MATHEMATICAL TOOLS - PROBLEM SET 3

Due Sunday, December 4th, 23:59, either in the course mailbox or through the Moodle. You may submit scanned files through the moodle, but please make sure they are crystal clear!

Problem 1. In this problem you'll show that the threshold for $G(n,n,p)$ containing a perfect matching is $\frac{\ln n}{n}$.

For a bipartite graph $H=(U,V,E)$ with $|U|=|V|$, a *perfect matching* is a vertex-disjoint set $M\subseteq E$ s.t. $|M|=|U|$. We say that H satisfies the *marriage condition* if for every $W\subseteq U$, $|N_H(W)|\geq|W|$ ($N_H(W)=\{v\in V:\exists w\in W,\{w,v\}\in E\}$ is the set of W's neighbors). Recall Hall's marriage theorem: H contains a perfect matching iff it satisfies the marriage condition.

(1) Show that if $c<1$ and $p\leq c\frac{\ln n}{n}$, a.a.s. $G(n,n,p)$ doesn't contain a perfect matching. You may rely on results from recitation.
(2) Show that if $H=(U,V,E)$ doesn't satisfy the marriage condition then there exists some $W\subseteq U$ s.t. $|N_H(W)|=|W|-1$.
(3) Henceforth, let $c>1$ and assume $p\geq c\frac{\ln n}{n}$. Let $(X,Y,E)=G\sim G(n,n,p)$, and for every $W\subseteq X$ let A_W be the event that $|N_G(W)|=|W|-1$. What is the distribution of $|N_G(W)|$? Use this knowledge to bound $\mathbb{P}[A_W]$.
(4) **Note (1/12): No need to hand in this clause. The result is correct, but doesn't follow straightforwardly from the preceeding clauses.** Conclude that a.a.s., G satisfies the marriage condition and therefore contains a perfect matching.

Problem 2. A fixed point of a permutation $\pi:[n]\to[n]$ is a value for which $\pi(x)=x$. Find the expectation and variance of the number of fixed points of a permutation chosen uniformly at random from all permutations.

Measure Concentration.

Problem 3. Consider the following setting: n balls are distributed into m bins, where each ball "chooses" a uniformly random bin independent of all other choices.

(1) Define an appropriate probability space that captures the situation above. For each $1\leq i\leq m$, let X_i be the number of balls in the mth bin.
(2) How is X_i distributed?
(3) Are $X_1,X_2,\ldots,X_m$ independent?
(4) Assume $3\sqrt{\frac{m\ln m}{n}}\leq 1$. Show that $\mathbb{P}\left[\forall i\in[m],\left|X_i-\frac{n}{m}\right|<3\sqrt{\ln m}\sqrt{\frac{n}{m}}\right]\geq 1-\frac{2}{m^2}$.
(5) No need to hand anything in: Consider what happens when $n\to\infty$. Part 4 implies that if $m=\omega(1)$, then a.a.s. the number of balls in every bin is tightly concentrated around the mean - the probability of even a single bin deviating by more than $3\sqrt{\ln m}\sqrt{\frac{n}{m}}$ tends to 0.

The Binary Entropy Function.

Problem 4. Define $f : [0,1] \to \mathbb{R}$ by:

$$f(x) = \begin{cases} -x\log_2 x - (1-x)\log_2(1-x) & x \in (0,1) \\ 0 & x = 0,1 \end{cases}$$

(1) Prove that f is continuous on $[0,1]$.
(2) Find $\max_{x\in[0,1]} f(x)$. Where does f attain its maximum?

An Exponential Bound on the Tail of Poisson Random Variables.

Problem 5. Let $\lambda > 0$ and $X \sim Poi(\lambda)$.

(1) For $t > 0$, calculate $\mathbb{E}\left[e^{tX}\right]$.
(2) Use Markov's inequality w.r.t. the random variable e^{tX}, to conclude that for any $t > 0$ and any $k \geq 0$:

$$\mathbb{P}\left[e^{tX} \geq e^{tk}\right] \leq \exp\left(\lambda e^t - tk - \lambda\right)$$

(3) Conclude that for any $k \geq \lambda$:

$$\mathbb{P}[X \geq k] \leq \left(\frac{\lambda e}{k}\right)^k e^{-\lambda}$$