MATHEMATICAL TOOLS - PROBLEM SET 7

Due Sunday, January 1st, 23:55, either in the course mailbox or through the Moodle.

Problem 1. In recitation we mentioned that if $A \in M_n(\mathbb{R})$ is symmetric with eigenvalues $\lambda_1, \ldots, \lambda_n$, then its singular values are $|\lambda_1|, \ldots, |\lambda_n|$.

Find a diagonalizable $A \in M_n(\mathbb{R})$ with eigenvalues $\lambda_1, \ldots, \lambda_n$ s.t. $\|A\|_{op} > \max\{|\lambda_1|, \ldots, |\lambda_n|\}$. In particular, A's singular values are not $|\lambda_1|, \ldots, |\lambda_n|$.

Problem 2. Find a SVD for:

$$A = \begin{pmatrix} -\frac{24}{25} & \frac{4}{5} \\ -\frac{32}{25} & -\frac{3}{5} \\ -\frac{6}{5} & 0 \end{pmatrix}, B = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

Hint: In recitation we solved this sort of problem by directly finding $x \in \mathbb{R}^n$ s.t. $\|A\|_{op} = \|Ax\|_2$. This is one approach. However, it might be easier to use the observation that $\sigma \geq 0$ is a singular value of A iff σ^2 is an eigenvalue of $A^T A$.

Problem 3. Let $A \in M_{m,n}(\mathbb{R})$ have rank k.

(1) Let $u, v \in \mathbb{R}^n$. Prove that $uv^T \in M_n(\mathbb{R})$ is of rank 1 or 0. Here, uv^T is the matrix multiplication of the column vector u with the row vector v^T.
(2) Prove directly (i.e. without SVD) that A is the sum of k matrices of rank 1.
(3) Prove that there exist orthonormal vectors $v_1, \ldots, v_k \in \mathbb{R}^n$, orthonormal vectors $u_1, \ldots, u_k \in \mathbb{R}^m$, and $\alpha_1, \ldots, \alpha_k > 0$ s.t. $A = \sum_{i=1}^k \alpha_i u_i v_i^T$. Here you are encouraged to use the SVD theorem.

Problem 4. Let $\sigma_1 \geq \ldots \geq \sigma_n \geq 0$ be the singular values of $A \in M_{m,n}(\mathbb{R})$. Show that for all $x \in \mathbb{R}^m$:

$$\|Ax\|_2 \geq \sigma_n \|x\|_2$$

and that there exists some $0 \neq x \in \mathbb{R}^n$ s.t. equality holds.

Problem 5. In this question you'll prove that the operator norms on $M_{m,n}(\mathbb{R})$ are well-defined, regardless of which norms are used on $\mathbb{R}^n, \mathbb{R}^m$.

Let $N_m : \mathbb{R}^m \to \mathbb{R}$ be a norm on $\mathbb{R}^m$ and $N_n : \mathbb{R}^n \to \mathbb{R}$ be a norm on $\mathbb{R}^n$. Let $A \in M_{m,n}(\mathbb{R})$. We want to show that

$$\|A\|_{N_n \to N_m} = \max_{x \in \mathbb{R}^n : N_n(x) = 1} N_m(Ax)$$

exists.

(1) Explain why it's enough to show that $S_A = \{N_m(Ax) : x \in \mathbb{R}^n, N_n(x) = 1\} \subseteq \mathbb{R}$ is closed and bounded.
(2) Show that S_A is bounded. In fact, show that there is some $C > 0$ s.t. for all $x \in \mathbb{R}^n$, $N_m(Ax) \leq CN_n(x)$.

(3) Show that S_A is closed:
Let $\{s_k\}_{k=1}^{\infty} \subseteq S_A$ be a sequence that converges to $s \in \mathbb{R}$. We want to show that $s \in S_A$. Let $x_k \in \mathbb{R}^n$ be s.t. $N_n(x_k) = 1$ and $N_m(Ax_k) = s_k$.
  (a) Show that $\{x_k\}_{k=1}^{\infty} \subseteq \mathbb{R}^n$ is ℓ_1-bounded. You might want to reference pset 6 problem 3.
  (b) Apply the Bolzano-Weierstrass theorem (pset 6 problem 1) to obtain a subsequence x_{k_ℓ} that converges in ℓ_1 to $x \in \mathbb{R}^n$.
  (c) Show that x_{k_ℓ} converges to x in N_n, as well. In other words, show that $\lim_{\ell\to\infty} N_n(x_{k_\ell} - x) = 0$. Pset 6, problem 3, might be useful here too.
  (d) Show that $N_n(x) = 1$.
  (e) Show that $N_m(Ax) = s$.