1. Let $P \subseteq \mathbb{R}^d$ be a polytope defined by $Ax \leq b$. Let $a_1, ..., a_m$ be the rows of A, and b=$(b_1, ..., b_m)$.
   Let $x \in P$, $I(x) = \{i \in [m] : \langle a_i, x\rangle < b_i\}$, let's denote $\triangle_{min} = min\left\{\frac{b_i - \langle a_i, x\rangle}{\|a_i\|} \mid i \in I(x)\right\}$.
   Take $\epsilon = \triangle_{min}$. Let $y \in B_\epsilon(x)$ in some norm of $\mathbb{R}^d$.

$$|\|x\| - \|y\|| \leq \|y - x\| \leq \triangle_{min} \leq$$

   For all $i \in [m]$:$\|a_i\| \triangle_{min} \geq \|a_i\| \|y - x\| \geq \langle a_i, y - x\rangle = \langle a_i, y\rangle - \langle a_i, x\rangle \geq \langle a_i, y\rangle - \triangle_{min}$

$$\langle a_i, y\rangle = \langle a_i, x + y - x\rangle \leq \langle a_i, x\rangle + \langle a_i, y - x\rangle \leq \langle a_i, x\rangle + \|a_i\| \|y - x\| \leq \langle a_i, x\rangle + \|a_i\| \frac{b_i - \langle a_i, x\rangle}{\|a_i\|} = \langle a_i$$

2. Let $P \subseteq \mathbb{R}^d$ be a polytope, suppose that P is defined by $Ax \leq b$.
   (a) Assume that (a) holds - $x \in P$ is a vertex of P and let's show that (b) holds. Since x is a vertex of P there exists some linear functional that x is it's unique maximizer :

$$f(y) = \langle c, y\rangle, \ \underset{y \in P}{argmax} f(y) = x, \ f(x) = f_{max}$$

   Suppose, by contradiction, that $\exists 0 \neq v \in \mathbb{R}^d$ such that $x + v \in P$ and $x - v \in P$. $\langle c, x + v\rangle = \langle c, x\rangle + \langle c, v\rangle$ If $\langle c, v\rangle \geq 0 \Rightarrow \langle c, x\rangle + \langle c, v\rangle \geq \langle c, x\rangle$ and so either x is not the maximizer or it's not a unique maximizer of f(y).
   If $\langle c, v\rangle < 0 \Rightarrow -\langle c, v\rangle > 0$ and so $\langle c, x - v\rangle = \langle c, x\rangle + \langle c, -v\rangle = \langle c, x\rangle - \langle c, v\rangle > \langle c, x\rangle$ and again x is not the maximizer of f(y) on P, in contradiction.
   (b) Assume that (b) holds : For every $0 \neq v \in \mathbb{R}^d$ either $x + v \notin P$ or $x - v \notin P$ or both. Let's show that (a) holds:
   Consider the linear functional $f(x) = \left\langle \sum_{i \in [m] \setminus I(x)} a_i, x \right\rangle$.Suppose, by contradiction, that x is not it's unique maximizer on P, then there exists some $v' \in \mathbb{R}^d$ such that $x + v' \in P$ and such that $f(x) \leq f(x + v') = \left\langle \sum_{i \in [m] \setminus I(x)} a_i, x + v' \right\rangle \Rightarrow$

$$\sum_{i \in [m] \setminus I(x)} \langle a_i, x\rangle \leq \sum_{i \in [m] \setminus I(x)} (\langle a_i, x\rangle + \langle a_i, v'\rangle) \Rightarrow \sum \langle a_i, v'\rangle \geq 0$$

$$x + v' \in P \Rightarrow A(x + v') \leq b \Rightarrow \forall 1 \leq i \leq d \quad a_i x + a_i v' \leq b_i$$

   $\forall 1 \leq i \leq d \ x \in P \Rightarrow a_i x \leq b_i$ and since $\sum \langle a_i, v'\rangle \geq 0$ we can zero every coordinate of v' such that $a_i v' < 0$ and we will get a vector with only non-negative coordinates, lets denote it v. $x + v \in P$ since for all coordinates we have zeroed out the condition that is required for $x + v$ to be in P is $a_i x \leq b_i$ which holds since $x \in P$. For coordinates we haven't zeroed the condition $a_i x + a_i v' \leq b_i$ still holds. The vector $x - v$ is also in P since $\forall 1 \leq i \leq d \quad a_i x - a_i v \leq a_i x + a_i v \leq b_i$. In contradiction to the assumption.

3. Let $G = (V, E)$ be a graph.

   (a) Let's denote the vertices of G as v_i $1 \leq i \leq |V|$.
   Define the vector x of size$|V|$ $x_i = \begin{cases} 1 & v_i \in C \\ 0 & v_i \notin C \end{cases}$

   $$m = \begin{array}{ll} min & \vec{\mathbf{1}} \cdot x \\ s.t. & x_i \in \{0,1\} \\ & \text{for each } v_i v_j \in E, x_i + x_j \geq 1 \end{array}$$

   (b) The functions $f : V \to \{0, 1\}$ such that for each $v_i v_j \in E, f(v_i) + f(v_j) \geq 1$ are a subset of the fractional vertex cover.
   Therefore The minimum weight of the functions $f : V \to \{0, 1\}$, $s.t.$ $v_i v_j \in E, f(v_i) + f(v_j) \geq 1$, let's call it: r, is smaller or equal to the minimum weight of the fractional vertex cover m^*.

   $$r = \begin{array}{ll} min & \sum f(v_i) \\ s.t. & \\ & \text{for each } v_i v_j \in E,\ f(v_i) + f(v_j) \geq 1 \end{array}$$

   If we denote the vector x to be f over all all vertices of G : $x = f(V)$

   $$\text{then}: \begin{array}{ll} min & \sum f(v_i) \\ s.t. & \\ & \text{for each } v_i v_j \in E,\ f(v_i) + f(v_j) \geq 1 \end{array} = \begin{array}{ll} min & \vec{\mathbf{1}} \cdot x \\ s.t. & x_i \in \{0,1\} \\ & \text{for each } v_i v_j \in E,\ x_i + x_j \geq 1 \end{array} = m$$

   and so we get that $m^* \leq m$.

   (c) $$m^* = \begin{array}{ll} min & \vec{\mathbf{1}} \cdot x \\ s.t. & x_i \geq 0 \\ & \text{for each } v_i v_j \in E, x_i + x_j \geq 1 \end{array}$$

   (d) Algorithm for finding a 2m vertex cover:

      i. Solve the LP that gives us $m^*$and denote it's argmin as x^* .

      ii. $x_i = \begin{cases} 1 & x_i^* \geq 0.5 \\ 0 & x_i^* < 0.5 \end{cases}$,
      claim: x is a vertex cover of size at most 2m.
      It is a vertex cover since for each edge $v_i v_j \in E$ $x_i^* + x_j^* \geq 1$, therefore at least one of x_i^* and x_j^* are at least 0.5 and so at least one of x_i and x_j are 1.
      $m \geq m^* = \sum_{i=1}^{|V|} x_i^*$.
      Since for each coordinate of x^* at the very most we have multiplied it by 2 then the size of the provided vertex cover by the algorithm is : $\sum_{i=1}^{|V|} x_i \leq \sum_{i=1}^{|V|} 2x_i^* = 2m^* \leq 2m$.

4. Let $x \in conv(S)$ therefore $x = \alpha_1 s_1 + ... + \alpha_k s_k$ such that $\alpha_i \geq 0$, $\sum_{i=1}^{k} \alpha_k = 1$
   If $k \leq d+1$ then we are done. Suppose $k > d+1$ then $s_1, ...s_k$ are linearly dependant and so :
   Therefore $s_2 - s_1, ..., s_k - s_1$ are also linearly dependant

5. Denote $v = \begin{array}{ll} \underset{x}{min} & \mathbf{1} \\ s.t. & \forall 1 \leq i \leq n, \ 0 \leq a_i \leq 1 \\ & a^T v = x \end{array}$