On the Convergence Parameters of the Prime-Composite Series

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1 Introduction

The prime-composite series is an infinite series of the form

$$\sum_{k=1}^{\infty} \frac{1}{p_k^u c_k^v} \tag{1.1}$$

where p_k is the k^{th} prime and c_k is the k^{th} composite. Let u and v be the convergence parameters of primes and composites respectively. We will show that whether this series converges or diverges is determined by the relationship of both convergence parameters.

2 The Harmonic Series of Primes and Composites

Euler [1] has proved that

$$\sum_{k=1}^{\infty} \frac{1}{p_k} = \infty. \tag{2.1}$$

Let $\tilde{c}_k$ be the k^{th} composite right before a prime. Thus, we can prove that

$$\sum_{k=1}^{\infty} \frac{1}{c_k} > \sum_{k=1}^{\infty} \frac{1}{\tilde{c}_k} = \sum_{k=3}^{\infty} \frac{1}{p_k - 1} > \sum_{k=3}^{\infty} \frac{1}{p_k} = \infty. \tag{2.2}$$

We have

$$\sum_{k=1}^{\infty} \frac{1}{p_k^u} > \sum_{k=1}^{\infty} \frac{1}{p_k} = \infty \tag{2.3}$$

where $u < 1$, and

$$\sum_{k=1}^{\infty} \frac{1}{p_k^u} < \sum_{k=1}^{\infty} \frac{1}{k^u} < \infty \tag{2.4}$$

where $u > 1$. We also have

$$\sum_{k=1}^{\infty} \frac{1}{c_k^v} > \sum_{k=1}^{\infty} \frac{1}{c_k} = \infty \tag{2.5}$$

where $v < 1$, and

$$\sum_{k=1}^{\infty} \frac{1}{c_k^v} < \sum_{k=1}^{\infty} \frac{1}{k^v} < \infty \tag{2.6}$$

where $v > 1$.

3 The General Prime-Composite Series

For the combination of two series, we will discuss eight separate cases.

3.1 Case One

We have

$$\sum_{k=1}^{\infty} \frac{1}{p_k^u c_k^v} = \sum_{k=1}^{\infty} p_k^{-u} c_k^{-v} = \infty \tag{3.1}$$

where $u < 0$ and $v < 0$. We conclude that the prime-composite series diverges where $u + v \le 1$ in this case.

3.2 Case Two

Since $p_k \gg c_k$ where k is sufficiently large, we have $p_k^{u+v} \gg p_k^u c_k^v \gg c_k^{u+v}$ and

$$\sum_{k=1}^{\infty} \frac{1}{p_k^{u+v}} < \sum_{k=1}^{\infty} \frac{1}{p_k^u c_k^v} < \sum_{k=1}^{\infty} \frac{1}{c_k^{u+v}} \tag{3.2}$$

where $u > 0$ and $v > 0$. From the results 2.1, 2.3, 2.6 and 3.2, we conclude that the prime-composite series diverges where $u + v \le 1$ and converges where $u + v > 1$ in this case.

3.3 Case Three

From the result 2.1, 2.3 and

$$\sum_{k=1}^{\infty} \frac{1}{p_k^u c_k^v} = \sum_{k=1}^{\infty} \frac{c_k^{-v}}{p_k^u} > \sum_{k=1}^{\infty} \frac{1}{p_k^u} \tag{3.3}$$

where $0 < u \le 1$ and $v < 0$, we conclude that the prime-composite series diverges where $u + v \le 1$ in this case.

3.4 Case Four

From the result 2.2, 2.5 and

$$\sum_{k=1}^{\infty} \frac{1}{p_k^u c_k^v} = \sum_{k=1}^{\infty} \frac{p_k^{-u}}{c_k^v} > \sum_{k=1}^{\infty} \frac{1}{c_k^v} \tag{3.4}$$

where $u < 0$ and $0 < v \le 1$, we conclude that the prime-composite series diverges where $u + v \le 1$ in this case.

3.5 Case Five

Since $p_k \gg c_k$ where k is sufficiently large, from the results 2.1, 2.2, 2.3, 2.5 and

$$\sum_{k=1}^{\infty} \frac{1}{p_k^u c_k^v} = \sum_{k=1}^{\infty} \frac{p_k^{-u}}{c_k^v} > \sum_{k=1}^{\infty} \frac{p_k^{-u}}{p_k^v} = \sum_{k=1}^{\infty} \frac{1}{p_k^{u+v}} \tag{3.5}$$

or

$$\sum_{k=1}^{\infty} \frac{1}{p_k^u c_k^v} = \sum_{k=1}^{\infty} \frac{p_k^{-u}}{c_k^v} > \sum_{k=1}^{\infty} \frac{c_k^{-u}}{c_k^v} = \sum_{k=1}^{\infty} \frac{1}{c_k^{u+v}} \tag{3.6}$$

where $u < 0$, $v > 1$ and $u + v \le 1$, we conclude that the prime-composite series diverges where $u + v \le 1$ in this case.

3.6 Case Six

Since $p_k \gg c_k$ where k is sufficiently large, from the results 2.4, 2.6 and

$$\sum_{k=1}^{\infty} \frac{1}{p_k^u c_k^v} = \sum_{k=1}^{\infty} \frac{c_k^{-v}}{p_k^u} < \sum_{k=1}^{\infty} \frac{p_k^{-v}}{p_k^u} = \sum_{k=1}^{\infty} \frac{1}{p_k^{u+v}} \tag{3.7}$$

or

$$\sum_{k=1}^{\infty} \frac{1}{p_k^u c_k^v} = \sum_{k=1}^{\infty} \frac{c_k^{-v}}{p_k^u} < \sum_{k=1}^{\infty} \frac{c_k^{-v}}{c_k^u} = \sum_{k=1}^{\infty} \frac{1}{c_k^{u+v}} \tag{3.8}$$

where $u > 1$, $v < 0$ and $u + v > 1$, we conclude that the prime-composite series converges where $u + v > 1$ in this case.

3.7 Case Seven

Bach and Shallit [2] have proved that

$$p_k < klnk + klnlnk \tag{3.9}$$

for $k \geq 6$. From the result 3.9, we have

$$\sum_{k=1}^{\infty} \frac{1}{p_k^u c_k^v} = \sum_{k=1}^{\infty} \frac{c_k^{-v}}{p_k^u} > \sum_{k=6}^{\infty} \frac{c_k^{-v}}{(klnk + klnlnk)^u} + C \tag{3.10}$$

where $u > 1$ and $u + v \leq 1$. Since $c_k \gg k$ where k is sufficiently large, we have

$$\sum_{k=6}^{\infty} \frac{c_k^{-v}}{(klnk + klnlnk)^u} > \sum_{k=6}^{\infty} \frac{k^{-v}}{(klnk + klnlnk)^u} > \sum_{k=6}^{\infty} \frac{k^{-v}}{(2klnk)^u} = \frac{1}{2^u} \sum_{k=6}^{\infty} \frac{k^{-v}}{k^u (lnk)^u}. \tag{3.11}$$

Since

$$lnk \ll k^{\epsilon} \tag{3.12}$$

where k is sufficiently large and $\epsilon > 0$, we set up $\epsilon = \frac{1-(u+v)}{2u}$ and have

$$\sum_{k=6}^{\infty} \frac{k^{-v}}{k^u (lnk)^u} > \sum_{k=6}^{\infty} \frac{k^{-v}}{k^{u+u\epsilon}} = \sum_{k=6}^{\infty} \frac{1}{k^{u+v+u\epsilon}} = \sum_{k=6}^{\infty} \frac{1}{k^{\frac{u+v+1}{2}}}. \tag{3.13}$$

From the results 3.10, 3.11 and 3.13, we conclude that the prime-composite series diverges where $u + v \leq 1$ in this case.

3.8 Case Eight

From the result 3.9, we have

$$\sum_{k=1}^{\infty} \frac{1}{p_k^u c_k^v} = \sum_{k=1}^{\infty} \frac{p_k^{-u}}{c_k^v} < \sum_{k=6}^{\infty} \frac{(klnk + klnlnk)^{-u}}{c_k^v} + C \tag{3.14}$$

where $u < 0$ and $u + v > 1$. Since $c_k \gg k$ where k is sufficiently large, we have

$$\sum_{k=6}^{\infty} \frac{(klnk + klnlnk)^{-u}}{c_k^v} < \sum_{k=6}^{\infty} \frac{(klnk + klnlnk)^{-u}}{k^v} < \sum_{k=6}^{\infty} \frac{(2klnk)^{-u}}{k^v} = \frac{1}{2^u} \sum_{k=6}^{\infty} \frac{(lnk)^{-u}}{k^{u+v}}. \tag{3.15}$$

From the result 3.12, we set up $\epsilon = \frac{1-(u+v)}{2u}$ and have

$$\sum_{k=6}^{\infty} \frac{(lnk)^{-u}}{k^{u+v}} < \sum_{k=6}^{\infty} \frac{k^{-u\epsilon}}{k^{u+v}} = \sum_{k=6}^{\infty} \frac{1}{k^{u+v+u\epsilon}} = \sum_{k=6}^{\infty} \frac{1}{k^{\frac{u+v+1}{2}}}. \tag{3.16}$$

From the results 3.14, 3.15 and 3.16, we conclude that the prime-composite series converges where $u+v>1$ in this case.

4 The Finite Multiple Summation Conjecture

It has been proved that the prime-composite series diverges where $u+v \leq 1$ and converges where $u+v>1$. Let w be the convergence parameter of natural numbers. For the prime-composite-natural series of the form

$$\sum_{k=1}^{\infty} \frac{1}{p_k^u c_k^v k^w}, \tag{4.1}$$

will it diverge where $u+v+w \leq 1$ and converge where $u+v+w>1$? Generally, for any finite multiple summation series of the form

$$\sum_{k=1}^{\infty} \frac{1}{\prod_{i=1}^{n} a_{ik}^{p_i}}, \tag{4.2}$$

if each series

$$\sum_{k=1}^{\infty} \frac{1}{a_{jk}^{p_j}} \tag{4.3}$$

diverges where $p_j \leq c$ and converges where $p_j > c$, will the finite multiple summation series diverge where $\sum_{i=1}^{n} p_i \leq c$ and converge where $\sum_{i=1}^{n} p_i > c$?

Remark. *There is a systematic way to find some qualified series to test this conjecture. Denote* P_{ad} *as a set of all primes such that* $p \equiv a \bmod d$ *where* a *is coprime to* d*. The reciprocals of primes in each set can form a divergent series proved by Dirichlet's theorem on arithmetic progressions [3] and its analytic statement. Their convergence parameters are all 1 which can be proved by the comparison test.*

References

[1] L. Euler, *Variae Observationes Circa Series Infinitas*, Commentarii Academiae Scientiarum Imperialis Petropolitanae. **9** (1744), 160–188.

[2] E. Bach and J. Shallit, *Algorithmic Number Theory*, Foundations of Computing Series, Cambridge: MIT Press. **1** (1996), 233.

[3] P. G. L. Dirichlet, *Beweis des Satzes, dass jede unbegrenzte arithmetische Progression, deren erstes Glied und Differenz ganze Zahlen ohne gemeinschaftlichen Factor sind, unendlich viele Primzahlen enthält*, Abhandlungen der Königlichen Preußischen Akademie der Wissenschaften zu Berlin. **48** (1837), 45–71.