MATHEMATICAL TOOLS - SOLUTION 5

Problem 1.

(1) From recitation we know that the solution space is of dimension $k+1$, so it's enough to show that the solutions $\{\lambda_i^n\}_{n=0}^{\infty}$ are linearly independent. For this it's enough to show that the vectors $\begin{pmatrix} 1 \\ \lambda_i \\ \dots \\ \lambda_i^k \end{pmatrix}$ are linearly independent, and this follows from problem 2.

(2) We prove this inductively:

$$\det(xI - A) = (x - \alpha_k)\det\begin{pmatrix} x & 0 & \dots & 0 \\ -1 & x & & \\ & -1 & \dots & \\ & & -1 & x \end{pmatrix} + \det\begin{pmatrix} -\alpha_{k-1} & \dots & & -\alpha_0 \\ -1 & x & & \\ & & \dots & \\ & & -1 & x \end{pmatrix}$$

$$= x^{k+1} - \alpha_k x^k - \alpha_{k-1}\det\begin{pmatrix} x & 0 & \dots & 0 \\ -1 & x & & \\ & -1 & \dots & \\ & & -1 & x \end{pmatrix} + \det\begin{pmatrix} -\alpha_{k-2} & \dots & & -\alpha_0 \\ -1 & x & & \\ & & \dots & \\ & & -1 & x \end{pmatrix}$$

$$= \dots = x^{k+1} - \alpha_k x^k - \dots - \alpha_1 x + \det(-\alpha_0) = p(x)$$

(3) The roots of the characteristic polynomial are $1, 2, 3, 4$, so the solution space is $span\left\{\{1\}_{n=0}^{\infty}, \{2^n\}_{n=0}^{\infty}, \{3^n\}_{n=0}^{\infty}, \{4^n\}_{n=0}^{\infty}\right\}$.

(4) $a_n = 1 + 2^n$

Problem 2.

(1) Take $p_i(x) = \prod_{i \neq j} \frac{(x-\alpha_j)}{(\alpha_i - \alpha_j)}$.

(2) If the α_i-s aren't distinct there are two identical rows so the determinant is 0. Otherwise: Note that if $p(x) = \sum_{i=0}^{k-1} c_i x^i$ then $A\begin{pmatrix} c_0 \\ c_1 \\ \dots \\ c_{k-1} \end{pmatrix} = \begin{pmatrix} p(\alpha_1) \\ p(\alpha_2) \\ \dots \\ p(\alpha_k) \end{pmatrix}$. So for each i, let $c_{i,0}, \dots, c_{i,k-1}$ be the coefficients of $p_i(x)$. Then:

$$A\begin{pmatrix} c_{1,0} & c_{2,0} & \dots & c_{k,0} \\ c_{1,1} & c_{2,1} & \dots & c_{k,1} \\ \dots & & & \dots \\ c_{1,k-1} & c_{2,k-1} & \dots & c_{k,k-1} \end{pmatrix} = I$$

Since A is invertible, it's determinant is non-zero.

Problem 3.

(1) This is a straightforward verification.
(2) For every $\lambda \in \mathbb{R}$, $D\left(e^{\lambda x}\right) = \lambda e^{\lambda x}$.
(3) Straightforward verification.
(4) Note that $\frac{d^i}{dx^i} e^{\lambda x} = D^i\left(e^{\lambda x}\right) = \lambda^i e^{\lambda x}$. Thus: $\sum_{i=0}^{k-1} \alpha_i D^i\left(e^{\lambda x}\right) = e^{\lambda x} \sum_{i=0}^{k-1} \alpha_i \lambda^i$. By assumption this is equal to $\lambda^k e^{\lambda x} = D^k\left(e^{\lambda x}\right)$, as desired.
(5) We can use the Vandermonde matrix to show linear independence.

Problem 4.

(1) This is really just a special case of problem 1. The characteristic polynomial is $x^n - 1$, whose roots are $e^{\frac{2\pi i}{n}k}, k = 0, 1, \dots, n-1$. The eigenvectors are $v_k = \begin{pmatrix} 1 \\ e^{\frac{2\pi i}{n}k} \\ e^{2\frac{2\pi i}{n}k} \\ e^{3\frac{2\pi i}{n}k} \\ \dots \\ e^{(n-1)\frac{2\pi i}{n}k} \end{pmatrix}$. Now, for $\ell \neq k$ (applying the formula for sum of geometric sequences):
$$\langle v_k, v_\ell \rangle = \sum_{j=0}^{n-1} e^{\frac{2\pi i}{n} j(k-\ell)} = \frac{\left(e^{\frac{2\pi i}{n}(k-\ell)}\right)^n - 1}{e^{\frac{2\pi i}{n}(k-\ell)}} = 0$$
.
(2) Let $\omega = e^{\frac{\pi i}{2}}$. Let $U_{k,\ell} = \frac{1}{2}\omega^{-k\ell}$. U is unitary: $UU*_{k,\ell} = \sum_{j=0}^{3} U_{k,j}\overline{U_{\ell,j}} = \frac{1}{4}\sum_{j=0}^{3} \omega^{j(\ell-k)}$. If $k = \ell$ this equals 1. Otherwise, by applying the formula for geometric sums, this equals 0. Furthermore, by what we've observed, column k of U is an eigenvector with eigenvalue ω^ℓ of T (with $n = 4$), so we may write:

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2}\omega^{-1} & \frac{1}{2}\omega^{-2} & \frac{1}{2}\omega^{-3} \\ \frac{1}{2} & \frac{1}{2}\omega^{-2} & \frac{1}{2} & \frac{1}{2}\omega^{-2} \\ \frac{1}{2} & \frac{1}{2}\omega^{-3} & \frac{1}{2}\omega^{-2} & \frac{1}{2}\omega^{-1} \end{pmatrix} \begin{pmatrix} 1 & & & \\ & \omega^1 & & \\ & & \omega^2 & \\ & & & \omega^3 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2}\omega & \frac{1}{2}\omega^2 & \frac{1}{2}\omega^3 \\ \frac{1}{2} & \frac{1}{2}\omega^2 & \frac{1}{2} & \frac{1}{2}\omega^2 \\ \frac{1}{2} & \frac{1}{2}\omega^3 & \frac{1}{2}\omega^2 & \frac{1}{2}\omega \end{pmatrix}$$