1. Let $C_d \subseteq \mathbb{R}^d$ be the unit ball in l_1 norm

   (a) Let's consider the polytope P derived from the following set of inequalities: $a_i x \leq 1$ where $a_i \in \{-1,1\}^d, 1 \leq i \leq 2^d$ the set of a_i contain all possibilities of each coordinate being either 1 or -1. I claim that $P = C_d$
   First, let's suppose that $x \in C_d$, then $\sum_{i=1}^{n} |x_i| \leq 1$
   Suppose, by contradiction, that there exist two subsets $M, N \subset [d]$ such that $M \cup N = [d]$ and such that:

   $$1 < \sum_{i \in M} -x_i + \sum_{i \in N} x_i$$

   Since $\sum\limits_{i \in M} -x_i + \sum\limits_{i \in N} x_i \leq \sum\limits_{i \in [d]} |x_i|$ we get $1 < \sum\limits_{i \in [d]} |x_i|$ in contradiction to $x \in C_d$.
   So we know that for every two subsets $M, N \subset [d]$ such that $M \cup N = [d] : \sum_{i \in M} -x_i + \sum_{i \in N} x_i \leq 1$. Therefore, for all $1 \leq i \leq 2^d$ $a_i x \leq 1$
   Now suppose that $x \in P$, therefore $1 \leq i \leq 2^d$ $a_i x \leq 1$, and suppose, by contradiction, that $\sum_{i=1}^{n} |x_i| > 1$, then there exist two subsets $M, N \subset [d]$ such that $M \cup N = [d]$ and such that:

   $$1 < \sum_{i \in M} -x_i + \sum_{i \in N} x_i$$

   But that means that there exist a vector $a \in \{-1,1\}^d$ such that $a_i x > 1$ in contradiction to the assumption.

   (b) Let's first show $C_d \subseteq conv(\{e_1, ..., e_d, -e_1, ..., -e_d\})$:
   Let $x \in C_d$, $\Rightarrow x = \sum_{i=1}^{d} x_i e_i$, $\sum_{i=1}^{d} |x_i| \leq 1$. We wish to find a convex combination of $e_1, ..., e_d, -e_1, ..., -e_d$ that will give us x.
   Let's denote $\sum_{i=1}^{d} |x_i| = p$ and so:

   $$x = \sum_{i=1}^{d} x_i e_i = \sum_{i \in [n]\ x_i \geq 0} |x_i| e_i + \sum_{i \in [n]\ x_i < 0} |x_i| (-e_i) =$$

   $$\sum_{i \in [n]\ x_i \geq 0} |x_i| e_i + \sum_{i \in [n]\ x_i < 0} |x_i| (-e_i) + \frac{(1-p)}{2} e_k + \frac{(1-p)}{2} (-e_k), \ k \in [n]$$

   This is a convex combination of $e_1, ..., e_d, -e_1, ..., -e_d$ since:

      i. All coefficients in $[0,1]$:$\sum_{i=1}^{d} |x_i| \leq 1 \Rightarrow 1 \leq i \leq d, |x_i| \leq 1 \Rightarrow |x_i| \in [0,1], p \in [0,1] \Rightarrow \frac{(1-p)}{2} \in [0,1]$

      ii. Sum of coefficients is 1:

   $$\sum_{i=1}^{d} |x_i| + \frac{(1-p)}{2} + \frac{(1-p)}{2} = p + 2\frac{(1-p)}{2} = p + 1 - p = 1$$

Now we will show that $conv\left(\{e_1, ..., e_d, -e_1, ..., -e_d\}\right) \subseteq C_d$:
Let
$conv\left(\{e_1, ..., e_d, -e_1, ..., -e_d\}\right) \ni v = \sum_{i=1}^{n} \alpha_i e_i + \sum_{i=1}^{n} \beta_i \left(-e_i\right)$
such that $0 \leq \alpha_i, \beta_i \leq 1,\ \sum_{i=1}^{n} \alpha_i + \beta_i = 1$

$$v = \sum_{i=1}^{n} \alpha_i e_i + \sum_{i=1}^{n} \beta_i \left(-e_i\right) = \sum_{i=1}^{n} \left(\alpha_i - \beta_i\right) e_i$$

Notice that
$\|v\|_1 = \sum_{i=1}^{n} |\alpha_i - \beta_i| \leq \sum_{i=1}^{n} |\alpha_i| + |\beta_i| = \sum_{i=1}^{n} \alpha_i + \beta_i = 1$
and so $v \in C_d$

(c) Claim: the set of vertices of C_d is $e_1, ..., e_d, -e_1, ..., -e_d$.
Proof: Let's consider the vector e_i. The following set $L\left(e_i, -1\right) = \{x \in \mathbb{R}^d : \langle e_i, x\rangle = -1\}$ is a supporting hyper-plane of C_d since:

i. Let $x \in C_d$ then $\langle e_i, x\rangle = \sum_{j=1}^{d} x_j e_{i,j} = x_i$ and since $x \in C_d$ we know that $|x_i| \leq 1 \Rightarrow -1 \leq x_i \leq 1$. So $\langle e_i, x\rangle \geq -1$ and so $C_d \subseteq H\left(e_i, -1\right)$

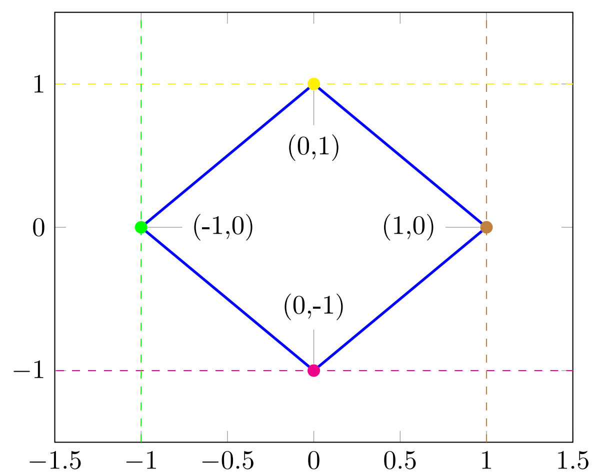
ii. $-e_i \in C_d$ and since $\langle e_i, -e_i\rangle = -1$ we get that $L\left(e_i, -1\right) \cap P \neq \emptyset$
Furthermore, let $x \in C_d$ s.t $\langle e_i, x\rangle = -1$. It is clear that x must be equal to$-e_i$ and so we have shown that $-e_i$ is a vertex of C_d.

Taking the exact same steps we can show that $L\left(-e_i, -1\right) = \{x \in \mathbb{R}^d : \langle -e_i, x\rangle = -1\}$ is a supporting hyper-plane of C_d:

i. Let $x \in C_d$ then $\langle -e_i, x\rangle = -\sum_{j=1}^{d} x_j e_{i,j} = -x_i$ and since $x \in C_d$ we know that $|x_i| \leq 1 \Rightarrow -1 \leq -x_i \leq 1$. So $\langle -e_i, x\rangle \geq -1$ and so $C_d \subseteq H\left(-e_i, -1\right)$

ii. $e_i \in C_d$ and since $\langle -e_i, e_i\rangle = -1$ we get that $L\left(-e_i, -1\right) \cap P \neq \emptyset$
Furthermore, let $x \in C_d$ s.t $\langle -e_i, x\rangle = -1$. It is clear that x must be equal to e_i and so we have shown that e_i is a vertex of C_d.

(d) C_2 drawing:



2. Let $H_d \subseteq \mathbb{R}^d$ be the unit ball in l_∞ norm

   (a) $H_d = \{x \in \mathbb{R} : max\,\{|x_i| : i \in [d]\} \leq 1\} = \cap_{j=1}^{d} \{x \in \mathbb{R} : -1 \leq x_i \wedge x_i \leq 1\}$
   Therefore H_d is the intersection of d halfspaces and is bounded in all dimensions and so it is a Polytope.

   (b) $S = \left\{x \in \mathbb{R}^d : \forall i \in [d],\, x_i \in \{-1,1\}\right\}$First let's show $H_d \subseteq conv(S)$
   We will prove by induction on the dimension d that the hypercube $H_d \subseteq conv(S_d)$, $S_d = \left\{x \in \mathbb{R}^d : \forall i \in [d],\, x_i \in \{-1,1\}\right\}$:

      i. Base: For d=1:
      $H_1 = \{x \in \mathbb{R} : |x| \leq 1\} = [-1,1]$, $S = \{-1,1\}$
      So $x \in H_1 \Rightarrow |x| \leq 1 \Rightarrow$ there exists $a_1, a_2 \in [0,1]$ such that $a_1 + a_2 = 1$ and such that $x = a_1 - a_2$:

$$\begin{cases} a_1 = \frac{1+x}{2}, a_2 = \frac{1-x}{2} & x \geq 0 \\ a_1 = \frac{1-x}{2}, a_2 = \frac{1+x}{2} & x < 0 \end{cases}$$

      ii. Step: Take H_d and transform it in the following way: given a vector
      $x \in H_d, x = [x_1 \quad x_2 \quad ... \quad x_d]$ get the vector : $y = [x_1 \quad x_2 \quad ... \quad x_{d-1}]$, denote the received set H_{d-1}.
      This set is a hypercube of dimension d-1 because for all $x \in H_{d-1}$ $\forall 1 \leq i \leq d-1 \; |x_i| \leq 1$
      By the induction assumption $H_{d-1} = conv(S_{d-1})$ where $S_{d-1} = \left\{x \in \mathbb{R}^{d-1} : \forall i \in [d-1],\, x_i \in \{-1,1\}\right\}$ and so each vector in H_{d-1} is a convex combination of all ± 1 vectors in $\mathbb{R}^{d-1}$.
      Now, let $x \in H_d, x = \begin{bmatrix} x_1 & x_2 & ... & x_d \end{bmatrix}$ the vector $y = \begin{bmatrix} x_1 & x_2 & ... & x_{d-1} & 1 \end{bmatrix}$ can be expressed as a convex combination of$\{s \in S_d : s_d = 1\}$ because there exists a convex combination of S_{d-1} that gives $\begin{bmatrix} x_1 & x_2 & ... & x_{d-1} \end{bmatrix}$ and the same coefficients, since their sum is 1, will give us 1 in the d coordinate.

In the same way, the vector $z = \begin{bmatrix} x_1 & x_2 & ... & x_{d-1} & -1 \end{bmatrix}$ can be expressed as a convex combination of$\{s \in S_d : s_d = -1\}$ because there exists a convex combination of S_{d-1} that gives $\begin{bmatrix} x_1 & x_2 & ... & x_{d-1} \end{bmatrix}$ and the same coefficients, since their sum is 1, will give us -1 in the d coordinate.

Now, x is a convex combination of y and z: $x = a_1 y + a_2 z$, $\begin{cases} a_1 = \frac{1+x_d}{2}, a_2 = \frac{1-x_d}{2} & x_d \geq 0 \\ a_1 = \frac{1-x_d}{2}, a_2 = \frac{1+x_d}{2} & x_d < 0 \end{cases}$

this is true since the sum of a_1, a_2 is 1 and y and z are equal for all coordinates other than d then for all coordinates different than d, the convex combination $a_1 y + a_2 z$ will give us the same value. For coordinate d this is easily checked.

So x is a convex combination of two vectors in $\mathbb{R}^d$ one which is a convex combination of ± 1 vectors where the d coordinate is 1 and the other which is a convex combination of ± 1 vectors where the d coordinate is -1, therefore x is a convex combination of ± 1 vectors where the d coordinate is 1 and ± 1 vectors where the d coordinate is -1 or simply all the± 1 vectors in $\mathbb{R}^d$. so

$$x \in conv(S_d)$$

Now let's show that $conv(S) \subseteq H_d$

Let $x \in conv(S)$, then x is equal to some convex combination of ± 1 vectors, let the coefficients of this convex combination be $\alpha_1, ..., \alpha_n$

Such that

$$\sum_{i=1}^{n} \alpha_i = 1$$

Now let's consider some coordinate of x : $x_j = \sum_{i=1}^{n} \alpha_i a_i$ where a_i is a ± 1 vector in $\mathbb{R}^n$.

$$|x_j| = \left| \sum_{i=1}^{n} \alpha_i a_i \right| \leq \sum_{i=1}^{n} |\alpha_i a_i| = \sum_{i=1}^{n} \alpha_i |a_i| = \sum_{i=1}^{n} \alpha_i = 1$$

This is true for every coordinate of the vector x and so $\|x\|_\infty \leq 1$ so $x \in H_d$.

(c) Claim : The set of vertices of H_d are $V = \{x \in \mathbb{R}^d : \ 1 \leq i \leq d \ x_i \in \{1, -1\}\}$

Proof: Let $v \in V$, let us consider the hyperplane $L(v, d) = \left\{x \in \mathbb{R}^d : \langle v, x \rangle = d\right\}$. First let's show that this is a supporting hyperplane of H_d

i. Let $x \in H_d$ then $\langle x, v \rangle = \sum_{i=1}^{d} x_i v_i \leq \sum_{i=1}^{d} |x_i v_i| = \sum_{i=1}^{d} |x_i| |v_i| \underset{|x_i| \leq 1, |v_i| = 1}{\leq} \sum_{i=1}^{d} 1 = d$.

ii. $\langle v, v\rangle = d$, which means $L(v,d) \cap H_d \neq \emptyset$
FInally let's show that for $x \in H_d$ if $\langle v, x\rangle = d \Rightarrow x = v$:

$$\langle x, v\rangle = \sum_{i=1}^{d} x_i v_i = d$$

We have d terms in the sum and the sum is equal to d, each term in the sum is at most equal to 1 and therefore all terms in the sum must be equal to 1. So x=v.
This means that $L(v,d) \cap H_d = v$ and so v is a vertex of H_d.

3. $\triangle_d \subseteq \mathbb{R}^{d+1} = \left\{x \in [0,1]^{d+1} : \sum_{i=1}^{d+1} x_i = 1\right\}$

   (a) $\triangle_d = \left\{x \in \mathbb{R}^{d+1} : \forall i \in [d+1],\ 0 \leq x_i \leq 1, \sum_{i=1}^{d+1} x_i = 1\right\}$ So $\triangle_d$ is a set of $x \in \mathbb{R}^{d+1}$ which satisfies a finite set of equalities and inequalities and is bounded in all coordinates. By the definition we saw in class this is a Polytope.

   (b) Claim$\triangle_d$'s vertices $V = \{e_1, ..., e_d, e_{d+1}\}$
   Let $e_i \in V$ let's consider the hyperplane : $L(v,d) = \left\{x \in \mathbb{R}^{d+1} : \langle e_i, x\rangle = 1\right\}$, this is a supporting hyperplane of $\triangle_d$ since:

      i. Let $x \in \triangle_d$ then $\langle x, e_i\rangle = x_i \underset{x \in \triangle_d}{\leq} 1$

      ii. $e_i \in \triangle_d$, $\langle e_i, e_i\rangle = \sum_{i=1}^{d+1} e_i e_i = 1 \Rightarrow \triangle_d \cap L(v,d) \neq \emptyset$
      Finally $x \in \triangle_d,\ \langle x, e_i\rangle = 1 \Rightarrow \sum_{i=1}^{d+1} x_i e_i = x_i = 1 \Rightarrow x_i = 1$ and since $\sum_{i=1}^{d+1} x_i = 1 \Rightarrow \forall j \in [d+1],\ j \neq i\ x_j = 0$ and so

$$x = v$$

      So $L(v,d) \cap \triangle_d = v$ and so v is a vertex $\triangle_d$.

   (c) First let's show that $\triangle_d \subseteq conv(V)$:
   Let $x \in \triangle_d$ $x = \sum_{i=1}^{d} x_i e_i$ and this is exactly the convex combination of V that is equal to x, since $0 \leq x_i \leq 1$, $\sum_{i=1}^{d+1} x_i = 1$
   Now let's show that $conv(V) \subseteq \triangle_d$:
   Let $v \in conv(V)$ so $v = \sum_{i=1}^{d+1} v_i e_i$ where $\sum_{i=1}^{d+1} v_i = 1,\ 0 \leq v_i \leq 1$, and this is by defintion of $\triangle_d$ means that $v \in \triangle_d$

4. $x_1, ...x_t \mathbb{R}^d$ and let $P = conv\left(\{x_1, ..., x_t\}\right)$. Let $f : \mathbb{R}^d \longrightarrow \mathbb{R},\ c \in \mathbb{R}^d,\ f(x) = \langle c, x\rangle$

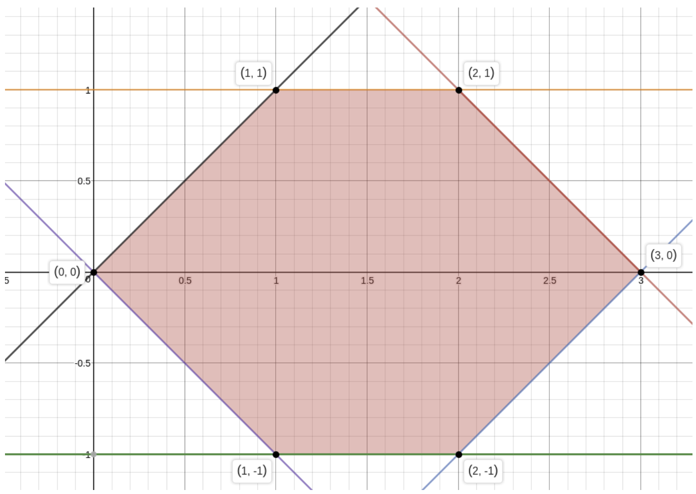
   let $x \in P,\ \ x = \sum_{i=1}^{t} a_i x_i,\ \ a_i \in [0,1],\ \ \sum_{i=1}^{t} a_i = 1$
   $f(x) = \langle c, x\rangle = \left\langle c, \left(\sum_{i=1}^{t} a_i x_i\right)\right\rangle = \sum_{i=1}^{t} a_i \langle c, x_i\rangle$

$max_{x \in P} f(x) = \underset{x \in P}{max} \sum_{i=1}^{t} a_i \langle c, x_i \rangle$
To maximize this sum we must choose what weights to give each inner product of a vertex with c.
The solution is to take the maximal such inner product and to give it the maximal weight, i.e 1, therefore

$$max_{x \in P} f(x) = \underset{x \in P}{max} \sum_{i=1}^{t} a_i \langle c, x_i \rangle = \underset{i=1,\ldots,t}{max} \langle c, x_i \rangle = \underset{i=1,\ldots,t}{max} f(x_i)$$

5. Max 3x - y

   (a) Feasible Solutions:



   (b) $max\, 3x - y$ on the feasible set will be received on one of the politope's vertices, let's compute the objective function on each:

$$f\left(\begin{bmatrix}0\\0\end{bmatrix}\right) = 0, f\left(\begin{bmatrix}1\\1\end{bmatrix}\right) = 2, f\left(\begin{bmatrix}2\\1\end{bmatrix}\right) = 5, f\left(\begin{bmatrix}3\\0\end{bmatrix}\right) = 9, f\left(\begin{bmatrix}2\\-1\end{bmatrix}\right) = 7, f\left(\begin{bmatrix}1\\-1\end{bmatrix}\right) = 4$$

   Therefore $max\, 3x - y$=9

   (c) $$\begin{array}{cc} max & 3x-y \\ & x+y \leq 3 \\ & x-y \leq 3 \\ s.t & -y \leq 1 \\ & y \leq 1 \\ & x+y \geq 0 \\ & x-y \geq 0 \end{array} \Leftrightarrow \begin{array}{cc} max & 3x-y \\ & -x-y=-3+z_1 \\ & y-x=-3+z_2 \\ s.t & y=-1+z_3 \\ & -y=-1+z_4 \\ & x+y=z_5 \\ & x-y=z_6 \\ & z_1,\ldots,z_6 \geq 0 \end{array} \Leftrightarrow \begin{array}{cc} max & 3x-y \\ & -x-y=-3+z_1 \\ & y-x=-3+z_2 \\ s.t & y=-1+z_3 \\ & -y=-1+z_4 \\ & x+y=z_5 \\ & x-y=z_6 \end{array} \Leftrightarrow$$

$$max \quad \begin{bmatrix} 3 & -1 & 0 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \end{bmatrix}$$

$$s.t. \quad \begin{bmatrix} -1 & -1 & -1 \\ -1 & 1 & -1 \\ 0 & 1 & -1 \\ 0 & -1 & -1 \\ 1 & 1 & -1 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \end{bmatrix} = \begin{bmatrix} -3 \\ -3 \\ -1 \\ -1 \\ 0 \\ 0 \end{bmatrix}$$

$$z_1, ..., z_6 \geq 0$$