MATHEMATICAL TOOLS IN COMPUTER SCIENCE - QUIZ 1

Problem 1. Let $X_1, X_2, \ldots$ be identically distributed and independent random variables with $\mathbb{E}[X_1] = \mu < \infty, Var[X_1] = \sigma^2 < \infty$. Show that:

$$\lim_{n\to\infty} \mathbb{P}\left[\left|\frac{1}{n}\sum_{i=1}^{n} X_i - \mu\right| > \sqrt{\frac{\log n}{n}}\right] = 0$$

Solution: We first note that for any random variable X with finite variance and any $\alpha \in \mathbb{R}$, $Var[\alpha X] = \alpha^2 Var[X]$. Also, if X, Y are independent random variables with finite variance, then $Var[X+Y] = Var[X] + Var[Y]$. Thus: $Var\left[\frac{1}{n}\sum_{i=1}^{n} X_i\right] = \frac{1}{n^2} n Var[X_1] = \frac{\sigma^2}{n}$. By linearity of expectation we also have: $\mathbb{E}\left[\frac{1}{n}\sum_{i=1}^{n} X_i\right] = \frac{1}{n} n\mu = \mu$. Chebychev's inequality tells us that for any $\lambda > 0$:

$$\mathbb{P}\left[\left|\frac{1}{n}\sum_{i=1}^{n} X_i - \mu\right| > \lambda\right] \le \frac{Var\left[\frac{1}{n}\sum_{i=1}^{n} X_i\right]}{\lambda^2} = \frac{\sigma^2}{n\lambda^2}$$

Taking $\lambda = \sqrt{\frac{\log n}{n}}$, we have:

$$\mathbb{P}\left[\left|\frac{1}{n}\sum_{i=1}^{n} X_i - \mu\right| > \sqrt{\frac{\log n}{n}}\right] \le \frac{\sigma^2}{\log n} \to_{n\to\infty} 0$$

Problem 2.

(1) Let X be a random variable s.t. $\mathbb{E}[X] < \infty$ taking values only in $\{0, 1, 2, \ldots\}$. Show that $\mathbb{P}[X > 0] \le \mathbb{E}[X]$.

(2) Let $n \in \mathbb{N}$ and let $k = \lceil 2\log_2 n \rceil + 3$. Let $G \sim G\left(n, \frac{1}{2}\right)$. Show that:

$$\mathbb{P}[G \textit{ contains a } k-\textit{clique}] < \frac{1}{2}$$

Solution:

(1) Since X is non-negative with finite mean taking only integer values, we can apply Markov's inequality as follows:

$$\mathbb{P}[X > 0] = \mathbb{P}\left[X \ge \frac{1}{\mathbb{E}[X]}\mathbb{E}[X]\right] \le \mathbb{E}[X]$$

(2) Let X be the number of k-cliques in G. There are $m = \binom{n}{k}$ potential k cliques, and the probability of a given k vertices being a clique is $2^{-\binom{k}{2}}$. Thus, using linearity of expectation:

$$\mathbb{E}[X] = \binom{n}{k} 2^{-\binom{k}{2}} \le \left(\frac{n}{2^{\frac{k-1}{2}}}\right)^k = \left(\frac{n\sqrt{2}}{2^{\frac{k}{2}}}\right)^k \le \left(\frac{n\sqrt{2}}{2^{\log_2 n + \frac{3}{2}}}\right)^k$$

$$\leq \left(\frac{1}{2}\right)^k \leq \frac{1}{4} < \frac{1}{2}$$

By applying part 1:

$$\mathbb{P}\left[G\ contains\ a\ k-clique\right] = \mathbb{P}\left[X > 0\right] \leq \mathbb{E}\left[X\right] < \frac{1}{2}$$