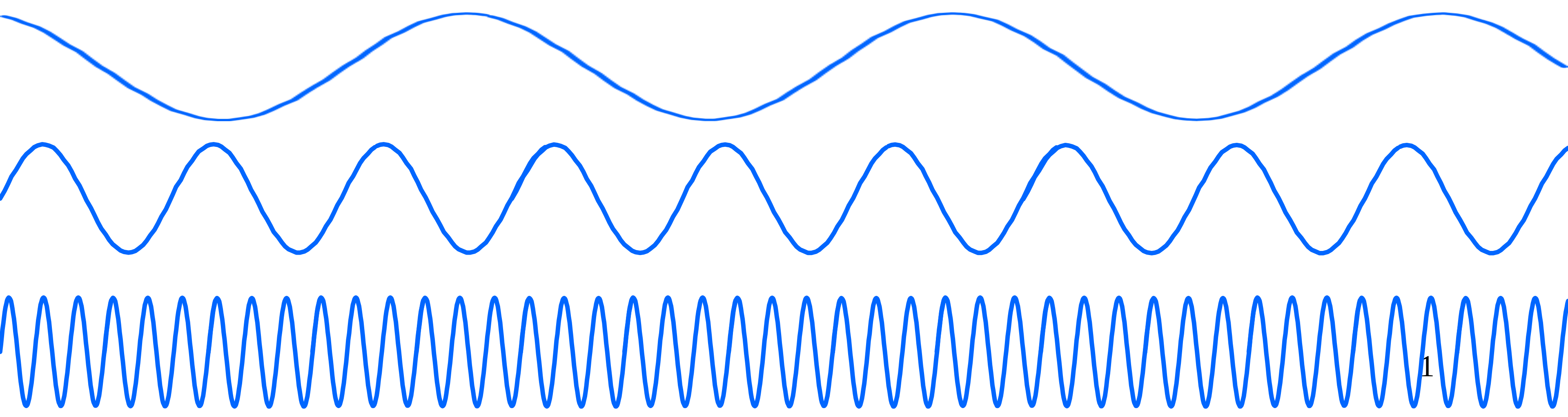
Fourier Transform

Easy Equations but Hard to Understand

- Another representation of images
- Original representation: Collection of pixels
  - One value gives the grey level at one pixel
- Fourier representation: Sum of sine waves
  - One value gives the strength of a specific sine



Basis of a Vector Space

- Every vector in a vector space is a linear combination of basis vectors

$$\begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} = 2\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 1\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + 1\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + 0\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

- Eq. above uses standard (natural) basis
- *k* <u>independent</u> vectors in a *k*-dimensional vector space form a basis
- <u>Orthogonal</u> Basis: every two basis vectors are orthogonal
- <u>Orthonormal</u> Basis: the absolute value of all basis vectors is 1

Transforms: Change of Basis

Standard Basis: Image location

$$\begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} = 2\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 1\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + 1\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + 0\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

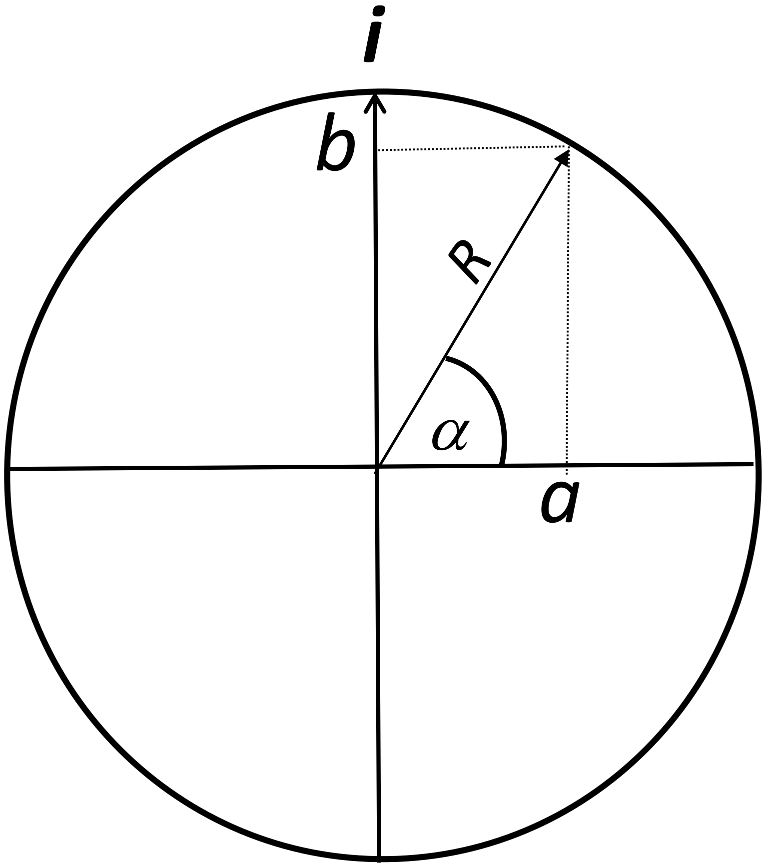
Hadamard Basis (Orthonormal):

$$\begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} = 2\begin{bmatrix} \tfrac{1}{2} & \tfrac{1}{2} \\ \tfrac{1}{2} & \tfrac{1}{2} \end{bmatrix} + 1\begin{bmatrix} \tfrac{1}{2} & -\tfrac{1}{2} \\ \tfrac{1}{2} & -\tfrac{1}{2} \end{bmatrix} + 1\begin{bmatrix} \tfrac{1}{2} & \tfrac{1}{2} \\ -\tfrac{1}{2} & -\tfrac{1}{2} \end{bmatrix} + 0\begin{bmatrix} \tfrac{1}{2} & -\tfrac{1}{2} \\ -\tfrac{1}{2} & \tfrac{1}{2} \end{bmatrix}$$

Jean Baptiste Joseph Fourier (1768-1830)

had crazy idea (1807):

- ***Any*** *periodic function can be rewritten as a weighted sum of sines and cosines of different frequencies.*
- Don’t believe it?
  - Neither did Lagrange, Laplace, Poisson etc.
  - Not translated into English until 1878!
- But it’s true!
  - Called Fourier Series
- Are pictures periodic?
  - If we tile them…



Complex Numbers

$$i^2 = -1$$

$$a + bi = R \cdot e^{i\alpha}$$

$$e^{i\alpha} = \cos(\alpha) + i\sin(\alpha)$$

Absolute Value:

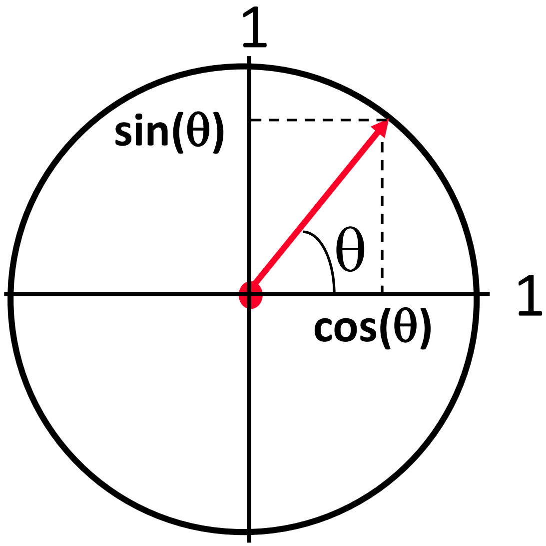
$$R = \sqrt{(a^2 + b^2)}$$

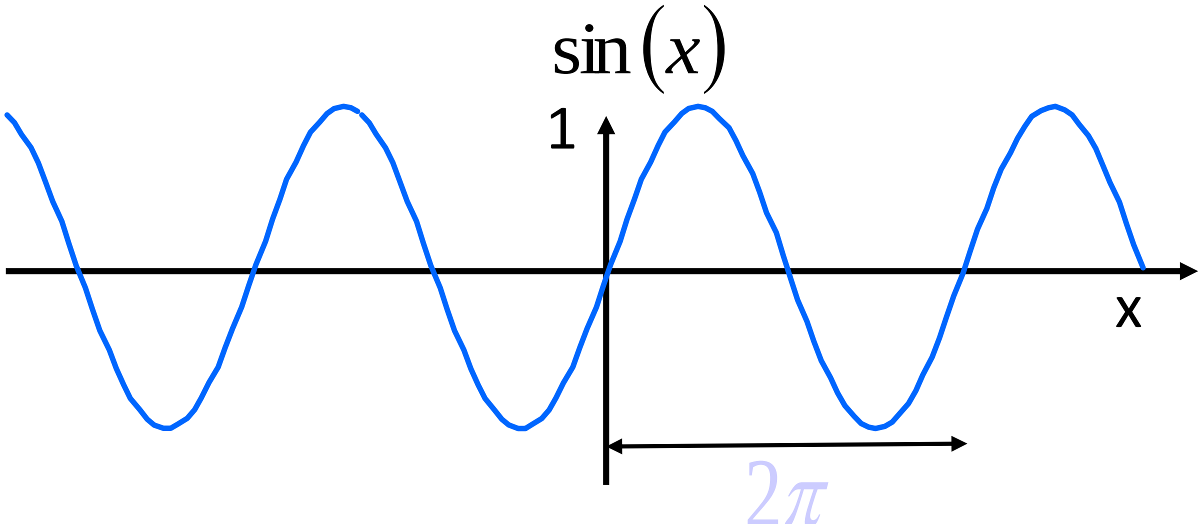
Phase:

$$\alpha = \tan^{-1}\left(\frac{b}{a}\right)$$

$$R_1 e^{i\alpha_1} \cdot R_2 e^{i\alpha_2} = R_1 R_2 e^{i(\alpha_1 + \alpha_2)}$$

Wavelength and Frequency of Co/Sine

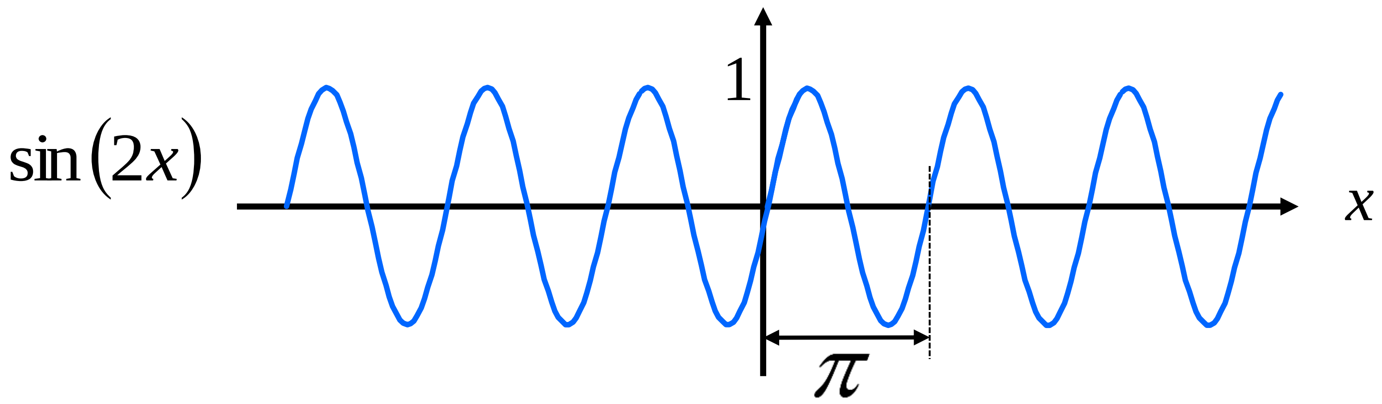


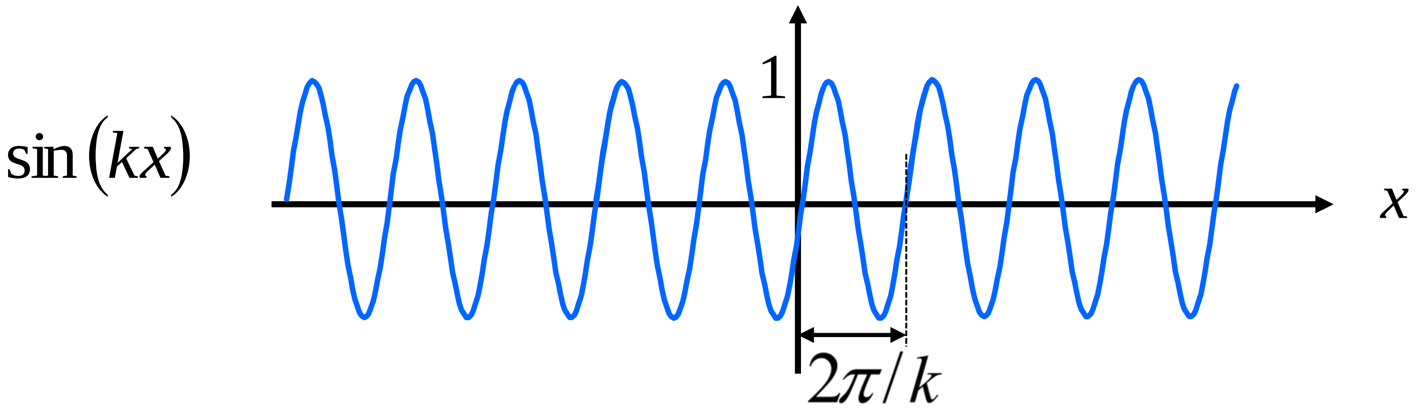


– The wavelength of *sin(x)* is 2π .

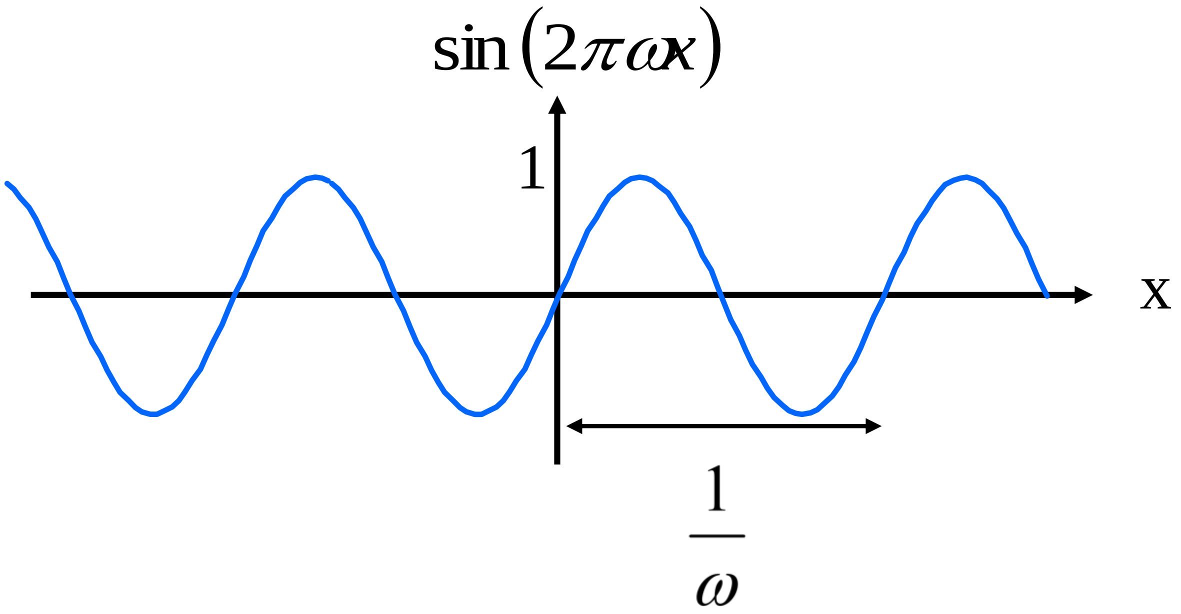
– The frequency is $1/(2\pi)$

- How many waves between 0 and 1





- The wavelength of *sin(ax)* is $2\pi/a$
- The frequency of *sin(ax)* is $a/(2\pi)$

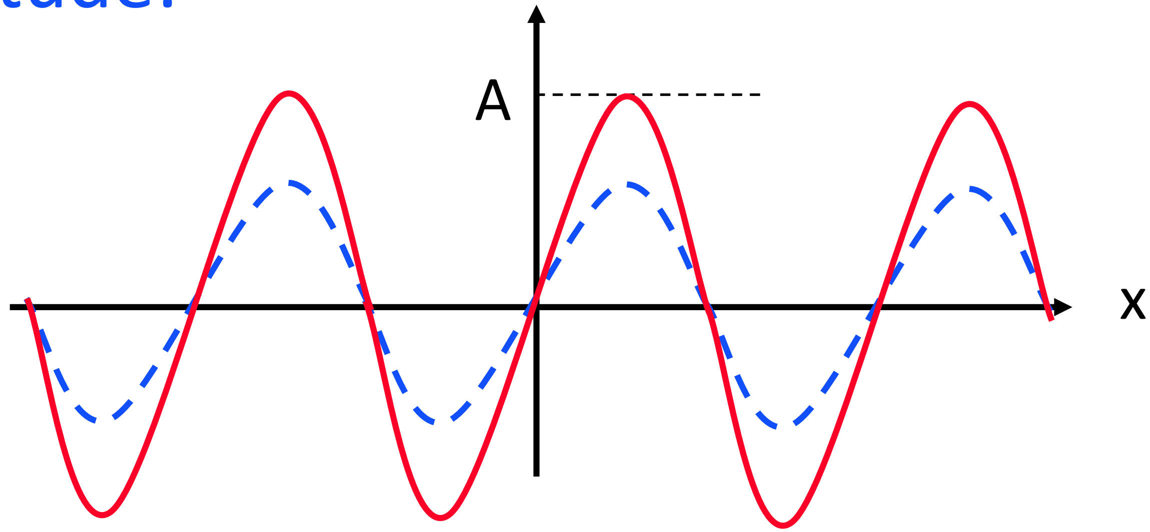


– The wavelength of $\sin(2\pi\omega x)$ is $1/\omega$

– The frequency of $\sin(2\pi\omega x)$ is ω

– Changing Amplitude:

$$A\sin\left(2\pi\omega x\right)$$



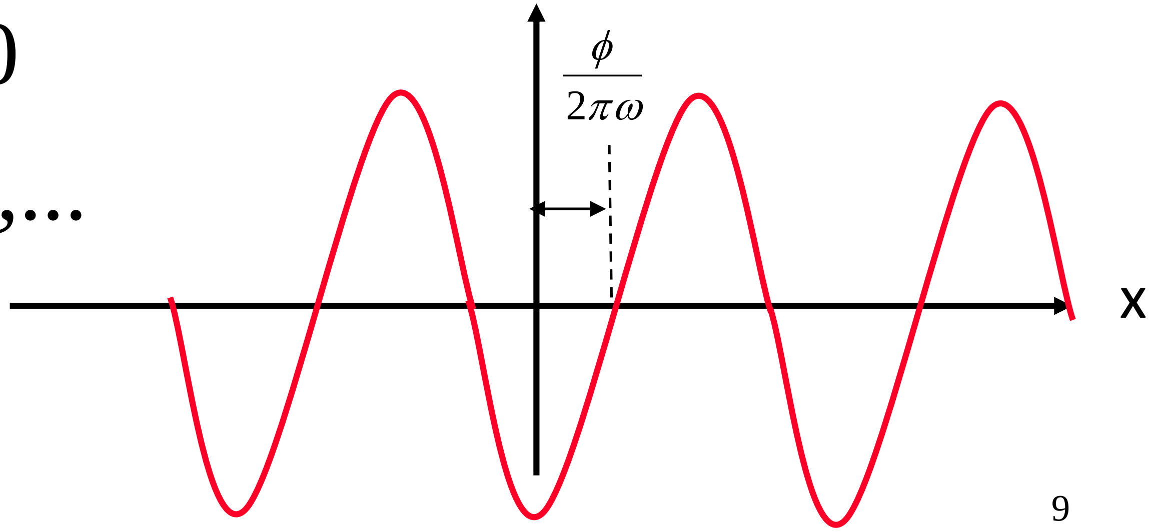
– Changing Phase:

$$A\sin\left(2\pi\omega x-\phi\right)$$

$$\sin(2\pi\omega x-\phi)=0$$

$$2\pi\omega x-\phi=0,\ 2\pi,\ldots$$

$$x=\frac{\phi}{2\pi\omega}$$



1-D Discrete Fourier Transform

$$\big(f(0), f(1), \ldots, f(N-1)\big) \;\Rightarrow\; \big(F(0), F(1), \ldots, F(N-1)\big)$$

1D Fourier Transform

$$F(u) = \frac{1}{N} \sum_{x=0}^{N-1} f(x) e^{\frac{-2\pi i u x}{N}}$$

$$F(0) = \tfrac{1}{N} \sum_{x=0}^{N-1} f(x) e^{0} = \bar{f}$$

1D Inverse Fourier Transform

$$f(x) = \tfrac{1}{1} \sum_{u=0}^{N-1} F(u) e^{\frac{2\pi i u x}{N}}$$

f is a sum of sines and cosines

Complexity: $O(N^2)$ $(10^6 \rightarrow 10^{12})$

FFT: $O(N \log N)$ $(10^6 \rightarrow 10^7)$

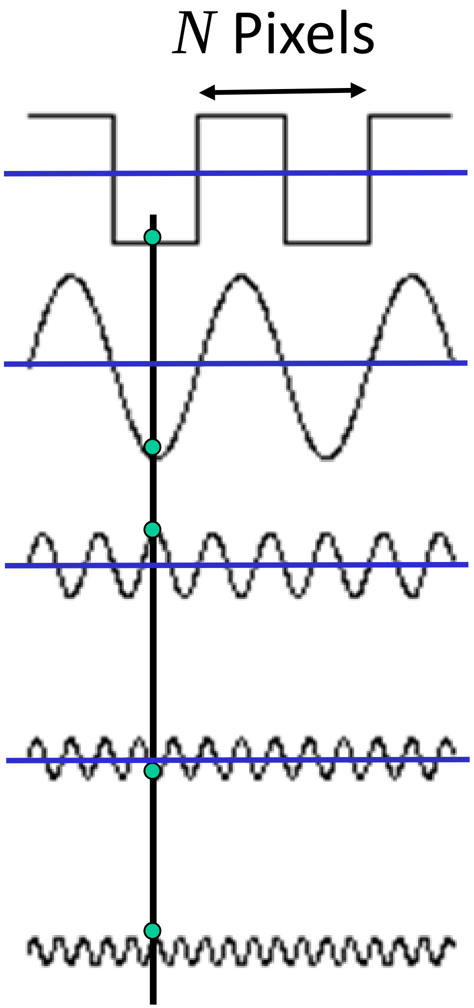
Fourier Basis Vectors

- Computing f from F $\boxed{f(x) = \sum_{u=0}^{N-1} F(u) e^{\frac{2\pi i u x}{N}}}$

- Fourier Basis $\boxed{e^{\frac{2\pi i u x}{N}} = \cos(\frac{2\pi u x}{N}) + i \sin(\frac{2\pi u x}{N})}$

- For each frequency $0 \le u \le N-1$ we have the <u>basis vector</u> above $(x = 0, 1, \ldots N-1)$

1D Example: Square Wave



- A square wave as a sum of:

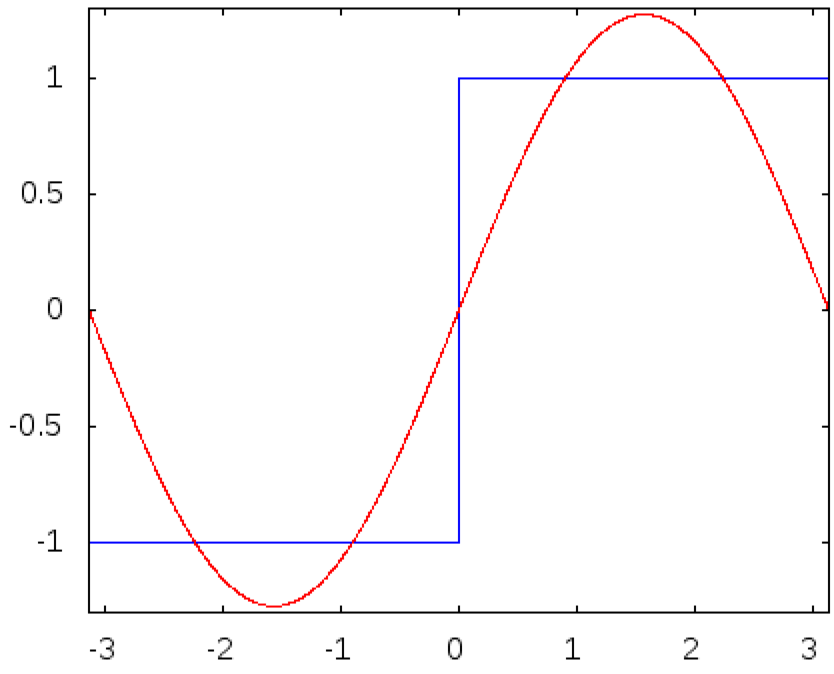
$$\sin\frac{2\pi x}{N}$$

$$-\tfrac{1}{3}\sin\frac{2\pi 3x}{N}$$

$$+\tfrac{1}{5}\sin\frac{2\pi 5x}{N}$$

$$-\tfrac{1}{7}\sin\frac{2\pi 7x}{N}$$

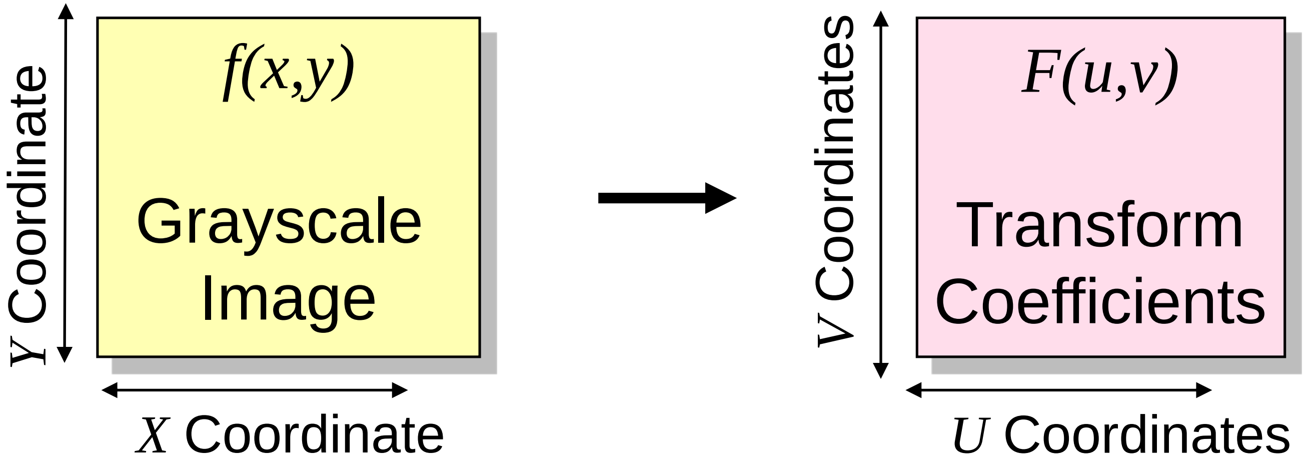
Trans Fourier is (0, 1, 0, -1/3, 0, 1/5, 0 -1/7, ...)


1
0.5
0
-0.5
-1
-3
-2
-1
0
1
2
3

Fourier Transform of Pictures: Change of Basis in 2D

Image Domain
Natural Basis

Frequency Domain
Fourier Basis



The transform coefficients are complex numbers arranged in a 2D array.

2D Discrete Fourier

Fourier Transform

$$F(u,v) = \tfrac{1}{N}\sum_{x=0}^{N-1}\sum_{y=0}^{N-1} f(x,y)\, e^{\frac{-2\pi i(ux+vy)}{N}}$$

Inverse Fourier Transform

$$f(x,y) = \tfrac{1}{N}\sum_{u=0}^{N-1}\sum_{v=0}^{N-1} F(u,v)\, e^{\frac{2\pi i(ux+vy)}{N}}$$

$$F(0,0) = \tfrac{1}{N}\sum_{x=0}^{N-1}\sum_{y=0}^{N-1} f(x,y)\, e^{\frac{-2\pi i(0x+0y)}{N}} = \tfrac{1}{N}\sum_{x=0}^{N-1}\sum_{y=0}^{N-1} f(x,y)$$

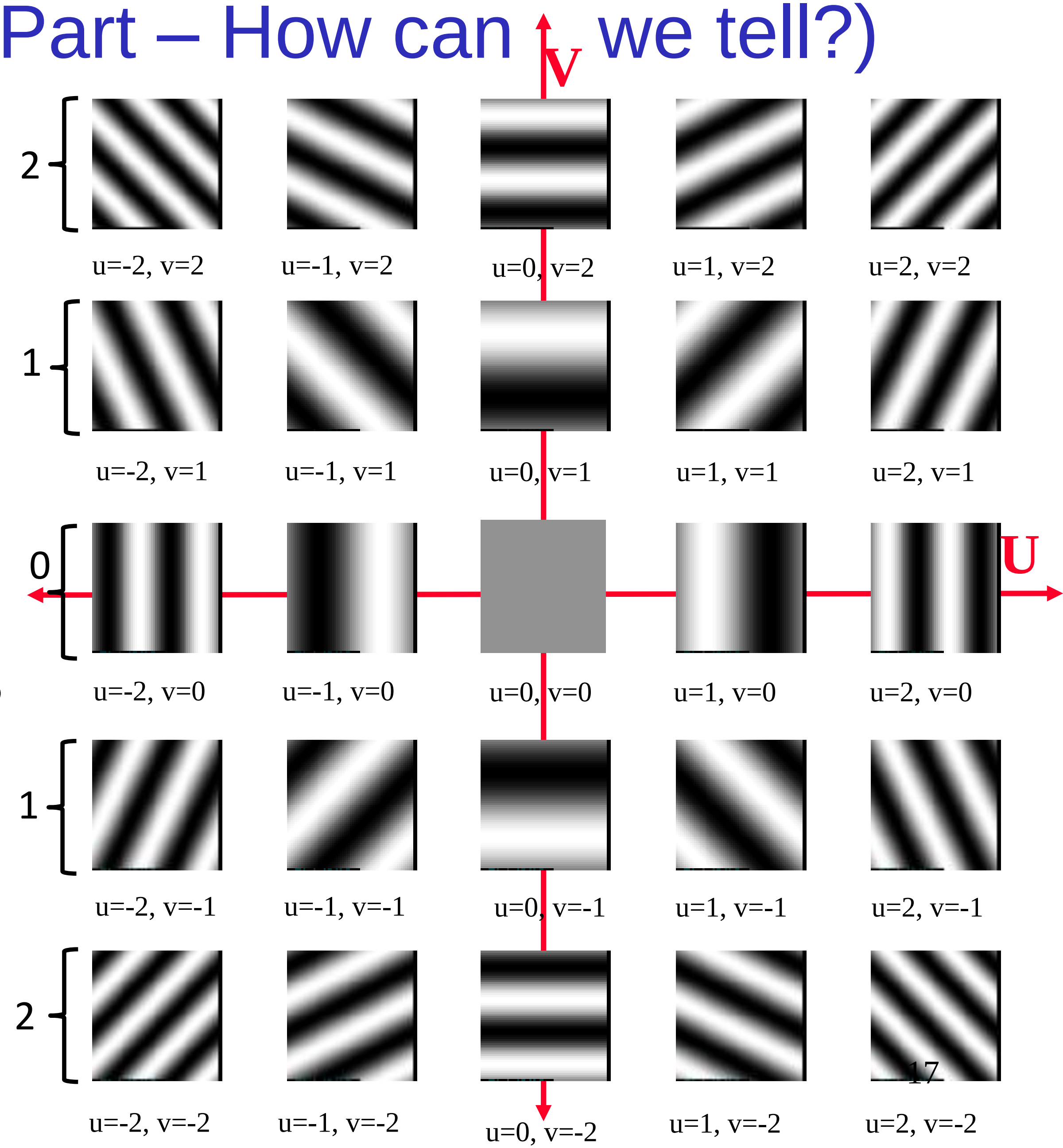
$$= N \cdot \bar{f}$$

2D Fourier Basis Functions

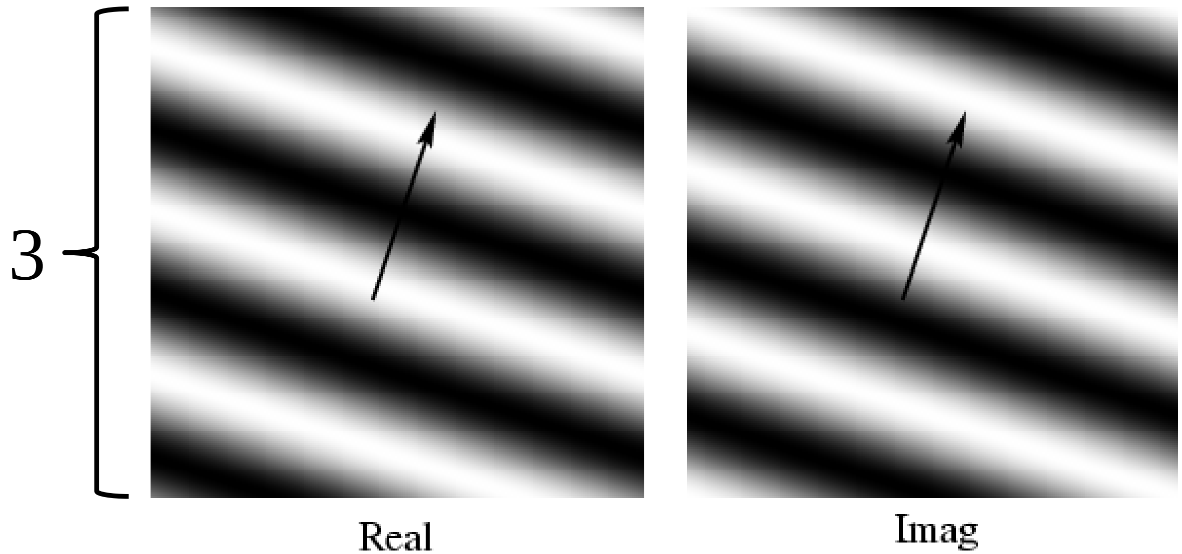
(Imaginary Part – How can we tell?)

$$e^{\frac{2\pi i(ux+vy)}{N}}$$

Every basis function with frequency *(u,v)* is multiplied by *F(u,v)* specifying
- Amplitude
- Phase (Shift)

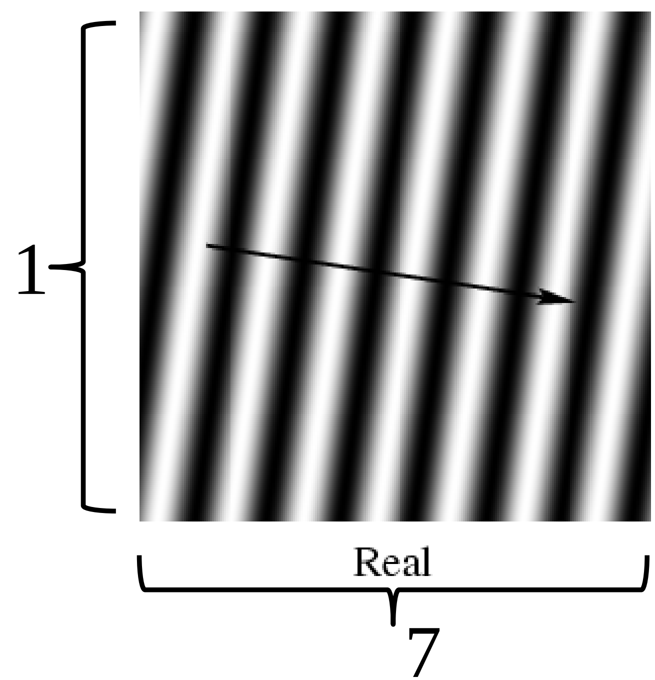


(u,v)=(1,-3)

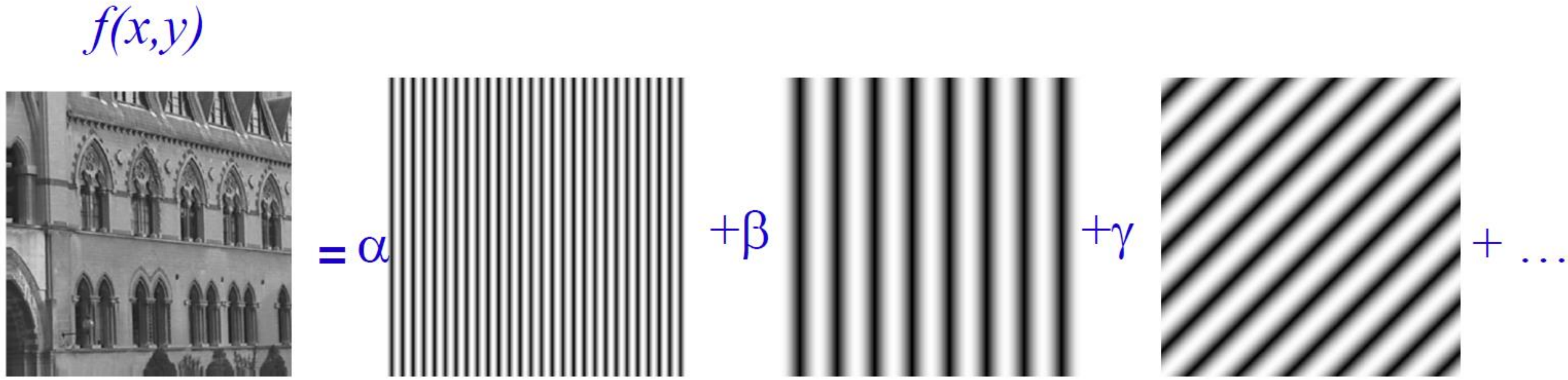


$$\boxed{e^{\frac{2\pi i(ux+vy)}{N}}}$$

(u,v)=(7,1)



Summary



Fourier Spectrum

Fourier (complex number): $$\boxed{F(u) = R(u) + iI(u)}$$

Fourier Spectrum $$\boxed{|F(u)| = \sqrt{R^2(u) + I^2(u)}}$$

Fourier Phase $$\boxed{\theta(u) = \tan^{-1}\left(I(u)/R(u)\right)}$$

Fourier: $$\boxed{F(u) = |F(u)|\exp(i\theta)}$$

Display Fourier Spectrum as Picture

1. Compute $\log\left(\left|F(u)\right|+1\right)$

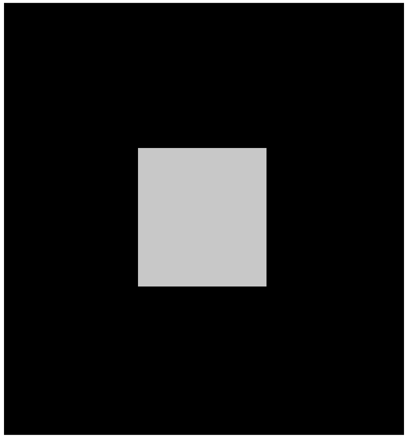
2. Scale to full grey-level range

3. Move (u=0,v=0) to center of image
   (Shift by *N/2*)

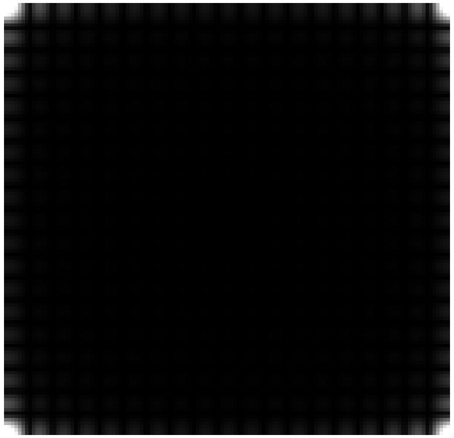
Example for downscaling [0..100] to [0..10]:

Original f	0	1	2	4	100
Divide by 10	0	0	0	0	10
Log (1+ f)	0	0.69	1.01	1.61	4.62
Scaled to 10	0	1	2	4	10

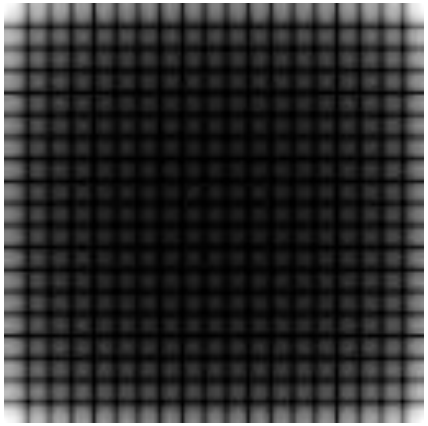
Display Fourier Spectrum



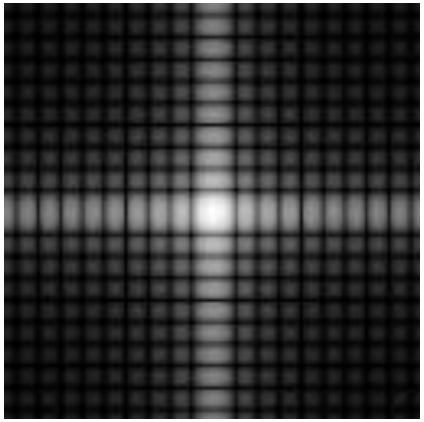
Original



$|F(u,v)|$



$\log(1 + |F(u,v)|)$



Shift (u=0, v=0) to center

- Question: When does $|F(0,0)|$ have the highest value in the Fourier spectrum?

Decomposition

$$F(u,v)=\frac{1}{N}\sum_{x=0}^{N-1}\sum_{y=0}^{N-1}f(x,y)e^{\frac{-2\pi i(ux+vy)}{N}}$$

$$F(u,v)=\frac{1}{N}\sum_{x=0}^{N-1}\left(e^{\frac{-2\pi iux}{N}}\cdot\sum_{y=0}^{N-1}f(x,y)e^{\frac{-2\pi ivy}{N}}\right)=$$

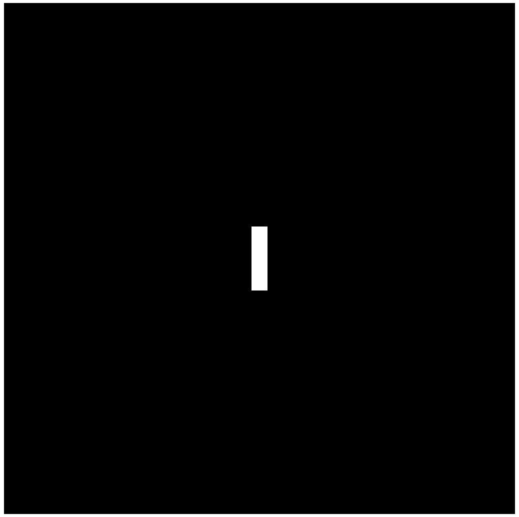
$$F(u,v)=\frac{1}{N}\sum_{x=0}^{N-1}\left(e^{\frac{-2\pi iux}{N}}\cdot F(x,v)\right)$$

Decomposition (II)

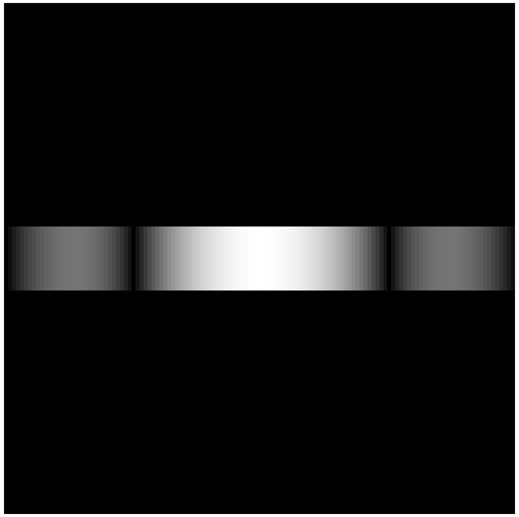
- 2-D Fourier Transform can be computed using 1-D Fourier
  - Compute 1-D Fourier on each column

    On result:
  - Compute 1-D Fourier on each row
  - (Multiply by *N*?)
- 1-D Fourier Transform is enough to compute Fourier of ANY dimension

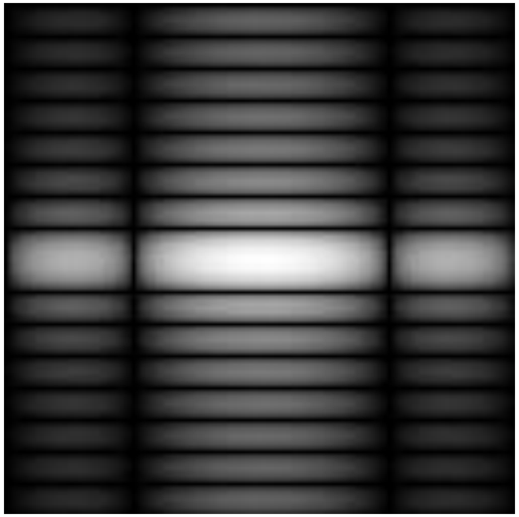
Decomposition Example



Original picture

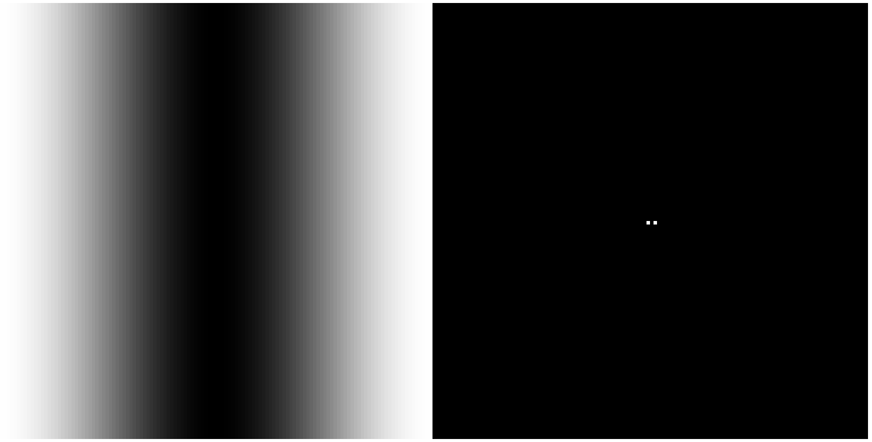


Fourier in rows

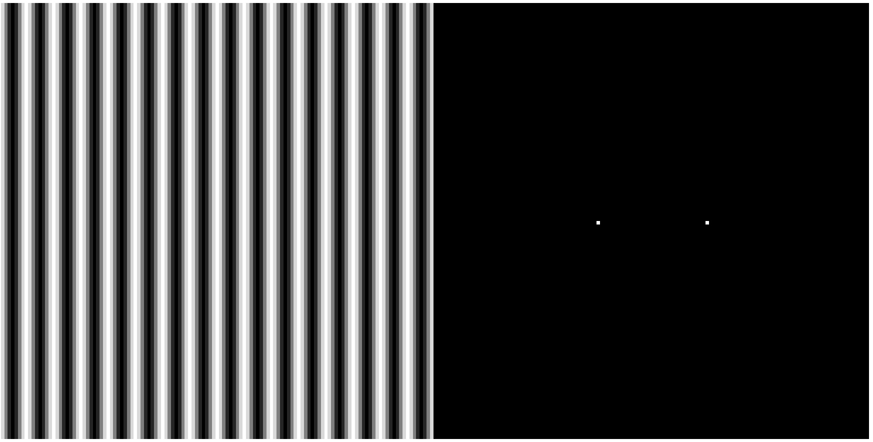


Fourier in columns

Fourier of Cos($2\pi nx/N$)

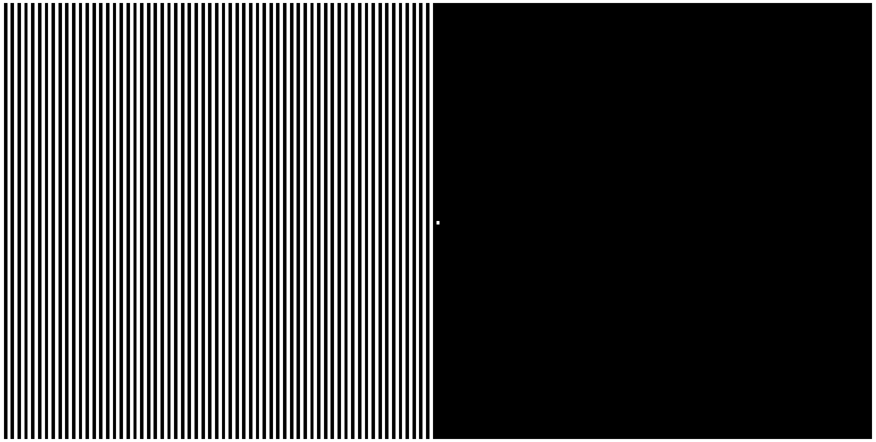


$n=1$



$n=N/8$

Why
2 points?



$n=N/2$

Periodicity & Symmetry

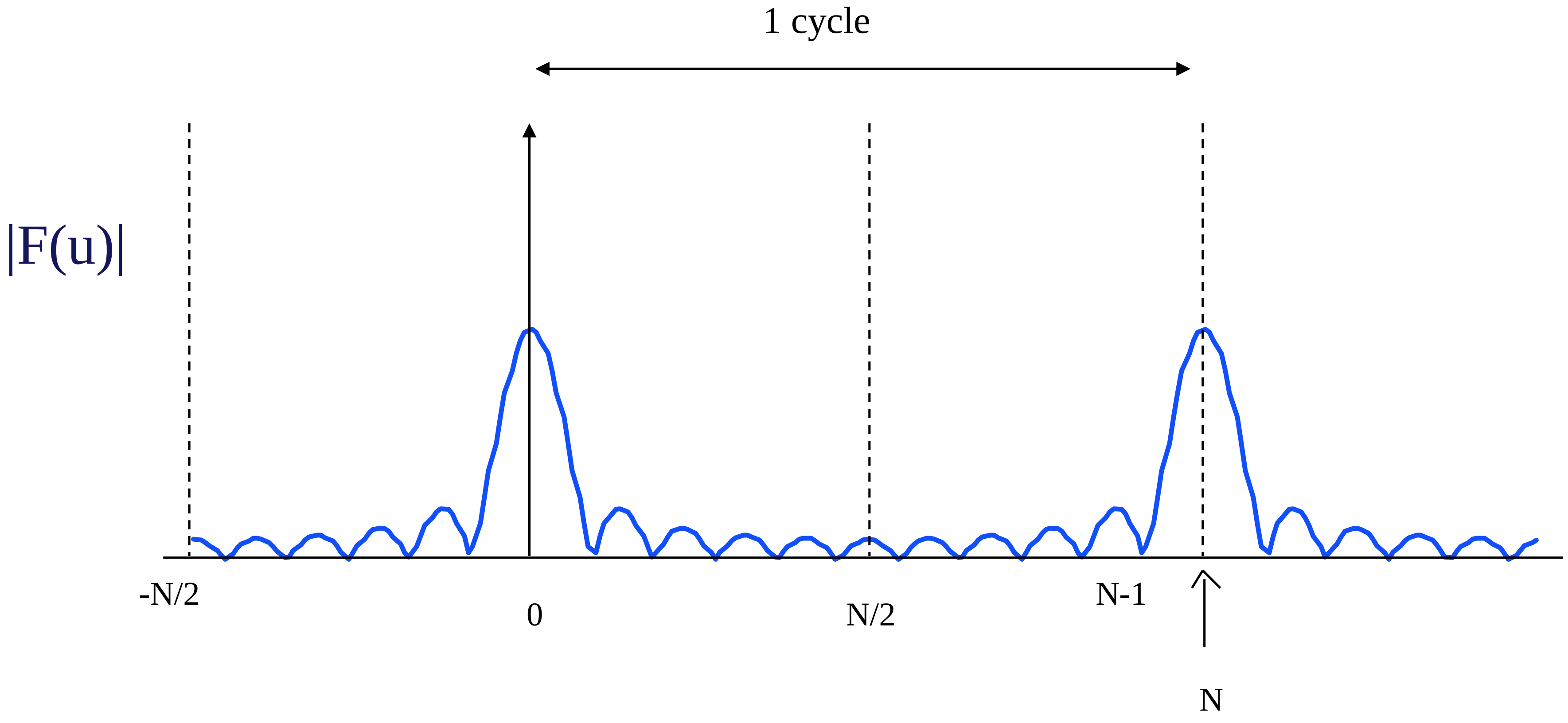
$$F(u,v) = F(u+N,v) = F(u,v+N) =$$
$$= F(u+N,v+N)$$

$$F(u,v) = F^*(-u,-v)$$

$$(a+bi)^* = (a-bi)$$

$$|F(u,v)| = |F(-u,-v)|$$

Periodicity & Symmetry (1D)



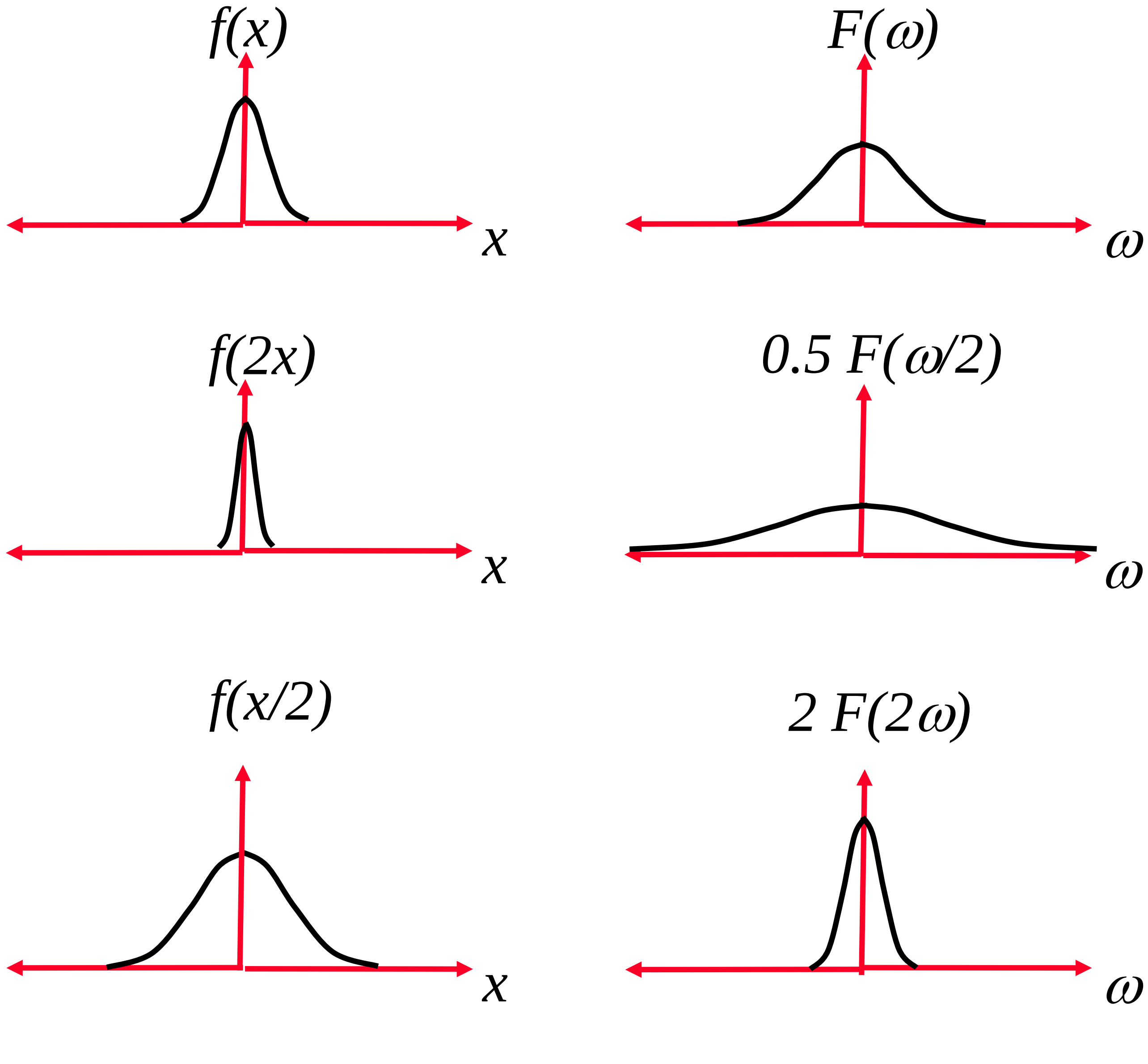
Linearity (Φ is Trans Fourier)

$$\Phi\left(f_1(x,y)+f_2(x,y)\right)=\Phi\left(f_1(x,y)\right)+\Phi\left(f_2(x,y)\right)$$

$$\Phi\left(a\cdot f(x,y)\right)=a\cdot\Phi\left(f(x,y)\right)$$

$$\Phi\left(f(ax,by)\right)=\frac{1}{|ab|}F\left({}^{u}\!/_{a},{}^{v}\!/_{b}\right)$$

Change Scale: Examples



Derivatives Using Fourier

Inverse Fourier Transform

$$f(x) = \boxed{\sum_u F(u) e^{\frac{2\pi i u x}{N}}}$$

$$f'(x) = \left(\sum_u F(u) e^{\frac{2\pi i u x}{N}} \right)' = \sum_u F(u) \left(e^{\frac{2\pi i u x}{N}} \right)' =$$

$$= \frac{2\pi i}{N} \boxed{\sum_u u F(u) e^{\frac{2\pi i u x}{N}}}$$

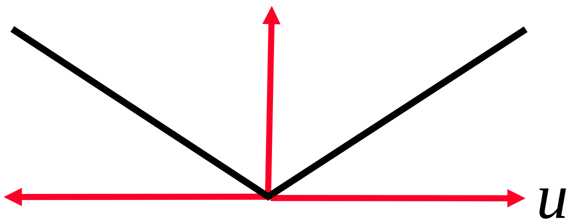
Derivatives II

- To compute the *x* derivative of *f* **(up to a constant)**:
  1. Compute the Fourier Transform *F*
  2. Multiply each Fourier coefficient *F(u,v)* by *u*
  3. Compute the Inverse Fourier Transform
- To compute the *y* derivative of *f* **(up to a constant)**:
  1. Compute the Fourier Transform *F*
  2. Multiply each Fourier coefficient *F(u,v)* by *v*
  3. Compute the Inverse Fourier Transform

Derivative as a Fourier Filter

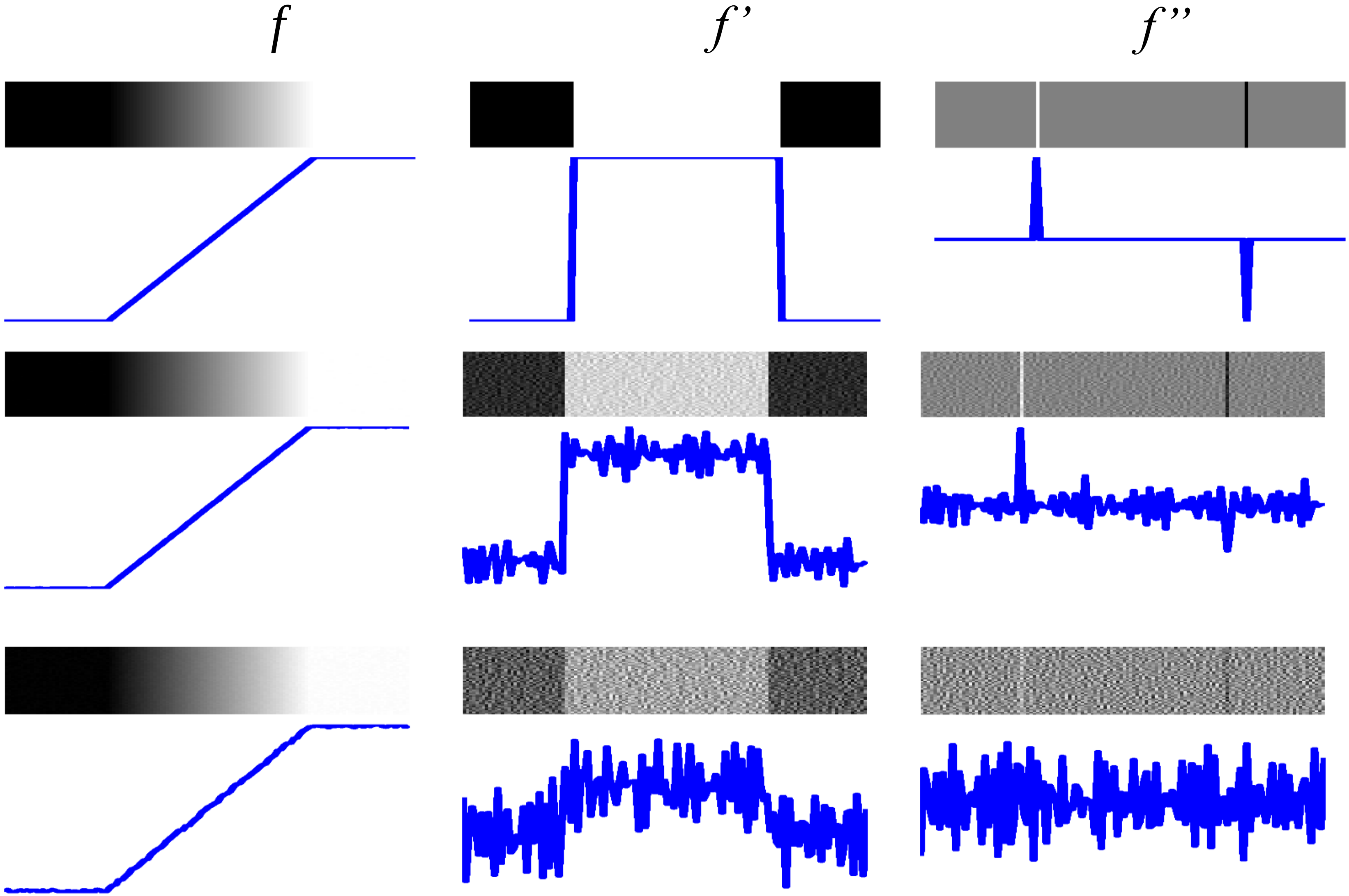
$$f'(x) = \frac{2\pi i}{N} \sum_u u F(u) e^{\frac{2\pi i u x}{N}}$$

- Multiply Fourier with:



- Amplifies higher frequencies
  - **Noise** has more high frequency than normal image.
  - Derivatives amplify noise
- Cancels DC (*F(0)*)

Effect of Noise on Derivatives



Convolution Theorem

$$\Phi\left(f * g\right) = F \cdot G$$

$$\Phi\left(f \cdot g\right) = F * G$$

Convolution in spatial domain (**$f(x,y)$**) is equivalent to **pointwise multiplication** in frequency domain (**$F(u,v)$**)

Convolution by Fourier:

$$f * g = \Phi^{-1}\left(F \cdot G\right) = \Phi^{-1}\left(\Phi\left(f\right) \cdot \Phi\left(g\right)\right)$$

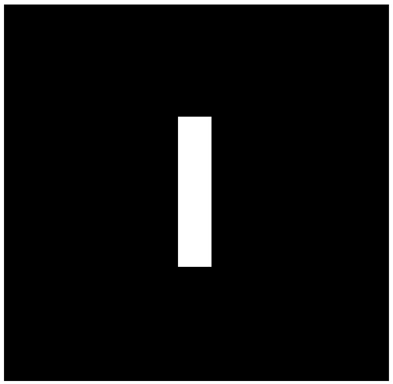
FFT reduces complexity of convolution:

$O(N^2) \rightarrow O(N \log N)$

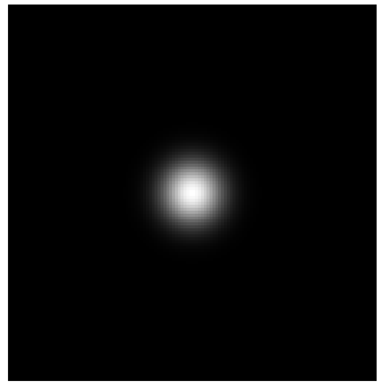
Filtering in the Frequency Domain

Image

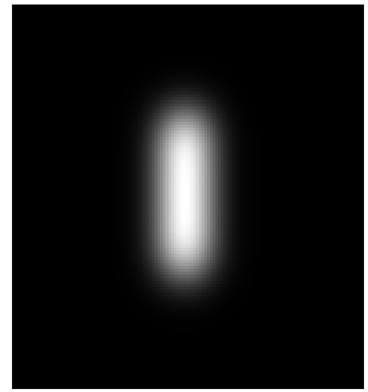
$f(x,y)$



$g(x,y)$

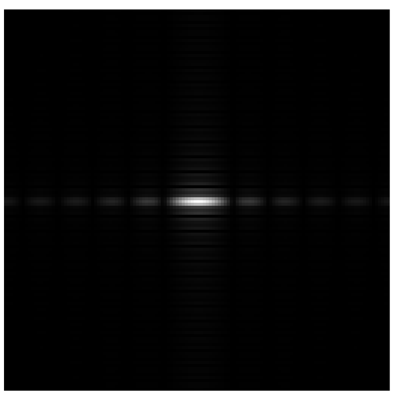


$f * g\ (x,y)$

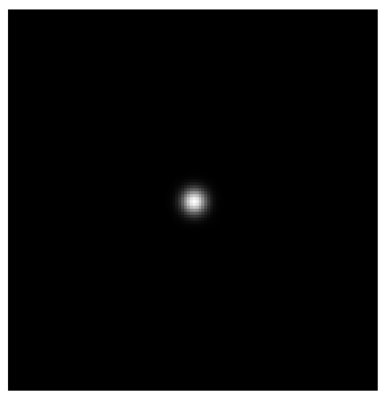


Fourier Spectrum

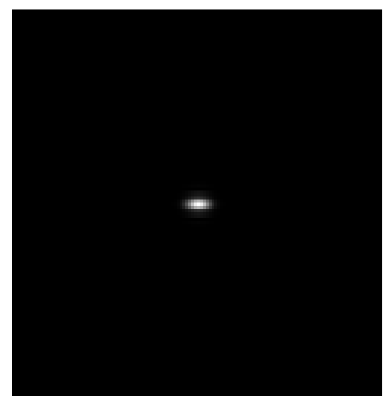
$F(u,v)$



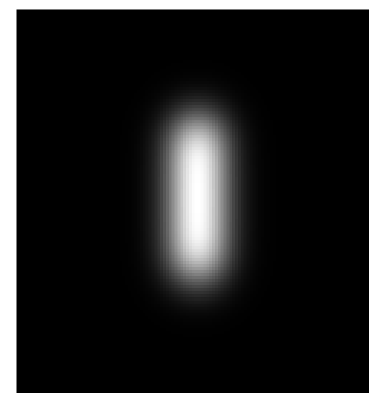
$G(u,v)$



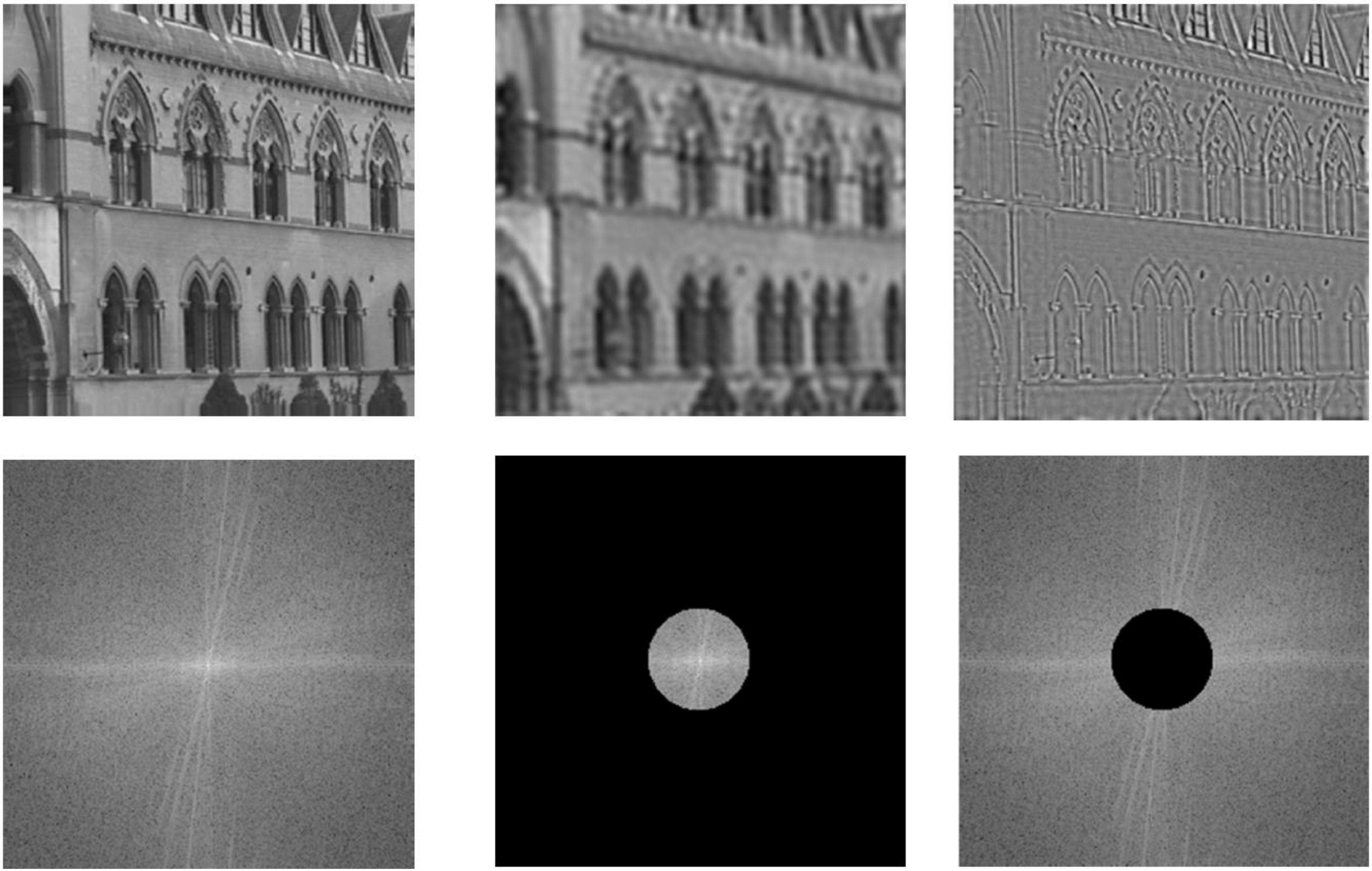
$F(u,v) \cdot G(u,v)$



$\Phi^{-1}[F(u,v) \cdot G(u,v)]$

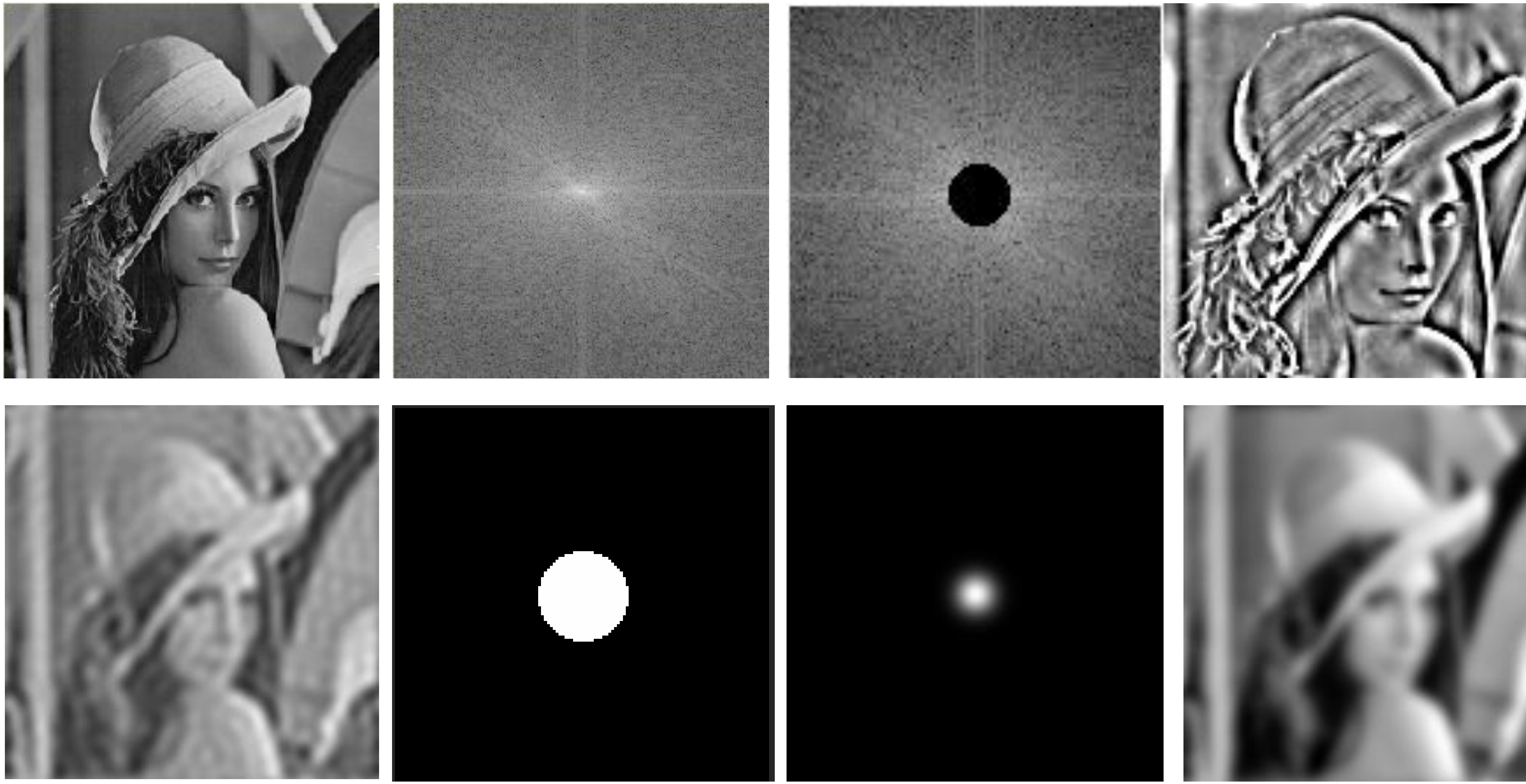


original | low pass | high pass



Compare Filters

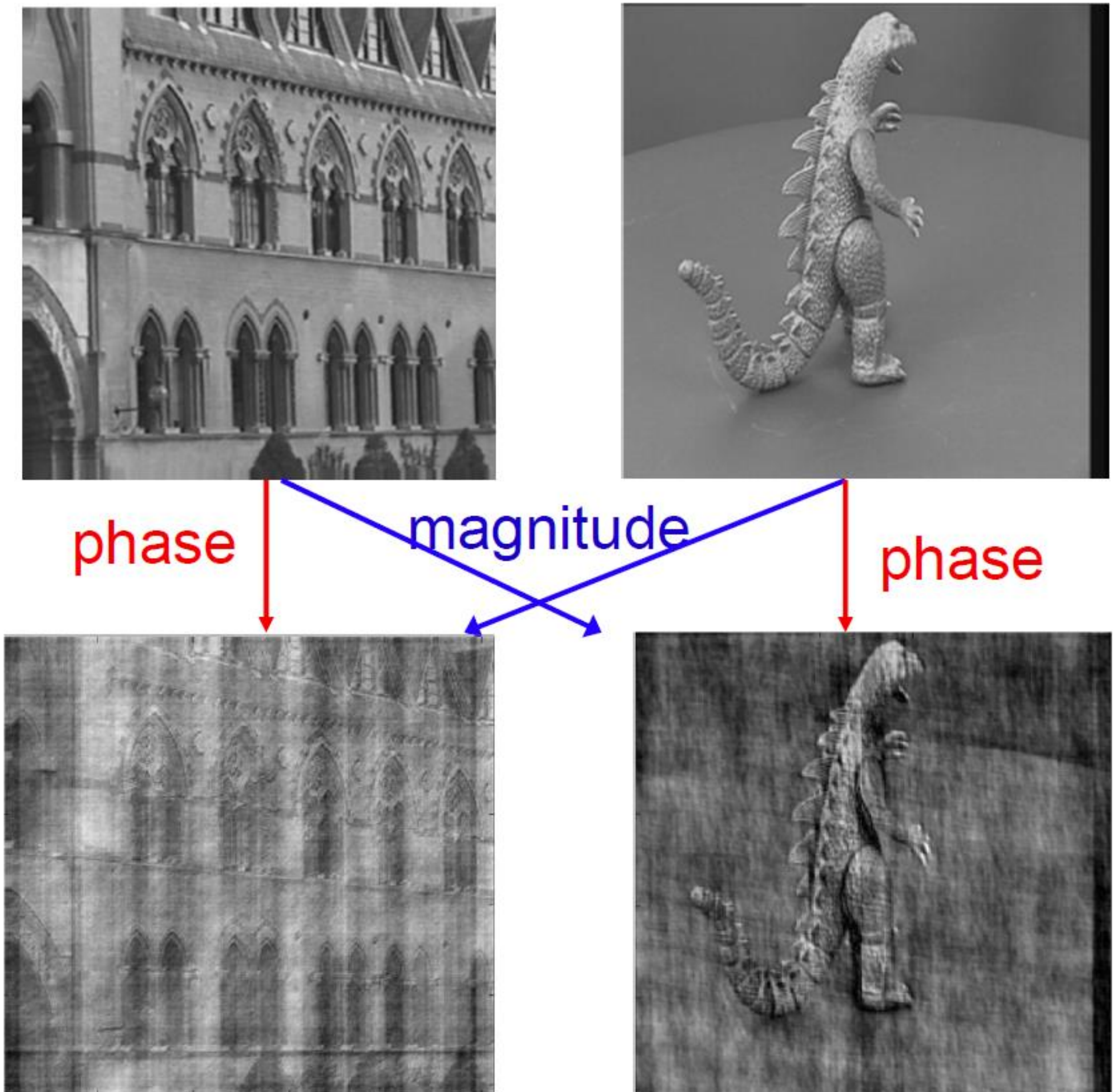
Original Fourier Hi-Pass Box Filter



Low Pass
Box Filter

Low Pass
Gaussian Filter

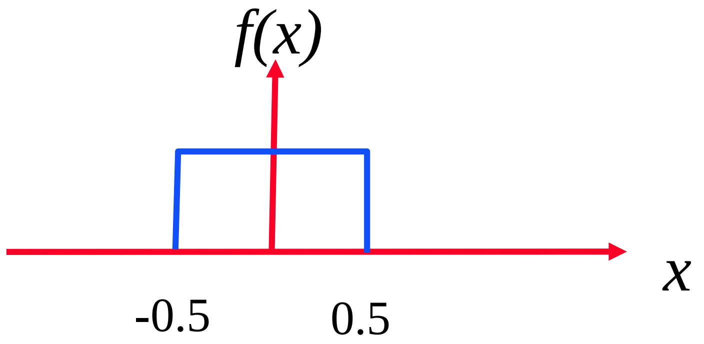
Effects of Phase and Magnitude

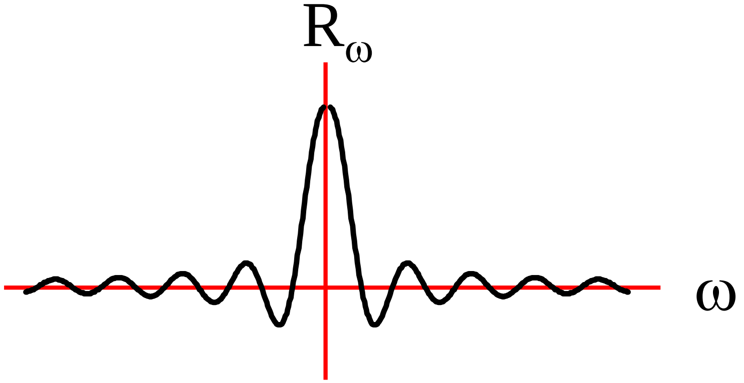


Fourier of Special Functions

The Window (Box) Function (Rect):

$$\operatorname{rect}(x) = \begin{cases} 1 & \text{if} \quad |x| < \dfrac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$

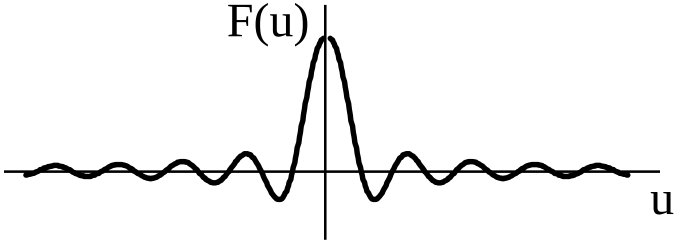




$$F(\omega) = \frac{\sin(\pi\omega)}{\pi\omega} = \operatorname{sinc}(\pi\omega)$$

Low Pass: Fourier & Image

- Image: convolve by box Fourier: Multiply by Sinc
- (0 0 1 1 0 0)



- Image: (0 0 1 1 0 0) * (0 0 1 1 0 0) = (0 1 2 1 0 0)
- Fourier: Multiply by Sinc^2
- Image: Gaussian $\Leftrightarrow$ Fourier: Gaussian
- Multiply Fourier by box $\Leftrightarrow$ Convolve image with Sinc
- Blur image w. Gaussian $\Leftrightarrow$ Multiply Fourier Gaussian

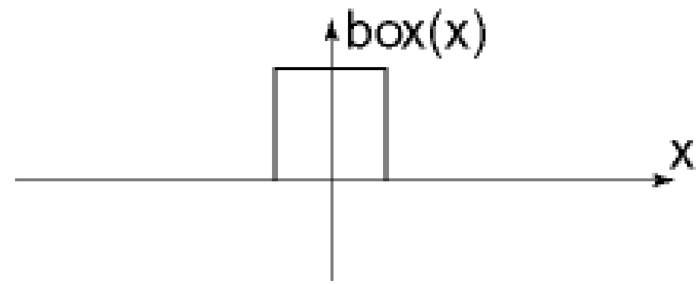
Fourier Transform Pairs

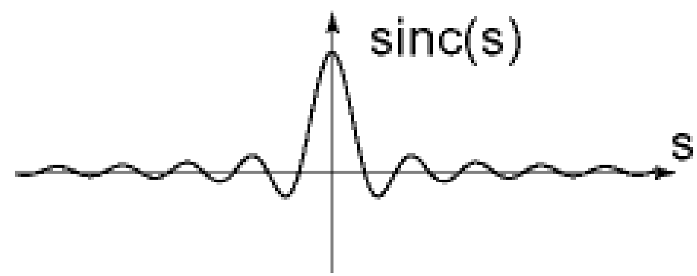
$f(x)$

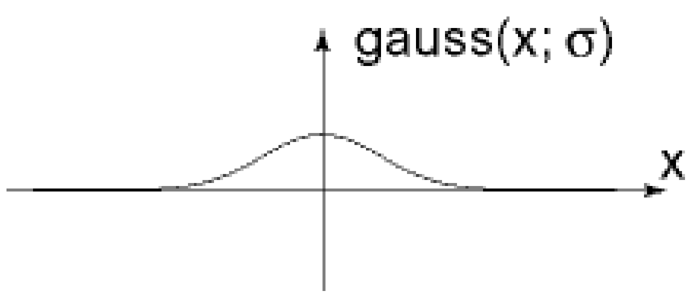
Spatial domain

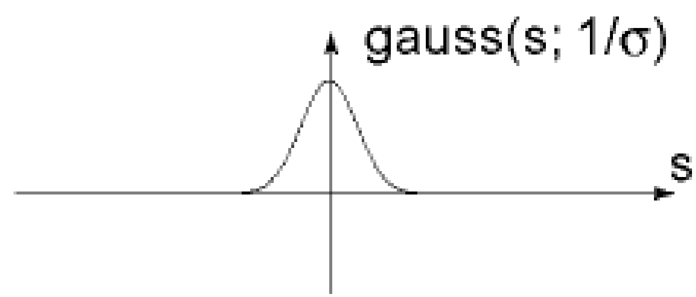
$F(s) = \int_{-\infty}^{\infty} f(x) e^{-i2\pi sx} dx$

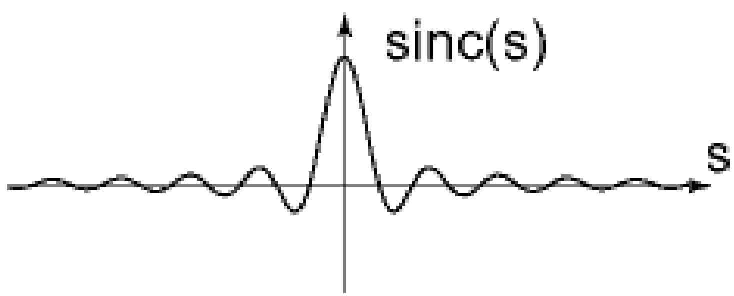
Frequency domain

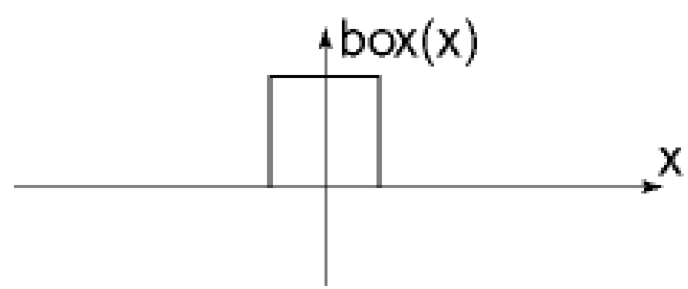








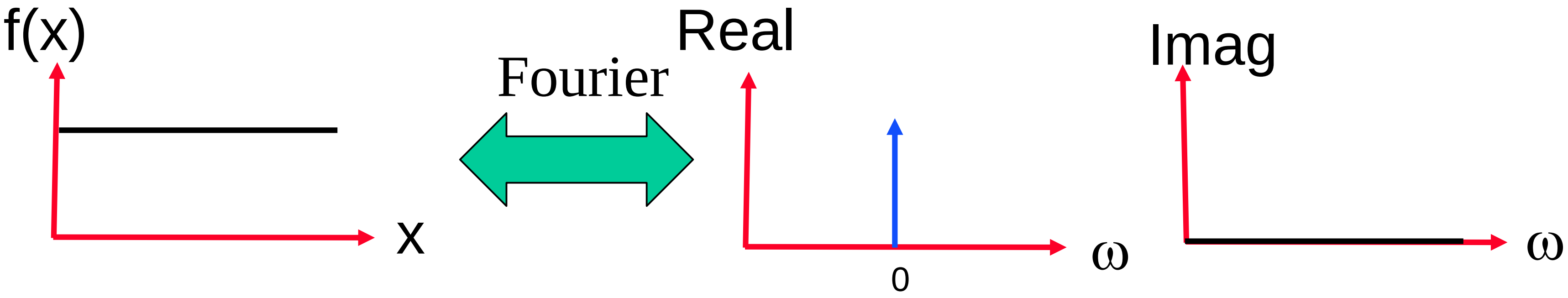




Fourier of Special Functions

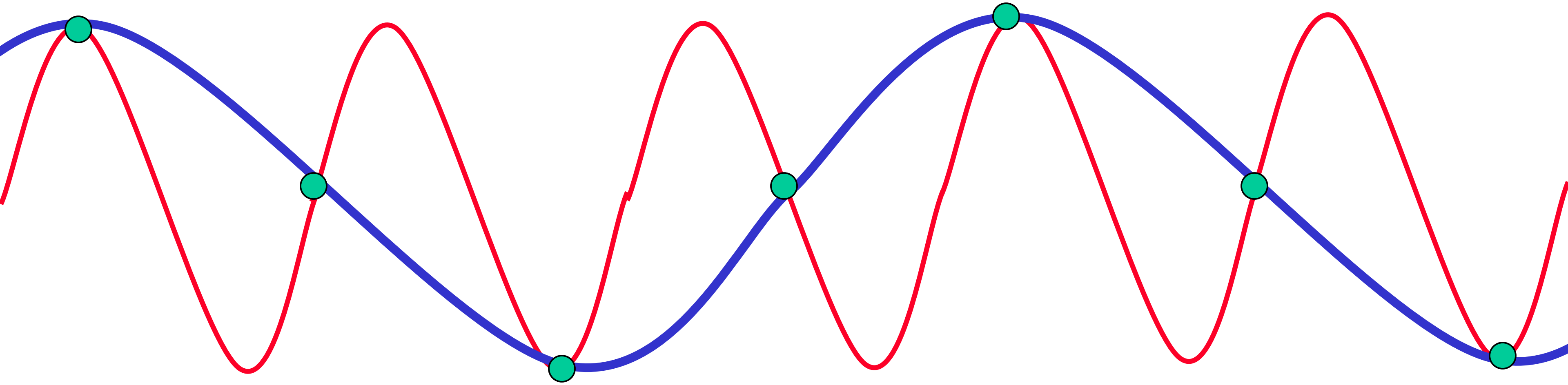
The Constant Function:

$$f(x) = 1 \qquad F(u) = \frac{1}{N}\sum_{x=0}^{N-1} 1\, e^{\frac{-2\pi i u x}{N}}$$



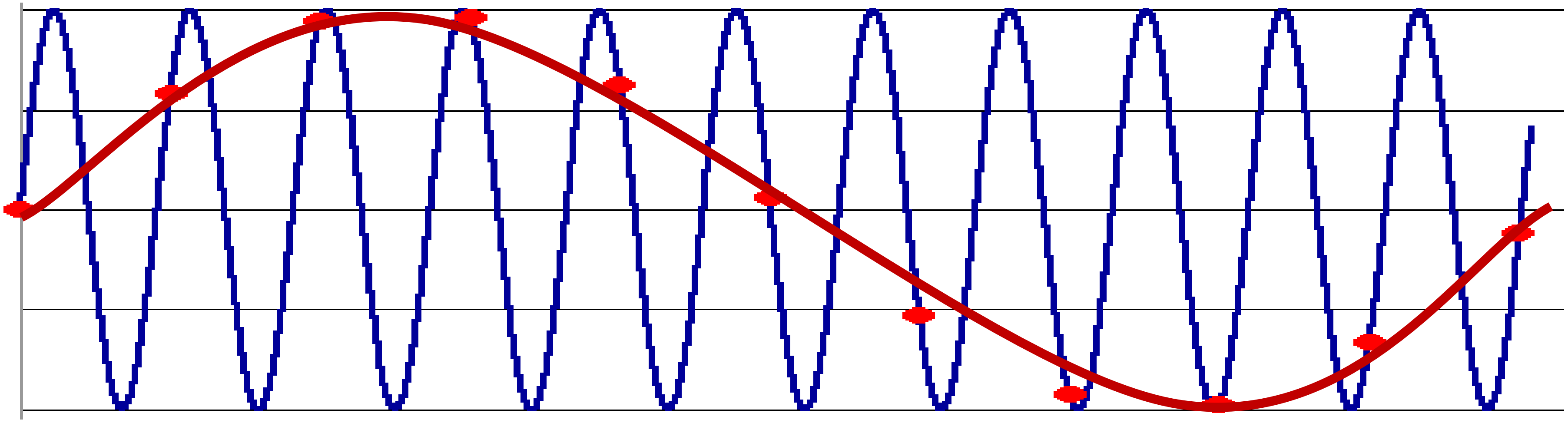
Aliasing

- Sampling can result in aliasing.
- Example: Sampling at 1.5π



- Sampling distance should be less than ½ of wavelength

Aliasing



Sampling

The most important topic in course!

- **Blur before you sample** (Low-pass filter: reduce the highest frequencies)
- Sampling without low-pass results in aliasing.

- How NOT to shrink an image:
  - sample every other pixel

- Blur before you sample!

Image Aliasing Example

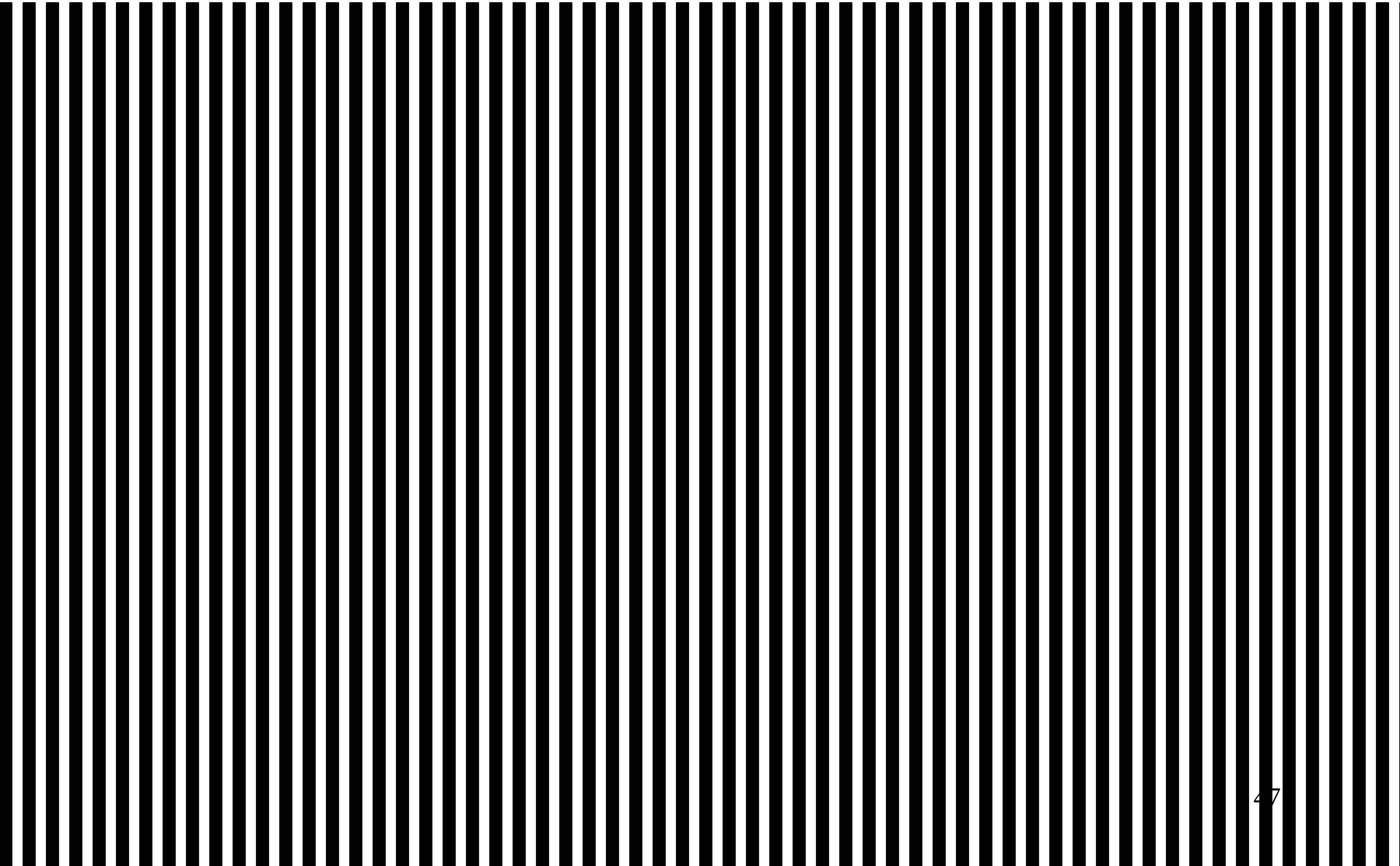
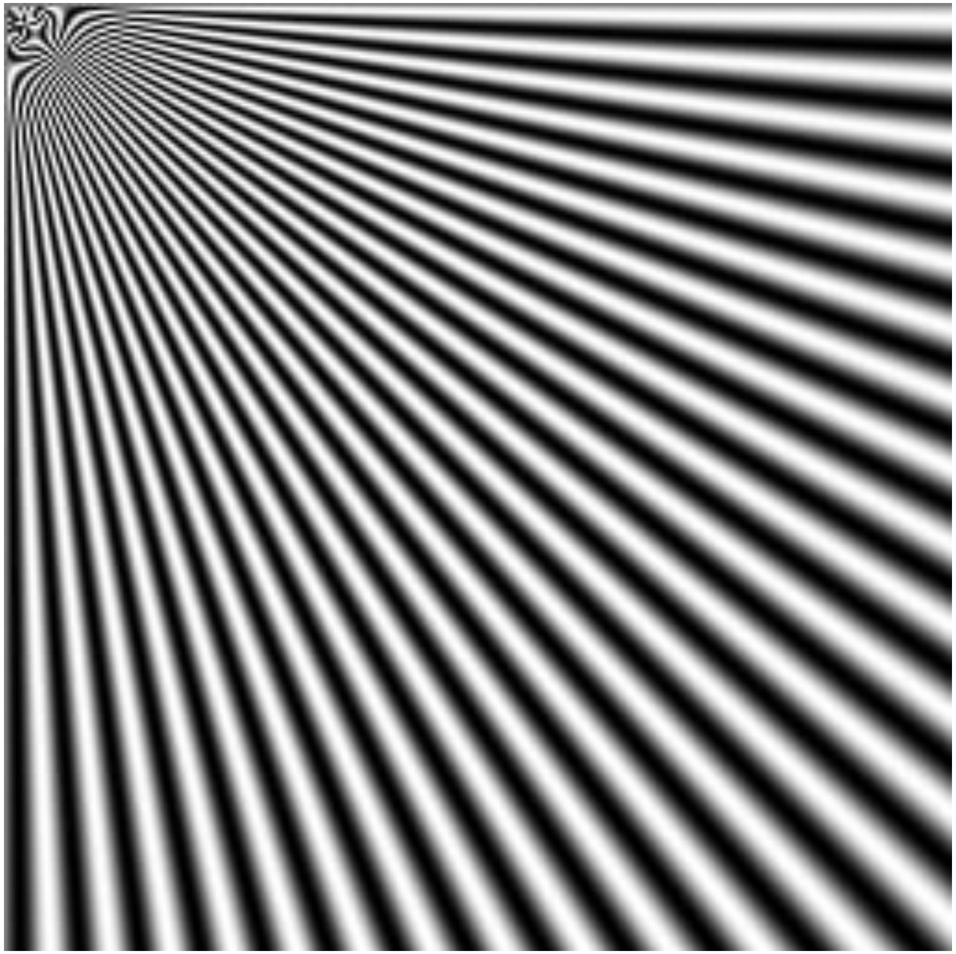


Image Aliasing Example

- Assume a column is 1 pixel wide.
- Sampling every second column will give either a solid black or a solid white.
- Blur before sample will give a solid gray regardless of shift.

Aliasing



Aliasing

Image Reduction ($N \times N \rightarrow N/2 \times N/2$)

1. **Blur** and Subsample every 2nd pixel
2. Use Fourier
   1. Compute Fourier ($N \times N$)
   2. Crop Fourier ($N/2 \times N/2$): Ideal low pass
   3. Compute Inverse Fourier

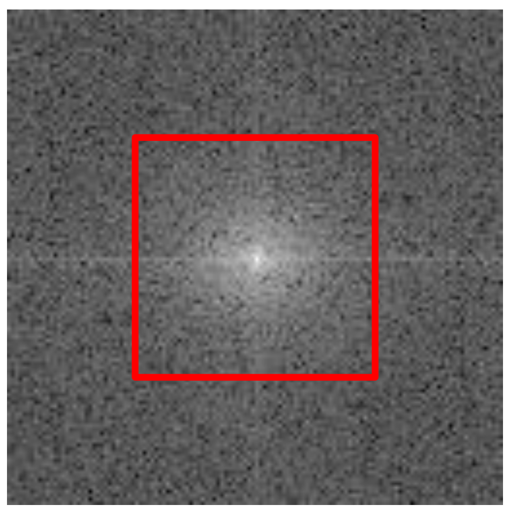
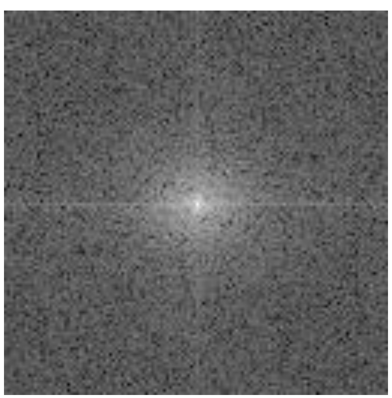
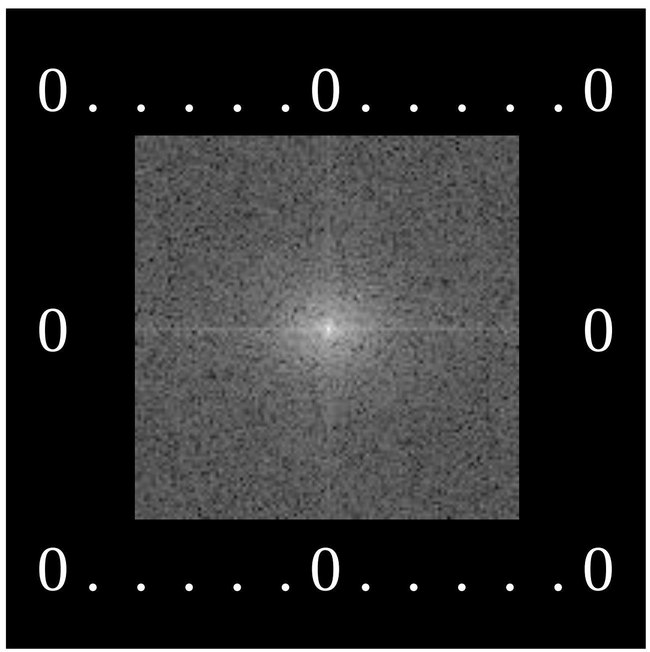
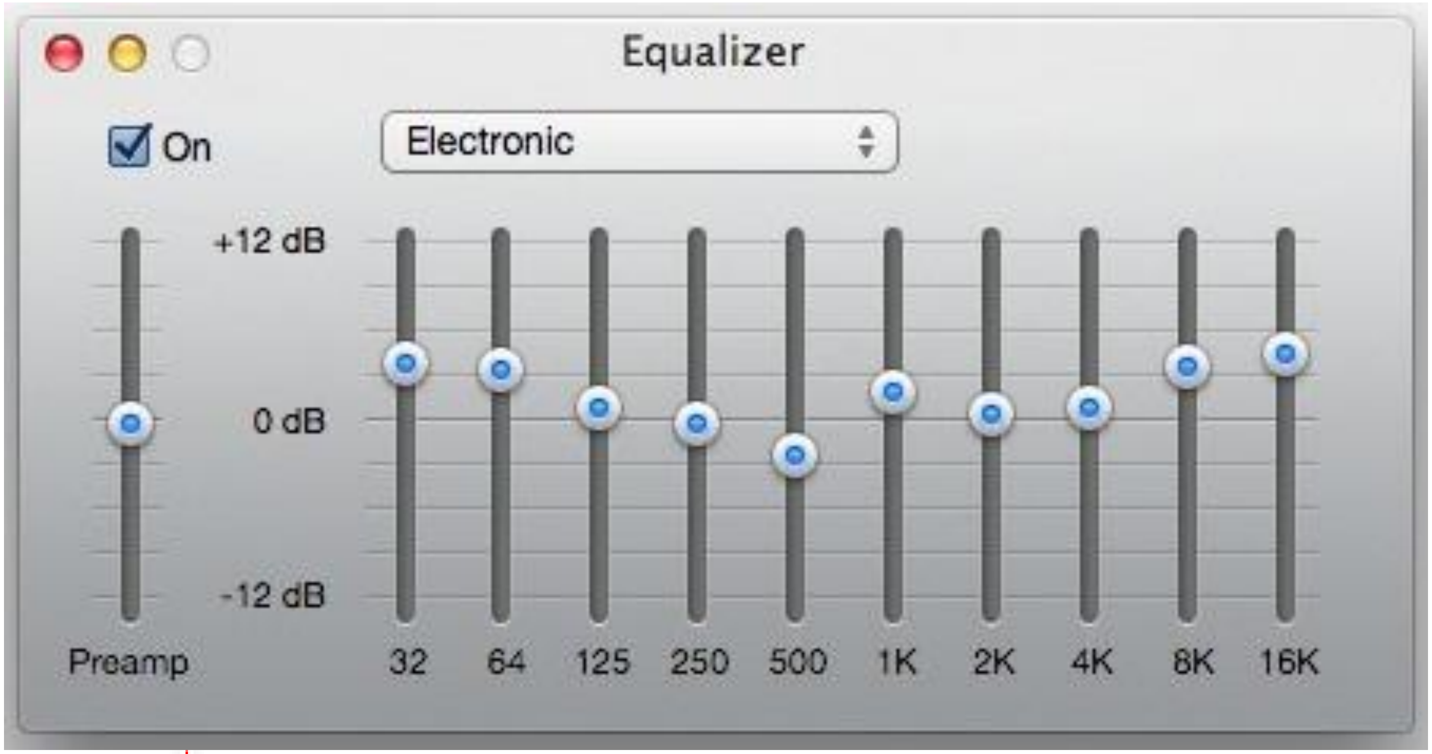


Image Expansion

1. Compute Fourier ($N \times N$)
2. Pad Fourier with zeros ($2N \times 2N$)
3. Compute Inverse Fourier ($2N \times 2N$)






Equalizer
On
Electronic
+12 dB
0 dB
-12 dB
Preamp
32
64
125
250
500
1K
2K
4K
8K
16K


Advanced
Equalizer
Effects
Freq. Morpher
Formant Morpher
Voice Equalizer
+20 dB
0
-20 dB
+20 dB
0
-20 dB
30Hz
75Hz
140Hz
280Hz
560Hz
1kHz
2kHz
4kHz
9kHz
16kHz
Equalizer
(1)
On/Off
Preset
Load
Save
Reset

End Fourier