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Mathematical Tools in Computer Science (67865), Winter 2008/9		
	Final Exam	
Lecturer:Nati Linial		*Date: March 8, 2009*
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On your submission write: Your ID
Your login
The questions you decided to solve.

You should answer four questions, one from each section.
You are allowed to use any written material but you must not consult with any other person. You are required to submit a signed letter stating you solved the exam by yourself (A form letter will be posted on the course's homepage).
You should return the solution to the secretary till next Sunday (first day of the semester 8/3).
Give a full account of your solutions. Strive to make your answers coherent, formal, and readable (Typed solutions are of course preferred but this is not mandatory)
Those of you that choose the type the solution, can mail the solution to mathtool2@cs till the same deadline. (Scans of solutions are usually not readable and hence not acceptable)

BEHATZLAHA

1 Probability

Question 1
Prove that there is a constant $c > 0$ such that the following holds: If $X \geqslant 0$ is a non negative (discrete) random variable with finite expectation and variance, then

$$\Pr[X \geqslant \frac{1}{2}\mathbb{E}X] \geqslant c\frac{(\mathbb{E}X)^2}{\mathbb{E}[X^2]}$$

What is the largest value of c you can achieve?
Hint: Recall first the Cauchy-Schwarz inequality for expectation

$$\mathbb{E}[XY] \leqslant \sqrt{\mathbb{E}[X^2]}\sqrt{\mathbb{E}[Y^2]}$$

Why does this hold?
Then split the probability space according to where $X \geqslant \frac{1}{2}\mathbb{E}X$, resp. $X < \frac{1}{2}\mathbb{E}X$.

Question 2
A 3-uniform hypergraph $G = (V, E)$ has vertex set V and edge set $E \subseteq \binom{V}{3}$. Namely, E is a collection of triples from V. A vertex $x \in V$ that does not belong to any edge is said to be **isolated**.

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Given two positive integers n, k we generate a random 3-uniform hypergraph $G = (V, E)$ as follows: We let $V = [n]$ and generate E by sampling uniformly and independently k members from $\binom{V}{3}$. (Notice that this allows repeated edges).
Find the threshold $\hat{k}$ (as a function of n) for the property of G having isolated vertices. Namely,

- If $k(n) = o(\hat{k}(n))$, then G almost surely has an isolated vertex.
- If $k(n) = \omega(\hat{k}(n))$, then G almost surely has no isolated vertices.

2 Linear Algebra

Question 3
If $G = (V, E)$ is an undirected graph, then its **line graph** $L(G) = (\bar{V}, \bar{E})$ is defined as follows:

$$\bar{V} = E$$
$$\text{for } (x, y), (u, v) \in E \ : \ ((x, y), (u, v)) \in \bar{E} \iff |\{x, y\} \cap \{u, v\}| = 1.$$

(2 points) Show that all eigenvalues of $L(G)$ are $\geqslant -2$.

(3 points) Show that if $|E| > |V|$ then (-2) is an eigenvalue of $L(G)$.

(10 points) Show that if G is d-regular ($d \geqslant 1$) with spectrum $spec(G) = \{\lambda_1, \ldots, \lambda_n\}$ then

$$spec(L(G)) = \begin{cases} \{0\} & d = 1 \\ \{\lambda_i\}_{i=1}^n & d = 2 \\ \{\lambda_i + d - 2\}_{i=1}^n \cup \{-2\} & d > 2 \end{cases}$$

(10 points) Let A be a real $\binom{n}{2} \times \binom{n}{3}$ matrix defined by $A_{S,T} = \begin{cases} 1 & S \subset T \\ 0 & S \not\subset T \end{cases}$ for $|S| = 2$ and $|T| = 3$
Find the singular values of A.
Hint: use the result of the former paragraph.

Question 4
A k-coloring of a graph $G = (V, E)$ is a function $\phi : V \to \{1, 2, \ldots, k\}$ such that if $(x, y) \in E$ then $\phi(x) \neq \phi(y)$. The chromatic number of G is defined as

$$\chi(G) = \min\{k \mid G \text{ has a k coloring}\}$$

Let G be a d-regular graph

(6 points) Prove

$$\chi(G) \leqslant d + 1$$

and show that the inequality is tight

Let $\alpha = \max \frac{|\lambda|}{d}$, where the maximum is over all $\lambda \neq d$ that are eigenvalues of G.

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(19 points) Prove

$$\chi(G) \geqslant \frac{1}{\alpha}$$

Hint: Let us denote the incidence matrix of G by A. For a subset $X \subset V$ let us denote by $e(X,X) \triangleq \#\{(x,y) \in E \mid x,y \in X\}$, i.e., twice the number of edges both of whose vertices are in X. Show that

$$e(X,X) = 1_X A 1_X$$
$$|e(X,X) - d\tfrac{|X|^2}{n}| \leqslant \alpha|X|$$

3 Linear Programming & Optimization

Question 5

Let $n,k \geqslant 1$, $p \in [0,1]$, and let $B_1, B_2, \ldots, B_n, B$ be random variables s.t.

$$i = 1,\ldots,n \quad Pr[B_i = 1] = 1 - \Pr[B_i = 0] = p \quad i.i.d$$
$$B = \sum_{i=1}^{n} B_i$$

- a(n,k,p)
  Let $A(n,k,p)$ be the set of all distributions of random variables $X_1, X_2, \ldots, X_n, X$ s.t.

$$i = 1,\ldots,n \quad Pr[X_i = 1] = 1 - \Pr[X_i = 0] = p$$
$$\mathbb{E}\left[X^j\right] = \mathbb{E}\left[B^j\right] \quad \text{for } j = 0,1,\ldots,k$$
$$X = \sum_{i=1}^{n} X_i$$

  $a(n,k,p)$ is defined to be

$$a(n,k,p) = \max\{\Pr[X = n] \mid (X_1, X_2, \ldots, X_n, X) \in A(n,k,p)\}$$

- b(n,k,p)
  Define $P(n,k)$ as the set of all real polynomials P of degree $\leq k$ for which $P(i) \geqslant 0$ for $i = 0,1,\ldots,n-1$ and $P(n) \geqslant 1$

  Next define

$$b(n,k,p) = \min\{\mathbb{E}\left[f(B)\right] \mid f \in P(n,k)\}$$

(5 points) Prove $a(n,1,p) = b(n,1,p)$

(20 points) Prove $a(n,k,p) = b(n,k,p)$ for every choice of p,n,k.
Hint:Use the duality theorem for LP problems.

(Partial score will be given for a proof for $p = \frac{1}{2}$)

Question 6

Let $z_1, \ldots, z_n$ be vectors in $\Re^d$ for some integer d. Associate with these vectors an $n \times n$ matrix A where $A_{i,j} = \|z_i - z_j\|_1$.
Let $\mathcal{L}_n$ be the collection of all $n \times n$ matrices that can be created in this way.

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(7 points) Prove

$$A, B \in \mathcal{L}_n \;\Rightarrow\; A + B \in \mathcal{L}_n$$

Recall that if Z is a subset of a real vector space, we define *cone*(Z) as the set of all finite nonnegative linear combinations of elements in Z. Namely,

$$cone(Z) = \{\sum_{i}^{N} a_i z_i \mid \forall i \;\; z_i \in Z \;\; and \;\; a_i \geq 0\}.$$

(18 points) Let Z be the set of all (0, 1)-matrices in $\mathcal{L}_n$. Show

$$\mathcal{L}_n = cone(Z)$$

4 Harmonic Analysis

Question 7
The functions $P_S : \{0,1\}^n \to \mathbb{C}$ are defined for $S \subseteq [n]$ by

$$P_\emptyset(x) = 1$$
$$P_S(x) = \prod_{i \in S} x_i$$

(4 points) Show that $\{P_S\}_{S \subseteq [n]}$ is a basis of the linear space of all functions from $\{0,1\}^n$ to $\mathbb{C}$.

Given a function $g : \{0,1\}^n \to \mathbb{C}$, we define $\check{g}(S)$ as the coefficient of P_S in the expansion of g in the above basis. We also recall that $\hat{g}(S)$ is the S-th Fourier coefficient in the expansion of g. We define the two corresponding notions of degree:

$$deg_M(g) = \max\{d \mid \exists S \in \tbinom{[n]}{d} \;\; \check{g}(S) \neq 0\}$$
$$deg_F(g) = \max\{d \mid \exists S \in \tbinom{[n]}{d} \;\; \hat{g}(S) \neq 0\}$$

(4 points) Find $deg_M(g)$ and $deg_F(g)$ for the following functions

- Constant function $g(x) = 5$
- Dictatorship of the $7^{\underline{th}}$ voter $h(x) = x_7$

(17 points) Show that for any $g : \{0,1\}^n \to \mathbb{C}$

$$deg_M(g) = deg_F(g)$$

Hint: First show that the equation holds for the characters

Question 8
A Boolean function is a function $f : \{0,1\}^n \to \{0,1\}$.
Such a function is called **balanced** if

$$\#\{x \in \{0,1\}^n \mid f(x) = 0\} = \#\{x \in \{0,1\}^n \mid f(x) = 1\}$$

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For $u, v \in \{0,1\}$, we say that $u > v$ if for every $i = 1, \ldots n$ there holds $u_i \geq v_i$. The Boolean function f is called **monotone** if $f(u) \geq f(v)$ whenever $u > v$. The influence of i for a Boolean function $f : \{0,1\}^n \to \{0,1\}$ is defined to be

$$I_i(f) = 2^{-n}\#\{x \in \{0,1\}^n | f(x) \neq f(x + e_i)\}$$

(3 points) Show that for any balanced Boolean function

$$\sum_{i \in [n]} I_i(f) \geqslant 1$$

(18 points) Show that for any balanced monotone Boolean function

$$\sum_{i \in [n]} I_i^2(f) \leqslant 1$$

(4 points) Show that the two inequalities above are tight

Question 9

A k-majority is a Boolean function $f : \{0,1\}^n \to \{0,1\}$ defined by $f(x) = \begin{cases} 1 & |x| \geqslant k \\ 0 & |x| < k \end{cases}$

(18 points) Find the Fourier coefficients of the 2-majority Boolean function

(7 points) Find the Fourier coefficients of the 3-majority Boolean function