MATHEMATICAL TOOLS - SOLUTION 3

Problem 1.

(1) In this case we know that a.a.s. $G(n,p)$ contains isolated vertices (we saw this in recitation). Clearly, having isolated vertices precludes the possibility of finding a perfect matching.

(2) Assume H doesn't satisfy the marriage condition. Let $W \subseteq U$ be a set of minimal size s.t. $|N_H(W)| < |W|$. We claim that $|N_H(W)| = |W| - 1$. Indeed, if this isn't the case, let $w \in W$ and set $W' = W \setminus \{w\}$. We then have $N_H(W') \subseteq N_H(W) \implies |N_H(W')| \leq |N_H(W)| < |W| - 1 = |W'|$. Thus W' violates the marriage condition, but this contradicts our choice of W as having minimal size with this property.

Problem 2. For every $i \in [n]$, let X_i be the indicator of the event $\pi(i) = i$, and let $X = \sum_{i=1}^{n} X_i$ be the number of fixed points. Then $\mathbb{P}[X_i = 1] = \frac{1}{n}$. Thus $\mathbb{E}[X] = \sum_{i=1}^{n} \mathbb{E}[X_i] = 1$. For the variance, note that $\mathbb{E}[X^2] = \sum_{i \neq j} \mathbb{E}[X_i X_j] + \sum_{i=1}^{n} \mathbb{E}[X_i^2]$. Now, for $i \neq j$, the probability that $\pi(i) = i$ and $\pi(j) = j$ is $\frac{1}{n(n-1)}$. Thus $\mathbb{E}[X^2] = n(n-1)\frac{1}{n(n-1)} + 1 = 2$. Therefore $Var[X] = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = 2 - 1 = 1$.

Problem 3.

(1) Take $\Omega = [m]^n$, and the uniform distribution.

(2) $X_i \sim Bin\left(n, \frac{1}{m}\right)$.

(3) No. One way to see this is to note that $\mathbb{P}[X_1 = n \wedge X_2 = n] = 0 \neq m^{-2n} = \mathbb{P}[X_1 = n]\,\mathbb{P}[X_2 = n]$.

(4) Let $\delta \in (0,1)$. By Chernoff, we know that for all i, $\mathbb{P}\left[\left|X_i - \frac{n}{m}\right| \geq \delta\frac{n}{m}\right] \leq 2\exp\left(-\frac{\delta^2}{3}\frac{n}{m}\right)$. If we take $\delta = 3\sqrt{\frac{m\ln m}{n}}$, then $\delta\frac{n}{m} = 3\sqrt{\frac{n\ln m}{m}}$, and we have $\mathbb{P}\left[\left|X_i - \frac{n}{m}\right| \geq 3\sqrt{\frac{n\ln m}{m}}\right] \leq 2\exp\left(-\frac{9n\ln m}{3m}\frac{n}{m}\right) = 2\exp(-3\ln m) = \frac{2}{m^3}$. Now:

$$\mathbb{P}\left[\exists i \in [m], \left|X_i - \frac{n}{m}\right| \geq 3\sqrt{\frac{n\ln m}{m}}\right] \leq m\frac{2}{m^3} = \frac{2}{m^2}$$

Thus:

$$\mathbb{P}\left[\forall i \in [m], \left|X_i - \frac{n}{m}\right| < 3\sqrt{\frac{n\ln m}{m}}\right] \geq 1 - \frac{2}{m^2}$$

Problem 4.

(1) f is clearly continuous on $(0,1)$. Note that for all $x \in (0,1)$, $f(1-x) = f(x)$. Thus it's enough to show that $\lim_{x\to 0^+} f(x) = 0$. Well, clearly $\lim_{x\to 0}(1-x)\log_2(1-x) = 0$. In order to analyze $\lim_{x\to 0^+} -x\log_2 x$, we'll apply L'Hospital's rule:

$$\lim_{x\to 0^+} x\log_2\left(\frac{1}{x}\right) = \frac{1}{\ln 2}\lim_{x\to\infty}\frac{1}{x}\ln x$$

$$\frac{d}{dx}\ln x=\frac{1}{x}\to 0,\frac{d}{dx}x=1$$

Therefore:

$$\lim_{x\to\infty}\frac{1}{x}\ln x=0$$

and so $\lim_{x\to 0^+} f(x)=0$, and f is continuous.

(2) f is continuously differentiable on $[0,1]$, but we'll use only the more obvious fact that f is differentiable on $(0,1)$. We have: $f(0)=f(1)=0$. $f'(x)=-\frac{1}{\ln 2}(\ln x+1-\ln(1-x)-1)=\log_2\left(\frac{1-x}{x}\right)$. Thus $f'(x)=0 \iff \frac{1-x}{x}=1 \iff x=\frac{1}{x}$. In this case we have $f\left(\frac{1}{2}\right)=1$.

Problem 5.

(1) $\mathbb{E}\left[e^{tX}\right]=e^{-\lambda}\sum_{k=0}^{\infty}\frac{\lambda^k}{k!}e^{tk}=e^{-\lambda}e^{\lambda e^t}=\exp\left(\lambda\left(e^t-1\right)\right)$.

(2) This is an immediate application of Markov.

(3) Take $t=\ln\left(\frac{k}{\lambda}\right)$ and plug it into the inequality from the previous clause.