MATHEMATICAL TOOLS - SOLUTION 2

Problem 1. These properties follow straightforwardly from the definitions.

Problem 2.

(1) $\mathbb{E}[X] = \sum_{k=0}^{\infty} k e^{-\lambda} \frac{\lambda^k}{k!} = e^{-\lambda} \lambda \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} = \lambda$.

(2) $\mathbb{E}\left[X^2\right] = e^{-\lambda} \sum_{k=1}^{\infty} k^2 \frac{\lambda^k}{k!} = e^{-\lambda}\lambda + e^{-\lambda} \sum_{k=2}^{\infty} \lambda^k \left(\frac{1}{(k-2)!} + \frac{1}{(k-1)!}\right) = \lambda^2 + e^{-\lambda} \sum_{k=1}^{\infty} \lambda^k \frac{1}{(k-1)!} = \lambda^2 + \lambda$. Thus: $Var(X) = \mathbb{E}\left[X^2\right] - \mathbb{E}[X]^2 = \lambda$.

Problem 3. First, clearly $X+Y$ takes on only values in $\mathbb{N} \cup \{0\}$. Second, if $k \in \mathbb{N} \cup \{0\}$ then $\mathbb{P}[X+Y=k] = \sum_{\ell=0}^{k} \mathbb{P}[X=\ell \wedge Y=k-\ell] = \sum_{\ell=0}^{k} e^{-\alpha} \frac{\alpha^\ell}{\ell!} e^{-\beta} \frac{\beta^{k-\ell}}{(k-\ell)!} = e^{-(\alpha+\beta)} \frac{(\alpha+\beta)^k}{k!}$.

Problem 4. If $\mathbb{E}\left[Y^2\right] = 0$ then $Y = 0$ with probability one and equality holds.Otherwise it follows from taking $\alpha = -\frac{\mathbb{E}[XY]}{\mathbb{E}[Y^2]}$.

Problem 5. $\mathbb{E}[X]^2 = \mathbb{E}[X 1_X]^2 \leq \mathbb{E}\left[X^2\right] \mathbb{E}\left[1_X^2\right] = \mathbb{E}\left[X^2\right] \mathbb{P}[X > 0]$.

Problem 6.

(1) First note that $Z_n \leq 2^n$ with probability 1. We have:

$$\mathbb{E}[Z_n] = \sum_{k=0}^{2^{n-1}} \mathbb{P}[Z_{n-1} = k] k\mu = \mu \mathbb{E}[Z_{n-1}]$$

Since $\mathbb{E}[Z_n] = 1 = \mu^0$, if we assume inductively that $\mathbb{E}[Z_{n-1}] = \mu^{n-1}$, we conclude $\mathbb{E}[Z_n] = \mu^n$ for all n.

(2) If $p < \frac{1}{2}$ then $\mu < 1$, so $\lim_{n\to\infty} \mathbb{P}[Z_n > 0] \leq \lim_{n\to\infty} \mathbb{E}[Z_n] = \lim_{n\to\infty} \mu^n = 0$.

(3) $\mathbb{E}\left[Z_n^2\right] = \sum_{k=0}^{2^{n-1}} \mathbb{P}[Z_{n-1} = k] \left(2kp(1-p) + 4k^2p^2\right) = \sigma^2\mu^{n-1} + \mu^2 \mathbb{E}\left[Z_{n-1}^2\right]$. We have $Var[Z_0] = 0$. If we assume inductively that $Var[Z_{n-1}] = \sigma^2\left(\mu^{n-2} + \ldots + \mu^{2n-4}\right)$ then $\mathbb{E}\left[Z_n^2\right] = \sigma^2\left(\mu^{n-2} + \ldots + \mu^{2n-4}\right) + \mu^{2n-2}$. Thus:

$$\mathbb{E}\left[Z_n^2\right] = \sigma^2\mu^{n-1} + \mu^2\left(\sigma^2\left(\mu^{n-2} + \ldots + \mu^{2n-4}\right) + \mu^{2n-2}\right) = \sigma^2\left(\mu^{n-1} + \mu^n + \ldots + \mu^{2n-2}\right) + \mu^{2n}$$

And so:

$$Var[Z_n] = \mathbb{E}\left[Z_n^2\right] - \mathbb{E}[Z_n]^2 = \sigma^2\left(\mu^{n-1} + \mu^n + \ldots + \mu^{2n-2}\right)$$

(4) Well:

$$\mathbb{P}[Z_n > 0] \geq \frac{\mathbb{E}[Z_n]^2}{\mathbb{E}\left[Z_n^2\right]} = \frac{\mu^{2n}}{\sigma^2\left(\mu^{n-1} + \mu^n + \ldots + \mu^{2n-2}\right) + \mu^{2n}} = \frac{1}{1 + \sigma^2\left(\mu^{-n-1} + \mu^{-n} + \ldots + \mu^{-2}\right)}$$

Now, because $\mu > 1$, $\sum_{k=2}^{n-2} \mu^{-k}$ is uniformly bounded by some $C > 0$. Thus:

$$\mathbb{P}[Z_n > 0] \geq \frac{1}{1 + \sigma^2 C} > 0$$

(5) No. For all n, $\mathbb{P}[Z_n > 0] \leq \mathbb{P}[Z_n > 0] = 1 - (1-p)^2 < 1$.