1. Let G = ([n], E) be a k-regular graph on n vertices.
   $A_{i,j} = A_{ji} \Rightarrow A \in S_m(\Re)$

   (a) Let v be an eigenvector of A corrosponding to eigenvalue λ.
   Let's assume $i \in [n]$ s.t. $|v_i|$ is maximal.

   $$Av = \lambda v$$

   On the one hand

   $$[Av]_i = \sum_{k=1}^{n} [A]_{i,k} v_k \leq \sum_{k=1}^{n} [A]_{i,k} |v_i| = |v_i| \sum_{k=1}^{n} [A]_{i,k} = k|v_i|$$

   And on the other hand

   $$[Av]_i = \lambda v_i$$

   Therefore, any such eigenvector satisfies

   $$\lambda v_i \leq k|v_i|$$

   if $v_i > 0 \Rightarrow \lambda \leq k$
   if $v_i < 0$ then let's consider the vector $-v$ which satisfies

   $$A(-v) = (-\lambda)v.$$

   For $-v$ it still holds that $|v_i|$ is maximal and so from the above equations we get

   $$-\lambda v_i \leq k|v_i| \Rightarrow \lambda \geq -k$$

   so we conclude

   $$|\lambda| \leq k$$

   (b) Let's take the vector v where $\forall i \in [n]$, $v_i = 1$

   $$[Av]_i = \sum_{j=1}^{n} A_{ij} v_j = \sum_{j=1}^{n} A_{ij} = k \Rightarrow Av = kv$$

   We have shown some eigenvalue $\lambda = k$. Since $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ and $|\lambda| \leq k$ we can conclude that

   $$\lambda_1 = k$$

   (c) Lemma: Let G be a graph with a connected component C and let v be the component's indicator vector, meaning:

   $$v_i = \begin{cases} 1 & i \in V(C) \\ 0 & otherwise \end{cases}$$

then

$$Av = kv$$

Proof:

if $a \in V(C)$ since a has k neighbors and they're all in C it follows that:

$$[Av]_a = \sum_{j=1}^{n} A_{kj} v_j = k$$

if $a \notin V(C)$then non of a's neighbors are in C and it follows that:

$$[Av]_a = \sum_{j=1}^{n} A_{kj} v_j = 0$$

Therefore

$$Av = kv$$

v_{!,}
Let's suppose that G has m connected components, $C_1, \ldots C_m$, the indicator vectors for these components ,$v_1, \ldots, v_m$, are orthogonal since they do not share any indices, and so there are at least m orthogonal eigenvectors corrosponding to eigenvalue k, meaning the geometric multiplicity of k is $\geq m$.
For any real symmetric matrix the algebraic multiplicity of an eigenvalue is equals to the geometric multiplicty and so we have shown that $l \geq m$

Let's show now that $l \leq m$
Let $u = (u_1, \ldots, u_n)$ be an eigenvector of A which corrosponds to eigenvalue k.
For each $i \in [m]$ we can take $u_{i*} = max\{|u_j| \mid j \in C_i\}$. lets suppose w.l.o.g that $u_{i*} > 0$.

$$ku_{i*} = [Au]_{i*} = \sum_{j=1}^{n} A_{i*,j} u_j = \sum_{(i*,j) \in E(G)} u_j$$

The sum on the right side has exactly k terms and from the left side we know that their average is v_{i*}, since v_{i*} is also their maximum we know that

$$\forall j \quad s.t \quad (i*, j) \in E(G), \, u_j = u_{i*}$$

So we get that for each $i \in [m]$, $v_j = v_{i*}$ for all $j \in C_i$. This means that

$$u = \alpha_1 v_1 + \ldots \alpha_m v_m$$

where

$$\alpha_i = u_{i*}$$

meaning u is a linear combination of $v_1, \ldots v_m$ and there are exactly m eigenvectors of eigenvalue k.

(d) Suppose that G has a bipartite connected component C, denote the two components comprising C as $C_1$and C_2.

Consider the following vector $v \in \Re^n : v_i = \begin{cases} 1 & i \in C_1 \\ -1 & i \in C_2 \\ 0 & otherwise \end{cases}$

$$[Av]_i = \sum_{j=1}^{n} A_{ij} v_j = \begin{cases} -k & i \in C_1 \\ k & i \in C_2 \\ 0 & otherwise \end{cases} \Rightarrow [Av]_i = -kv_i$$

$$Av = -kv$$

So there exists an eigenvalue that is equal to $-k$. It follows from $|\lambda_i| \leq k$ and from $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ that

$$\lambda_n = -k$$

Now suppose that $-k$ is an eigenvalue of A:
Let's assume that G has no bipartite connected component, therefore each of the connected components of G has an odd cycle. Let's consider one such connected component C of G.
Suppose that $|v_{i*}| = max\{|v_j| \mid j \in [n] \, , \, j \in C\}$.
Again we know that

$$-kv_{i*} = [Av]_{i*} = \sum_{j=1}^{n} A_{i*,j} v_j = \sum_{(i*,j) \in E(G)} v_j$$

So $-v_{i*}$ is the average of v_j but also $|v_j| \leq |v_{i*}|$ therefore $\forall j \in C, (i*, j) \in E(G)\, , \, v_j = -v_{i*}$. Since we can get from $i_*$to any other vertice in C we can conclude by applying this logic recursively that for each vertice i we choose in C, $\forall j \in C, (i, j) \in E(G)\, , \, v_j = -v_{i*}$. Now if we apply this on the odd cycle in C we will conclude that

$$v_{i*} = -v_{i*} \Rightarrow v_{i*} = 0 \Rightarrow \forall i \in C \, , \, v_i = 0$$

Now apply this logic for all connected components of G and we get that $v = 0$ in contradiction of v being an eigenvector of A.

(e) Suppose that λ is an eigenvalue of A and we need to prove that $-\lambda$ is also an eigenvalue.

$$\exists v \, s.t \, Av = \lambda v$$

G is bipartite, let's denote the sides of G as C_1, C_2. Now let's consider the following vector:

$$v'_i = \begin{cases} v_i & i \in C_1 \\ -v_i & i \in C_2 \end{cases}$$

$$[Av']_r = \sum_{i=1}^{n} A_{ri} v'_i = \sum_{i \in C_1} A_{ri} v_i - \sum_{i \in C_2} A_{ri} v_i = \begin{cases} -\sum_{i \in C_2} A_{ri} v_i = -\lambda v_r = -\lambda v'_r & r \in C_1 \\ \sum_{i \in C_1} A_{ri} v_i = [Av]_r = \lambda v_r = -\lambda v'_r & r \in C_2 \end{cases}$$

$$Av' = -\lambda v'$$

2. Let $G = K_n$

$$G_A = \begin{bmatrix} 0 & 1 & \dots & 1 & 1 \\ 1 & 0 & 1 & \vdots & 1 \\ \vdots & \vdots & \ddots & 0 & \vdots \\ 1 & 1 & \dots & 1 & 0 \end{bmatrix}$$

$$det(G_A - \lambda I) = \begin{vmatrix} -\lambda & 1 & \dots & 1 \\ 1 & -\lambda & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & -\lambda \end{vmatrix} =$$

$$\underset{1 \le i \le n-1, R_i \Leftarrow R_i - R_n}{=} \begin{vmatrix} (-\lambda - 1) & 0 & 0 & \dots & 0 & (1+\lambda) \\ 0 & (-\lambda - 1) & 0 & \dots & 0 & (1+\lambda) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & 1 & \dots & 1 & -\lambda \end{vmatrix} =$$

$$\underset{1 \le i \le n-1, R_n \Leftarrow C_n C_n}{=} \begin{vmatrix} (-\lambda - 1) & 0 & 0 & \dots & 0 & 0 \\ 0 & (-\lambda - 1) & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & 1 & \dots & 1 & (-\lambda + n - 1) \end{vmatrix} =$$

$$(-\lambda - 1)^{n-1}(-\lambda + n - 1)$$

Therefore:

$$\lambda_1 = -1, \, \lambda_2 = (n-1)$$

3. We have seen in recitation that $A^k_{Gi,j}$ is the number of length-k paths from i to j in G. A triangle is a length-3 path from a vertex to itself. We can count the number of length-3 paths that start from each vertex and end in that same vertex but this will count each triangle 6 times since there are 6 ways decide on the order of the vertices in the triangle. Therefore :

$$\#\text{ triangles in G}=\frac{1}{6}tr(A_G^3)$$

Since A is a real symmetric matrix by the spectral decomposition theorem we can say:

$$A_G = U\Lambda U^T$$

Where U is an orthogonal matrix and Λ is a diagonal $\Lambda = diag(\lambda_1, \dots, \lambda_n)$

$$\#\text{ triangles in G}=\frac{1}{6}tr(A_G^3) = \frac{1}{6}tr(U\Lambda U^T U\Lambda U^T U\Lambda U^T) =$$

$$= \frac{1}{6}tr(U\Lambda\Lambda\Lambda U^T) = \frac{1}{6}tr(U^T U\Lambda^3) = \frac{1}{6}tr(\Lambda^3) = \frac{1}{6}\sum_{i=1}^{n}\lambda_i$$

4. $A \in M_n(\Re),\ S = \{x \in \Re^n : \|x\| = 1\}$

   (a) $f : S \to \Re\,,\ f(x) = x^T Ax$
   f is bounded:

   $$f(x) = \sum_{i,j} x_i a_{ij} x_j \underset{x_i,x_j\le 1}{\le} \sum_{ij} a_{ij}$$

   f is closed:
   f is bounded therefore it has a supremum.

   $$L = sup\{x^T Ax \,:\, ||x|| = 1\}$$

   $\forall\epsilon\exists x\, s.t.\, f(x) > L - \epsilon$, lets denote $\epsilon_n = \frac{1}{n}$,so we know

   $$\exists x_n\, s.t.\, f(x_n) > L - \frac{1}{n}$$

   denote $f(x_n) = f_n$

   $$f_n = x_n^T Ax_n \underset{n\to\infty}{\to} L$$

   $\{x_n\}_{n\in\mathbb{N}}$ is a sequence in S and S is compact therefore it has a convergant subsequence , $\{x_{n_k}\} \to x_0 \in S$

   $$f_{n_k} \overset{def}{=} f(x_{n_k}) \to L$$

and since f is continuous it follows that $f(x_0) = L$ and so f obtains a maximum.

(b) Let $v \in S$ be a vector on which f is maximal.
Let $u \in span(\{v\})^{\perp}$, $g : \mathbb{R} \to \mathbb{R}$, $g(t) = f\left(\frac{v+tu}{||v+tu||}\right)$
$g(t) = f\left(\frac{v+tu}{||v+tu||}\right) = \frac{1}{||v+tu||^2}((v+tu)^T A(v+tu)) = \frac{1}{||v+tu||^2}(v^T Av + (tu)^T Av + v^T Atu + (tu)^T Atu) \underset{\text{v maximizes f}}{\leq} v^T Av$
$||v+tu||^2 = \langle v, v\rangle + \langle v, tu\rangle + \langle tu, v\rangle + \langle tu, tu\rangle = 1 + \langle tu, tu\rangle$
Therefore

$$(v^T Av + (tu)^T Av + v^T Atu + (tu)^T Atu) \leq (1 + \langle tu, tu\rangle)v^T Av$$

$$\frac{(tu)^T Av + v^T Atu}{\langle tu, tu\rangle} + \frac{(tu)^T Atu}{\langle tu, tu\rangle} \leq v^T Av$$

$$\frac{2u^T Av}{t\langle u, u\rangle} + \frac{u^T Au}{\langle u, u\rangle} \leq v^T Av$$

this holds $\forall t \in \mathbb{R}$, if we take small enough t the inequality will not hold unless $u^T Av = 0$.

(c) Let $v_2, \ldots, v_n \in \Re^n$ be an orthonormal basis of $span(v)^{\perp}$ then since $v_2, \ldots, v_n \in span(v)^{\perp}$ then from section (b) we have

$$2 \leq i \leq n \quad , \quad v_i^T Av = 0$$

and so $v, v_1, v_2, \ldots, v_n$ is an orthonormal basis of $\Re^n$.
$Av = \langle Av, v\rangle v + \sum_{i=2}^{n}\langle Av, v_i\rangle v_i)$ since $v_i \in span(\{v\})^{\perp}$ then by section (b) $\langle Av, v_i\rangle = 0$ and so we get

$$Av = \langle Av, v\rangle v$$

Therefore v is an eigenvector of A.

(d) Let's suppose that $v_1, \ldots, v_k \in \Re^n$ are orthonormal eigenvectors of A and find v_{k+1}which is orthogonal to $v_1, \ldots, v_k$ and is an eigenvector of A.
Define $S = \{x \in span(\{v_1, \ldots, v_k\})^{\perp} : ||x|| = 1\}$, suppose that f is maximized on S by some vector v_{k+1}.

$$v_{k+1} \in span(\{v_1, \ldots, v_k\})^{\perp} \Rightarrow v_1, \ldots, v_k \in span(\{v_{k+1}\})^{\perp}$$

By section (b) we know that $1 \leq i \leq k$, $v_i^T v_{k+1} = 0$ and so $v_1, \ldots, v_k, v_{k+1} \in \Re^n$ are orthonormal vectors, we can complete these set to an orthonormal base of $\Re^n$: $v_1, \ldots, v_k, v_{k+1}, \ldots, v_n$, and so $v_1, \ldots, v_k, v_{k+2}, \ldots, v_n \in span(\{v_{k+1}\})^{\perp}$ and by section (b) again we know

$$1 \leq i \leq k, k+2 \leq i \leq n \ \ , \ \ v_i^T v_{k+1} = 0$$

$Av_{k+1} = \sum_{i=1}^{n} \langle Av_{k+1}, v_i \rangle v_i = \langle Av_{k+1}, v_{k+1} \rangle v_{k+1}$