TIME SERIES ANALYSIS OF THE SABR MODEL PARAMETERS

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Abstract

In real practice, the calibration process of the stochastic-alpha-beta-rho (SABR) model is time-consuming and the results can only be applied within a short amount of time due to dynamics of markets. We introduce the times series and machine learning models to predict future parameters in the SABR model, which could be used to price options in advance.

SECTION 1

SABR Model Calibration

Given the SP500 option prices, we only choose those with the trading volumes higher than 500 and the expiry equal to three months. Also, we only use out-of-the-money prices because the out-of-the-money options are more liquid. The time ranges from Jan 1st, 2014 to Dec 31st, 2018. The calibrated parameters before 2018 are used as training data, while the parameters after 2018 are used as testing data. The SABR model is driven by two Brownian motions W_1 and W_2 such that

$$dS_t = rS_t\,dt + \sigma_t S_t^{\beta}\,dW_t^1, \tag{1.1}$$

$$d\sigma_t = \alpha\sigma_0\,dW_2, \tag{1.2}$$

$$\mathrm{Cov}\left(dW_t^1, dW_t^2\right) = \rho\,dt. \tag{1.3}$$

The price process follows the CEV model, in which β represents the skewness of the volatility surface. The volatility process is log-normal, in which α represents the growth rate of the volatility and σ_0 is the initial volatility. The parameter ρ is the correlation between two Brownian motions. The probability density function could become negative when the strikes are very low, and thus it is very important to check the no-arbitrage assumption after the calibration.
The optimization problem is

$$\min_{\alpha,\rho,\sigma_0} \left(\sigma\left(\alpha,\rho,\sigma_0\right) - \tilde{\sigma}\right)^2. \tag{1.4}$$

Setting up $\beta = 0.5$ in general will lead to the best result, but we still test different betas just in case. During the tests, as the beta varies, alpha and rho will approximately remain the same. The scale of the sigma will change but the shape will remain the same.
Given the implied volatility, we extract three parameters α, β, and σ_0 by optimization. The relation between the implied volatility and the parameters is

$$\tilde{\sigma}_t = \frac{\alpha \ln\left(\frac{F_t}{K}\right)}{x(z)} \cdot \frac{A(F_t,K)}{B(F_t,K)}, \tag{1.5}$$

where

$$F_t = S_t e^{rt}, \tag{1.6}$$

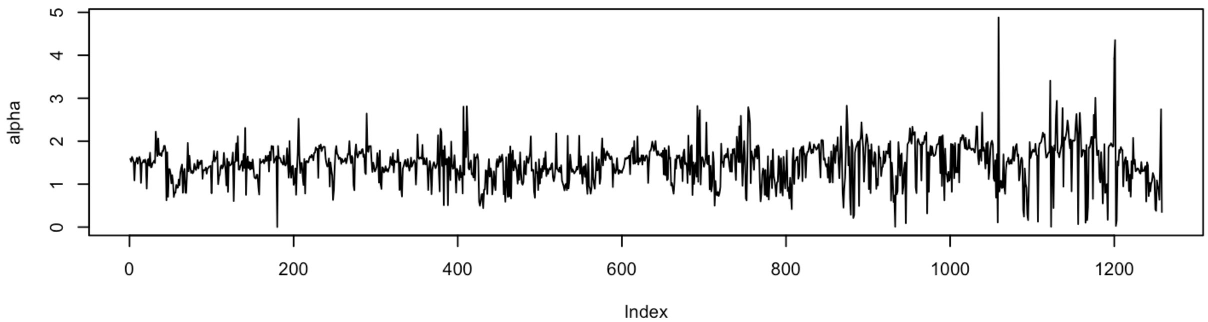
$$A(F_t,K) = 1 + \left[\frac{\sigma_0^2(1-\beta)^2}{24(F_tK)^{1-\beta}} + \frac{\alpha\beta\rho\sigma_0}{4(F_tK)^{\frac{1-\beta}{2}}} + \alpha^2\frac{2-3\rho^2}{24}\right] t + \cdots \tag{1.7}$$

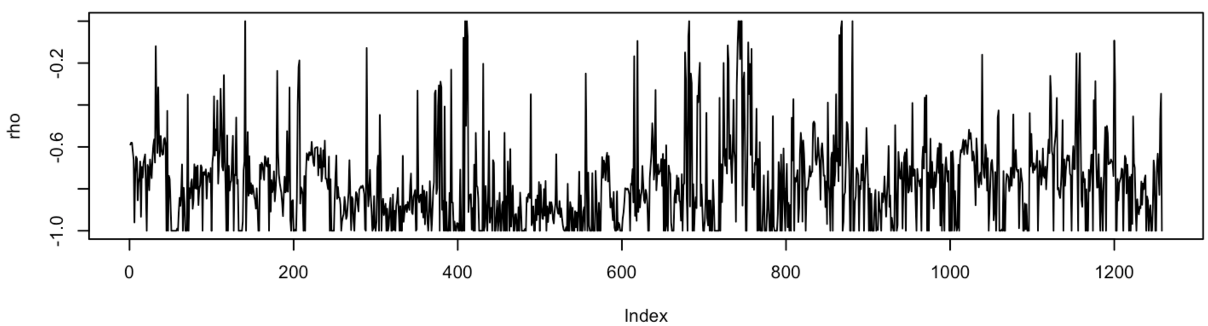
$$B(F_t,K) = 1 + \frac{1}{24}\left[(1-\beta)\ln\left(\frac{F_t}{K}\right)\right]^2 + \frac{1}{1920}\left[(1-\beta)\ln\left(\frac{F_t}{K}\right)\right]^4 + \cdots \tag{1.8}$$

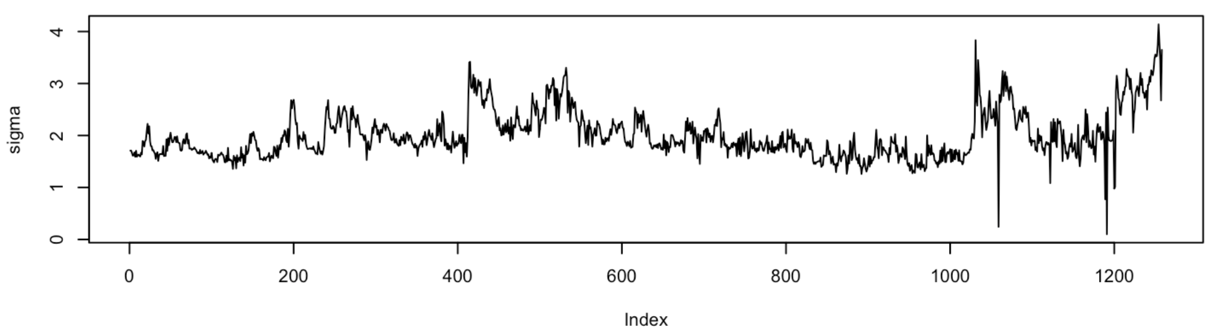
$$x(z) = \ln\left(\frac{\sqrt{1-2\rho z+z^2}+z-\rho}{1-\rho}\right), \tag{1.9}$$

$$z = \frac{\alpha}{\sigma_0}(F_tK)^{\frac{1-\beta}{2}}\ln\left(\frac{F_t}{K}\right). \tag{1.10}$$

The time series are plotted as follows.







SECTION 2

No-Arbitrage Tests

After the calibration, we also need to verify the no-arbitrage assumption. We use the Breeden-Litzenberger method to estimate the approximate probability density function and check if the value at every strike is non-negative. Given the calibrated SABR model, we simulate thousands of paths to acquire the option price at every strike. Since there are too many parameter tuples, we just randomly selected and verified 12 cases and displayed them in the appendix.

In most cases, the probability density is always above zero which implies no arbitrage opportunities. But in a few cases, the probability density may still fall below zero slightly at low strikes. It is acceptable for our model as long as it does not occur often and we can still claim that there is no arbitrage.

SECTION 3

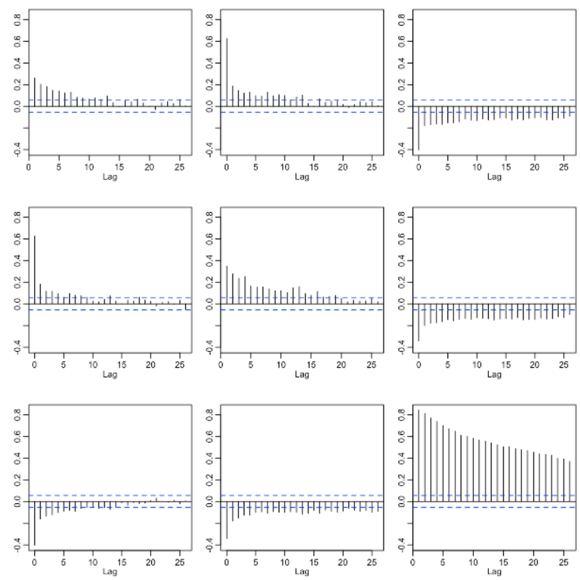
VARMA Model

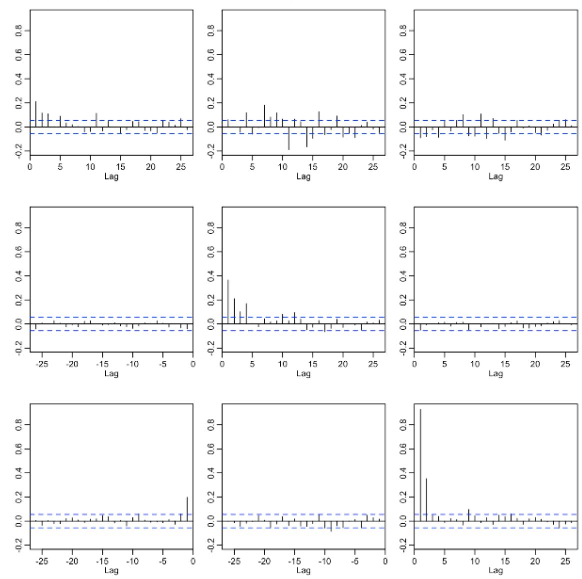
We will use the VARMA model, a multivariate time series model, to predict future parameters which is shown as follows.

$$y_t = \varphi_0 + \epsilon_t + \sum_{i=1}^{p} \varphi_i y_{t-i} + \sum_{j=1}^{q} \theta_j \epsilon_{t-i}. \tag{3.1}$$

3.1 Data Visualization

We plot ACFs on the left and PACFs on the right as follows. They both have shown that three time series have neither drift nor trend nor fluctuations and are likely to be stationary.





The left figure above demonstrates that there are some autocorrelations for each parameter, especially for sigma, and correlations between them. Thus, to capture those relations, a multivariate time series model would be suitable. The right figure above tells us that there may be an autocorrelation effect of 3 or 4 lags or a moving average effect since some parameters in the PACFs have a small number of significant lags which decrease gradually.

3.2 Stationary Tests

Unless specified otherwise, the cutoff p-value is 0.05 for all conducted tests.

3.2.1 ADF Test

The ADF test is one of the most widely used tests to determine whether the tested series are stationary by conducting auxiliary regressions. The null hypothesis is that the tested series is stationary.

	ADF statistics	crirical value	stationary
α	-3.00	-1.95	no
ρ	-1.89	-1.95	yes
σ	-0.77	-1.95	yes

3.2.2 Phillips–Perron Unit Root Test

The Phillips–Perron (PP) Test also has an auxiliary regression to detect the unit root in a time series data as ADF test does. However, the PP test transforms the residuals so that it would be more reasonable to detect the unit root from the residuals. The alternative hypothesis for the PP Test is that the time series is stationary. The test results are shown as follows.

	lags	p-value	stationary
α	23	0.01	yes
	7	0.01	yes
ρ	23	0.01	yes
	7	0.01	yes
σ	23	0.01	yes
	7	0.01	yes

3.2.3 Breitung Variance-Ratio Test

The Breitung variance-ratio test is another test for the unit root. The alternative hypothesis is that the time series is stationary. The test results are shown as follows.

	p-value	stationary
α	0.001	yes
ρ	0.001	yes
σ	0.007	yes

To conclude, although the ADF test shows that α is non-stationary, the other two tests illustrate that all series are stationary so we still consider all parameter series are stationary in this project.

3.3 Estimation Results

We use the Kronecker index to find optimal lags. The first step is to calculate the Kronecker index for each series. The results are shown as follows.

	α	ρ	σ
Kronecker Index	1	1	1

Given the Kronecker indices, the order for each VARMA series should be (1, 1), and the estimated parameters are shown as follows.

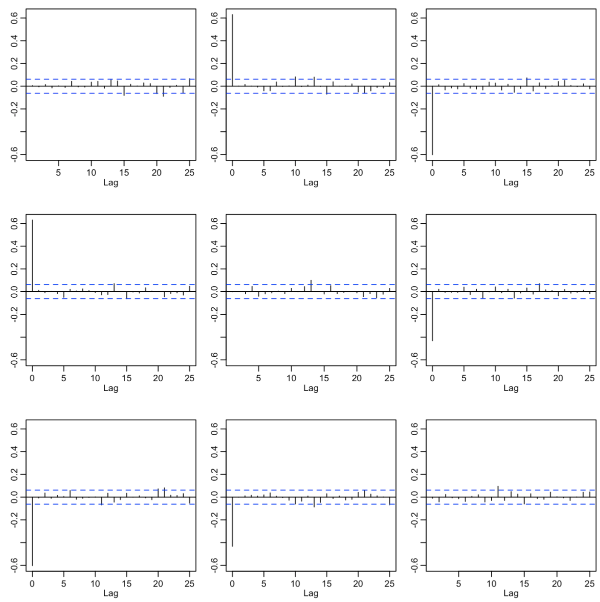
	φ_1^α	θ_1^α	φ_1^ρ	θ_1^ρ	φ_1^σ	φ_1^σ	φ_0
α	0.0731	0.487833	0.1493	0.3075	-0.0425	-0.0171	0.5915
ρ	-0.0144	0.029423	0.8640	0.6421	-0.0168	0.1094	-0.0538
σ	0.0294	-0.123709	0.0413	-0.0158	0.9679	0.1197	0.0518

The measures of the goodness of fitness are shown as follows.

	R^2 score	MSE	MAE
α	0.9382	0.1409	0.2723
ρ	0.9460	0.0357	0.1335
β	0.9945	0.0215	0.1091

3.4 Model Diagnosis

To confirm that the VARMA models explain all relations among all time series, we conduct the residual diagnosis and plot the ACFs of residuals as follows.



From the figure above, there is no statistically significant lags among all series. To be more discrete, three univariate and one multivariate Portmenteau Tests are conducted to determine whether the residuals are white noises. The results are shown as follows.

residual	p-value	white noise
α	0.231	yes
ρ	0.341	yes
σ	0.119	yes
all	0.079	yes

It is sufficient to conclude that the residuals of the VARMA model are white noises, and that means the VARMA model can fully explain the relations among model parameters.

3.5 Out-of-Sample Predictions

Although the VARMA model performs well in in-sample data, we also need to check the out-of-sample predictions to ensure that there is no apparent overfitting.

From the figure above, this model tends to overreact to the large shocks. The measure of the goodness of fitness are shown as follows.

	R^2 score	MSE	MAE
α	0.6095	1.1478	0.6781
ρ	0.8647	0.0777	0.2073
β	0.9526	0.2644	0.2984

SECTION 4

Machines Learning Models

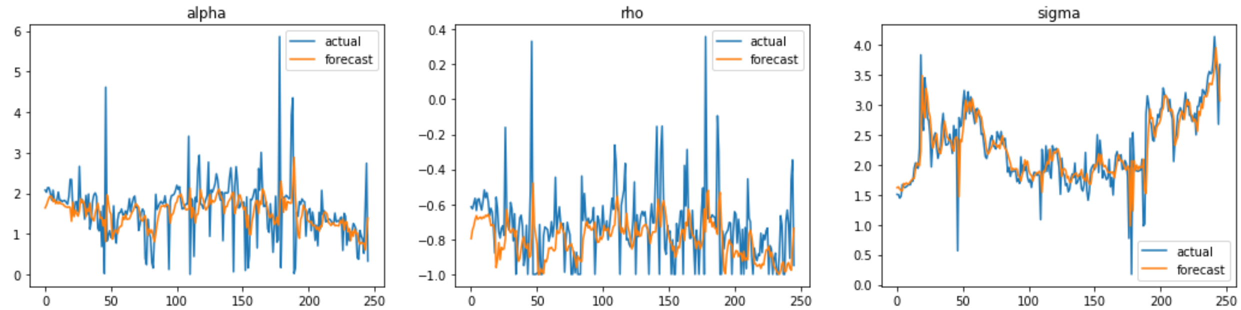
To forecast the parameters of the SABR model, we can also use several regression machine learning methods such as Support Vector Regression (SVR) and Adaboost Regression (AdaR) with different kernels such as the linear kernel, the polynomial kernel, and the radial basis function (rbf) kernel. For the technical details, the penalty parameter of the error c in SVR is set to 3.0 to increase the precision of the regression. The tolerance parameter of the error ϵ is set to 0.1 for α and σ, and it is set to 0.01 for ρ, considering the various magnitudes of parameters. The in-sample results are shown as follows.

		R^2 score		MSE		MAE	
		SVR	SVR + AdaR	SVR	SVR + AdaR	SVR	SVR + AdaR
linear	α	0.1129	-0.1535	0.1454	0.1891	0.2675	0.3352
	ρ	0.1832	-1.1420	0.0382	0.1002	0.1340	0.2802
	σ	0.8402	0.7768	0.0217	0.0304	0.1092	0.1390
polynomial	α	0.1443	-0.0756	0.1403	0.1763	0.2597	0.3328
	ρ	0.2075	-0.1807	0.0371	0.0552	0.1281	0.1990
	σ	0.8472	0.8234	0.0208	0.0240	0.1077	0.1262
rbf	α	0.1753	-0.0227	0.1352	0.1676	0.2532	0.3285
	ρ	0.2431	0.0549	0.0354	0.0442	0.1220	0.1744
	σ	0.8568	0.8555	0.0195	0.0197	0.1050	0.1148

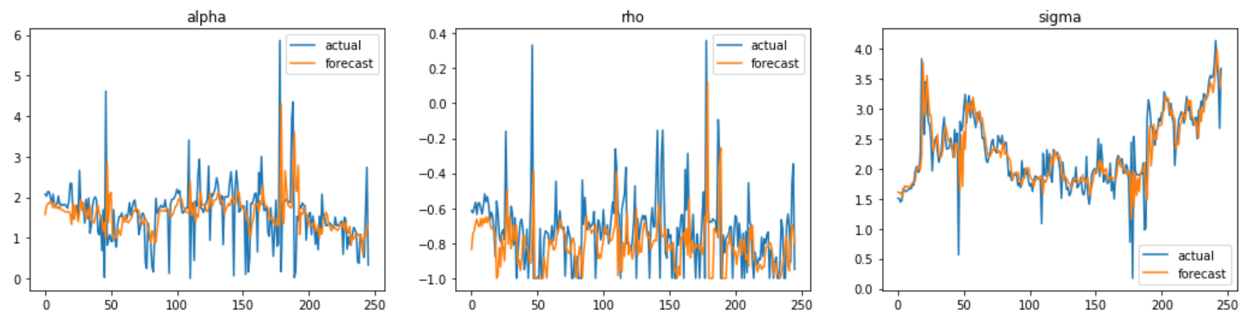
Given the in-sample results, SVR models are much better than SVR + AdaR models, and thus we only SVR models later. Among three kernels, rbf is the best, followed by polynomial kernel and the linear kernel. However, it seems that SVR models with different kernels do not have significant differences. The out-of-sample results are shown as follows.

		R^2 score	MSE	MAE
linear	α	-0.0147	0.5220	0.4586
	ρ	-0.1095	0.0460	0.1513
	σ	0.6324	0.1361	0.2273
polynomial	α	-0.1747	0.6043	0.4699
	ρ	-0.8927	0.0518	0.1586
	σ	0.6256	0.1386	0.2329
rbf	α	0.0672	0.4799	0.4377
	ρ	-0.4340	0.0538	0.1662
	σ	0.6333	0.1358	0.2361

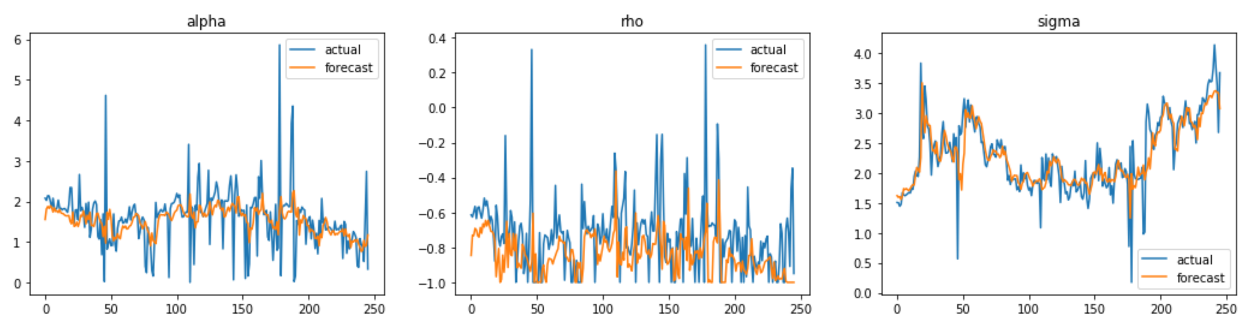
The comparisons of forecast parameters and actual ones are shown as follows.



(a) linear kernel



(b) polynomial kernel



(c) rbf kernel

Given the out-of-sample results, the rbf kernel performs the best, especially for the parameters α and σ, followed by the linear kernel and the polynomial kernel. For the parameter ρ, the linear kernel perform the best, followed by the polynomial kernel and the rbf kernel. From the figure above, it seems that the SVR model with the polynomial kernel is the best overall to forecast the volatility in the time series. However, it also causes more noises that may lead to errors.

SECTION 5

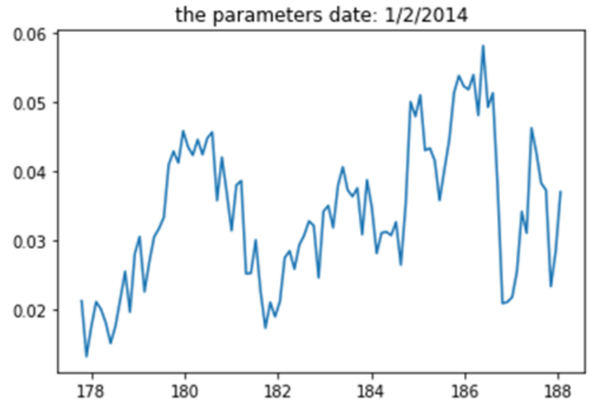
Conclusion

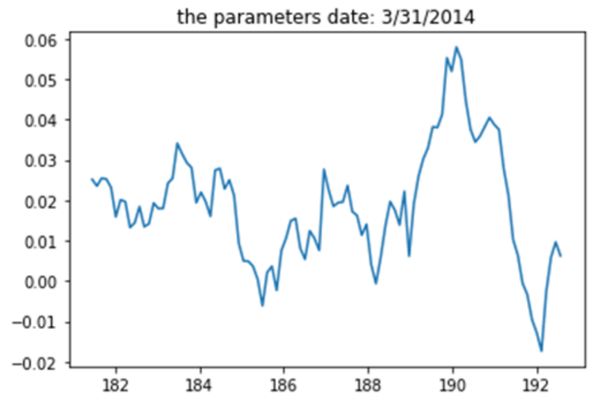
	SVR	VARMA	Previous Parameters
MSE	0.1961	0.1946	0.1977
MAE	0.2540	0.2521	0.2545

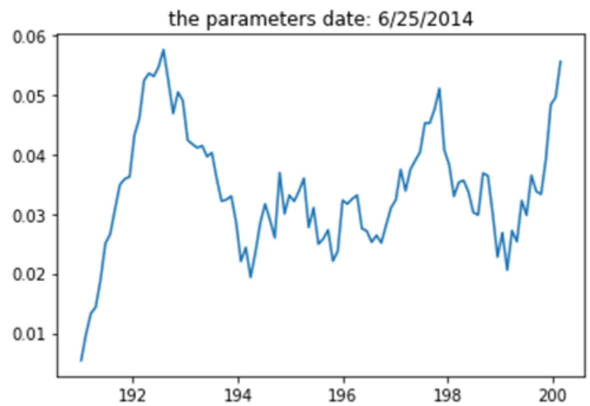
Assuming the SABR model performs well in simulating market dynamics, compared with adopting parameters calibrated from the last day, both SVR and VARMA model improve the accuracy in prediction, and VARMA does better due to the smaller MSE and MAE.

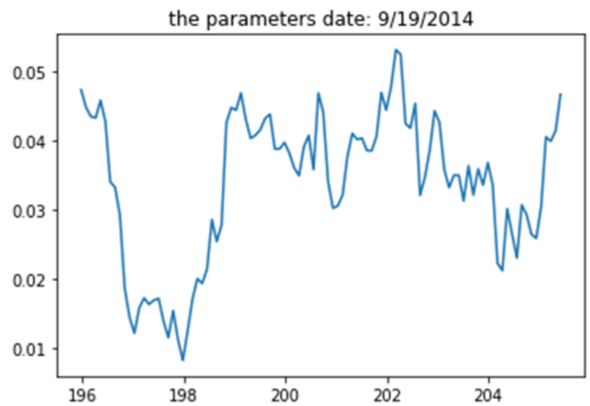
SECTION A

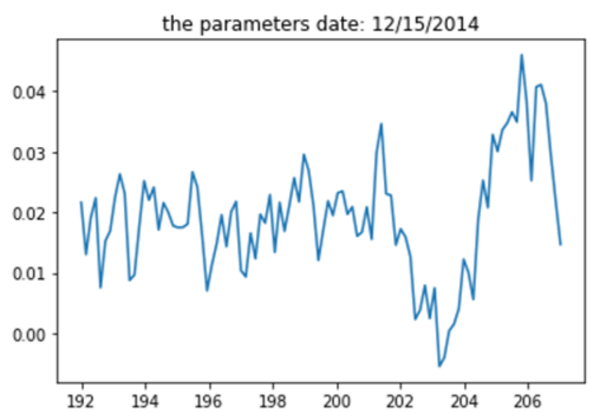
Figures of No-Arbitrage Tests

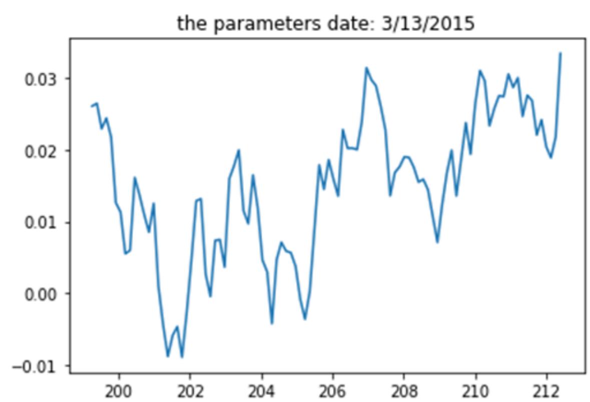


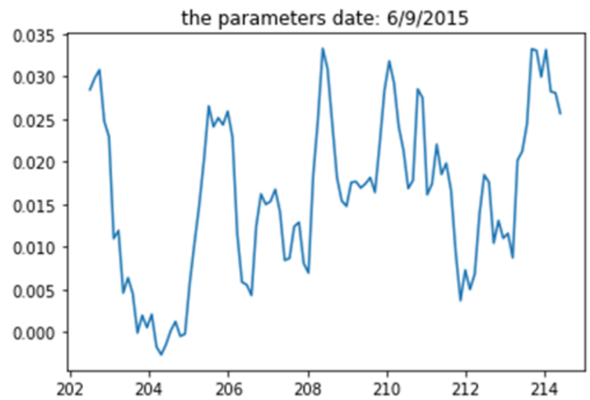








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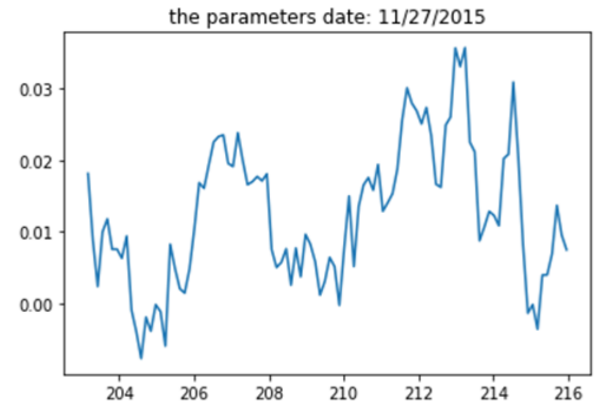
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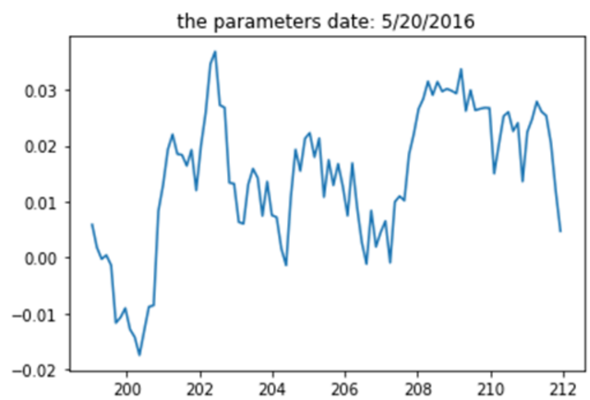
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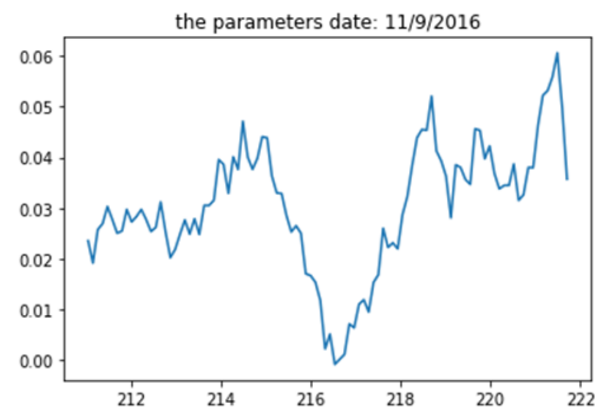
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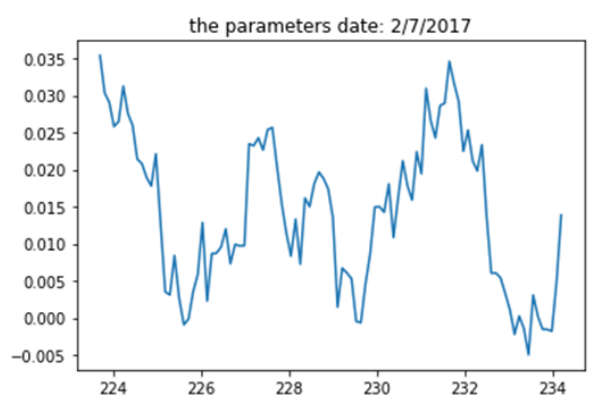
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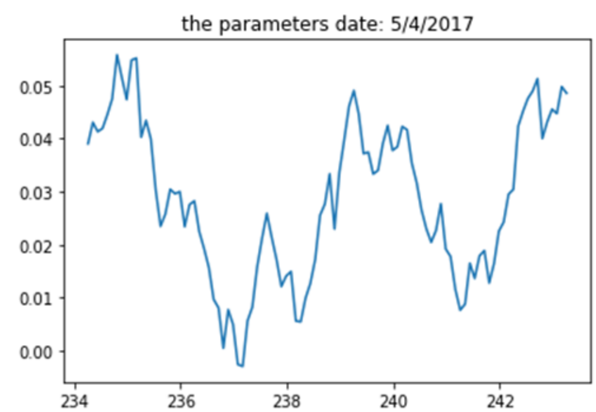
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SECTION B

Reference

[1] Nan Zhang. (2011). Properties of the SABR Model.
[2] Hagan, Patrick Kumar, Deep Lesniewski, Andrew Woodward, Diana. (2014). Arbitrage-free SABR. Wilmott.
[3] Marco Bianchetti, Mattia Carlicchi. (2012). Interest Rates After The Credit Crunch: Multiple-Curve Vanilla Derivatives and SABR.
[4] Bossu S. (2014). Advanced Equity Derivatives: Volatility and Correlation. John Wiley & Sons, Inc. 176.