MATHEMATICAL TOOLS - PROBLEM SET 5

Due Sunday, December 18th, 23:55, either in the course mailbox or through the Moodle.

Linear Homogenous Recurrence Relations.

Problem 1. Let $\alpha_0, \alpha_1, \ldots, \alpha_k \in \mathbb{C}$ and let $a_{n+k+1} = \sum_{i=0}^{k} \alpha_i a_{n+i}$ be a recurrence relation. Let $A = \begin{pmatrix} \alpha_k & \alpha_{k-1} & \ldots & \alpha_1 & \alpha_0 \\ 1 & 0 & \ldots & & 0 \\ 0 & 1 & 0 & \ldots & 0 \\ \ldots & & \ldots & \ldots & \ldots \\ 0 & \ldots & 0 & 1 & 0 \end{pmatrix}$ and let $p(x) = x^{k+1} - \sum_{i=0}^{k} \alpha_i x^i$.
In recitation we showed that $\lambda \in \mathbb{C}$ is an eigenvalue of A iff $p(\lambda) = 0$. We also showed that if λ is an eigenvalue of A, then $a_n = \lambda^n$ is a solution to the recurrence relation.

(1) Show that if A has $k+1$ distinct eigenvalues $\lambda_0, \lambda_1, \ldots, \lambda_k \in \mathbb{C}$, then every solution $\{a_n\}_{=0}^{\infty}$ can be written in a unique way as $a_n = \sum_{i=0}^{k} A_i \lambda_i^n$, for $A_0, \ldots, A_k \in \mathbb{C}$.
(2) Show that $p_A(x) = p(x)$.
(3) Find the solution space of the recurrence relation $a_{n+4} = 10a_{n+3} - 35a_{n+2} + 50a_{n+1} - 24a_n$.
(4) Let $a_0 = 2, a_1 = 3, a_2 = 5, a_3 = 9$. Find an explicit (non-inductive) formula for a_n satisfying the recurrence relation from the previous part.

Remark. We can actually give a complete description of the solution space of any recurrence relation. Assume $\alpha_0 \neq 0$ (if this isn't the case the problem is degenerate, and can be reformulated as a lower-order recurrence relation). By the fundamental theorem of algebra, there exist $\lambda_1, \ldots, \lambda_\ell \in \mathbb{C}$ and $p_1, \ldots, p_\ell \in \mathbb{N}$ s.t. $p(x) = \prod_{i=1}^{\ell} (x - \lambda_i)^{p_i}$ and $\sum_{i=1}^{\ell} p_i = k+1$. In this case, for every $0 \leq j < p_i$, $a_n = n^j \lambda_i^n$ is also a solution. From here we can conclude that every solution can be written uniquely as

$$a_n = \left(A_{1,1} + A_{1,2}n + \ldots + A_{1,p_1} n^{p_1 - 1}\right) \lambda_1^n + \ldots + \left(A_{\ell,1} + A_{\ell,2}n + \ldots + A_{\ell,p_\ell}\right) \lambda_\ell^n$$

Proving this is not difficult, but is a little technical and too far from the course's subject matter.

Vandermonde matrix.

Problem 2. Let $\mathbb{F}$ be a field and let $\alpha_1, \ldots, \alpha_k \in \mathbb{F}$.

(1) Assume the α_i-s are distinct. Show that for all $1 \leq i \leq k$ there exists a polynomial $p_i(x)$ of degree $k-1$ with coefficients in $\mathbb{F}$ s.t. for all $1 \leq j \leq k$, $p_i(\alpha_j) = \begin{cases} 1 & j = i \\ 0 & j \neq i \end{cases}$.

(2) Let:

$$A=\begin{pmatrix} 1 & \alpha_1 & \alpha_1^2 & \dots & \alpha_k^{k-1} \\ 1 & \alpha_2 & \alpha_2^2 & \dots & \alpha_2^{k-1} \\ \dots & & & & \dots \\ 1 & \alpha_k & \alpha_k^2 & \dots & \alpha_k^{k-1} \end{pmatrix}$$

Show that $\det A\neq 0$ iff the α_i-s are distinct.

Linear Homogenous Ordinary Differential Equations. Let $\alpha_0,\alpha_1,\dots,\alpha_{k-1}\in\mathbb{R}$. A solution to the linear homogenous ordinary differential equation (ODE) $F^{(k)}(x)=\sum_{i=0}^{k-1}\alpha_i F^{(i)}(x)$ is a function $f:\mathbb{R}\to\mathbb{R}$ that is smooth (i.e. differentiable infinitely many times), and that satisfies: $\forall x\in\mathbb{R}, f^{(k)}(x)=\sum_{i=0}^{k-1}\alpha_i f^{(i)}(x)$.

Let $C^{\infty}(\mathbb{R})=\{f:\mathbb{R}\to\mathbb{R}|f\ is\ smooth\}$. We define operations of addition and scalar multiplication in the obvious ways: For $f,g\in C^{\infty}$ and $a\in\mathbb{R}$, $(f+g)(x)=f(x)+g(x)$ and $(af)(x)=af(x)$.

Problem 3.

(1) Show that C^{∞}, together with the operations above, is a vector space over $\mathbb{R}$.
(2) Let $D:C^{\infty}\to C^{\infty}$ be: $D(f)=f'$. Show that D is a linear operator. Find D's eigenvalues and matching eigenvectors.
(3) Let $V\subseteq C^{\infty}$ be the set of solutions to the ODE above. Show that V is a vector subspace of C^{∞}.
(4) Let $p(x)=x^k-\sum_{i=0}^{k-1}\alpha_i x^i$. Show that if $p(\lambda)=0$, then $f(x)=e^{\lambda x}$ is a solution to the ODE.
(5) Conclude that if p has k distinct roots $\lambda_1,\dots,\lambda_k$, then for every $a_0,\dots,a_{k-1}\in\mathbb{R}$ there exists $f\in V$ s.t. for all $0\le i\le k-1$, $f^{(i)}(0)=a_i$.

Hint: Define $T:V\to\mathbb{R}^k$ by $T(f)=\begin{pmatrix} f(0) \\ f'(0) \\ \dots \\ f^{(k-1)}(0)\end{pmatrix}$. Show that T is surjective by showing that $T\left(e^{\lambda_1 x}\right),T\left(e^{\lambda_2 x}\right),\dots,T\left(e^{\lambda_k x}\right)$ are linearly independent.

The Fourier Basis.

Problem 4.

(1) Let $T:\mathbb{C}^n\to\mathbb{C}^n$ be defined by:

$$T\begin{pmatrix} z_1 \\ z_2 \\ \dots \\ z_n\end{pmatrix}=\begin{pmatrix} z_n \\ z_1 \\ \dots \\ z_{n-1}\end{pmatrix}$$

Find T's eigenvalues, with appropriate eigenvectors. Show that these eigenvectors form an orthogonal basis for $\mathbb{C}^n$.
(2) Find a unitary matrix U and a diagonal matrix D s.t:

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0\end{pmatrix}=UDU^*$$