MATHEMATICAL TOOLS - SOLUTION 9

Problem 1.

(1) $A$is symmetric so this follows from the spectral theorem for symmetric real matrices.

(2)

(a) $(1, 1, \ldots, 1)$ is a (left) eigenvector with eigenvalue d.

(b) Let λ be an eigenvalue with right eigenvector f. Let $v \in V$ maximize $|f|$. Then, on the one hand $|(Af)(v)| = |\lambda f(v)|$, and on the other $|(Af)(v)| \leq \sum_{w \in N(v)} |f(w)| \leq d|f(v)|$. Thus $|\lambda| \leq d$.

(c) For each of the k connected components, the indicator function for the component is an eigenvector with eigenvalue d. These vectors span the set of functions that are constant on every connected component. We'll show that this is the entire eigenspace corresponding to the eigenvalue d: Let f be an eigenvector with eigenvalue d. Let v maximize $f(v)$ on v's connected component. We know that $df(v) = \sum_{w \in N(v)} f(w)$, but $f(w) \leq f(v)$ and therefore in fact $f(w) = f(v)$. So if v maximized f in its connected component, so do all its neighbors. By induction, f is constant on the entire component.

(d) Let f be a right eigenvector corresponding to eigenvalue λ_i. Let U, W be a partition of V into parts with no internal edges. Define $\tilde{f}(v) = \begin{cases} f(v) & v \in U \\ -f(v) & v \in W \end{cases}$. We claim that $\tilde{f}$ is an eigenvector corresponding to $-\lambda$. Indeed: $\left(A\tilde{f}\right)(v) = \sum_{w \in N(v)} \tilde{f}(w)$. If $v \in U$ then $N(v) \subseteq W$ in which case $\sum_{w \in N(v)} \tilde{f}(w) = -\sum_{w \in N(v)} f(w) = -\lambda f(v) = -\lambda \tilde{f}(v)$. Otherwise $v \in W$ and $N(v) \subseteq U$, in which case $\sum_{w \in N(v)} \tilde{f}(w) = \sum_{w \in N(v)} f(w) = \lambda f(v) = -\lambda \tilde{f}(v)$.
Now let k be the largest integer s.t. $\lambda_k > 0$. Let $u_1, \ldots, u_k$ be corresponding orthogonal eigenvectors. Then, by what we've shown $\tilde{u_1}, \ldots, \tilde{u_k}$ are eigenvectors for $-\lambda_1, \ldots -\lambda_k$, respectively. Furthermore these vectors are orthogonal and correspond to eigenvalues distinct from $\lambda_1, \ldots, \lambda_k$, and so $u_1, \ldots, u_k, \tilde{u_1}, \ldots, \tilde{u_k}$ is an orthogonal system. Because of the ordering we know that $\lambda_n = -\lambda_1, \ldots, \lambda_{n-k+1} = -\lambda_k$. By a symmetric argument the remaining eigenvalues are 0, which completes the proof.

(e) Assume $\lambda_n = -d$. Let f be a corresponding eigenvector. Let $v \in V$ maximize $|f|$. Note that $|f(v)| > 0$. Then $(Af)(v) = -df(v)$, but on the other hand $(Af)(v) = \sum_{w \in N(v)} f(w)$. But $|f(w)| \leq |f(v)|$ for all vertices w in v's connected component. Thus $f(w) = -f(v)$ for all $w \in N(v)$. But then all of v's neighbors maximize $|f|$ as well, and by induction $|f|$ is maximized in the entire connected

component of v, and if u, w are adjacent in v's connected component, then $f(u) = -f(v)$. So let $U = \{u \in C(v) : f(u) < 0\}, W = \{w \in C(v) : f(w) > 0\}$ where $C(v)$ is v's connected component. Then U, W are a partition of $C(v)$, and none of them has internal edges. So G has a bipartite connected component.
Conversely, assume U, W are a bipartite partition of a connected component of G. Define $f : V \to \mathbb{R}$ by

$$f(v) = \begin{cases} 1 & v \in U \\ -1 & v \in W \\ 0 & otherwise \end{cases}$$

One may verify that f is an eigenvector with eigenvalue $-d$.

Problem 2. Let $B = \{i \in [n] : p_i > q_i\}, C = \{i \in [n] : q_i > p_i\}$. Then $2\delta(p, q) = \sum_{i \in B}(p_i - q_i) + \sum_{i \in C}(q_i - p_i)$. Also,

$$\sum_{i \in B} p_i + \sum_{i \in C} p_i + \sum_{i \in [n] \setminus (B \cup C)} p_i = \sum_{i \in B} q_i + \sum_{i \in C} q_i + \sum_{i \in [n] \setminus (B \cup C)} q_i$$

$$\implies \sum_{i \in B}(p_i - q_i) = \sum_{i \in C}(q_i - p_i) + \sum_{i \in [n] \setminus (B \cup C)}(q_i - p_i)$$

But for all $i \in [n] \setminus (B \cup C), q_i = p_i$, and so:

$$\sum_{i \in B}(p_i - q_i) = \sum_{i \in C}(q_i - p_i) = \delta(p, q)$$

Let $A \subseteq [n]$ maximize $|P(A) - Q(A)|$. Assume w.l.o.g. that $P(A) \geq Q(A)$. Then $|P(A) - Q(A)| = \sum_{i \in A}(p_i - q_i) \leq \sum_{i \in A \cap B}(p_i - q_i) = |P(A \cap B) - Q(A \cap B)|$. By maximality of A equality must hold, so $A = B$. But then: $|P(A) - Q(A)| = \sum_{i \in A \cap B}(p_i - q_i) = \delta(p, q)$, as desired.

Problem 3. Assume $G = (V, E)$. For $v \in V$, define $P(v) = \frac{\deg(v)}{2|E|}$. We'll prove that this is the stationary distribution. For this we need to show that for all $v \in V$, $P(v) = \sum_{w \in V} A_{w,v} P(w)$ where $A_{w,v}$ is the transition probability from w to v. This is defined by:

$$A_{w,v} = \begin{cases} \frac{1}{\deg(w)} & v \in N(w) \\ 0 & otherwise \end{cases}$$

So, we need to show:

$$P(v) = \frac{\deg(v)}{2|E|} = \sum_{w \in N(v)} \frac{1}{\deg(w)} \frac{\deg(w)}{2|E|} = \sum_{w \in N(v)} \frac{1}{2|E|} = \frac{\deg(v)}{2|E|} = P(v)$$

(This was enough to answer the question. We can also give a quick proof that this is the unique stationary distribution. For this we need to show that the dimension of the left-eigenspace corresponding to 1 is 1. This is equivalent to showing that the row-rank of $A - I$ is 1. Row-rank is equal to column-rank. And in fact, if $f : V \to \mathbb{R}$ is a right-eigenvector with eigenvalue 1, let v maximize $f(v)$. Then $f(v) = Af(v) = \sum_{w \in N(v)} \frac{1}{\deg(v)} f(w) \leq f(v)$ so we must have equality. Thus f is constant on all of v's connected component, which is all of G. So the column rank of $A - I$ is one, as desired)

Problem 4. A is symmetric and therefore orthogonally diagonalizable. Furthermore, if $\lambda_1 \geq \ldots \geq \lambda_n$ is B's spectrum with orthonormal basis $\left(\frac{1}{\sqrt{n}}, \ldots, \frac{1}{\sqrt{n}}\right) u_1, \ldots, u_n$, then $Au_i = \frac{1}{2}(I+B)u_i = \frac{1}{2}(1+\lambda_i)$. Now G is connected, so 1 appears exactly once in B's spectrum. Since $\lambda \mapsto \frac{1}{2}(1+\lambda)$ is bijective, and $1 \mapsto 1$, 1 appears exactly once in A's spectrum. Furthermore $[-1,1]$ is mapped to $[0,1]$, so all eigenvalues of A other than 1 are strictly smaller than 1 in absolute value. Let $1 = \sigma_1 \geq \ldots \geq \sigma_n$ be A's spectrum. Then:

$$x^0 = \sum_i \langle x, u_i \rangle u_i \implies \left\| x^0 A^t - \left(\frac{1}{n}, \ldots, \frac{1}{n}\right) \right\|_2^2 = \left\| \left\langle x^0, u_1 \right\rangle u_1 - \left(\frac{1}{n}, \ldots, \frac{1}{n}\right) \right\|_2^2 + \sum_{i>1} \langle x, u_i \rangle^2 \sigma_i^{2t}$$

Now the last sum tends to zero because for all $i > 1$, $|\sigma_i| < 1$. Finally, $\left\langle x^0, u_1 \right\rangle u_1 = \frac{1}{\sqrt{n}}$, so $\left\| \left\langle x^0, u_1 \right\rangle u_1 - \left(\frac{1}{n}, \ldots, \frac{1}{n}\right) \right\|_2^2 = 0$. Therefore $\lim_{t\to\infty} \left\| x^0 A^t - \left(\frac{1}{n}, \ldots, \frac{1}{n}\right) \right\|_2 = 0$, and by equivalence of norms $\lim_{t\to\infty} \delta\left(x^0 A^t, \left(\frac{1}{n}, \ldots, \frac{1}{n}\right)\right) = \frac{1}{2} \lim_{t\to\infty} \left\| x^0 A^t - \left(\frac{1}{n}, \ldots, \frac{1}{n}\right) \right\|_2 = 0$.

Problem 5.

(1) Every edge in G moves between two $0,1$-vectors with different parity for the number of 1s. So the vertices may be partitioned into those with an even number of 1s and those with an odd number of 1s.

(2) Given that all coordinates have been chosen, consider the times t_i corresponding to the last time coordinate i was chosen. $(X_t)_i$ is equal to the value set in coordinate i at time i, which are i.i.d. uniformly $\{0,1\}$. Thus $X_t \sim \mathcal{U}\left(\{0,1\}^n\right)$. Since $\left|\{0,1\}^n\right| = \frac{1}{2^n}$, the result follows.

(3) The probability that coordinate i hasn't been chosen by time t is $\left(1 - \frac{1}{n}\right)^t$. By a union bound, the probability that *some* coordinate hasn't been chosen by time t is $n\left(1 - \frac{1}{n}\right)^t$. Thus $\mathbb{P}[A_t] \geq 1 - n\left(1 - \frac{1}{n}\right)^t$.

(4) For every $x \in \{0,1\}^n$, $\mathbb{P}[X_t = x] = \frac{1}{2^n}\mathbb{P}[A_t] + \mathbb{P}\left[X_t = x | \overline{A_t}\right](1 - \mathbb{P}[A_t])$. Thus:

$$\left| \mathbb{P}[X_t = x] - \frac{1}{2^n} \right| \leq \frac{1}{2^n}(1 - \mathbb{P}[A_t]) + (1 - \mathbb{P}[A_t]) \leq \left(1 + 2^{-n}\right) n \left(1 - \frac{1}{n}\right)^t$$

Thus:

$$\delta\left(p_t, \left(2^{-n}, \ldots, 2^{-n}\right)\right) \leq 2^{n-1}\left(1 + 2^{-n}\right) n \left(1 - \frac{1}{n}\right)^t$$

Now, assume $t \geq 3n^2$. Then:

$$\begin{aligned} 2^{n-1}\left(1 + 2^{-n}\right) n \left(1 - \frac{1}{n}\right)^t &\leq 2^{n-1}\left(1 + 2^{-n}\right) n \left(1 - \frac{1}{n}\right)^{3n^2} \\ &\leq 2^{n-1}\left(1 + 2^{-n}\right) e^{-3n} \leq \left(\frac{2}{e^3}\right)^n \leq \frac{1}{4} \end{aligned}$$