MATHEMATICAL TOOLS - PROBLEM SET 10

Due Sunday, January 22nd, 23:55, either in the course mailbox or through the Moodle.

The words “polyhedron”, “polytope”, and “face” refer to the definitions from recitation.

Problem 1. Let $\Delta_n = conv\{e_1, \ldots, e_{n+1}\} \subseteq \mathbb{R}^{n+1}$, where $e_1, \ldots, e_{n+1}$ is the standard basis of $\mathbb{R}^{n+1}$. In recitation we described Δ_n as the finite intersection of half-spaces, thus demonstrating that it is a polytope. Let $S \subseteq \{e_1, \ldots, e_{n+1}\}$. Show that $conv(S)$ is a face of Δ_n.

Problem 2. Let $P_n = \{x \in \mathbb{R}^n : \|x\|_1 \leq 1\}$. P_n is known as the cross-polytope.

(1) Show that P_n is the intersection of finitely many half-spaces, and thus a polyhedron. Since P_n is bounded by definition, it is in fact a polytope.
(2) Find P_n's vertex set.
(3) Let V be P_n's vertex set. Show that $P_n = conv(V)$.
(4) Let $c \in \mathbb{R}^n$. Describe as explicitly as possible the set $M = \left\{x \in P_n : c^T x \text{ } is \text{ } maximal\right\}$.

Problem 3. Let $\{v_1, \ldots, v_n\} \subseteq \mathbb{R}^n$. Let $P = conv\{v_1, \ldots, v_n\}$. Let $c \in \mathbb{R}^n$. Show that there exists $i \in [n]$ s.t. $c^T v_i = \max_{x \in P} c^T x$.

Problem 4. Let $H = (X, Y, Z, T)$ be a finite, tripartite, three-uniform, hypergraph. In other words, X, Y, Z are disjoint finite sets (whose elements are called vertices) and $T \subseteq X \times Y \times Z$ is a collection of *triangles.*

A *vertex cover* of H is a set $U \subseteq X \cup Y \cup Z$ s.t. every triangle $t \in T$ contains at least one vertex in U.

We're interested in determining the size of the smallest vertex cover. Unfortunately, this problem is NP-hard so we're unlikely to find an efficient algorithm for it. In this question you'll show how linear programming can be used to find a vertex cover that is at most three times the optimal size.

A *fractional cover* of H is a weight function $w : X \cup Y \cup Z \to [0, \infty)$ s.t. for every triangle in T the sum of the weights of its vertices is at least 1. We want $w^* = \sum_{a \in X \cup Y \cup Z} w(a)$ to be minimal.

(1) Show that an optimal such w can be found efficiently, and that w^* is at most the size of the minimal vertex cover.
(2) Use w to construct a vertex cover U' of size at most $3w^*$, and conclude that U' is at most three times as large as the optimal vertex cover.

Problem 5. Recall that a linear program in standard form is: Maximize $c^T x$ for $x \geq 0$, $Ax = b$.

Consider the following LP:

Maximize $3x - y$

Subject to

$$\begin{aligned} x+y &\leq 3 \\ x-y &\leq 3 \\ -y &\leq 1 \\ y &\leq 1 \\ x+y &\geq 0 \\ x-y &\geq 0 \end{aligned}$$

(1) Solve the program.
(2) Present the program in standard form.