MATHEMATICAL TOOLS - PROBLEM SET 6

Due Monday, December 26th, 23:55, either in the course mailbox or through the Moodle.

High-Dimensional Bolzano-Weierstrass.

Problem 1. Prove the BW++ theorem from recitation: Let $\{v_n\}_{n\in\mathbb{N}} \subseteq \mathbb{R}^d$, and assume there exists some $M > 0$ s.t. for all n, $\|v_n\|_1 < M$. Then v_n has an ℓ_1-convergent subsequence (in other words there exists $v \in \mathbb{R}^d$ and a subsequence $\{v_{n_k}\}_{k\in\mathbb{N}} \subseteq \{v_n\}_{n\in\mathbb{N}}$ s.t. $\lim_{k\to\infty} \|v_{n_k} - v\|_1 = 0$).

You may rely on the one-dimensional Bolzano-Weierstrass theorem (i.e. the theorem above with $d = 1$).

Norms.

For $A \in M_n(\mathbb{R})$, let

$$\|A\|_{op} = \max_{x\in\mathbb{R}^n:\|x\|_2=1} \|Ax\|_2$$

and

$$\|A\|_F = \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} A_{i,j}^2}$$

Problem 2.

(1) Show that if $D = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & \dots \\ \dots & \dots & \dots & 0 \\ 0 & \dots & 0 & \lambda_n \end{pmatrix}$ then $\|D\|_{op} = \max_{i=1}^n |\lambda_i|$.

(2) Show that if $U \in M_n(\mathbb{R})$ is orthogonal (i.e. $UU^T = I$) then for every $A \in M_n(\mathbb{R})$, $\|UA\|_{op} = \|A\|_{op}$.

Problem 3. In recitation we proved that if $|\cdot|$ is a norm on $\mathbb{R}^n$, then there exist $C, D > 0$ s.t. for all $v \in \mathbb{R}^n$, $C\|v\|_1 \le |v| \le D\|v\|_1$.

(1) Prove that if $|\cdot|_1, |\cdot|_2$ are norms on $\mathbb{R}^n$, there are $C, D > 0$ s.t. for all $v \in \mathbb{R}^n$, $C|v|_1 \le |v|_2 \le D|v|_1$.

(2) Find the best possible constants for ℓ_1, ℓ_2 on $\mathbb{R}^n$. In other words, find $C_n, D_n > 0$ s.t. for all $v \in \mathbb{R}^n$, $C_n\|v\|_1 \le \|v\|_2 \le D_n\|v\|_1$, and there exist $0 \ne v, w \in \mathbb{R}^n$ s.t. $C_n\|v\|_1 = \|v\|_2, \|w\|_2 = D_n\|w\|_1$.

(3) Find the best possible constants $C_n, D_n > 0$ s.t. for all $A \in M_n(\mathbb{R})$, $C_n\|A\|_F \le \|A\|_{op} \le D_n\|A\|_F$.

Not all Norms are Induced by Metrics. Let V be a vector space and let d be a metric on V and let $\|\cdot\|$ be a norm on V. We say that d *is induced by* $\|\cdot\|$ if for all $u, v \in V$, $d(u,v) = \|u - v\|$.

Problem 4. Show that there exists a metric d on $\mathbb{R}^n$ that isn't induced by a norm.

Orthonormality and Inner Products. Let $V = \mathbb{R}^n$ and let $\langle \cdot, \cdot \rangle$ be the standard inner product on V. A set of vectors $v_1, \ldots, v_k \in V$ are called orthonormal if for all i, j, $\langle v_i, v_j \rangle = \delta_{i,j}$ (where $\delta_{i,j}$ is the *Kronecker delta*).

For $S \subseteq V$, we set: $S^{\perp} = \{v \in V : \forall s \in S, \langle v, s \rangle = 0\}$.

Problem 5. The Grahm-Schmidt process: Let $v_1, \ldots, v_k \in V$ be linearly independent. Show that there exist orthonormal $w_1, \ldots, w_k \in V$ s.t. for all $1 \leq i \leq k$, $\mathit{span}\{w_1, \ldots, w_i\} = \mathit{span}\{v_1, \ldots, v_i\}$.

Hint/partial solution: Assume $w_1, \ldots, w_\ell$ have been defined. Set $u_{\ell+1} = v_{\ell+1} - \sum_{i=1}^{\ell} \langle v_{\ell+1}, w_i \rangle w_i$ and $w_{\ell+1} = \frac{u_{\ell+1}}{\sqrt{\langle u_{\ell+1}, u_{\ell+1} \rangle}}$.

Problem 6. Let $S \subseteq V$.

(1) Show that $S^{\perp}$ is a linear subspace of V.
(2) Let $W \subseteq V$ be a subspace. Show that for every $v \in V$ there exist unique $w \in W, u \in W^{\perp}$ s.t. $v = w + u$.
Hint: Use Grahm-Schmidt to produce orthonormal bases for W and $W^{\perp}$.
(3) Show that $(\mathit{span}(S))^{\perp} = S^{\perp}$.
(4) Show that $\left(S^{\perp}\right)^{\perp} = \mathit{span}(S)$.