MATHEMATICAL TOOLS - SOLUTION 6

Problem 1. Sketch: Let $w_n \in \mathbb{R}^{d-1}$ be the projection of v_n onto the first $d-1$ coordinates. Then $\{w_n\}$ is ℓ_1-bounded, so by an inductive hypothesis contains a convergent subsequence w_{n_k}. Now, $\left\{v_{n_k}^d\right\} \subseteq \mathbb{R}$ (i.e. the dth coordinate of v_{n_k}) is a bounded sequence, so by the one-dimensional BW theorem contains a convergent subsequence $\left\{v_{n_{k_\ell}}^d\right\}$. A simple calculation shows that $\left\{v_{n_{k_\ell}}\right\}$ converges.

Problem 2.

(1) Let $v \in \mathbb{R}^n$. Then:

$$\|Dv\|_2 = \left\|\sum_{i=1}^n \langle v, e_i\rangle De_i\right\|_2 = \sqrt{\sum_{i=1}^n \langle v, e_i\rangle^2 \lambda_i^2} \leq \sqrt{\sum_{i=1}^n \langle v, e_i\rangle^2} \max_{i=1}^n |\lambda_i| = \|v\|_2 \max_{i=1}^n |\lambda_i|$$

and equality is attained for $v = e_1$.

(2) Let $v \in \mathbb{R}^n$. Then $\|Uv\|_2 = \sqrt{\langle Uv, Uv\rangle} = \sqrt{\langle U^T Uv, v\rangle} = \sqrt{\langle v, v\rangle} = \|v\|_2$. In particular:

$$\max_{x\in\mathbb{R}^n : \|x\|_2 = 1} \|UAx\|_2 = \max_{x\in\mathbb{R}^n : \|x\|_2=1} \|Ax\|_2$$

Problem 3.

(1) This is an immediate corollary of the result from recitation.

(2) Well, let $v \in \mathbb{R}^n$. Then:

$$\|v\|_1 = \sum_{i=1}^n |v_i| \leq \sqrt{\sum_{i=1}^n v_i^2 \sum_{i=1}^n 1} = \sqrt{n}\,\|v\|_2$$

where the inequality follows from Cauchy-Schwarz. This is tight, because for $v = (1, 1, \ldots, 1)^T$:

$$\|v\|_1 = n = \sqrt{n}\sqrt{n} = \sqrt{n}\sqrt{\sum_{i=1}^n 1} = \sqrt{n}\,\|v\|_2$$

For the other direction:

$$\|v\|_2 = \sqrt{\sum_{i=1}^n v_i^2} \leq \sqrt{\sum_{i=1}^n v_i^2 + \sum_{i\neq j} |v_i v_j|} = \sum_{i=1}^n |v_i| = \|v\|_1$$

And this is tight because $\|e_1\|_2 = \|e_1\|_1$.

(3) Let $A \in M_n(\mathbb{R})$. Let $x \in \mathbb{R}^n$ s.t. $\|x\|_2 = 1$. Then:

$$\|Ax\|_2 = \sqrt{\sum_{i=1}^n \left(\sum_{j=1}^n A_{i,j} x_j\right)^2} \leq \sqrt{\sum_{i=1}^n \left(\sum_{j=1}^n |A_{i,j} x_j|\right)^2}$$

Applying Cauchy-Schwarz:

$$\sqrt{\sum_{i=1}^{n}\left(\sum_{j=1}^{n}\left|A_{i,j}x_j\right|\right)^2} \leq \sqrt{\sum_{i=1}^{n}\left(\sum_{j=1}^{n}A_{i,j}^2\right)\left(\sum_{j=1}^{n}x_j^2\right)} = \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n}A_{i,j}^2} = \|A\|_F$$

Thus $\|A\|_{op} \leq \|A\|_F$, and this is tight: Let $A = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & \dots & & \dots \\ \dots & & & \\ 0 & & \dots & 0 \end{pmatrix}$.

Then $\|A\|_{op} = \|A\|_F = 1$.
For the other direction:

$$\|A\|_F = \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n}A_{j,i}^2} = \sqrt{\sum_{i=1}^{n}\|Ae_i\|_2^2} \leq \sqrt{n\|A\|_{op}^2} = \sqrt{n}\|A\|_{op}$$

And this is tight:

$$\|I\|_F = \sqrt{n} = \sqrt{n}\|I\|_{op}$$

Problem 4. Take, for example, the discrete metric on $\mathbb{R}^n$: $d(v,w) = \begin{cases} 1 & v \neq w \\ 0 & v = w \end{cases}$.
This clearly isn't induced by a norm: d can only take on $0,1$, whereas, by homogeneity, a metric induced by a norm can take on non-negative real value.

Problem 5. Standard tricks of inner product spaces...

Problem 6.

(1) Straightforward verification.

(2) First uniqueness: Assume $w, w' \in W, u, u' \in W^{\perp}$ and $w + u = w' + u' = v$. Then: $0 = \langle w + u - (w' + u'), w - w'\rangle = \langle w - w', w - w'\rangle + \langle u - u', w - w'\rangle = \langle w - w', w - w'\rangle = \|w - w'\|_2^2$. Thus $w = w'$. A similar calculation shows uniqueness of u.
Existence: Let $w_1, \dots, w_k$ be an orthonormal basis for W (exists by Grahm-Schmidt). Complete it with $w_{k+1}, \dots, w_n$ to an orthonormal basis for V (again, this is possible by Grahm-Schmidt). Note that for every $i \geq k+1$, $w_{k+1} \in W^{\perp}$. Let $v \in V$. We then have: $v = \sum_{i=1}^{k}\langle w_i, v\rangle w_i + \sum_{i=k+1}^{n}\langle w_i, v\rangle w_i \in W + W^{\perp}$.

(3) Let $w \in S^{\perp}$ and let $s \in span(S)$. Then $s = \sum \alpha_i s_i$ for $s_i \in S$. Thus $\langle w, s\rangle = \sum_i \alpha_i \langle w, s_i\rangle = 0$, and so $w \in span(S)^{\perp}$.
Conversely, let $w \in span(S)^{\perp}$. Then, in particular, since $S \subseteq span(S)$, $w \in S^{\perp}$.

(4) First, $\left(S^{\perp}\right)^{\perp} = \left(span(S)^{\perp}\right)^{\perp}$.
Let $s_1, \dots, s_k$ be an orthonormal basis for $span(S)$. Complete it with $s_{k+1}, \dots, s_n$ to an orthonormal basis for V. Note that by part 2, $s_{k+1}, \dots, s_n$ is an orthonormal basis for $S^{\perp}$. A symmetric argument shows that $s_1, \dots, s_k$ is an orthonormal basis for $\left(span(S)^{\perp}\right)^{\perp}$. But then $span(S) = \left(span(S)^{\perp}\right)^{\perp}$ since they have the same basis.