MATHEMATICAL TOOLS - SOLUTION 4

Problem 1.

(1) We know that $e(n,p) \sim Bin\left(\binom{n}{2}, p\right)$. Thus,

$$\mathbb{P}[e(n,p)=0] = \binom{\binom{n}{2}}{0} p^0 (1-p)^{\binom{n}{2}} = \left(1-\frac{c}{n^2}\right)^{\frac{n^2}{2}-\frac{n}{2}}$$

We have $\lim_{n\to\infty}\left(1-\frac{c}{n^2}\right)^{\frac{n^2}{2}} = e^{-\frac{c}{2}}$ and $\lim_{n\to\infty}\left(1-\frac{c}{n^2}\right)^{-\frac{n}{2}} = 1$. Thus $\lim_{n\to\infty}\mathbb{P}[e(n,p)=0] = e^{-\frac{c}{2}}$.

(2) Note that if $e(n,p) = 0 < \frac{1}{2}\frac{n^2}{2}p$, then $e(n,p) \notin \left(1 \pm \frac{1}{2}\right)\frac{n^2}{2}p$. Let $\{n_k\}_{k=1}^{\infty}$ be a subsequence of indices s.t. $p(n_k) \leq \frac{c}{n_k^2}$ for all k. Then:

$$\limsup_{n\to\infty}\mathbb{P}[e(n,p(n))=0] \geq \limsup_{k\to\infty}\mathbb{P}[e(n_k,p(n_k))=0] \geq \lim_{k\to\infty}\mathbb{P}\left[e\left(n_k,\frac{c}{n_k^2}\right)\right] = e^{-\frac{c}{2}} > 0$$

Problem 2. Let $u,v \in V$. For $w \in V \setminus \{u,v\}$, let X_w be the indicator random variable for the event that w is a common neighbor of u and v. Then $X_w \sim Ber\left(p^2\right)$. Also, these events are independent, so $codeg(u,v) = \sum_{w\in V\setminus\{u,v\}} X_w \sim Bin\left(n-2,p^2\right)$. Thus $\mathbb{E}[codeg(u,v)] = (n-2)p$. By applying the Chernoff bound we saw in recitation:

$$\mathbb{P}\left[codeg(u,v) \notin (1\pm\varepsilon)(n-2)p^2\right] \leq 2\exp\left(-\frac{\varepsilon^2}{3}(n-2)p^2\right)$$

Now, there are $\binom{n}{2}$ pairs $u \neq v \in V$. Thus we may use the union bound to conclude:

$$\mathbb{P}\left[\exists u\neq v \in V : codeg(u,v) \notin (1\pm\varepsilon)(n-2)p^2\right] \leq 2\binom{n}{2}\exp\left(-\frac{\varepsilon^2}{3}(n-2)p^2\right)$$

$$\leq n^2\exp\left(-\frac{\varepsilon^2}{3}(n-2)p^2\right) = \exp\left(-\frac{\varepsilon^2}{3}(n-2)p^2 - 2\ln n\right)$$

$$= \exp\left(-\frac{\varepsilon^2}{3}\frac{n}{\ln n}p^2\left(\ln n - 2\frac{\ln n}{n} + \ln n\frac{2\ln n}{np^2}\frac{3}{\varepsilon^2}\right)\right) = \exp\left(-\frac{\varepsilon^2}{3}\frac{n}{\ln n}p^2\ln(n)(1+o(1))\right) \to 0$$

where we used the fact that $p = \omega\left(\sqrt{\frac{\ln n}{n}}\right) \implies p^2\frac{n}{\ln n} \to \infty$.

Problem 3. Let $\{a_1,\ldots,a_m\} \subseteq \mathbb{R}$ be the set of values that X_1 takes on with positive probability, and let $p_i = \mathbb{P}[X_i = a_i]$. For $i \in [m]$, let Y_i be the number of X_js that took the value a_i. We can apply the "usual" Chernoff bound to conclude that with probability at most $2\exp\left(-\frac{\varepsilon^2}{3}np_i\right)$, $Y_i \notin (1\pm\varepsilon)np_i$. Thus $Y_i \in (1\pm\varepsilon)np_i$ for all i with probability at least $1-\sum_{i=1}^m 2\exp\left(-\frac{\varepsilon^2}{3}np_i\right)$. If this happens, then $Y = \sum_{i=1}^m Y_i a_i \in (1\pm\varepsilon)n\sum_{i=1}^m a_i p_i = (1\pm\varepsilon)\mathbb{E}[X]$.

Now, let $P = \min_{i\in[m]} p_i$. Then:

$$\sum_{i=1}^{m} 2\exp\left(-\frac{\varepsilon^2}{3}np_i\right) \le 2m\exp\left(-\frac{\varepsilon^2}{3}Pn\right)$$

The conclusion follows from setting $C = \max\left\{2m, \frac{3}{P}\right\}$.

Problem 4.

(1) If the stock doubled itself k times and halved itself $n-1-k$ times, its value is $2^k\left(\frac{1}{2}\right)^{n-1-k}$. Thus:

$$\mathbb{E}\left[X_n\right] = \sum_{k=0}^{n-1}\left(\begin{array}{c} n-1 \\ k \end{array}\right)\left(\frac{3}{4}\right)^k\left(\frac{1}{4}\right)^{n-1-k} 2^k\left(\frac{1}{2}\right)^{n-1-k} = \sum_{k=0}^{n-1}\left(\begin{array}{c} n-1 \\ k \end{array}\right)\left(\frac{3}{2}\right)^k\left(\frac{1}{8}\right)^{n-1-k}$$

$$= \left(\frac{3}{2}+\frac{1}{8}\right)^{n-1} = \left(\frac{13}{8}\right)^{n-1}$$

(2) For example, $\mathbb{E}\left[X_2|X_1\right] = \frac{3}{4}2 + \frac{1}{4}\frac{1}{2} = \frac{13}{8} \ne 1 = X_1$.

(3) $\mathbb{E}\left[Y_{n+1}|Y_1,\ldots,Y_n\right] = \left(\frac{8}{13}\right)^n\left(\frac{3}{4}2X_n + \frac{1}{4}\frac{1}{2}X_n\right) = \left(\frac{8}{13}\right)^n X_n\frac{13}{8} = \left(\frac{8}{13}\right)^{n-1}X_n = Y_n$

(4) $\mathbb{E}\left[Z_{n+1}|Z_1,\ldots,Z_n\right] = \frac{3}{4}\log_2 2X_n + \frac{1}{4}\log_2\frac{X_n}{2} - \frac{n}{2} = \log_2 X_n + \frac{3}{4} - \frac{1}{4} - \frac{n}{2} = \log_2 X_n - \frac{n-1}{2} = Z_n$. Further, $\left|Z_{n+1}-Z_n\right| = \left|\log_2 X_{n+1} - \frac{n}{2} - \log_2 X_n + \frac{n-1}{2}\right| = \left|\log_2\frac{X_{n+1}}{X_n} - \frac{1}{2}\right| \le \max\left\{\left|1-\frac{1}{2}\right|, \left|-1-\frac{1}{2}\right|\right\} = \frac{3}{2}$.

(5) This is immediate from the Azuma-Hoeffding inequality applied to $Z_1,\ldots,Z_n$, and the observation that $Z_1 = 0$.

(6) Since decreasing ε only decreases the probability that the desired event happens, we may assume that $\lim_{n\to\infty}\varepsilon(n) = 0$.Well:

$$\left((1-\varepsilon)\sqrt{2}\right)^{n-1} \le X_n \le \left((1+\varepsilon)\sqrt{2}\right)^{n-1}$$

$$\iff (n-1)\log_2(1-\varepsilon) \le \log_2 X_n - \frac{n-1}{2} = Z_n \le (n-1)\log_2(1+\varepsilon)$$

This is implied by $|Z_n| \le (n-1)\max\left\{\left|\log_2(1+\varepsilon)\right|, \left|\log_2(1-\varepsilon)\right|\right\}$. For large enough n, this is less than $(n-1)\frac{\varepsilon}{2}$. We have:

$$\mathbb{P}\left[|Z_n| < (n-1)\frac{\varepsilon}{2}\right] \le 2\exp\left(-\frac{2(n-1)^2\varepsilon^2}{36(n-1)}\right) = 2\exp\left(-\frac{(n-1)\varepsilon^2}{18}\right)$$

By assumption, $\varepsilon = \omega\left(\frac{1}{\sqrt{n}}\right)$, so $\varepsilon^2 = \omega\left(\frac{1}{n}\right)$ hence $\lim_{n\to\infty}\varepsilon^2(n-1) = \infty$, so the probability tends to 0, as desired.