MATHEMATICAL TOOLS - PROBLEM SET 4

Due Sunday, December 11th, 23:55, either in the course mailbox or through the Moodle. You may submit scanned files through the moodle, but please make sure they are crystal clear!

Problem 1. In recitation we showed that if $p = \omega\left(\frac{1}{n^2}\right)$ then for every $\varepsilon > 0$, a.a.s. $G(n,p)$ has $(1 \pm \varepsilon)\frac{n^2}{2}p$ edges. In this exercise you'll show that the assumption of $p = \omega\left(\frac{1}{n^2}\right)$ is necessary.
Let $e(n,p)$ be a random variable equal to the number of edges in $G \sim G(n,p)$.

(1) Let $c > 0$, and let $p = \frac{c}{n^2}$. Calculate $\lim_{n\to\infty} \mathbb{P}[e(n,p) = 0]$.
(2) Now assume that $0 \neq p \neq \omega\left(\frac{1}{n^2}\right)$. This means that there exists some $c > 0$ s.t. for infinitely many n, $0 < p(n) \leq \frac{c}{n^2}$. Conclude that there exists some $\varepsilon > 0$ s.t. $\limsup_{n\to\infty} \mathbb{P}\left[e(n,p(n)) \notin (1 \pm \varepsilon)\frac{n^2}{2}p\right] > 0$.

Problem 2. Let $(V,E) = G \sim G(n,p)$. For $u \neq v \in V$, we'll define the *codegree* of u, v as the number of their common neighbors. In other words: $codeg(u,v) = |\{w \in V \setminus \{u,v\} : uw, vw \in E\}|$. Show that if $p = \omega\left(\sqrt{\frac{\ln n}{n}}\right)$ then for every $\varepsilon > 0$, a.a.s. for all $u \neq v \in V$ we have $codeg(u,v) \in (1 \pm \varepsilon) np^2$.

Problem 3. Let $X_1, \dots, X_n$ be independent and identically distributed random variables taking only finitely many values, and let $X = \sum_{i=1}^n X_i$. Show that there exists some $C > 0$ that doesn't depend on n, such that for all $\varepsilon \in (0,1)$:

$$\mathbb{P}[X \notin (1 \pm \varepsilon)\mathbb{E}[X]] \leq C \exp\left(-\frac{1}{C}\varepsilon^2 n\right)$$

Martingales. A (finite or countably infinite) sequence of random variables $X_1, X_2, \dots$ defined on the same probability space is called a *martingale* if for all n:

$$\mathbb{E}[X_{n+1} | X_1, X_2, \dots, X_n] = X_n$$

Note that this definition generalizes the one you saw in class for Doob martingales.

If $X_1, X_2, \dots$ is a martingale, and there exist positive constants $c_1, \dots, c_{n-1}$ s.t. for all k, $|X_{k+1} - X_k| \leq c_k$, the Azuma-Hoeffding inequality tells us that for all $\lambda > 0$ and all n:

$$\mathbb{P}[|X_n - X_1| \geq \lambda] \leq 2\exp\left(-\frac{\lambda^2}{2\sum_{k=1}^{n-1} c_k^2}\right)$$

Problem 4. Consider the following simplistic (and optimistic!) model for the fluctuations of a share value in the stock market: On the first day, the share is valued at 1. On each subsequent day, with probability $\frac{3}{4}$ the share doubles its value and with probability $\frac{1}{4}$ the share loses half its value. The fluctuation on each day is independent of what happened on all other days. Let $X_1 = 1$ be the value of the share on the first day, and let X_n be the value of the share on the nth day.

Consider the following intuition: If we look at a period of four days, we'd expect the value of the share to rise on three of them and to fall on one of them. Thus it should have multiplied its value by $2^3 \cdot \frac{1}{2} = 4$, or on (geometric) average $4^{\frac{1}{4}} = \sqrt{2}$ each day. Thus, we expect that $X_n \approx \sqrt{2}^{n-1}$. Our goal is to use martingales to ground this intuition.

(1) Calculate $\mathbb{E}\left[X_n\right]$.
(2) Show that the sequence $X_1, X_2, \ldots$ isn't a martingale.
(3) Define $Y_n = \left(\frac{8}{13}\right)^{n-1} X_n$. Show that the sequence $Y_1, Y_2, \ldots$ is a martingale. Although $Y_1, Y_2, \ldots,$ is a martingale, the differences $|Y_{n+1} - Y_n|$ can be very large (with positive probability $|Y_{n+1} - Y_n| = \frac{9}{13}\left(\frac{16}{13}\right)^{n-1}$). Thus, working with Azuma's inequality directly on this sequence is difficult. On the other hand, the multiplicative difference $\frac{1}{2} \leq \frac{X_{n+1}}{X_n} \leq 2$ is bounded. In order to take advantage of this we'll define the sequence $Z_n = \log_2 X_n - \frac{n-1}{2}$.
(4) Show that $Z_1, Z_2, \ldots,$ is a martingale, and that for every $n \geq 1$, $|Z_{n+1} - Z_n| \leq \frac{3}{2}$.
(5) Conclude that for all $n \geq 1, \lambda > 0$: $\mathbb{P}\left[|Z_n| \geq \lambda\right] \leq 2\exp\left(-\frac{2\lambda^2}{9(n-1)}\right)$.
(6) Let $\varepsilon(n) = \omega\left(\frac{1}{\sqrt{n}}\right)$. Conclude that asymptotically almost surely

$$\left((1 - \varepsilon(n))\sqrt{2}\right)^{n-1} \leq X_n \leq \left((1 + \varepsilon(n))\sqrt{2}\right)^{n-1}$$