Chapter **4**

Functions of Several Variables

4.1 INTRODUCTION

The students have studied in the lower classes the concept of partial differentiation of a function of more than one variable. They were also exposed to Homogeneous functions of several variables and Euler's theorem associated with such functions. In this chapter, we discuss some of the applications of the concept of partial differentiation, which are frequently required in engineering problems.

4.2 TOTAL DIFFERENTIATION

In partial differentiation of a function of two or more variables, it is assumed that only one of the independent variables varies at a time. In total differentiation, all the independent variables concerned are assumed to vary and so to take increments simultaneously.

Let $z = f(x, y)$, where x and y are continuous functions of another variable t.

Let Δt be a small increment in t. Let the corresponding increments in x, y, z be Δx, Δy and Δz respectively.

Then
$$\begin{aligned}\Delta z &= f(x+\Delta x, y+\Delta y) - f(x, y)\\ &= \{f(x+\Delta x, y+\Delta y) - f(x, y+\Delta y)\} + \{f(x, y+\Delta y) - f(x, y)\}\end{aligned}$$

$\therefore$
$$\begin{aligned}\frac{\Delta z}{\Delta t} &= \left\{\frac{f(x+\Delta x, y+\Delta y) - f(x, y+\Delta y)}{\Delta x}\right\}\cdot\frac{\Delta x}{\Delta t}\\ &\quad + \left\{\frac{f(x, y+\Delta y) - f(x, y)}{\Delta y}\right\}\cdot\frac{\Delta y}{\Delta t} \qquad (1)\end{aligned}$$

We note that Δx and $\Delta y \to 0$ as $\Delta t \to 0$ and hence $\Delta z \to 0$ as $\Delta t \to 0$

Taking limits on both sides of (1) as $\Delta t \to 0$, we have $\dfrac{dz}{dt} = \dfrac{\partial f}{\partial x}\cdot\dfrac{dx}{dt} + \dfrac{\partial f}{\partial y}\cdot\dfrac{dy}{dt}$

($\because$ x, y and z are functions of t only and f is a function of x and y).

i.e.,
$$\frac{\mathrm{d}z}{\mathrm{d}t} = \frac{\partial z}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial z}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} \text{ [since } f(x, y) \equiv z(x, y)\text{]}. \tag{2}$$

$\frac{\mathrm{d}z}{\mathrm{d}t}\left(\text{and also } \frac{\mathrm{d}x}{\mathrm{d}t} \text{ and } \frac{\mathrm{d}y}{\mathrm{d}t}\right)$ is called the *total differential coefficient* of z.

This name is given to distinguish it from the partial differential coefficients $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$. Thus to differentiate z, which is directly a function of x and y, (where x and y are functions of t) with respect to t, we need not express z as a function of t by substituting for x and y. We can differentiate z with respect to t via x and y using the result (2).

Corollary 1: In the differential form, result (2) can be written as

$$\mathrm{d}z = \frac{\partial z}{\partial x}\mathrm{d}x + \frac{\partial z}{\partial y}\mathrm{d}y \tag{3}$$

$\mathrm{d}z$ is called the *total differential* of z.

Corollary 2: If z is directly a function of two variables u and v, which are in turn functions of two other variables x and y, clearly z is a function of x and y ultimately. Hence the total differentiation of z is meaningless. We can find only $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ by using the following results which can be derived as result (2) given above.

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x} \tag{4}$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y} \tag{5}$$

We note that the partial differentiation of z is performed via the intermediate variables u and v, which are functions of x and y. Hence $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ are called *partial derivatives of a function of two functions.*

Note ☑ Results (2), (3), (4) and (5) can be extended to a function z of several intermediate variables.

4.2.1 Small Errors and Approximations

Since $\lim\limits_{\Delta x \to 0}\left(\frac{\Delta y}{\Delta x}\right) = \frac{\mathrm{d}y}{\mathrm{d}x}$, $\frac{\Delta y}{\Delta x} = \frac{\mathrm{d}y}{\mathrm{d}x}$ approximately or $\Delta y \simeq \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)\Delta x$ (1)

If we assume that $\mathrm{d}x$ and $\mathrm{d}y$ are approximately equal to Δx and Δy respectively, result (1) can be derived from the differential relation.

$$dy = \left(\frac{dy}{dx}\right) dx \tag{2}$$

Though (2) is an exact relation, it can be made use of to get the approximate relation (1), by replacing dx and dy by Δx and Δy respectively.

Let $y = f(x)$. If we assume that the value of x is obtained by measurement, it is likely that there is a small error Δx in the measured value of x. This error in the value of x will contribute a small error Δy in the calculated value of y, as x and y are functionally related. The small increments Δx and Δy can be assumed to represent the small errors Δx and Δy. Thus the relation between the errors Δx and Δy can be taken as

$$\Delta y \simeq f'(x)\,\Delta x$$

This concept can be extended to a function of several variables.

If $u = u(x, y, z)$ or $f(x, y, z)$ and if the value of u is calculated on the measured values of x, y, z, the likely errors $\Delta x, \Delta y, \Delta z$ will result in an error Δu in the calculated value of u, given by

$$\Delta u \simeq \frac{\partial u}{\partial x}\Delta x + \frac{\partial u}{\partial y}\Delta y + \frac{\partial u}{\partial z}\Delta z,$$

which can be assumed as the approximate version of the total differential relation

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz$$

Note ☑ The error Δx in x is called *the absolute error* in x, while $\frac{\Delta x}{x}$ is called the *relative* or *proportional error* in x and $\frac{100\,\Delta x}{x}$ is called *the percentage error* in x.

4.2.2 Differentiation of Implicit Functions

When x and y are connected by means of a relation of the form $f(x, y) = 0$, x and y are said to be implicitly related or y is said to be an *implicit function* of x. When x and y are implicitly related, it may not be possible in many cases to express y as a single valued function of x explicitly. However $\frac{dy}{dx}$ can be found out in such cases as a mixed function of x and y using partial derivatives as explained below:

Since $$f(x, y) = 0,\ df = 0$$

i.e., $\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = 0$, by definition of total differential. Dividing by dx, we have

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dx} = 0$$

$$\therefore \qquad \frac{dy}{dx} = -\frac{\left(\frac{\partial f}{\partial x}\right)}{\left(\frac{\partial f}{\partial y}\right)} \tag{1}$$

If we denote $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial x \partial y}$ and $\frac{\partial^2 f}{\partial y^2}$ by the letters p, q, r, s, t respectively, then

$$\frac{dy}{dx} = -\frac{p}{q} \tag{2}$$

We can express the second order derivative $\frac{d^2 y}{dx^2}$ in terms of p, q, r, s, t as given below. Noting that p and q are functions of x and y and differentiating both sides of (2) with respect to x totally, we have

$$\frac{d^2 y}{dx^2} = -\left(\frac{q\frac{dp}{dx} - p\frac{dq}{dx}}{q^2}\right)$$

$$= \frac{p\left\{\frac{\partial q}{\partial x} + \frac{\partial q}{\partial y}\cdot\frac{dy}{dx}\right\} - q\left\{\frac{\partial p}{\partial x} + \frac{\partial p}{\partial y}\cdot\frac{dy}{dx}\right\}}{q^2}$$

$$= \frac{p\left\{s + t\left(\frac{-p}{q}\right)\right\} - q\left\{r + s\left(\frac{-p}{q}\right)\right\}}{q^2},$$

since $$\frac{\partial p}{\partial x} = \frac{\partial^2 f}{\partial x^2} = r; \quad \frac{\partial p}{\partial y} = \frac{\partial q}{\partial x} = \frac{\partial^2 f}{\partial x\,\partial y} = s; \quad \frac{\partial q}{\partial y} = \frac{\partial^2 f}{\partial y^2} = t.$$

$$= \frac{p(qs - pt) - q(qr - ps)}{q^3}$$

$$= -\frac{(p^2 t - 2pqs + q^2 r)}{q^3}$$

WORKED EXAMPLE 4(a)

Example 4.1

(i) If $u = xy + yz + zx$, where $x = e^t$, $y = e^{-t}$ and $z = \frac{1}{t}$, find $\frac{du}{dt}$

(ii) If $u = \sin^{-1}(x - y)$, where $x = 3t$ and $y = 4t^3$, show that $\frac{du}{dt} = \frac{3}{\sqrt{1 - t^2}}$

(i) $u = xy + yz + zx$

$$\therefore \qquad \frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial u}{\partial z}\cdot\frac{dz}{dt}$$

$$= (y+z)e^t + (z+x)(-e^{-t}) + (x+y)\left(-\frac{1}{t^2}\right)$$

$$= \left(e^{-t}+\frac{1}{t}\right)e^t - \left(\frac{1}{t}+e^t\right)e^{-t} - \left(e^t+e^{-t}\right).\frac{1}{t^2}$$

$$= 1+\frac{1}{t}e^t - \frac{1}{t}e^{-t} - 1 - \frac{1}{t^2}e^t - \frac{1}{t^2}e^{-t}$$

$$= \frac{2}{t}\sinh t - \frac{2}{t^2}\cosh t.$$

(ii) $u = \sin^{-1}(x-y)$

$$\therefore \qquad \frac{du}{dt} = \frac{1}{\sqrt{1-(x-y)^2}}\frac{dx}{dt} + \frac{1}{\sqrt{1-(x-y)^2}}\left(-\frac{dy}{dt}\right)$$

$$= \frac{1}{\sqrt{1-(x-y)^2}}(3-12t^2) \tag{1}$$

Now $1-(x-y)^2 = 1-(3t-4t^3)^2$

$$= 1-9t^2+24t^4-16t^6$$

$$= (1-t^2)(1-8t^2+16t^4)$$

$$= (1-t^2)(1-4t^2)^2 \tag{2}$$

Using (2) in (1), we get

$$\frac{du}{dt} = \frac{1}{(1-4t^2)\sqrt{1-t^2}} \times 3(1-4t^2)$$

$$= \frac{3}{\sqrt{1-t^2}}.$$

Example 4.2 If $u = f\left(\frac{x}{y}, \frac{y}{z}, \frac{z}{x}\right)$, prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0$.

Let $$r = \frac{x}{y},\; s = \frac{y}{z} \text{ and } t = \frac{z}{x} \tag{1}$$

$\therefore\ u = f(r, s, t)$, where r, s, t are functions of x, y, z as assumed in (1)

$$\therefore \qquad \frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial x} + \frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial x}$$

$$= \frac{1}{y}\cdot\frac{\partial u}{\partial r} + \frac{\partial u}{\partial s}\cdot 0 - \frac{z}{x^2}\cdot\frac{\partial u}{\partial t} \tag{2}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial y} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial y} + \frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial y}$$

$$= -\frac{x}{y^2}\cdot\frac{\partial u}{\partial r} + \frac{1}{z}\cdot\frac{\partial u}{\partial s} \tag{3}$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial z} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial z} + \frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial z}$$

$$= -\frac{y}{z^2}\cdot\frac{\partial u}{\partial s} + \frac{1}{x}\cdot\frac{\partial u}{\partial t} \tag{4}$$

From (2), (3) and (4), we have

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = \left(\frac{x}{y}\frac{\partial u}{\partial r} - \frac{z}{x}\frac{\partial u}{\partial t}\right)$$

$$+\left(-\frac{x}{y}\frac{\partial u}{\partial r} + \frac{y}{z}\frac{\partial u}{\partial s}\right) + \left(-\frac{y}{z}\frac{\partial u}{\partial s} + \frac{z}{x}\frac{\partial u}{\partial t}\right)$$

$$= 0.$$

Example 4.3 If z be a function of x and y, where $x = e^u + e^{-\upsilon}$ and $y = e^{-u} - e^{\upsilon}$, prove that

$$\frac{\partial z}{\partial u} - \frac{\partial z}{\partial \upsilon} = x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}$$

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}$$

$$= e^u\frac{\partial z}{\partial x} - e^{-u}\frac{\partial z}{\partial y} \tag{1}$$

$$\frac{\partial z}{\partial \upsilon} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial \upsilon} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial \upsilon}$$

$$= -e^{-\upsilon}\frac{\partial z}{\partial x} - e^{\upsilon}\cdot\frac{\partial z}{\partial y} \tag{2}$$

From (1) and (2), we have

$$\frac{\partial z}{\partial u} - \frac{\partial z}{\partial \upsilon} = (e^u + e^{-\upsilon})\frac{\partial z}{\partial x} - (e^{-u} - e^{\upsilon})\frac{\partial z}{\partial y}$$

$$= x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}$$

Example 4.4 If $u = f(x, y)$, where $x = r\cos\theta$ and $y = r\sin\theta$, prove that

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = \left(\frac{\partial u}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2.$$

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial r}$$

$$= \cos\theta\cdot\frac{\partial u}{\partial x} + \sin\theta\frac{\partial u}{\partial y} \tag{1}$$

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial \theta}$$

$$= -r\sin\theta\cdot\frac{\partial u}{\partial x} + r\cos\theta\cdot\frac{\partial u}{\partial y}$$

i.e.,
$$\frac{1}{r}\frac{\partial u}{\partial \theta} = -\sin\theta\frac{\partial u}{\partial x} + \cos\theta\cdot\frac{\partial u}{\partial y} \tag{2}$$

Squaring both sides of (1) and (2) and adding, we get

$$\left(\frac{\partial u}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2 = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2$$

Example 4.5 Find the equivalent of $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2}$ in polar co-ordinates.

$$u = u(x, y), \text{ where } x = r\cos\theta \text{ and } y = r\sin\theta$$

$\therefore$ u can also be considered as $u(r, \theta)$, where

$$r = \sqrt{x^2 + y^2} \quad \text{and} \quad \theta = tan^{-1}\left(\frac{y}{x}\right)$$

Now we proceed to find $\dfrac{\partial u}{\partial x}$ and $\dfrac{\partial u}{\partial y}$ via r and θ.

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}\cdot\frac{\partial \theta}{\partial x}$$

$$= \frac{x}{\sqrt{x^2 + y^2}}\frac{\partial u}{\partial r} + \frac{1}{1 + \dfrac{y^2}{x^2}}\cdot\left(-\frac{y}{x^2}\right)\cdot\frac{\partial u}{\partial \theta}$$

$$= \cos\theta\frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial \theta} \tag{1}$$

From (1), we can infer that

$$\frac{\partial}{\partial x} \equiv \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta} \tag{2}$$

Now
$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)$$
$$= \left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right)\left(\cos\theta\frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial\theta}\right)$$
$$= \cos^2\theta\cdot\frac{\partial^2 u}{\partial r^2} - \sin\theta\cos\theta\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial u}{\partial\theta}\right)$$
$$-\frac{\sin\theta}{r}\cdot\frac{\partial}{\partial\theta}\left(\cos\theta\frac{\partial u}{\partial r}\right) + \frac{\sin\theta}{r^2}\cdot\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right)$$

$\left(\because r \text{ and } \theta \text{ are independent and } \frac{\partial u}{\partial r} \text{ and } \frac{\partial u}{\partial\theta} \text{ are functions of } r \text{ and } \theta\right)$.

$$= \cos^2\theta\frac{\partial^2 u}{\partial r^2} - \sin\theta\cos\theta\left(\frac{1}{r}\frac{\partial^2 u}{\partial r\,\partial\theta} - \frac{1}{r^2}\frac{\partial u}{\partial\theta}\right)$$
$$-\frac{\sin\theta}{r}\left(\cos\theta\frac{\partial^2 u}{\partial\theta\,\partial r} - \sin\theta\frac{\partial u}{\partial r}\right) + \frac{\sin\theta}{r^2}\left(\sin\theta\frac{\partial^2 u}{\partial\theta^2} + \cos\theta\frac{\partial u}{\partial\theta}\right) \tag{3}$$

Now
$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial y} + \frac{\partial u}{\partial\theta}\cdot\frac{\partial\theta}{\partial y}$$
$$= \frac{y}{\sqrt{x^2+y^2}}\frac{\partial u}{\partial r} + \frac{1}{1+\frac{y^2}{x^2}}\cdot\left(\frac{1}{x}\right)\frac{\partial u}{\partial\theta}$$
$$= \sin\theta\frac{\partial u}{\partial r} + \frac{\cos\theta}{r}\frac{\partial u}{\partial\theta} \tag{4}$$

From (4) we infer that

$$\frac{\partial}{\partial y} \equiv \sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta} \tag{5}$$

$\therefore$
$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)$$
$$= \left(\sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}\right)\left(\sin\theta\frac{\partial u}{\partial r} + \frac{\cos\theta}{r}\frac{\partial u}{\partial\theta}\right)$$
$$= \sin^2\theta\frac{\partial^2 u}{\partial r^2} + \sin\theta\cos\theta\left(\frac{1}{r}\frac{\partial^2 u}{\partial r\,\partial\theta} - \frac{1}{r^2}\frac{\partial u}{\partial\theta}\right)$$
$$+\frac{\cos\theta}{r}\left(\sin\theta\frac{\partial^2 u}{\partial\theta\,\partial r} + \cos\theta\frac{\partial u}{\partial r}\right) + \frac{\cos\theta}{r^2}\left(\cos\theta\frac{\partial^2 u}{\partial\theta^2} - \sin\theta\frac{\partial u}{\partial\theta}\right) \tag{6}$$

Adding (3) and (6), we have

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$$

Example 4.6 Given the transformations $u = e^x \cos y$ and $\upsilon = e^x \sin y$ and that f is a function of u and υ and also of x and y, prove that

$$\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}=(u^2+\upsilon^2)\left(\frac{\partial^2 f}{\partial u^2}+\frac{\partial^2 f}{\partial \upsilon^2}\right)$$

$$\frac{\partial f}{\partial x}=\frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial f}{\partial \upsilon}\cdot\frac{\partial \upsilon}{\partial x}$$

$$=e^x\cos y\cdot\frac{\partial f}{\partial u}+e^x\sin y\cdot\frac{\partial f}{\partial \upsilon}$$

$$=u\frac{\partial f}{\partial u}+\upsilon\frac{\partial f}{\partial \upsilon} \tag{1}$$

$$\frac{\partial}{\partial x}\equiv u\frac{\partial}{\partial u}+\upsilon\frac{\partial}{\partial \upsilon} \tag{2}$$

$$\frac{\partial^2 f}{\partial x^2}=\left(u\frac{\partial}{\partial u}+\upsilon\frac{\partial}{\partial \upsilon}\right)\left(u\frac{\partial f}{\partial u}+\upsilon\frac{\partial f}{\partial \upsilon}\right)$$

$$=u\left(u\cdot\frac{\partial^2 f}{\partial u^2}+\frac{\partial f}{\partial u}\right)+u\upsilon\frac{\partial^2 f}{\partial u\,\partial \upsilon}+u\upsilon\frac{\partial^2 f}{\partial \upsilon\,\partial u}$$

$$+\upsilon\left(\upsilon\frac{\partial^2 f}{\partial \upsilon^2}+\frac{\partial f}{\partial \upsilon}\right) \tag{3}$$

$$\frac{\partial f}{\partial y}=\frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial f}{\partial \upsilon}\cdot\frac{\partial \upsilon}{\partial y}$$

$$=-e^x\sin y\cdot\frac{\partial f}{\partial u}+e^x\cos y\frac{\partial f}{\partial \upsilon}$$

$$=-\upsilon\frac{\partial f}{\partial u}+u\frac{\partial f}{\partial \upsilon} \tag{4}$$

$$\therefore \qquad \frac{\partial}{\partial y}\equiv-\upsilon\frac{\partial}{\partial u}+u\frac{\partial}{\partial \upsilon} \tag{5}$$

$$\frac{\partial^2 f}{\partial y^2}=\left(-\upsilon\frac{\partial}{\partial u}+u\frac{\partial}{\partial \upsilon}\right)\left(-\upsilon\frac{\partial f}{\partial u}+u\frac{\partial f}{\partial \upsilon}\right)$$

$$=\upsilon^2\frac{\partial^2 f}{\partial u^2}-\upsilon\left(u\frac{\partial^2 f}{\partial u\,\partial \upsilon}+\frac{\partial f}{\partial \upsilon}\right)-u\left(\upsilon\frac{\partial^2 f}{\partial \upsilon\,\partial u}+\frac{\partial f}{\partial u}\right)$$

$$+u^2\frac{\partial^2 f}{\partial \upsilon^2} \tag{6}$$

Adding (3) and (6), we get

$$\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}=(u^2+v^2)\left(\frac{\partial^2 f}{\partial u^2}+\frac{\partial^2 f}{\partial v^2}\right)$$

Example 4.7 If $z=f(u,v)$, where $u=\cosh x\cos y$ and $v=\sinh x\sin y$, prove that

$$\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=\left(\sinh^2 x+\sin^2 y\right)\left(\frac{\partial^2 z}{\partial u^2}+\frac{\partial^2 z}{\partial v^2}\right)$$

$$z_x = z_u\cdot u_x + z_v\cdot v_x,\ \text{where}\ z_x\equiv\frac{\partial z}{\partial x}\ \text{etc.}$$

$$= \sinh x\cdot\cos y\cdot z_u + \cosh x\sin y\cdot z_v$$

Since z is a function of u and v, z_u and z_v are also functions of u and v. Hence to differentiate z_u and z_v with respect to x or y, we have to do it via u and v.

$$\therefore\quad z_{xx} = \cos y\left[\cosh x\cdot z_u + \sinh x\left\{z_{uu}\cdot\sinh x\cos y + z_{uv}\cosh x\sin y\right\}\right]$$
$$+\sin y\left[\sinh x\cdot z_v + \cosh x\left\{z_{vu}\cdot\sinh x\cos y + z_{vv}\cdot\cosh x\sin y\right\}\right]$$

i.e.,
$$z_{xx} = \cosh x\cos y\cdot z_u + \sinh x\cdot\sin y\, z_v + \sinh^2 x\cos^2 y\cdot z_{uu}$$
$$+2\sinh x\cosh x\sin y\cos y\cdot z_{uv} + \cosh^2 x\sin^2 y\cdot z_{vv} \tag{1}$$

$$z_y = -z_u\cdot\cosh x\sin y + z_v\sinh x\cos y$$

$$z_{yy} = -\cosh x[\cos y\cdot z_u + \sin y\{z_{uu}\cdot(-\cosh x\sin y)$$
$$+z_{uv}\cdot\sinh x\cdot\cos y\} + \sinh x[-\sinh y\cdot z_v$$
$$+\cos y\{-z_{vu}\cdot\cosh x\sin y + z_{vv}\cdot\sinh x\cos y\}]$$

i.e.,
$$z_{yy} = -\cosh x\cos y\cdot z_u - \sinh x\cdot\sin y\cdot z_v$$
$$+\cosh^2 x\sin^2 y\cdot z_{uu} - 2\sinh x\cosh x\sin y\cos y\cdot z_{uv}$$
$$+\sinh^2 x\cos^2 y\cdot z_{vv} \tag{2}$$

Adding (1) and (2), we get

$$z_{xx}+z_{yy} = (\sinh^2 x\cos^2 y + \cosh^2 x\sin^2 y)(z_{uu}+z_{vv})$$
$$= \{\sinh^2 x(1-\sin^2 y) + (1+\sinh^2 x)\sin^2 y\}(z_{uu}+z_{vv})$$
$$= (\sinh^2 x + \sin^2 y)(z_{uu}+z_{vv})$$

Example 4.8 Find $\frac{dy}{dx}$, when (i) $x^3 + y^3 = 3ax^2y$ and (ii) $x^y + y^x = c$.

(i) $f(x, y) = x^3 + y^3 - 3ax^2y$

$$p = \frac{\partial f}{\partial x} = 3x^2 - 6axy$$

$$q = \frac{\partial f}{\partial y} = 3y^2 - 3ax^2$$

$$\frac{dy}{dx} = -\frac{p}{q} = -\frac{3(x^2 - 2axy)}{3(y^2 - ax^2)} = \frac{x(2ay - x)}{y^2 - ax^2}$$

(ii) $f(x, y) = x^y + y^x - c$

$$p = \frac{\partial f}{\partial x} = yx^{y-1} + y^x \log y$$

$$q = \frac{\partial f}{\partial y} = x^y \log x + xy^{x-1}$$

$$\frac{dy}{dx} = -\frac{p}{q} = -\frac{yx^{y-1} + y^x \log x}{xy^{x-1} + x^y \log x}.$$

Example 4.9 If $ax^2 + 2hxy + by^2 = 1$, show that $\frac{d^2y}{dx^2} = \frac{h^2 - ab}{(hx + by)^3}$.

$f(x, y) = ax^2 + 2hxy + by^2 - 1$

$$p = \frac{\partial f}{\partial x} = 2(ax + hy);\ q = \frac{\partial f}{\partial y} = 2(hx + by)$$

$$r = \frac{\partial^2 f}{\partial x^2} = 2a;\ s = \frac{\partial^2 f}{\partial x \partial y} = 2h;\ t = \frac{\partial^2 f}{\partial y^2} = 2b$$

$$\frac{dy}{dx} = -\frac{p}{q} = -\frac{(ax + hy)}{hx + by}$$

$$\frac{d^2y}{dx^2} = \frac{-(p^2t - 2pqs + q^2r)}{q^3}$$

(Refer to differentiation of implicit functions)

$$= \frac{-\{8b(ax + hy)^2 - 16h(ax + hy)(hx + by) + 8a(hx + by)^2\}}{8(hx + by)^3}$$

$$= \frac{1}{(hx + by)^3}[2h\{ahx^2 + (ab + h^2)xy + bhy^2\}$$
$$-\{a^2bx^2 + 2abhxy + h^2by^2\} - \{ah^2x^2 + 2abhxy + ab^2y^2\}]$$

$$= \frac{1}{(hx+by)^3}[a(h^2-ab)x^2+2h(h^2-ab)xy+b(h^2-ab)y^2]$$

$$= \frac{(h^2-ab)}{(hx+by)^3}(ax^2+2hxy+by^2) = \frac{(h^2-ab)\cdot 1}{(hx+by)^3} = \frac{h^2-ab}{(hx+by)^3}.$$

Example 4.10 Find $\frac{du}{dx}$ if (i) $u=\sin(x^2+y^2)$, where $a^2x^2+b^2y^2=c^2$ (i), $u=\tan^{-1}\left(\frac{y}{x}\right)$ where $x^2+y^2=a^2$, by treating u as function of x and y only.

(i) $u = \sin(x^2+y^2)$

$$\therefore \quad \frac{du}{dx} = \frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}\cdot\frac{dy}{dx}$$

$$= 2x\cos\left(x^2+y^2\right)+2y\cos\left(x^2+y^2\right)\frac{dy}{dx} \tag{1}$$

Now $a^2x^2 + b^2y^2 = c^2$

Differentiating with respect to x,

$$2a^2x+2b^2y\frac{dy}{dx} = 0$$

or
$$\frac{dy}{dx} = -\frac{a^2x}{b^2y} \tag{2}$$

Using (2) in (1), we get

$$\frac{du}{dx} = 2x\cos\left(x^2+y^2\right)+2y\cos\left(x^2+y^2\right)\times\left(\frac{-a^2x}{b^2y}\right)$$

$$= 2x\cos\left(x^2+y^2\right)\left(b^2-a^2\right)/b^2$$

(ii)
$$u = \tan^{-1}\left(\frac{y}{x}\right)$$

$$\frac{du}{dx} = \frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}\cdot\frac{dy}{dx}$$

$$= \frac{1}{1+\frac{y^2}{x^2}}\left(-\frac{y}{x^2}\right)+\frac{1}{1+\frac{y^2}{x^2}}\left(\frac{1}{x}\right)\cdot\frac{dy}{dx}$$

$$= -\frac{y}{x^2+y^2}+\frac{x}{x^2+y^2}\cdot\frac{dy}{dx} \tag{3}$$

$$x^2+y^2 = a^2$$

$$\therefore \quad 2x+2y\frac{dy}{dx} = 0$$

or
$$\frac{dy}{dx} = -\frac{x}{y} \tag{4}$$

Using (4) in (3), we get

$$\frac{du}{dx} = -\frac{y}{x^2+y^2} + \frac{x}{x^2+y^2}\left(-\frac{x}{y}\right)$$
$$= -\frac{1}{y}.$$

Example 4.11 If $u = x^2 - y^2$ and $v = xy$, find the values of $\frac{\partial x}{\partial u}, \frac{\partial x}{\partial v}, \frac{\partial y}{\partial u}$ and $\frac{\partial y}{\partial v}$.

x and y cannot be easily expressed as single valued functions or u and v.

Given
$$x^2 - y^2 = u \tag{1}$$
and
$$xy = v \tag{2}$$

Nothing that x and y are functions of u and v and differentiating both sides of (1) and (2) partially with respect to u, we have

$$2x\frac{\partial x}{\partial u} - 2y\frac{\partial y}{\partial u} = 1 \tag{3}$$

$$y\frac{\partial x}{\partial u} + x\frac{\partial y}{\partial u} = 0 \tag{4}$$

Solving (3) and (4), we get

$$\frac{\partial x}{\partial u} = \frac{x}{2\left(x^2+y^2\right)} \text{ and } \frac{\partial y}{\partial u} = -\frac{y}{2\left(x^2+y^2\right)}$$

Differentiating both sides of (1) and (2) partially with respect to v, we have

$$2x\frac{\partial x}{\partial v} - 2y\frac{\partial y}{\partial v} = 0 \tag{5}$$

$$y\frac{\partial x}{\partial v} + x\frac{\partial y}{\partial v} = 1 \tag{6}$$

Solving (5) and (6), we get

$$\frac{\partial x}{\partial v} = \frac{y}{x^2+y^2} \text{ and } \frac{\partial y}{\partial v} = \frac{x}{x^2+y^2}.$$

Example 4.12 If $x^2 + y^2 + z^2 - 2xyz = 1$, show that $\dfrac{dx}{\sqrt{1-x^2}} + \dfrac{dy}{\sqrt{1-y^2}} + \dfrac{dz}{\sqrt{1-z^2}} = 0.$

Let
$$\phi \equiv x^2 + y^2 + z^2 - 2xyz - 1 = 0 \tag{1}$$

$\therefore$
$$d\phi = 0$$

i.e.,
$$\frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz = 0 \tag{2}$$

i.e.,
$$2(x - yz)dx + 2(y - zx)dy + 2(z - xy)dz = 0$$

Now
$$\begin{aligned}(x - yz)^2 &= x^2 - 2xyz + y^2z^2 \\ &= 1 - y^2 - z^2 + y^2z^2, \text{ from (1)} \\ &= (1 - y^2)(1 - z^2)\end{aligned}$$

$\therefore$
$$x - yz = \sqrt{(1-y^2)(1-z^2)}$$

Similarly,
$$y - zx = \sqrt{(1-z^2)(1-x^2)}$$

and
$$z - xy = \sqrt{(1-x^2)(1-y^2)}$$

Using these values in (2), we have
$$\sqrt{(1-y^2)(1-z^2)}\,dx + \sqrt{(1-z^2)(1-x^2)}\,dy + \sqrt{(1-x^2)(1-y^2)}\,dz = 0$$

Dividing by $\sqrt{(1-x^2)(1-y^2)(1-z^2)}$, we get
$$\frac{dx}{\sqrt{1-x^2}} + \frac{dy}{\sqrt{1-y^2}} + \frac{dz}{\sqrt{1-z^2}} = 0.$$

Example 4.13 The specific gravity s of a body is given by $s = \dfrac{W_1}{W_1 - W_2}$ where W_1 and W_2 are the weights of the body in air and in water respectively. Show that if there is an error of 1% in each weighing, s is not affected. But if there is an error of 1% in W_1 and 2% in W_2, show that the percentage error in s is $\dfrac{W_2}{W_1 - W_2}$.

$$s = \frac{W_1}{W_1 - W_2}$$

$\therefore$
$$\log s = \log W_1 - \log(W_1 - W_2)$$

Taking differentials on both sides,

$$\frac{1}{s}\,\mathrm{d}s = \frac{1}{W_1}\,\mathrm{d}W_1 - \frac{1}{W_1 - W_2}\left(dW_1 - dW_2\right)$$

$\therefore$ The relation among the errors is nearly

$$\frac{1}{s}\,\Delta s = \frac{1}{W_1}\,\Delta W_1 - \frac{1}{W_1 - W_2}\left(\Delta W_1 - \Delta W_2\right) \tag{1}$$

or
$$\frac{100\,\Delta s}{s} = \frac{100\,\Delta W_1}{W_1} - \frac{1}{W_1 - W_2}\left(100\,\Delta W_1 - 100\,\Delta W_2\right) \tag{2}$$

(i) Given that $\dfrac{100\,\Delta W_1}{W_1} = 1$ and $\dfrac{100\,\Delta W_2}{W_2} = 1$

Using these values in (2), we have

$$\frac{100\,\Delta s}{s} = 1 - \frac{1}{W_1 - W_2}\left(W_1 - W_2\right) = 0$$

$\therefore$ s is not affected, viz., there is no error in s.

(ii) Given that $\dfrac{100\,\Delta W_1}{W_1} = 1$ and $\dfrac{100\,\Delta W_2}{W_2} = 2.$ Using these values in (2), we have

$$\begin{aligned}\frac{100\,\Delta s}{s} &= 1 - \frac{1}{W_1 - W_2}\left(W_1 - 2W_2\right)\\ &= \frac{W_2}{W_1 - W_2}\end{aligned}$$

i.e., % error in $s = \dfrac{W_2}{W_1 - W_2}.$

Example 4.14 The work that must be done to propel a ship of displacement D for a distance s in time t is proportional to $s^2 D^{3/2} \div t^2$. Find approximately the percentage increase of work necessary when the distance is increased by 1%, the time is diminished by 1% and the displacement of the ship is diminished by 3%.

Given that $W = ks^2 D^{3/2}/t^2$, where k is the constant of proportionality.

$\therefore$
$$\log W = \log k + 2\log s + \frac{3}{2}\log D - 2\log t.$$

Taking differentials on both sides,

$$\frac{\mathrm{d}W}{W} = 2\frac{\mathrm{d}s}{s} + \frac{3}{2}\frac{\mathrm{d}D}{D} - 2\frac{\mathrm{d}t}{t}$$

∴ The relation among the percentage errors is approximately,

$$\frac{100\,\Delta W}{W} = 2\times\frac{100\,\Delta s}{s}+\frac{3}{2}\cdot\frac{100\,\Delta D}{D}-2\times\frac{100\,\Delta t}{t} \tag{1}$$

Given that $\dfrac{100\,\Delta s}{s} = 1, \dfrac{100\,\Delta t}{t} = -1$ and $\dfrac{100\,\Delta D}{D} = -3.$

Using these values in (1), we have

$$\begin{aligned}\frac{100\,\Delta W}{W} &= 2\times 1+\frac{3}{2}\times(-3)-2\times(-1)\\ &= -0.5\end{aligned}$$

i.e., percentage decrease of work = 0.5.

Example 4.15 The period T of a simple pendulum with small oscillations is given by $T = 2\pi\sqrt{\dfrac{l}{g}}$. If T is computed using $l = 6$ cm and $g = 980$ cm/sec^2, find approximately the error in T, if the values are $l = 5.9$ cm and $g = 981$ cm/sec^2. Find also the percentage error.

$$T = 2\pi\sqrt{\frac{l}{g}}$$

$$\therefore \qquad \log T = \log 2+\log\pi+\frac{1}{2}\log l-\frac{1}{2}\log g$$

Taking differentials on both sides,

$$\frac{1}{T}\,\mathrm{d}T=\frac{1}{2l}\,\mathrm{d}l-\frac{1}{2g}\,\mathrm{d}g \tag{1}$$

$$\begin{aligned}\therefore \qquad \mathrm{d}T &= 2\pi\sqrt{\frac{l}{g}}\left\{\frac{1}{2l}\,\mathrm{d}l-\frac{1}{2g}\,\mathrm{d}g\right\}\\ &=\pi\left\{\frac{1}{\sqrt{l}\,\mathrm{g}}\,\mathrm{d}l-\frac{\sqrt{l}}{g\sqrt{g}}\,\mathrm{d}g\right\}\\ &= \pi\left\{\frac{0\cdot 1}{\sqrt{5.9\times 981}}-\frac{\sqrt{5\cdot 9}}{981\sqrt{981}}\times(-1)\right\}\end{aligned}$$

i.e., Error in $T = 0.0044$ sec.

$$\% \text{ error in } T = \frac{100\,\mathrm{d}T}{T}$$

$$= 50\left\{\frac{\mathrm{d}l}{l} - \frac{\mathrm{d}g}{g}\right\}, \text{by (1)}$$

$$= 50\left\{\frac{0\cdot 1}{5\cdot 9} - \frac{(-1)}{981}\right\}$$

$$= 0.8984$$

Example 4.16 The base diameter and altitude of a right circular cone are measured as 4 cm and 6 cm respectively. The possible error in each measurement is 0.1 cm. Find approximately the maximum possible error in the value computed for the volume and lateral surface.

Volume of the right circular cone is given by $V = \frac{1}{3}\pi\cdot\left(\frac{D}{2}\right)^2 h$

$$\therefore \qquad \mathrm{d}V = \frac{\pi}{12}(D^2\cdot\mathrm{d}h + 2\,Dh\cdot\mathrm{d}D)$$

$$= \frac{\pi}{12}\{16\times 0.1 + 2\times 4\times 6\times 0.1\}$$

i.e., Error in $V = 1.6755\ \text{cm}^3$.

Lateral surface area of the right circular cone is given by

$$S = \pi\frac{D}{2}l$$

$$= \frac{\pi}{4}D\sqrt{D^2 + 4h^2}$$

$$\therefore \qquad \mathrm{d}S = \frac{\pi}{4}\left[D\cdot\frac{1}{2\sqrt{D^2+4h^2}}(2D\,\mathrm{d}D + 8h\,\mathrm{d}h + \sqrt{D^2+4h^2}\ \mathrm{d}D\right]$$

$$= \frac{\pi}{4}\left[\frac{4}{\sqrt{16+144}}\{4\times 0\cdot 1 + 24\times 0\cdot 1\} + \sqrt{16+144}\times 0\cdot 1\right]$$

$$= 1.6889\,\text{cm}^2.$$

Example 4.17 The side c of a triangle ABC is calculated by using the measured values of its sides a, b and the angle C. Show that the error in the side c is given by

$$\Delta c = \cos B \cdot \Delta a + \cos A \cdot \Delta b + a \sin B \cdot \Delta C.$$

The side c is given by the formula

$$c^2 = a^2 + b^2 - 2ab \cos C \tag{1}$$

Taking the differentials on both sides of (1),

$$\begin{aligned} 2c\,\Delta c = 2a\,\Delta a + 2b\,\Delta b - 2\{&b \cos C \cdot \Delta a \\ &+ a \cos C \cdot \Delta b - ab \sin C \cdot \Delta C\}, \text{ nearly} \end{aligned}$$

i.e.,
$$\Delta c = \frac{(a - b\cos C)\Delta a + (b - a\cos C)\Delta b + ab \sin C \cdot \Delta C}{c} \tag{2}$$

Now $b \cos C + c \cos B = a$

$$\therefore \quad \frac{a - b\cos C}{c} = \cos B \tag{3}$$

$$a \cos C + c \cos A = b$$

$$\therefore \quad \frac{b - a\cos C}{c} = \cos A \tag{4}$$

Also
$$\frac{b}{\sin B} = \frac{c}{\sin C}$$

$$\therefore \quad b \sin C = c \sin B$$

$$\therefore \quad \frac{ab \sin C}{c} = a \sin B \tag{5}$$

Using (3), (4) and (5) in (2), we get

$$\Delta c = \cos B \cdot \Delta a + \cos A \cdot \Delta b + a \sin B \cdot \Delta C$$

Example 4.18 The angles of a triangle ABC are calculated from the sides a, b, c. If small changes δa, δb, δc are made in the measurement of the sides, show that

$$\delta A = \frac{a}{2\Delta}(\delta a - \delta b \cos C - \delta c \cos B)$$

and δB and δC are given by similar expressions, where Δ is the area of the triangle. Verify that $\delta A + \delta B + \delta C = 0$.
In triangle ABC,

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} \tag{1}$$

Taking differentials on both sides of (1),

$$-2\sin A\cdot\delta A=\left[bc\left\{2b\,\delta b+2c\delta c-2a\delta a\right\}-\left(b^2+c^2-a^2\right)\left(b\delta c+c\delta b\right)\right]\div b^2c^2$$

$$=\frac{\left(b^2c-c^3+a^2c\right)\delta b+\left(bc^2-b^3+a^2b\right)\delta c-2abc\,\delta a}{b^2c^2}$$

$$=\frac{c\left(a^2+b^2-c^2\right)\delta b+b\left(c^2+a^2-b^2\right)\delta c-2abc\delta a}{b^2c^2}$$

$$=\frac{c\left(2\,ab\cos C\right)\delta b+b\left(2\,ca\cos B\right)\delta c-2abc\,\delta a}{b^2c^2},$$

by formulas similar to (1)

$$=\frac{2a}{bc}\left(\cos\,C\cdot\delta b+\cos B\cdot\delta c-\delta a\right)$$

$$\therefore\qquad \delta A=\frac{a}{bc\sin A}\left(\delta a-\cos C\cdot\delta b-\cos B\cdot\delta c\right)$$

$$=\frac{a}{2\Delta}\left(\delta a-\cos C\cdot\delta b-\cos B\cdot\delta c\right),\text{ since }\Delta=\frac{1}{2}bc\sin A \tag{2}$$

Similarly,

$$\delta B=\frac{b}{2\Delta}\left(\delta b-\cos A\cdot\delta c-\cos C\cdot\delta a\right) \tag{3}$$

$$\delta C=\frac{c}{2\Delta}\left(\delta c-\cos B\cdot\delta a-\cos A\cdot\delta b\right) \tag{4}$$

Adding (2), (3) and (4), we get

$$\begin{aligned}2\Delta\left(\delta A+\delta B+\delta C\right)&=\left(a-b\cos C-c\cos B\right)\delta a\\&\quad+\left(b-a\cos C-c\cos A\right)\delta b+\left(c-a\cos B-b\cos A\right)\delta c\\&=\left(a-a\right)\delta a+\left(b-b\right)\delta b+\left(c-c\right)\delta c\\&\qquad\qquad\left(\because\ b\cos C+c\cos B=a\text{ etc.}\right)\\&=0\end{aligned}$$

$$\therefore\qquad \delta A+\delta B+\delta C=0.$$

Example 4.19 The area of a triangle ABC is calculated from the lengths of the sides a, b, c. If a is diminished and b is increased by the same small amount k, prove that the consequent change in the area is given by

$$\frac{\delta\Delta}{\Delta} = \frac{2(a-b)k}{c^2 - (a-b)^2}$$

The area of triangle *ABC* is given by

$$\Delta = \sqrt{s(s-a)(s-b)(s-c)}\ , \text{ where } 2s = a+b+c$$

$$\therefore \qquad \log\Delta = \frac{1}{2}\{\log s + \log(s-a) + \log(s-b) + \log(s-c)\}$$

Taking differentials on both sides, we get

$$\frac{\delta\Delta}{\Delta} = \frac{1}{2}\left\{\frac{\delta s}{s} + \frac{\delta s - \delta a}{s-a} + \frac{\delta s - \delta b}{s-b} + \frac{\delta s - \delta c}{s-c}\right\} \qquad (1)$$

Since $\qquad 2s = a+b+c,\ 2\delta s = \delta a + \delta b + \delta c$

i.e.; $2\delta s = -k + k + 0 = 0$, by the given data.

$$\therefore \qquad \delta s = 0 \qquad (2)$$

Using (2) in (1), we have

$$\begin{aligned}\frac{\delta\Delta}{\Delta} &= \frac{1}{2}\left(\frac{k}{s-a} - \frac{k}{s-b}\right)\\ &= \frac{k}{2}\left\{\frac{2}{b+c-a} - \frac{2}{c+a-b}\right\} (\because 2s = a+b+c)\\ &= \frac{k}{2}\times 2\left\{\frac{(c+a-b)-(b+c-a)}{[c-(a-b)][c+(a-b)]}\right\}\\ &= \frac{2k(a-b)}{c^2-(a-b)^2}\end{aligned}$$

EXERCISE 4(a)

Part A

(Short Answer Questions)

1. What is meant by total differential? Why it is called so?
2. If $u = \sin(xy^2)$, express the total differential of u in terms of those of x and y.
3. If $u = x^y \cdot y^x$, express du in terms of dx and dy.
4. If $u = xy \log xy$, express du in terms of dx and dy.
5. If $u = a^{xy}$, express du in terms of dx and dy.

6. Find $\frac{\mathrm{d}u}{\mathrm{d}t}$, if $u = x^3y^2 + x^2y^3$, where $x = at^2$, $y = 2at$.

7. Find $\frac{\mathrm{d}u}{\mathrm{d}t}$, if $u = e^{xy}$, where $x = \sqrt{a^2 - t^2}$, $y = \sin^3 t$.

8. Find $\frac{\mathrm{d}u}{\mathrm{d}t}$, if $u = \log(x + y + z)$, where $x = e^{-t}$, $y = \sin t$, $z = \cos t$.

9. Find $\frac{\mathrm{d}y}{\mathrm{d}x}$, using partial differentiation, if $x^3 + 3x^2y + 6xy^2 + y^3 = 1$.

10. If $x \sin(x - y) - (x + y) = 0$, use partial differentiation to prove that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y + x^2 \cos(x - y)}{x + x^2 \cos(x - y)}.$$

11. Find $\frac{\mathrm{d}y}{\mathrm{d}x}$, when $u = \sin(x^2 + y^2)$, where $x^2 + 4y^2 = 9$.

12. Find $\frac{\mathrm{d}y}{\mathrm{d}x}$, if $u = x^2y$, where $x^2 + xy + y^2 = 1$.

13. Define absolute, relative and percentage errors.
14. Using differentials, find the approximate value of $\sqrt{15}$.
15. Using differentials, find the approximate value of $2x^4 + 7x^3 - 8x^2 + 3x + 1$ when $x = 0.999$.
16. What error in the common logarithm of a number will be produced by an error of 1% in the number?
17. The radius of a sphere is found to be 10 cm with a possible error of 0.02 cm. Find the relative errors in computing the volume and surface area.
18. Find the percentage error in the area of an ellipse, when an error of 1% is made in measuring the lengths of its axes.
19. Find the approximate error in the surface of a rectangular parallelopiped of sides a, b, c if an error of k is made in measuring each side.
20. If the measured volume of a right circular cylinder is 2% too large and the measured length is 1% too small, find the percentage error in the calculated radius.

Part B

21. If $u = f(x - y, y - z, z - x)$, prove that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$.

22. If f is a function of u, v, w, where $u = \sqrt{yz}$, $v = \sqrt{zx}$, and $w = \sqrt{xy}$ show that

$$\sum u \frac{\partial f}{\partial u} = \sum x \frac{\partial f}{\partial x}.$$

23. If $f = f\left(\dfrac{y-x}{xy}, \dfrac{z-x}{zx}\right)$, show that $x^2\dfrac{\partial f}{\partial x} + y^2\dfrac{\partial f}{\partial y} + z^2\dfrac{\partial f}{\partial z} = 0.$

24. If $u = f(x^2 + 2yz, y^2 + 2zx)$, prove that

$$\left(y^2 - zx\right)\frac{\partial u}{\partial x} + \left(x^2 - yz\right)\frac{\partial u}{\partial y} + \left(z^2 - xy\right)\frac{\partial u}{\partial z} = 0.$$

25. If $f(cx - az, cy - bz) = 0$, where z is a function of x and y, prove that

$$a\frac{\partial z}{\partial x} + b\frac{\partial z}{\partial y} = c.$$

26. If $z = f(u, \upsilon)$, where $u = x + y$ and $\upsilon = x - y$, show that $2\dfrac{\partial z}{\partial u} = \dfrac{\partial z}{\partial x} + \dfrac{\partial z}{\partial y}\cdot$

27. If $z = f(x, y)$, where $x = u^2 + \upsilon^2$, $y = 2u\upsilon$, prove that

$$u\frac{\partial z}{\partial u} - \upsilon\frac{\partial z}{\partial \upsilon} = 2\sqrt{(x^2 - y^2)}\,\frac{\partial z}{\partial x}.$$

28. If $z = f(x, y)$, where $x = u + \upsilon$, $y = u\upsilon$, prove that

$$u\frac{\partial z}{\partial u} + \upsilon\frac{\partial z}{\partial \upsilon} = x\frac{\partial z}{\partial x} + 2y\frac{\partial z}{\partial y}\cdot$$

29. If $x = r\cos\theta$, $y = r\sin\theta$, prove that the equation $\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} = 0$ is equivalent to $\dfrac{\partial u}{\partial r} + \dfrac{1}{r}\tan\left(\dfrac{\pi}{4} - \theta\right)\dfrac{\partial u}{\partial \theta} = 0.$

30. If $z = f(u, \upsilon)$, where $u = x^2 - 2xy - y^2$ and $\upsilon = y$, show that the equation $(x+y)\dfrac{\partial z}{\partial x} + (x-y)\dfrac{\partial z}{\partial y} = 0$ is equivalent to $\dfrac{\partial z}{\partial \upsilon} = 0.$

31. If $z = f(u, \upsilon)$, where $u = x^2 - y^2$ and $\upsilon = 2xy$, prove that

$$\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = 4(x^2 + y^2)\left\{\left(\frac{\partial z}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial \upsilon}\right)^2\right\}.$$

32. If $z = f(u, \upsilon)$, where $u = x^2 - y^2$ and $\upsilon = 2xy$, show that

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 4(x^2 + y^2)\left(\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial \upsilon^2}\right).$$

33. If $z = f(x, y)$ where $x = X\cos a - Y\sin\alpha$ and $y = X\sin\alpha + Y\cos\alpha$, show that
$$\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=\frac{\partial^2 z}{\partial X^2}+\frac{\partial^2 z}{\partial Y^2}.$$

34. If $z = f(u, \upsilon)$, where $u = lx + my$ and $\upsilon = ly - mx$, show that
$$\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=(l^2+m^2)\left(\frac{\partial^2 z}{\partial u^2}+\frac{\partial^2 z}{\partial \upsilon^2}\right).$$

35. By changing the independent variables x and t to u and υ by means of the transformations $u = x - at$ and $\upsilon = x + at$, show that $a^2\dfrac{\partial^2 y}{\partial x^2}-\dfrac{\partial^2 y}{\partial t^2}=4a^2\dfrac{\partial^2 y}{\partial u\,\partial \upsilon}$.

36. By using the transformations $u = x + y$ and $\upsilon = x - y$, change the independent variables x and y in the equation $\dfrac{\partial^2 z}{\partial x^2}-\dfrac{\partial^2 z}{\partial y^2}=0$ to u and υ.

37. Transform the equation $\dfrac{\partial^2 z}{\partial x^2}+2\dfrac{\partial^2 z}{\partial x\,\partial y}+\dfrac{\partial^2 z}{\partial y^2}=0$ by changing the independent variables using $u = x - y$ and $\upsilon = x + y$.

38. Transform the equation $x^2\dfrac{\partial^2 z}{\partial x^2}+2xy\dfrac{\partial^2 z}{\partial x\,\partial y}+y^2\dfrac{\partial^2 z}{\partial y^2}=0$, by changing the independent variables using $u = x$ and $\upsilon = \dfrac{y^2}{x}$.

39. Transform the equation $\dfrac{\partial^2 z}{\partial x^2}-5\dfrac{\partial^2 z}{\partial x\,\partial y}+6\dfrac{\partial^2 z}{\partial y^2}=0$, by changing the independent variables using $u = 2x + y$ and $\upsilon = 3x + y$.

40. Transform the equation $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}=0$, by changing the independent variables using $z = x + iy$ and $z^* = x - iy$, where $i=\sqrt{-1}$.

41. Use partial differentiation to find $\dfrac{dy}{dx}$, when (i) $x^y = y^x$; (ii) $x^m y^n = (x+y)^{m+n}$; (iii) $(\cos x)^y = (\sin y)^x$; (iv) $(\sec x)^y = (\cot y)^x$; (v) $x^y = e^{x-y}$.

42. Use partial differentiation to find $\dfrac{d^2 y}{dx^2}$, when $x^3 + y^3 - 3axy = 0$.

43. Use partial differentiation to find $\dfrac{d^2y}{dx^2}$, when $x^4 + y^4 = 4a^2xy$.

44. Use partial differentiation to prove that $\dfrac{d^2y}{dx^2} = \dfrac{abc + 2fgh - af^2 - bg^2 - ch^2}{(hx + by + f)^3}$, when $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$.

45. Use partial differentiation to prove that $\dfrac{d^2y}{dx^2} = \dfrac{b^2 - ac}{(ay + b)^3}$, when $ay^2 + 2by + c = x^2$.

46. If $x^2 - y^2 + u^2 + 2v^2 = 1$ and $x^2 + y^2 - u^2 - v^2 = 2$, prove that $\dfrac{\partial u}{\partial x} = \dfrac{3x}{u}$ and $\dfrac{\partial v}{\partial x} = -\dfrac{2x}{v}$.

47. The deflection at the centre of a rod of length l and diameter d supported at its ends and loaded at the centre with a weight w is proportional to wl^3/d^4. What is the percentage increase in the deflection, if the percentage increases in w, l and d are 3, 2 and 1 respectively.

48 The torsional rigidity of a length of wire is obtained from the formula $N = \dfrac{8\pi Il}{t^2r^4}$. If l is decreased by 2%, t is increased by 1.5% and r is increased by 2%, show that the value of N is decreased by 13% approximately.

49. The Current C measured by a tangent galvanometer is given by the relation $C = k \tan\theta$, where θ is the angle of deflection. Show that the relative error in C due to a given error in θ is minimum when $\theta = 45°$.

50. The range R of a projectile projected with velocity v at an elevation θ is given by $R = \dfrac{v^2}{g}\sin 2\theta$. Find the percentage error in R due to errors of 1% in v and $\dfrac{1}{2}$% in θ, when $\theta = \dfrac{\pi}{6}$.

51. The velocity v of a wave is given by $v^2 = \dfrac{g\lambda}{2\pi} + \dfrac{2\pi T}{\rho\lambda}$, where g and λ are constants and ρ and T are variables. Prove that, if ρ is increased by 1% and T is decreased by 2%, then the percentage decrease in v is approximately $\dfrac{3\pi T}{\lambda\rho v^2}$.

52. The focal length of a mirror is given by the formula $\dfrac{1}{f} = \dfrac{1}{v} - \dfrac{1}{u}$. If equal errors k are made in the determination of u and v, show that the percentage error in f is $100k\left(\dfrac{1}{u} + \dfrac{1}{v}\right)$.

53. A closed rectangular box of dimensions a, b, c has the edges slightly altered in length by amounts Δa, Δb and Δc respectively, so that both its volume and surface area remain unaltered. Show that $\dfrac{\Delta a}{a^2(b-c)} = \dfrac{\Delta b}{b^2(c-a)} = \dfrac{\Delta c}{c^2(a-b)}$.

    [**Hint:** Solve the equations $dV = 0$ and $dS = 0$ for Δa, Δb, Δc using the method of cross-multiplication]

54. If a triangle ABC is slightly disturbed so as to remain inscribed in the same circle, prove that

$$\frac{\Delta a}{\cos A} + \frac{\Delta b}{\cos B} + \frac{\Delta c}{\cos C} = 0.$$

55. The area of a triangle ABC is calculated using the formula $\Delta = \dfrac{1}{2}\, bc \sin A$. Show that the relative error in Δ is given by

$$\frac{\delta\Delta}{\Delta} = \frac{\delta b}{b} + \frac{\delta c}{c} + \cot A\,\delta A.$$

    If an error of $5'$ is made in the measurement of A which is taken as $60°$, find the percentage error in Δ.

56. Prove that the error in the area Δ of a triangle ABC due to a small error in the measurement of c is given by

$$\delta\Delta = \frac{\Delta}{4}\left(\frac{1}{s} + \frac{1}{s-a} + \frac{1}{s-b} - \frac{1}{s-c}\right)\delta c.$$

57. The area of a triangle ABC is determined from the side a and the two angles B and C. If there are small errors in the values of B and C, show that the resulting error in the calculated value of the area Δ will be $\dfrac{1}{2}(c^2\Delta B + b^2 \Delta C)$.

    $\left[\textbf{Hint}: \Delta = \dfrac{1}{2}\dfrac{a^2 \sin B \sin C}{\sin(B+C)}\right]$

4.2.3 Taylor's Series Expansion of a Function of Two Variables

Students are familiar with Taylor's series of a function of one variable viz. $f(x+h) = f(x) + \dfrac{h}{1!}f'(x) + \dfrac{h^2}{2!}f''(x) + \cdots \infty$, which is an infinite series of powers of h. This idea can be extended to expand $f(x+h, y+k)$ in an infinite series of powers of h and k.

Statement

If $f(x, y)$ and all its partial derivatives are finite and continuous at all points (x, y), then

$$f(x+h, y+k) = f(x, y)+\frac{1}{1!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)f+\frac{1}{2!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^2 f$$

$$+\frac{1}{3!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^3 f+\cdots\infty$$

Proof:

If we assume y to be a constant, $f(x + h, y + k)$ can be treated as a function of x only.

Then $$f(x+h, y+k) = f(x, y+k) + \frac{h}{1!}\frac{\partial f(x, y+k)}{\partial x}+\frac{h^2}{2!}\frac{\partial^2 f(x, y+k)}{\partial x^2}+\cdots \tag{1}$$

Now treating x as a constant,

$$f(x, y+k) = f(x, y)+\frac{k}{1!}\frac{\partial f(x, y)}{\partial y}+\frac{k^2}{2!}\frac{\partial^2 f(x, y)}{\partial y^2}+\cdots \tag{2}$$

Using (2) in (1), we have

$$f(x+h, y+k) = \left[f(x, y)+\frac{k}{1!}\frac{\partial f(x, y)}{\partial y}+\frac{k^2}{2!}\frac{\partial^2 f(x, y)}{\partial y^2}+\cdots\right]$$

$$+\frac{h}{1!}\frac{\partial}{\partial x}\left\{f(x, y)+\frac{k}{1!}\frac{\partial f(x, y)}{\partial y}+\frac{k^2}{2!}\frac{\partial^2 f(x, y)}{\partial y^2}+\cdots\right\}$$

$$+\frac{h^2}{2!}\frac{\partial^2}{\partial x^2}\left\{f(x, y)+\frac{k}{1!}\frac{\partial f(x, y)}{\partial y}+\frac{k^2}{2!}\frac{\partial^2 f(x, y)}{\partial y^2}+\cdots\right\}+\cdots\infty$$

$$= f(x, y)+\frac{1}{1!}\left(h\frac{\partial f}{\partial x}+k\frac{\partial f}{\partial y}\right)+\frac{1}{2!}\left(h^2\frac{\partial^2 f}{\partial x^2}+2hk\frac{\partial^2 f}{\partial x\,\partial y}+k^2\frac{\partial^2 f}{\partial y^2}+\cdots\infty\right)$$

$$= f(x, y)+\frac{1}{1!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)f+\frac{1}{2!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^2 f+\cdots\infty \tag{3}$$

Interchanging x and h and also y and k in (3) and then putting $h = k = 0$, we have

$$f(x, y) = f(0,0)+\frac{1}{1!}\left\{x\frac{\partial f(0,0)}{\partial x}+y\frac{\partial f(0,0)}{\partial y}\right\}+\frac{1}{2!}\left\{x^2\frac{\partial^2 f(0,0)}{\partial x^2}\right.$$

$$\left.+2xy\frac{\partial^2 f(0,0)}{\partial x\,\partial y}+y^2\frac{\partial^2 f(0,0)}{\partial y^2}\right\}+\cdots \tag{4}$$

Series in (4) is the *Maclarin's series* of the function $f(x, y)$ in powers of x and y.
Another form of Taylor's series of $f(x, y)$

$$
\begin{aligned}
f(x,y) &= f(a+\overline{x-a},\ b+\overline{y-b}) \\
&= f(a+h),\ (b+k), \text{ say} \\
&= f(a,b)+\frac{1}{1!}\left\{h\frac{\partial f(a,b)}{\partial x}+k\frac{\partial f(a,b)}{\partial y}\right\} \\
&\quad +\frac{1}{2!}\left\{h^2\frac{\partial^2 f(a,b)}{\partial x^2}+2kh\frac{\partial^2 f(a,b)}{\partial x\,\partial y}+k^2\frac{\partial^2 f(a,b)}{\partial y^2}\right\}+\cdots, \text{ by (3)} \\
&= f(a,b)+\frac{1}{1!}\left[(x-a)\frac{\partial f(a,b)}{\partial x}+(y-b)\frac{\partial f(a,b)}{\partial y}\right] \\
&\quad +\frac{1}{2!}\left[(x-a)^2\frac{\partial^2 f(a,b)}{\partial x^2}+2(x-a)(y-b)\frac{\partial^2 f(a,b)}{\partial x\,\partial y}+(y-b)^2\frac{\partial^2 f(a,b)}{\partial y^2}\right]+\cdots \qquad (5)
\end{aligned}
$$

(5) is called the Taylor's series of $f(x, y)$ at or near the point (a, b).

Thus the Taylor's series of $f(x, y)$ at or near the point (0, 0) is Maclaurins series of $f(x, y)$.

4.3 JACOBIANS

If u and v are functions of two independent variables x and y, then the determinant

$$
\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}
$$

is called *the Jacobian* or *functional determinant* of u, v with respect to x and y and is written as

$$
\frac{\partial(u,v)}{\partial(x,y)} \text{ or } J\left(\frac{u,v}{x,y}\right).
$$

Similarly the Jacobian of u, v, w with respect to x, y, z is defined as

$$
\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}
$$

Note ☑

1. To define the Jacobian of n dependent variables, each of these must be a function of n independent variables.
2. The concept of Jacobians is used when we change the variables in multiple integrals. (See property 4 given below)

4.3.1 Properties of Jacobians

1. If u and v are functions of x and y, then $\dfrac{\partial(u,\, v)}{\partial(x,\, y)} \times \dfrac{\partial(x,\, y)}{\partial(u,\, v)} = 1.$

Proof:

Let $u = f(x, y)$ and $v = g(x, y)$. When we solve for x and y, let $x = \phi(u,\, v)$ and $y = \psi(u,\, v)$.

Then

$$\left.\begin{aligned}
\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial u} &= \frac{\partial u}{\partial u} = 1\\
\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial v} &= \frac{\partial u}{\partial v} = 0\\
\frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial u} &= \frac{\partial v}{\partial u} = 0\\
\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial v} &= \frac{\partial v}{\partial v} = 1
\end{aligned}\right\} \tag{1}$$

Now
$$\begin{aligned}
\frac{\partial(u,v)}{\partial(x,y)} \times \frac{\partial(x,y)}{\partial(u,v)} &= \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \times \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}\\
&= \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \times \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix}, \quad \text{by interchanging the rows and columns of the second determinant.}\\
&= \begin{vmatrix} \left(\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial u}\right) & \left(\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial v}\right) \\ \left(\frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial u}\right) & \left(\frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial v}\right) \end{vmatrix}\\
&= \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \quad \text{[by (1)]}\\
&= 1
\end{aligned}$$

2. If u and υ functions of r and s, where r and s are functions of x and y, then

$$\frac{\partial(u,\upsilon)}{\partial(x,y)} = \frac{\partial(u,\upsilon)}{\partial(r,s)} \times \frac{\partial(r,s)}{\partial(x,y)}$$

Proof:

$$\frac{\partial(u,\upsilon)}{\partial(r,s)} \times \frac{\partial(r,s)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial r} & \frac{\partial u}{\partial s} \\ \frac{\partial \upsilon}{\partial r} & \frac{\partial \upsilon}{\partial s} \end{vmatrix} \times \begin{vmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial s}{\partial x} & \frac{\partial s}{\partial y} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial u}{\partial r} & \frac{\partial u}{\partial s} \\ \frac{\partial \upsilon}{\partial r} & \frac{\partial \upsilon}{\partial s} \end{vmatrix} \times \begin{vmatrix} \frac{\partial r}{\partial x} & \frac{\partial s}{\partial x} \\ \frac{\partial r}{\partial y} & \frac{\partial s}{\partial y} \end{vmatrix}, \text{ by rewriting the second determinant.}$$

$$= \begin{vmatrix} \left(\frac{\partial u}{\partial r}.\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}.\frac{\partial s}{\partial x}\right) & \left(\frac{\partial u}{\partial r}.\frac{\partial r}{\partial y} + \frac{\partial u}{\partial s}.\frac{\partial s}{\partial y}\right) \\ \left(\frac{\partial \upsilon}{\partial r}.\frac{\partial r}{\partial x} + \frac{\partial \upsilon}{\partial s}.\frac{\partial s}{\partial x}\right) & \left(\frac{\partial \upsilon}{\partial r}.\frac{\partial r}{\partial y} + \frac{\partial \upsilon}{\partial s}.\frac{\partial s}{\partial y}\right) \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial \upsilon}{\partial x} & \frac{\partial \upsilon}{\partial y} \end{vmatrix} = \frac{\partial(u,\upsilon)}{\partial(x,y)}.$$

Note ☑ The two properties given above hold good for more than two variables too.

3. If u, υ, w are functionally dependent functions of three independent variables x, y, z then $\frac{\partial(u,\upsilon,w)}{\partial(x,y,z)} = 0.$

Note ☑ The functions u, υ, w are said to be functionally dependent, if each can be expressed in terms of the others or equivalently $f(u, \upsilon, w) = 0$. Linear dependence of functions is a particular case of functional dependence.

Proof:

Since u, υ, w are functionally dependent, $f(u, \upsilon, w) = 0$ (1)

Differentiating (1) partially with respect to x, y and z, we have

$$f_u \,.\, u_x + f_\upsilon \,.\, \upsilon_x + f_w \,.\, w_x = 0 \quad (2)$$

$$f_u \,.\, u_y + f_\upsilon \,.\, \upsilon_y + f_w \,.\, w_y = 0 \quad (3)$$

$$f_u \,.\, u_z + f_\upsilon \,.\, \upsilon_z + f_w \,.\, w_z = 0 \quad (4)$$

Equations (2), (3) and (4) are homogeneous equations in the unknowns f_u, f_v, f_w. At least one of f_u, f_v and f_w is not zero, since if all of them are zero, then $f(u, v, w) \equiv$ constant, which is meaningless.

Thus the homogeneous equations (2), (3) and (4) possess a non-trivial solution.
∴ Matrix of the coefficients of f_u, f_v, f_w is singular.

i.e., $$\begin{vmatrix} u_x & v_x & w_x \\ u_y & v_y & w_y \\ u_z & v_z & w_z \end{vmatrix} = 0$$

i.e., $$\frac{\partial(u, v, w)}{\partial(x, y, z)} = 0$$

Note ☑ The converse of this property is also true. viz., if u, v, w are functions of x, y, z such that $\dfrac{\partial(u, v, w)}{\partial(x, y, z)} = 0$ then u, v, w are functionally dependent. i.e., there exists a relationship among them.

4. If the transformations $x = x(u, v)$ and $y = y(u, v)$ are made in the double integral $\iint f(x, y)\,dx\,dy$, then $f(x, y) = F(u, v)$ and $dx\,dy = |J|\,du\,dv$, where $J = \dfrac{\partial(x, y)}{\partial(u, v)}$.

Proof:
$dx\,dy$ = Elemental area of a rectangle with vertices (x, y), $(x + dx, y)$, $(x + dx, y + dy)$ and $(x, y + dy)$

This elemental area can be regarded as equal to the area of the parallelogram with vertices (x,y), $\left(x + \dfrac{\partial x}{\partial u}du, y + \dfrac{\partial y}{\partial u}du\right)$, $\left(x + \dfrac{\partial x}{\partial u}du + \dfrac{\partial x}{\partial v}dv, y + \dfrac{\partial y}{\partial u}du + \dfrac{\partial y}{\partial v}dv\right)$ and $\left(x + \dfrac{\partial x}{\partial v}dv, y + \dfrac{\partial y}{\partial v}dv\right)$, since dx and dy are infinitesimals.

Now the area of this parallelogram is equal to 2× area of the triangle with vertices (x, y), $\left(x + \dfrac{\partial x}{\partial u}du, y + \dfrac{\partial y}{\partial u}du\right)$ and $\left(x + \dfrac{\partial x}{\partial v}dv, y + \dfrac{\partial y}{\partial v}dv\right)$

$$\therefore \qquad dx\,dy = 2 \times \frac{1}{2}\begin{vmatrix} x & y & 1 \\ x + \dfrac{\partial x}{\partial u}du & y + \dfrac{\partial y}{\partial u}du & 1 \\ x + \dfrac{\partial x}{\partial v}dv & y + \dfrac{\partial y}{\partial v}dv & 1 \end{vmatrix}$$

$$= \begin{vmatrix} x & y & 1 \\ \dfrac{\partial x}{\partial u}\mathrm{d}u & \dfrac{\partial y}{\partial u}\mathrm{d}u & 0 \\ \dfrac{\partial x}{\partial v}\mathrm{d}v & \dfrac{\partial y}{\partial v}\mathrm{d}v & 0 \end{vmatrix} = \begin{vmatrix} \dfrac{\partial x}{\partial u}\mathrm{d}u & \dfrac{\partial y}{\partial u}\mathrm{d}u \\ \dfrac{\partial x}{\partial v}\mathrm{d}v & \dfrac{\partial y}{\partial v}\mathrm{d}v \end{vmatrix}$$

$$= \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial y}{\partial u} \\ \dfrac{\partial x}{\partial v} & \dfrac{\partial y}{\partial v} \end{vmatrix} \mathrm{d}u\,\mathrm{d}v$$

i.e., $$\mathrm{d}x\,\mathrm{d}y = \frac{\partial(x, y)}{\partial(u, v)}\mathrm{d}u\,\mathrm{d}v.$$

Since both dx dy and du dv are positive, $\mathrm{d}x\,\mathrm{d}y = |J|\,\mathrm{d}u\,\mathrm{d}v$, where $J = \dfrac{\partial(x, y)}{\partial(u, v)}$

Similarly, if we make the transformations

$$x = x(u, v, w),\ y = y(u, v, w) \text{ and } z = z(u, v, w)$$

in the triple integral $\iiint f(x, y, z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$, then $\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = |J|\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}w$, where $J = \dfrac{\partial(x, y, z)}{\partial(u, v, w)}$.

4.4 DIFFERENTIATION UNDER THE INTEGRAL SIGN

When a function $f(x, y)$ of two variables is integrated with respect to y partially, viz., treating x as a parameter, between the limits a and b, then $\int_a^b f(x, y)\,\mathrm{d}y$ will be a function of x.

Let it be denoted by $F(x)$.

Now to find $F'(x)$, if it exists, we need not find $F(x)$ and then differentiate it with respect to x. $F'(x)$ can be found out without finding $F(x)$, by using *Leibnitz's* rules, given below:

1. Leibnitz's rule for constant limits of integration

If $f(x, y)$ and $\dfrac{\partial f(x, y)}{\partial x}$ are continuous functions of x and y, then

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\int_a^b f(x, y)\mathrm{d}y\right] = \int_a^b \frac{\partial f(x, y)}{\partial x}\mathrm{d}y, \text{ where}$$

a and b are constants independent of x.

Proof:

Let $$\int_a^b f(x,y)\,\mathrm{d}y = F(x).$$

Then $$F(x+\Delta x) - F(x) = \int_a^b f(x+\Delta x,\, y)\mathrm{d}y - \int_a^b f(x,y)\mathrm{d}y$$
$$= \int_a^b [f(x+\Delta x,\, y) - f(x,y)]\mathrm{d}y$$
$$= \Delta x \int_a^b \frac{\partial f(x+\theta\Delta\, x,\, y)}{\partial x}\mathrm{d}y,\ 0<\theta<1,$$

$$\left[\text{by Mean Value theorem, viz., } f(x+h) - f(x) = h\frac{\mathrm{d}f(x+\theta h)}{\mathrm{d}x},\ 0<\theta<1\right]$$

$$\therefore \qquad \frac{F(x+\Delta x) - F(x)}{\Delta x} = \int_a^b \frac{\partial f(x+\theta\cdot\Delta x, y)}{\partial x}\mathrm{d}y \tag{1}$$

Taking limits on both sides of (1) as $\Delta x \to 0$,

$$F'(x) = \int_a^b \frac{\partial f(x,y)}{\partial x}\,\mathrm{d}y$$

i.e., $$\frac{\mathrm{d}}{\mathrm{d}x}\left[\int_a^b f(x,y)\mathrm{d}y\right] = \int_a^b \frac{\partial f(x,y)}{\partial x}\,\mathrm{d}y$$

2. Leibnitz's rule for variable limits of integration

If $f(x, y)$ and $\dfrac{\partial f(x,y)}{\partial x}$ are continuous functions of x and y, then $\dfrac{\mathrm{d}}{\mathrm{d}x}\left[\displaystyle\int_{a(x)}^{b(x)} f(x,y)\mathrm{d}y\right]$

$= \displaystyle\int_{a(x)}^{b(x)} \frac{\partial f(x,y)}{\partial x}\mathrm{d}y + f\{x,b(x)\}\frac{\mathrm{d}b}{\mathrm{d}x} - f\{x,a(x)\}\frac{\mathrm{d}a}{\mathrm{d}x}$, provided $a(x)$ and $b(x)$ possess continuous first order derivatives.

Proof:

Let $\int f(x,y)\mathrm{d}y = F(x,y)$, so that $\dfrac{\partial}{\partial y}F(x,y) = f(x,y)$ (1)

$$\therefore \qquad \int_{a(x)}^{b(x)} f(x,y)\mathrm{d}y = F\{x,b(x)\} - F\{x,a(x)\}$$

$$\frac{d}{dx}\left[\int_{a(x)}^{b(x)} f(x,y)dy\right] = \frac{d}{dx}F\{x,b(x)\} - \frac{d}{dx}F\{x,a(x)\}$$

$$= \left[\frac{d}{dx}F(x,y)\right]_{y=b(x)} - \left[\frac{d}{dx}F(x,y)\right]_{y=a(x)}$$

$$= \left[\frac{\partial}{\partial x}F(x,y) + \frac{\partial}{\partial y}F(x,y)\cdot\frac{dy}{dx}\right]_{y=b(x)}$$

$$- \left[\frac{\partial}{\partial x}F(x,y) + \frac{\partial}{\partial y}F(x,y)\frac{dy}{dx}\right]_{y=a(x)}$$

by differentiation of implicit functions

$$= \left[\frac{\partial}{\partial x}F(x,y)\right]_{y=a(x)}^{y=b(x)} + \left[f(x,y)\frac{\mathrm{d}y}{\mathrm{d}x}\right]_{y=b(x)}$$

$$- \left[f(x,y)\frac{\mathrm{d}y}{\mathrm{d}x}\right]_{y=a(x)} \qquad \text{by (1)}$$

$$= \left[\frac{\partial}{\partial x}\int f(x,y)\,\mathrm{d}y\right]_{y=a(x)}^{y=b(x)} + f\{x,\, b(x)\}b'(x) - f\{x,\, a(x)\}a'(x)$$

$$= \left[\int \frac{\partial f(x,y)}{\partial x}\mathrm{d}y\right]_{y=a(x)}^{y=b(x)} + f\{x,\, b(x)\}b'(x) - f\{x,\, a(x)\}a'(x)$$

$$= \int_{a(x)}^{b(x)} \frac{\partial f(x,y)}{\partial x}\mathrm{d}y + f\{x,\, b(x)\}b'(x) - f\{x,\, a(x)\}a'(x)$$

WORKED EXAMPLE 4(b)

Example 4.1 Expand $e^x \cos y$ in powers of x and y as far as the terms of the third degree.

$$f(x,y) = e^x \cos y; \; \frac{\partial f(x,y)}{\partial x} = f_x(x,y) = e^x \cos y;$$

$$\frac{\partial f(x,y)}{\partial y} = f_y(x,y) = -e^x \sin y.$$

$$\frac{\partial^2 f(x,y)}{\partial x^2} = f_{xx}(x,y) = e^x \cos y; \ \frac{\partial^2 f(x,y)}{\partial x \, \partial y} = f_{yx}(x,y) = -e^x \sin y$$

$$\frac{\partial^2 f(x,y)}{\partial y^2} = f_{yy}(x,y) = -e^x \cos y.$$

Similarly $f_{xxx}(x,y) = e^x \cos y; \ f_{xxy}(x,y) = -\ e^x \sin y;$

$f_{xyy}(x,y) = -\ e^x \cos y; \ f_{yyy}(x,y) = e^x \sin y$

$\therefore$ $f(0,0) = 1; \ f_x(0,0) = 1; \ f_y(0,0) = 0;$

$f_{xx}(0,0) = 1; \ f_{xy}(0,0) = 0; \ f_{yy}(0,0) = -1;$

$f_{xxx}(0,0) = 1; \ f_{xxy}(0,0) = 0; \ f_{xyy}(0,0) = -1; \ f_{yyy}(0,0) = 0$

Taylor's series of $f(x, y)$ in powers of x and y is

$$\begin{aligned} f(x,y) = f(0,0) &+ \frac{1}{1!}\{xf_x(0,0) + yf_y(0,0)\} + \\ &\frac{1}{2!}\{x^2 f_{xx}(0,0) + 2xyf_{xy}(0,0) + y^2 f_{yy}(0,0)\} + \\ &\frac{1}{3!}\{x^3 f_{xxx}(0,0) + 3x^2 yf_{xxy}(0,0) + 3xy^2 f_{xyy}(0,0) + y^3 f_{yyy}(0,0)\} + \cdots \end{aligned}$$

$$\begin{aligned} \therefore\ e^x \cos y &= 1 + \frac{1}{1!}\{x \cdot 1 + y \cdot 0\} + \frac{1}{2!}\{x^2 \cdot 1 + 2xy \cdot 0 + y^2(-1)\} \\ &+ \frac{1}{3!}\{x^3 \cdot 1 + 3x^2 y \cdot 0 + 3xy^2 \cdot (-1) + y^3 \cdot 0\} + \cdots \\ &= 1 + \frac{x}{1!} + \frac{1}{2!} + (x^2 - y^2) + \frac{1}{3!}(x^3 - 3xy^2) + \cdots \end{aligned}$$

4.4.1 Verification

$e^x \cos y$ = Real part of e^{x+iy}

$$= \text{R.P. of}\left[1 + \frac{x+iy}{1!} + \frac{(x+iy)^2}{2!} + \frac{(x+iy)^3}{3!} + \cdots\right],$$

by exponential theorem

$$= 1 + \frac{x}{1!} + \frac{1}{2!}(x^2 - y^2) + \frac{1}{3!}(x^3 - 3y^2) + \cdots$$

Example 4.2 Expand $\dfrac{(x+h)(y+k)}{x+h+y+k}$ in a series of powers of h and k upto the second degree terms.

Let $$f(x+h, y+k) = \frac{(x+h)(y+k)}{x+h+y+k}$$

$\therefore$ $$f(x, y) = \frac{xy}{x+y}.$$

Taylor's series of $f(x+h, y+k)$ in powers of h and k is

$$f(x+h, y+k) = f(x, y) + \frac{1}{1!}\left(h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y}\right) + \frac{1}{2!}\left(h^2\frac{\partial^2 f}{\partial x^2} + 2hk\frac{\partial^2 f}{\partial x\,\partial y} + k^2\frac{\partial^2 f}{\partial y^2}\right) + \cdots \tag{1}$$

Now $$f_x = y\left\{\frac{(x+y)-x}{(x+y)^2}\right\} = \frac{y^2}{(x+y)^2}$$

$$f_y = x\left\{\frac{(x+y)-y}{(x+y)^2}\right\} = \frac{x^2}{(x+y)^2}$$

$$f_{xx} = -\frac{2y^2}{(x+y)^3};\; f_{xy} = \frac{(x+y)^2\cdot 2y - y^2\cdot 2(x+y)}{(x+y)^4}$$

$$= \frac{2\{y(x+y)-y^2\}}{(x+y)^3} = \frac{2xy}{(x+y)^3}$$

$$f_{yy} = -\frac{2x^2}{(x+y)^3}.$$

Using these values in (1), we have

$$\frac{(x+h)(y+k)}{x+h+y+k} = \frac{xy}{x+y} + \frac{hy^2}{(x+y)^2} + \frac{kx^2}{(x+y)^2} - \frac{h^2y^2}{(x+y)^3} + \frac{2hkxy}{(x+y)^3} - \frac{k^2x^2}{(x+y)^3} + \cdots$$

Example 4.3 Find the Taylor's series expansion of x^y near the point (1, 1) upto the second degree terms.

Taylor's series of $f(x, y)$ near the point (1, 1) is $f(x,y) = f(1,1)+\dfrac{1}{1!}$

$$\left\{(x-1)f_x(1,1)+(y-1)f_y(1,1)\right\}+\frac{1}{2!}\left\{(x-1)^2 f_{xx}(1,1)+2(x-1)(y-1)f_{xy}(1,1)\right.$$
$$\left.+(y-1)^2 f_{yy}(1,1)\right\}+\cdots \tag{1}$$

$$f(x,y) = x^y;\; f_x(x,y) = yx^{y-1};\; f_y(x,y) = x^y \log x;$$
$$f_{xx}(x,y) = y(y-1)x^{y-2};\; f_{xy}(x,y) = x^{y-1}+yx^{y-1}\log x$$
$$f_{yy}(x,y) = x^y\cdot(\log x)^2\cdot$$
$$f_{xx}(1,1) = 1;\; f_x(1,1) = 1;\; f_y(1,1) = 0;$$
$$f_{xx}(1,1) = 0;\; f_{xy}(1,1) = 1;\; f_{yy}(1,1) = 0$$

Using these values in (1), we get

$$x^y = 1+(x-1)+(x-1)(y-1)+\cdots$$

Example 4.4 Find the Taylor's series expansion of $e^x \sin y$ near the point $\left(-1,\dfrac{\pi}{4}\right)$ upto the third degree terms.

Taylor's series of $f(x, y)$ near the point $\left(-1,\dfrac{\pi}{4}\right)$ is

$$f(x,y) = f\left(-1,\frac{\pi}{4}\right)+\frac{1}{1!}\left\{(x+1)f_x\left(-1,\frac{\pi}{4}\right)+\left(y-\frac{\pi}{4}\right)f_y\left(-1,\frac{\pi}{4}\right)\right\}$$
$$+\frac{1}{2!}\left\{(x+1)^2 f_{xx}\left(-1,\frac{\pi}{4}\right)+2(x+1)\left(y-\frac{\pi}{4}\right)f_{xy}\left(-1,\frac{\pi}{4}\right)\right.$$
$$\left.+\left(y-\frac{\pi}{4}\right)^2 f_{yy}\left(-1,\frac{\pi}{4}\right)\right\}+\cdots \tag{1}$$

$$f(x,y) = e^x \sin y;\; f_x = e^x \sin y;\; f_y = e^x \cos y;$$
$$f_{xx} = e^x \sin y; f_{xy} = e^x \cos y;\; f_{yy} = -e^x \sin y;$$
$$f_{xxx} = e^x \sin y;\; f_{xxy} = e^x \cos y;\; f_{xyy} = -e^x \sin y;$$
$$f_{yyy} = -e^x \cos y.$$

$$\therefore \quad f\left(-1,\frac{\pi}{4}\right) = \frac{1}{e\sqrt{2}};\; f_x\left(-1,\frac{\pi}{4}\right) = \frac{1}{e\sqrt{2}};\; f_y\left(-1,\frac{\pi}{4}\right) = \frac{1}{e\sqrt{2}};$$
$$f_{xx}\left(-1,\frac{\pi}{4}\right) = \frac{1}{e\sqrt{2}};\; f_{xy}\left(-1,\frac{\pi}{4}\right) = \frac{1}{e\sqrt{2}};\; f_{yy}\left(-1,\frac{\pi}{4}\right) = -\frac{1}{e\sqrt{2}};$$

$$f_{xxx}\left(-1,\frac{\pi}{4}\right)=\frac{1}{e\sqrt{2}};\ f_{xxy}\left(-1,\frac{\pi}{4}\right)=\frac{1}{e\sqrt{2}};\ f_{xyy}\left(-1,\frac{\pi}{4}\right)$$
$$=\frac{1}{e\sqrt{2}};\ f_{yyy}\left(-1,\frac{\pi}{4}\right)=-\frac{1}{e\sqrt{2}}.$$

Using these values in (1), we get

$$\begin{aligned}
e^x\sin y=\frac{1}{e\sqrt{2}}\Bigg[&1+\frac{1}{1!}\left\{(x+1)+\left(y-\frac{\pi}{4}\right)\right\}\\
&+\frac{1}{2!}\left\{(x+1)^2+2(x+1)\left(y-\frac{\pi}{4}\right)-\left(y-\frac{\pi}{4}\right)^2\right\}\\
&+\frac{1}{3!}\left\{(x+1)^3+3(x+1)^2\left(y-\frac{\pi}{4}\right)-3(x+1)\left(y-\frac{\pi}{4}\right)^2-\left(y-\frac{\pi}{4}\right)^3\right\}+\cdots
\end{aligned}$$

Example 4.5 Find the Taylor's series expansion of $x^2y^2 + 2x^2y + 3xy^2$ in powers of $(x + 2)$ and $(y - 1)$ upto the third powers.

Taylor's series of $f(x, y)$ in powers of $(x + 2)$ and $(y - 1)$ or near $(-2, 1)$ is

$$\begin{aligned}
f(x,y)=f(-2,1)&+\frac{1}{1!}\{(x+2)f_x(-2,1)+(y-1)f_y(-2,1)\}\\
&+\frac{1}{2!}\{(x+2)^2f_{xx}(-2,1)+2(x+2)(y-1)f_{xy}(-2,1)\\
&+(y-1)^2f_{yy}(-2,1)\}+\cdots \qquad (1)
\end{aligned}$$

$$\begin{aligned}
f(x,y)&=x^2y^2+2x^2y+3xy^2 & f(-2,1)&=6\\
f_x&=2xy^2+4xy+3y^2 & f_x(-2,1)&=-9\\
f_y&=2x^2y+2x^2+6xy & f_y(-2,1)&=4\\
f_{xx}&=2y^2+4y & f_{xx}(-2,1)&=6\\
f_{xy}&=4xy+4x+6y & f_{xy}(-2,1)&=-10\\
f_{yy}&=2x^2+6x & f_{yy}(-2,1)&=-4\\
f_{xxx}&=0 & f_{xxx}(-2,1)&=0\\
f_{xxy}&=4y+4 & f_{xxy}(-2,1)&=8\\
f_{xyy}&=4x+6 & f_{xyy}(-2,1)&=-2\\
f_{yyy}&=0 & f_{yyy}(-2,1)&=0
\end{aligned}$$

Using these values in (1), we have

$$
\begin{aligned}
x^2y^2+2x^2y+3xy^2 &= 6+\frac{1}{1!}\{-9(x+2)+4(y-1)\} \\
&\quad +\frac{1}{2!}\left\{6(x+2)^2-20(x+2)(y-1)-4(y-1)^2\right\} \\
&\quad +\frac{1}{3!}\left\{24(x+2)^2(y-1)-6(x+2)(y-1)^2\right\}+\cdots
\end{aligned}
$$

Example 4.6 Using Taylor's series, verify that

$$\log(1+x+y)=(x+y)-\frac{1}{2}(x+y)^2+\frac{1}{3}(x+y)^3-\cdots$$

The series given in the R.H.S. is a series of powers of x and y.

So let us expand $f(x, y) = \log(1 + x + y)$ as a Taylor's series near (0, 0) or Maclaurin's series.

$$f_x=\frac{1}{1+x+y};\ f_y=\frac{1}{1+x+y}$$

$$f_{xx}=-\frac{1}{(1+x+y)^2}=f_{xy}=f_{yy}$$

$$f_{xxx}=\frac{2}{(1+x+y)^3}=f_{xxy}=f_{xyy}=f_{yyy}$$

$$f(0,0)=0;\ f_x(0,0)=f_y(0,0)=1;$$

$$f_{xx}(0,0)=f_{xy}(0,0)=f_{yy}(0,0)=-1;$$

$$f_{xxx}(0,0)=f_{xxy}(0,0)=f_{xyy}(0,0)=f_{yyy}(0,0)=2\,.$$

Maclaurin's series of $f(x, y)$ is given by

$$
\begin{aligned}
f(x,y) &= f(0,0)+\frac{1}{1!}\{xf_x(0,0)+yf_y(0,0)\} \\
&\quad +\frac{1}{2!}\{x^2f_{xx}(0,0)+2xyf_{xy}(0,0)+y^2f_{yy}(0,0)\}+\cdots \qquad (1)
\end{aligned}
$$

Using the relevant values in (1), we have

$$
\begin{aligned}
\log(1+x+y) &= (x+y)+\frac{1}{2}\left\{-x^2-2xy-y^2\right\}+ \\
&\quad +\frac{1}{6}\left\{2x^3+6x^2y+6xy^2+2y^3\right\}+\cdots
\end{aligned}
$$

$$= (x+y) - \frac{1}{2}(x+y)^2 + \frac{1}{3}(x+y)^3 - \cdots$$

Example 4.7 If $x = r\cos\theta$, $y = r\sin\theta$, verify that $\dfrac{\partial(x,y)}{\partial(r,\theta)} \times \dfrac{\partial(r,\theta)}{\partial(x,y)} = 1$.

$$x = r\cos\theta;\ y = r\sin\theta$$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r\left(\cos^2\theta + \sin^2\theta\right) = r.$$

Now $$r^2 = x^2 + y^2 \text{ and } \theta = \tan^{-1}\frac{y}{x}$$

$\therefore$ $$2r\frac{\partial r}{\partial x} = 2x$$

$\therefore$ $$\frac{\partial r}{\partial x} = \frac{x}{r}$$

Similarly, $$\frac{\partial r}{\partial y} = \frac{y}{r}$$

$$\frac{\partial \theta}{\partial x} = \frac{1}{1+\frac{y^2}{x^2}} \times \frac{-y}{x^2}$$

$$= -\frac{y}{x^2+y^2} = \frac{-y}{r^2}$$

Similarly $\dfrac{\partial \theta}{\partial y} = \dfrac{x}{r^2}$

$$\frac{\partial(r,\theta)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{x}{r} & \frac{y}{r} \\ -\frac{y}{r^2} & \frac{x}{r^2} \end{vmatrix} = \frac{x^2+y^2}{r^3} = \frac{r^2}{r^3} = \frac{1}{r}$$

$\therefore$ $$\frac{\partial(x,y)}{\partial(r,\theta)} \times \frac{\partial(r,\theta)}{\partial(x,y)} = r \times \frac{1}{r} = 1.$$

Example 4.8 If we transform from three dimensional cartesian co-ordinates (x, y, z) to spherical polar co-ordinates (r, θ, ϕ), show that the Jacobian of x, y, z with respect to r, θ, ϕ is $r^2 \sin\theta$.

The transformation equations are

$$x = r\sin\theta\cos\phi,\ y = r\sin\theta\sin\phi,\ z = r\cos\theta.$$

$$\frac{\partial x}{\partial r} = \sin\theta\cos\phi,\ \frac{\partial y}{\partial r} = \sin\theta\sin\phi,\ \frac{\partial z}{\partial r} = \cos\theta$$
$$\frac{\partial x}{\partial \theta} = r\cos\theta\cos\phi,\ \frac{\partial y}{\partial \theta} = r\cos\theta\sin\phi,\ \frac{\partial z}{\partial \theta} = -r\sin\theta$$
$$\frac{\partial x}{\partial \phi} = -r\sin\theta\sin\phi,\ \frac{\partial y}{\partial \phi} = r\sin\theta\cos\phi,\ \frac{\partial z}{\partial \phi} = 0.$$

Now $$\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} & \frac{\partial z}{\partial r} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} & \frac{\partial z}{\partial \theta} \\ \frac{\partial x}{\partial \phi} & \frac{\partial y}{\partial \phi} & \frac{\partial z}{\partial \phi} \end{vmatrix}$$

$$= \begin{vmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ r\cos\theta\cos\phi & r\cos\theta\sin\phi & -r\sin\theta \\ -r\sin\theta\sin\phi & r\sin\theta\cos\phi & 0 \end{vmatrix}$$

$$= r^2[\sin\theta\cos\phi\left(0+\sin^2\theta\cos\phi\right)-\sin\theta\sin\phi$$
$$\times(0-\sin^2\theta\sin\phi)+\cos\theta(\sin\theta\cos\theta\cos^2\phi+\sin\theta\cos\theta\sin^2\phi)]$$

$$= r^2[\sin^3\theta\cos^2\phi+\sin^3\theta\sin^2\phi+\sin\theta\cos^2\theta]$$

$$= r^2(\sin^3\theta+\sin\theta\cos^2\theta)$$

$$= r^2\sin\theta.$$

Example 4.9 If $u = 2xy$, $v = x^2 - y^2$, $x = r\cos\theta$ and $y = r\sin\theta$, compute $\frac{\partial(u,v)}{\partial(r,\theta)}$.

By the property of Jacobians,

$$\frac{\partial(u,v)}{\partial(r,\theta)} = \frac{\partial(u,v)}{\partial(x,y)} \times \frac{\partial(x,y)}{\partial(r,\theta)}$$

$$= \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} \times \begin{vmatrix} x_r & x_\theta \\ y_r & y_\theta \end{vmatrix}$$

$$= \begin{vmatrix} 2y & 2x \\ 2x & -2y \end{vmatrix} \times \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= -4(y^2+x^2)\times r(\cos^2\theta+\sin^2\theta)$$

$$= -4r^3.$$

Example 4.10 Find the Jacobian of y_1, y_2, y_3 with respect to x_1, x_2, x_3, if

$$y_1 = \frac{x_2x_3}{x_1},\ y_2 = \frac{x_3x_1}{x_2},\ y_3 = \frac{x_1x_2}{x_3}$$

$$\frac{\partial(y_1,y_2,y_3)}{\partial(x_1,x_2,x_3)} = \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \frac{\partial y_1}{\partial x_3} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \frac{\partial y_2}{\partial x_3} \\ \frac{\partial y_3}{\partial x_1} & \frac{\partial y_3}{\partial x_2} & \frac{\partial y_3}{\partial x_3} \end{vmatrix}$$

$$= \begin{vmatrix} -\frac{x_2x_3}{x_1^2} & \frac{x_3}{x_1} & \frac{x_2}{x_1} \\ \frac{x_3}{x_2} & -\frac{x_3x_1}{x_2^2} & \frac{x_1}{x_2} \\ \frac{x_2}{x_3} & \frac{x_1}{x_3} & -\frac{x_1x_2}{x_3^2} \end{vmatrix}$$

$$= \frac{1}{x_1^2x_2^2x_3^2}\begin{vmatrix} -x_2x_3 & x_3x_1 & x_1x_2 \\ x_2x_3 & -x_3x_1 & x_1x_2 \\ x_2x_3 & x_3x_1 & -x_1x_2 \end{vmatrix}$$

$$= \frac{x_1^2x_2^2x_3^2}{x_1^2x_2^2x_3^2}\begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$$

$$= \begin{vmatrix} -1 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{vmatrix} = 4$$

Example 4.11 Express $\iiint \sqrt{xyz(1-x-y-z)}\ \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$ in terms of u, v, w given that $x+y+z=u$, $y+z=uv$ and $z=uvw$.

The given transformations are

$$x + y + z = u \tag{1}$$

$$y + z = u\upsilon \tag{2}$$

and

$$z = u\upsilon w \tag{3}$$

Using (3) in (2), we have $y = u\upsilon(1 - w)$
Using (2) in (1), we have $x = u(1 - \upsilon)$

$$\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = |J|\,\mathrm{d}u\,\mathrm{d}\upsilon\,\mathrm{d}w, \text{ where}$$

$$J = \frac{\partial(x,y,z)}{\partial(u,\upsilon,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial \upsilon} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial \upsilon} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial \upsilon} & \frac{\partial z}{\partial w} \end{vmatrix}$$

$$= \begin{vmatrix} 1-\upsilon & -u & 0 \\ \upsilon(1-w) & u(1-w) & -u\upsilon \\ \upsilon w & wu & u\upsilon \end{vmatrix}$$

$$= (1-\upsilon)\{u^2\upsilon\,(1-w)+u^2\upsilon w\}+u\{u\upsilon^2\ (1-w)+u\upsilon^2 w\}$$

$$= u^2\upsilon\,(1-\upsilon)+u^2\upsilon^2$$

$$= u^2\upsilon$$

$$\therefore \quad \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = u^2\upsilon\ \mathrm{d}u\,\mathrm{d}\upsilon\,\mathrm{d}w \tag{4}$$

Using (1), (2), (3) and (4) in the given triple integral I, we have

$$I = \iiint \sqrt{u^3\upsilon^2\,w(1-\upsilon)(1-w)(1-u)}\ u^2\upsilon\,\mathrm{d}u\,\mathrm{d}\upsilon\,\mathrm{d}w$$

$$= \iiint u^{7/2}\,\upsilon^2\,w^{1/2}\,(1-u)^{\frac{1}{2}}\,(1-\upsilon)^{\frac{1}{2}}\,(1-w)^{\frac{1}{2}}\,\mathrm{d}u\,\mathrm{d}\upsilon\,\mathrm{d}w$$

Example 4.12 Examine if the following functions are functionally dependent. If they are, find also the functional relationship.

(i) $u = \sin^{-1}x+\sin^{-1}y;\ \upsilon = x\sqrt{1-y^2}+y\sqrt{1-x^2}$

(ii) $u = y+z;\ \upsilon = x+2z^2;\ w = x-4yz-2y^2$

(i) $u = \sin^{-1}x+\sin^{-1}y;\ \upsilon = x\sqrt{1-y^2}+y\sqrt{1-x^2}$

$$\frac{\partial u}{\partial x} = \left(\frac{1}{\sqrt{1-x^2}}\right); \ \frac{\partial u}{\partial y} = \left(\frac{1}{\sqrt{1-y^2}}\right); \ \frac{\partial v}{\partial x} = \sqrt{1-y^2} - \frac{xy}{\sqrt{1-x^2}};$$

$$\frac{\partial v}{\partial y} = -\frac{xy}{\sqrt{1-y^2}} + \sqrt{1-x^2}$$

Now $$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \dfrac{1}{\sqrt{1-x^2}} & \dfrac{1}{\sqrt{1-y^2}} \\ \sqrt{1-y^2} - \dfrac{xy}{\sqrt{1-x^2}} & -\dfrac{xy}{\sqrt{1-y^2}} + \sqrt{1-x^2} \end{vmatrix}$$

$$= \frac{-xy}{\sqrt{(1-x^2)(1-y^2)}} + 1 - 1 + \frac{xy}{\sqrt{(1-x^2)(1-y^2)}}$$

$$= 0.$$

$\therefore$ u and v are functionally dependent by property (3).

Now $$\sin u = \sin\,(\sin^{-1} x + \sin^{-1} y)$$

$$= \sin\,(\sin^{-1} x)\cos\,(\sin^{-1} y) + \cos\,(\sin^{-1} x)\sin\,(\sin^{-1} y)$$

$$= x\cdot\cos\left\{\cos^{-1}(\sqrt{1-y^2})\right\} + \cos\left\{\cos^{-1}(\sqrt{1-x^2})\right\}\cdot y$$

$$= x\sqrt{1-y^2} + y\sqrt{1-x^2}$$

$$= v.$$

$\therefore$ The functional relationship between u and v is $v = \sin u$.

(ii) $u = y + z;\ v = x + 2z^2;\ w = x - 4yz - 2y^2$

$$\frac{\partial u}{\partial x} = 0;\ \frac{\partial v}{\partial x} = 1;\ \frac{\partial w}{\partial x} = 1$$

$$\frac{\partial u}{\partial y} = 1;\ \frac{\partial v}{\partial y} = 0;\ \frac{\partial w}{\partial y} = -4y - 4z$$

$$\frac{\partial u}{\partial z} = 1;\ \frac{\partial v}{\partial z} = 4z;\ \frac{\partial w}{\partial z} = -4y$$

Now $$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 4z \\ 1 & -4y-4z & -4y \end{vmatrix}$$

$$= -\{-4y + 4y + 4z\} + 4z = 0.$$

$\therefore$ u, v and w are functionally dependent

Now $$v - w = 2z^2 + 4yz + 2y^2$$
$$= 2(y+z)^2 = 2u^2$$

$\therefore$ The functional relationship among u, v and w is $2u^2 = v - w$.

Example 4.13 Given that $\int_0^{\pi} \frac{dx}{a + b\cos x} = \frac{\pi}{\sqrt{a^2 - b^2}} (a > b)$, find

$\int_0^{\pi} \frac{dx}{(a + b\cos x)^2}$ and $\int_0^{\pi} \frac{\cos x \, dx}{(a + b\cos x)^2}$

$$\int_0^{\pi} \frac{dx}{a + b\cos x} = \frac{\pi}{\sqrt{a^2 - b^2}} \tag{1}$$

Differentiating both sides of (1) with respect to a, we get

$\int_0^{\pi} \frac{\partial}{\partial a}\left(\frac{1}{a + b\cos x}\right) dx = \frac{\partial}{\partial a} \cdot \frac{\pi}{\sqrt{a^2 - b^2}}$, since the limits of integration are constants

i.e., $$\int_0^{\pi} \frac{-dx}{(a + b\cos x)^2} = \pi \times -1/2\left(a^2 - b^2\right)^{-3/2} 2a$$

i.e., $$\int_0^{\pi} \frac{dx}{(a + b\cos x)^2} = \frac{\pi a}{\left(a^2 - b^2\right)^{3/2}}$$

Differentiating both sides of (1) with respect to b, we get

$$\int_0^{\pi} \frac{\partial}{\partial b}\left(\frac{1}{a + b\cos x}\right) dx = \frac{\partial}{\partial b} \cdot \frac{\pi}{\sqrt{a^2 - b^2}}$$

i.e., $$\int_0^{\pi} -\frac{1}{(a + b\cos x)^2} \times \cos x \, dx = \pi \times -1/2\left(a^2 - b^2\right)^{-3/2} (-2b)$$

i.e., $$\int_0^{\pi} \frac{\cos x}{(a + b\cos x)^2} dx = -\frac{\pi b}{\left(a^2 - b^2\right)^{3/2}}.$$

Example 4.14 By differentiating inside the integral, find the value of $\int_0^x \frac{\log(1+xy)}{1+y^2}\,dy$. Hence find the value of $\int_0^1 \frac{\log(1+x)}{1+x^2}\,dx$.

Let $$f(x) = \int_0^x \frac{\log(1+xy)}{1+y^2}\,dy \tag{1}$$

Differentiating both sides of (1) with respect to x, we have

$$f'(x) = \frac{d}{dx}\int_0^x \frac{\log(1+xy)}{1+y^2}\,dx$$

$$= \int_0^x \frac{\partial}{\partial x}\left\{\frac{\log(1+xy)}{1+y^2}\right\}dx + \frac{\log(1+x^2)}{1+x^2}\cdot\frac{d(x)}{dx}$$

(by Leibnitz's rule)

$$= \int_0^x \frac{y}{(1+xy)(1+y^2)}\,dy + \frac{\log(1+x^2)}{1+x^2}$$

$$= \int_0^x \left[\frac{-x}{(1+x^2)(1+xy)} + \frac{1}{1+x^2}\left(\frac{y+x}{1+y^2}\right)\right]dy + \frac{\log(1+x^2)}{1+x^2},$$

by resolving the integrand in the first term into partial fractions

$$= \left[-\frac{1}{1+x^2}\log(1+xy) + \frac{1}{2}\cdot\frac{1}{1+x^2}\log(1+y^2) + \frac{x}{1+x^2}\tan^{-1}y\right]_0^x + \frac{\log(1+x^2)}{1+x^2}$$

$$= \frac{1}{2}\cdot\frac{1}{1+x^2}\log(1+x^2) + \frac{x}{1+x^2}\tan^{-1}x \tag{2}$$

Integrating both sides of (2) with respect to x, we have

$$f(x) = \frac{1}{2}\int \log(1+x^2)\,d(\tan^{-1}x) + \int \frac{x}{1+x^2}\tan^{-1}x\,dx + c$$

$$= \frac{1}{2}\left[\tan^{-1}x\log(1+x^2) - \int \tan^{-1}x\cdot\frac{2x}{1+x^2}\,dx\right]$$

$$+ \int \frac{x\tan^{-1}x}{1+x^2}\,dx + c$$

$$= \frac{1}{2}\tan^{-1} x \cdot \log\left(1+x^2\right)+c \tag{2}$$

Now putting $x = 0$ in (2), we get

$$c = f(0) = 0, \text{ by (1)}$$

$\therefore$

$$f(x) = \int_0^x \frac{\log(1+xy)}{1+y^2}\,\mathrm{d}y = \frac{1}{2}\tan^{-1} x \cdot \log\left(1+x^2\right) \tag{3}$$

Putting $x = 1$ in (3), we get

$$\int_0^1 \frac{\log(1+y)}{1+y^2}\,\mathrm{d}y = \frac{1}{2}\tan^{-1}(1) \cdot \log 2$$

$$= \frac{\pi}{8}\log 2$$

Since y is only a dummy variable,

$$\int_0^1 \frac{\log(1+x)}{1+x^2}\,\mathrm{d}x = \frac{\pi}{8}\log 2$$

Example 4.15 Show that $\dfrac{\mathrm{d}}{\mathrm{d}a}\displaystyle\int_0^{a^2} \tan^{-1}\left(\frac{x}{a}\right)\mathrm{d}x = 2a\tan^{-1}(a) - \frac{1}{2}\log\left(a^2+1\right).$

$$\frac{\mathrm{d}}{\mathrm{d}a}\int_0^{a^2} \tan^{-1}\left(\frac{x}{a}\right)\mathrm{d}x = \int_0^{a^2} \frac{\partial}{\partial a}\tan^{-1}\left(\frac{x}{a}\right)\mathrm{d}x + \tan^{-1}\left(\frac{a^2}{a}\right)\cdot\frac{\mathrm{d}}{\mathrm{d}a}\left(a^2\right),$$

by Leibnitz's rule

$$= \int_0^{a^2} \frac{1}{1+\dfrac{x^2}{a^2}}\cdot\left(\frac{-x}{a^2}\right)\mathrm{d}x + 2a\tan^{-1} a$$

$$= -\int_0^{a^2} \frac{x}{x^2+a^2}\,\mathrm{d}x + 2a\tan^{-1} a$$

$$= -\frac{1}{2}\Big[\log\left(x^2+a^2\right)\Big]_0^{a^2} + 2a\tan^{-1} a$$

$$= -\frac{1}{2}\log\left(\frac{a^4+a^2}{a^2}\right)+2a\,\tan^{-1} a$$

$$= 2a\tan^{-1} a-\frac{1}{2}\log\left(a^2+1\right)$$

Example 4.16 If $I = \int_0^\infty e^{-x^2-\left(\frac{a}{x}\right)^2}\,\mathrm{d}x$, prove that $\frac{\mathrm{d}I}{\mathrm{d}a} = -2I$. Hence find the value of I.

$$I = \int_0^\infty e^{-x^2-\left(\frac{a}{x}\right)^2}\,\mathrm{d}x \tag{1}$$

Differentiating both sides of (1) with respect to a, we have

$$\frac{\mathrm{d}I}{\mathrm{d}a} = \int_0^\infty \frac{\partial}{\partial a}\left\{e^{-x^2-\frac{a^2}{x^2}}\right\}\mathrm{d}x$$

$$= \int_0^\infty e^{-x^2-\frac{a^2}{x^2}}\cdot\left(-\frac{2a}{x^2}\right)\mathrm{d}x$$

$$= \int_\infty^0 e^{-\frac{a^2}{y^2}-y^2}\,2\,\mathrm{d}y,\text{ on putting } x = \frac{a}{y} \text{ or } y = \frac{a}{x}$$

$$=-2\int_0^\infty e^{-y^2-\left(\frac{a}{y}\right)^2}\,\mathrm{d}y$$

i.e., $$\frac{\mathrm{d}I}{\mathrm{d}a} =-2I \tag{2}$$

$\therefore$ $$\frac{\mathrm{d}I}{I} =-2\,\mathrm{d}a$$

Solving, we get $\log I = \log c - 2a$

$\therefore$ $$I = ce^{-2a} \tag{3}$$

When $$a = 0,\ I = \int_0^\infty e^{-x^2}\,\mathrm{d}x = \frac{\sqrt{\pi}}{2} \tag{4}$$

Using (4) in (3), we get $c = \frac{\sqrt{\pi}}{2}$

Hence
$$I = \frac{\sqrt{\pi}}{2} e^{-2a}.$$

EXERCISE 4(b)

Part A

(Short Answer Question)

1. Write down the Taylor's series expansion of $f(x + h, y + k)$ in a series of (i) powers of h and k (ii) power of x and y.
2. Write down the Maclaurin's series expansion of (i) $f(x, y)$, (ii) $f(x + h, y + k)$.
3. Write down the Taylor's series expansion of $f(x, y)$ near the point (a, b).
4. Write down the Maclaurin's series for e^{x+y}.
5. Write down the Maclaurin's series for $\sin(x + y)$.
6. Define Jacobian.
7. State any three properties of Jacobians.
8. State the condition for the functional dependence of three functions $u(x, y, z)$, $v(x, y, z)$ and $w(x, y, z)$.
9. Prove that $\iint f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \iint f(r\cos\theta, r\sin\theta)\cdot r\,\mathrm{d}r\,\mathrm{d}\theta.$
10. Show that $\iint f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \iint f\{u(1-v),\, uv\}\cdot u\,\mathrm{d}u\,\mathrm{d}v.$
11. If $x = u(1 + v)$ and $y = v(1 + u)$, find the Jacobian of x, y with respect to u, v.
12. State the Leibnitz's rule for differentiation under integral sign, when both the limits of integration are variables.
13. Write down the Leibnitz's formula for $\frac{\mathrm{d}}{\mathrm{d}x}\int_a^{b(x)} f(x,y)\,\mathrm{d}y$, where a is a constant.
14. Write down the Leibnitz's formula for $\frac{\mathrm{d}}{\mathrm{d}x}\int_{a(x)}^{b} f(x,y)\,\mathrm{d}y$, where b is a constant.
15. Evaluate $\frac{\mathrm{d}}{\mathrm{d}y}\int_0^1 \log(x^2+y^2)\,\mathrm{d}x$, without integrating the given function.

Part B

16. Expand $e^x \sin y$ in a series of powers of x and y as far as the terms of the third degree.
17. Find the Taylor's series expansion of $e^x \cos y$ in the neighbourhood of the point $\left(1, \frac{\pi}{4}\right)$ upto the second degree terms.

18. Find the Maclaurin's series expansion of $e^x \log(1 + y)$ upto the terms of the third degree.
19. Find the Taylor's series expansion of $\tan^{-1}\left(\frac{y}{x}\right)$ in powers of $(x - 1)$ and $(y - 1)$ upto the second degree terms.
20. Expand $x^2y + 3y - 2$ in powers of $(x - 1)$ and $(y + 2)$ upto the third degree terms.
21. Expand $xy^2 + 2x - 3y$ in powers of $(x + 2)$ and $(y - 1)$ upto the third degree terms.
22. Find the Taylor's series expansion of y^x at (1, 1) upto the second degree terms.
23. Find the Taylor's series expansion of e^{xy} at (1, 1) upto the third degree terms.
24. Using Taylor's series, verify that

$$\cos(x+y) = 1-\frac{(x+y)^2}{2!}+\frac{(x+y)^4}{4!}-\cdots\infty$$

25. Using Taylor's series, verify that

$$\tan^{-1}(x+y) = (x+y)+\frac{1}{3}(x+y)^3\cdots\infty$$

26. If $x = u(1 - v)$, $y = uv$, verify that

$$\frac{\partial(x,y)}{\partial(u,v)}\times\frac{\partial(u,v)}{\partial(x,y)}=1$$

27. (i) if $x = u^2 - v^2$ and $y = 2uv$, find the Jacobian of x and y with respect to u and v.
    (ii) if $u = x^2$ and $v = y^2$, find $\frac{\partial(u,v)}{\partial(x,y)}$
28. If $x = a\cosh u \cos v$ and $y = a \sinh u \cdot \sin v$, show that

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{a^2}{2}(\cosh 2u-\cos 2v).$$

29. If $x = r\cos\theta$, $y = r\sin\theta$, $z = z$, find $\frac{\partial(x,y,z)}{\partial(r,\theta,z)}$
30. If $F = xu + v - y$, $G = u^2 + vy + w$ and $H = zu - v + vw$, compute $\frac{\partial(F,G,H)}{\partial(u,v,w)}$.
31. If $u = xyz$, $v = xy + yz + zx$ and $w = x + y + z$, find $\frac{\partial(u,v,w)}{\partial(x,y,z)}$.
32. Examine the functional dependence of the functions $u = \frac{x+y}{x-y}$ and $v = \frac{xy}{(x-y)^2}$. If they are dependent, find the relation between them.

33. Are the functions $u = \dfrac{x+y}{1-xy}$ and $v = \tan^{-1} x + \tan^{-1} y$ functionally dependent? If so, find the relation between them.
34. Are the functions $f_1 = x + y + z$, $f_2 = x^2 + y^2 + z^2$ and $f_3 = xy + yz + zx$ functionally dependent? If so, find the relation among f_1, f_2 and f_3.
35. If $\int_0^x \lambda e^{-\lambda(x-y)} f(y)\mathrm{d}y = \lambda^2 \cdot xe^{-\lambda x}$ prove that $f(x) = \lambda e^{-\lambda x}$. [**Hint:** Differentiate both sides with respect to x].

Use the concept of differentiation under integral sign to evaluate the following:

36. $\int_0^x \dfrac{\mathrm{d}x}{(x^2+a^2)^2}$ $\left[\textbf{Hint:}\text{ Use} \int_0^x \dfrac{\mathrm{d}x}{x^2+a^2}\right.$
37. $\int_0^1 x^m (\log x)^n \,\mathrm{d}x$ $\left[\textbf{Hint:}\text{ Use} \int_0^1 x^m \,\mathrm{d}x\right.$
38. $\int_0^\infty e^{-x^2} \cos 2ax \,\mathrm{d}x$
39. $\int_0^\infty \dfrac{e^{-ax}\sin x}{x}\mathrm{d}x$ and hence $\int_0^\infty \dfrac{\sin x}{x}\mathrm{d}x$
40. $\int_0^1 \dfrac{x^m - 1}{\log x}\mathrm{d}x,\ m \geq 0.$

4.5 MAXIMA AND MINIMA OF FUNCTIONS OF TWO VARIABLES

Students are familiar with the concept of maxima and minima of a function of one variable. Now we shall consider the maxima and minima of a function of two variables.

A function $f(x, y)$ is said to have a *relative maximum* (or simply maximum) at $x = a$ and $y = b$, if $f(a, b) > f(a + h, b + k)$ for all small values of h and k.

A function $f(x, y)$ is said to have a *relative maximum* (or simply maximum) at $x = a$ and $y = b$, if $f(a, b) < f(\mathrm{a} + h, b + k)$ for all small values of h and k.

A maximum or a minimum value of a function is called its *extreme value.* We give below *the working rule to find the extreme values of a function* $f(x, y)$:

(1) Find $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$.

(2) Solve the equations $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$ simultaneously. Let the solutions be (a, b); (c, d);...

Note ☑ The points like (a, b) at which $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$ are called *stationary points* of the function $f(x, y)$. The values of $f(x, y)$ at the stationary points are called *stationary values* of $f(x, y)$.

(3) For each solution in step (2), find the values of $A = \frac{\partial^2 f}{\partial x^2}$, $B = \frac{\partial^2 f}{\partial x\, \mathrm{d}y}$, $C = \frac{\partial^2 f}{\partial y^2}$ and $\Delta = AC - B^2$.

(4) (i) If $\Delta > 0$ and A (or C) < 0 for the solution (a, b) then $f(x, y)$ has a maximum value at (a, b).

(ii) If $\Delta > 0$ and A (or C) > 0 for the solution (a, b) then $f(x, y)$ has a minimum value at (a, b).

(iii) If $\Delta < 0$ for the solution (a, b), then $f(x, y)$ has neither a maximum nor a minimum value at (a, b). In this case, the point (a, b) is called *a saddle point* of the function $f(x, y)$.

(iv) If $\Delta = 0$ or $A = 0$, the case is doubtful and further investigations are required to decide the nature of the extreme values of the function $f(x, y)$.

4.5.1 Constrained Maxima and Minima

Sometimes we may require to find the extreme values of a function of three (or more) variables say $f(x, y, z)$ which are not independent but are connected by some given relation $\phi(x, y, z) = 0$. The extreme values of $f(x, y, z)$ in such a situation are called *constrained extreme values.*

In such situations, we use $\phi(x, y, z) = 0$ to eliminate one of the variables, say z from the given function, thus converting the function as a function of only two variables and then find the unconstrained extreme values of the converted function. [Refer to examples (4.8), (4.9), (4.10)].

When this procedure is not practicable, we use Lagrange's method, which is comparatively simpler.

4.5.2 Lagrange's Method of Undetermined Multipliers

Let
$$u = f(x, y, z) \tag{1}$$

be the function whose extreme values are required to be found subject to the constraint

$$\phi(x, y, z) = 0 \tag{2}$$

The necessary conditions for the extreme values of u are $\frac{\partial f}{\partial x} = 0, \frac{\partial f}{\partial y} = 0$ and $\frac{\partial f}{\partial z} = 0$

$$\therefore \qquad \frac{\partial f}{\partial x}\mathrm{d}x+\frac{\partial f}{\partial y}\mathrm{d}y+\frac{\partial f}{\partial z}\mathrm{d}z = 0 \tag{3}$$

From (2), we have

$$\frac{\partial \phi}{\partial x}\mathrm{d}x+\frac{\partial \phi}{\partial y}\mathrm{d}y+\frac{\partial \phi}{\partial z}\mathrm{d}z = 0 \tag{4}$$

Now (3) + λ × (4), where λ is an unknown multiplier, called *Langrange multiplier*, gives

$$\left(\frac{\partial f}{\partial x}+\lambda\frac{\partial \phi}{\partial x}\right)\mathrm{d}x+\left(\frac{\partial f}{\partial y}+\lambda\frac{\partial \phi}{\partial y}\right)\mathrm{d}y+\left(\frac{\partial f}{\partial z}+\lambda\frac{\partial \phi}{\partial z}\right) = 0 \tag{5}$$

Equation (5) holds good, if

$$\frac{\partial f}{\partial x}+\lambda\frac{\partial \phi}{\partial x} = 0 \tag{6}$$

$$\frac{\partial f}{\partial y}+\lambda\frac{\partial \phi}{\partial y} = 0 \tag{7}$$

$$\frac{\partial f}{\partial z}+\lambda\frac{\partial \phi}{\partial z} = 0 \tag{8}$$

Solving the Equations (2), (6), (7) and (8), we get the values of x, y, z, λ, which give the extreme values of u.

Note ☑

(1) The Equations (2), (6), (7) and (8) are simply the necessary conditions for the extremum of the auxiliary function $(f+\lambda\phi)$, where λ is also treated as a variable.

(2) Lagrange's method does not specify whether the extreme value found out is a maximum value or a minimum value. It is decided from the physical consideration of the problem.

WORKED EXAMPLE 4(c)

Example 4.1 Examine $f(x,y) = x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x$ for extreme values.

$$f(x,y) = x^3+3xy^2-15x^2-15y^2+72x$$

$$f_x = 3x^2+3y^2-30x+72$$

$$f_y = 6xy-30y$$

$$f_{xx} = 6x-30;\ f_{xy} = 6y;\ f_y = 6x-30$$

The stationary points are given by $f_x = 0$ and $f_y = 0$

i.e., $$3(x^2 + y^2 - 10x + 24) = 0 \quad (1)$$

and $$6y(x-5) = 0 \quad (2)$$

From (2), $x = 5$ or $y = 0$
When $x = 5$, from (1), we get $y^2 - 1 = 0$; $\therefore\ y = \pm 1$
When $y = 0$, from (1), we get $x^2 - 10x + 24 = 0$

$\therefore$ $$x = 4, 6.$$

The stationary points are (5, 1), (5, –1), (4, 0) and (6, 0)
At the point (5, ±1), $A = f_{xx} = 0$; $B = f_{xy} = \pm 6$; $C = f_{yy} = 0$
Though $AC - B^2 < 0$, $A = 0$
$\therefore$ Nothing can be said about the maxima or minima of $f(x, y)$ at (5, ±1).
At the point (4, 0), $A = -6$, $B = 0$, $C = -6$

$\therefore$ $$AC - B^2 = 36 > 0 \quad \text{and} \quad A < 0$$

$\therefore$ $f(x, y)$ is maximum at (4, 0) and maximum value of $f(x, y) = 112$.
At point (6, 0), $A = 6$, $B = 0$, $C = 6$

$\therefore$ $$AC - B^2 = 36 > 0 \text{ and } A > 0.$$

$\therefore$ $f(x, y)$ is minimum at (6, 0) and the minimum value of $f(x, y) = 108$.

Example 4.2 Examine the function $f(x, y) = x^3y^2(12 - x - y)$ for extreme values.

$$\begin{aligned} f(x,y) &= 12x^3y^2 - x^4y^2 - x^3y^3 \\ f_x &= 36x^2y^2 - 4x^3y^2 - 3x^2y^3 \\ f_y &= 24x^3y - 2x^4y - 3x^3y^2 \\ f_{xx} &= 72xy^2 - 12x^2y^2 - 6xy^3 \\ f_{xy} &= 72x^2y - 8x^3y - 9x^2y^2 \\ f_{yy} &= 24x^3 - 2x^4 - 6x^3y \end{aligned}$$

The stationary points are given by $f_x = 0$; $f_y = 0$

i.e., $$x^2y^2(36 - 4x - 3y) = 0 \quad (1)$$

and $$x^3y(24 - 2x - 3y) = 0 \quad (2)$$

Solving (1) and (2), the stationary points are (0, 0), (0, 8), (0, 12), (12, 0), (9, 0) and (6, 4).

At the first five points, $AC - B^2 = 0$.

$\therefore$ Further investigation is required to investigate the extremum at these points. At the point (6, 4), $A = -2304$, $B = -1728$, $C = -2592$ and $AC - B^2 > 0$.

Since $AC - B^2 > 0$ and $A < 0$, $f(x, y)$ has a maximum at the point (6, 4).

Maximum value of $f(x, y) = 6912$.

Example 4.3 Discuss the maxima and minima of the function $f(x, y) = x^4 + y^4 - 2x^2 + 4xy - 2y^2$.

$$f(x,y) = x^4 + y^4 - 2x^2 + 4xy - 2y^2.$$
$$f_x = 4(x^3 - x + y)$$
$$f_y = 4(y^3 + x - y)$$
$$f_{xx} = 4(3x^2 - 1);\ f_{xy} = 4;\ f_{yy} = 4(3y^2 - 1)$$

The possible extreme points are given by

$$f_x = 0 \text{ and } f_y = 0$$

i.e.,
$$x^3 - x + y = 0 \quad (1)$$

and
$$y^3 + x - y = 0 \quad (2)$$

Adding (1) and (2), $\quad x^3 + y^3 = 0 \therefore y = -x \quad (3)$

Using (3) in (1): $\quad x^3 - 2x = 0$

i.e., $\quad x(x^2 - 2) = 0 \therefore x = 0, +\sqrt{2}, -\sqrt{2}$

and the corresponding values of y are $0, -\sqrt{2}, +\sqrt{2}$.

$\therefore$ The possible extreme points of $f(x, y)$ are (0, 0), $(+\sqrt{2}, -\sqrt{2})$ and $(-\sqrt{2}, \sqrt{2})$.

At the point (0, 0), $A = -4$, $B = 4$ and $C = -4$

$$AC - B^2 = 0$$

$\therefore$ The nature of $f(x, y)$ is undecided at (0, 0). At the points $(\pm\sqrt{2}, \mp\sqrt{2})$, $A = 20$, $B = 4$, $C = 20$

$$AC - B^2 > 0$$

$\therefore f(x, y)$ is minimum at the points $(\pm\sqrt{2}, \mp\sqrt{2})$, and minimum value of $f(x, y) = 8$.

Example 4.4 Examine the extrema of $f(x,y) = x^2 + xy + y^2 + \frac{1}{x} + \frac{1}{y}$.

$$f(x,y) = x^2 + xy + y^2 + \frac{1}{x} + \frac{1}{y}$$

$$f_x = 2x + y - \frac{1}{x^2}$$

$$f_y = x + 2y - \frac{1}{y^2}$$

$$f_{xx} = 2+\frac{2}{x^3};\ f_{xy} = 1;\ f_{yy} = 2+\frac{2}{y^3}$$

The possible extreme points are given by $f_x = 0$ and $f_y = 0$.

i.e., $$2x+y-\frac{1}{x^2} = 0 \qquad (1)$$

and $$x+2y-\frac{1}{y^2} = 0 \qquad (2)$$

(1) – (2) gives $$x-y+\frac{1}{y^2}-\frac{1}{x^2} = 0$$

i.e., $$x-y+\frac{x^2-y^2}{x^2y^2} = 0$$

i.e., $$(x-y)(x^2y^2+x+y) = 0$$

$\therefore$ $$x = y \qquad (3)$$

Using (3) in (1), $3x^3 - 1 = 0$

$\therefore$ $$x = \left(\frac{1}{3}\right)^{\frac{1}{3}} = y$$

At the point $\left\{\left(\frac{1}{3}\right)^{\frac{1}{3}}, \left(\frac{1}{3}\right)^{\frac{1}{3}}\right\}$, $A = 8$, $B = 1$ and $C = 8$

$\therefore$ $$AC - B^2 > 0$$

$\therefore$ $f(x, y)$ is minimum at $\left\{\left(\frac{1}{3}\right)^{\frac{1}{3}}, \left(\frac{1}{3}\right)^{\frac{1}{3}}\right\}$ and minimum value of $f(x,y) = 3^{\frac{4}{3}}$.

Example 4.5 Discuss the extrema of the function $f(x, y) = x^2 - 2xy + y^2 + x^3 - y^3 + x^4$ at the origin

$$f(x,y) = x^2-2xy+y^2+x^3-y^3+x^4.$$

$$f_x = 2x-2y+3x^2+4x^3$$

$$f_y = -2x+2y-3y^2$$

$$f_{xx} = 2+6x+12x^2$$

$$f_{xy} = -2;\quad f_{yy} = 2-6y$$

The origin (0, 0) satisfies the equations $f_x = 0$ and $f_y = 0$.
$\therefore$ (0, 0) is a stationary point of $f(x, y)$.
At the origin, $A = 2$, $B = -2$ and $C = 2$

$$\therefore \qquad AC - B^2 = 0$$

Hence further investigation is required to find the nature of the extrema of $f(x, y)$ at the origin.

Let us consider the values of $f(x, y)$ at three points close to (0, 0), namely at $(h, 0)$, $(0, k)$ and (h, h) which lie on the x-axis, the y-axis and the line $y = x$ respectively.

$$f(h,0) = h^2 + h^3 + h^4 > 0.$$
$$f(0,k) = k^2 - k^3 = k^2(1-k) > 0, \text{ when } 0 < k < 1$$
$$f(h,h) = h^4 > 0$$

Thus $f(x, y) > f(0, 0)$ in the neighbourhood of (0, 0).
$\therefore$ (0, 0) is a minimum point of $f(x, y)$ and minimum value of $f(x, y) = 0$.

Example 4.6 Find the maximum and minimum values of

$$f(x, y) = \sin x \sin y \sin(x + y); 0 < x, y < \pi.$$

$$f(x, y) = \sin x \sin y \sin(x + y)$$
$$f_x = \cos x \sin y \sin(x + y) + \sin x \sin y \cos(x + y)$$
$$f_y = \sin x \cos y \sin(x + y) + \sin x \sin y \cos(x + y)$$

i.e.,
$$f_x = \sin y \sin(2x + y)$$

and
$$f_y = \sin x \cdot \sin(x + 2y)$$
$$f_{xx} = 2\sin y \cos(2x + y)$$
$$f_{xy} = \sin y \cos(2x + y) + \cos y \cdot \sin(2x + y)$$
$$= \sin(2x + 2y)$$
$$f_{yy} = 2\sin x \cos(x + 2y)$$

For maximum or minimum values of $f(x, y)$, $f_x = 0$ and $f_y = 0$
i.e., $\sin y \sin(2x + y) = 0$ and $\sin x \cdot \sin(x + 2y) = 0$

i.e., $\frac{1}{2}[\cos 2x - \cos(2x + 2y)] = 0$ and $\frac{1}{2}[\cos 2y - \cos(2x + 2y)] = 0$

i.e.,
$$\cos 2x - \cos(2x + 2y) = 0 \qquad (1)$$

and
$$\cos 2y - \cos(2x + 2y) = 0 \qquad (2)$$

From (1) and (2), $\cos 2x = \cos 2y$. Hence $x = y$ (3)
Using (3) in (1), $\cos 2x - \cos 4x = 0$
i.e., $2 \sin x \sin 3x = 0$
$\therefore$ $\sin x = 0$ or $\sin 3x = 0$
$\therefore$ $x = 0, \pi$ and $3x = 0, \pi, 2\pi$ i.e., $x = 0, \frac{\pi}{3}, \frac{2\pi}{3}$

$\therefore$ The admissible values of x are $0, \dfrac{\pi}{3}, \dfrac{2\pi}{3}$.

Thus the maxima and minima of $f(x, y)$ are given by $(0, 0)$ $\left(\dfrac{\pi}{3}, \dfrac{\pi}{3}\right)$ and $\left(\dfrac{2\pi}{3}, \dfrac{2\pi}{3}\right)$

At the point $(0, 0)$, $A = B = C = 0$

$\therefore \qquad AC - B^2 = 0$

Thus the extremum of $f(x, y)$ at $(0, 0)$ is undecided.

At the point $\left(\dfrac{\pi}{3}, \dfrac{\pi}{3}\right)$, $A = -\sqrt{3}$, $B = -\dfrac{\sqrt{3}}{2}$ and $C = -\sqrt{3}$ and $AC - B^2 = 3 - \dfrac{3}{4} > 0$.

As $AC - B^2 > 0$ and $A < 0$, $f(x, y)$ is maximum at $\left(\dfrac{\pi}{3}, \dfrac{\pi}{3}\right)$.

Maximum value of $f(x, y) = \dfrac{\sqrt{3}}{2} \cdot \dfrac{\sqrt{3}}{2} \cdot \dfrac{\sqrt{3}}{2} = \dfrac{3\sqrt{3}}{8}$.

At the point $\left(\dfrac{2\pi}{3}, \dfrac{2\pi}{3}\right)$, $A = \sqrt{3}, B = \dfrac{\sqrt{3}}{2}$ and $C = \sqrt{3}$ and $AC - B^2 = 3 - \dfrac{3}{4} > 0$.

As $AC - B^2 > 0$ and $A > 0$, $f(x, y)$ is maximum at $\left(\dfrac{2\pi}{3}, \dfrac{2\pi}{3}\right)$.

Minimum value of $f(x, y) = -\dfrac{3\sqrt{3}}{8}$.

Example 4.7 Identify the saddle point and the extremum points of

$$f(x, y) = x^4 - y^4 - 2x^2 + 2y^2.$$

$$f(x, y) = x^4 - y^4 - 2x^2 + 2y^2$$

$$f_x = 4x^3 - 4x; \; f_y = 4y - 4y^3$$

$$f_{xx} = 12x^2 - 4; \; f_{xy} = 0; \quad f_{yy} = 4 - 12y^2.$$

The stationary points of $f(x, y)$ are given by $f_x = 0$ and $f_y = 0$

i.e., $$4(x^3 - x) = 0 \text{ and } 4(y - y^3) = 0$$

i.e., $$4x(x^2 - 1) = 0 \text{ and } 4y(1 - y^2) = 0$$

$\therefore$ $x = 0$ or ± 1 and $y = 0$ or ± 1.

At the points $(0, 0)$, $(\pm 1, \pm 1)$, $AC - B^2 < 0$

$\therefore$ The points $(0, 0)$, $(1, 1)$, $(1, -1)$, $(-1, 1)$ and $(-1, -1)$ are saddle points of the function $f(x, y)$.

At the point $(\pm 1, 0)$, $AC - B^2 > 0$ and $A > 0$

$\therefore$ $f(x, y)$ attains its minimum at $(\pm 1, 0)$ and the minimum value is -1.

At the point $(0, \pm 1)$, $AC - B^2 > 0$ and $A < 0$

$\therefore f(x, y)$ attains its maximum at $(0, \pm 1)$ and the maximum value is $+ 1$.

Example 4.8 Find the minimum value of $x^2 + y^2 + z^2$, when $x + y + z = 3a$.

Here we try to find the conditional minimum of $x^2 + y^2 + z^2$, subject to the condition

$$x + y + z = 3a \tag{1}$$

Using (1), we first express the given function as a function of x and y.

From (1), $z = 3a - x - y$.

Using this in the given function, we get

$$f(x, y) = x^2 + y^2 + (3a - x - y)^2$$

$$f_x = 2x - 2(3a - x - y)$$

$$f_y = 2y - 2(3a - x - y)$$

$$f_{xx} = 4;\ f_{xy} = 2;\ f_{yy} = 4$$

The possible extreme points are given by $f_x = 0$ and $f_y = 0$.

i.e., $$2x + y = 3a \tag{2}$$

and $$x + 2y = 3a \tag{3}$$

Solving (2) and (3), we get the only extreme point as (a, a)

At the point (a, a), $AC - B^2 > 0$ and $A > 0$

$\therefore f(x, y)$ is minimum at (a, a) and the minimum value of $f(x, y) = 3a^2$.

Example 4.9 Show that, if the perimeter of a triangle is constant, its area is maximum when it is equilateral.

Let the sides of the triangle be a, b, c.

Given that $$a + b + c = \text{constant}$$

$$= 2k, \text{ say} \tag{1}$$

Area of the triangle is given by

$$A = \sqrt{s(s-a)(s-b)(s-c)} \tag{2}$$

where $$s = \frac{a+b+c}{2}$$

Using (1) in (2),

$$A = \sqrt{k(k-a)(k-b)(k-c)} \tag{3}$$

A is a function of three variables a, b , c

Again using (1) in (3), we get

$$A = \sqrt{k(k-a)(k-b)(a+b-k)}$$

A is maximum or minimum, when $f(a, b) = \dfrac{A^2}{k} = (k - a)(k - b)(a + b - k)$ is maximum or minimum.

$$\begin{aligned} f_a &= (k-b)\{(k-a)\cdot 1+(a+b-k)\cdot(-1)\} \\ &= (k-b)(2k-2a-b) \\ f_b &= (k-a)\{(k-b)\cdot 1+(a+b-k)\cdot(-1)\} \\ &= (k-a)(2k-a-2b) \\ f_{aa} &= -2(k-b);\ f_{ab} = -3k+2a+2b; \\ f_{ab} &= -2(k-a) \end{aligned}$$

The possible extreme points of $f(a, b)$ are given by

$f_a = 0$ and $f_b = 0$

i.e., $(k - b)(2k - 2a - b) = 0$ and $(k - a)(2k - a - 2b) = 0$

$\therefore\ b = k$ or $2a + b = 2k$ and $a = k$ or $a + 2b = 2k$

Thus the possible extreme points are given by

(i) $a = k$, $b = k$; (ii) $b = k$, $a + 2b = 2k$; (iii) $a = k$, $2a + b = 2k$ and (iv) $2a + b = 2k$, $a + 2b = 2k$.

(i) gives $a = k$, $b = k$ and hence $c = 0$.
(ii) gives $a = 0$, $b = k$ and hence $c = k$.
(iii) gives $a = k$, $b = 0$ and hence $c = k$.

All these lead to meaningless results.

Solving $2a + b = 2k$ and $a + 2b = 2k$, we get

$$a = \frac{2k}{3} \text{ and } b = \frac{2k}{3}$$

At the point $\left(\dfrac{2k}{3}, \dfrac{2k}{3}\right)$,

$$A = f_{aa} = -\frac{2k}{3};\ B = f_{ab} = -\frac{k}{3};\ C = f_{bb} = -\frac{2k}{3}$$

$AC - B^2 > 0$ and $A < 0$

$\therefore f(a, b)$ is maximum at $\left(\dfrac{2k}{3}, \dfrac{2k}{3}\right)$

Hence the area of the triangle is maximum when $a = \dfrac{2k}{3}$ and $b = \dfrac{2k}{3}$.

When $a = \dfrac{2k}{3}$, $b = \dfrac{2k}{3}$; $c = 2k - (a+b) = \dfrac{2k}{3}$

Thus the area of the triangle is maximum, when $a = b = c = \dfrac{2k}{3}$, i.e., when the triangle is equilateral.

Example 4.10 In a triangle ABC, find the maximum value of $\cos A \cos B \cos C$. In triangle ABC, $A + B + C = \pi$.

Using this condition, we express the given function as a function of A and B

Thus $\cos A \cos B \cos C = \cos A \cos B \cos \{\pi - (A + B)\}$

$$= -\cos A \cos B \cos (A + B)$$

Let
$$f(A, B) = -\cos A \cos B \cos (A + B)$$
$$f_A = -\cos B\{-\sin A \cos(A + B) - \cos A \sin(A + B)\}$$
$$= \cos B \sin(2A + B)$$
$$f_B = -\cos A\{-\sin B \cos(A + B) - \cos B \sin(A + B)\}$$
$$= \cos A \sin(A + 2B)$$
$$f_{AA} = 2\cos B \cos(2A + B)$$
$$f_{AB} = \cos B \cos(2A + B) - \sin B \sin(2A + B)$$
$$= \cos(2A + 2B)$$
$$f_{BB} = 2\cos A \cos(A + 2B)$$

The possible extreme points are given by

$$f_A = 0 \text{ and } f_B = 0$$

i.e.,
$$\cos B \sin(2A + B) = 0 \tag{1}$$

and
$$\cos A \sin (A + 2B) = 0$$

Thus the possible values of A and B are given by (i) $\cos B = 0$, $\cos A = 0$; (ii) $\cos B = 0$, $\sin (A + 2B) = 0$; (iii) $\sin (2A + B) = 0$, $\cos A = 0$ and (iv) $\sin (2A + B) = 0$, $\sin (A + 2B) = 0$

i.e., (i) $A = \frac{\pi}{2}$, $B = \frac{\pi}{2}$; (ii) $B = \frac{\pi}{2}$, $A = 0$ or π, (iii) $A = \frac{\pi}{2}$, $B = 0$ or π and (iv) $2A + B = \pi$, $A + 2B = \pi$ or

$$A = \frac{\pi}{3}, B = \frac{\pi}{3}$$

The first three sets of values of A and B lead to meaningless results.

Hence $A = \frac{\pi}{3}$, $B = \frac{\pi}{3}$ give the extreme point.

At this point $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$, $A = f_{AA} = -1$; $B = f_{AB} = -\frac{1}{2}$; $f_{BB} = -1$ and $AC = B^2 > 0$.

Also $A < 0$

$\therefore f(A, B)$ is maximum at $A = B = \frac{\pi}{3}$ and the maximum value

$$= -\cos\frac{\pi}{3} \cdot \cos\cdot\frac{\pi}{3}\cos\frac{2\pi}{3} = \frac{1}{8}.$$

Example 4.11 Find the maximum value of $x^m y^n z^p$, when $x + y + z = a$.

Let $f = x^m y^n z^p$ and $\phi = x + y + z - a$.

Using the Lagrange multiple λ, the auxiliary function is $g = (f + \lambda\phi)$.
This stationary points of $g = (f + \lambda\phi)$ are given by $g_x = 0$, $g_y = 0$, $g_z = 0$ and $g_\lambda = 0$

i.e.,
$$mx^{m-1} y^n z^p + \lambda = 0 \tag{1}$$
$$nx^m y^{n-1} z^p + \lambda = 0 \tag{2}$$
$$px^m y^n z^{p-1} + \lambda = 0 \tag{3}$$
$$x + y + z - a = 0 \tag{4}$$

From (1), (2) and (3), we have
$$-\lambda = mx^{m-1} y^n z^p = nx^m y^{n-1} z^p = px^m y^n z^{p-1}.$$

i.e.,
$$\frac{m}{x} = \frac{n}{y} = \frac{p}{z} = \frac{m+n+p}{x+y+z}$$
$$= \frac{m+n+p}{a}, \text{ by (4)}$$

$\therefore$ Maximum value of f occurs,

when $x = \dfrac{am}{m+n+p}, \; y = \dfrac{an}{m+n+p}, \; z = \dfrac{ap}{m+n+p}$

Thus maximum value of $f = \dfrac{a^{m+n+p} \cdot m^m \cdot n^n \cdot p^p}{(m+n+p)^{m+n+p}}$

Example 4.12 A rectangular box, open at the top, is to have a volume of 32 c.c. Find the dimensions of the box, that requires the least material for its construction.

Let, x, y, z be the length, breadth and height of the respectively.
The material for the construction of the box is least, when the area of surface of the box is least.
Hence we have to minimise
$$S = xy + 2yz + 2zx,$$
subject to the condition that the volume of the box, i.e., $xyz = 32$.
Here $f = xy + 2yz + 2zx$; $\phi = xyz - 32$.
The auxiliary function is $g = f + \lambda\phi$, where λ is the Lagrange multiplier. The stationary points of g are given by $g_x = 0$, $g_y = 0$, $g_z = 0$ and $g_\lambda = 0$

i.e.,
$$y + 2z + \lambda yz = 0 \tag{1}$$
$$x + 2x + \lambda zx = 0 \tag{2}$$
$$2x + 2y + \lambda xy = 0 \tag{3}$$
$$xyz - 32 = 0 \tag{4}$$

From (1), (2) and (3), we have

$$\frac{1}{z}+\frac{2}{y} = -\lambda \tag{5}$$

$$\frac{1}{z}+\frac{2}{x} = -\lambda \tag{6}$$

$$\frac{2}{y}+\frac{2}{x} = -\lambda \tag{7}$$

Solving (5), (6) and (7), we get

$$x = -\frac{4}{\lambda},\ y = -\frac{4}{\lambda} \text{ and } z = -\frac{2}{\lambda}$$

Using these values in (4), we get

$$-\frac{32}{\lambda^3}-32 = 0$$

i.e., $$\lambda = -1$$

$\therefore$ $$x = 4,\ y = 4,\ z = 2.$$

Thus the dimensions of the box and 4 cm; 4 cm and 2 cm.

Example 4.13 Find the volume of the greatest rectangular parallelopiped inscribed in the ellipsoid whose equation is $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} = 1.$

Let $2x$, $2y$, $2z$ be the dimensions of the required rectangular parallelopiped.

By symmetry, the centre of the parallelopiped coincides with that of the ellipsoid, namely, the origin and its faces are parallel to the co-ordinate planes.

Also one of the vertices of the parallelopiped has co-ordinates (x, y, z), which satisfy the equation of the ellipsoid.

Thus, we have to maximise $V = 8xyz$, subject to the condition $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} = 1$

Here $f = 8xyz$ and $\phi = \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1$

The auxiliary function is $g = f + \lambda\phi$, where λ is the Lagrange multiplier. The stationary points of g are given by

$$g_x = 0,\ g_y = 0,\ g_z = 0 \text{ and } g_\lambda = 0$$

i.e., $$8yz+\frac{2\lambda x}{a^2} = 0 \tag{1}$$

$$8zx+\frac{2\lambda y}{b^2} = 0 \tag{2}$$

$$8xy + \frac{2\lambda z}{c^2} = 0 \tag{3}$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \tag{4}$$

Multiplying (1) by x, $\dfrac{2\lambda x^2}{a^2} = -8xyz$

Similarly $\dfrac{2\lambda y^2}{b^2} = \dfrac{2\lambda z^2}{c^2} = -8xyz$ from (2) and (3)

Thus $\dfrac{x^2}{a^2} = \dfrac{y^2}{b^2} = \dfrac{z^2}{c^2} = k$ say

Using in (4), $3k = 1 \quad \therefore \ k = \dfrac{1}{3}$

$$\therefore x = \frac{a}{\sqrt{3}}, \ y = \frac{b}{\sqrt{3}} \text{ and } z = \frac{c}{\sqrt{3}}$$

$$\therefore \text{Maximum volume} = \frac{8abc}{3\sqrt{3}}.$$

Example 4.14 Find the shortest and the longest distances from the point (1, 2, –1) to the sphere $x^2 + y^2 + z^2 = 24$.

Let (x, y, z) be any point on the sphere. Distance of the point (x, y, z) from $(1, 2, -1)$ is given by $d = \sqrt{(x-1)^2 + (y-2)^2 + (z+1)^2}$.

We have to find the maximum and minimum values of d or equivalently

$$d^2 = (x-1)^2 + (y-2)^2 + (z+1)^2,$$

subject to the constant $x^2 + y^2 + z^2 - 24 = 0$

Here $\quad f = (x-1)^2 + (y-2)^2 + (z+1)^2$ and

$\quad \phi = x^2 + y^2 + z^2 - 24$

The auxiliary function is $g = f + \lambda\phi$, where λ is the Lagrange multiplier. The stationary points of g are given by $g_x = 0$, $g_y = 0$, $g_z = 0$ and $g_\lambda = 0$.

i.e.,

$$2(x-1) + 2\lambda x = 0 \tag{1}$$

$$2(y-2) + 2\lambda y = 0 \tag{2}$$

$$2(z+1) + 2\lambda z = 0 \tag{3}$$

$$x^2 + y^2 + z^2 = 24 \tag{4}$$

From (1), (2) and (3), we get

$$x = \frac{1}{1+\lambda},\ y = \frac{2}{1+\lambda},\ z = \frac{1}{1+\lambda}$$

Using these values in (4), we get

$$\frac{6}{(1+\lambda)^2} = 24 \text{ i.e., } (1+\lambda)^2 = \frac{1}{4}$$

$\therefore$ $$\lambda = -\frac{1}{2} \text{ or } -\frac{3}{2}.$$

When $\lambda = -\frac{1}{2}$, the point on the sphere is (2, 4, –2)

When $\lambda = -\frac{3}{2}$, the point on the sphere is (–2, –4, 2)

When the point is (2, 4, –2), $d = \sqrt{(1)^2 + (2)^2 + (-1)^2} = \sqrt{6}$

When the point is (–2, –4, 2), $d = \sqrt{(-3)^2 + (-6)^2 + 3^2} = 3\sqrt{6}$

$\therefore$ Shortest and longest distances are $\sqrt{6}$ and $3\sqrt{6}$ respectively.

Example 4.15 Find the point on the curve of intersection of the surfaces $z = xy + 5$ and $x + y + z = 1$ which is nearest to the origin.

Let (x, y, z) be the required point.

It lies on both the given surfaces.

$\therefore$ $$xy - z + 5 = 0 \quad \text{and} \quad x + y + z = 1$$

Distance of the point (x, y, z) from the origin is given by $d = \sqrt{x^2 + y^2 + z^2}$.
We have to minimize d or equivalently

$$d^2 = x^2 + y^2 + z^2,$$

subject to the constraints $xy - z + 5 = 0$ and $x + y + z - 1 = 0$.

Note ☑ Here we have two constraint conditions. To find the extremum of $f(x, y, z)$ subject to the conditions $\phi_1(x, y, z) = 0$ and $\phi_2(x, y, z) = 0$, we form the auxiliary function
$g = f + \lambda_1\phi_1 + \lambda_2\phi_2$, where λ_1 and λ_2 are two Lagrange multipliers.

The stationary points of g are given by $g_x = 0$, $g_y = 0$, $g_z = 0$, $g_{\lambda 1} = 0$ and $g_{\lambda 2} = 0$.

In this problem, $f = x^2 + y^2 + z^2$, $\phi_1 = xy - z + 5$ and $\phi_2 = x + y + z - 1$.

The auxiliary function is $g = f + \lambda_1\phi_1 + \lambda_2\phi_2$, where λ_1, λ_2 are Lagrange multipliers.
The stationary points of g are given by

$$2x + \lambda_1 y + \lambda_2 = 0 \tag{1}$$

$$2y + \lambda_1 x + \lambda_2 = 0 \tag{2}$$

$$2z - \lambda_1 + \lambda_2 = 0 \tag{3}$$

$$xy - z + 5 = 0 \tag{4}$$

$$x + y + z - 1 = 0 \tag{5}$$

Eliminating λ_1, λ_2 from (1), (2), (3), we have

$$\begin{vmatrix} 2x & y & 1 \\ 2y & x & 1 \\ 2z & -1 & 1 \end{vmatrix} = 0$$

i.e., $x(x+1) - y(y-2) - (y + zx) = 0$

i.e., $x^2 - y^2 + x - y - z(x - y) = 0$

i.e., $(x - y)(x + y - z + 1) = 0$

$\therefore$ $x = y$ or $x + y - z + 1 = 0$

Using $x = y$ in (4) and (5), we have

$$z = x^2 + 5 \tag{6}$$

and $$z = 1 - 2x \tag{7}$$

From (6) and (7), $x^2 + 2x + 4 = 0$, which gives only imaginary values for x.

Hence $$x + y - z + 1 = 0 \tag{8}$$

Solving (5) and (8), we get $x + y = 0$ (9)

and $$z = 1 \tag{10}$$

Using (10) in (4), we get $xy = -4$ (11)

Solving (9) and (11), we get $x = \pm 2$ and $y = \pm 2$.

$\therefore$ The required points are (2, –2, 1) and (–2, 2 ,1) and the shortest distance is 3.

EXERCISE 4(c)

Part A

(Short Answer Questions)

1. Define relative maximum and relative minimum of a function of two variables.
2. State the conditions for the stationary point (a, b) of $f(x, y)$ to be (i) a maximum point (ii) a minimum point and (iii) a saddle point.
3. Define saddle point of a function $f(x, y)$.
4. Write down the conditions to be satisfied by $f(x, y, z)$ and $\phi(x, y, z)$, when we extremise $f(x, y, z)$ subject to the condition $\phi(x, y, z) = 0$.
5. Find the minimum point of $f(x, y) = x^2 + y^2 + 6x + 12$.
6. Find the stationary point of $f(x, y) = x^2 - xy + y^2 - 2x + y$.
7. Find the stationary point of $f(x, y) = 4x^2 + 6xy + 9y^2 - 8x - 24y + 4$.

8. Find the possible extreme point of $f(x, y) = x^2 + y^2 + \frac{2}{x} + \frac{2}{y}$.
9. Find the nature of the stationary point (1, 1) of the function $f(x, y)$, if $f_{xx} = 6xy^3$, $f_{xy} = 9x^2 y^2$ and $f_{yy} = 6x^3 y$.
10. Given $f_{xx} = 6x$, $f_{xy} = 0$, $f_{yy} = 6y$, find the nature of the stationary point (1, 2) of the function $f(x, y)$.

Part B

Examine the following functions for extreme values:

11. $x^3 + y^3 - 3axy$
12. $x^3 + y^3 - 12x - 3y + 20$
13. $x^4 + 2x^2 y - x^2 + 3y^2$
14. $x^3 y - 3x^2 - 2y^2 - 4y - 3$
15. $x^4 + x^2 y + y^2$ at the origin
16. $x^3 y^2 (a - x - y)$
17. $x^3 y^2 (12 - 3x - 4y)$
18. $xy + 27\left(\frac{1}{x} + \frac{1}{y}\right)$
19. $\sin x + \sin y + \sin(x + y), 0 \le x, y \le \frac{\pi}{2}$.
20. Identify the saddle points and extreme points of the function $xy(3x + 2y + 1)$.
21. Find the minimum value of $x^2 + y^2 + z^2$, when (i) $xyz = a^3$ and (ii) $xy + yz + zx = 3a^2$.
22. Find the minimum value of $x^2 + y^2 + z^2$, when $ax + by + cz = p$.
23. Show that the minimum value of $(a^3 x^2 + b^3 y^2 + c^3 z^2)$, when $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{1}{k}$, is $k^2 (a + b + c)^3$.
24. Split 24 into three parts such that the continued product of the first, square of the second and cube of the third may be minimum.
25. The temperature at any point (x, y, z) in space is given by $T = k\,x\,y\,z^2$, where k is a constant. Find the highest temperature on the surface of the sphere $x^2 + y^2 + z^2 = a^2$.
26. Find the dimensions of a rectangular box, without top, of maximum capacity and surface area 432 square meters.
27. Show that, of all rectangular parallelopipeds of given volume, the cube has the least surface.
28. Show that, of all rectangular parallelopipeds with given surface area, the cube has the greatest volume.
29. Prove that the rectangular solid of maximum volume which can be inscribed in a sphere is a cube.
30. Find the points on the surface $z^2 = xy + 1$ whose distance from the origin is minimum.
31. If the equation $5x^2 + 6xy + 5y^2 = 8$ represents an ellipse with centre at the origin, find the lengths of its major and minor axes.

(**Hint**: The longest distance of a point on the ellipse from its centre gives the length of the semi-major axis. The shortest distance of a point on the ellipse from its centre gives the length of the semi-minor axis).

32. Find the point on the surface $z = x^2 + y^2$, that is nearest to the point $(3, -6, 4)$.
33. Find the minimum distance from the point $(3, 4, 15)$ to the cone $x^2 + y^2 = 4z^2$.
34. Find the points on the ellipse obtained as the curve of intersection of the surfaces $x + y = 1$ and $x^2 + 2y^2 + z^2 = 1$, which are nearest to and farthest from the origin.
35. Find the greatest and least values of z, where (x, y, z) lies on the ellipse formed by the intersection of the plane $x + y + z = 1$ and the ellipsoid $16x^2 + 4y^2 + z^2 = 16$.

ANSWERS

Exercise 4(a)

(2) $du = \cos(xy^2)\,(y^2dx + 2xy\,dy)$

(3) $du = x^{y-1} \cdot y^x\,(y + x\log y)\,dx + x^y y^{x-1}\,(x + y\log x)\,dy$

(4) $du = y\,(1 + \log xy)\,dx + x(1 + \log xy)\,dy$

(5) $du = (y \log a)\,a^{xy}\,dx + (x\log a)\,a^{xy}\,dx$

(6) $8\,a^5t^6\,(4t + 7)$;

(7) $e^{\sqrt{a^2-t^2}}\sin^3 t\left\{3\sqrt{a^2-t^2}\sin^2 t\cos t - t\sin^3 t/\sqrt{a^2-t^2}\right\}$

(8) $(\cos t - e^{-t} - \sin t)/(e^{-t} + \sin t + \cos t)$

(9) $-\dfrac{x^2+2xy+2y^2}{x^2+4xy+y^2}$;

(11) $\dfrac{3}{2}x\cos(x^2+y^2)$

(12) $x(xy + 4y^2 - 2x^2)/(x + 2y)$;

(14) 3.875

(15) 4.984

(16) 0.0043

(17) 0.006 cm^3; 0.004 cm^2

(18) 2

(19) $4(a + b + c)k$

(20) 1.5

(36) $\dfrac{\partial^2 z}{\partial u\,\partial v} = 0$;

(37) $\dfrac{\partial^2 z}{\partial v^2} = 0$

(38) $\dfrac{\partial^2 z}{\partial u^2} = 0$

(39) $\dfrac{\partial^2 z}{\partial u\partial v} = 0$;

(40) $\dfrac{\partial^2 u}{\partial z\cdot\partial z^*} = 0$

(41) (i) $\dfrac{y(y-x\log y)}{x(x-y\log x)}$

(ii) $\dfrac{y}{x}$

(iii) $\dfrac{y\tan x+\log\sin y}{\log\cos x-x\cot y}$

(iv) $\dfrac{\log\cot y-y\tan x}{\log\sec x+x\sec y\,\mathrm{cosec}\,y}$

(v) $\dfrac{x-y}{x(1+\log x)}$

(42) $2a^3xy/(ax-y^2)^3$

(43) $2a^2xy\,(3a^4+x^2y^2)/(a^2x-y^3)^3$

(47) 5%

(50) 2.3%

(55) $\dfrac{5\sqrt{3}\pi}{324}$.

Exercise 4(b)

(4) $1+(x+y)+\dfrac{(x+y)^2}{2}+\cdots$

(5) $(x+y)-\dfrac{1}{3!}(x+y)^3+\ldots$

(11) $u+\upsilon+1$;

(15) $2\tan^{-1}\left(\dfrac{1}{y}\right)$

(16) $y+xy+\dfrac{x^2y}{2}-\dfrac{y^3}{6}+\cdots$

(17) $\dfrac{e}{\sqrt{2}}\left\{1+(x-1)-\left(y-\dfrac{\pi}{4}\right)+\dfrac{(x-1)^2}{2}-(x-1)\left(y-\dfrac{\pi}{4}\right)-\dfrac{1}{2}\left(y-\dfrac{\pi}{4}\right)^2+\cdots\right\}$

(18) $y+xy-\dfrac{y^2}{2}+\dfrac{1}{2}x^2y-\dfrac{1}{2}xy^2+\dfrac{1}{3}y^3+\cdots$

(19) $\dfrac{\pi}{4}-\dfrac{1}{2}(x-1)+\dfrac{1}{2}(y-1)+\dfrac{1}{4}(x-1)^2-\dfrac{1}{4}(y-1)^2+\cdots$

(20) $-10-4(x-1)+4(y+2)-2(x-1)^2+2(x-1)(y+2)+(x-1)^2(y+2)$

(21) $-9+3(x+2)-7(y-1)+2(x+2)(y-1)-2(y-1)^2+(x+2)(y-1)^2$

(22) $1+(y-1)+(x-1)(y-1)+\cdots$

(23) $e\left[1+(x-1)+(y-1)+\dfrac{1}{2}(x-1)^2+2(x-1)(y-1)+(y-1)^2+\dfrac{1}{6}(x-1)^3\right.$
$\left.+\dfrac{3}{2}(x-1)^2(y-2)+\dfrac{3}{2}(x-1)(y-2)^2+\dfrac{1}{6}(y-2)^3\right]$

(27) (i) $4(u^2+\upsilon^2)$

(ii) $4xy$

(28) r

(30) $x(y\upsilon+1-w)+z-2u\upsilon$

(31) $(x-y)(y-z)(z-x)$

(32) $u^2=\upsilon+1$

(33) $u\tan\upsilon$

(34) $f_1^2 = f_2 + 2f_3$

(36) $\dfrac{1}{2a^3}\{\tan^{-1}\dfrac{x}{a} + (ax)/(x^2 + a^2)\}$

(37) $\dfrac{(-1)^n\, n!}{(m+1)^{n+1}}$

(38) $\dfrac{1}{2}\sqrt{\pi}e^{-a^2}$

(39) $\tan^{-1}\left(\dfrac{1}{a}\right); \dfrac{\pi}{2}$

(40) log (1 + m)

Exercise 4(c)

(5) (–3, 0). (6) (1, 0). (7) $\left(0, \dfrac{4}{3}\right)$

(8) (1, 1). (9) Saddle point. (10) Minimum point.

(11) Maximum at (a, a) if $a < 0$ and minimum at (a, a) if $a > 0$.

(12) Minimum at (2, 1) and maximum at (–2, –1).

(13) Minimum at $\left(\pm\dfrac{\sqrt{3}}{2}, -\dfrac{1}{4}\right)$

(14) Maximum at (0, –1). (15) Minimum at (0, 0).

(16) Maximum at $\left(\dfrac{a}{2}, \dfrac{a}{3}\right)$ (17) maximum at (2, 1). (18) Minimum at (3, 3).

(19) Maximum at $\left(\dfrac{\pi}{3}, \dfrac{\pi}{3}\right)$ and minimum at $\left(-\dfrac{\pi}{3}, -\dfrac{\pi}{3}\right)$.

(20) Saddle point are (0, 0), $\left(-\dfrac{1}{3}, 0\right)$ and $\left(0, -\dfrac{1}{2}\right)$; maximum at $\left(-\dfrac{1}{9}, -\dfrac{1}{6}\right)$.

(21) $3a^2$; $3a^2$. (22) $\dfrac{p^2}{a^2 + b^2 + c^2}$ (24) 4, 8, 12.

(25) $\dfrac{ka^4}{8}$. (26) 12, 12 and 6 metres.

(30) (0, 0, 1) and (0, 0, –1). (31) 4, 2. (32) (1, –2, 5).

(33) $5\sqrt{5}$. (34) $\left(\dfrac{1}{3}, \dfrac{2}{3}, 0\right)$; (1, 0, 0). (35) $\dfrac{8}{3}; -\dfrac{8}{7}$.