Final Exam

Math tools

February 2, 2014

Regulations

- **You must solve the exam on your own**. You may not consult with any other person.
- Do not distribute the exam answers.
- Write your id and the questions you decided to solve on your submission.
- You have until Thursday, Feb. 20 to solve the exam.
- Hand your solution in both electronically (to the course email at mathtools.huji@gmail.com) and physically on the due date in room 422A.
- You must solve five questions, including at least one from each topic.
- You may consult any written text. If you rely on a written source, you have to cite it.
- Grading will depend on correctness, of course, but also on clarity and brevity.

Enjoy!

1 Probability

1.1 A more inclusive Chernoff bound

The aim of this question is to prove a Chernoff-like bound under weaker conditions.

Let $1 > \alpha > 0$, and let $X_1, ..., X_n$ be (not necessarily independent) indicator random variables with the property that for every subset $A \subseteq [n]$,

$$\Pr\left(\prod_{j \in A} X_j = 1\right) \leq \alpha^{|A|}.$$

(a) Let $S \subseteq [n]$ be a randomly chosen subset, where each element of $[n]$ is chosen with probability q independently of the other elements. Show that the probability that $X_j = 1$ for every $j \in S$ is at most $(\alpha q + 1 - q)^n$.

(b) Let β be some constant such that $0 < \alpha < \beta < 1$, and let S be chosen at random as above. Define

$$p = \Pr\left(\sum_{i=1}^{n} X_i \geq \beta n\right).$$

Show that

$$\Pr\left(\prod_{j \in S} X_j = 1\right) \geq p(1-q)^{(1-\beta)n}.$$

The probability is over the choice of S and the distribution of $X_1, ..., X_n$.

(c) Show that $p \leq \gamma^n$ for some $\gamma < 1$ that depends on α, β.
Remark: It is even possible to show that the proper choice of q leads to $\gamma = e^{-2(\beta-\alpha)^2}$.

1.2 The vertex of maximal degree

Let G be sampled at random from $G_{n,p}$ with $p = \frac{1}{n}$, and let $D(G)$ be the maximal degree of a vertex in G.

(a) Show that there is a constant $c > 0$ such that the expected number of vertices whose degree is at least $k(n) = \frac{c \log(n)}{\log(\log(n))}$ tends to infinity.

(b) Show that $\Pr(D(G) \geq k(n))$ tends to one when n tends to infinity.

2 Linear Algebra

2.1 Singular values of bipartite graphs

Let $G = \langle U \cup V, E \rangle$ be a bipartite graph with n vertices on each side. Define the $n \times n$ matrix A by $A(i,j) = 1$ if $\{u_i, v_j\} \in E$ and $A(i,j) = 0$ otherwise, and let $(\sigma_1, ..., \sigma_n)$ be A's singular values.

(a) Prove that if $d(G)$ is the maximal degree of G, then $\sqrt{d(G)} \leq \sigma_1 \leq d(G)$.

(b) Prove that if G is d-regular, then G is connected iff $\sigma_1 > \sigma_2$.

2.2 Orthogonal optimization

(a) Let A be an $n \times n$ real matrix whose SVD is UDV^T. Prove that the orthogonal matrix R that minimizes $\|A - R\|_F$ is UV^T.

(b) Let $p_1, ..., p_k$ be k points in $\mathbb{R}^n$, and let $t_1, ..., t_k \in \mathbb{R}^n$ be k additional points. The goal of this question is to rotate, reflect and move the points p_i so that they are as close as possible to the $t_i's$.

For an orthogonal matrix Q and a vector $t \in \mathbb{R}^n$, we define $q_i = Qp_i + t$. Find the orthogonal matrix Q and t that minimize $\sum_{i=1}^{k} \|q_i - t_i\|_2^2$.

2.3 An inequality

Let $x_1, ..., x_k$ be k vectors in $\mathbb{R}^n$. Prove that:

$$\left(\sum_{i=1}^{n} |(x_1)_i \cdot ... \cdot (x_k)_i|^{\frac{1}{k}} \right)^k \leq \prod_{j=1}^{k} \|x_j\|_1$$

3 Markov chains

3.1 The stationary distribution

Let P be the transition matrix of a Markov chain over a finite state space Ω, and let π be P's stationary distribution. Define

$$d(t) = \max_{\mu} \{ d_{TV} \left(\mu P^t, \pi \right) \}$$

Where the maximum is taken over all distributions μ over Ω.

(a) Show that

$$d(t) = \max_{x \in \Omega} \{ d_{TV} \left(P^t(x, \cdot), \pi \right) \}.$$

Here $A(i, \cdot)$ denotes the i'th row of A.

(b) Assume that P is symmetric. Show that for every $t \geq 0$, $d(t+1) \leq d(t)$.

3.2 Properties of expander graphs

Let G be a d-regular graph on n vertices whose adjacency matrix has the eigenvalues $\lambda_1, ..., \lambda_n$, ordered so that $d = |\lambda_1| > |\lambda_2| \geq ... \geq |\lambda_n|$.

(a) Consider a random walk on G that starts at a random vertex v_0 chosen from some distribution μ. Let P denote the walk's transition matrix, and let π denote the uniform distribution. Prove that

$$\|\mu P^t - \pi\|_2 \leq \|\mu - \pi\|_2 \left(\frac{|\lambda_2|}{d}\right)^t.$$

(b) Let $\chi(G)$ denote the minimal number of colors needed to color G's vertices so that every two adjacent vertices have different colors. Show that

$$\chi(G) \geq \frac{d}{|\lambda_2|}.$$

4 Linear Programming

4.1 Integer linear programming

Formulate the following problems as integer linear programs.
Remarks: In this formulation, the number of variables and constraints is not necessarily polynomial in the input size, and some of the problems may have efficient algorithms that do not use linear programming.

(a) Find the edge-chromatic number of a graph G. That is, find the fewest number of colors necessary to color each edge of G such that no two edges incident on the same vertex have the same color.

(b) Find the minimal-weight connected subgraph of a given connected graph G with (not necessarily nonnegative) weights on the edges.

(c) Find the minimal weight cut in a graph with nonnegative weights on the edges.

4.2 The closest point on a hyperplane

Let $A = \{v \in \mathbb{R}^n | a^T v = 0\}$ be the hyperplane consisting of all of the vectors orthogonal to some vector a, and let $x \in \mathbb{R}^n$ be a point that isn't in A. Describe an algorithm to find a point $y \in A$ in such that:

(a) $\|x - y\|_1$ is minimal.

(b) $\|x - y\|_2$ is minimal.

(c) $\|x - y\|_\infty$ is minimal.

You may assume that linear programs are efficiently solvable in this question.