Mathematical Methods in Computer Science

Home Exam

Lecturer: Nati Linial *Hand in date: February 18, 2006*

Note: You are required to submit a signed letter stating you solved the exam by yourself (A form letter will be posted on the course's homepage). You are allowed to use any written material but you must not consult with any other person. Give a full account of your solutions.
3 points course: Answer 4 questions, at least one question from each subject.
4 points course: Answer 5 questions, at least one question from each subject. You may answer a bonus (sixth) question, and the best five will be considered in the final grade.

1. **Fourier Analysis**

   (a) Let $L \subset \{0,1\}^n$ be an affine subspace. Consider the indicator function $f_L(\mathbf{x}) = 1$ if $\mathbf{x} \in L$ and zero otherwise. Determine $\hat{f}$, the Fourier transform of f. (Hint: Consider first the case when L is a *linear* subspace.)

   (b) Consider $f : \mathbb{Z}/n\mathbb{Z} \to \mathbb{R}$ and the function $g(n) = f(n)$ and $g(i) = f(i) + f(i+1)$ for $i < n$. Express $\hat{g}$ as a function of $\hat{f}$.

2. **Linear Algebra**

   (a) i. Let $G = (V, E)$ be a d-regular graph on n vertices, and let $\lambda_{\min}$ be the least eigenvalue of (the adjacency matrix of) G. We recall that an *anticlique* is a set of vertices no two of which are adjacent. Show that no anticlique in G has cardinality exceeding $-n\lambda_{\min}/(d - \lambda_{\min})$.

   ii. Let A be a real symmetric matrix with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Suppose N is an $m \times n$ real matrix such that $NN^\top = I_m$, where $m < n$. Let $B = NAN^\top$ and let $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_m$ be the eigenvalues of B. Show that for all $1 \leq i \leq m$

   $$\lambda_i \geq \mu_i \geq \lambda_{n-m+i}$$

   iii. Prove that the cardinality of an anticlique cannot exceed the number of non-negative eigenvalues, or the number of non-positive eigenvalues of G.

   (b) What can you say about the largest cardinality m of a subset $\{\mathbf{x}_1, ..., \mathbf{x}_m\} \subset \mathbb{R}^n$ such that for all $i \neq j$: $\|\mathbf{x}_i - \mathbf{x}_j\|_2 \in \{\delta_1, \delta_2\}$, for some two reals δ_1, δ_2?

   i. Show that there is such a set of cardinality $m \geq \binom{n}{2}$

   ii. Prove that no such set exists with cardinality exceeding $\frac{(n+4)(n+1)}{2}$. Hint: Define the polynomial $F(\mathbf{y}, \mathbf{z}) = (\|\mathbf{y} - \mathbf{z}\|_2^2 - \delta_1^2)(\|\mathbf{y} - \mathbf{z}\|_2^2 - \delta_2^2)$ and the polynomials $f_i(\mathbf{y}) = F(\mathbf{y}, \mathbf{x}_i)$. Consider the vectors space spanned by the polynomials $f_i(\mathbf{y})$ over $\mathbb{R}$ (why it is a vector space?) and show that $\frac{(n+4)(n+1)}{2}$ is an upper bound for its dimension.

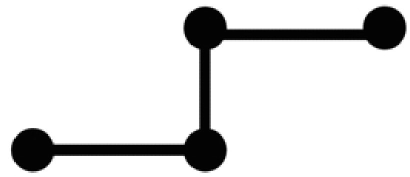

Figure 1: A simple path on 4 vertices

3. **Probability**

   (a) Consider the probability space of random graphs $G(n, p)$. The random variable $X(G)$ counts the number of simple paths on 4 (distinct) vertices in the graph G. What is the expectation and variance of $X(G)$?

   (b) Choose independently and uniformly $m = (1 + \epsilon)^n$ binary vectors $\mathbf{x}_1, ..., \mathbf{x}_m$ of length n. Prove that for $\epsilon = 0.1$ it holds with probability greater than zero that every pair of vectors are far, i.e., $\|\mathbf{x}_i - \mathbf{x}_j\|_1 \geq \epsilon n$ for every $i \neq j$. What is the largest ϵ for which you can prove this statement.

4. **Convex Optimization**

   (a) Will be published tomorrow.

   (b) Will be published tomorrow.