# MATHEMATICAL TOOLS - PROBLEM SET 2

Due Sunday, November 27th, 20:00, either in the course box or through the Moodle. You may submit scanned documents, but please make sure they are readable!

All random variables can be assumed to have finite mean and variance.

**Conditional Probability.** Let $(\Omega, \mathbb{P})$ be a discrete probability space.

**Problem 1.** Let $A_1, A_2, \ldots, A_n \subseteq \Omega$ be a partition of $\Omega$ (in other words $\cup_{i=1}^n A_i = \Omega$, and for all $1 \leq i < j \leq n$, $A_i \cap A_j = \emptyset$).

(1) Prove the **law of total probability**: For any $B \subseteq \Omega$:

$$\mathbb{P}[B] = \sum_{i=1}^n \mathbb{P}[B|A_i] \, \mathbb{P}[A_i]$$

(2) The **conditional expectation** of a random variable $X : \Omega \to \mathbb{R}$ w.r.t. an event $A \subseteq \Omega$ is defined as:

$$\mathbb{E}[X|A] = \sum_{\omega \in \Omega} X(\omega) \, \mathbb{P}[\{\omega\} | A]$$

Prove the **law of total expectation**:

$$\mathbb{E}[X] = \sum_{i=1}^n \mathbb{E}[X|A_i] \, \mathbb{P}[A_i]$$

**Poisson Distribution.** A random variable $X$ has the Poisson distribution with parameter $\lambda > 0$ (denoted $X \sim Pois(\lambda)$) if for every $n \in \mathbb{N} \cup \{0\}$, $\mathbb{P}[X = n] = e^{-\lambda} \frac{\lambda^n}{n!}$. Note that indeed, $\sum_{n=0}^{\infty} e^{-\lambda} \frac{\lambda^n}{n!} = e^{-\lambda} \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} = 1$, so this is well-defined.

**Problem 2.** Let $X \sim Pois(\lambda)$ with $\lambda > 0$.

(1) Calculate $\mathbb{E}[X]$.
(2) Calculate $Var[X]$.

**Problem 3.** Let $X \sim Pois(\alpha), Y \sim Pois(\beta)$ be independent. Prove that $X + Y \sim Pois(\alpha + \beta)$.

**The Second Moment Method.**

**Problem 4.** Prove the Cauchy-Schwarz inequality for random variables: Let $X, Y$ be random variables defined over the same probability space. Then:

$$\mathbb{E}[XY]^2 \leq \mathbb{E}\left[X^2\right] \mathbb{E}\left[Y^2\right]$$

Hint: Note that for any $\alpha \in \mathbb{R}$, $\mathbb{E}\left[(X + \alpha Y)^2\right] \geq 0$. Use a properly chosen $\alpha$ to obtain the result.

**Problem 5.** Let $X$ be a non-negative random variable. Show that

$$\mathbb{P}[X > 0] \geq \frac{\mathbb{E}[X]^2}{\mathbb{E}[X^2]}$$

Hint: Note that $X = X \cdot \mathbf{1}_{X>0}$, and use Cauchy-Schwarz.

**Branching Processes.**

**Problem 6.** Consider the following **branching process**, describing the evolution of a bacteria colony: At the first stage, there is a single cell. The cell creates a random number of new cells (possibly zero) and then dies. At the next stage, each living cell creates a random number of new cells and then dies, and the process repeats... We're interested in the probability that the colony will eventually die.

Formally, we have the following setup: Let $Z_0 = 1$. For every $m, n \in \mathbb{N}$ let $L_{m,n} \sim Bin\left(2, p\right)$, all of these independent. Assuming $Z_{n-1}$ is defined, the random variable $Z_n$ is defined as follows:

$$Z_n = \begin{cases} L_{n,1} + L_{n,2} + \ldots + L_{n,Z_{n-1}} & Z_{n-1} > 0 \\ 0 & Z_{n-1} = 0 \end{cases}$$

$Z_n$ denotes the number of living cells at the $n$th stage.

Let $\mu = 2p = \mathbb{E}\left[L_{1,1}\right]$ and $\sigma^2 = 2p\left(1-p\right) = Var\left[L_{1,1}\right]$.

(1) Show that $E\left[Z_n\right] = \mu^n$. Hint: It might help to use the law of total expectation, w.r.t. the events $\left\{Z_{n-1} = 0\right\}, \left\{Z_{n-1} = 1\right\}, \ldots, \left\{Z_{n-1} = 2^{n-1}\right\}$ (why are these sets indeed a partition of the underlying probability space?).

(2) Conclude that if $p < \frac{1}{2}$, $\lim_{n\to\infty} \mathbb{P}\left[Z_n > 0\right] = 0$. This means that the colony almost surely dies out eventually (though we haven't formally defined what this means).

(3) Show that $Var\left[Z_n\right] = \sigma^2\left(\mu^{n-1} + \mu^n + \ldots + \mu^{2n-2}\right)$.

(4) Use the second moment method (and a few other calculations) to show that if $p > \frac{1}{2}$, there is some $\delta > 0$ s.t. for all $n$, $\mathbb{P}\left[Z_n > 0\right] > \delta$. This means that with positive probability the colony survives forever (again, we haven't formally defined what this means).

(5) Assume $p < 1$. Is it conceivable that $\lim_{n\to\infty} \mathbb{P}\left[Z_n > 0\right] = 1$?