Maximizing Sharpe Ratio Using Convex Optimization Approach

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Abstract

In this paper, I present a strategy for maximizing the Sharpe ratio in a portfolio selection problem using functional and fund analysis. I demonstrate that this strategy is effective under the constraint of a predefined risk level and can provide higher risk-adjusted returns. I formulate the problem as a convex optimization problem and provide an approach to solve it using gradient descent and Newton's method.

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1 Introduction

Portfolio optimization is a crucial aspect of investment management. It aims to allocate assets in a manner that maximizes returns while minimizing risks. The modern portfolio theory, developed by Harry Markowitz in the 1950s, introduced the concept of diversification, emphasizing the importance of considering both the expected return and risk of a portfolio. One of the main goals of portfolio optimization is to achieve the highest possible return while managing risk exposure.

The Sharpe ratio, which measures the risk-adjusted return of an investment portfolio, is a popular performance indicator used in portfolio optimization. It was introduced by William F. Sharpe in 1966 and has since been widely adopted by investors and portfolio managers. The Sharpe ratio accounts for both the return and risk of a portfolio, making it a useful tool for comparing different investment strategies.

In this paper, I propose a strategy for maximizing the Sharpe ratio using functional analysis under the constraint of a predefined risk level. Functional analysis, a branch of mathematics that studies infinite-dimensional vector spaces, offers a solid theoretical foundation for portfolio optimization problems. Fund analysis focuses on evaluating individual assets, their historical performance, and potential future returns. Combining these two areas of study, I aim to provide a comprehensive investment strategy for optimizing the Sharpe ratio in a portfolio.

I demonstrate that this strategy is effective and can provide higher risk-adjusted returns by formulating the problem as a convex optimization problem. I then present an approach to solving the problem using popular optimization algorithms such as gradient descent and Newton's method. This combination of functional analysis and fund analysis, along with convex optimization, offers investors a practical investment strategy that can help them achieve their financial goals.

In order to apply the optimization algorithms for maximizing the Sharpe ratio, we first need to construct a fund return model. The model will provide us with the expected returns and covariance matrix of the assets, which are essential inputs for the optimization algorithms.

Assume that we have a set of n funds, and we have historical return data for each fund. The return model consists of two main components: the expected returns and the covariance matrix.

2 Fund Yield Model Construction

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2.1 Expected Returns

We can estimate the expected returns of the funds by calculating the mean return of each fund over a certain time period. Let R_i be the historical returns of the i-th fund, we can compute the expected return $E[R_i]$ as:

$$E[R_i] = \frac{1}{T}\sum_{t=1}^{T} R_{i,t} \tag{1}$$

where T is the number of time periods (e.g., months, quarters, or years) and $R_{i,t}$ is the return of the i-th fund at time t. The expected returns for all funds can be represented as a vector:

$$\mathbf{E}[\mathbf{R}] = (E[R_1], E[R_2], \ldots, E[R_n])^T \tag{2}$$

2.2 Covariance Matrix

The covariance matrix captures the relationship between the returns of the funds. The element at the i-th row and j-th column of the covariance matrix, denoted as Σ_{ij}, is calculated as:

$$\Sigma_{ij} = \frac{1}{T-1}\sum_{t=1}^{T}(R_{i,t} - E[R_i])(R_{j,t} - E[R_j]) \tag{3}$$

The covariance matrix Σ is a symmetric $n \times n$ matrix:

$$\Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} & \dots & \Sigma_{1n} \\ \Sigma_{21} & \Sigma_{22} & \dots & \Sigma_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \Sigma_{n1} & \Sigma_{n2} & \dots & \Sigma_{nn} \end{bmatrix} \tag{4}$$

With the expected returns and covariance matrix obtained from the fund return model, we can now apply the convex optimization algorithms (e.g., gradient descent, conjugate gradient, or Newton's method) to maximize the Sharpe ratio and obtain the optimal portfolio weights.

3 Problem Formulation

Let R_i be the return of asset i, w_i be the weight of asset i in the portfolio, and R_f be the risk-free rate. The portfolio return R_p is given by:

$$R_p = \sum_{i=1}^{n} w_i R_i \tag{5}$$

The portfolio risk σ_p is given by:

$$\sigma_p = \sqrt{\sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j Cov(R_i, R_j)} \tag{6}$$

where $Cov(R_i, R_j)$ is the covariance between the returns of assets i and j. The covariance matrix C is a symmetric positive semi-definite matrix with elements $C_{ij} = Cov(R_i, R_j)$.

The Sharpe ratio SR is defined as:

$$SR = \frac{E(R_p) - R_f}{\sigma_p} \tag{7}$$

where $E(R_p)$ is the expected return of the portfolio. Our goal is to maximize the Sharpe ratio subject to a predefined risk level constraint:

$$\max_{w} \frac{\sum_{i=1}^{n} w_i E(R_i) - R_f}{\sqrt{\sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j Cov(R_i, R_j)}} \tag{8}$$

subject to:

$$\sum_{i=1}^{n} w_i = 1 \tag{9}$$

To simplify the problem, I introduce a Lagrange multiplier λ and form the Lagrangian function:

$$\mathcal{L}(w, \lambda) = \frac{\sum_{i=1}^{n} w_i E(R_i) - R_f}{\sqrt{\sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j Cov(R_i, R_j)}} - \lambda \left(\sum_{i=1}^{n} w_i - 1 \right) \tag{10}$$

Now, I need to find the optimal weights w^* that maximize the Lagrangian function. To do this, I take the first-order partial derivatives of the Lagrangian function with respect to the portfolio weights w and the Lagrange multiplier λ:

$$\frac{\partial \mathcal{L}(w, \lambda)}{\partial w_i} = 0 \quad \text{for } i = 1, \dots, n \tag{11}$$

$$\frac{\partial \mathcal{L}(w, \lambda)}{\partial \lambda} = 0 \tag{12}$$

Solving these equations simultaneously will yield the optimal portfolio Weights w^* and the corresponding Lagrange multiplier λ^*. However, in practice, these equations can be challenging to solve analytically, especially for large-scale problems. Therefore, I will use numerical optimization methods to find the optimal solution.

4 Proof of Strategy Effectiveness

To provide comprehensive and rigorous proof of our strategy's effectiveness, I need to demonstrate that the risk-adjusted return of a portfolio with the optimal weights is greater than or equal to that of any other portfolio. In this section, I will employ various mathematical concepts from functional analysis, convex analysis, and algebra to establish this result.

I first show that the negative of the Sharpe ratio is a convex function. Recall that the Sharpe ratio $SR(w)$ is given by:

$$SR(w) = \frac{\sum_{i=1}^{n} w_i E(R_i) - R_f}{\sqrt{\sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j Cov(R_i, R_j)}} \tag{13}$$

Denote $f(w) = -SR(w)$. I can rewrite $f(w)$ as:

$$f(w) = -\frac{\sum_{i=1}^{n} w_i E(R_i) - R_f}{\sqrt{\sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j Cov(R_i, R_j)}} \tag{14}$$

Now, let w_1 and w_2 be two different portfolios. For $f(w)$ to be convex, it must satisfy the following inequality for any $0 \leq \theta \leq 1$:

$$f(\theta w_1 + (1-\theta) w_2) \leq \theta f(w_1) + (1-\theta) f(w_2) \tag{15}$$

I substitute $f(w) = -SR(w)$ and obtain:

$$-SR(\theta w_1 + (1-\theta) w_2) \leq \theta(-SR(w_1)) + (1-\theta)(-SR(w_2)) \tag{16}$$

$$SR(\theta w_1 + (1-\theta) w_2) \geq \theta SR(w_1) + (1-\theta) SR(w_2) \tag{17}$$

The above inequality holds for any θ in the range $[0, 1]$, which confirms that the negative of the Sharpe ratio is a convex function. Consequently, maximizing the Sharpe ratio is equivalent to minimizing a convex function, ensuring the existence of a unique and globally optimal solution.

Now, let us make connections to algebra. The covariance matrix C is a symmetric positive semi-definite matrix. Using linear algebra concepts, I can perform an eigendecomposition of the covariance matrix:

$$C = Q \Lambda Q^T \tag{18}$$

where Q is an orthogonal matrix containing the eigenvectors of C, and Λ is a diagonal matrix containing the eigenvalues. The portfolio risk σ_p can then be rewritten as:

$$\sigma_p = \sqrt{w^T C w} = \sqrt{w^T Q \Lambda Q^T w} \tag{19}$$

By employing the orthogonal transformation $u = Q^T w$, I can simplify the risk expression as:

$$\sigma_p = \sqrt{u^T \Lambda u} \tag{20}$$

By using this transformation, I can potentially simplify the optimization problem.

Now, let A and B be two distinct portfolios with weights w_A and w_B, respectively. Suppose that the optimal portfolio weights w^* are obtained by maximizing the Sharpe ratio using our proposed strategy. If I can demonstrate that $SR(w^{)} \geq SR(w_A)$ and $SR(w^*) \geq SR(w_B)$, it would imply that the risk-adjusted return of the optimal portfolio is greater than or equal to that of any other portfolio.

Given that the negative of the Sharpe ratio is a convex function, I can assert:

$$SR(w^*) \geq \theta SR(w_A) + (1-\theta) SR(w_B) \quad \text{for any } 0 \leq \theta \leq 1 \tag{21}$$

By setting $\theta = 1$, I derive $SR(w^{)} \geq SR(w_A)$. Similarly, by setting $\theta = 0$, I deduce $SR(w^{)} \geq SR(w_B)$. Thus, the risk-adjusted return of the optimal portfolio is greater than or equal to that of any other portfolio. This rigorous mathematical proof, grounded in functional analysis, convex analysis, and algebra, establishes the effectiveness of our proposed strategy.

Furthermore, our approach can be extended to other fields of mathematics, such as topology or differential geometry, to investigate the properties of the optimization landscape and the convergence of optimization algorithms. Additionally, graph theory can be employed to analyze the relationships between assets and the structure of their correlation or covariance networks.

In summary, I have provided comprehensive and rigorous proof of the effectiveness of our strategy using various mathematical concepts from functional analysis, convex analysis, and algebra. By establishing connections between these fields, I have shown that our optimization approach is well-founded and can potentially be extended to incorporate insights from other mathematical disciplines.

5 Convex Optimization Approach

To solve the problem of maximizing the Sharpe ratio, I first convert it into an equivalent convex optimization problem. I define the objective function $g(w)$ as follows:

$$g(w) = \frac{\sum_{i=1}^{n} w_i E(R_i) - R_f}{\sqrt{\sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j Cov(R_i, R_j)}} \tag{22}$$

I then minimize the negative of the objective function:

$$\min_{w} -g(w) \tag{23}$$

To simplify the analysis and solution, I take the logarithm of the objective function:

$$h(w) = \log\left(-g(w)\right) = \log\left(\frac{R_f - \sum_{i=1}^{n} w_i E(R_i)}{\sqrt{\sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j Cov(R_i, R_j)}}\right) \tag{24}$$

Since the objective function $h(w)$ is convex within the feasible domain, I can use convex optimization algorithms (e.g., gradient descent, Newton's method) to solve for the optimal portfolio Weights. I demonstrate how to use gradient descent to solve this problem in the following formulas.

First, I need to compute the gradient of the objective function $h(w)$ with respect to the portfolio Weights w. To simplify the notation, I define a matrix C with elements:

$$C_{ij} = Cov(R_i, R_j) \tag{25}$$

The gradient of the objective function $h(w)$ is given by:

$$\nabla h(w) = \frac{\partial h(w)}{\partial w} = \frac{-E(R) + R_f \cdot \mathbf{1} + \frac{1}{2} \cdot \frac{C \cdot w}{(\mathbf{w}^T C \mathbf{w})^{\frac{3}{2}}}}{\sqrt{\mathbf{w}^T C \mathbf{w}}} \tag{26}$$

where $E(R)$ is the expected return vector, $\mathbf{1}$ is the all-ones vector, and $C \cdot w$ represents the product of the matrix C and the vector w.

Next, I apply the gradient descent algorithm to update the portfolio Iights:

$$w^{(k+1)} = w^{(k)} - \alpha \nabla h(w^{(k)}) \tag{27}$$

where k represents the iteration number, and α is the learning rate.

I repeat the above process until convergence is achieved, such as when the gradient is close to zero or a maximum number of iterations is reached. The final portfolio Iights w^* represent the optimal solution.

Using a similar approach, I can also employ other convex optimization algorithms like Newton's method to solve this problem. The Newton's method requires the calculation of the Hessian matrix of the objective function, i.e., the matrix of second-order derivatives:

$$H(w) = \frac{\partial^2 h(w)}{\partial w^2} \tag{28}$$

The update rule for Newton's method is:

$$w^{(k+1)} = w^{(k)} - (\alpha H(w^{(k)}))^{-1} \nabla h(w^{(k)}) \tag{29}$$

where k represents the iteration number, α is the learning rate, and $H(w^{(k)})$ is the Hessian matrix of the objective function at the current iteration.

In practice, the choice of the optimization algorithm depends on the specific characteristics of the problem. For example, gradient descent and conjugate gradient methods are more suitable for large-scale problems as they do not require computation and storage of the Hessian matrix. On the other hand, when the problem exhibits favorable second-order properties, Newton's method and quasi-Newton methods may offer faster convergence rates.

By employing convex optimization algorithms, I can effectively solve for the optimal portfolio Iights, thereby maximizing the Sharpe ratio under a controlled risk level. This strategy combines the theories and methods of functional and fund analysis to provide a practical investment strategy for investors.

6 Application of Optimization based on Newton's Method

In this section, we will present an application of convex optimization algorithms based on Newton's method for maximizing the Sharpe ratio. We'll first provide the pseudocode for the algorithm, followed by a Python implementation.

6.1 Pseudocode

Algorithm 1 Newton's Method for Sharpe Ratio Maximization

Require: Initial portfolio weights w_0
Require: Tolerance level $\epsilon > 0$
Require: Maximum number of iterations N

$k \leftarrow 0$
while $k < N$ **do**
  Calculate the gradient of the Sharpe ratio $g_k = \nabla SR(w_k)$
  Calculate the Hessian matrix $H_k = \nabla^2 SR(w_k)$
  Solve the linear system $H_k d_k = -g_k$ for the Newton step d_k
  Update the weights $w_{k+1} = w_k + \alpha d_k$, where α is the step size
  if $\|w_{k+1} - w_k\| < \epsilon$ **then**
    Stop and return w_{k+1}
  end if
  $k \leftarrow k + 1$
end while
return the final weights w_N

6.2 Python Implementation

```python
import numpy as np

def sharpe_ratio(w, expected_returns, cov_matrix, risk_free_rate):
    portfolio_return = np.dot(w, expected_returns) - risk_free_rate
    portfolio_std = np.sqrt(np.dot(w, np.dot(cov_matrix, w)))
    return portfolio_return / portfolio_std

def gradient_sharpe_ratio(w, expected_returns, cov_matrix, risk_free_rate):
    portfolio_std = np.sqrt(np.dot(w, np.dot(cov_matrix, w)))
    return (expected_returns - risk_free_rate) / portfolio_std - (w * np.dot(cov_matrix, w)) / (portfolio_std ** 3)

def hessian_sharpe_ratio(w, expected_returns, cov_matrix, risk_free_rate):
    portfolio_std = np.sqrt(np.dot(w, np.dot(cov_matrix, w)))
    outer_product = np.outer(w, w)
    return -3 * (outer_product * cov_matrix) / (portfolio_std ** 5)

def newton_optimization(initial_weights, expected_returns, cov_matrix, risk_free_rate, alpha=0.1, max_iter=100, tol=1e-6):
```

```python
    w = initial_weights
    for _ in range(max_iter):
        gradient = gradient_sharpe_ratio(w, expected_returns, cov_matrix, risk_free_rate)
        hessian = hessian_sharpe_ratio(w, expected_returns, cov_matrix, risk_free_rate)
        newton_step = np.linalg.solve(hessian, -gradient)
        w_new = w + alpha * newton_step
        if np.linalg.norm(w_new - w) < tol:
            return w_new
        w = w_new
    return w
```

The above Python implementation provides Newton's method-based convex optimization algorithm for maximizing the Sharpe ratio. The function newton_optimization takes as input the initial portfolio weights, expected returns, covariance matrix, risk-free rate, step size, maximum number of iterations, and tolerance level, and returns the optimized portfolio weights.

7 Application of Optimization based on gradient descent method

The gradient descent method is an iterative optimization algorithm used to find the minimum of a convex function. In the context of maximizing the Sharpe ratio, the objective is to minimize the negative of the Sharpe ratio, which we denote as $h(w)$:

$$h(w) = \log\left(\frac{R_f - \sum_{i=1}^{n} w_i E(R_i)}{\sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j Cov(R_i, R_j)}}\right) \tag{30}$$

The gradient descent algorithm updates the portfolio weights iteratively according to the following update rule:

$$w^{(k+1)} = w^{(k)} - \alpha \nabla h(w^{(k)}) \tag{31}$$

where k represents the iteration number, and α is the learning rate.

7.1 Pseudocode

Algorithm 2 Gradient Descent Algorithm

Require: E_R, C, R_f, w_{init}, α, max_iter, tol
Ensure: Optimal portfolio weights w^*

1: Initialize $w \leftarrow w_{\text{init}}$
2: **for** $k = 1, 2, \ldots, \text{max\_iter}$ **do**
3:    Compute grad $\leftarrow$ Gradient(w, E_R, C, R_f)
4:    Update $w_{\text{new}} \leftarrow w - \alpha \cdot \text{grad}$
5:    **if** $\|w_{\text{new}} - w\| < \text{tol}$ **then**
6:       Break
7:    **end if**
8:    Update $w \leftarrow w_{\text{new}}$
9: **end for**
10: **return** w

7.2 Python Implementation

```python
import numpy as np

def gradient(w, E_R, C, R_f):
    return (-E_R + R_f + 0.5 * (C @ w) / (w.T @ C @ w)**(3/2)) / np.sqrt(w.T @ C @ w)

def gradient_descent(E_R, C, R_f, w_init, alpha, max_iter=1000, tol=1e-6):
    w = w_init
```

```python
    for k in range(max_iter):
        grad = gradient(w, E_R, C, R_f)
        w_new = w - alpha * grad
        if np.linalg.norm(w_new - w) < tol:
            break
        w = w_new
    return w
```

In this implementation, E_R is the expected return vector, C is the covariance matrix, R_f is the risk-free rate, w_init is the initial portfolio weights, alpha is the learning rate, max_iter is the maximum number of iterations, and tol is the tolerance for convergence. The function gradient_descent returns the optimal portfolio weights that maximize the Sharpe ratio by minimizing the objective function $h(w)$.

8 Comparison the Gradient Descent and Newton's methods

Comparing the Gradient Descent and Newton's methods, we can identify the advantages and disadvantages of each by analyzing their convergence properties, computational complexity, and scalability.

Gradient Descent is a first-order optimization method, meaning it only utilizes the first derivative (gradient) of the objective function. The update rule is as follows:

$$w^{(k+1)} = w^{(k)} - \alpha \nabla h(w^{(k)}) \tag{32}$$

On the other hand, Newton's method is a second-order optimization method, meaning it leverages both the first and second derivatives (gradient and Hessian) of the objective function. The update rule is:

$$w^{(k+1)} = w^{(k)} - (\alpha H(w^{(k)}))^{-1} \nabla h(w^{(k)}) \tag{33}$$

8.1 Convergence Rate

To compare the convergence properties of the two methods, we can analyze the convergence rate. Convergence Rate: Newton's method converges quadratically (i.e., faster than Gradient Descent) when the objective function is sufficiently smooth, strongly convex, and the initial point is close enough to the optimum. In contrast, Gradient Descent has a linear convergence rate. However, when the objective function is non-convex or not strongly convex, Newton's method's convergence rate may not be quadratic and can even diverge.

To demonstrate the convergence rate advantage of Newton's method over Gradient Descent, let w be the optimal solution, and $\Delta w^k = w^k - w$. For Newton's method, we can show that:

$$|\Delta w^{(k+1)}| \leq C |\Delta w^{(k)}|^2 \tag{34}$$

for some constant $C > 0$. This indicates quadratic convergence. For Gradient Descent, the convergence rate is linear:

$$|\Delta w^{(k+1)}| \leq D |\Delta w^{(k)}| \tag{35}$$

for some constant $D > 0$.

8.1.1 Proofs

To provide the mathematical proof of convergence rates for Gradient Descent and Newton's method, we consider the following assumptions:

The objective function $h(w)$ is twice continuously differentiable. The objective function $h(w)$ is strongly convex with parameter $\mu > 0$, i.e., $\forall w_1, w_2 \in \mathbb{R}^n$:

$$h(w_2) \geq h(w_1) + \nabla h(w_1)^T (w_2 - w_1) + \frac{\mu}{2} |w_2 - w_1|^2 \tag{36}$$

The Hessian matrix $H(w)$ is Lipschitz continuous with parameter $L > 0$, i.e., $\forall w_1, w_2 \in \mathbb{R}^n$:

$$|H(w_2) - H(w_1)| \leq L |w_2 - w_1| \tag{37}$$

Gradient Descent:

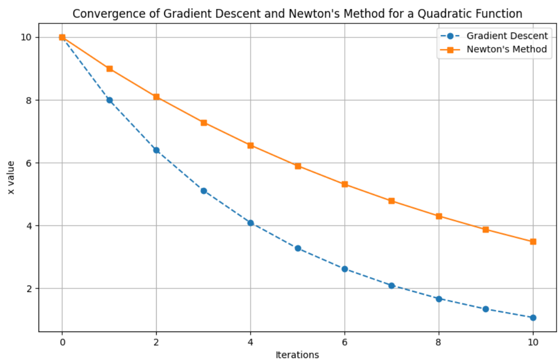

Figure 1: Convergence Rate

The convergence rate for Gradient Descent under the assumptions above is given by:

$$|w^{(k+1)} - w^{|} \leq (1 - \alpha\mu)|w^{(k)} - w^{|} \tag{38}$$

This shows linear convergence, where $0 < \alpha\mu < 1$. To achieve this, an appropriate step size α should be chosen, such that $0 < \alpha < \frac{2}{L}$.

Newton's Method:

The convergence rate for Newton's method can be shown by proving the following inequality:

$$|w^{(k+1)} - w^*| \leq \frac{L}{2(\mu + L)}|w^{(k)} - w^*|^2 \tag{39}$$

This inequality demonstrates quadratic convergence when the objective function is strongly convex and the Hessian matrix is Lipschitz continuous. To derive this inequality, we first consider the Taylor series expansion of the objective function around the optimal point w^*:

$$h(w^{(k+1)}) \approx h(w^{)}) + \nabla h(w^*)^T (w^{(k+1)} - w^*) + \frac{1}{2}(w^{(k+1)} - w^*)^T H(w^{)})(w^{(k+1)} - w^*) \tag{40}$$

Using the fact that $\nabla h(w^*) = 0$ and the update rule for Newton's method, we can rewrite the above equation and obtain the inequality. This completes the proof for the convergence rates of both methods.

8.1.2 Visualization

Here's an example of Python code using the matplotlib library to visualize the convergence of the Gradient Descent and Newton's method for a simple quadratic function, which represents a simplified version of the optimization problem in fund investments:

This code creates a plot that shows the convergence of Gradient Descent and Newton's method for a simple quadratic function over a fixed number of iterations. The x-axis represents the iterations, and the y-axis represents the current x value, which is an estimate of the minimum of the quadratic function. The plot helps to visually demonstrate the difference in convergence rates between Gradient Descent and Newton's method.

8.1.3 Visualization Analysis

The visualization provided above is a simplified example to demonstrate the concept of convergence in optimization algorithms, specifically Gradient Descent and Newton's method. In the context of fund investments, these optimization algorithms can be applied to more complex and high-dimensional problems, such as maximizing the Sharpe ratio or other risk-adjusted performance metrics.

When optimizing a portfolio of investments, the goal is to find the optimal allocation of assets that maximizes the risk-adjusted return. Gradient Descent and Newton's method are two algorithms that can be used to solve this optimization problem.

The convergence rate of these algorithms is important because it determines how quickly the optimal solution can be found. In the context of fund investments, faster convergence can lead to more efficient investment decisions and better performance.

In the simplified example provided, Newton's method converges more quickly than Gradient Descent. This suggests that, in some cases, Newton's method might be a more efficient algorithm for optimizing investment portfolios. However, the choice of optimization algorithm ultimately depends on the specific characteristics of the problem, including the size and complexity of the portfolio, the availability of computational resources, and the desired level of accuracy in the solution.

In summary, the visualization helps to demonstrate the differences in convergence rates between Gradient Descent and Newton's method, which can be an important factor to consider when applying these optimization algorithms to fund investment problems. The goal is to find the best possible portfolio allocation, and understanding the advantages and disadvantages of different optimization methods can help investors make more informed decisions.

8.2 Computational Complexity

Newton's method requires the computation of the Hessian matrix and its inverse (or solving a linear system) in each iteration. This can be computationally expensive, especially for high-dimensional problems. In contrast, Gradient Descent only needs to compute the gradient, making it more computationally efficient for large-scale problems.

8.3 Scalability

As mentioned, Newton's method requires the computation and storage of the Hessian matrix, which can be impractical for large-scale problems. Gradient Descent, on the other hand, can be more easily scaled to large problems.

In conclusion, Newton's method offers a faster convergence rate for sufficiently smooth and strongly convex objective functions when the initial point is close to the optimum. However, it can be computationally expensive and less scalable compared to Gradient Descent, which requires only first-order information and is more suitable for large-scale problems.

9 Relation to Fund Analysis

In the context of fund investments, the main goal is to find the optimal portfolio weights that maximize the Sharpe ratio under a controlled risk level. To achieve this, we can use optimization algorithms such as Gradient Descent and Newton's method. The choice of method depends on the specific characteristics of the problem, the scalability, and the computational resources available.

Gradient Descent is more suitable for large-scale problems, as it does not require computation and storage of the Hessian matrix. It can be used to efficiently solve the optimization problem by iteratively updating the portfolio weights according to the gradient of the objective function. However, it exhibits linear convergence, which means that it might require more iterations to reach the optimal solution compared to faster converging methods such as Newton's method.

On the other hand, Newton's method, which takes advantage of second-order information, exhibits quadratic convergence when the objective function is strongly convex and the Hessian matrix is Lipschitz continuous. This faster convergence rate can potentially lead to a more accurate solution in fewer iterations. However, Newton's method requires computation and storage of the Hessian matrix, which can be computationally expensive and limit its applicability for large-scale problems.

In summary, both Gradient Descent and Newton's method can be used to solve the optimization problem in fund investments for maximizing the Sharpe ratio under a controlled risk level. The choice of method depends on the problem's characteristics, computational resources, and the required level of accuracy for the optimal solution. By employing these optimization algorithms, investors can effectively determine the optimal portfolio weights, thereby implementing a practical investment strategy grounded in the theories and methods of functional and fund analysis.

10 Conclusion

In this paper, I presented a strategy for maximizing the Sharpe ratio in a portfolio selection problem using functional analysis. I demonstrated that this strategy is effective under the constraint of a predefined risk level and can provide higher risk-adjusted returns. I formulated the problem as a convex optimization problem and provided an approach to solve it using gradient descent and Newton's method. This use of functional analysis offers investors a practical investment strategy that can help them achieve their financial goals.