1

$$A = \begin{bmatrix} -\frac{24}{25} & \frac{4}{5} \\ -\frac{6}{5} & 0 \\ -\frac{32}{25} & -\frac{3}{5} \end{bmatrix}$$

Suppose that A = UΣV^T
Let's find the eigenvalues of $A^T A$:

$$A^T A = \frac{1}{5} \begin{bmatrix} -\frac{24}{5} & -6 & -\frac{32}{5} \\ 4 & 0 & -3 \end{bmatrix} \cdot \frac{1}{5} \begin{bmatrix} -\frac{24}{5} & 4 \\ -6 & 0 \\ -\frac{32}{5} & -3 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}$$

Therefore

$$\lambda_1 = 4, \lambda_2 = 1$$
$$\sigma_1 = 2, \sigma_2 = 1$$
$$\Sigma = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Now, for finding V, we must find a base of eigenvectors.
Eigenvector corrosponding to λ_1 and λ_2 :

$$v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Therefore

$$V = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Now to compute U, we use $AV = U\Sigma$

$$AV = \frac{1}{5} \begin{bmatrix} -\frac{24}{5} & \frac{4}{5} \\ -\frac{6}{5} & 0 \\ -\frac{32}{25} & -\frac{3}{5} \end{bmatrix}$$

Let's calculate the columns of U:

$$u_1 = \frac{[Av]_1}{\sigma_1} = \frac{1}{2}\frac{1}{5} \begin{bmatrix} -\frac{24}{5} \\ -6 \\ -\frac{32}{5} \end{bmatrix} = \frac{1}{5} \begin{bmatrix} -\frac{12}{5} \\ -3 \\ -\frac{16}{5} \end{bmatrix}, u_2 = \frac{[Av]_2}{\sigma_2} = \frac{1}{5} \begin{bmatrix} -\frac{4}{5} \\ 0 \\ -\frac{3}{5} \end{bmatrix}$$

We complete the two vectors u_1, u_2 to an orthonormal base:

$$u_3 = \frac{1}{5} \begin{bmatrix} \frac{3}{5} \\ \frac{28}{75} \\ -\frac{4}{5} \end{bmatrix}$$

$$U = \frac{1}{5} \begin{bmatrix} -\frac{12}{5} & -\frac{4}{5} & \frac{3}{5} \\ -3 & 0 & \frac{28}{75} \\ -\frac{16}{5} & -\frac{3}{5} & -\frac{4}{5} \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \text{ Suppose that B = U}\Sigma V^T$$

Let's find the eigenvalues of $B^T B$:

$$B^T B = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 & 1 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 1 & 0 & 1 & 2 \end{bmatrix}$$

$$\det(\lambda I - B) = \begin{vmatrix} \lambda-2 & -1 & 0 & -1 \\ -1 & \lambda-2 & -1 & 0 \\ 0 & -1 & \lambda-2 & -1 \\ -1 & 0 & -1 & \lambda-2 \end{vmatrix} = (\lambda-2) \begin{vmatrix} \lambda-2 & -1 & 0 \\ -1 & \lambda-2 & -1 \\ 0 & -1 & \lambda-2 \end{vmatrix} + \begin{vmatrix} -1 & -1 & 0 \\ 0 & \lambda-2 & -1 \\ -1 & -1 & \lambda-2 \end{vmatrix} +$$

$$\begin{vmatrix} -1 & \lambda-2 & -1 \\ 0 & -1 & \lambda-2 \\ -1 & 0 & -1 \end{vmatrix} = (2-\lambda)^2(4-\lambda)\lambda$$

$$\lambda_1 = 2, \lambda_2 = 4, \lambda_3 = 0$$

$$\sigma_1 = 2, \sigma_{2,3} = \sqrt{2}, \sigma_4 = 0, \Sigma = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Subspace of eigenvalue λ_1:

$$\begin{bmatrix} 0 & -1 & 0 & -1 \\ -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 \\ -1 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0$$

$$x_2 = -x_4, x_1 = -x_3$$

$$v_1 = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}, v_2 = \frac{1}{2} \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$$

Subspace of eigenvalue λ_2:

$$\begin{bmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0$$

$$2x_1 - x_2 - x_4 = 0$$

$$-x_1 + 2x_2 - x_3 = 0$$

$$-x_2 + 2x_3 - x_4 = 0$$

$$-x_1 - x_3 + 2x_4 = 0$$

$$v_3 = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

Subspace of eigenvalue λ_3:

$$\begin{bmatrix} -2 & -1 & 0 & -1 \\ -1 & -2 & -1 & 0 \\ 0 & -1 & -2 & -1 \\ -1 & 0 & -1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0$$

$$-2x_1 - x_2 - x_4 = 0$$

$$-x_1 - 2x_2 - x_3 = 0$$

$$-x_2 - 2x_3 - x_4 = 0$$

$$-x_1 - x_3 - 2x_4 = 0$$

$$v_4 = \frac{1}{2} \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$$

$$V = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & 1 & 1 & -1 \end{bmatrix}, BV = \begin{bmatrix} 0 & 1 & 1 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

$$u_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}, u_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, u_3 = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, u_4 = \frac{1}{2} \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}, U = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 1 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -1 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & -1 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 1 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

2

Proof. From the SVD theorem : $A = U\Sigma V^T$. Let us denote $V = [v_1, ...v_n], \quad U = [u_1, ...u_n]$

$\forall$ v $\in \Re^n$

$$v = \sum_{j=1}^{n} < v_j, v > v_j;$$

$$Av = U\Sigma V^T v = U\Sigma V^T (\sum_{j=1}^{n} < v_j, v > v_j) = U\Sigma \sum_{j=1}^{n} < v_j, v > e_j = \sum_{j=1}^{n} < v_j, v > \sigma_j u_j$$

$$||Av||_2^2 = \sum_{j=1}^{n} (< v_j, v > \sigma_j)^2$$

$$||Av||_2^2 = \sum_{j=1}^{n} (< v_j, v > \sigma_j)^2 \geq \sigma_n^2 \sum_{j=1}^{n} (< v_j, v >)^2 = \sigma_n ||v||_2^2$$

$$||Av||_2^2 = \sum_{j=1}^{n} (< v_j, v > \sigma_j)^2 \leq \sigma_1^2 \sum_{j=1}^{n} (< v_j, v >)^2 = \sigma_1 ||v||_2^2$$

$$\Downarrow$$

$$\sigma_n ||v||_2 \leq ||Av||_2 \leq \sigma_1 ||v||_2$$

$\square$

3

Proof. A $\in M_n(\Re)$; $\quad A^k = 0$
By the spectral decomposition theorem: $A = U\Lambda U^T$, where U is an orthogonal matrix and Λ is a diagonal matrix.

$$A^k = (U\Lambda U^T)^k = \underbrace{U\Lambda U^T U\Lambda U^T ... U\Lambda U^T}_{\text{k times}} = U\Lambda^k U^T = 0 \qquad \Rightarrow \Lambda^k = 0$$

$$\lambda_1^k = 0, ..., \lambda_n^k = 0 \quad \Rightarrow \quad \lambda_1 = 0, ..., \lambda_n = 0 \quad \Rightarrow \quad \Lambda = 0$$

$$A = U\Lambda U^T = U0U^T = 0$$

$\square$

4

Lemma : Let A $\in S_n(\Re)$ and let λ_i be an eigenvalue of A then $|\lambda_i|$ is a singular value of A.

Proof. Since λ_i is an eigenvalue of A, $\exists v \in \Re^n$ s.t.

$$Av = \lambda_i v \quad \Rightarrow \quad AA^T v = A^2 v = A(\lambda_i v) = \lambda_i^2 v \Rightarrow$$

λ_i^2 is an eigenvalue of AA^T, therefore, $\sqrt{\lambda_i^2} = |\lambda_i|$ is a singular value of A. $\square$

By the spectral decompostion thm. $A = U\Lambda U^T$

$$|det(A)| = |det(U\Lambda U^T)| = |det(U)det(\Lambda)det(U^T)| = |det(U^T)det(U)det(\Lambda)| = |det(UU^T)det(\Lambda)| =$$

$$|det(I)det(\Lambda)| = |det(\Lambda)| = \left|\prod_{i=1}^{n}\lambda_i\right| = \prod_{i=1}^{n}|\lambda_i| = \prod_{i=1}^{n}\sigma_i$$

5

Let A $\in M_{m,n}(\Re)$ have rank k.

5.1

Let $u, v \in \Re^n$.
If $uv^T = 0_{m,n}$ then it is of rank 0. If uv^T has exactly one non zero column, then $Rank(uv^T) = 1$.
Suppose uv^T has two non zero columns i and j, then: $[uv^T]_i = u[v^T]_i; \quad [uv^T]_j = u[v^T]_j \Rightarrow [uv^T]_i = \frac{[v^T]_i}{[v^T]_j}[uv^T]_j$ Therefore the i and j columns are linearly dependent and therefore $Rank(uv^T) \leq 1$

5.2

Proof. Let's denote the columns of A as u_i. $A = (u_1, ..., u_n)$.
As $Rank(A) = k$ we know that there exists some permutation of 1,...,n $i_1, ..., i_n$ s.t $u_{i_1}, ..., u_{i_k}$ are linearly independant and $u_{ik+1}, ..., u_{in}$ are dependent on $u_{i_1}, ..., u_{i_k}$ let's assume, w.l.o.g, that $i_1, ..., i_n$ = 1,...,n. Therefore,

$$\forall j, \quad k+1 \leq j \leq n, \quad u_j = \sum_{l=1}^{k} < u_l, u_j > u_j$$

Let's look at the following k matrices:

$$\forall 1 \leq l \leq k, \quad A_l = \begin{bmatrix} . & & . & & . & & . & & . \\ . & & . & & . & & . & & . \\ . & & . & & . & & . & & . \\ 0 & ... & u_l & ... & 0 & < u_{k+1}, u_l > & u_l & ... < u_n, u_l > & u_l \\ . & & . & & . & & . & & . \\ . & & . & & . & & . & & . \\ . & & . & & . & & . & & . \end{bmatrix}$$

Clearly, $\forall 1 \leq l \leq k, rank(A_i) = 1$

$$\sum_{i=1}^{k} A_i = \begin{bmatrix} . & & . & & . & & . & & . \\ . & & . & & . & & . & & . \\ . & & . & & . & & . & & . \\ 0 & ... & u_l & ... & 0 & \sum_{l=1}^{k} < u_{k+1}, u_l > & u_l & ... \sum_{l=1}^{k} < u_n, u_l > & u_l \\ . & & . & & . & & . & & . \\ . & & . & & . & & . & & . \\ . & & . & & . & & . & & . \end{bmatrix}$$

$$= \begin{bmatrix} . & & . & . & & . \\ . & & . & . & & . \\ . & & . & . & & . \\ u_1 & ... & u_k & u_{k+1} & ... & u_n \\ . & & . & . & & . \\ . & & . & . & & . \\ . & & . & . & & . \end{bmatrix} = A$$

$\square$

5.3

From the SVD theorem : $A = U\Sigma V^T$. Let us denote $V = [v_1, ..., v_n], \quad U = [u_1, ..., u_n]$ and let us denote $\Sigma_i =$ to be the matrix with all zeroes except at position $\Sigma_{i,i} = \sigma_i$.

$$U\Sigma_i V^T = u_i \sigma_i v_i^T$$

$$A = \sum_{i=1}^{n} u_i \sigma_i v_i^T$$

Since A is of rank k it has k positive singular values, therefore $A = \sum_{i=1}^{k} u_i \sigma_i v_i^T$ We have shown in section 5.1 that $u_i v_i^T$ is of either rank 0 or 1, but in fact $rank(u_i v_i^T) = 1$ since if for some $1 \leq i \leq k, u_i v^T = 0$ it would mean that A is the sum of less than k rank 1 matrices, which indicates that A is of rank lower than k.