_1 v_1$, $Av_2 = \lambda_2 v_2$

Then, if $\lambda_1 \neq \lambda_2$, then v_1 and v_2 are linearly independent.

[Theorem] ·· Diagonalizable

⇒ For $A_{n\times n}$, if there are n linearly independent eigenvectors.

⇒ For any λ, there are k linearly independent eigenvectors.

Multiplicity,

Diagonalization