Real Analysis: A Long-Form Mathematics Textbook Solutions

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Please contact me to correct any mistakes that will inevitably appear in my attempt at solution. Also, excuse the less-than-elegant solutions as I favored elaboration for self reference.

1 THE REALS

1.1 Line 3 does not imply line 4 due to $x - y = 0$, a division by zero error.

1.2 (a) True, by Archimedean Principle.
(b) False, as by well-ordering principle, take $m = 1$ which is the smallest element of $\mathbb{N}$.
(c) True, take $m = n$.
(d) True, take $m = n$.
(e) True, take $m = n$.
(f) False, take $x = 1, y = 2$. There is no integer in between.
(g) True, take $z = \frac{x+y}{2} \in \mathbb{R}$, where it is easily shown that $x < z < y$.

1.3 (a) Take all items that are also in B out of A.

(b) Put each combination of items that can be taken out of A (including taking "nothing" out as a combination) into a separate box. Then, put all those boxes into a big box.

(c) Count the total number of distinct items in A.

1.4 (a) Take everything out of each and every one of the boxes, and put them into a big box.

(b) Check to see what items are common in every one of the boxes. Put those items into a box.

1.5 (a) $A \cup B = A$ means (1) if $x \in A$ or $x \in B$, then $x \in A$ (2) if $x \in A$ then $x \in A$ or $x \in B$ which is trivially true. $B \subseteq A$ means if $x \in B$ then $x \in A$. ($\implies$) We want to prove the statement $B \subseteq A$ i.e. $x \in B$ then $x \in A$. This is already given by (1) of our hypothesis.

($\Longleftarrow$) We want to prove (1) and (2). Since (2) is trivially true as mentioned, we only prove (1). But for (1), if $x \in A$ then $x \in A$ is again trivially true, and if $x \in B$ then $x \in A$ is given by our hypothesis as well. So (1) is also true (i.e "if a or b, then c" is proven by "if a then c" and "if b then c" separately).

(b), (c), (d) is proven similarly by the above method.

1.6 (a) Since $f^{-1}(B) = \{x \in X : f(x) \in B\}$, then by definition $f(f^{-1}(x)) \in B$.

(b) Consider non-surjective functions. Let $X = \{1, 2, 3\}$, $B = \{1, 2\}$ and $f(x) = 1$. Clearly, f is not surjective, as nothing gets mapped to $2 \in B$. That is, the pre-image of 2 is just the null set. So, $f(f^{-1}(B)) = \{1\} \neq B$.

(c) By definition, $f(A) = \{f(a) \in Y : a \in A\}$, so $f^{-1}(f(A)) = \{x \in X : f(x) \in f(A)\}$. Now suppose for contradiction that $x \notin A$ but $f(x) \in f(A)$. Then $x \notin A$ yet $x \in A$ by definition of $f(A)$. Thus, it must be true that $x \in A$.

(d) Consider $X = \{1, 2, 3, 4\}, A = \{1, 2\}, Y = \{1\}$ and $f : X \to Y$ by defined as $f(x) = 1$. We see that $f(A) = 1$, but $f^{-1}(f(A)) = \{1, 2, 3, 4\} = X \neq A = \{1, 2\}$ since it's a constant function for the entire domain X, not just the subset A.

1.7 To show f is one-to-one, we want to show if $f(x) = f(y)$ then $x = y$. Since $f(x) = f(y)$, then $g(f(x)) = g(f(y))$. Thus, $x = y$.

To show g is onto, we want to show for all $b \in X$, there exists an $a \in Y$ such that $f(a) = b$. Take $a = f(b) \in Y$. Then, a is in the domain of g, then we get $g(f(b)) = b$.

1.8 Proof done in Grimmett, Chapter 1, question 1.

1.9 (a) Use the same proof to show $\sqrt(2)$ is irrational.

(b) It does not work because $4|m^2$ does not imply $4|m$. This is exactly the case when $m = 2$. And if we do do the proper assumptions, we get $n = k$, which shows $n = 1$.

(c)

Lemma 1.1. *If m is rational, n is irrational, then mn is irrational.*

Proof. Suppose for contradiction that mn is rational. Then

$$mn = \frac{p}{q}$$

but since m is rational, $m = \frac{a}{b}$. So we have

$$n = \frac{pb}{qa}$$

which shows n is rational also, which is a contradiction. □

Now we begin the main part. Since $(\sqrt{3}-\sqrt{5})(\sqrt{3}+\sqrt{5}) = -2$, which is rational, then by Lemma 1.1, they are either both rational or both irrational. Now consider

$$(\sqrt{3}-\sqrt{5})+(\sqrt{3}+\sqrt{5})$$

This equals $\sqrt{3}$, which is irrational. If they were both rational, then by property of the ordered field $\mathbb{Q}$, the sum must also be rational. But clearly it is not, which means they must both be irrational.

1.10 Let 1 and $1'$ be multiplicative inverses of a field. See that

$$1'a = a \tag{1}$$
$$1'a - a = 0 \tag{2}$$
$$\tag{3}$$

Since $1a = a$ also, substitute into the above.

$$1'a - 1a = 0 \tag{4}$$
$$a(1'-1) = 0 \tag{5}$$

The only way for this to be true for all $a \in \mathbb{F}$ is for $1' - 1 = 0$. So $1' = 1$.

1.11 (a) $\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$ satisfies (i) but not (ii) since both a and $-a$ is in P. So only one "sign" is allowed.

(b) The half line

$$[0, \infty)$$

satisfies (ii) but not (i) because, for example,

$$(-2)(-4) = (8).$$

1.12 (a) Note that $a < b \implies a - b < 0$ and $c < d \implies 0 < d - c$. So $a - b < d - c$ by Order Axiom, and thus $a + c < b + d$ by re-arranging.
(b) It suffices to give a counter-example. Take $a = -100, b = 1, c = -1, d = 1$. We have $a < b$ and $c < d$ but $100 > 1$ i.e. $ac > bd$. The key here is to think of a very small negative numbers being flipped back to a very large positive number. In other words, having a large absolute value. Also, Order Axiom has been used to define the inequality symbols.

1.13 (a) Proof 1. For the sake of contradiction, suppose $a < b + \epsilon$ for every $\epsilon > 0$ yet $a > b$. Since $a - b = \epsilon_0 > 0$, take $\epsilon = \epsilon_0$. Then $a - b = \epsilon_0$ and $a - b < \epsilon_0$, which is impossible. Thus, the original statement must be true.
Proof 2. This proof I came up with initially myself. However, they are exactly the same. Suppose for contradiction $a < b + \epsilon$ for every $\epsilon > 0$ yet $a > b$. Take $\epsilon = a - b > 0$ (by Order Axiom since $a > b$). Then $a < b + (a - b) = a$, i.e. $a < a$, again a contradiction. For some reason, this slight change of wording makes more intuitive sense to me.
(b) If $a - b > 0$. Then $a - b < \epsilon$ for all $\epsilon > 0$. By (a), we know $a \leq b$. If

$a - b < 0$, i.e. $b - a > 0$, then $b < a + \epsilon$ for all ϵ, again by (a) we have $b \leq a$. Since we asserted that both must be true, we deduce $a = b$.

1.14 Simply check all the cases, which I'm too lazy to type out.

1.15 TODO

1.16 By Order Axiom, we have either $a > b$, $a = b$ or $a < b$. For $\max\{x, y\}$, if $y \geq x$, then $|y - x| = y - x$. $\max\{x, y\} = \frac{x+y+y-x}{2} = y$. Otherwise, $|y - x| = -(y - x) = x - y$, thus $\max\{x, y\} = \frac{x+y+x-y}{2} = x$. The proof is similar for $\min\{x, y\}$.

For $\max\{x, y, z\}$, we expand $\max\{x, \max\{y, z\}\}$.

1.17

Lemma 1.2. *If $a, b, c, d \geq 0$, then $ac < bd$.*

Proof. By Note 1.9, $ac < bc$. Similarly, $bc < bd$. Thus $ac < bd$ by transitivity. □

We proceed by induction. For $n = 2$, $a^2 < b^2$ is true by Lemma 1.2. Now suppose it is true for n. That is, $a^n < b^n$. Applying Lemma 1.2 again gives $a^{n+1} < b^{n+1}$. Thus, by induction, the statement is true.

1.18 We proceed by induction. The base case $n = 1$ is straight-forward. For $n = 2$, it is true by triangle inequality. Now assume it is true for n, that is

$$|a_1 + \cdots + a_n| \leq |a_1| + \cdots + |a_n|.$$

Then

$$\begin{aligned}
|a_1 + \cdots + a_n + a_{n+1}| &= |(a_1 + \cdots + a_n) + a_{n+1}| && (6)\\
&\leq |a_1 + \cdots + a_n| + |a_{n+1}| && (7)\\
&\leq |a_1| + \cdots + |a_n| + |a_{n+1}| && (8)
\end{aligned}$$

And the result follows from induction.

1.19 The result follows from standard induction techniques.

1.20 Prime numbers.

1.21 If $f(x) \in f(A_1 \cap A_2)$ iff $x \in A_1 \cap A_2$. Then $x \in A_1$ and $x \in A_2$, so $f(x) \in f(A_1)$ and $f(x) \in f(A_2)$. That is, $f(x) \in f(A_1) \cap f(A_2)$.

1.22 Consider $A = \{1, 2, 3\}$, $B = \{2, 3, 4\}$. Then $A \cap B = \{2, 3\}$. Suppose $f(2) = f(3) = 1$, and $f(1) = f(4) = 2$. Then $f(A) = f(B) = \{1, 2\}$. So $f(A) \cap f(B) = \{1, 2\} \neq \{1\} = f(A \cap B)$. Intuitively, the intersection may have the intersection range, but the non-intersection may also have the intersecting range.

1.23 Since $A \subseteq B$, we have $A = B$ or $A \subset B$. The first is true since suprema are unique, as proved by Proposition 1.2.2., hence $\sup(A) = \sup(B)$. For the second, $\sup(B)$ is an upper bound for A. Since $\sup(A)$ must be the least of these upper bounds by definition, $\sup(A) \leq \sup(B)$.

1.24 (a) We use the supremum property. By assumption, since $x \leq M$ for all $x \in A$, M is an upper bound for A. Then, given any $\epsilon > 0$, consider $M - \epsilon$. Since $M \in A \geq M - \epsilon$, $M - \epsilon$ is not an upper bound. QED .

(b) The proof is similar. We use the infimum property. Since $m \leq x$ for all $x \in B$, m is a lower bound of B. Also, given any $\epsilon > 0$, consider $m + \epsilon$. Since $m \in B \leq m + \epsilon$, $m + \epsilon$ is not a lower bound. QED .

Remark. *This exercise proves that if the maximum (minimum) exists, then it equals the supremum (infimum). The converse is not always true.*

1.25 We proceed by induction on an n element set A_n. Consider the base case $n = 2$, where a set A_2 only has two elements. There exists an maximal element, which can be found by $\max\{x, y\}$ as in Exercise 1.16, where $x, y \in A_2$. Moreover, $\max\{x, y\}$ is either x or y. Then, suppose it is true for an n element set A_n, i.e. the maximal element exists and is in A_n. Just like in Exercise 1.16, since $A_{n+1} = A_n \cup z$ where z is a discrete real number, $\max\{A_n + 1\} = \max\{\max(A_n), z\}$. This exists, and the maximal element is either in A_n, or z, in both cases it is in A_{n+1}. By mathematical induction, we are done.

1.26 (Prove that $\mathbb{N}$ is complete). If a non-empty set A in $\mathbb{N}$ is bounded above, then it is finite. By Exercise 1.25, there is a maximal element M and it is in A. By Exercise 1.24, $\sup A = M$.

1.27 (a) $\sup A = \frac{1}{2}$ (requires proof).

(b) $\inf B = 0$ (requires proof).

(c) $\sup C$ does not exist since C is not bounded above (requires proof).

1.28 (Prove the infimum property).

($\implies$) Assume $\inf(A) = \beta$. By definition of infimum, β is a lower bound of A. Fix an $\epsilon > 0$. Since $\beta + \epsilon > \beta$, so $\beta + \epsilon$ cannot be a lower bound, since β by definition is the greatest of all lower bounds. Then since β is not a lower bound of A, by definition there must be an $x \in A < \beta + \epsilon$.

($\impliedby$) Assume (i) and (ii). We want to show that β is the infimum of A, i.e. it is a lower bound (given already by (i)) and it is the greatest of all lower bounds. To show β is the greatest, suppose there exists another lower bound β' such that $\beta' > \beta$. We know that $\beta' - \beta = \epsilon_0 > 0$. But by (ii), $\beta + \epsilon_0 = \beta + (\beta' - \beta) = \beta'$ is not a lower bound of A. This is a contradiction. Therefore, β must be the greatest lower bound.

1.29 For the first one, $\frac{n}{n+1} < 1$ for all $n \geq 1$, so 1 is an upper bound. Also, observe that

$$1-\epsilon < \frac{n}{n+1} \tag{9}$$

$$\frac{\epsilon}{1-\epsilon} > \frac{1}{n} \tag{10}$$

This n always exists by Archimedean principle. So, $1-\epsilon$ is not an upper bound. By suprema analytically theorem, 1 is the supremum.

For the second one, note that $\frac{n}{n+1} \geq \frac{1}{2}$ whenever $n \geq \frac{1}{2}$ by simple computation. With the fact that $\frac{1}{2}$ is an element of the set, it follows from suprema analytically theorem that $\frac{1}{2}$ is the infimum.

1.30 (a) Take $\epsilon = \sup(B) - \sup(A) > 0$. By supremum property, there exists $b \in B$ such that $b > \sup(B) - \epsilon = \sup(B) - (\sup(B) + \sup(A)) = \sup(A)$. Thus, b is an upper bound for A.

(b) Consider $A = \{0\}$ and $B = \{-\frac{1}{n}\}$. It is easily proven that

$$\sup(A) = \sup(B) = 0$$

However, all elements in B are strictly smaller than the only element in A.

1.31 $\sup(A \cup B) = \max(\sup A, \sup B)$. Since $a \leq \sup A$ and $b \leq \sup B$ for all $a \in A, b \in B$. Thus, $a, b \leq \max(\sup A, \sup B)$ for all a, b. And for either case, by suprema analytically theorem, we can find the corresponding element in A or B.

1.32 (a) Since $a \leq \sup A$ for all a, then $a + c \leq \sup A + c$ for all a. So $\sup A + c$ is an upper bound. Then see that there exists an a s.t. $\sup A - \epsilon < a$ so adding c to both sides does not change the inequality. I.e. $\sup A + c - \epsilon < a + c$ shows it's not an upper bound. Thus, it is the supremum by suprema analytically theorem.

(b) $c > 0$.

1.33 Since there is an element $x \in A$, then by definition $-x \in -A$. Thus $-A$ is non-empty. And A is bounded below by $\inf A$, that is

$$\inf A \leq x$$

for all $x \in A$. Multiplying both sides by -1, we get

$$-\inf A \geq -x$$

for all $x \in A$. So $-A$ is bounded above by $-\inf A$. Now we show it is the supremum. By infimum analytically theorem, there exists $x \in A$ such that

$$\inf A + \epsilon > x$$

so multiplying -1 to both sides gets

$$-\inf A - \epsilon < -x.$$

By suprema analytically theorem, $-\inf A$ is indeed $\sup -A$.

1.34 (Nested interval property). Consider the set $A = \{a_n : n \in \mathbb{N}\}$. Every b_n is greater than every a_n, since they are nested. Thus, we know that A is bounded. Since $\mathbb{R}$ is complete, there exists an $x = \sup(A) \geq a_n$ for all a_n. Also, by definition of supremum, $x \leq b_n$ for all b_n. Hence, we see that $a_n \leq x \leq b_n$ for all n. Equivalently, $x \in [a_n, b_n] = I_n$ for all n. Again, this is equivalent to $\bigcap_{n=1}^{\infty} I_n$. It follows that the intersection is non-empty.

1.35 The counter-example $(0, \frac{1}{n})$ proves this. Every positive number is eventually excluded, yet 0 is not in the set either.

1.36 (a) $\{-2, 3, 6\}$ (b) It is straight-forward to show that $\sup A + \sup B$ is an upper bound. By suprema analytically theorem,

$$\sup(A) + \sup(B) - \epsilon = \sup(A) + \sup(B) - \frac{\epsilon}{2} - \frac{\epsilon}{2} \tag{11}$$

$$= (\sup(A) - \frac{\epsilon}{2}) + (\sup(B) - \frac{\epsilon}{2}) \tag{12}$$

$$< a + b \tag{13}$$

where such elements a and b must both exist.

1.37 (a) {-3,0,5}

(b) $A = \{-1, -3, -5\}$, $B = \{-2, -4, -6\}$.

2 CARDINALITY

2.1 (a) $\{\phi\}, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}$

Remark. *Note that the cardinality is* $2^3 = 8$*.*

(b) 2^n.

2.2 Recall Fact 2.4. Define $f : \mathbb{N} \to \{e^n : n \in \mathbb{N}\}$ as $n \mapsto e^n$. (One-to-one) If $e^x = e^y$, then taking log on both sides gives $x \ln e = y \ln e$, so $x = y$. (Surjective) For all $b = e^n$ in the codomain, take $a = n$ in the domain.

2.3 (a) $f(x) = x + 1$

(b) $f(x) = -x + 3$

(c)

(d)

(e)

(f)

(g)

(h)

Remark. *This exercise showcases that strict subsets can have the cardinality, even for uncountably infinite sets.*

2.4 (a) The identity function $id : A \to A, a \mapsto a$ is a bijection that always exists. Thus, $A \sim A$.

(b) If $A \sim B$, then there exists a bijection $f : A \to B$, which is the same as the bijection $f^{-1} : B \to A$. Thus, $B \sim A$.

(c) If $A \sim B$ and $B \sim C$, then there are bijections $f : A \to B$ and $g : B \to C$. Thus, $g \circ f$ is a bijection from $A \to C$. Thus, $A \sim C$.

Remark. *Clearly the above proof has some missing details. For (c), we could prove that composition of bijections is also a bijection by showing injectivity and surjectivity.*

2.5 (a) Since A and B are countable, we can enumerate them both as

$$A \cup B = \{a_1, b_1, a_2, b_2, a_3, b_3, \dots\}.$$

(b) Use Cantor's diagonalization argument to represent $\bigcup_{n=1}^{\infty} A_n$ as

$$\begin{array}{ccccc} A_1 & a_{11} & a_{12} & a_{13} & \dots \\ A_2 & a_{21} & a_{22} & a_{23} & \dots \\ A_3 & a_{31} & a_{32} & a_{33} & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{array}$$

to see that it must be countable.

2.6 $\{1, 2, 3, 4, 5, 6, \dots\}$ to $\{1, -1, 2, -2, 3, -3, \dots\}$. That is, every odd number of $\mathbb{N}$ maps to every positive number of $\mathbb{Z}$, every even number of $\mathbb{N}$ maps to every negative number of $\mathbb{Z}$.

2.7 (a) $A_1 = \{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}\}$ and $B_1 = \{2, 3, 4\}$. Then $A \cdot B = \{1\}$.

(b) $A_1 = \{2, 5, 7\}$ and $B_1 = \{9, 11, 13\}$.

(c) $A_1 = \{1, 2, 3\}$ and $B_1 = \{1\}$. Then $A \cdot B = \{1, 2, 3\}$.

For (c).

2.8 (a) Let the subsets be $\{1, 7, 13, \dots\}$, $\{2, 8, 14, \dots\}$, $\{3, 9, 15, \dots\}$, $\{4, 10, 16, \dots\}$, $\{5, 11, 17, \dots\}$, $\{6, 12, 18, \dots\}$. So $6n - 5$, $6n - 4$, $\dots$, $6n$.

(b) ???

2.9 $|\mathbb{Z} \times \mathbb{N}|$ is countable by Cantor's diagonalization argument. Let the horizontal be natural numbers, the vertical be integers. Then the matrix would be each pairing of $(n \in \mathbb{N}, z \in \mathbb{Z})$.

2.10 S is uncountable, with the same proof to show $\mathbb{R}$ is uncountable.

2.11 (a) $\implies$ (b) If X is finite or countably infinite, then $|X| \leq |\mathbb{N}|$. This implies there exists a one-to-one function $f : X \to \mathbb{N}$.

(b) $\implies$ (c) If there is a one-to-one function $f : X \to \mathbb{N}$, then $|X| \leq |\mathbb{N}|$. Thus, there exists an onto function $g : \mathbb{N} \to X$.

(c) $\implies$ (a) If there is an onto function $g : \mathbb{N} \to X$, then $|N| \geq |X$. Thus, $|X|$ is finite or countably infinite.

We are done.

2.12 (a) $(n, n+1)$ for $n \in \mathbb{N}$.

(b) Suppose there were some collection of uncountably many disjoint open intervals. For any interval, say (a, b), by density of $\mathbb{Q}$ in $\mathbb{R}$, there always exists some rational q such that $a < q < b$. And since they are disjoint, each q must be different. Thus, we can enumerate all the intervals using this q. Hence, it is countable since there is a bijection with $\mathbb{Q}$ and hence $\mathbb{N}$. By contradiction, it must be countable.

2.13 Suppose for contradiction that irrational numbers $\mathbb{R} \setminus \mathbb{Q}$ was countable. Then, by Exercise 2.5, $\mathbb{R} = \mathbb{R} \cup (\mathbb{R} \setminus \mathbb{Q})$ is countable. But $\mathbb{R}$ is uncountable, a contradiction arises. Thus, irrational numbers must be uncountable.

2.14 (One-to-one) Suppose $f(p, q) = f(m, n)$, then

$$2^{n-1}(2m-1) = 2^{q-1}(2p-1) \tag{14}$$

$$2^{n}(2m-1) = 2^{q}(2p-1) \tag{15}$$

3 SEQUENCES

3.1 By the definition of convergence, for $\epsilon = 0.001$, there exists an $N \in \mathbb{N}$ such that for all $n > N$, $|a_n - 0.001| < \epsilon$. Equivalently,

$$0.001 - 0.001 < a_n < 0.001 + 0.001$$

So

$$0 < a_n < 0.002$$

Therefore, since there are finitely many terms for a_n with $n \leq N$, only finitely many $a_n < 0$.

3.2 (a) Possible. $\{1, 0, 1, 0, 1, 0, \dots\}$

(b) Impossible. Suppose there exists a sequence (a_n) with infinitely many 0s, but converges to a non-zero number. Then we have for all $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that for all $n > N$, we have $|a_n - L| < \epsilon$, where $L \neq 0$. Then we set $0 < \epsilon < L$. But since there are infinitely many 0s, we must also have $L < \epsilon$. Thus, we reach a contradiction.

(c) TO DO.

(d) TO DO.

3.3 (a) Consider the sequence $\{1, 0, 1, 0, 1, \dots\}$. The sequence does not converge to $a = 0$. However, for all $\epsilon > 0$, $|a_{2n} - 0| = |0 - 0| = 0 < \epsilon$ for $n = 1, 2, 3, \dots$. So it *Non*verges-type-1 to $a = 0$.

(b) Consider the sequence $\{1, 1, 1, 1, 1, 0, 0, 0, 0, 0, 1, 1, 1, 1, 1, \dots\}$, alternating five consecutive 1's and 0's. The sequence does not converge to $a = 1$. However, since for any $\epsilon > 0$, pick $N = 5$. There is some $n > N = 5$, which is 0, that satisfies this definition.

(c) Consider the sequence $\sin n$. This is divergent, as the sequence is periodic. However, for the limit $a = 0$, we can always pick any $\epsilon \leq 1$. Then for all $N \in \mathbb{N}$, there exists $n > N$ (in fact it is all n), $|\sin n - 0| \leq 1$.

(d) Consider the slightly modified sequence from (c), $n \sin n$ for $n < 1000$, n for $n \geq 1000$. Notice it diverges to positive infinity. However, if we fix some $\epsilon = 1000$ at the start, then clearly if we let $N = 1$ then at least $n = 1, 2, 3, 4 > N$ satisfies $|n \sin n - 0| < 1000$. Also, however, starting from $n = 1000$, it is no longer periodic and shoots off to infinity. Thus, it does not satisfy (c).

Remark. *(a) does not enforce the sequence to stay close to the limit forever after some point. It only needs to be close to the limit at some point.*
(b) is similar to (a).
(c) does not enfore the sequence to be close to the limit at all. However, it does enforce that we don't have some divergent sequence as there must always be some point in the ϵ-neighbourhood of the limit.
(d) is similar to (c), but it further allows divergence to infinity.

3.5 The sequence $\{-\frac{1}{n}\}$. All the terms are negative but it converges to 0.

Proof. For all $\epsilon > 0$, choose $N = \frac{1}{\epsilon}$. For all $n > N$,

$$\begin{aligned}
|-\frac{1}{n} - 0| &= \frac{1}{n} && (16)\\
&< \frac{1}{N} && (17)\\
&= \frac{1}{\frac{1}{\epsilon}} && (18)\\
&= \epsilon && (19)
\end{aligned}$$

□

The sketch work has been omitted.

3.6 (a) $L \in [0, 1]$.

(b) Consider enumerating $\mathbb{Q}$, which is countable, in the same manner as the diagonal counting argument. Notice that we can always encounter the same rational number infinitely many times (since there are there are more than one representation for each rational number). And by density of $\mathbb{Q}$ in $\mathbb{R}$, we can always find a subsequence that converges to any real number. This is because for any $\epsilon > 0$, and real number L, there is rational number q such that $a-\epsilon < q < a+\epsilon$, i.e. $|q-a| < \epsilon$. And this rational q will appear infinitely many times so we are sure to be able to construct some subsequence that converges to a.

Remark. *I think this showcases the significance of density of $\mathbb{Q}$ in $\mathbb{R}$. It captures the essence of a rational number being infinitely close (ϵ-close) to an irrational, but not equalling the irrational.*

3.7 (a) Not possible. Suppose it did, then it implies that for all $\epsilon > 0$, there is an N such that for all $n > N$, $|a_n - 3.5| < \epsilon$. Say we pick $\epsilon = 0.1$. Then the assumption implies that there will be some integer a_n such that $3.4 < a_n < 3.6$, which is impossible.

(b) For the sake of contradiction, suppose $a_n \neq a_m$ for all $n \neq m$, but (a_n) converges to a. Without loss of generality, assume $n > m$. Suppose $\epsilon = 0.25$, then by the definition of convergence, there exists an N such that for all $j > N$, $|a_j - a| < \epsilon$. Hence, if $n > m > N$, then $|a_n - a| < 0.25$ and $|a_m - a| < 0.25$. Then

$$\begin{aligned} |a_n - a_m| &= |a_n - a + a - a_m| && (20)\\ &\geq |a_n - a| + |a_m - a| && (21)\\ &< 0.25 + 0.25 && (22)\\ &= 0.5 && (23) \end{aligned}$$

However, this is impossible, since we must have $|a_n - a_m| \geq 1$ as they are both distinct integers. We have reached a contradiction.

Remark. *The choice for 0.25 was arbitrary. In fact, we could have picked any $\epsilon < \frac{1}{2}$. We just need to show that ϵ must be greater than 1 for distinct integers a_n, a_m.*

(c) The sequence must be constant after at most finite terms. It may be constant right from the beginning, or after finite terms.

3.8 (a) By assumption, for $\epsilon/2 > 0$, there exists N_1 such that for all $n > N_1$, $|a_n - a| < \epsilon/2$, and there exists N_2 such that for all $n > N_2$, $|b_n - b| < \epsilon/2$. Thus, for all $\epsilon > 0$, let $N = \max\{N_1, N_2\}$. For all $n > N$,

$$\begin{aligned} |(a_n + b_n) - (a + b)| &= |(a_n - a) + (b_n - b)| && (24)\\ &\leq |a_n - a| + |b_n - b| && (25)\\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} && (26)\\ &= \epsilon && (27) \end{aligned}$$

And we are done.

(b) First suppose $c \neq 0$. For $\frac{\epsilon}{|c|} > 0$, there exists N_1 such that for all $n > N_1$, $|a_n - a| < \frac{\epsilon}{|c|}$. Thus, for all $\epsilon > 0$, let $N = N_1$. Then for all $n > N$, we have

$$
\begin{aligned}
|ca_n - ca| &= |c||a_n - a| && (28)\\
&< |c|\frac{\epsilon}{|c|} && (29)\\
&= \epsilon && (30)
\end{aligned}
$$

If $c = 0$, then for all $\epsilon > 0$, for all n, $|0 - 0| = 0 < \epsilon$.

3.9 For any $M > \frac{M}{2} > 0$, there exists an N_0 such that $a_n > \frac{M}{2}$ for all $n > N$, and there exists an N_1 such that $b_n > \frac{M}{2}$ for all $n > N$. Thus, take $N = \max N_0, N_1$. Then for for all $n > N$,

$$
\begin{aligned}
a_n + b_n &> \frac{M}{2} + \frac{M}{2} && (31)\\
&> M. && (32)
\end{aligned}
$$

3.10 (Intertwining sequence theorem) If a_{2n} and a_{2n+1} both converge to L, then there exist an N_1 such that $|a_{2n} - L| < \epsilon$ and an N_2 such that $|a_{2n+1} - L| < \epsilon$. Take $N = \max N_1, N_2$. Then for all $n > N$, since a_n is an index of one of these subsequences, $|a_n - L| < \epsilon$.

3.11 Consider the counter-example $\{1 - \frac{1}{n} : n \in \mathbb{N}\}$. Clearly, every element is less than 1. However, by a straight-forward proof we can show $(1 - \frac{1}{n}) \to 1$. By another straight-forward proof, we can show that the above set has no maximum (but has a supremum). Thus, a_n converges but does not have a maximum. TO DO.

3.12 (a) If a_n diverges to ∞, then for all $M > 0$, there exists an N such that $a_n > M$ for all $n > N$. This implies that for all $n \leq N$, $a_n \leq M$. The set of these finite a_n will be so that $a_{f(n)} \leq M$. Since these are finite, we can find take $N_{\max} = \max\{f(n) : a_{f(n)} \leq M\}$. And with f being bijective, we are guaranteed that no other numbers n in the range will be cause $a_{f(n)} \leq M$. Thus, for all $f(n) > N_{\max}$, $a_{f(n)} > M$.

(b) Similar to (a). If $a_n \to L$, then for all $\epsilon > 0$, there exists an N such that $|a_n - L| < \epsilon$ for all $n > N$. Thus, there are finitely many $n \leq N$ such that $|a_n - L| \geq \epsilon$. Some of these may be less than ϵ, but that doesn't change the fact that it's finite. For those n, they will be mapped to $f(n)$, such that $|a_{f(n)} - L| \geq \epsilon$. Take $M = \max\{a_f(n) : |a_f(n) - L| \geq \epsilon\}$. Then for all $f(n) > M$, we have $|a_{f(n)} - L| < \epsilon$.

(c) TO DO.

3.13 (a) For $\epsilon/2 > 0$, there exists an N_1 such that $n > N_1$ implies $|a_n - a| < \epsilon/2$ and $n > N_2$ implies $|b_n - b| < \epsilon/2$. Let $N = \max\{N_1, N_2\}$, then for all $\epsilon > 0$,

$$|(a_n - b_n) - (a - b)| = |(a_n - a) - (b_n + b)| \leq |a_n - a| + |b_n - b| \leq 2 \cdot \epsilon/2 = \epsilon$$

(b) Since b_n is convergent, then $|b_n|$ is bounded by some real constant C. Thus, (by scratch work), for $\epsilon_1 = \frac{C\epsilon}{2}$ there is some N_1 such that $n > N_1$ implies $|a_n - a| < \epsilon_1$. And for $\epsilon_2 = \frac{C|b|\epsilon}{2|a|}$ there is some N_2 such that $n > N_2$ implies $|b_n - b| < \epsilon_2$. Thus, let $N = \max\{N_1, N_2\}$, then for all $\epsilon > 0$,

$$|\frac{a_n}{b_n} - \frac{a}{b} = |\frac{a_n}{b_n} - \frac{a}{b_n} + \frac{a}{b_n} - \frac{a}{b}| \leq |\frac{a_n - a}{b_n}| + |\frac{a(b_n - b)}{b_n b}| < \epsilon/2 + \epsilon/2 = \epsilon$$

And we are done.

3.14 For (a_n), we will show it is monotonically decreasing and bounded. For the former, we proceed by induction. Since $a_0 = 2\sqrt{3}$, and $a_1 =$ TO DO.

3.15
$a_n = \{1, -1, 1, -1, \dots\}$ and $b_n = \{-1, 1, -1, 1, \dots\}$, so $a_n + b_n = \{0, 0, 0, \dots\}$.

3.16 $a_n = \{1, 0, 1, 0, 1, 0, \dots\}, b_n = \{0, 1, 0, 1, \dots\}$, so $a_n b_n = \{0, 0, 0, 0, \dots\}$.

3.19 (a) This one is simple. For odd terms, let $a_n = 6 + \frac{1}{2n}$. For even terms, let $a_n = 7 - \frac{1}{2n}$.

Proof. We prove that $6 < a_n < 7$ and a_{2k+1}, a_{2k} converge respectively to 6 and 7. □

(b) Revisit Exercise 3.6. Let the sequence be

$$\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \frac{1}{5}, \dots$$

Proof. TO DO. □

(c) Intuition says that such a sequence is impossible, as if terms are getting closer and closer to any $\frac{1}{k}$, which itself is getting closer and closer to 0, then there will be terms getting closer to 0.

Proof. TO DO. □

(d) This is the same as Exercise 3.6(b). The sequence should be all rational numbers.

Proof. TO DO. □

3.20 (a) $\{\frac{4}{9}, -\frac{4}{9}, \frac{4}{9}, -\frac{4}{9}, \dots\}$ is bounded, does not converge to anything, but odd terms converge to $\frac{4}{9}$.

(b) If a_n does not converge to a, then it cannot have a subsequence that converges to a.

(c) False, consider $a_n = (-1)^{n+1}\frac{1}{n}$. Odd terms are decreasing, even terms are increasing, a_n converges to 0.

(d) By Monotone Convergence Theorem (Theorem 3.27), monotone sequences converge iff bounded. Thus, this is impossible.

(e) Consider $a_n = n$. The sequence is unbounded, thus is divergent. Any subsequence is also unbounded, thus is also divergent.

(f) Impossible. A bounded sequence means all terms $a_n \leq |C|$. Any term from a subsequence is also a term in the sequence. Thus, any term in the subsequence must also be bounded by $|C|$.

3.21 By Bolzano-Weierstrass theorem, (a_{n_k}) itself is a sequence. Thus, by Bolzano-Weierstrass theorem, since it is bounded, it must have a bounded subsequence $(a_{n_{k_j}})$. This sub-sub-sequence is a sub-seqeuence of a_n.

3.22 (a) We have $a_n \to L$ and $a_n \leq M$. For all $\epsilon > 0$, there exists an N such that $n > N$ implies $|a_n - L| < \epsilon$. Then

$$L - \epsilon < a_n < L + \epsilon \tag{33}$$

$$L - \epsilon < a_n \leq M \tag{34}$$

$$L - \epsilon < M \tag{35}$$

and so

$$L < M + \epsilon$$

for all $\epsilon > 0$. From the exercise in Chapter 1, this implies $L \leq M$ (proof by contradiction).

(b) Let $a_n \leq b_n$ for all n. Also, let $a_n \to L$ and $b_n \to M$. Similar to (a), we have

$$L - \epsilon < a_n < L + \epsilon$$

and

$$M - \epsilon < b_n < M + \epsilon.$$

Putting them together gives

$$L - \epsilon < M + \epsilon$$

so

$$L < M + 2\epsilon$$

for all ϵ. Suppose for contradiction $L > M$. Then choose $\epsilon = \frac{L-M}{2}$. Then $L < M + 2(\frac{L-M}{2}) = L$, a contradiction. So $L \leq M$.

3.23 (a) For all $\epsilon > 0$, there exists an N such that for all $n > N$, we have $|a_n - L| < \epsilon$. Choose the same N. Then, by reverse triangle inequality,

$$||a_n| - |L|| \leq |a_n - L| < \epsilon \tag{36}$$

(b) Let $a_n = (-1)^{n+1}$, an alternating sequence of 1s and -1s.

3.24 (Cauchy $\implies$ bounded). Fix $\epsilon = 1$. Then there is an N such that for all $m, n > N$, we have $|a_m - a_n| < 1$. Then for all $m > N$,

$$a_{N+1} - 1 < a_m < a_{N+1} + 1.$$

Let $U = \max\{a_1, \dots, a_N, a_{N+1} + 1\}$. Then $a_n \leq U$ for all n. For the lower bound, take $L = \min\{a_1, \dots, a_n, a_{N+1} - 1\}$.

3.25 (Monotone convergence theorem, decreasing case). Let (a_n) be a monotonically decreasing sequence. If (a_n) is bounded below by M, then $a_n \geq M$ for all n. Let $S = \{a_n : n \in \mathbb{N}\}$. It is non-empty and bounded, and so by completeness of $\mathbb{R}$, $\beta = \inf S$ exists. By infimum property, there exists an $a_N \in S$ and a corresponding N such that $a_N < \beta + \epsilon$. Since (a_n) is monotonically decreasing,

$$\beta - \epsilon < \cdots < a_{n+2} < a_{n+1} < a_n < a_N < \beta + \epsilon.$$

That is to say for all $n > N$, $|a_n - \beta| < \epsilon$.

Now suppose a_n was not bounded. Then for all lower bounds M, there is an n such that $a_n < M$. Since a_n is monotonically decreasing, we have

$$\cdots < a_{n+1} < a_n < M$$

which is to say that for all $n > N$, we have $a_n < M$. And so, this means it diverges to $-\infty$ and is thus not convergent.

3.26 If S is bounded below, and is nonempty, then by completeness of $\mathbb{R}$, $\inf S$ exists. By infima analytically thoerem, for all ϵ and thus for all n here, there exists a $b_n \in S$ such that

$$\inf S + \frac{1}{n} > b_n.$$

This forms a sequence. Now by squeeze theorem and limit laws,

$$\inf S + \frac{1}{n} \geq b_n \geq \inf S$$

for all n, so

$$\lim_{n \to \infty} b_n = \inf S.$$

3.27 $\{1, 1, 2, 1, 3, 1, 4, 1, 5, 1, 6, 1, \dots\}$

3.28 We prove them by cases. The cases $r = 0, r = 1$ are trivial as they are constant sequences. First consider $0 < r < 1$. We claim that r^n will be bounded and monotonically decreasing. It is clear that r^n is bounded by 0 and 1, as they are all positive numbers, and $a^n < b^n$ for all n since $a < b$ thus $a^n/b^n < 1$. To show they are decreasing monotonically, consider $(a^{n+1}/b^{n+1})/(a^n/b^n)$ which gives $a/b < 1$, thus it is monotonically decreasing (since the subsequent term is less than the previous term for all n). Thus, we conclude it converges to 0. Applying the same reasoning to $-1 < r < 0$, it converges to 0 as well.

For $r > 1$, it is monotone increasing since $r^{n+1}/r^n = r > 1$. It is unbounded. To prove this, pick $N = \log_r M$ for all M, then $r^n > M$ for all $n > N$ as

$$r^n > r^N = r^{\log_r M} = M$$

Finally, use the same reasoning to show $r < -1$ diverges.

3.29 <u>Sketch</u>.

$$\begin{aligned}
|b_n - a| &= |\frac{a_1 + \cdots + a_n}{n} - a| && (37)\\
&\leq |\frac{a_1 + \cdots + a_n - a}{n}| && (38)\\
&= \frac{1}{n}|a_1 + \cdots + a_n - a| && (39)\\
&\leq \frac{1}{n}(|a_1| + \cdots + |a_n| + |a|) && (40)\\
&\leq \frac{1}{n}(n|C| + |a|) && (41)\\
&= |C| + \frac{|a|}{n} && (42)
\end{aligned}$$

TO DO.

3.30 If (a_n) is bounded, then $\sup\{a_n : n \in \mathbb{N}\}$ exists. Thus, b_n is a monotonically decreasing sequence by Exercise 1.23. Also, since $\sup\{a_n\} \geq a_n \geq \inf\{a_n\}$, then $\sup\{a_n\} \geq \inf\{a_n\}$ always and so b_n is bounded below by $\inf\{a_n\}$. Thus, (b_n) converges.

3.31 A trivial example of a sequence is $\{1, 17, -\pi, 1, 17, -\pi, \dots\}$.

3.32 (Subsequence proof, every $a_{n_k} \implies a_n \to a$). If every subsequence $a_{n_k} \to a$, since a_n is itself a subsequence, $a_n \to a$.

Remark. *This gives us a good way to check if a sequence diverges.*

1. *If there exists a divergent subsequence a_{n_k}, then a_n diverges. Conversely, if a_n diverges, there exists a divergent subsequence.*
2. *In fact, if a_n diverges to ∞, every subsequence a_{n_k} diverges to ∞!*

3.33 TO DO.

3.34 TO DO.

3.35 Consider the sequence $\{n : n \in \mathbb{N}\}$. Every subsequence is strictly monotonically increasing. Every subsequence is not bounded. Thus, every subsequence diverges to infinity.

3.41 (*Hint: Bolzano-Weierstrass theorem, countable union theorem*) TO DO.

3.42 TO DO.

3.43 TO DO.

3.44 Since a_n is bounded, by Bolzano-Weierstrass theorem, there exists a convergent subsequence (at least one). TO DO.

3.45 Clearly, a_n is a monotonically increasing sequence, since $a_n \leq a_{n+1}$ for all n. Thus, by monotone convergence theorem, since a_n converges to $\frac{\pi^2}{6}$ by Euler's result, we have $\sup\{a_n : n \in \mathbb{N}\} = \frac{\pi^2}{6}$, so $a_n \leq \frac{\pi^2}{6}$ for all n.

Notice that $b_n \leq a_n$ for all n, since we are adding on $\frac{1}{n^3} \leq \frac{1}{n^2}$ at each step. So now we have

$$b_n \leq a_n \leq \frac{\pi^2}{6}$$

for all n. Since b_n is monotonically increasing also, by monotone convergence theorem, since b_n is bounded above by $\frac{\pi^2}{6}$, b_n is convergent.

Remark. *We had to do all these steps because we needed a* fixed *upper bound* M.

4 SERIES

4.1 (a) Converges conditionally by alternating series test.
(b) Diverges by k^{th}-term test.
(c) Converges absolutely by geometric series test.
(d) Converges by ratio test.

Remark. *Try to do this without ratio test (direct comparison).*

(e) Diverges by "telescoping series test".
(f) Diverges by k^{th}-term test.
(g) Converges conditionally by alternating series test.
(h) Diverges by comparison test to harmonic series.
(i) Diverges by comparison test to harmonic series.

4.2 Diverges by Proposition 3.32. The subsequence S_{2n} and S_{2n+1} converge to 0 and -1 respectively, thus S_n must diverge.

4.3 Since it is always true that $-|a_k| \leq a_k \leq |a_k|$ from Chapter 1, then we must also have $\sum -|a_k| \leq \sum a_k \leq \sum |a_k|$. Thus, proof idea: Split the sum into positive and negative parts, both converge to finite value, ...Actually, I was correct after looking at solutions online, but I'm currently far too lazy to write it out. Check proof from Paul's Online Math Notes.

4.4 (a) $\frac{1}{n}$ diverges but $\frac{1}{n^2}$

(b) Since $\sum a_k$ converges, it follows that $a_k \to 0$. Thus, for $\epsilon = 1$, there exists some $N \in \mathbb{N}$ where $|a_k| < \epsilon = 1$ for all $k > N$. Also note that $|a_k| = a_k$. Consider $\sum_{k=N+1}^{\infty} a_k = L$ (this is true since we ignored finitely many terms in the beginning). Since for $0 < a_k < 1$, we have $a_k > a_k^2$, since all terms are positive we can use the comparison test to see that $\sum a_k^2$ converges (ignoring finitely many terms in the beginning, but they must sum to a finite value anyway).

(c) Consider the counter-example $\sum_{k=1}^{\infty} -1^{k+1}\frac{1}{\sqrt{n}}$. Its square's series diverges.

4.5 The series $\sum (-1)^n \frac{1}{n}$.

4.6 Consider partial sums $S_n = \sum_{k=1}^{n} a_k - a_{k+1}$. Notice that this is a telescoping series, and cancels out to give $S_n = a_1 - a_{n+1}$. As $n \to \infty$, the latter term goes to 0 by assumption. The result follows.

4.7 Take $\sum (-1)^{k+1} \frac{1}{\sqrt{n}}$.

4.8 By the fact presented in this chapter, it sums to $\frac{\ln 2}{7}$.

4.9 It can diverge to ∞ or $-\infty$, depending on a.

4.10 <u>Sketch</u>.

$$\begin{aligned}
|S_m - S_n| &= |\sum_{k=0}^{m} r^k - \sum_{k=0}^{n} r^k| && (43)\\
&= |\sum_{k=n+1}^{m} r^k| && (44)\\
&= |r^{n+1} + r^{n+2} + \cdots + r^m| && (45)\\
&\leq |r|^{n+1} + \cdots + |r|^m && (46)\\
&= \frac{|r|^{n+1}(1 - |r|^{m-n})}{1 - |r|} && (47)\\
&= \frac{|r|^{n+1} - |r|^{m+1}}{1 - |r|} && (48)\\
&\leq \frac{|r|^n - |r|^m}{1 - |r|} && (49)
\end{aligned}$$

So we want to set $\frac{|r|^n}{1-|r|} < \frac{\epsilon}{2}$ which means we want $N > \log_{|r|} \frac{\epsilon \cdot (1-|r|)}{2}$.

Then we proceed with the proof as standard (omitted here since I'm lazy ...).

4.11 (a) $70 + 7\sum_{k=0}^{\infty} \frac{1}{10}^k$

(b) $70\sum_{k=0}^{\infty} \frac{1}{10}^k$

We get the answer $\frac{700}{9}$ from both (a) and (b).

(c) If a number q has a repeating decimal m every n digits, then represent the decimal as $a + \sum_{k=1}^{\infty} m(\frac{1}{10^n})^k = \frac{m - am + a10^n}{10^n - m}$ which means q is rational.

4.12 (a) If $\sum a_k$ converges absolutely, meaning $\sum |a_k|$ converges, since b_k is a subsequence of a_k, the number of terms of b_k is at most that of a_k. Thus, it follows that

$$\sum |b_k| \leq \sum |a_k|$$

and so by comparison test, since $\sum |a_k|$ converges, $\sum |b_k|$ converges, and the result follows from Proposition 4.18.

(b) Consider the example from Exercise 4.5. That is, consider $(a_k) = (-1)^{k+1}\frac{1}{n}$. This converges conditionally, and if we pick the subsequence (a_{2k}), it diverges to $-\infty$.

5 THE TOPOLOGY OF $\mathbb{R}$

5.1 (a) $\mathbb{Z}$ is not open, is closed, is not compact.

(b) $\{1, \frac{1}{2}, \frac{1}{3}, \ldots\} \cup \{0\}$ is not open, is closed, is compact.

(c) $\mathbb{R}$ is open, is closed, is not compact.

(d) $(0,1) \cup [3,4]$ is not open, is not closed, is not compact.

(e) $\mathbb{Q}$ is not open, is not closed, is not compact.

Remark. *Why is it not closed? Recall that a set is closed iff it contains all limit points. Consider the sequence $a_n = \frac{[10^n\pi]}{10^n}$ where each term is rational and not equal to π but it converges to π by squeeze theorem. Then $\mathbb{Q}$ does not contain all of the limit points.*

Another proof is that its complement, i.e. the set of irrationals, is not open either, since every neighbourhood of an irrational contains a rational by density.

(f) $\{17\}$ is not open, is closed, and is compact.

5.2 (a) ϕ. That is, $\mathbb{Z}$ has no limit points.

(b) $\{0\}$.

(c) $\mathbb{R}$.

(d) $[0,1] \cup [3,4]$.

(e) $\mathbb{R}$.

(f) ϕ.

5.3 (a) Yes. If A and B are compact, for every open cover of A there is a finite subcover of A, namely $\{U_1, \ldots, U_k\}$ and for every open cover of B there is a finite subcover of B namely $\{M_1, \ldots, M_k\}$. Clearly

$$A \cup B \subseteq \{U_1, \ldots, U_k\} \cup \{M_1, \ldots, M_k\}.$$

(b) Yes. Since $A \cap B \subseteq A \cup B$, applying (a) gives the result.

Remark. *Using Heine-Borel theorem works as well.*

5.4 (a) $A \setminus B = A \cap B^c$. Its complement is $A^c \cup B$ by De Morgan's laws. Since A is closed, then A^c is open. Thus, $A^c \cup B$ is also open. Therefore, $A \setminus B$ is closed.

(b) Similarly, consider $A \cap B^c$. A is open, and B is closed then B^c is open. Thus, finite intersection of open sets is open so $A \setminus B$ is also open.

5.5 (a) $\cup_{n\in\mathbb{Z}}(n, n+1)$ (b) Suppose there did exist this collection. For each arbitrary open interval (a, b), where $a, b \in \mathbb{R}$, we can find a rational number $q \in (a, b)$ by density of $\mathbb{Q}$ in $\mathbb{R}$. And since they are disjoint, q is in only one of these intervals (for bijection). Thus, we can enumerate them with $q_1, q_2, \ldots,$ and must therefore be countable.

5.6 (a) $\cap_{n\in\mathbb{N}}(-\frac{1}{n}, \frac{1}{n}) = \{0\}$, which is not open.

(b) Each $[1, n]$ is closed. But $\cup_{n\in\mathbb{N}}[1, n] = 1, \infty$ is not closed (not open either).

(c) $\cup_{n\in\mathbb{N}}[-n, n] = \mathbb{R}$ which is not compact.

5.7 ($\implies$). If x is a limit point of A, then there exists a sequence $a_1, a_2, \ldots$ from $A \setminus \{x\}$ such that $a_n \to x$. That is, for all $\epsilon > 0$, there is an N such that $|a_n - x| < \epsilon$ for all $n > N$. That is, for all $n > N$, $a_n \neq x$ is in the ϵ-neighbourhood of x.

($\impliedby$). Fix an $\epsilon > 0$. If every ϵ-neighbourhood of x intersects A at some point other than x, then for ϵ, there is an $a_1 \in (x-\epsilon, x+\epsilon)$ and $a_1 \in A\setminus\{x\}$. Similarly for $\frac{\epsilon}{2}$, we can find some a_2. This continues for $a_3, a_4, \ldots$ Thus, we can always find a sequence $a_n \neq x$ such that for all n, it is in that ϵ-neighbourhood of x.

5.8 $\cdots \cup (-2 + \frac{1}{2n}) \cup (-1 + \frac{1}{2n}) \cup (\frac{1}{2n}) \cup (1 + \frac{1}{2n}) \cup (2 + \frac{1}{2n}) \cup \cdots$ where $n \in \mathbb{N}$.

5.9

5.10 Let A be a set, and let B be the set of the limit points of A. We want to show that B is closed, that is, B^c is open. Equivalently, if B contains all of its limit points.

5.11 Simple proofs by De Morgan's laws.

(a)

$$\bigcup_{k=1}^{n} U_k = \bigcap_{k=1}^{n} (U_k)^c$$

which is a finite intersection of open sets. Thus, it is open and so the original set is closed.

(b)

$$\bigcap_{\alpha} U_\alpha = \bigcup_{\alpha} (U_\alpha)^c$$

is an arbitrary union of open sets, so it is still open, and thus the original set is closed.

5.12 (a) Suppose there was a non-empty open set that is a subset of $\mathbb{Q}$. Then it must be uncountable. However, an uncountable set cannot be a subset of a countable set. Thus, this cannot exist.

(b) Take any singleton set, say $\{\frac{1}{2}\}$. This is closed because its complement is open.

(c)

(d) $\mathbb{Z}$.

(e)

(f) $\cup_{n=1}^{\infty}[n, n+1]$ is not bounded and is thus not compact.

(g) By Heine-Borel theorem, a set is compact iff it is closed and bounded. Since each compact set is closed, infinite intersection must also be closed. Also, since each set is bounded, their intersection must also be bounded. Thus, infinite intersection of compact sets must be bounded.

5.13 (a) Denote the open set as a countable union of open intervals. Now, for each point a, we must have $a \in (c, d)$ for some open interval. Removing it gives $(c, a) \cup (a, d)$ which is still a union of open intervals. Do this for all the (finite) points and we will still have an open set.

(b) No it seems. My first answer was yes, but I could not prove it. I was not being... *destructive enough*. Consider the open set $\mathbb{R}$ and the countable set $\mathbb{Q}$. The new set $\mathbb{R} \setminus \mathbb{Q}$ is the set of all irrationals. However, for any irrational p and for all $\delta > 0$, there is always a rational q such that $p - \delta < q < p + \delta$, by density of $\mathbb{Q}$ in $\mathbb{R}$. Thus, that δ-neighbourhood certainly is not contained in $\mathbb{R} \setminus \mathbb{Q}$. Thus, we have a counter-example that is not open.

(c) No. Let (a, b) be an open interval. Let $c < d$ be elements inside that open interval. Consider

$$(a, b) \setminus ((a, c) \cup (d, b)) = [c, d]$$

which is closed.

5.14