1. Since we have seen in class the LP and SoCP are special cases of SDP then if we show that SDP is a special case of Linear Conic Programming then we will have shown that LP SoCP are also special cases of Linear Conic Programming.
   Linear Conic Programming problem:

$$\begin{aligned} &\min_x \ c^T x \\ s.t.\ &f_i(x) \in K, \forall i \in \{1, ..., m\} \\ &h_i(x) = 0 \forall i \in \{1, ..., k\} \end{aligned}$$

   SDP problem:

$$\begin{aligned} &\min_x = c^T x \\ s.t.\ \ &F_0 + \sum x_i F_i \succeq 0 \\ &Ax = b \end{aligned}$$

   S_+^n is a proper cone, therefore if we define:

$$f(x) = F_0 + \sum x_i F_i \in S_+^n, \ \ h(x) = Ax - b$$

   then the following Linear Conic Programming problem is equivalent to an SDP problem:

$$\begin{aligned} &\min_x = c^T x \\ s.t.\ \ &f(x) \in S_+^n \\ &h(x) = 0 \end{aligned}$$

2.

   (a) LP:

$$\begin{aligned} &\min_x \quad c^T x \\ &s.t \quad Ax = b \\ &\qquad\ \ Fx \le d \end{aligned}$$

   Let's define the set:

$$\mathcal{A}' = \left\{ A \in \mathbb{R}^{nxm} : A_{ij} \in \left\{ \hat{A}_{ij} - V_{ij} \ , \ \hat{A}_{ij} + V_{ij} \right\} \right\}$$

   And the following linear problem:

$$\begin{aligned} &\min_x \quad c^T x \\ &s.t \quad\ Ax \preceq b \ \ \forall A \in \mathcal{A}' \end{aligned}$$

Now given that x satisfies the conditions of the linear problem, since $\mathcal{A}^{'} \subseteq \mathcal{A}$ then clearly x satisfies the conditions of the robust LP.
Now suppose that x satisfies the conditions of this linear problem, that is $Ax \preceq b \ \ \forall A \in \mathcal{A}^{'}$.
Let's take some B in $\mathcal{A}$, $\hat{A}_{ij} - V_i \leq B_{ij} \leq \hat{A}_{ij} + V_i$.
Let's consider the following element in $\mathcal{A}^{'}$:

$$\forall i \in [n], j \in [m], \, C^{'}_{ij} = \begin{cases} \hat{A}_{ij} + V_{ij} & x_j \geq 0 \\ \hat{A}_{ij} - V_{ij} & x_j < 0 \end{cases}$$

Since we know that $C^{'}x \preceq b \Rightarrow \left[C^{'}x\right]_i \leq b_i \Rightarrow \sum_{j=1}^{m} C^{'}_{ij}x_j \leq b_i \Rightarrow \sum_{j=1}^{m} C^{'}_{ij}x_j \leq b_i$

$$B_{ij}x_j \leq C^{'}_{ij}x_j \Rightarrow [Bx]_i = \sum_{j=1}^{m} B_{ij}x_j \leq \sum_{j=1}^{m} C^{'}_{ij}x_j \leq b_i$$

(b) $\sum\limits_i x_i P_i + G \preceq 0$, Since P_i and G are simultaneously diagonalizable there exists $Q \in M_{nxn}(\mathbb{R})$ such that :
$Q^{-1}P_iQ = D_i$, $Q^{-1}GQ = F$

$$\sum_i x_i P_i + G \preceq 0 \Leftrightarrow Q^{-1}\left(\sum_i x_i P_i + G\right) Q \preceq 0 \Leftrightarrow \sum_i x_i Q^{-1}P_iQ + Q^{-1}GQ \preceq 0 \Leftrightarrow \sum_i x_i D_i + F \preceq 0$$

We know that a symmetric matrix is PSD iff it's eigenvalues are all non-negative, and since require a diagonal matrix to be a PSD this is equivalent to it's diagonal elements being non-negative:

$$-\sum_i x_i D_i - F \succeq 0 \ \Leftrightarrow \ \forall j \ -\sum_i x_i \left[D_i\right]_{jj} - \left[F_i\right]_{jj} \geq 0$$

And so the given problem is equivalent to the following problem:

$$\begin{array}{ll} \underset{x}{min} & c^T x \\ s.t. & \forall j \ -\sum_i x_i \left[D_i\right]_{jj} - \left[F_i\right]_{jj} \geq 0 \end{array}$$

3. For the l_2 norm:

$$\begin{array}{ll} \underset{x}{max} & \|x\|_2 \\ s.t. & Ax = b \end{array}$$

Define the vector $z \in \mathbb{R}^{2n}$, $z = \begin{bmatrix} R(x) & I(x) \end{bmatrix}$

$$\|x\|_2 = \left(\sum_{i=1}^{n} \left(R^2(x_i) + I^2(x_i) \right) \right)^{\frac{1}{2}} = \left(\sum_{i=1}^{n} \left(R^2(x_i) + \sum_{i=1}^{n} I^2(x_i) \right) \right)^{\frac{1}{2}} = \|z\|_2$$

Additionally:
$Ax = b \Leftrightarrow Re(A) \cdot Re(x) - Im(A) \cdot Im(x) = Re(b) \wedge Re(A) \cdot Im(x) + Im(A) \cdot Re(x) = Im(b) \Leftrightarrow$
$\Leftrightarrow Re(A) [z]_{1..n} - Im(A) [z]_{n+1..2n} = Re(b) \wedge Re(A) [z]_{n+1..2n} + Im(A) [z]_{1..n} = Im(b)$

Define the matrix $A' = \begin{bmatrix} Re(A) & -Im(A) \\ Im(A) & Re(A) \end{bmatrix}$ and the vector $b^{'} = \begin{bmatrix} Re(b) \\ Im(b) \end{bmatrix}$ and we get that:

$$A^{'} z = b^{'}$$

$$\begin{aligned} \max_{x} \quad & \|x\|_2 \qquad \Leftrightarrow \\ s.t. \quad & Ax = b \end{aligned}$$

$$\begin{aligned} \max_{z} \quad & \|z\|_2 \qquad \Leftrightarrow \\ s.t. \quad & A^{'} z = b^{'} \end{aligned}$$

$$\begin{aligned} \min_{z} \quad & - \|z\|_2 \qquad \Leftrightarrow \\ s.t. \quad & A^{'} z = b^{'} \end{aligned}$$

$$\begin{aligned} \min_{t,z} \quad & t \qquad \Leftrightarrow \\ s.t. \quad & A^{'} z = b^{'} \\ & t = - \|z\|_2 \end{aligned}$$

$$\begin{aligned} \min_{t,z} \quad & t \\ s.t. \quad & A^{'} z = b^{'} \\ & \|z\|_2 \geq -t \end{aligned}$$

For l_1 norm:
Define the vectors $z_i \in \mathbb{R}^2$, $z_i = \begin{bmatrix} R(x_i) & I(x_i) \end{bmatrix}$ and the vector $z \in \mathbb{R}^{2n} = \begin{bmatrix} R(x) & I(x) \end{bmatrix}$

$$z_i \in \mathbb{R}^2, z_i = \begin{bmatrix} R(x_i) & I(x_i) \end{bmatrix}$$

$\|x\|_1 = \sum_{i=1}^{n} |x_i| = \sum_{i=1}^{n} \sqrt{Re^2(x_i) + Im^2(x_i)} = \sum_{i=1}^{n} \|z_i\|_2$
$Ax = b \Leftrightarrow Re(A)\cdot[z_i]_1 - Im(A)\cdot[z_i]_2 = Re(b) \wedge Re(A)\cdot[z_i]_2 + Im(A)\cdot[z_i]_1 = Im(b)$

Define the matrix $A' = \begin{bmatrix} Re(A) & -Im(A) \\ Im(A) & Re(A) \end{bmatrix}$ and the vector $b^{'} = \begin{bmatrix} Re(b) \\ Im(b) \end{bmatrix}$ and we get that:

$$\begin{aligned} \underset{x}{max} \quad & \|x\|_1 \qquad \Leftrightarrow \\ s.t. \quad & Ax = b \end{aligned}$$

$$\begin{aligned} \underset{z_i}{min} \quad & -\sum_{i=1}^{n} \|z_i\|_2 \qquad \Leftrightarrow \\ s.t. \quad & Re(A)\,[z]_1 - Im(A)\,[z]_2 = Re(b) \\ & Re(A)\,[z]_2 - Im(A)\,[z]_1 = Im(b) \end{aligned}$$

$$\begin{aligned} \underset{z_i, t_i}{min} \quad & -\sum_{i=1}^{n} t_i \qquad \Leftrightarrow \\ s.t. \quad & t_i \leq \|z_i\|_2 \\ & A^{'} z = b^{'} \end{aligned}$$

For l_∞ norm:
Define the vectors $z_i \in \mathbb{R}^2$, $z_i = \begin{bmatrix} R(x_i) & I(x_i) \end{bmatrix}$,and the vector $z \in \mathbb{R}^{2n} = \begin{bmatrix} R(x) & I(x) \end{bmatrix}$
$\|x\|_\infty = max\,|x_i| = max\,\|z_i\|_2$

$$\begin{aligned} \underset{x}{max} \quad & \|x\|_\infty \qquad \Leftrightarrow \\ s.t. \quad & A^{'} z = b^{'} \end{aligned}$$

$$\begin{aligned} \underset{x}{max} \quad & \|x\|_\infty \qquad \Leftrightarrow \\ s.t. \quad & A^{'} z = b^{'} \end{aligned}$$

$$\begin{aligned} \underset{z_i}{max} \quad & max\,\|z_i\|_2 \qquad \Leftrightarrow \\ s.t. \quad & A^{'} z = b^{'} \end{aligned}$$

$$\begin{aligned} \max_{z_i} \quad & \|z_i\|_2 & \Leftrightarrow \\ s.t. \quad & A^{'}z = b^{'} \end{aligned}$$

$$\begin{aligned} \min_{z_i} \quad & -\|z_i\|_2 & \Leftrightarrow \\ s.t. \quad & A^{'}z = b^{'} \end{aligned}$$

$$\begin{aligned} \min_{z_i, t_i} \quad & t_i & \Leftrightarrow \\ s.t. \quad & A^{'}z = b^{'} \\ & t_i \geq -\|z_i\|_2 \end{aligned}$$

4. $\min_{x} \quad \max_{\delta A, \delta b : \left\| \begin{matrix} \delta A & \delta b \end{matrix} \right\|_F \leq 1} \quad \|(A+\delta A)x - (b-\delta b)\|_2$

$(A+\delta A)x = A(1+\delta)x = A\delta\frac{1+\delta}{\delta}x$, $(b-\delta b) = b\delta\frac{1-\delta}{\delta}$

define $A^{'} = \begin{bmatrix} \delta A & \delta b \end{bmatrix}$, $q = \begin{bmatrix} \frac{1+\delta}{\delta}x \\ -\frac{1-\delta}{\delta} \end{bmatrix}$ and it holds that :

$$\left\|A^{'}q\right\|_2 = \|(1+\delta)Ax - b(1-\delta)\|_2 = \|(A+\delta A)x - (b-\delta b)\|_2 \Rightarrow$$

$$\max_{\delta A, \delta b : \left\| \begin{matrix} \delta A & \delta b \end{matrix} \right\|_F \leq 1} \quad \|(A+\delta A)x - (b-\delta b)\|_2 \quad =$$

$$\max_{\delta A, \delta b : \left\| \begin{matrix} \delta A & \delta b \end{matrix} \right\|_F \leq 1} \quad \left\|A^{'}q\right\|_2 \quad = \|q\|_2$$

Where the optimal A^* is given by:$A^* = \begin{bmatrix} \delta A & \delta b \end{bmatrix} = \frac{uq^T}{\|u\|_2\|q\|_2}$