MATHEMATICAL TOOLS - SOLUTION 10

Problem 1. Let $T \subseteq [n+1]$. We need to show that $S = \{e_i; i \in T\}$ is a face of Δ_n. Let $c_T \in \mathbb{R}^{n+1}$ be T's indicator (i.e. $c_{T_i i} = 1$ if $i \in T$ and 0 otherwise). Consider the hyperplane $\left\{x \in \mathbb{R}^{n+1} : c_T^T x = 1\right\}$. First, this is a supporting hyperplane. If $x \in \Delta_n$ then $\sum_{i=1}^{n+1} x_{i+1} c_{T i+1} \leq \sum x_i = 1$. Second, its intersection with Δ_n is exactly $convS$: Let $x \in convS$. Then $c_T^T x = \sum_{i \in T} x_i = 1$. On the other hand, if $c_T^T x = 1$ then $1 = \sum_{i=1}^{n+1} x_i = \sum_{i \in T} x_i$ and since $x_i \geq 0$ for all i, we have $i \notin T \implies x_i = 0$. Thus $x \in convS$.

Problem 2.

(1)

(2) The vertices are $\{\pm e_1, \ldots, \pm e_n\}$: Consider the hyperplane $\left\{x \in \mathbb{R}^n : e_i^T x = 1\right\}$. This is supporting, because if $x \in P_n$ then $e_i^T x = x_i \leq \|x\|_1 \leq 1$. Furthermore, $e_i^T e_i = 1$. Finally, if for $x \in P_n$, $e_i^T x = 1$, then $x_i = 1$. But then $1 \geq \|x\|_1 = 1 + \sum_{j \neq i} |x_j| \implies \forall j \neq i, x_j = 0 \implies e_i = x$. A similar argument, with the hyper plane $\left\{x : e_i^T x = -1\right\}$ shows that $-e_i$ is also a vertex.

(3) First, if $x \in convV$, then $x = \sum_{i=1}^n (\lambda_i - \mu_i) e_i$, and $\sum (\lambda_i + \mu_i) = 1, \lambda_i, \mu_i \geq 1$. Thus $\|x\|_1 \leq \sum_{i=1}^n (\lambda_i + \mu_i) = 1$. So $x \in P_n$.
For the converse, let $x \in P_n$. Write $x = x_+ - x_-$ for x's positive and negative decomposition (i.e. x_+ consists of the vector that is equal to x on positive coordinates and 0 otherwise, and a similar definition for x_-). Then $\left\|\frac{x_+}{\|x_+\|_1}\right\|_1 = 1$ which implies $y_+ = \frac{x_+}{\|x_+\|_1} \in conv\{e_1, \ldots, e_n\} \subseteq convV$. Similarly, $-y_- = -\frac{x_-}{\|x_-\|_1} \in conv\{-e_1, \ldots, -e_n\} \subseteq convV$. Note that $1 = \|x\|_1 = \|x_+\|_1 + \|x_-\|_1$. Thus $x = x_+ - x_- = \|x_+\|_1 y_+ + \|x_-\|_1 (-y_-) \in convS$.

Problem 3.

Problem 4.

(1) We'll define the linear program: Minimize $\sum_{a \in X \cup Y \cup Z} w(a)$ subject to $w \geq 0$, and the inequalities $\forall t \in T, \sum_{a \in t} w(a) \geq 1$. First note that if we take $w \equiv 1$, this is certainly a fractional vertex cover, so the program is feasible. Second, note that the program is bounded below, for example by 0. So an optimal solution in fact exists. Hence it can be found efficiently in the size of the LP, which is polynomial (in fact, linear) in the size of the input. Now, let U be a minimal vertex cover and let w be U's indicator function. Then w is clearly a feasible solution, and so $|U| = \sum w$ is an upper bound on w^*.

(2) Let w be an optimal fractional solution, and let $U = \left\{a \in V : w(a) \geq \frac{1}{3}\right\}$. First, U is a vertex cover: Let $t \in T$. Then $\sum_{a \in t} w(a) \geq 1$ and since $w \geq 0$ this means that at least one vertex in t has weight $\frac{1}{3}$ or more. This vertex is in U. Second, $|U| = \sum_{a \in U} 3 \times \frac{1}{3} \leq 3 \sum_{a \in U} w(a) \leq 3 \sum w = 3w^*$.