Math tools, Hebrew U.

Final exam

Lecturer	Teaching assistant
Guy Kindler	Ilya Soloveychik

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Instructions

- The exam is three hours long, there will be no extensions
- There is no choice – try to solve all questions
- Manage your time carefully – do not linger too long on any particular question
- You may use statements proven in class, but you should cite exactly which statement you are using. Of course this doesn't hold when you are asked to prove these very statements or special cases of thereof. When in doubt, ask.
- No use of any material except writing equipment is allowed. You may not use or even have available any electronic devices during the test for any purpose.
- The questions in each section will have equal weight. The relative weights of items inside questions will be decided on when grading.

Short questions (30 points)

In this section we expect the answers to be short and simple, so please answer minimally (but in a formal and exact manner).

(1) Let X be a random variable such that $\mathbb{E}[(X-1)^4] = 1$. Prove that

$$\Pr[X > 11] \leq \frac{1}{10000} .$$

(2) In this question it is ok to only write final expressions. Let $G \sim G(n, 1/2)$ be a random graph with vertices $V = \{1, 2, \ldots, n\}$. For every set S of three vertices in G let X_S be the indicator of the event that S is a triangle in G.

a. Find $\mathbb{E}\left[\sum_{S\subseteq V,\ |S|=3} X_S\right]$ as a function of n.

b. For every two sets S, T of three vertices each, compute $\mathrm{Cov}[X_S, X_T]$ (partition your answer to cases where $\left|S\cap T\right| = 0, 1, 2$ or 3).

(3) Here your final answers can be written as a formula, no need to complete the computation. Let A be a 2×2 matrix with real entries, and with characteristic polynomial $x^2 - 3x + 2$. It is known that the vectors $\binom{3}{4}$ and $\binom{5}{6}$ are eigenvectors of A. What can the matrix A be?

(4) Let u_1, u_2, u_3 be an orthonormal basis for an inner product space V, and let $v \in \mathbb{R}^3$ be a vector which satisfies $\langle u_1, v\rangle = 2$, $\langle u_2, v\rangle = 3$, and $\langle u_3, v\rangle = 6$. Compute $\|v\|$.

(5) Let $(U, \|\cdot\|_U)$, $(V, \|\cdot\|_V)$, and $(W, \|\cdot\|_W)$ be finite dimensional normed spaces over $\mathbb{R}$, and let $T : U \to V$, and $S : V \to W$ be linear operators.

a. Define the norm $\|T\|_{U\to V}$.

b. Prove that $\|S\circ T\|_{U\to W} \le \|S\|_{V\to W}\cdot \|T\|_{U\to V}$.

Proof questions (34 points)

The following statements were proven in class. Now it's your turn to prove them...

(6) Let $K \subseteq \mathbb{R}^n$ be a closed convex set, and let $\bigwedge(u, a)$ be a supporting half-space for K (namely $K \subseteq \bigwedge(u, a)$, and there is a point $x \in K$ such that $\langle u, x\rangle = a$).

a. Define at extremal point in K.

b. Let $H = \{x \in R^n : \langle u, x\rangle = a\}$, and show that if x is an extremal point of $H \cap K$ then it is also an extremal point of K.

c. Draw a 2 dimensional example showing that the previous statement is not necessarily true if $\bigwedge(u, a)$ is supporting K.

(7) Let A be a symmetric $n\times n$ matrix over $\mathbb{R}$, and define a function $T : \mathbb{R}^n \to \mathbb{R}$ by $T(v) = \frac{\langle v, Av\rangle}{\|v\|_2}$. Let $v \in \mathbb{R}^n \setminus \{0\}$ be a vector where T attains its maximum. Show that for every $u \in \mathbb{R}^n$ such that $\langle u, v\rangle = 0$ it also holds that $\langle u, Av\rangle = 0$.

Just questions (36 points)

This section contains questions. Please try to solve them.

(8) Let $G = (V, E)$ be a d-regular undirected graph. We define a *random walkwalk* on G to be similar to a usual random walk, except that at each time slot we make two consecutive steps of the usual random walk. Let p^0 be an initial distribution over

the vertices of G and let p^t be the distribution after t walkwalk steps on G (namely after $2t$ regular random-walk steps).

Show that p^t converges to some distribution p. Moreover, show that p is stationary with respect to the random walkwalk (that is, if $p^0 = p$ then $p^t = p$ for all t).

(9) For two n-bit binary strings $x, y \in \{0,1\}^n$, we define the Hamming distance between them by $H(x,y) = \sharp\{i \in \{1,2,\ldots,n\} : x_i \neq y_i\}$. So $H(x,y)$ is the number of coordinates on which x and y differ. We now randomly choose n^{100} binary strings $x^{(1)}, \ldots, x^{(n^{100})} \in \{0,1\}^n$, where each $x^{(k)}$ is distributed uniformly and they are all independent. Show that

$$\Pr\left[\exists\, k_1 \neq k_2 \quad H(x^{(k_1)}, x^{(k_2)}) < \frac{n}{4}\right] \xrightarrow{n\to\infty} 0.$$

(10) Let $f : \{-1,1\}^n \to \mathbb{R}$ be a function, and let $i, j \in \{1,2,\ldots,n\}$ be different indices. Write the expression $\left\|\frac{f(x)-f(\sigma_i\sigma_j x)}{2}\right\|_2^2$ in terms of the Fourier-Walsh coefficients of f. (hint: first write the expression inside the norm in terms of the Fourier-Walsh representation of f).