MATHEMATICAL TOOLS - SOLUTION 1

Problem 1.

(1) By definition:

$$\mathbb{E}[XY] = \sum_{\omega\in\Omega} \mathbb{P}[\omega] X(\omega) Y(\omega) = \sum_{x\in Im(x)} \sum_{\omega\in X^{-1}(x)} \mathbb{P}[\omega] xY(\omega) = \sum_{x\in Im(x)} x \sum_{\omega\in X^{-1}(x)} \mathbb{P}[\omega] Y(\omega)$$

$$\sum_{x\in Im(x)} x \sum_{y\in Im(Y)} \mathbb{P}[X = x \wedge Y = y] y$$

By independence:

$$= \sum_{x\in Im(x)} x \sum_{y\in Im(Y)} \mathbb{P}[X = x] \mathbb{P}[Y = y] y = \sum_{x\in Im(x)} x\mathbb{P}[X = x] \sum_{y\in Im(Y)} \mathbb{P}[Y = y] y = \sum_{x\in Im(x)} x\mathbb{P}[X = x] \mathbb{E}[Y$$

(2) Let X, Y be given by the following probability density:

	$X = -1$	$X = 0$	$X = 1$
$Y = -1$	0	$\frac{1}{6}$	0
$Y = 1$	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{3}$

Now, X, Y aren't independent because (for example) $\mathbb{P}[X = 1, Y = 1] = \frac{1}{3} \neq \frac{1}{3} \cdot \frac{5}{6} = \mathbb{P}[X = 1] \mathbb{P}[Y = 1]$. On the other hand $\mathbb{E}[XY] = 0$, and since $\mathbb{E}[X] = 0$ so is $\mathbb{E}[X]\mathbb{E}[Y]$.

(3) We have:

$$Var\left[\sum_{i=1}^{n} X_i\right] = \mathbb{E}\left[\left(\sum_{i=1}^{n} X_i\right)^2\right] - \left(\mathbb{E}\left[\sum_{i=1}^{n} X_i\right]\right)^2 = \sum_{i\neq j} \mathbb{E}[X_iX_j] + \sum_{i=1}^{n} \mathbb{E}\left[X_i^2\right] - \sum_{i=1}^{n} \mathbb{E}[X_i]^2 - \sum_{i\neq j} \mathbb{E}[X_i]\mathbb{E}[X_j]$$

$$= \sum_{i=1}^{n} Var[X_i]$$

where the last equality follows from the previous clause,

(4) Let $X, Y \sim Ber\left(\frac{1}{2}\right)$ independently, and $Z = X \oplus Y$ (that is, Z is the indicator of $X \neq Y$). So $Z \sim Ber\left(\frac{1}{2}\right)$. Now, $\mathbb{P}[X = Y = Z = 1] = 0 \neq \frac{1}{2^3}$, so the random variables aren't independent. But they are pairwise independent: X, Y are independent by construction, and $\mathbb{P}[Z = 1 \wedge X = 1] = \mathbb{P}[X = 1 \wedge Y = 0] = \frac{1}{4}$, and similar calculations hold for the other possibilities.

Problem 2.

Problem 3.

(1) We can use Chebychev's inequality: Note that $\mathbb{E}[X_n] = pn$, $\sigma_n^2 = Var[X_n] = np(1-p)$. Now:

$$\lim_{n\to\infty} \mathbb{P}[X_n \leq \alpha n] \leq \lim_{n\to\infty} \mathbb{P}[|X_n - pn| \geq (p-\alpha) n] \leq \lim_{n\to\infty} \frac{\sigma_n^2}{(p-\alpha)^2 n^2} = \lim_{n\to\infty} \frac{np(1-p)}{(p-\alpha)^2 n^2} = 0$$

A similar calculation hold for $\mathbb{P}[X_n \geq \beta n]$.

(2) By Chebychev:

$$\mathbb{P}\left[X_{100} \leq \frac{1}{4}100\right] \leq \frac{100\frac{1}{4}}{\left(\frac{100}{4}\right)^2} = \frac{4}{100} = \frac{1}{25} = 0.04 = 4 \cdot 10^{-2}$$

(3) $\mathbb{P}\left[X_{100} \leq 25\right] \approx 2.8 \cdot 10^{-7}$. Obviously, this result is orders of magnitude off from the one from the previous clause. The reason is that Chebychev applies to *all* random variables. In our case, we're summing 100 i.i.d. random variables. This is a very strong condition, and allows us to obtain better bounds, such as Chernoff, which we'll see later in the course.

(4) $A_n \sim Bin\left(\binom{n}{2}, p\right)$, so this follows immediately from part 1, with $\mathbb{E}\left[A_n\right] = \binom{n}{2} p$.

Problem 4.

(1) Note that $\mathbb{P}\left[T\left(G_n\right) = 0\right] = 1 - \mathbb{P}\left[T\left(G_n\right) \geq 1\right]$. We have: $\mathbb{E}\left[T\left(G_n\right)\right] = \binom{n}{3}\frac{1}{n^3} < \frac{1}{6}$. Thus, by Markov, $\mathbb{P}\left[T\left(G_n\right) \geq 1\right] < \frac{1}{6} \implies \mathbb{P}\left[T\left(G_n\right) = 0\right] > \frac{5}{6}$.

(2) $A_n \sim Bin\left(\binom{n}{2}, \frac{1}{n}\right)$. Thus $\mathbb{E}\left[A_n\right] = \frac{n-1}{2}$ and $Var\left[A_n\right] \leq \frac{n-1}{2}$.We'll use Chebychev: $\mathbb{P}\left[A_n \leq \frac{n}{2} - \sqrt{n}\right] \leq \mathbb{P}\left[\left|A_n - \frac{n-1}{2}\right| \geq \sqrt{n} - \frac{1}{2}\right] \leq \frac{n-1}{2\left(\sqrt{n}-\frac{1}{2}\right)^2} \leq \frac{5}{6}$, where the last inequality follows from elementary calculus.

(3) Using a union bound, $\mathbb{P}\left[T\left(G_n\right) = 0 \wedge A_n \geq \frac{n}{2} - \sqrt{n}\right] \geq 1 - \mathbb{P}\left[T\left(G_n\right) > 0\right] - \mathbb{P}\left[A_n < \frac{n}{2} - \sqrt{n}\right] > 1 - \frac{1}{6} - \frac{5}{6} = 0$.

(4) Take a complete bipartite graph with parts of $\left\lfloor\frac{n}{2}\right\rfloor$ and $\left\lceil\frac{n}{2}\right\rceil$ vertices. Then there are no triangles (as any three vertices have at least two on the same side), and $\left\lfloor\frac{n}{2}\right\rfloor\left\lceil\frac{n}{2}\right\rceil \geq \frac{n^2}{4} - \frac{1}{4}$ edges.