Chapter 3

Differential Calculus

3.1 CURVATURE AND RADIUS OF CURVATURE

Consider the two circles shown in the Fig. 3.1. It is obvious that the ways in which the two circles bend or 'curve' at the point P are not the same. The smaller circle 'curves' or changes its direction more rapidly than the bigger circle. In other words the smaller circle is said to have greater curvature than the other. This concept of curvature which holds good for any curve is formally defined as follows:

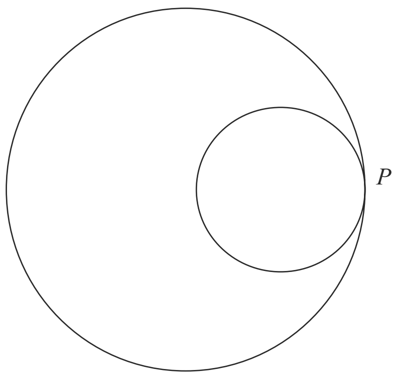

Fig. 3.1

3.1.1 Definition of Curvature

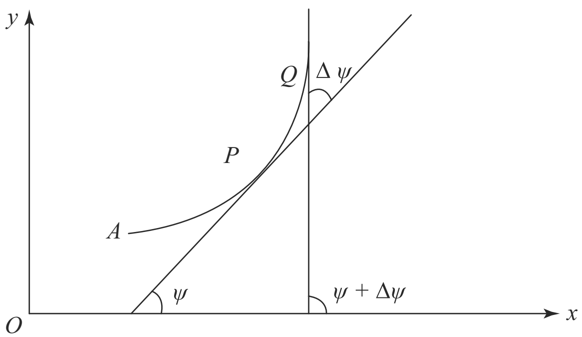

Fig. 3.2

Let P and Q be any two close points on a plane curve. Let the arcual distances of P and Q measured from a fixed point A on the given curve be s and $s + \Delta s$, so that $\overparen{PQ}$ (the arcual length of PQ) is Δs. [Refer to Fig. 3.2]

Let the tangents at P and Q to the curve make angles ψ and $\psi + \Delta\psi$ with a fixed line in the plane of the curve, say, the x-axis.
Then the angle between the tangents at P and $Q = \Delta\psi$.

Thus for a change of Δs in the arcual length of the curve, the direction of the tangent to the curve changes by $\Delta\psi$.

Hence $\frac{\Delta\psi}{\Delta s}$ is the average rate of bending of the curve (or average rate of change of direction of the tangent to the curve in the arcual interval $\overset{\frown}{PQ}$) or average curvature of the arc PQ.

$\therefore \lim\limits_{\Delta s \to 0}\left(\frac{\Delta\psi}{\Delta s}\right) = \frac{d\psi}{ds}$ is the rate of bending of the curve with respect to arcual distance at P or *the curvature* of the curve at the point P. The curvature is denoted by k.

For example, let us find the curvature of a circle of radius at any point on it. [Refer to Fig. 3.3]

Let the arcual distances of points on the circle be measured from A, the lowest point of the circle and let the tangent at A be chosen as the x-axis. Let $AP = s$ and let the tangent at P make an angle ψ with the x-axis.

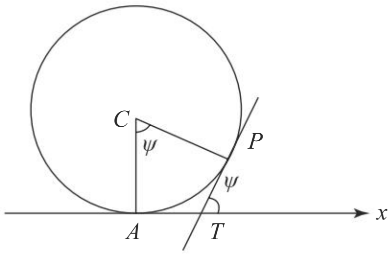

Fig. 3.3

Then $s = a\,A\hat{C}P$

$= a\,\psi$ [$\because$ the angle between CA and CP equals the angle between the respective perpendiculars AT and PT.]

or $\psi = \frac{1}{a}s$

$$\therefore \frac{d\psi}{ds} = \frac{1}{a}$$

Thus the curvature of a circle at any point on it equals the reciprocal of its radius. Equivalently, the radius of a circle equals the reciprocal of the curvature at any point on it. It is this property of the circle that has led to the definition of radius of curvature.

Radius of curvature of a curve at any point on it is defined as the reciprocal of the curvature of the curve at that point and denoted by ρ. Thus $\rho = \frac{1}{k} = \frac{ds}{d\psi}$.

Note ☑ To find k or ρ of a curve at any point on it, we should know the relation between s and ψ for that curve, which is not easily derivable in most cases.

Generally curves will be defined by means of their cartesian, parametric or polar equations. Hence formulas for ρ in terms of cartesian, parametric or polar co-ordinates are necessary, which are derived below:

Some Basic Results: Let $P(x, y)$ and $Q(x + \Delta x, y + \Delta y)$ be any two close points on a curve $y = f(x)$. [Refer to Fig. 3.4.] Let $\overset{\frown}{AP} = s$ and $\overset{\frown}{AQ} = s + \Delta s$ where A is a fixed point on the curve. Let the chord PQ make an angle θ with the x-axis.

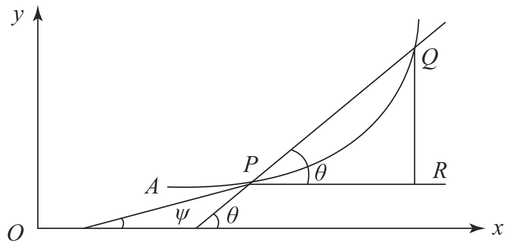

Fig. 3.4

From ΔPQR, $\sin\theta = \dfrac{RQ}{PQ} = \dfrac{RQ}{\Delta s}\cdot\dfrac{\Delta s}{PQ}$, where $\overparen{PQ} = \Delta s$ (1)

$$= \frac{\Delta y}{\Delta s}\cdot\frac{\Delta s}{PQ}$$

and $$\cos\theta = \frac{PR}{PQ} = \frac{PR}{\Delta s}\cdot\frac{\Delta s}{PQ}$$

$$= \frac{\Delta x}{\Delta s}\cdot\frac{\Delta s}{PQ} \quad (2)$$

When Q approaches P, chord $PQ \to$ tangent at P and hence $\theta \to \psi$. Also $\dfrac{\Delta s}{PQ} \to 1.$

Thus in the limiting case when $Q \to P$, (1) and (2) become $\sin\psi = \dfrac{dy}{ds}$ and $\cos\psi = \dfrac{dx}{ds}$.

$\therefore$ $$\tan\psi = \frac{dy}{dx}.$$

3.1.2 Formula for Radius of Curvature in Cartesian Co-ordinates

Let ψ be the angle made by the tangent at any point (x, y) on the curve $y = f(x)$.

Then $$\tan\psi = \frac{dy}{dx} \quad (1)$$

Differentiating both sides of (1) w.r.t. x, we get,

$$\sec^2\psi\,\frac{d\psi}{dx} = \frac{d^2y}{dx^2}$$

i.e. $$\sec^2\psi\,\frac{d\psi}{ds}\cdot\frac{ds}{dx} = \frac{d^2y}{dx^2}$$

i.e. $$\sec^2\psi\cdot\frac{1}{\rho}\cdot\sec\psi = \frac{d^2y}{dx^2} \qquad \left[\because \cos\psi = \frac{dx}{ds}\right]$$

$\therefore$ $$\rho = \frac{\sec^3\psi}{\dfrac{d^2y}{dx^2}}$$

$$= \frac{(1+\tan^2\psi)^{3/2}}{\dfrac{d^2y}{dx^2}} = \frac{\left\{1+\left(\dfrac{dy}{dx}\right)^2\right\}^{3/2}}{\dfrac{d^2y}{dx^2}}, \text{ by (1).}$$

Note ☑ As curvature (and hence radius of curvature) of a curve at any point is independent of the choice of x and y-axis, x and y can be interchanged in the formula for ρ derived above. Thus ρ is also given by

$$\rho = \frac{\left[1+\left(\dfrac{dx}{dy}\right)^2\right]^{3/2}}{\dfrac{d^2x}{dy^2}}$$

This formula will be of use, when $\dfrac{dy}{dx}$ is infinite at a point.

3.1.3 Formula for Radius of Curvature in Parametric Co-ordinates

Let the parametric equations of the curve be

$$x = f(t) \quad \text{and} \quad y = g(t).$$

Then $\quad \dot{x} = \dfrac{dx}{dt} = f'(t) \quad$ and $\quad \dot{y} = \dfrac{dy}{dt} = g'(t).$

$\therefore$

$$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{\dot{y}}{\dot{x}}\right) \times \frac{dt}{dx}$$

$$= \left(\frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{\dot{x}^2}\right) \cdot \frac{1}{\dot{x}} = \frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{\dot{x}^3}$$

Now

$$\rho = \frac{\left\{1+\left(\dfrac{dy}{dx}\right)^2\right\}^{3/2}}{\left(\dfrac{d^2y}{dx^2}\right)}$$

$$= \frac{\left\{1+\left(\dfrac{\dot{y}}{\dot{x}}\right)^2\right\}^{3/2}}{\left(\dfrac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{\dot{x}^3}\right)}$$

$$= \frac{(\dot{x}^2 + \dot{y}^2)^{3/2}}{\dot{x}^3} \times \frac{\dot{x}^3}{\dot{x}\ddot{y} - \dot{y}\ddot{x}}$$

$$= \frac{(\dot{x}^2 + \dot{y}^2)^{3/2}}{\dot{x}\ddot{y} - \dot{y}\ddot{x}}$$

3.1.4 Formula for Radius of Curvature in Polar Co-ordinates

Students are familiar with the following transformations from cartesian co-ordinates (x, y) to polar co-ordinates (r, θ):

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta \tag{1}$$

We shall make use of (1) and the formula for the radius of curvature in cartesian co-ordinates, namely,

$$\rho = \frac{\left\{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2\right\}^{3/2}}{\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}} \tag{2}$$

and derive the corresponding formula for ρ at the point (r, θ) which lies on the curve $r = f(\theta)$

Now $$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y/\mathrm{d}\theta}{\mathrm{d}x/\mathrm{d}\theta} = \frac{r\cos\theta + r'\sin\theta}{-r\sin\theta + r'\cos\theta}$$

where $r' = \frac{\mathrm{d}r}{\mathrm{d}\theta}$ (3)

[$\because$ r is a function of θ]

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \frac{\mathrm{d}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)}{\mathrm{d}x} = \frac{\mathrm{d}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)}{\mathrm{d}\theta} \div \frac{\mathrm{d}x}{\mathrm{d}\theta}$$

$$= \frac{[(-r\sin\theta + r'\cos\theta)(-r\sin\theta + 2r'\cos\theta + r''\sin\theta) - (r\cos\theta + r'\sin\theta)(-r\cos\theta - 2r'\sin\theta + r''\cos\theta)]}{(-r\sin\theta + r'\cos\theta)^3}$$

$$= \frac{[(r^2\sin^2\theta - 3rr'\sin\theta\cos\theta - rr''\sin^2\theta + 2r'^2\cos^2\theta + r'r''\sin\theta\cos\theta) + (r^2\cos^2\theta + 3rr'\sin\theta\cos\theta - rr''\cos^2\theta + 2r'^2\sin^2\theta - r'r''\sin\theta\cos\theta)]}{(-r\sin\theta + r'\cos\theta)^3}$$

$$= \frac{r^2 - rr'' + 2r'^2}{(-r\sin\theta + r'\cos\theta)^3}, \text{where } r'' = \frac{\mathrm{d}^2 r}{\mathrm{d}\theta^2} \tag{4}$$

Also $$1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 = 1 + \left(\frac{r\cos\theta + r'\sin\theta}{-r\sin\theta + r'\cos\theta}\right)^2$$

$$= \frac{r^2 + r'^2}{(-r\sin\theta + r'\cos\theta)^2} \tag{5}$$

Using (4) and (5) in (2), we get

$$\rho = \frac{(r^2 + r'^{\,2})^{3/2}}{r^2 - rr'' + 2r'^{\,2}}$$

3.2 CENTRE AND CIRCLE OF CURVATURE

Let $P(x, y)$ be a point on the curve $y = f(x)$. On the inward drawn normal to the curve at P, cut off a length PC = radius of curvature of the curve at P (namely ρ). The point C is called *the centre of curvature* at P for the curve. [Refer to Fig. 3.5]

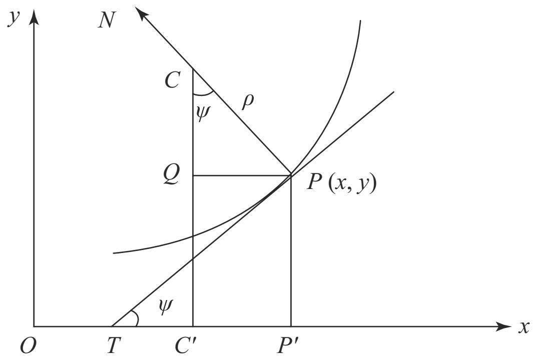

Fig. 3.5

The circle whose centre is C and radius ρ is called *the circle of curvature* at P for the curve.

Let $(\bar{x}, \bar{y})$ be the co-ordinates of C.

Then $\bar{x} = OC'$

$= OP' - QP$

$= x - \rho \sin \psi$ ($\because$ angle between CP and CQ = angle between the respective perpendiculars PT and OP')

$$= x - \frac{\rho}{\operatorname{cosec} \psi}$$

$$= x - \frac{\rho}{\sqrt{1 + \cot^2 \psi}}$$

$$= x - \frac{\rho y'}{\sqrt{1 + y'^2}} \qquad \left(\because \cot \psi = \frac{dx}{dy} = \frac{1}{y'}\right)$$

$$= x - \frac{(1 + y'^{\,2})^{3/2}}{y''} \cdot \frac{y'}{\sqrt{1 + y'^{\,2}}} \qquad \left(\text{where } y' = \frac{dy}{dx} \text{ and } y'' = \frac{d^2 y}{dx^2}\right)$$

i.e. $$\bar{x} = x - \frac{y'}{y''}(1 + y'^2)$$

Now $\bar{y} = C'C$

$= P'P + QC$

$$= y + \rho \cos \psi$$

$$= y + \frac{\rho}{\sec \psi}$$

$$= y + \frac{\rho}{\sqrt{1 + \tan^2 \psi}}$$

$$= y + \frac{(1 + y'^2)^{\frac{3}{2}}}{y''} \cdot \frac{1}{\sqrt{1 + y'^2}}$$

$$= y + \frac{(1 + y'^2)}{y''}$$

Having found out the co-ordinates of the centre of curvature, the equation of the circle of curvature is written as $(x - \bar{x})^2 + (y - \bar{y})^2 = \rho^2$.

WORKED EXAMPLE 3(a)

Example 3.1 Find the radius of curvature at the point $\left(\frac{3a}{2}, \frac{3a}{2}\right)$ on the curve $x^3 + y^3 = 3axy$.

Differentiating the equation of the curve with respect to x,

$$3\left(x^2 + y^2 \frac{dy}{dx}\right) = 3a\left(x\frac{dy}{dx} + y\right)$$

i.e. $$(y^2 - ax)\frac{dy}{dx} = ay - x^2$$

$\therefore$ $$\frac{dy}{dx} = \frac{ay - x^2}{y^2 - ax} \tag{1}$$

Again differentiating with respect to x,

$$\frac{d^2y}{dx^2} = \frac{(y^2 - ax)\left(a\frac{dy}{dx} - 2x\right) - (ay - x^2)\left(2y\frac{dy}{dx} - a\right)}{(y^2 - ax)^2} \tag{2}$$

$\therefore$ $$\left(\frac{dy}{dx}\right)_{\left(\frac{3a}{2}, \frac{3a}{2}\right)} = \frac{\frac{3a^2}{2} - \frac{9a^2}{4}}{\frac{9a^2}{4} - \frac{3a^2}{2}} = -1 \quad \text{and}$$

$$\left(\frac{d^2y}{dx^2}\right)_{\left(\frac{3a}{2},\frac{3a}{2}\right)} = \frac{\frac{3}{4}a^2(-a-3a)-\left(-\frac{3a^2}{4}\right)(-3a-a)}{\left(\frac{9a^4}{16}\right)}$$

Note ☑ It is not necessary to express $\frac{dy}{dx}$ as a function of x and y from (1) and then evaluate $\frac{d^2y}{dx^2}$. When $x=\frac{3a}{2}$ and $y=\frac{3a}{2}$, $\frac{dy}{dx}=-1$, which may be used in (2).

i.e.
$$\left(\frac{d^2y}{dx^2}\right)_{\left(\frac{3a}{2},\frac{3a}{2}\right)} = -\frac{6a^3}{9a^4}\times 16 = -\frac{32}{3a}$$

$$\rho = \frac{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{3/2}}{\frac{d^2y}{dx^2}}$$

$\therefore$
$$(\rho)_{\left(\frac{3a}{2},\frac{3a}{2}\right)} = \frac{(1+1)^{3/2}}{\left(-\frac{32}{3a}\right)}$$

$\therefore$
$$|\rho| = \frac{3\sqrt{2}a}{16}.$$

Example 3.2 Find the radius of curvature at $(a, 0)$ on the curve $xy^2 = a^3 - x^3$.

The equation of the curve is

$$y^2 = \frac{a^3-x^3}{x} \tag{1}$$

Differentiating w.r.t. x,

$$2yy' = \frac{x(-3x^2)-(a^3-x^3)}{x^2} = \frac{-(2x^3+a^3)}{x^2}$$

i.e.
$$y' = -\frac{(2x^3+a^3)}{2x^2y} \tag{2}$$

Now $(y')_{(a,0)} = \infty$.

$\therefore$ The formula $\rho = \dfrac{\left\{1+\left(\frac{dx}{dy}\right)^2\right\}^{3/2}}{\left(\frac{d^2x}{dy^2}\right)}$ should be used.

From (2), $$\frac{dx}{dy} = -\frac{2x^2y}{2x^3+a^3} \tag{3}$$

Differentiating (3) w.r.t. y,

$$\frac{d^2x}{dy^2} = -2\left[\frac{\left(2x^3+a^3\right)\left(x^2+y\cdot 2x\frac{dx}{dy}\right) - x^2\,y\cdot 6x^2\frac{dx}{dy}}{\left(2x^3+a^3\right)^2}\right] \tag{4}$$

From (3), we get $\left(\dfrac{dx}{dy}\right)_{(a,0)} = 0$

From(4), we get $\left(\dfrac{d^2x}{dy^2}\right)_{(a,0)} = -\dfrac{2\times 3a^5}{9a^6} = -\dfrac{2}{3a}$

$$\therefore \qquad |\rho| = \frac{(1+0)^{3/2}}{2/3a} = \frac{3a}{2}$$

Example 3.3 If ρ is the radius of curvature at any point $(x,\ y)$ on the curve $y = \dfrac{ax}{a+x}$, show that $\left(\dfrac{2\rho}{a}\right)^{2/3} = \left(\dfrac{x}{y}\right)^2 + \left(\dfrac{y}{x}\right)^2$.

$$y = \frac{ax}{a+x} \tag{1}$$

Differentiating w.r.t. x,

$$y' = \frac{a\,(a+x-x)}{(a+x)^2} = \frac{a^2}{(a+x)^2} \tag{2}$$

Differentiating again w.r.t. x,

$$y'' = \frac{-2a^2}{(a+x)^3} \tag{3}$$

$$\rho = \frac{\left(1+y'^2\right)^{3/2}}{|y''|}$$ (only the numerical value of ρ is considered)

$$= \frac{\left\{1+\dfrac{a^4}{(a+x)^4}\right\}^{3/2}(a+x)^3}{2a^2}$$

$$\therefore \qquad \frac{2\rho}{a} = \frac{\left\{(a+x)^4+a^4\right\}^{3/2}}{a^3\,(a+x)^3}$$

$$\therefore \qquad \left(\frac{2\rho}{a}\right)^{2/3} = \frac{(a+x)^4+a^4}{a^2\,(a+x)^2} = \left(\frac{a+x}{a}\right)^2 + \left(\frac{a}{a+x}\right)^2$$

$$=\left(\frac{x}{y}\right)^2+\left(\frac{y}{x}\right)^2 \quad \left[\because \text{ the point } (x, y) \text{ lies on the curve } \frac{y}{x}=\frac{a}{a+x}\right]$$

Example 3.4 Show that the measure of curvature of the curve $\sqrt{\frac{x}{a}}+\sqrt{\frac{y}{b}}=1$ at any point (x, y) on it is $\dfrac{ab}{2(ax+by)^{\frac{3}{2}}}$.

The equation of the curve is

$$\frac{1}{\sqrt{a}}\sqrt{x}+\frac{1}{\sqrt{b}}\sqrt{y}=1 \tag{1}$$

Differentiating w.r.t. x,

$$\frac{1}{2\sqrt{a}\sqrt{x}}+\frac{1}{2\sqrt{b}\sqrt{y}}y'=0$$

$\therefore$

$$y'=-\frac{\sqrt{b}\sqrt{y}}{\sqrt{a}\sqrt{x}} \tag{2}$$

Differentiating further w.r.t. x,

$$y''=-\frac{\sqrt{b}}{\sqrt{a}}\left\{\frac{\sqrt{x}\cdot\frac{1}{2\sqrt{y}}y'-\sqrt{y}\cdot\frac{1}{2\sqrt{x}}}{x}\right\}$$

$$=-\frac{\sqrt{b}}{\sqrt{a}}\cdot\frac{1}{x}\left\{-\frac{\sqrt{b}}{2\sqrt{a}}-\frac{\sqrt{y}}{2\sqrt{x}}\right\}, \text{ using } (2)$$

$$=\frac{\sqrt{b}}{2ax^{\frac{3}{2}}}\left\{\sqrt{bx}+\sqrt{ay}\right\}$$

$$=\frac{\sqrt{b}}{2ax^{\frac{3}{2}}}\times\sqrt{ab}, \text{ using } (1)$$

$$=\frac{b}{2\sqrt{a}\,x^{\frac{3}{2}}}$$

Now,

$$\rho=\frac{\left(1+y'^2\right)^{\frac{3}{2}}}{y''}=\frac{\left(1+\frac{by}{ax}\right)^{\frac{3}{2}}}{b}\times 2\sqrt{a}\,x^{\frac{3}{2}}$$

$$=\frac{2}{ab}(ax+by)^{\frac{3}{2}}$$

$\therefore$

$$\text{Curvature } k=\frac{1}{\rho}=\frac{ab}{2(ax+by)^{3/2}}$$

Example 3.5 Find the co-ordinates of the real points on the curve $y^2 = 2x(3 - x^2)$, the tangents at which are parallel to the x-axis. Show that the radius of curvature at each of these point is $\frac{1}{3}$.

$$y^2 = 2x(3 - x^2) \tag{1}$$

Differentiating w.r.t. x,

$$2yy' = 2[3 - 3x^2]$$

i.e.
$$yy' = 3(1 - x^2) \tag{2}$$

The points at which the tangents are parallel to the x-axis are given by $y' = 0$.

i.e. $\quad 3(1 - x^2) = 0$, from (2)

i.e. $\quad x = \pm 1.$

Putting $x = -1$ in (1), we get y^2 = negative

i.e. y is imaginary.

$\therefore$ The real points are given by $x = 1$.

Putting $x = 1$ in (1), we get $y^2 = 4$. $\therefore\ y = \pm 2$.

$\therefore$ The points, the tangents at which are parallel to the x-axis, are (1, 2) and (1, − 2).

From(2),
$$y' = \frac{3(1 - x^2)}{y}$$

$$= \frac{3(1 - x^2)}{\sqrt{2}\sqrt{3x - x^3}}$$

Differentiating w.r.t. x,

$$y'' = \frac{3}{\sqrt{2}}\left\{\frac{\sqrt{3x - x^3}\,(-2x) - (1 - x^2)\frac{d}{dx}\sqrt{3x - x^3}}{3x - x^3}\right\}$$

$$\therefore \qquad (y'')_{(1,\pm 2)} = \frac{3}{\sqrt{2}}\cdot\frac{(-2\sqrt{2})}{2} = -3$$

$$\rho = \frac{(1 + y'^2)^{3/2}}{|y''|}$$

$$\therefore \qquad (\rho)_{(1,\pm 2)} = \frac{(1 + 0)^{3/2}}{|-3|} = \frac{1}{3}$$

Example 3.6 Show that the curves $y = c\cosh\frac{x}{c}$ and $x^2 = 2c(y - c)$ have the same curvature at the points where they cross the y-axis.

The point at which the curve $y = c\cosh\frac{x}{c}$ crosses the y-axis is got by solving the equation of the curve with $x = 0$.

Thus the point is $(0, c)$.

Similarly the point of intersection of the second curve with the y-axis is also found to be $(0, c)$

Equation of the first curve is $y = c \cosh \dfrac{x}{c}$.

Differentiating w.r.t. x twice, we get

$$y' = c \sinh \frac{x}{c} . \frac{1}{c} = \sinh \frac{x}{c}$$

$$y'' = \frac{1}{c} \cosh \frac{x}{c}$$

$$\rho = \frac{(1+y'^2)^{3/2}}{y''} = \frac{\left(1+\sinh^2 \dfrac{x}{c}\right)^{3/2}}{\dfrac{1}{c} \cosh \dfrac{x}{c}}$$

$$\therefore \qquad (\rho)_{(0,c)} = c.$$

Equation of the second curve is $y = \dfrac{x^2}{2c} + c$

Differentiating w.r.t. x twice, we get

$$y' = \frac{x}{c} \quad \text{and} \quad y'' = \frac{1}{c}$$

$$\therefore \qquad (\rho)_{(0,c)} = \left[\frac{\left(1+\dfrac{x^2}{c^2}\right)^{3/2}}{\dfrac{1}{c}}\right]_{x=0} = c$$

Thus $(\rho)_{(0,c)}$ is the same for both curves.

$\therefore$ $(k)_{(0,c)}$ is the same $\left(=\dfrac{1}{c}\right)$ for both curves.

Example 3.7 Find the radius of curvature at the point $(a \cos^3 \theta, a \sin^3 \theta)$ on the curve $x^{2/3} + y^{2/3} = a^{2/3}$.

The parametric equations of the given curve are $x = a \cos^3 \theta$ and $y = a \sin^3 \theta$. Differentiating twice w.r.t. θ,

$$\dot{x} = \frac{dx}{d\theta} = 3a \cos^2 \theta(-\sin \theta); \dot{y} = \frac{dy}{d\theta} = 3a \sin^2 \theta(\cos \theta)$$

$$\ddot{x} = \frac{d^2x}{d\theta^2} = -3a\left(\cos^3\theta - 2 \cos \theta \sin^2 \theta\right)$$

$$\ddot{y} = \frac{d^2y}{d\theta^2} = 3a\left(2 \sin \theta \cos^2 \theta - \sin^3\theta\right)$$

$$\rho = \frac{(\dot{x}^2+\dot{y}^2)^{3/2}}{\dot{x}\ddot{y}-\dot{y}\ddot{x}} = \frac{(9a^2\cos^4\theta\sin^2\theta + 9a^2\sin^4\theta\cos^2\theta)^{3/2}}{9a^2\left\{-\cos^2\theta\sin\theta\left(2\sin\theta\cos^2\theta - \sin^3\theta\right) + \sin^2\theta\cos\theta\left(\cos^3\theta - 2\cos\theta\sin^2\theta\right)\right\}}$$

$$= \frac{27a^3\sin^3\theta\cos^3\theta\left(\cos^2\theta+\sin^2\theta\right)^{3/2}}{9a^2\sin^2\theta\cos^2\theta\left\{-\left(2\cos^2\theta-\sin^2\theta\right)+\left(\cos^2\theta-2\sin^2\theta\right)\right\}}$$

$$= \frac{3a\sin\theta\cos\theta}{-\left(\cos^2\theta+\sin^2\theta\right)} = -3a\sin\theta\cos\theta$$

$\therefore$ $$|\rho| = 3a\sin\theta\cos\theta\,.$$

Example 3.8 Show that the radius of curvature at the point 'θ' on the curve $x = 3a\cos\theta - a\cos 3\theta$, $y = 3a\sin\theta - a\sin 3\theta$ is $3a\sin\theta$.

$$x = 3a\cos\theta - a\cos 3\theta;\ y = 3a\sin\theta - a\sin 3\theta.$$

Differentiating w.r.t. θ,

$$\dot{x} = 3a\sin 3\theta - 3a\sin\theta;\ \dot{y} = 3a\cos\theta - 3a\cos 3\theta;$$

Now, $$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{3a(\cos\theta - \cos 3\theta)}{3a(\sin 3\theta - \sin\theta)}$$

$$= \frac{2\sin 2\theta\cdot\sin\theta}{2\cos 2\theta\cdot\sin\theta} = \tan 2\theta$$

$$\frac{d^2y}{dx^2} = \frac{d}{d\theta}\left(\tan 2\theta\right)\cdot\frac{d\theta}{dx}$$

$$= 2\sec^2 2\theta\cdot\frac{1}{3a\,(\sin 3\theta - \sin\theta)}$$

$$= \frac{2\sec^2 2\theta}{6a\cos 2\theta\sin\theta}$$

$$= \frac{\sec^3 2\theta}{3a\sin\theta}$$

$$\rho = \frac{(1+y'^2)^{3/2}}{y''} = \frac{(1+\tan^2 2\theta)^{3/2}}{\sec^3 2\theta}\cdot 3a\sin\theta$$

$$= 3a\sin\theta.$$

Example 3.9 Find the radius of curvature of the curve $r = a\,(1+\cos\theta)$ at the point $\theta = \frac{\pi}{2}\cdot$

$$r = a\,(1+\cos\theta)$$

$\therefore$ $$r' = -a\sin\theta \quad\text{and}\quad r'' = -a\cos\theta$$

$$\rho = \frac{(r^2+r'^2)^{3/2}}{r^2 - r'' + 2r'^2}$$

$$= \frac{\left[a^2(1+\cos\theta)^2 + a^2 \sin^2\theta\right]^{3/2}}{a^2(1+\cos\theta)^2 + a^2\cos\theta(1+\cos\theta) + 2a^2\sin^2\theta}$$

$$= \frac{a^3\left[2(1+\cos\theta)\right]^{3/2}}{a^2\left[3(1+\cos\theta)\right]}$$

$$= \frac{2\sqrt{2}}{3}a(1+\cos\theta)^{1/2} = \frac{4a}{3}\cos\frac{\theta}{2}$$

$$\therefore \qquad (\rho)_{\theta=\frac{\pi}{2}} = \frac{4a}{3}\cos\frac{\pi}{4} = \frac{2\sqrt{2}}{3}a.$$

Example 3.10 Show that the radius of curvature of the curve $r^n = a^n \sin n\theta$ at the pole is $\dfrac{a^n}{(n+1)r^{n-1}}$.

$$r^n = a^n \sin n\,\theta \tag{1}$$

$$\therefore \qquad n\log r = n\log a + \log\sin n\,\theta.$$

Differentiating w.r.t. θ,

$$\frac{n}{r}r' = n\cot n\theta$$

$$\therefore \qquad r' = r\cot n\theta \tag{2}$$

$$r'' = r'\cot n\theta - nr\,\mathrm{cosec}^2\, n\theta.$$

$$= r\cot^2 n\theta - nr\,\mathrm{cosec}^2\, n\theta. \tag{3}$$

$$(\rho) = \frac{(r^2 + r'^2)^{3/2}}{r^2 - rr'' + 2r'^2}$$

$$= \frac{\left(r^2 + r^2\cot^2 n\theta\right)^{3/2}}{r^2 - r^2\cot^2 n\theta + nr^2\,\mathrm{cosec}^2\, n\theta + 2r^2\cot^2 n\theta}, \text{ using (2) and (3)}$$

$$= \frac{\left(r^2\,\mathrm{cosec}^2 n\theta\right)^{3/2}}{r^2\,\mathrm{cosec}^2 n\theta\,(n+1)}$$

$$= \frac{r\,\mathrm{cosec}\, n\theta}{n+1}$$

$$= \frac{a^n}{(n+1)r^{n-1}}$$

Example 3.11 Find the radius of curvature at the point (r, θ) on the curve $r^2\cos 2\theta = a^2$.

$$r^2 = a^2\sec 2\theta \tag{1}$$

$$\therefore \qquad 2\log r = 2\log a + \log\sec 2\,\theta.$$

Differentiating w.r.t. θ ,

$$\frac{2}{r}r' = 2\tan 2\theta.$$

i.e. $\quad r' = r\tan 2\theta \qquad (2)$

$\therefore \quad r'' = r'\tan 2\theta + 2r\sec^2 2\theta$

$\quad = r\tan^2 2\theta + 2r\sec^2 2\theta$, using (2).

$$\rho = \frac{(r^2 + r'^2)^{3/2}}{r^2 - rr'' + 2r'^2}$$

$$= \frac{(r^2 + r^2\tan^2 2\theta)^{3/2}}{r^2 - r^2\tan^2 2\theta - 2r^2\sec^2 2\theta + 2r^2\tan^2 2\theta}, \text{ using (1) and (2),}$$

$$= \frac{r^3\sec^3 2\theta}{-r^2\sec^2 2\theta}$$

$$\therefore \quad |\rho| = r\sec 2\theta = \frac{r^3}{a^2}$$

Example 3.12 Show that at the points of intersection of the curves $r = a\theta$ and $r = \dfrac{a}{\theta}$, their curvatures are in the ratio 3:1.

$$r = a\theta \qquad (1)$$

and $$r = \frac{a}{\theta} \qquad (2)$$

Solving (1) and (2), $a\theta = \dfrac{a}{\theta}$

i.e. $\quad \theta = \pm 1.$

$\therefore$ The points of intersection of the two curves are given by $\theta = \pm 1$.

For curve (1), $r' = a$ and $r'' = 0$.

$$\therefore \quad \rho_1 = \frac{(a^2\theta^2 + a^2)^{3/2}}{a^2\theta^2 + 2a^2}$$

$$\therefore \quad (\rho_1)_{\theta=\pm 1} = \frac{(2a^2)^{3/2}}{3a^2} = \frac{2\sqrt{2}}{3}a.$$

For curve (2), $r' = -\dfrac{a}{\theta^2}$ and $r'' = \dfrac{2a}{\theta^3}$

$$\therefore \quad \rho_2 = \frac{\left(\dfrac{a^2}{\theta^2} + \dfrac{a^2}{\theta^4}\right)^{3/2}}{\dfrac{a^2}{\theta^2} - \dfrac{2a^2}{\theta^4} + 2\dfrac{a^2}{\theta^4}}$$

$$\therefore \qquad (\rho_2)_{\theta=\pm 1} = \frac{(2a^2)^{3/2}}{a^2} = 2\sqrt{2a}$$

$$\therefore \qquad \rho_1 : \rho_2 = 1 : 3$$

$\therefore$ Ratio of their curvatures = 3:1.

Example 3.13 If the centre of curvature of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at one end of the minor axis lies at the other end, prove that the eccentricity of the ellipse is $\frac{1}{\sqrt{2}}$.

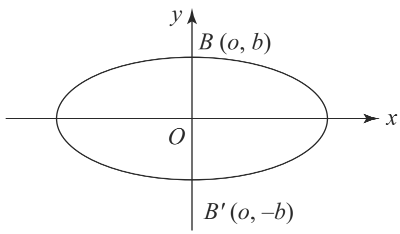

Fig. 3.6

The centre of curvature of the ellipse at $B(0, b)$ lies at B' $(0, -b)$. [Refer to Fig. 3.6]

We recall that if the centre of curvature of any curve at a point P is C, then PC equals the radius of curvature of the curve at P.

$\therefore$ Radius of curvature of the ellipse at

$$B = BB' = 2b. \tag{1}$$

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

Differentiating w.r.t. x,

$$\frac{x}{a^2}+\frac{y}{b^2}\cdot y' = 0$$

$$\therefore \qquad y' = -\frac{b^2x}{a^2y} \tag{2}$$

Differentiating again w.r.t. x,

$$y'' = -\frac{b^2}{a^2}\left(\frac{y-xy'}{y^2}\right) \tag{3}$$

From (2) and (3), we get

$$(y')_{(o,b)} = 0 \quad \text{and} \quad (y'')_{(o,b)} = -\frac{b}{a^2}$$

Now,
$$\rho = \frac{\left(1+y'^2\right)^{\frac{3}{2}}}{|y''|}$$

$$\therefore \qquad (\rho)_{(0,b)} = \frac{1}{\left|-\frac{b}{a^2}\right|} = \frac{a^2}{b}$$

From (1)
$$\frac{a^2}{b} = 2b \text{ i.e. } a^2 = 2b^2 \tag{4}$$

The eccentricity e of the ellipse is given by

$$b^2 = a^2(1 - e^2) \quad \text{or} \quad e^2 = \frac{a^2 - b^2}{a^2}$$

Using (4), we get, $e^2 = \dfrac{b^2}{2b^2} = \dfrac{1}{2}$ $\qquad \therefore\ e = \dfrac{1}{\sqrt{2}}$

Example 3.14 Find the centre of curvature at $\theta = \dfrac{\pi}{2}$ on the curve

$$x = 2\cos t + \cos 2t,\ y = 2\sin t + \sin 2t.$$

$$x = 2\cos t + \cos 2t; \quad y = 2\sin t + \sin 2t$$

Differentiating w.r.t. t,

$$\dot{x} = -\left(2\sin t + 2\sin 2t\right);\ \dot{y} = \left(2\cos t + 2\cos 2t\right)$$

$$\therefore \qquad y' = \frac{-2\left(\cos 2t + \cos t\right)}{2\left(\sin 2t + \sin t\right)}$$

$$= -\frac{2\cos\dfrac{3t}{2}\cos\dfrac{t}{2}}{2\sin\dfrac{3t}{2}\cos\dfrac{t}{2}}$$

$$= -\cot\frac{3t}{2}$$

$$y'' = \frac{d}{dx}\left(y'\right) = \frac{d}{dt}\left(-\cot\frac{3t}{2}\right)\cdot\frac{dt}{dx}$$

$$= \frac{3}{2}\operatorname{cosec}^2\frac{3t}{2}\cdot\frac{1}{-2\left(\sin 2t + \sin t\right)}$$

$$= -\frac{3}{8\sin^3\left(\dfrac{t}{2}\right)\cdot\cos\left(\dfrac{t}{2}\right)}$$

$$\left(x\right)_{\theta=\frac{\pi}{2}} = -1;\ \left(y\right)_{\theta=\frac{\pi}{2}} = 2;\ \left(y'\right)_{\theta=\frac{\pi}{2}} = -\cot\frac{3\pi}{4} = 1;$$

$$\left(y''\right)_{\theta=\frac{\pi}{2}} = -\frac{3}{8\sin^3\dfrac{\pi}{4}\cos\dfrac{\pi}{4}} = -\frac{3}{2}$$

Now $\qquad \bar{x} = x - \dfrac{y'}{y''}\left(1 + y'^2\right)$

$$\therefore \qquad \left(\bar{x}\right)_{\theta=\frac{\pi}{2}} = -1 - \frac{(1+1)}{\left(-\dfrac{3}{2}\right)} = -1 + \frac{4}{3} = \frac{1}{3}$$

$$\bar{y} = y + \frac{1}{y''}\left(1 + y'^2\right)$$

$\therefore \qquad (\bar{y})_{\theta=\frac{\pi}{2}} = 2 + \dfrac{(1+1)}{\left(-\dfrac{3}{2}\right)} = 2 - \dfrac{4}{3} = \dfrac{2}{3}$

$\therefore$ Required centre of curvature is $\left(\dfrac{1}{3}, \dfrac{2}{3}\right)$.

Example 3.15 Find the equation of the circle of curvature of the parabola $y^2 = 12x$ at the point (3, 6).

$$y^2 = 12x$$

Differentiating w.r.t. x,

$$2yy' = 12 \qquad \therefore\ y' = \frac{6}{y}$$

Differentiating again w.r.t. x,

$$y'' = -\frac{6}{y^2}\, y'$$

$$(y')_{(3,6)} = 1 \quad \text{and} \quad (y'')_{(3,6)} = -\frac{1}{6}$$

$$\rho = \frac{(1+y'^2)^{3/2}}{|y''|} \qquad \therefore\ (\rho)_{(3,6)} = \frac{2\sqrt{2}}{\frac{1}{6}} = 12\sqrt{2}$$

$$\bar{x} = x - \frac{y'}{y''}\left(1 + y'^2\right)$$

$\therefore \qquad (\bar{x})_{(3,6)} = 3 - \dfrac{1}{\left(-\dfrac{1}{6}\right)}(1+1) = 15.$

$$\bar{y} = y + \frac{1}{y''}\left(1 + y'^2\right)$$

$$= 6 + \frac{1}{\left(-\frac{1}{6}\right)}(1+1) = -6$$

The equation of the circle of curvature is

$$(x - \bar{x})^2 + (y - \bar{y})^2 = \rho^2$$

$\therefore$ The equation of the circle of curvature at the point (3, 6) is

$$(x-15)^2 + (y+6)^2 = (12\sqrt{2})^2$$

i.e. $\qquad x^2 - 30x + 225 + y^2 + 12y + 36 = 288$

i.e. $\qquad x^2 + y^2 - 30x + 12y - 27 = 0.$

Example 3.16 Find the equation of the circle of curvature of the curve $\sqrt{x}+\sqrt{y}=\sqrt{a}$ at $\left(\frac{a}{4},\frac{a}{4}\right)$.

$$\sqrt{x}+\sqrt{y}=\sqrt{a}$$

Differentiating w.r.t. x, $\frac{1}{2\sqrt{x}}+\frac{1}{2\sqrt{y}}y'=0$

$$\therefore \qquad y'=-\frac{\sqrt{y}}{\sqrt{x}}$$

Differentiating again w.r.t. x,

$$y''=-\left\{\frac{\sqrt{x}\cdot\frac{1}{2\sqrt{y}}y'-\sqrt{y}\cdot\frac{1}{2\sqrt{x}}}{x}\right\}$$

$$\therefore \qquad (y')_{\left(\frac{a}{4},\frac{a}{4}\right)}=-1 \quad \text{and}$$

$$(y'')_{\left(\frac{a}{4},\frac{a}{4}\right)}=-\left\{\frac{-\frac{1}{2}-\frac{1}{2}}{\frac{a}{4}}\right\}=\frac{4}{a}$$

$$\rho=\frac{\left(1+y'^2\right)^{3/2}}{y''} \qquad \therefore \quad (\rho)_{\left(\frac{a}{4},\frac{a}{4}\right)}=\frac{2\sqrt{2}}{\left(\frac{4}{a}\right)}=\frac{a}{\sqrt{2}}$$

$$\overline{x}=x-\frac{y'}{y''}\left(1+y'^2\right)$$

$$\therefore \qquad (\overline{x})_{\left(\frac{a}{4},\frac{a}{4}\right)}=\frac{a}{4}+\frac{1}{\left(\frac{4}{a}\right)}(1+1)=\frac{3a}{4}$$

$$\overline{y}=y+\frac{1}{y''}\left(1+y'^2\right);(\overline{y})_{\left(\frac{a}{4},\frac{a}{4}\right)}=\frac{a}{4}+\frac{1}{\left(\frac{4}{a}\right)}(1+1)=\frac{3a}{4}$$

The equation of the circle of curvature is

$$(x-\overline{x})^2+(y-\overline{y})^2=\rho^2$$

$\therefore$ The equation of the circle of curvature at $\left(\frac{a}{4},\frac{a}{4}\right)$ is

$$\left(x-\frac{3a}{4}\right)^2+\left(y-\frac{3a}{4}\right)^2=\frac{a^2}{2}$$

EXERCISE 3(a)

Part A

(Short Answer Questions)

1. Define curvature and radius of curvature.
2. Prove that the radius of curvature of a circle is its radius.
3. Find the curvature of the curve given by $s = c \tan \psi$ at $\psi = 0$.

Find the radius of curvature of each of the following curves at the points indicated:

4. $y = e^x$ at $x = 0$.
5. $y = e^{\sqrt{3}x}$ at $x = 0$.
6. $y = \log \sec x$ at any point on it.
7. $y = \log \sin x$ at $x = \dfrac{\pi}{2}$.
8. $xy = c^2$ at (c, c).
9. $y^2 = 4ax$ at $y = 2a$.
10. $x = t^2, y = t$ at $t = 1$.
11. $r = a\theta$ at the pole.
12. $r\theta = a$ at any point on it.
13. $r = a \cos \theta$ at any point on it.
14. $r = e^\theta$ at any point on it.

Part B

Find the radius of curvature of the following curves at the points specified:

15. $x^3 + xy^2 - 6y^2 = 0$ at $(3, 3)$.
16. $4ay^2 = (2a - x)^3$ at $\left(a, \dfrac{a}{2}\right)$.
17. $x^3 + y^3 = (y - x)(y - 2x)$ at $(0, 0)$.
18. $xy^2 = a^2(a - x)$ at $(a, 0)$
19. $4ay^2 = 27(x - 2a)^3$ at $\left(\dfrac{7}{3}a, \dfrac{a}{2}\right)$
20. $y = x^2(x - 3)$ at the points where the tangent is parallel to the x-axis.
21. $y = c \cosh \dfrac{x}{c}$ at the point where it is minimum.
22. $x^2 = 4ay$ at the point where the slope of the tangent is $\tan \theta$.
23. $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ at $(a \cos \theta, b \sin \theta)$

Find the radius of curvature of the following curves at the points specified:

24. $x = a(\cos t + t \sin t), y = a(\sin t - t \cos t)$ at 't'.
25. $x = a(\theta - \sin \theta), y = a(1 - \cos \theta)$ at 'θ'.
26. $x = e^t \cos t, y = e^t \sin t$ at $(1, 0)$.

27. $x = a \log\left(\frac{\pi}{4} + \frac{\theta}{2}\right)$, $y = a \sec\theta$ at 'θ'.
28. $x = a \log\left(\cot\frac{\theta}{2} - \cos\theta\right)$, $y = a \sin\theta$ at 'θ'.
29. Find the radius of curvature at any point on the equiangular spiral $r = a\, e^{\theta \cot \alpha}$.
30. Find the radius of curvature of the curve $r = a(1 - \cos\theta)$ at any point on it.
31. Find the radius of curvature of the curve $r^n = a^n \cos n\theta$ at any point (r, θ). Hence prove that the radius of curvature of the lemniscate $r^2 = a^2 \cos 2\theta$ is $\frac{a^2}{3r}$.
32. Find the radius of curvature at any point (r, θ) on the curve $\sqrt{r}\cos\frac{\theta}{2} = \sqrt{a}$.
33. Find the radius of curvature at any point (r, θ) on the curve $r(1 + \cos\theta) = a$.
34. If ρ_1 and ρ_2 be the radii of curvature at the ends of any chord of the cardioid $r = a(1 + \cos\theta)$, that passes through the pole, prove that $\rho_1^2 + \rho_2^2 = \frac{16a^2}{9}$.

    [*Hint:* The ends of any chord that passes through the pole are given by $\theta = \theta_1$ and $\theta = \pi + \theta_1$. Use the result $(\rho)_\theta = \frac{4a}{3}\cos\frac{\theta}{2}$.]
35. Find the centre of curvature of the curve $y = x^3 - 6x^2 + 3x + 1$ at the point $(1, -1)$
36. Find the centre of curvature of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the point $(a\sec\theta, b\tan\theta)$.
37. Show that the line joining any point 't' on the cycloid $x = a(t + \sin t)$. $y = a(1 - \cos t)$ and its centre of curvature is bisected by the line $y = 2a$.
38. Find the equation of the circle of curvature of the parabola $y^2 = 4ax$ at the positive end of the latus rectum.
39. Find the equation of the circle of curvature of the rectangular hyperbola $xy = 12$ at the point $(3, 4)$.
40. Find the equation of the circle of curvature of the curve $x^3 + y^3 = 3axy$ at the point $\left(\frac{3a}{2}, \frac{3a}{2}\right)$.

3.3 EVOLUTES AND ENVELOPES

Let Q be the centre of curvature of a given curve C at the point P on it. When P moves on the curve C and takes different positions, Q will also take different positions and move on another curve C'. This curve C' is called *the evolute* of the curve C. Thus evolute can be defined as the locus of the centre of curvature.

When C' is the evolute of the curve C, C is called *the involute* of the curve C'.

The procedure to find the equation of the evolute of a given curve is given below:

Let the equation of the given curve be $y = f(x)$ (1)

If $(\bar{x}, \bar{y})$ is the centre of curvature corresponding to the point (x, y) on (1), then

$$\bar{x} = x - \frac{y'}{y''}\left(1 + y'^2\right) \tag{2}$$

$$\bar{y} = y + \frac{1}{y''}\left(1 + y'^2\right) \tag{3}$$

By eliminating x and y from (1), (2), (3), we get a relation between $\bar{x}$ and $\bar{y}$, which is the equation of the evolute.

Note ☑ If the parametric co-ordinates of any point on the given curve are assumed, then we have to eliminate the parameter from Equations (2) and (3), which will simplify the procedure.

Evolute of a given curve can also be defined in a different manner, using the concept of envelope of a family of curves, which is discussed below:

Consider the equation $f(x, y, c) = 0$, where c is a constant. If c takes a particular value, the equation represents a single curve. If c is an arbitrary constant or parameter which takes different values, then the equation $f(x, y, c) = 0$ represents a family of similar curves.

If we assign two consecutive values for c, we get two close curves of the family. The locus of the limiting positions of the points of intersection of consecutive members of a family of curves is called *the envelope* of the family.

It can be proved that the envelope of a family of curves touches every member of the family of curves.

3.3.1 Method of Finding the Equation of the Envelope of a Famly of Curves

Let $f(x, y, c) = 0$ be the equation of the given family of curves, where c is the parameter. Two consecutive members of the family (corresponding to two close values of c) are given by

$$f(x, y, c) = 0 \tag{1}$$

and

$$f(x, y, c + \Delta c) = 0 \tag{2}$$

The co-ordinates of the points of intersection of (1) and (2) will satisfy (1) and (2) and hence satisfy $\dfrac{f(x, y, c + \Delta c) - f(x, y, c)}{\Delta c} = 0$

Hence the co-ordinates of the limiting positions of the points of intersection of (1) and (2) will satisfy the equation

$$\lim_{\Delta c \to 0}\left\{\frac{f(x, y, c + \Delta c) - f(x, y, c)}{\Delta c}\right\} = 0$$

i.e.

$$\frac{\partial f}{\partial c}(x, y, c) = 0 \tag{3}$$

These limiting points will continue to lie on (1) and satisfy

$$f(x, y, c) = 0$$

If we eliminate c between (1) and (3), we get the equation of a curve, which is the locus of the limiting positions of the points of intersection of consecutive members of the given family, i.e. we get the equation of the envelope.

Thus the equation of the envelope of the family of curves $f(x, y, c) = 0$ (c is the parameter) is obtained by eliminating c between the equations

$$f(x, y, c) = 0 \quad \text{and} \quad \frac{\partial f}{\partial c}(x, y, c) = 0\cdot$$

Equation of the envelope of the family $A\alpha^2 + B\alpha + C = 0$, where α is the parameter and A, B, C are functions of x and y:

Very often the equation of the family of curves will be a quadratic equation in the parameter. In such cases, the equation of the envelope may be remembered as a formula.

Let the equation of the family of curves be

$$A\alpha^2 + B\alpha + C = 0 \tag{1}$$

Differentiating partially w.r.t. α,

$$2A\alpha + B = 0 \tag{2}$$

From (2), $\alpha = -\dfrac{B}{2A}$

Substituting this values of α in (1), we get the eliminant of α as

$$A\left(-\frac{B}{2A}\right)^2 + B\left(-\frac{B}{2A}\right) + C = 0$$

i.e.

$$\frac{B^2}{4A} - \frac{B^2}{2A} + C = 0$$

i.e. $B^2 - 4AC = 0$, which is the equation of the envelope of the family (1).

3.3.2 Evolute as the Envelope of Normals

The normals to a curve form a family of straight lines. The envelope of this family of normals is the locus of the limiting position of the point of intersection of consecutive normals. But the point of intersection of consecutive normals of a curve is the centre of curvature of the curve. Hence the locus of centre of curvature is the same as the envelope of normals.

Thus the evolute of a curve is the envelope of the normals of that curve.

WORKED EXAMPLE 3(b)

Example 3.1 Find the evolute of the parabola $x^2 = 4ay$.

The parametric co-ordinates of any point on the parabola $x^2 = 4ay$ are

$$x = 2\,at \quad \text{and} \quad y = a\,t^2$$

$$\dot{x} = 2a\,;\ \dot{y} = 2\ at \qquad \therefore\ \frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = t$$

$$\frac{d^2y}{dx^2} = \frac{d}{dt}(t) \times \frac{dt}{dx} = \frac{1}{2a}$$

Let $(\bar{x}, \bar{y})$ be the centre of curvature at the point 't'

$$\bar{x} = x - \frac{y'}{y''}(1 + y'^2)$$

$$= 2at - \frac{t}{\left(\frac{1}{2a}\right)}(1 + t^2)$$

$$= -2\,at^3 \tag{1}$$

$$\bar{y} = y + \frac{1}{y''}(1 + y'^2)$$

$$= at^2 + \frac{1}{\left(\frac{1}{2a}\right)}(1 + t^2) = 3at^2 + 2a \tag{2}$$

To get the relation between $\bar{x}$ and $\bar{y}$, we have to eliminate t from (1) and (2).

From (1), we get $t^3 = -\dfrac{\bar{x}}{2a}$ (3)

From (2), we get $t^2 = \dfrac{\bar{y} - 2a}{3a}$ (4)

From (3) and (4), we get

$$\left(-\frac{\bar{x}}{2a}\right)^2 = \left(\frac{\bar{y} - 2a}{3a}\right)^3$$

i.e.

$$27a\,\bar{x}^2 = 4(\bar{y} - 2a)^3.$$

$\therefore$ Locus of $(\bar{x}, \bar{y})$, i.e. the equation of the evolute is $27a\,x^2 = 4(y - 2a)^3$.

Example 3.2 Find the evolute of the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1.$

The parametric co-ordinates of any point on the hyperbola are

$$x = a\sec\theta \quad \text{and} \quad y = b\tan\theta$$

$$\therefore \quad \dot{x} = a\sec\theta\tan\theta; \quad \dot{y} = b\sec^2\theta$$

$$y' = \frac{\dot{y}}{\dot{x}} = \frac{b\sec\theta}{a\tan\theta} = \frac{b}{a\sin\theta}$$

$$\therefore \quad y'' = -\frac{b}{a\sin^2\theta}\cos\theta \cdot \frac{d\theta}{dx} = -\frac{b\cos\theta}{a\sin^2\theta} \cdot \frac{\cos^2\theta}{a\sin\theta}$$

$$= -\frac{b\cos^3\theta}{a^2\sin^3\theta}$$

$$\bar{x} = x - \frac{y'}{y''}\left(1 + y'^{\,2}\right)$$

$$= a\sec\theta + \frac{b}{a\sin\theta}\cdot\frac{a^2\sin^3\theta}{b\cos^3\theta}\left(1 + \frac{b^2}{a^2\sin^2\theta}\right)$$

$$= \frac{a}{\cos\theta} + \frac{1}{a\cos^3\theta}\left(a^2\sin^2\theta + b^2\right)$$

$$= \frac{a^2\cos^2\theta + a^2\sin^2\theta + b^2}{a\cos^3\theta}$$

$$= \frac{a^2 + b^2}{a\cos^3\theta} \qquad (1)$$

$$\bar{y} = y + \frac{1}{y''}\left(1 + y'^2\right)$$

$$= b\tan\theta - \frac{a^2\sin^3\theta}{b\cos^3\theta}\left(1 + \frac{b^2}{a^2\sin^2\theta}\right)$$

$$= \frac{b\sin\theta}{\cos\theta} - \frac{\sin\theta}{b\cos^3\theta}\left(a^2\sin^2\theta + b^2\right)$$

$$= \frac{\sin\theta}{b\cos^3\theta}\left(b^2\cos^2\theta - a^2\sin^2\theta - b^2\right)$$

$$= -\frac{\left(a^2 + b^2\right)}{b}\tan^3\theta \qquad (2)$$

From (1), $\sec^3\theta = \dfrac{a\bar{x}}{a^2 + b^2}$ and

From (2), $\tan^3\theta = -\dfrac{b\bar{y}}{a^2 + b^2}$

To eliminate θ, we use the identity $\sec^2\theta - \tan^2\theta = 1$

$$\therefore \quad \left(\frac{a\bar{x}}{a^2 + b^2}\right)^{2/3} - \left(\frac{-b\bar{y}}{a^2 + b^2}\right)^{2/3} = 1$$

$\therefore$ The locus of $\left(\bar{x}, \bar{y}\right)$ i.e. the evolute of the hyperbola is

$$(ax)^{2/3} - (by)^{2/3} = (a^2 + b^2)^{2/3} \qquad [\because (-by)^{2/3} = (by)^{2/3}]$$

Example 3.3 Find the evolute of the curve $x^{2/3} + y^{2/3} = a^{2/3}$.

The parametric co-ordinates of any point on the curve are $x = a\cos^3\theta$ and $y = a\sin^3\theta$

$\therefore \qquad \dot{x} = -3a\cos^2\theta\sin\theta; \qquad \dot{y} = 3a\sin^2\theta\cos\theta$

$\therefore \qquad y' = \dfrac{\dot{y}}{\dot{x}} = -\tan\theta$

$$y'' = -\sec^2\theta \cdot \frac{d\theta}{dx} = -\sec^2\theta\left(\frac{1}{-3a\cos^2\theta\sin\theta}\right)$$

$$= \frac{1}{3a\cos^4\theta\sin\theta}$$

$$\bar{x} = x - \frac{y'}{y''}\left(1 + y'^2\right)$$

$$= a\cos^3\theta + \frac{\tan\theta\left(1+\tan^2\theta\right)}{\left(\dfrac{1}{3a\cos^4\theta\sin\theta}\right)}$$

$$= a\cos^3\theta + \frac{\sin\theta}{\cos\theta}\cdot\frac{1}{\cos^2\theta}\cdot 3a\cos^4\theta\sin\theta$$

$$= a\cos^3\theta + 3a\cos\theta\sin^2\theta \tag{1}$$

$$\bar{y} = y + \frac{1}{y''}\left(1 + y'^2\right)$$

$$= a\sin^3\theta + \sec^2\theta\cdot 3a\cos^4\theta\sin\theta$$

$$= a\sin^3\theta + 3a\sin\theta\cos^2\theta \tag{2}$$

Now $\quad \bar{x} + \bar{y} = a\left(\cos^3\theta + \sin^3\theta + 3\cos^2\theta\sin\theta + 3\cos\theta\sin^2\theta\right)$

$$= a\left(\cos\theta + \sin\theta\right)^3 \tag{3}$$

and $\quad \bar{x} - \bar{y} = a\left(\cos^3\theta - 3\cos^2\theta\sin\theta + 3\cos\theta\sin^2\theta - \sin^3\theta\right)$

$$= a\left(\cos\theta - \sin\theta\right)^3 \tag{4}$$

Now $\quad \left(\dfrac{\bar{x}+\bar{y}}{a}\right)^{2/3} + \left(\dfrac{\bar{x}-\bar{y}}{a}\right)^{2/3} = \left(\cos\theta + \sin\theta\right)^2 + \left(\cos\theta - \sin\theta\right)^2$

$$= 2$$

i.e. $\quad \left(\bar{x}+\bar{y}\right)^{2/3} + \left(\bar{x}-\bar{y}\right)^{2/3} = 2a^{2/3}$

$\therefore$ The equation of the evolute is

$$\left(x+y\right)^{2/3} + \left(x-y\right)^{2/3} = 2a^{2/3}.$$

Example 3.4 Show that the evolute of the cycloid $x = a(\theta - \sin\theta)$, $y = a(1-\cos\theta)$ is another cycloid.

Any point on the cycloid is given by

$$x = a\,(\theta - \sin\theta) \quad \text{and} \quad y = a(1 - \cos\theta)$$

$\therefore \qquad \dot{x} = a\left(1 - \cos\theta\right); \qquad \dot{y} = a\sin\theta$

$$y' = \frac{a \sin \theta}{a\left(1-\cos \theta\right)} = \frac{2 \sin \frac{\theta}{2} \cdot \cos \frac{\theta}{2}}{2 \sin^2 \frac{\theta}{2}} = \cot \frac{\theta}{2}$$

$$y'' = -\frac{1}{2} \operatorname{cosec}^2 \frac{\theta}{2} \cdot \frac{d\theta}{dx} = -\frac{1}{2 \sin^2 \frac{\theta}{2}} \cdot \frac{1}{2 a \sin^2 \frac{\theta}{2}}$$

$$= -\frac{1}{4 a \sin^4 \frac{\theta}{2}}$$

$$\bar{x} = x - \frac{y'}{y''}\left(1+y'^2\right)$$

$$= a\left(\theta - \sin \theta\right) + \cot \frac{\theta}{2} \cdot \operatorname{cosec}^2 \frac{\theta}{2} \cdot 4a \sin^4 \frac{\theta}{2}$$

$$= a\left(\theta - \sin \theta\right) + 4a \sin \frac{\theta}{2} \cdot \cos \frac{\theta}{2}$$

$$= a\left(\theta - \sin \theta\right) + 2a \sin \theta$$

$$= a\left(\theta + \sin \theta\right) \tag{1}$$

$$\bar{y} = y + \frac{1}{y''}\left(1+y'^2\right)$$

$$= a\left(1-\cos \theta\right) - \operatorname{cosec}^2 \frac{\theta}{2} \cdot 4a \sin^4 \frac{\theta}{2}$$

$$= a\left(1-\cos \theta\right) - 2a\left(1-\cos \theta\right)$$

$$= -a\left(1-\cos \theta\right) \tag{2}$$

Elimination of θ from (1) and (2) is not easy.

$\therefore$ The locus of $\left(\bar{x}, \bar{y}\right)$ is given by the parametric equations $x = a\ (\theta + \sin \theta)$ and $y = -a\ (1 - \cos \theta)$, which represent another cycloid.

Example 3.5 Find the equation of the evolute of the curve $x = a\ (\cos t + t \sin t)$, $y = a\ (\sin t - t \cos t)$

$$x = a(\cos t + t \sin t);\ y = a\ (\sin t - t \cos t)$$

$$\therefore \quad \dot{x} = a\left(-\sin t + \sin t + t \cos t\right);\ \dot{y} = a\left(\cos t - \cos t + t \sin t\right)$$

$$\therefore \quad y' = \frac{at \sin t}{at \cos t} = \tan t$$

$$y'' = \sec^2 t \cdot \frac{dt}{dx} = \sec^2 t \cdot \frac{1}{at \cos t} = \frac{1}{at \cos^3 t}$$

$$\bar{x} = x - \frac{y'}{y''}\left(1+y'^2\right)$$

$$= a\left(\cos t + t \sin t\right) - \tan t \cdot \sec^2 t \cdot at \cos^3 t$$

$$= a\left(\cos t + t \sin t\right) - a\ t \sin t$$

$$= a \cos t \tag{1}$$

$$\bar{y} = y + \frac{1}{y''}(1 + y'^2)$$
$$= a(\sin t - t\cos t) + \sec^2 t \cdot at\cos^3 t$$
$$= a\sin t \qquad (2)$$

Eliminating t between (1) and (2), we get

$$\bar{x}^2 + \bar{y}^2 = a^2$$

$\therefore$ The evolute of the given curve is $x^2 + y^2 = a^2$.

Example 3.6 Prove that the evolute of the curve $x = a\left(\cos\theta + \log\tan\frac{\theta}{2}\right)$, $y = a\sin\theta$ is the catenary $y = a\cosh\frac{x}{a}$.

$$x = a\left(\cos\theta + \log\tan\frac{\theta}{2}\right);\ y = a\sin\theta.$$

$$\dot{x} = a\left(-\sin\theta + \frac{1}{\tan\frac{\theta}{2}}\cdot\sec^2\frac{\theta}{2}\cdot\frac{1}{2}\right)$$

$$= a\left(-\sin\theta + \frac{1}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}\right)$$

$$= a\left(-\sin\theta + \frac{1}{\sin\theta}\right) = \frac{a\cos^2\theta}{\sin\theta}$$

and $\qquad \dot{y} = a\cos\theta$

$\therefore \qquad y' = a\cos\theta\cdot\frac{\sin\theta}{a\cos^2\theta} = \tan\theta$

$$y'' = \sec^2\theta\cdot\frac{d\theta}{dx} = \sec^2\theta\cdot\frac{\sin\theta}{a\cos^2\theta} = \frac{\sin\theta}{a\cos^4\theta}$$

$$\bar{x} = x - \frac{y'}{y''}\left(1 + y'^2\right)$$
$$= a\left(\cos\theta + \log\tan\frac{\theta}{2}\right) - \tan\theta\cdot\sec^2\theta\cdot\frac{a\cos^4\theta}{\sin\theta}$$
$$= a\left(\cos\theta + \log\tan\frac{\theta}{2}\right) - a\cos\theta$$
$$= a\log\tan\frac{\theta}{2} \qquad (1)$$

$$\bar{y} = y + \frac{1}{y''}\left(1 + y'^2\right)$$

$$= a \sin\theta + \sec^2\theta \cdot \frac{a\cos^4\theta}{\sin\theta}$$

$$= a\sin\theta + \frac{a\cos^2\theta}{\sin\theta}$$

$$= \frac{a}{\sin\theta} \tag{2}$$

Now $$\bar{y} = \frac{a\left(1+\tan^2\frac{\theta}{2}\right)}{2\tan\frac{\theta}{2}} \tag{3}$$

and $$\tan\frac{\theta}{2} = e^{\bar{x}/a} \qquad \text{(4) [from (1)]}$$

From (3) and (4), we get,

$$\bar{y} = \frac{a}{2}\left\{\frac{1+e^{2x/a}}{e^{x/a}}\right\}$$

$$= \frac{a}{2}\left\{e^{\bar{x}/a} + e^{-\bar{x}/a}\right\} = a\cosh\frac{\bar{x}}{a}$$

$\therefore$ The evolute is $y = a\cosh\dfrac{x}{a}$

Example 3.7 Find the envelope of the family of straight lines given by (i) $y = mx \pm \sqrt{a^2m^2 - b^2}$, where m is the parameter, (ii) $x \cos\alpha + y \sin\alpha = a \sec\alpha$, where α is the parameter, (iii) the family of parabolas given by $y = x\tan\alpha - \dfrac{gx^2}{2u^2\cos^2\alpha}$, where α is the parameter.

(i) Rewriting the given equation, we have

$$a^2 m^2 - b^2 = (y - mx)^2$$
$$= y^2 - 2xym + m^2x^2$$

i.e. $$(x^2 - a^2)\, m^2 - 2xym + (y^2 + b^2) = 0$$

This is a quadratic equation in 'm'. $\therefore$ The envelope is given by the equation '$B^2 - 4AC = 0$'

i.e. $$4x^2y^2 - 4(x^2 - a^2)(y^2 + b^2) = 0$$

i.e. $$x^2y^2 - (x^2y^2 + b^2x^2 - a^2y^2 - a^2b^2) = 0$$

i.e. $$b^2x^2 - a^2y^2 = a^2b^2$$

i.e. $$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$ which is the standard hyperbola.

Note ☑ The envelope touches every member of the given family of straight lines and vice versa. This is, in fact, obvious as the given family represents the family of tangents to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

(ii) $x \cos \alpha + y \sin \alpha = a \sec \alpha$
Dividing throughout by $\cos \alpha$, we have

$$x + y \tan \alpha = a \sec^2 \alpha$$

i.e. $x + yt = a(1 + t^2)$, where $t = \tan \alpha$ can be treated as the new parameter.

i.e. $$at^2 - yt + (a - x) = 0$$

This is a quadratic equation in 't'.
∴ The envelope is given by

$$y^2 - 4a(a - x) = 0 \quad \text{i.e.} \quad y^2 = -4a(x - a)$$

(iii) $y = x \tan \alpha - \dfrac{gx^2}{2u^2 \cos^2 \alpha}$
Putting $t = \tan \alpha$, we get

$$\frac{gx^2}{2u^2}\left(1 + t^2\right) - xt + y = 0$$

i.e. $$gx^2t^2 - 2u^2xt + (gx^2 + 2u^2y) = 0$$

If we treat 't' as the parameter, we see that this equation is a quadratic equation in the parameter
∴ The envelope is given by

$$4u^4x^2 - 4gx^2(gx^2 + 2u^2y) = 0$$

i.e. $$g^2x^2 + 2u^2gy - u^4 = 0$$

i.e. $$x^2 = -\frac{2u^2}{g}\left(y - \frac{u^2}{2g}\right).$$

Example 3.8 Find the envelope of the family of straight lines (i) $y \cos \theta - x \sin \theta = a \cos 2\theta$, θ being the parameter, (ii) $x \cos \alpha + y \sin \alpha = c \sin \alpha \cos \alpha$, α being the parameter, (iii) $x \sec^2 \theta + y \operatorname{cosec}^2 \theta = c$, θ being the parameter.

(i) $y \cos \theta - x \sin \theta = a \cos 2\theta$ (1)

Differentiating (1) partially w.r.t. θ,

$$-y \sin \theta - x \cos \theta = -2a \sin 2\theta \tag{2}$$

(1) × $\cos \theta$ − (2) × $\sin \theta$ gives

$$\begin{aligned} y &= a(\cos 2\theta \cos \theta + 2 \sin 2\theta \sin \theta) \\ &= a(\cos \theta + \sin 2\theta \sin \theta) \end{aligned} \tag{3}$$

(1) × $\sin \theta$ + (2) × $\cos \theta$ gives

$$\begin{aligned} x &= -a(\cos 2\theta \sin \theta - 2 \sin 2\theta \cos \theta) \\ &= -a(-\sin \theta - \sin 2\theta \cos \theta) \\ &= a(\sin \theta + \sin 2\theta \cos \theta) \end{aligned} \tag{4}$$

Adding (3) and (4), we get

$$\begin{aligned} x + y &= a\{(\sin\theta + \cos\theta) + \sin 2\theta \,.\, (\sin\theta + \cos\theta)\} \\ &= a(\sin\theta + \cos\theta)\,(1 + \sin 2\theta) \\ &= a(\sin\theta + \cos\theta)\,(\sin^2\theta + \cos^2\theta + 2\sin\theta\cos\theta) \\ &= a(\sin\theta + \cos\theta)\,(\sin\theta + \cos\theta)^2 \\ &= a(\sin\theta + \cos\theta)^3 \end{aligned} \tag{5}$$

Subtracting (3) from (4), we get

$$\begin{aligned} x - y &= a\{(\sin\theta - \cos\theta) - \sin 2\theta\,(\sin\theta - \cos\theta)\} \\ &= a(\sin\theta - \cos\theta)^3 \end{aligned} \tag{6}$$

From (5) and (6), we get

$$\begin{aligned} \left(\frac{x+y}{a}\right)^{2/3} + \left(\frac{x-y}{a}\right)^{2/3} &= (\sin\theta + \cos\theta)^2 + (\sin\theta - \cos\theta)^2 \\ &= 2 \end{aligned}$$

$\therefore$ The envelope is

$$(x + y)^{2/3} + (x - y)^{2/3} = 2a^{2/3}$$

Note ☑ This is the evolute of the astroid $x^{2/3} + y^{2/3} = a^{2/3}$

[Refer to Example (3.3) above].

In this problem, we have found out the envelope of the normals of the astroid.

(ii) $x\cos\alpha + y\sin\alpha = c\sin\alpha\cos\alpha.$

Dividing by $\sin\alpha\cos\alpha$, we get

$$\frac{x}{\sin\alpha} + \frac{y}{\cos\alpha} = c \tag{1}$$

Differentiating (1) w.r.t. α,

$$-\frac{x}{\sin^2\alpha}\cos\alpha + \frac{y}{\cos^2\alpha}\sin\alpha = 0 \tag{2}$$

From (2), $\dfrac{x\cos\alpha}{\sin^2\alpha} = \dfrac{y\sin\alpha}{\cos^2\alpha}$

i.e. $$\frac{x}{\sin^3\alpha} = \frac{y}{\cos^3\alpha} = k \text{ say.}$$

$\therefore$ $$\sin^3\alpha = \frac{x}{k} \quad\text{and}\quad \cos^3\alpha = \frac{y}{k} \tag{3}$$

$$\sin^2\alpha + \cos^2\alpha = 1$$

$\therefore$ $$\left(\frac{x}{k}\right)^{2/3} + \left(\frac{y}{k}\right)^{2/3} = 1$$

i.e. $$k^{2/3} = x^{2/3} + y^{2/3} \tag{4}$$

$\therefore$ From (3), we have

$$\sin\alpha = \frac{x^{1/3}}{k^{1/3}} \quad \text{and} \quad \cos\alpha = \frac{y^{1/3}}{k^{1/3}} \tag{5}$$

Using (5) in (1), the equation of the envelope is $k^{1/3}(x^{2/3} + y^{2/3}) = c$

i.e. $$\left(x^{2/3} + y^{2/3}\right)^{1/2} \cdot \left(x^{2/3} + y^{2/3}\right) = c, \qquad \text{from (4)}$$

i.e. $$\left(x^{2/3} + y^{2/3}\right)^{3/2} = c$$

i.e. $$x^{2/3} + y^{2/3} = c^{2/3}$$

(iii) $x \sec^2\theta + y \operatorname{cosec}^2\theta = c$ (1)

Differentiating (1) partially w.r.t. θ,

$$2x\sec^2\theta\tan\theta - 2y\operatorname{cosec}^2\theta\cot\theta = 0$$

i.e. $$\frac{x\sin\theta}{\cos^3\theta} - \frac{y\cos\theta}{\sin^3\theta} = 0 \tag{2}$$

From (2), $\dfrac{x}{\cos^4\theta} = \dfrac{y}{\sin^4\theta} = k$. say.

$\therefore$ $$\cos^4\theta = \frac{x}{k} \quad \text{and} \quad \sin^4\theta = \frac{y}{k} \tag{3}$$

Using the identity $\cos^2\theta + \sin^2\theta = 1$, we have

$$\sqrt{\frac{x}{k}} + \sqrt{\frac{y}{k}} = 1$$

i.e. $$k = (\sqrt{x} + \sqrt{y})^2 \tag{4}$$

Using (3) and (4) in (1), we get

$$x \cdot \frac{\left(\sqrt{x} + \sqrt{y}\right)}{\sqrt{x}} + y \cdot \frac{\left(\sqrt{x} + \sqrt{y}\right)}{\sqrt{y}} = c$$

i.e. $\left(\sqrt{x} + \sqrt{y}\right)^2 = c$

i.e. $\sqrt{x} + \sqrt{y} = \sqrt{c}$, which is the equation of the required envelope.

Example 3.9 Find the envelope of the straight line $\dfrac{x}{a} + \dfrac{y}{b} = 1$, where a and b are parameters that are connected by the relation $a + b = c$.

$$\frac{x}{a} + \frac{y}{b} = 1 \tag{1}$$

$$a + b = c \tag{2}$$

From (2), $b = c - a$.

Using in (1), $\dfrac{x}{a} + \dfrac{y}{c-a} = 1$, where a is the only parameter. (3)

Differentiating (3) w.r.t. a, we get

$$\frac{-x}{a^2}+\frac{y}{(c-a)^2}=0 \tag{4}$$

From (3) $$\frac{x}{a^2}=\frac{y}{(c-a)^2}$$

$\therefore$ $$\frac{\sqrt{x}}{a}=\frac{\sqrt{y}}{c-a}=\frac{\sqrt{x}+\sqrt{y}}{c}$$

$\therefore$ $$\frac{1}{a}=\frac{\sqrt{x}+\sqrt{y}}{c\sqrt{x}} \quad \text{and} \quad \frac{1}{c-a}=\frac{\sqrt{x}+\sqrt{y}}{c\sqrt{y}} \tag{5}$$

Using (5) in (3), the equation of the envelope is $\frac{\sqrt{x}}{c}\left(\sqrt{x}+\sqrt{y}\right)+\frac{\sqrt{y}}{c}\left(\sqrt{x}+\sqrt{y}\right)=1$

i.e. $$\left(\sqrt{x}+\sqrt{y}\right)^2=c$$

or $$\sqrt{x}+\sqrt{y}=\sqrt{c}\ .$$

3.4 ALITER

Without eliminating one of the parameters, we may treat both a and b as functions of a third parameter t and proceed as follows:
Differentiating (1) w.r.t. *t,*

$$-\frac{x}{a^2}\frac{\mathrm{d}a}{\mathrm{d}t}-\frac{y}{b^2}\frac{\mathrm{d}b}{\mathrm{d}t}=0$$

i.e. $$\frac{x}{a^2}\frac{\mathrm{d}a}{\mathrm{d}t}=-\frac{y}{b^2}\frac{\mathrm{d}b}{\mathrm{d}t} \tag{3}$$

Differentiating $a + b = c$ w.r.t. t

$$\frac{\mathrm{d}a}{\mathrm{d}t}=-\frac{\mathrm{d}b}{\mathrm{d}t} \tag{4}$$

Dividing (3) by (4), we have

$$\frac{x}{a^2}=\frac{y}{b^2}$$

$\therefore$ $$\frac{\sqrt{x}}{a}=\frac{\sqrt{y}}{b}=\frac{\sqrt{x}+\sqrt{y}}{c} \tag{5}$$

$$(\because\ a+b=c)$$

Using (5) in (1), we get

$$\frac{\sqrt{x}}{c}\left(\sqrt{x}+\sqrt{y}\right)+\frac{\sqrt{y}}{c}\left(\sqrt{x}+\sqrt{y}\right)=1$$

i.e. $\left(\sqrt{x}+\sqrt{y}\right)^2=c$

or $\sqrt{x}+\sqrt{y}=\sqrt{c}$.

Example 3.10 Find the envelope of the system of lines $\frac{x}{l}+\frac{y}{m}=1,$, where l and m are connected by the relation $\frac{l}{a}+\frac{m}{b}=1$ (l and m are the parameters).

$$\frac{x}{l}+\frac{y}{m}=1 \tag{1}$$

and
$$\frac{l}{a}+\frac{m}{b}=1 \tag{2}$$

Differentiating (1) and (2) w.r.t. t,

$$-\frac{x}{l^2}\frac{\mathrm{d}l}{\mathrm{d}t}-\frac{y}{m^2}\frac{\mathrm{d}m}{\mathrm{d}t}=0 \tag{3}$$

and
$$\frac{1}{a}\frac{\mathrm{d}l}{\mathrm{d}t}+\frac{1}{b}\frac{\mathrm{d}m}{\mathrm{d}t}=0 \tag{4}$$

From (3) and (4), we have

$$\frac{x}{l^2/a}=\frac{y}{m^2/b}$$

i.e.
$$\frac{\left(\frac{x}{l}\right)}{\left(\frac{l}{a}\right)}=\frac{\left(\frac{y}{m}\right)}{\left(\frac{m}{b}\right)}=\frac{\frac{x}{l}+\frac{y}{m}}{\frac{l}{a}+\frac{m}{b}}=\frac{1}{1}$$

or
$$\frac{ax}{l^2}=\frac{by}{m^2}=1$$

$\therefore$
$$l=\sqrt{ax} \quad \text{and} \quad m=\sqrt{by} \tag{5}$$

Using (5) in (1), we get the envelope as $\sqrt{\frac{x}{a}}+\sqrt{\frac{y}{b}}=1$.

Example 3.11 Find the envelope of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, where a and b are connected by the relation $a^2 + b^2 = c^2$, c being a constant.

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \tag{1}$$

and
$$a^2 + b^2 = c^2 \tag{2}$$

Eliminating b from (1) and (2), we get

$$\frac{x^2}{a^2}+\frac{y^2}{c^2-a^2}=1$$

i.e. $(c^2 - a^2)x^2 + a^2y^2 = a^2(c^2 - a^2)$

i.e. $a^4 - a^2(c^2 + x^2 - y^2) + c^2x^2 = 0$ (3)

(3) is a quadratic equation in a^2, which may be regarded as the parameter.

$\therefore$ The envelope is given by $B^2 - 4AC = 0$

i.e. $(c^2 + x^2 - y^2)^2 - 4c^2x^2 = 0$

i.e. $[(c^2 + x^2 - y^2) + 2cx]\,[c^2 + x^2 - y^2 - 2cx] = 0$

i.e. $(x + c)^2 - y^2 = 0;\quad (x - c)^2 - y^2 = 0$

$\therefore$ $x + c = \pm y$ and $x - c = \pm y$

i.e. $x = -c \pm y$ and $x = c \pm y$

i.e. $x \pm y = \pm c.$

Example 3.12 Find the envelope of a system of concentric ellipses with their axes along the co-ordinate axes and of constant area.

The equation of the system of ellipses is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \tag{1}$$

The condition satisfied by a and b is $\pi\, ab = c$ (2)

Differentiating (1) and (2) w.r.t. 't',

$$-\frac{2x^2}{a^3}\frac{\mathrm{d}a}{\mathrm{d}t}-\frac{2y^2}{b^3}\frac{\mathrm{d}b}{\mathrm{d}t}=0 \tag{3}$$

$$\pi b\frac{\mathrm{d}a}{\mathrm{d}t}+\pi a\frac{\mathrm{d}b}{\mathrm{d}t}=0 \tag{4}$$

From (3) and (4), we have

$$\frac{x^2}{a^3 b}=\frac{y^2}{ab^3} \quad \text{or} \quad \frac{x^2}{a^2}=\frac{y^2}{b^2}$$

or $$\frac{x}{a}=\frac{y}{b}=k\text{, say} \tag{5}$$

From (5), $a=\dfrac{x}{k}$ and $b=\dfrac{y}{k}$

Using in (2), $\dfrac{\pi xy}{k^2}=c$ or $k=\sqrt{\dfrac{\pi xy}{c}}$ (6)

Using (5) and (6) in (1), the equation of the envelope is $\dfrac{\pi xy}{c}+\dfrac{\pi xy}{c}=1$

i.e. $2\,\pi\, xy = c.$

Example 3.13 Find the evolute of the parabola $y^2 = 4ax$, considering it as the envelope of its normals.

The normal at any point $(at^2, 2at)$ on the parabola $y^2 = 4ax$ is

$$y + xt = 2at + at^3 \tag{1}$$

(1) represents the family of normals, where t is the parameter.
Differentiating (1) w.r.t. 't',

$$x = 2a + 3at^2 \tag{2}$$

From (2),
$$t = \left(\frac{x - 2a}{3a}\right)^{\frac{1}{2}} \tag{3}$$

Substituting (3) in (1), we get

$$y = -(x - 2a)\left(\frac{x - 2a}{3a}\right)^{\frac{1}{2}} + a\left(\frac{x - 2a}{3a}\right)^{\frac{3}{2}}$$

$$= \frac{-(x - 2a)^{\frac{3}{2}}}{(3a)^{\frac{1}{2}}} + \frac{1}{3}\cdot\frac{(x - 2a)^{\frac{3}{2}}}{(3a)^{\frac{1}{2}}}$$

$$= -\frac{2}{3}\cdot\frac{(x - 2a)^{\frac{3}{2}}}{(3a)^{\frac{1}{2}}}$$

i.e.
$$y^2 = \frac{4}{27a}(x - 2a)^3$$

$\therefore$ The evolute of the parabola is

$$27ay^2 = 4\,(x - 2a)^3$$

Example 3.14 Find the envelope of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, treating it as the envelope of its normals.

The normal at any point $(a \cos \theta, b \sin \theta)$ on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is

$$\frac{ax}{\cos \theta} - \frac{by}{\sin \theta} = a^2 - b^2 \tag{1}$$

where θ is the parameter.
Differentiating (1) w.r.t. 'θ',

$$\frac{ax}{\cos^2 \theta}\sin \theta + \frac{by \cos \theta}{\sin^2 \theta} = 0 \tag{2}$$

From (2), $\frac{ax}{\cos^3 \theta} = -\frac{by}{\sin^3 \theta} = k$, say

$\therefore$
$$\cos \theta = \left(\frac{ax}{k}\right)^{\frac{1}{3}} \quad \text{and} \quad \sin \theta = \left(-\frac{by}{k}\right)^{\frac{1}{3}} \tag{3}$$

Using the identity $\cos^2\theta + \sin^2\theta = 1$, we have

$$\left(\frac{ax}{k}\right)^{\frac{2}{3}} + \left(\frac{by}{k}\right)^{\frac{2}{3}} = 1$$

$\therefore$ $$k^{\frac{2}{3}} = (ax)^{\frac{2}{3}} + (by)^{\frac{2}{3}}$$

i.e. $$k^{\frac{1}{3}} = \left\{(ax)^{\frac{2}{3}} + (by)^{\frac{2}{3}}\right\}^{\frac{1}{2}} \tag{4}$$

Using (3) and (4) in (1), we have

$$\left\{(ax)^{\frac{2}{3}} + (by)^{\frac{2}{3}}\right\} k^{\frac{1}{3}} = a^2 - b^2$$

i.e. $$\left\{(ax)^{\frac{2}{3}} + (by)^{\frac{2}{3}}\right\}^{\frac{3}{2}} = a^2 - b^2$$

i.e. $$(ax)^{\frac{2}{3}} + (by)^{\frac{2}{3}} = \left(a^2 - b^2\right)^{\frac{2}{3}}$$

Example 3.15 Find the evolute of the tractrix $x = a\left(\cos\theta + \log\tan\frac{\theta}{2}\right)$, $y = a\sin\theta$, treating it as the envelope of its normals.

$$x = a\left(\cos\theta + \log\tan\frac{\theta}{2}\right),\quad y = a\sin\theta\,.$$

Differentiating w.r.t. θ,

$$\dot{x} = a\left(-\sin\theta + \frac{1}{\tan\frac{\theta}{2}\cdot\cos^2\frac{\theta}{2}\cdot\frac{1}{2}}\right)$$

$$= a\left(-\sin\theta + \frac{1}{\sin\theta}\right) = \frac{a\cos^2\theta}{\sin\theta}$$

and $$\dot{y} = a\cos\theta$$

$\therefore$ $$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = \tan\theta\,.$$

$\therefore$ Slope of the normal at 'θ' $= -\cot\theta$

Now the equation of the normal at 'θ' is

$$y - a\sin\theta = -\cot\theta\left\{x - a\left(\cos\theta + \log\tan\frac{\theta}{2}\right)\right\} \tag{1}$$

(1) represents the family of normals of the tractrix, where θ is the parameter.

The evolute of the tractrix is the envelope of (1).
Differentiating (1) w.r.t. 'θ',

$$-a\cos\theta = x\,\mathrm{cosec}^2\,\theta + a\cot\theta\cdot\frac{\cos^2\theta}{\sin\theta} - a\,\mathrm{cosec}^2\,\theta\times\left(\cos\theta + \log\tan\frac{\theta}{2}\right)$$

$$= \frac{x}{\sin^2\theta} + \frac{a\cos^3\theta}{\sin^2\theta} - \frac{a\cos\theta}{\sin^2\theta} - a\,\mathrm{cosec}^2\,\theta\times\log\tan\frac{\theta}{2}$$

$$= \frac{x}{\sin^2\theta} - a\cos\theta - \frac{a}{\sin^2\theta}\log\tan\frac{\theta}{2}$$

$$\therefore \qquad x = a\log\tan\frac{\theta}{2} \tag{2}$$

Rewriting (1), we have

$$y = a\sin\theta - x\cot\theta + a\frac{\cos^2\theta}{\sin\theta} + a\cot\theta\cdot\log\tan\frac{\theta}{2} \tag{3}$$

Using (2) in (3), we get

$$y = a\sin\theta + \frac{a\cos^2\theta}{\sin\theta} - a\cot\theta\log\tan\frac{\theta}{2} + a\cot\theta\log\tan\frac{\theta}{2}$$

i.e.

$$y = \frac{a}{\sin\theta}$$

$$= \frac{a\left(1+\tan^2\frac{\theta}{2}\right)}{2\tan\frac{\theta}{2}}$$

$$= \frac{a}{2}\left\{\tan\frac{\theta}{2} + \frac{1}{\tan\frac{\theta}{2}}\right\}$$

$$= \frac{a}{2}\left\{e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right\}, \text{ again using (2)}$$

i.e. $y = a\cosh\dfrac{x}{a}$, which is the equation of the evolute of the tractrix.

EXERCISE 3(b)

Part A
(Short Answer Questions)

1. Define evolute and involute.
2. Explain briefly the procedure to find the evolute of a given curve $y = f(x)$.

3. Define envelope of a family of curves.
4. Give the working rule to find the equation of the envelope of the family $f(x, y, \alpha) = 0$, α being the parameter.
5. Obtain the equation of the envelope of the family $f_1(x, y)\alpha^2 + f_2(x, y)\alpha + f_3(x, y) = 0$, where α is the parameter.
6. Define evolute of a curve as an envelope.
7. If the centre of curvature of a curve at a variable point 't' on it is $(2a + 3at^2, -2at^3)$, find the evolute of the curve.
8. If the centre of curvature of a curve at a variable point 't' on it is $\left(\frac{c}{a}\cos^3 t, -\frac{c}{b}\sin^3 t\right)$, find the evolute of the curve.
9. If the centre of curvature of curve at a variable point 'θ' on it is $\left(a \log \cot \frac{\theta}{2}, \frac{a}{\sin\theta}\right)$, find the evolute of the curve.
10. Find the envelope of the family of lines $y = mx \pm a\sqrt{1+m^2}$, m being the parameter.
11. Find the envelope of the family of lines $y = mx + \frac{a}{m}$, m being the parameter.
12. Find the envelope of the family of lines $y = mx + am^2$, m being the parameter.
13. Find the envelope of the family of lines $y = mx \pm \sqrt{a^2m^2 + b^2}$, m being the parameter.
14. Find the envelope of the family of lines $\frac{x}{t} + yt = 2c$, t being the parameter.
15. Find the envelope of the lines $x \cos \alpha + y \sin \alpha = p$, α being the parameter.
16. Find the envelope of the lines $\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$, θ being the parameter.
17. Find the envelope of the lines $\frac{x}{a}\sec\theta - \frac{y}{b}\tan\theta = 1$, θ being the parameter.
18. Find the envelope of the lines $x \sec \theta - y \tan \theta = a$, θ being the parameter.
19. Find the envelope of the lines $x \operatorname{cosec} \theta - y \cot \theta = a$, θ being the parameter.
20. Show that the family of circles $(x - a)^2 + y^2 = a^2$ (a is the parameter) has no envelope.

Part B

21. Find the evolute of the parabola $y^2 = 4ax$.
22. Find the evolute of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.
23. Find the evolute of the rectangular hyperbola $xy = c^2$.
24. Show that the evolute of the cycloid $x = a(\theta + \sin \theta)$, $y = a(1 - \cos \theta)$ is another cycloid, given by $x = a(\theta - \sin \theta)$, $y - 2a = a(1 + \cos \theta)$.
25. Find the envelope of the family of lines $\frac{x}{a} + \frac{y}{b} = 1$, where the parameters a and b are connected by the relation $a^2 + b^2 = c^2$.

26. Find the envelope of the family of lines $\frac{x}{a}+\frac{y}{b}=1$, where the parameters a and b are connected by the relation $ab = c^2$.
27. Find the envelope of the family of ellipses $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, where the parameters a and b are connected by the relation $a + b = c$.
28. From a point on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, perpendiculars are drawn to the axis and the feet of these perpendiculars are joined. Find the envelope of the line thus formed.
29. Find the evolute of the parabola $x^2 = 4ay$, treating it as the envelope of its normals.
30. Find the evolute of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, treating it as the envelope of its normals.

ANSWERS

Exercise 3(a)

(5) $\frac{1}{c}$. (4) $2\sqrt{2}$. (5) $\frac{8}{3}$.

(6) $\sec x$. (7) 1. (8) $c\sqrt{2}$.

(9) $4a\sqrt{2}$. (10) $\frac{5\sqrt{5}}{2}$. (11) $\frac{a}{2}$.

(12) $\frac{a\left(1+\theta^2\right)^{\frac{3}{2}}}{\theta^4}$. (13) $\frac{a}{2}$. (14) $\sqrt{2}\ r$.

(15) $5\sqrt{5}$. (16) $\frac{125a}{24}$. (17) $\frac{5\sqrt{5}}{18}$.

(18) $\frac{a}{2}$. (19) $\frac{97\sqrt{97}\ a}{216}$. (20) $\frac{1}{6}$.

(21) c. (22) $2a\sec^3\theta$.

(23) $\left(a^2\sin^2\theta+b^2\cos^2\theta\right)^{\frac{3}{2}}/ab$. (24) at.

(25) $4a\sin\frac{\theta}{2}$. (26) $\sqrt{2}$. (27) $a\sec^2\theta$.

(28) $a\cot\theta$. (29) r cosec α. (30) $\frac{2}{3}\sqrt{2ar}$.

(31) $\frac{a^n}{(n+1)r^{n-1}}$. (32) $\frac{2r^{\frac{3}{2}}}{\sqrt{a}}$. (33) $\frac{(2r)^{\frac{3}{2}}}{\sqrt{a}}$.

(35) $\left(-36,-\frac{43}{6}\right)$. (36) $\left[\frac{(a^2+b^2)}{a}\sec^3\theta, \frac{-(a^2+b^2)}{b}\tan^3\theta\right]$.

(37) $x^2+y^2-10ax+4ay-3a^2=0$

(39) $\left(x-\frac{43}{6}\right)^2+\left(y-\frac{57}{8}\right)^2=\left(\frac{125}{24}\right)^2$.

(40) $\left(x-\frac{21a}{16}\right)^2+\left(y-\frac{21a}{16}\right)^2=\frac{9a^2}{128}$.

Exercise 3(b)

(5) $f_2^2-4f_1f_3=0$. (7) $27ay^2=4(x-2a)^3$.

(8) $(ax)^{\frac{2}{3}}+(by)^{\frac{2}{3}}=c^{\frac{2}{3}}$. (9) $y=a\cosh\frac{x}{a}$.

(10) $x^2+y^2=a^2$. (11) $y^2=4ax$. (12) $x^2+4ay=0$.

(13) $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. (14) $xy=c^2$. (15) $x^2+y^2=p^2$.

(16) $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. (17) $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$. (18) $x^2-y^2=a^2$.

(19) $x^2-y^2=a^2$. (21) $4(x-2a)^3=27ay^2$.

(22) $(ax)^{\frac{2}{3}}+(by)^{\frac{2}{3}}=(a^2-b^2)^{\frac{2}{3}}$.

(23) $(x+y)^{\frac{2}{3}}-(x-y)^{\frac{2}{3}}=(4c)^{\frac{2}{3}}$. (25) $x^{\frac{2}{3}}+y^{\frac{2}{3}}=c^{\frac{2}{3}}$.

(26) $4xy=c^2$. (27) $x^{\frac{2}{3}}+y^{\frac{2}{3}}=c^{\frac{2}{3}}$

(28) $\left(\frac{x}{a}\right)^{\frac{2}{3}}+\left(\frac{y}{b}\right)^{\frac{2}{3}}=1$. (29) $27ax^2=4(y-2a)^3$.

(30) $(ax)^{\frac{2}{3}}-(by)^{\frac{2}{3}}=(a^2+b^2)^{\frac{2}{3}}$.