Please indicate at the beginning of your notebook which questions you decided to answer (this is worth 2 points). The remaining instructions are detailed in the Hebrew document. Pay particular attention to the grading scheme.

Problem 1.

(1) Define the stationary distribution of a Markov chain with transition matrix $P \in M_n(\mathbb{R})$ and state space $[n]$.

(2) Give an example of a simple random walk on a connected graph that doesn't converge to its stationary distribution.

(3) Let $n \geq 2$, and let $G = (V, E)$ be the graph with $V = [2n+1]$ and:

$$E = \{\{i,j\} : 1 \leq i < j \leq n\} \bigcup \{\{i,j\} : n+1 \leq i < j \leq 2n\} \bigcup \{\{2n+1, i\} : 1 \leq i \leq 2n\}$$

(a) What is the stationary distribution of a simple random walk on G? Don't forget to prove your answer.

(b) A simple random walk on G is started at vertex 1 at time 0. Denote the walk's position at time t by X_t. Prove that at time $t = \frac{n}{3}$:

$$\mathbb{P}[n+1 \leq X_t \leq 2n] \leq \frac{2}{5}$$

Remark: The claim is true for all n, but it's enough to prove it for large enough n.

Problem 2. Let $n \in \mathbb{N}$. Let $v_1, \ldots, v_n \in \mathbb{R}^n$ be random vectors, where in each the value of every coordinate is chosen uniformly at random from $\{-1, 1\}$, and all choices are independent.

For $1 \leq i < j \leq n$ define $X_{i,j} = \langle v_i, v_j \rangle$ (so that there are $\binom{n}{2}$ random variables).

(1) For $1 \leq i < j \leq n$, find $\mathbb{E}[X_{i,j}]$.

(2) For $1 \leq i < j \leq n$, find $Var[X_{i,j}]$.

(3) Prove:

$$\lim_{n \to \infty} \mathbb{P}\left[\exists 1 \leq i < j \leq n : |X_{i,j} - \mathbb{E}[X_{i,j}]| \geq \sqrt{n} \ln n\right] = 0$$

Problem 3. Let $G = (V, E)$ be a finite graph. For $v \in V$, let $N(v)$ be v's neighbors.

A **fractional cover** of G is a function $f : V \to [0, \infty)$ s.t. for every $e = \{u, v\}$, $f(u) + f(v) \geq 1$. Let P be the set of fractional covers of G. For $f \in P$, let $\tau(f) = \sum_{v \in V} f(v)$. $\tau(f)$ is called f's weight.

A **fractional matching** of G is a function $g : E \to [0, \infty)$ s.t. for every $v \in V$, $\sum_{u \in N(v)} g(\{u, v\}) \leq 1$. Let Q be the set of fractional matchings of G. For $g \in Q$, let $\nu(g) = \sum_{e \in E} g(e)$.

(1) Prove that Q is a bounded convex polytope.

(2) Prove that $\tau^* = \min\{\tau(f) : f \in P\}$ and $\nu^* = \max\{\nu(g) : g \in Q\}$ exist and are finite.

(3) Write a linear program that finds a fractional cover with value τ^*.

(4) Write the dual program to the one from the previous clause.

(5) Prove that $\nu^* = \tau^*$.

Problem 4. For $A \in M_{m,n}(\mathbb{R})$ define:

$$\|A\|_{op} = \max_{0 \neq x \in \mathbb{R}^n} \frac{\|Ax\|_2}{\|x\|_2}$$

(1) Prove that $\|A\|_{op} = \left\|A^T\right\|_{op}$.
(2) Assume $m \leq n$. Prove that $\sigma \in [0, \infty)$ is a singular value of A iff σ^2 is a singular value of AA^T.
(3) Henceforth, assume $m = n$ and A is symmetric. Prove that σ is a singular value of A iff at least one of $\{-\sigma, \sigma\}$ is an eigenvalue of A.
(4) Let $\lambda_1 \geq \ldots \geq \lambda_n$ be A's eigenvalues. Prove that for every $\alpha \in \mathbb{R}$, the eigenvalues of $A + \alpha I$ are $\lambda_1 + \alpha \geq \ldots \geq \lambda_n + \alpha$.
(5) Reminder: A symmetric $B \in M_n(\mathbb{R})$ is called positive sem-definite if for all $x \in \mathbb{R}^n$, $\langle Bx, x \rangle \geq 0$.
Prove that for all $\alpha \geq -\lambda_n$, $A + \alpha I$ is positive semi-definite.