1.6 EIGENVALUES AND EIGENVECTORS

1.6.1 Definition

Let $A = [a_{ij}]$ be a square matrix of order n. If there exists a *non-zero* column vector X and a scalar λ, such that

$$AX = \lambda X$$

then λ is called an eigenvalue of the matrix A and X is called the eigenvector corresponding to the eigenvalue λ.

To find the eigenvalues and the corresponding eigenvectors of a square matrix A, we proceed as follows:

Let λ be an eigenvalue of A and X be the corresponding eigenvector. Then, by definition,

$AX = \lambda X = \lambda IX$, where I is the unit matrix of order n.

i.e. $$(A - \lambda I)\, X = 0 \tag{1}$$

i.e. $$\left\{\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \hdashline a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & \dots & \dots & 0 \\ 0 & 1 & \dots & \dots & 0 \\ \hdashline 0 & 0 & 0 & \dots & 1 \end{bmatrix}\right\} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

i.e. $$\begin{aligned} (a_{11} - \lambda)x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\ a_{21}x_1 + (a_{22} - \lambda)x_2 + \cdots + a_{2n}x_n &= 0 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots (a_{nn} - \lambda)\, x_n &= 0 \end{aligned} \tag{2}$$

Equations (2) are a system of homogeneous linear equations in the unknowns x_1, $x_2, \dots, x_n$.

Since $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ is to be a non-zero vector,

$x_1, x_2, \dots, x_n$ should not be all zeros. In other words, the solution of the system (2) should be a non-trivial solution.

The condition for the system (2) to have a non-trivial solution is

$$\begin{vmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & \cdots & a_{2n} \\ \hdashline a_{n1} & a_{n2} & \cdots & a_{nn} - \lambda \end{vmatrix} = 0 \tag{3}$$

i.e. $$|A - \lambda I| = 0 \tag{4}$$

The determinant $|A - \lambda I|$ is a polynomial of degree n in λ and is called *the characteristic polynomial* of A.

The equation $|A - \lambda I| = 0$ or the equation (3) is called *the characteristic equation* of A.

When we solve the characteristic equation, we get n values for λ. These n roots of the characteristic equation are called the *characteristic roots* or *latent roots* or *eigenvalues* of A.

Corresponding to each value of λ, the equations (2) possess a non-zero (non-trivial) solution X. X is called *the invariant vector* or *latent vector* or *eigenvector* of A corresponding to the eigenvalue λ.

Notes ☑

1. Corresponding to an eigenvalue, the non-trivial solution of the system (2) will be a one-parameter family of solutions. Hence the eigenvector corresponding to an eigenvalue is not unique.
2. If all the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of a matrix A are distinct, then the corresponding eigenvectors are linearly independent.
3. If two or more eigenvalues are equal, then the eigenvectors may be linearly independent or linearly dependent.

1.6.2 Properties of Eigenvalues

1. A square matrix A and its transpose A^T have the same eigenvalues.
   Let $A = (a_{ij})$; $i, j = 1, 2, \ldots, n$.
   The characteristic polynomial of A is

$$|A - \lambda I| = \begin{vmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \hdashline a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{vmatrix} \qquad (1)$$

   The characteristic polynomial of A^T is

$$|A^T - \lambda I| = \begin{vmatrix} a_{11}-\lambda & a_{21} & \ldots & a_{n1} \\ a_{12} & a_{22}-\lambda & \ldots & a_{n2} \\ \hdashline a_{1n} & a_{2n} & \ldots & a_{nn}-\lambda \end{vmatrix} \qquad (2)$$

   Determinant (2) can be obtained by changing rows into columns of determinant (1).

   $\therefore \qquad |A - \lambda I| = |A^T - \lambda I|$

   $\therefore$ The characteristic equations of A and A^T are identical.

   $\therefore$ The eigenvalues of A and A^T are the same.

2. The sum of the eigenvalues of a matrix A is equal to the sum of the principal diagonal elements of A. (The sum of the principal diagonal elements is called *the Trace* of the matrix.)

   The characteristic equation of an n^{th} order matrix A may be written as

$$\lambda^n - D_1\lambda^{n-1} + D_2\lambda^{n-2} - \cdots + (-1)^n D_n = 0, \qquad (1)$$

   where D_r is the sum of all the r^{th} order minors of A whose principal diagonals lie along the principal diagonal of A.

(*Note* ☑ $D_n = |A|$). We shall verify the above result for a third order matrix.

Let $$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

The characteristic equation of A is given by

$$\begin{bmatrix} a_{11}-\lambda & a_{12} & a_{13} \\ a_{21} & a_{22}-\lambda & a_{23} \\ a_{31} & a_{32} & a_{33}-\lambda \end{bmatrix} = 0 \tag{2}$$

Expanding (2), the characteristic equation is

$$(a_{11}-\lambda)\left\{\lambda^2 - (a_{22}+a_{33})\lambda + \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}\right\}$$
$$- a_{12}\left\{-a_{21}\lambda + \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}\right\} + a_{13}\left\{a_{31}\lambda + \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}\right\} = 0$$

i.e. $$-\lambda^3 + (a_{11}+a_{22}+a_{33})\lambda^2$$
$$-\left\{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}\right\}\lambda + |A| = 0$$

i.e. $\lambda^3 - D_1\lambda^2 + D_2\lambda - D_3 = 0$, using the notation given above.
This result holds good for a matrix of order n.

Note ☑ This form of the characteristic equation provides an alternative method for getting the characteristic equation of a matrix.

Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of A.
$\therefore$ They are the roots of equation (1).

$$\therefore \quad \lambda_1 + \lambda_2 + \cdots + \lambda_n = \frac{-(-D_1)}{1} = D_1$$
$$= a_{11} + a_{22} + \cdots + a_{nn}$$
$$= \text{Trace of the matrix } A.$$

3. The product of the eigenvalues of a matrix A is equal to $|A|$.
   If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of A, they are the roots of

$$\lambda^n - D_1\lambda^{n-1} + D_2\lambda^{n-2} - \cdots + (-1)^n D_n = 0.$$

$$\therefore \text{ Product of the roots } = \frac{(-1)^n \cdot (-1)^n D_n}{1}$$

i.e. $\lambda_1, \lambda_2 \ldots \lambda_n = D_n = |A|$.

1.6.3 Aliter

$\lambda_1, \lambda_2, \ldots, \lambda_n$ are the roots of $|A - \lambda I| = 0$

$\therefore$ $|A - \lambda I| \equiv (-1)^n (\lambda - \lambda_1)(\lambda - \lambda_2) \ldots (\lambda - \lambda_n)$, since L.S. is a n^{th} degree polynomial in λ whose leading term is $(-1)^n \lambda^n$.

Putting $\lambda = 0$ in the above identity, we get $|A| = (-1)^n (-\lambda_1)(-\lambda_2) \ldots (-\lambda_n)$
i.e. $\lambda_1 \lambda_2 \ldots \lambda_n = |A|$.

1.6.4 Corollary

If $|A| = 0$, i.e. A is a singular matrix, at least one of the eigenvalues of A is zero and conversely.

4. If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of a matrix A, then
   (i) $k\lambda_1, k\lambda_2, \ldots k\lambda_n$ are the eigenvalues of the matrix kA, where k is a non-zero scalar.
   (ii) $\lambda_1^p, \lambda_2^p, \ldots, \lambda_n^p$ are the eigenvalues of the matrix A^p, where p is a positive integer.
   (iii) $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \ldots \frac{1}{\lambda_n}$ are the eigenvalues of the inverse matrix A^{-1}, provided $\lambda_r \neq 0$ i.e. A is non-singular.

(i) Let λ_r be an eigenvalue of A and X_r the corresponding eigenvector. Then, by definition,

$$AX_r = \lambda_r x_r \tag{1}$$

Multiplying both sides of (1) by k,

$$(kA)X_r = (k\lambda_r) X_r \tag{2}$$

From (2), we see that $k\lambda_r$ is an eigenvalue of kA and the corresponding eigenvector is the same as that of λ_r, namely X_r.

(ii) Premultiplying both sides of (1) by A,

$$\begin{aligned} A^2 X_r &= A(AX_r) \\ &= A(\lambda_r X_r) \\ &= \lambda_r (AX_r) \\ &= \lambda_r^2 X_r \end{aligned}$$

Similarly $A^3 X_r = \lambda_r^3 X_r$ and so on.

In general, $A^p X_r = \lambda_r^p X_r$

From, (3), we see that λ_r^p is an eigenvalue of A^p with the corresponding eigenvector equal to X_r, which is the same for λ_r.

(iii) Premultiplying both sides of (1) by A^{-1},

$$A^{-1}(AX_r) = A^{-1}(\lambda_r X_r)$$

i.e. $$X_r = \lambda_r (A^{-1} X_r)$$

$\therefore$ $$A^{-1} X_r = \frac{1}{\lambda_r} X_r \tag{4}$$

From (4), we see that $\dfrac{1}{\lambda_r}$ is an eigenvalue of A^{-1} with the corresponding eigenvector equal to X_r, which is the same for λ_r.

5. The eigenvalues of a real symmetric matrix (i.e. a symmetric matrix with real elements) are real.

Let λ be an eigenvalue of the real symmetric matrix and X be the corresponding eigenvector.

Then $$AX = \lambda \qquad (1)$$

Premultiplying both sides of (1) by $\bar{X}^T$ (the transpose of the conjugate of X), we get

$$\bar{X}^T AX = \lambda\, \bar{X}^T X \qquad (2)$$

Taking the complex conjugate on both sides of (2),

$$X^T \bar{A}\bar{X} = \bar{\lambda}\, X^T \bar{X} \text{ (assuming that } \lambda \text{ may be complex)}$$

i.e. $$X^T A\bar{X} = \bar{\lambda}\, X^T \bar{X} \; (\because \bar{A} = A, \text{ as } A \text{ is real}) \qquad (3)$$

Taking transpose on both sides of (3),

$$\bar{X}^T A^T X = \bar{\lambda}\, \bar{X}^T X \qquad \left[\because (AB)^T = B^T A^T\right]$$

i.e. $$\bar{X}^T A X = \bar{\lambda}\, \bar{X}^T X \qquad \left[\because (A)^T = A, \text{ as } A \text{ is symmetric}\right] \qquad (4)$$

From (2) and (4), we get

$$\lambda \bar{X}^T X = \bar{\lambda} \bar{X}^T X$$

i.e. $$(\lambda - \bar{\lambda}) \bar{X}^T X = 0$$

$\bar{X}^T X$ is an 1×1 matrix, i.e. a single element which is positive

$\therefore$ $$\lambda - \bar{\lambda} = 0$$

i.e. λ is real.

Hence all the eigenvalues are real.

6. The eigenvectors corresponding to distinct eigenvalues of a real symmetric matrix are orthogonal.

Note ☑ Two column vectors $X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ and $Y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$ are said to be orthogonal, if their inner product $(x_1 y_1 + x_2 y_2 + \cdots x_n y_n) = 0$ i.e. if $X^T Y = 0$.

Let λ_1, λ_2 be any two distinct eigenvalues of the real symmetric matrix A and X_1, X_2 be the corresponding eigenvectors respectively.

Then $$AX_1 = \lambda_1 X_1 \qquad (1)$$

and $$AX_2 = \lambda_2 X_2 \qquad (2)$$

Premultiplying both sides of (1) by, X_2^T we get

$$X_2^T AX_1 = \lambda_1 X_2^T X_1$$

Taking the transpose on both sides,

$$X_1^T AX_2 = \lambda_1 X_1^T X_2 \qquad (\because A^T = A) \tag{3}$$

Premultiplying both sides of (2) by X_1^T, we get

$$X_1^T AX_2 = \lambda_2 X_1^T X_2 \tag{4}$$

From (3) and (4), we have

$$\lambda_1 X_1^T X_2 = \lambda_2 X_1^T X_2$$

i.e. $$(\lambda_1 - \lambda_2) X_1^T X_2 = 0$$

Since $$\lambda_1 \neq \lambda_2, X_1^T X_2 = 0$$

i.e. the eigenvectors X_1 and X_2 are orthogonal.

WORKED EXAMPLE 1(b)

Example 1.1 Given that $A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$, verify that the eigenvalues of A^2 are the squares of those of A.

Verify also that the respective eigenvectors are the same.

The characteristic equation of A is $\begin{vmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{vmatrix} = 0$

i.e. $$(5 - \lambda)(2 - \lambda) - 4 = 0$$

i.e. $$\lambda^2 - 7\lambda + 6 = 0$$

$\therefore$ The eigenvalues of A are $\lambda = 1, 6$.

The eigenvector corresponding to any λ is given by $(A - \lambda I) X = 0$

i.e. $$\begin{vmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{vmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

When $\lambda = 1$, the eigenvector is given by the equations

$$4x_1 + 4x_2 = 0 \text{ and}$$

$$x_1 + x_2 = 0, \text{ which are one and the same.}$$

Solving, $x_1 = -x_2$. Taking $x_1 = 1, x_2 = -1$.

$\therefore$ The eigenvector is $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

When $\lambda = 6$, the eigenvector is given by

$$-x_1 + 4x_2 = 0$$

and $$x_1 - 4x_2 = 0$$

Solving, $x_1 = 4x_2$

Taking $x_2 = 1, x_1 = 4$

$\therefore$ The eigenvector is $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$

Now $$A^2 = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 29 & 28 \\ 7 & 8 \end{bmatrix}$$

The characteristic equation of A^2 is $\begin{vmatrix} 29-\lambda & 28 \\ 7 & 8-\lambda \end{vmatrix} = 0$

i.e. $$(29 - \lambda)(8 - \lambda) - 196 = 0$$

i.e. $$\lambda^2 - 37\lambda + 36 = 0$$

i.e. $$(\lambda - 1)(\lambda - 36) = 0$$

$\therefore$ The eigenvalues of A^2 are 1 and 36, that are the squares of the eigenvalues of A, namely 1 and 6. When $\lambda = 1$, the eigenvector of A^2 is given by

$$\begin{bmatrix} 28 & 28 \\ 7 & 7 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0. \quad \text{i.e.} \quad 28x_1 + 28x_2 = 0 \quad \text{and} \quad 7x_1 + 7x_2 = 0$$

Solving, $x_1 = -x_2$. Taking $x_1 = 1, x_2 = -1$.

When $\lambda = 36$, the eigenvector of A^2 is given by

$$\begin{bmatrix} -7 & 28 \\ 7 & -28 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0. \text{ i.e.} -7x_1 + 28x_2 = 0 \quad \text{and} \quad 7x_1 - 28x_2 = 0.$$

Solving, $x_1 = 4x_2$. Taking $x_2 = 1, x_1 = 4$.

Thus the eigenvectors of A^2 are

$\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 4 \\ 1 \end{bmatrix}$, which are the same as the respective eigenvectors of A.

Example 1.2 Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$$

The characteristic equation of A is

$$\begin{vmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{vmatrix} = 0$$

i.e. $(1 - \lambda)\{\lambda^2 - 6\lambda + 4\} - (1 - \lambda - 3) + 3(1 - 15 + 3\lambda) = 0$

i.e. $-\lambda^3 + 7\lambda^2 - 36 = 0$ or $\lambda^3 - 7\lambda^2 + 36 = 0$ (1)

i.e. $(\lambda + 2)(\lambda^2 - 9\lambda + 18) = 0$ [$\therefore \lambda = -2$ satisfies (1)]

i.e. $(\lambda + 2)(\lambda - 3)(\lambda - 6) = 0$

$\therefore$ The eigenvalues of A are $\lambda = -2, 3, 6$.

Case (i) $\lambda = -2$.

The eigenvector is given by

$$\begin{bmatrix} 3 & 1 & 3 \\ 1 & 7 & 1 \\ 3 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \qquad (2)$$

i.e.
$$x_1 + 7x_2 + x_3 = 0$$
$$3x_1 + x_2 + 3x_3 = 0$$

Solving these equations by the rule of cross-multiplication, we have

$$\frac{x_1}{21-1} = \frac{x_2}{3-3} = \frac{x_3}{1-21}$$

i.e.
$$\frac{x_1}{20} = \frac{x_2}{0} = \frac{x_3}{-20} \qquad (3)$$

Note☑ To solve for x_1, x_2, x_3, we have taken the equations corresponding to the second and third rows of the matrix in step (2). The proportional values of x_1, x_2, x_3 obtained in step (3) are the co-factors of the elements of the first row of the determinant of the matrix in step (2). This provides an alternative method for finding the eigenvector.

From step (3), $x_1 = k$, $x_2 = 0$ and $x_3 = -k$.

Usually the eigenvector is expressed in terms of the simplest possible numbers, corresponding to $k = 1$ or -1.

$\therefore$ $x_1 = 1, \quad x_2 = 0, \quad x_3 = -1$

Thus the eigenvector corresponding to $\lambda = -2$ is

$$X_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

Case (ii) $\lambda = 3$.

The eigenvector is given by $\begin{bmatrix} -2 & 1 & 3 \\ 1 & 2 & 1 \\ 3 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0.$

Values of x_1, x_2, x_3 are proportional to the co-factors of –2, 1, 3 (elements of the first row i.e. –5, 5, –5.

i.e.
$$\frac{x_1}{-5} = \frac{x_2}{5} = \frac{x_3}{-5} \quad \text{or} \quad \frac{x_1}{1} = \frac{x_2}{-1} = \frac{x_3}{1}$$

$\therefore$
$$X_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

Case (iii) $\lambda = 6$.

The eigenvector is given by $\begin{bmatrix} -5 & 1 & 3 \\ 1 & -1 & 1 \\ 3 & 1 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

$\therefore \qquad \dfrac{x_1}{4} = \dfrac{x_2}{8} = \dfrac{x_3}{4}$

or $\qquad \dfrac{x_1}{1} = \dfrac{x_2}{2} = \dfrac{x_3}{1}$

$$\therefore \qquad X_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

Note ☑ Since the eigenvalues of A are distinct, the eigenvectors X_1, X_2, X_3 are linearly independent, as can be seen from the fact that the equation $k_1X_1 + k_2X_2 + k_3X_3 = 0$ is satisfied only when $k_1 = k_2 = k_3 = 0$.

Example 1.3 Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

The characteristic equation is given by

$$\lambda^3 - D_1\lambda^2 + D_2\lambda - D_3 = 0, \text{ where}$$

D_1 = the sum of the first order minors of A that lie along the main diagonal of A
$= 0 + 0 + 0$
$= 0$

D_2 = the sum of the second order minors of A whose principal diagonals lie along the principal diagonal of A.

$$= \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}$$

$= -3$

$D_3 = |A| = 2$

Thus the characteristic equation of A is

$$\lambda^3 - 3\lambda - 2 = 0$$

i.e. $\qquad (\lambda + 1)^2 (\lambda - 2) = 0$

$\therefore$ The eigenvalues of A are $\lambda = -1, -1, 2$.

Case (i) $\qquad \lambda = -1$.

The eigenvector is given by

$$\begin{bmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

All the three equations reduce to one and the same equation $x_1 + x_2 + x_3 = 0$. There is one equation in three unknowns.

$\therefore$ Two of the unknowns, say, x_1 and x_2 are to be treated as free variables (parameters). Taking $x_1 = 1$ and $x_2 = 0$, we get $x_3 = -1$ and taking $x_1 = 0$ and $x_2 = 1$, we get $x_3 = -1$

$$\therefore \qquad X_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \text{ and } X_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

Case (ii) $\lambda = 2$.

The eigenvector is given by

$$\begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

Values of x_1, x_2, x_3 are proportional to the co-factors of elements in the first row.

i.e. $$\frac{x_1}{3} = \frac{x_2}{3} = \frac{x_3}{3}$$

or $$\frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{1}$$

$$\therefore \qquad x_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Note ☑ Though two of the eigenvalues are equal, the eigenvectors X_1, X_2, X_3 are found to be linearly independent.

Example 1.4 Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} 2 & -2 & 2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$$

The characteristic equation of A is

$$\begin{vmatrix} 2-\lambda & -2 & 2 \\ 1 & 1-\lambda & 1 \\ 1 & 3 & -1-\lambda \end{vmatrix} = 0$$

i.e. $$(2-\lambda)(\lambda^2 - 4) + 2(-1-\lambda-1) + 2(3-1+\lambda) = 0$$

i.e. $$(2-\lambda)(\lambda-2)(\lambda+2) = 0$$

$\therefore$ The eigenvalues of A are $\lambda = -2, 2, 2$.

Case (i) $\lambda = -2$

The eigenvector is given by

$$\begin{bmatrix} 4 & -2 & 2 \\ 1 & 3 & 1 \\ 1 & 3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$\therefore \dfrac{x_1}{-8} = \dfrac{x_2}{-2} = \dfrac{x_3}{14}$ (by taking the co-factors of elements of the third row) i.e. $$\frac{x_1}{-4} = \frac{x_2}{-1} = \frac{x_3}{7}$$

$\therefore$ $$X_1 = \begin{bmatrix} -4 \\ -1 \\ 7 \end{bmatrix}$$

Case (ii) $\lambda = 2$.

The eigenvector is given by

$$\begin{vmatrix} 0 & -2 & 2 \\ 1 & -1 & 1 \\ 1 & 3 & -3 \end{vmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$\therefore$ $$\frac{x_1}{0} = \frac{x_2}{4} = \frac{x_3}{4} \quad \text{or} \quad \frac{x_1}{0} = \frac{x_2}{1} = \frac{x_3}{1}$$

$\therefore$ $$X_2 = X_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Note ☑ Two eigenvalues are equal and the eigenvectors are linearly dependent.

Example 1.5 Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} 11 & -4 & -7 \\ 7 & -2 & -5 \\ 10 & -4 & -6 \end{bmatrix}$$

Can you guess the nature of A from the eigenvalues? Verify your answer.

The characteristic equation of A is

$$\begin{vmatrix} 11-\lambda & -4 & -7 \\ 7 & -2-\lambda & -5 \\ 10 & -4 & -6-\lambda \end{vmatrix} = 0$$

i.e. $$(11 - \lambda)(\lambda^2 + 8\lambda - 8) + 4(8 - 7\lambda) - 7(10\lambda - 8) = 0$$

i.e. $$\lambda^3 - 3\lambda^2 + 2\lambda = 0$$

$\therefore$ The eigenvalues of A are $\lambda = 0, 1, 2$.

Case (i) $\lambda = 0$.

The eigenvector is given by $$\begin{bmatrix} 11 & -4 & -7 \\ 7 & -2 & -5 \\ 10 & -4 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$\therefore$ $$\frac{x_1}{-8} = \frac{x_2}{-8} = \frac{x_3}{-8}$$

or $$\frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{1}$$

$\therefore$ $$X_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Case (ii) $\lambda = 1$.

The eigenvector is given by $\begin{bmatrix} 10 & -4 & -7 \\ 7 & -3 & -5 \\ 10 & -4 & -7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

$\therefore$ $$\frac{x_1}{-1} = \frac{x_2}{1} = \frac{x_3}{-2}$$

$\therefore$ $$X_2 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

Case (iii) $\lambda = 2$.

The eigenvector is given by $\begin{bmatrix} 9 & -4 & -7 \\ 7 & -4 & -5 \\ 10 & -4 & -8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

$\therefore$ $$\frac{x_1}{12} = \frac{x_2}{6} = \frac{x_3}{12}$$

or $$\frac{x_1}{2} = \frac{x_2}{1} = \frac{x_3}{2}$$

$\therefore$ $$X_3 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$$

Since one of the eigenvalues of A is zero, product of the eigenvalues = $|A| = 0$, i.e. A is non-singular. It is verified below:

$$\begin{vmatrix} 11 & -4 & -7 \\ 7 & -2 & -5 \\ 10 & -4 & -6 \end{vmatrix} = 11(12 - 20) + 4(-42 + 50) - 7(-28 + 20) = 0.$$

Example 1.6 Verify that the sum of the eigenvalues of A equals the trace of A and that their product equals $|A|$, for the matrix

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix}$$

The characteristic equation of A is

$$\begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 3-\lambda & -1 \\ 0 & -1 & 3-\lambda \end{vmatrix} = 0$$

i.e. $$(1-\lambda)(\lambda^2 - 6\lambda + 8) = 0$$

$\therefore$ The eigenvalues of A are $\lambda = 1, 2, 4$.
Sum of the eigenvalues = 7.
Trace of the matrix = 1 + 3 + 3 = 7
Product of the eigenvalues = 8.

$$|A| = 1 \times (9-1) = 8.$$

Hence the properties verified.

Example 1.7 Verify that the eigenvalues of A^2 and A^{-1} are respectively the squares and reciprocals of the eigenvalues of A, given that

$$A = \begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix}$$

The characteristic equation of A is

$$\begin{vmatrix} 3-\lambda & 1 & 4 \\ 0 & 2-\lambda & 6 \\ 0 & 0 & 5-\lambda \end{vmatrix} = 0$$

i.e. $$(3-\lambda)(2-\lambda)(5-\lambda) = 0$$

$\therefore$ The eigenvalues of A are $\lambda = 3, 2, 5$.

Now $$A^2 = \begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix}\begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix} = \begin{bmatrix} 9 & 5 & 38 \\ 0 & 4 & 42 \\ 0 & 0 & 25 \end{bmatrix}$$

The characteristic equation of A^2 is

$$\begin{vmatrix} 9-\lambda & 5 & 38 \\ 0 & 4-\lambda & 42 \\ 0 & 0 & 25-\lambda \end{vmatrix} = 0$$

i.e. $$(9-\lambda)(4-\lambda)(25-\lambda) = 0$$

$\therefore$ The eigenvalues of A^2 are 9, 4, 25, which are the squares of the eigenvalues of A.

Let $$A = \begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix} \equiv \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

A_{11} = Co-factor of $a_{11} = 10$; $A_{12} = 0$; $A_{13} = 0$;

$A_{21} = -5$; $A_{22} = 15$; $A_{23} = 0$; $A_{31} = -2$; $A_{32} = -18$; $A_{33} = 6$

$$|A| = 30.$$

$$\therefore \qquad A^{-1} = \frac{1}{30}\begin{bmatrix} 10 & -5 & -2 \\ 0 & 15 & -18 \\ 0 & 0 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{3} & -\frac{1}{6} & -\frac{1}{15} \\ 0 & \frac{1}{2} & -\frac{3}{5} \\ 0 & 0 & \frac{1}{5} \end{bmatrix}$$

The characteristic equation of A^{-1} is

$$\begin{vmatrix} \frac{1}{3}-\lambda & -\frac{1}{6} & -\frac{1}{15} \\ 0 & \frac{1}{2}-\lambda & -\frac{3}{5} \\ 0 & 0 & \frac{1}{5}-\lambda \end{vmatrix} = 0$$

i.e. $$\left(\frac{1}{3}-\lambda\right)\left(\frac{1}{2}-\lambda\right)\left(\frac{1}{5}-\lambda\right) = 0$$

$\therefore$ The eigenvalues of A^{-1} are $\frac{1}{3}, \frac{1}{2}, \frac{1}{5}$, which are the reciprocals of the eigenvalues of A.

Hence the properties verified.

Example 1.8 Find the eigenvalues and eigenvectors of (adj A), given that the matrix

$$A = \begin{bmatrix} 2 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 2 \end{bmatrix}$$

The characteristic equation of A is

$$\begin{vmatrix} 2-\lambda & 0 & -1 \\ 0 & 2-\lambda & 0 \\ -1 & 0 & 2-\lambda \end{vmatrix} = 0$$

i.e. $$(2-\lambda)^3 - (2-\lambda) = 0$$

i.e. $$(2-\lambda)(\lambda^2 - 4\lambda + 3) = 0$$

$\therefore$ The eigenvalues of A are $\lambda = 1, 2, 3$.

Case (i) $\lambda = 1$.

The eigenvector is given by $$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$\therefore \qquad \dfrac{x_1}{1} = \dfrac{x_2}{0} = \dfrac{x_3}{1}$

$$\therefore \qquad X_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Case (ii) $\lambda = 2$.

The eigenvector is given by $\begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

i.e. $-x_3 = 0$ and $-x_1 = 0$

$\therefore x_1 = 0, x_3 = 0$ and x_2 is arbitrary. Let $x_2 = 1$

$$\therefore \qquad X_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Case (iii) $\lambda = 3$.

The eigenvector is given by $\begin{bmatrix} -1 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0.$

$\therefore \qquad \dfrac{x_1}{1} = \dfrac{x_2}{0} = \dfrac{x_3}{-1}$

$$\therefore \qquad X_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

The eigenvalues of A^{-1} are $1, \dfrac{1}{2}, \dfrac{1}{3}$ with the eigenvectors X_1, X_2, X_3.

Now $\qquad \dfrac{\text{adj } A}{|A|} = A^{-1}$

i.e. adj $A = |A| \cdot A^{-1} = 6A^{-1}$ ($\because |A| = 6$ for the given matrix A)

$\therefore$ The eigenvalues of (adj A) are equal to 6 times those of A^{-1}, namely, 6, 3, 2. The corresponding eigenvectors are X_1, X_2, X_3 respectively.

Example 1.9 Verify that the eigenvectors of the real symmetric matrix

$$A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$$

are orthogonal in pairs.

The characteristic equation of A is

$$\begin{vmatrix} 3-\lambda & -1 & 1 \\ -1 & 5-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{vmatrix} = 0$$

i.e. $(3 - \lambda)(\lambda^2 - 8\lambda + 14) + (\lambda - 3 + 1) + (1 + \lambda - 5) = 0$

i.e. $\lambda^3 - 11\lambda^2 + 36\lambda - 36 = 0$

i.e. $(\lambda - 2)(\lambda - 3)(\lambda - 6) = 0$

$\therefore$ The eigenvalues of A are $\lambda = 2, 3, 6$.

Case (i) $\lambda = 2$.

The eigenvector is given by $\begin{bmatrix} 1 & -1 & 1 \\ -1 & 3 & -1 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

$\therefore$ $\dfrac{x_1}{2} = \dfrac{x_2}{0} = \dfrac{x_3}{-2}$ or $\dfrac{x_1}{1} = \dfrac{x_2}{0} = \dfrac{x_3}{-1}$

$\therefore$ $X_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

Case (ii) $\lambda = 3$.

The eigenvector is given by $\begin{bmatrix} 0 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

$\therefore$ $\dfrac{x_1}{-1} = \dfrac{x_2}{-1} = \dfrac{x_3}{-1}$

$\therefore$ $X_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Case (iii) $\lambda = 6$.

The eigenvector is given by $\begin{bmatrix} -3 & -1 & 1 \\ -1 & -1 & -1 \\ 1 & -1 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

$$\frac{x_1}{2} = \frac{x_2}{-4} = \frac{x_3}{2}$$

$$X_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

Now

$$X_1^T X_2 = \begin{bmatrix} 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 0$$

$$X_2^T X_3 = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = 0$$

$$X_3^T X_1 = \begin{bmatrix} 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = 0$$

Hence the eigenvectors are orthogonal in pairs.

Example 1.10 Verify that the matrix

$$A = \frac{1}{3}\begin{bmatrix} 2 & 2 & 1 \\ -2 & 1 & 2 \\ 1 & -2 & 2 \end{bmatrix}$$

is an orthogonal matrix. Also verify that $\frac{1}{\lambda}$ is an eigenvalue of A, if λ is an eigenvalue and that the eigenvalues of A are of unit modulus.

Note ☑ A square matrix A is said to be orthogonal if $AA^T = A^TA = I$.

Now

$$AA^T = \frac{1}{3}\begin{bmatrix} 2 & 2 & 1 \\ -2 & 1 & 2 \\ 1 & -2 & 2 \end{bmatrix} \times \frac{1}{3}\begin{bmatrix} 2 & -2 & 1 \\ 2 & 1 & -2 \\ 1 & 2 & 2 \end{bmatrix}$$

$$= \frac{1}{9}\begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

Similarly we can prove that $A^TA = I$.
Hence A is an orthogonal matrix.
The characteristic equation of $3A$ is

$$\begin{vmatrix} 2-\lambda & 2 & 1 \\ -2 & 1-\lambda & 2 \\ 1 & -2 & 2-\lambda \end{vmatrix} = 0$$

i.e. $(2 - \lambda)(\lambda^2 - 3\lambda + 6) - 2(2\lambda - 4 - 2) + (4 - 1 + \lambda) = 0$

i.e. $\lambda^3 - 5\lambda^2 + 15\lambda - 27 = 0$

i.e. $(\lambda - 3)(\lambda^2 - 2\lambda + 9) = 0$

$\therefore$ The eigenvalues of $3A$ are given by

$$\lambda = 3 \qquad \text{and} \qquad \lambda = \frac{2 \pm \sqrt{4-36}}{2} = 1 \pm i\,2\sqrt{2}$$

$\therefore$ The eigenvalues of A are

$$\lambda_1 = 1, \quad \lambda_2 = \frac{1 + i\,2\sqrt{2}}{3}, \quad \lambda_3 = \frac{1 - i\,2\sqrt{2}}{3}$$

Now $\qquad \dfrac{1}{\lambda_1} = 1 = \lambda_1$

$$\frac{1}{\lambda_2} = \frac{3}{1 + i2\sqrt{2}} = \frac{3\,(1 - i2\sqrt{2})}{(1 + i\,2\sqrt{2})\,(1 - i\,2\sqrt{2})} = \frac{1 - i\,2\sqrt{2}}{3} = \lambda_3$$

and similarly $\dfrac{1}{\lambda_3} = \lambda_2$.

Thus, if λ is an eigenvalue of an orthogonal matrix, $\dfrac{1}{\lambda}$ is also an eigenvalue.

Also $|\lambda_1| = |1| = 1$.

$$|\lambda_2| = \left|\frac{1}{3} + \frac{i\,2\sqrt{2}}{3}\right| = \sqrt{\frac{1}{9} + \frac{8}{9}} = 1$$

Similarly, $|\lambda_3| = 1$.

Thus the eigenvalues of an orthogonal matrix are of unit modulus.

EXERCISE 1(b)

Part A

(Short Answer Questions)

1. Define eigenvalues and eigenvectors of a matrix.
2. Prove that A and A^T have the same eigenvalues.
3. Find the eigenvalues of $2A^2$, if $A = \begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix}$.
4. Prove that the eigenvalues of $(-3A^{-1})$ are the same as those of $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$.
5. Find the sum and product of the eigenvalues of the matrix $A = \begin{bmatrix} 1 & 2 & -2 \\ 1 & 0 & 3 \\ -2 & -1 & -3 \end{bmatrix}$.
6. Find the sum of the squares of the eigenvalues of $A = \begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix}$.

7. Find the sum of the eigenvalues of $2A$, if $A=\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$.

8. Two eigenvalues of the matrix $A=\begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$ are equal to 1 each. Find the third eigenvalue.

9. If the sum of two eigenvalues and trace of a 3×3 matrix A are equal, find the value of $|A|$.

10. Find the eigenvectors of $A=\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$.

11. Find the sum of the eigenvalues of the inverse of $A=\begin{bmatrix} 3 & 0 & 0 \\ 8 & 4 & 0 \\ 6 & 2 & 5 \end{bmatrix}$.

12. The product of two eigenvalues of the matrix $A=\begin{bmatrix} 6 & -2 & 2 \\ -3 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$ is 16. Find the third eigenvalue.

Part B

13. Verify that the eigenvalues of A^{-1} are the reciprocals of those of A and that the respective eigenvectors are the same with respect to the matrix

$$A=\begin{bmatrix} 1 & -2 \\ -5 & 4 \end{bmatrix}.$$

14. Show that the eigenvectors of the matrix $A=\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$ are $\begin{bmatrix} 1 \\ i \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -i \end{bmatrix}$.

Find the eigenvalues and eigenvectors of the following matrices:

15. $\begin{bmatrix} 2 & 2 & 0 \\ 2 & 1 & 1 \\ -7 & 2 & -3 \end{bmatrix}$

16. $\begin{bmatrix} -1 & 2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}$

17. $\begin{bmatrix} 2 & 2 & -7 \\ 2 & 1 & 2 \\ 0 & 1 & -3 \end{bmatrix}$

18. $\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$

19. $\begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$

20. $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$

21. $\begin{bmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{bmatrix}$

22. $\begin{bmatrix} 2 & -2 & 2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$

23. $\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$

24. $\begin{bmatrix} 5 & 2 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 0 & 0 & 5 & -2 \\ 0 & 0 & -2 & 2 \end{bmatrix}$

25. Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$.

    What can you infer about the matrix A from the eigenvalues. Verify your answer.

26. Given that $A = \begin{bmatrix} -15 & 4 & 3 \\ 10 & -12 & 6 \\ 20 & -4 & 2 \end{bmatrix}$, verify that the sum and product of the eigenvalues of A are equal to the trace of A and $|A|$ respectively.

27. Verify that the eigenvalues of A^2 and A^{-1} are respectively the squares and reciprocals of the eigenvalues of A, given that $A = \begin{bmatrix} 3 & 0 & 0 \\ 8 & 4 & 0 \\ 6 & 2 & 5 \end{bmatrix}$.

28. Find the eigenvalues and eigenvectors of (adj A), when $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$.

29. Verify that the eigenvectors of the real symmetric matrix $A = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 1 & -2 \\ -1 & -2 & 1 \end{bmatrix}$.

    are orthogonal in pairs.

30. Verify that the matrix $A = \frac{1}{3}\begin{bmatrix} -1 & 2 & -2 \\ -2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$ is orthogonal and that its eigenvalues are of unit modulus.

1.7 CAYLEY-HAMILTON THEOREM

This theorem is an interesting one that provides an alternative method for finding the inverse of a matrix A. Also any positive integral power of A can be expressed, using this theorem, as a linear combination of those of lower degree. We give below the statement of the theorem without proof:

1.7.1 Statement of the Theorem

Every square matrix satisfies its own characteristic equation.

This means that, if $c_0 \lambda^n + c_1 \lambda^{n-1} + \cdots + c_{n-1} \lambda + c_n = 0$ is the characteristic equation of a square matrix A of order n, then

$$c_0 A^n + c_1 A^{n-1} + \cdots + c_{n-1} A + c_n I = 0 \tag{1}$$

Note: ☑ When λ is replaced by A in the characteristic equation, the constant term c_n should be replaced by $c_n I$ to get the result of Cayley-Hamilton theorem, where I is the unit matrix of order n.

Also 0 in the R.S. of (1) is a null matrix of order n.

1.7.2 Corollary

(1) If A is non-singular, we can get A^{-1}, using the theorem, as follows:
Multiplying both sides of (1) by A^{-1} we have

$$c_0 A^{n-1} + c_1 A^{n-2} + \cdots + c_{n-1} I + c_n A^{-1} = 0$$

$$\therefore \qquad A^{-1} = -\frac{1}{c_n}\left(c_0 A^{n-1} + c_1 A^{n-2} + \cdots + c_{n-1} I\right).$$

(2) If we multiply both sides of (1) by A, $c_0 A^{n+1} + c_1 A^n + \cdots + c_{n-1} A^2 + c_n A = 0$

$$\therefore \qquad A^{n+1} = -\frac{1}{c_0}\left(c_1 A^n + c_2 A^{n-1} + \cdots + c_{n+1} A^2 + c_n A\right)$$

Thus higher positive integral powers of A can be computed, if we know powers of A of lower degree.

1.7.3 Similar Matrices

Two matrices A and B are said to be similar, if there exists a non-singular matrix P such that $B = P^{-1} AP$.

When A and B are connected by the relation $B = P^{-1} AP$, B is said to be obtained from A by a similarity transformation.

When B is obtained from A by a similarity transformation, A is also obtained from B by a similarity transformation as explained below:

$$B = P^{-1} AP$$

Premultiplying both sides by P and postmultiplying by P^{-1}, we get

$$PBP^{-1} = PP^{-1} APP^{-1}$$
$$= A$$

Thus $$A = PBP^{-1}$$

Now taking $P^{-1} = Q$, we get $A = Q^{-1} BQ$.

1.8 PROPERTY

Two similar matrices have the same eigenvalues.

Let A and B be two similar matrices.

Then, by definition, $B = P^{-1} AP$

$$\therefore \qquad B - \lambda I = P^{-1}AP - \lambda I$$
$$= P^{-1} AP - P^{-1} \lambda IP$$

$$= P^{-1}(A - \lambda I)P$$

$$\therefore \quad |B - \lambda I| = |P^{-1}|\,|A - \lambda I|\,|P|$$
$$= |A - \lambda I|\,|P^{-1}P|$$
$$= |A - \lambda I|\,|I|$$
$$= |A - \lambda I|$$

Thus A and B have the same characteristic polynomials and hence the same characteristic equations.
$\therefore$ A and B have the same eigenvalues.

1.8.1 Diagonalisation of a Matrix

The process of finding a matrix M such that $M^{-1}AM = D$, where D is a diagonal matrix, is called diagonalisation of the matrix A. As $M^{-1}AM = D$ is a similarity transformation, the matrices A and D are similar and hence A and D have the same eigenvalues.

The eigenvalues of D are its diagonal elements. Thus, if we can find a matrix M such that $M^{-1}AM = D$, D is not any arbitrary diagonal matrix, but it is a diagonal matrix whose diagonal elements are the eigenvalues of A.

The following theorem provides the method of finding M for a given square matrix whose eigenvectors are distinct and hence whose eigenvectors are linearly independent.

1.8.2 Theorem

If A is a square matrix with distinct eigenvalues and M is the matrix whose columns are the eigenvectors of A, then A can be diagonalised by the similarity transformation $M^{-1}AM = D$, where D is the diagonal matrix whose diagonal elements are the eigenvalues of A.

Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the distinct eigenvalues of A and $X_1, X_2, \ldots, X_n$ be the corresponding eigenvectors.

Let $M = [X_1, X_2, \ldots, X_n]$, which is an $n \times n$ matrix, called the Modal matrix.
$\therefore$ $AM = [AX_1, AX_2, \ldots, AX_n]$ [*Note* ☑ Each AX_r is a $(n \times 1)$ column vector]
Since X_r is the eigenvector of A corresponding to the eigenvalue λ_r,

$$AX_r = \lambda X_r \quad (r = 1, 2, \ldots n)$$

$$\therefore \quad AM = [\lambda_1 X_1, \lambda_2 X_2, \ldots, \lambda_n X_n]$$

$$= [X_1, X_2, \ldots, X_n]\begin{bmatrix} \lambda_1 & 0 & 0 & -- & 0 \\ 0 & \lambda_2 & 0 & -- & 0 \\ - & - & - & - & - \\ 0 & 0 & 0 & -- & \lambda_n \end{bmatrix}$$

$$= MD \tag{1}$$

As $X_1, X_2, \ldots, X_n$ are linearly independent column vectors, M is a non-singular matrix Premultiplying both sides of (1) by M^{-1}, we get $M^{-1}AM = M^{-1}MD = D$.

Note ☑ For this diagonalisation process, A need not necessarily have distinct eigenvalues. Even if two or more eigenvalues of A are equal, the process holds good, provided the eigenvectors of A are linearly independent.

1.9 CALCULATION OF POWERS OF A MATRIX *A*

Assuming A satisfies the conditions of the previous theorem,

$$D = M^{-1} AM$$

$\therefore$ $$A = M D M^{-1}$$

$$A^2 = (M D M^{-1})\,(M D M^{-1})$$

$$= MD(M^{-1} M)DM^{-1}$$

$$= MD^2M^{-1}$$

Similarly, $$A^3 = MD^3\, M^{-1}$$

Extending, $$A^k = M D^k\, M^{-1}$$

$$= M \begin{bmatrix} \lambda_1^k & 0 & 0 & -- & 0 \\ 0 & \lambda_2^k & 0 & -- & 0 \\ - & - & - & -- & - \\ 0 & 0 & 0 & -- & \lambda_n^k \end{bmatrix} M^{-1}$$

1.10 DIAGONALISATION BY ORTHOGONAL TRANSFORMATION OR ORTHOGONAL REDUCTION

If A is a real symmetric matrix, then the eigenvectors of A will be not only linearly independent but also pairwise orthogonal. If we normalise each eigenvector X_r, i.e. divide each element of X_r by the square-root of the sum of the squares of all the elements of X_r and use the normalised eigenvectors of A to form the normalised modal matrix N, then it can be proved that N is an orthogonal matrix. By a property of orthogonal matrix, $N^{-1} = N^T$.

$\therefore$ The similarity transformation $M^{-1} A\, M = D$ takes the form $N^T AN = D$.

Transforming A into D by means of the transformation $N^T AN = D$ is known as orthogonal transformation or orthogonal reduction.

Note: ☑ Diagonalisation by orthogonal transformation is possible only for a real symmetric matrix.

WORKED EXAMPLE 1(c)

Example 1.1 Verify Cayley-Hamilton theorem for the matrix $A = \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}$ and also use it to find A^{-1}.

The characteristic equation of A is

$$\begin{vmatrix} 1-\lambda & 3 & 7 \\ 4 & 2-\lambda & 3 \\ 1 & 2 & 1-\lambda \end{vmatrix} = 0$$

i.e. $(1-\lambda)(\lambda^2-3\lambda-4)-3(4-4\lambda-3)+7(8-2+\lambda)=0$

i.e. $\lambda^3-4\lambda^2-20\lambda-35=0$

Cayley-Hamilton theorem states that

$$A^3-4A^2-20A-35I=0 \tag{1}$$

which is to be verified.

Now, $$A^2=\begin{bmatrix}1&3&7\\4&2&3\\1&2&1\end{bmatrix}\begin{bmatrix}1&3&7\\4&2&3\\1&2&1\end{bmatrix}=\begin{bmatrix}20&23&23\\15&22&37\\10&9&14\end{bmatrix}$$

$$A^3=A\cdot A^2=\begin{bmatrix}1&3&7\\4&2&3\\1&2&1\end{bmatrix}\begin{bmatrix}20&23&23\\15&22&37\\10&9&14\end{bmatrix}=\begin{bmatrix}135&152&232\\140&163&208\\60&76&111\end{bmatrix}$$

Substituting these values in (1), we get,

$$\text{L.S.}=\begin{bmatrix}135&152&232\\140&163&208\\60&76&111\end{bmatrix}-\begin{bmatrix}80&92&92\\60&88&148\\40&36&56\end{bmatrix}-\begin{bmatrix}20&60&140\\80&40&60\\20&40&20\end{bmatrix}-\begin{bmatrix}35&0&0\\0&35&0\\0&0&35\end{bmatrix}$$

$$=\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$$

$$=\text{R.S.}$$

Thus Cayley-Hamilton theorem is verified. Premultiplying (1) by A^{-1},

$$A^2-4A-20I-35A^{-1}=0$$

$$\therefore \qquad A^{-1}=\frac{1}{35}\left(A^2-4A-20I\right)$$

$$=\frac{1}{35}\left(\begin{bmatrix}20&23&23\\15&22&37\\10&9&14\end{bmatrix}-\begin{bmatrix}4&12&28\\16&8&12\\4&8&4\end{bmatrix}-\begin{bmatrix}20&0&0\\0&20&0\\0&0&20\end{bmatrix}\right)$$

$$=\frac{1}{35}\begin{bmatrix}-4&11&-5\\-1&-6&25\\6&1&-10\end{bmatrix}$$

Example 1.2 Verify that the matrix $A=\begin{bmatrix}2&-1&2\\-1&2&-1\\1&-1&2\end{bmatrix}$ satisfies its characteristic equation and hence find A^4.

The characteristic equation of A is

$$\begin{vmatrix}2-\lambda&-1&2\\-1&2-\lambda&-1\\1&-1&2-\lambda\end{vmatrix}=0$$

i.e. $(2-\lambda)(\lambda^2-4\lambda+3)+(\lambda-2+1)+2(1-2+\lambda)=0$

i.e. $$\lambda^3-6\lambda^2+8\lambda-3=0 \quad (1)$$

According to Cayley-Hamilton theorem, A satisfies (1), i.e.

$$A^3-6A^2+8A-3I=0 \quad (2)$$

which is to be verified.

Now $$A^2=\begin{bmatrix}2&-1&2\\-1&2&-1\\1&-1&2\end{bmatrix}\begin{bmatrix}2&-1&2\\-1&2&-1\\1&-1&2\end{bmatrix}=\begin{bmatrix}7&-6&9\\-5&6&-6\\5&-5&7\end{bmatrix}$$

$$A^3=A\cdot A^2=\begin{bmatrix}2&-1&2\\-1&2&-1\\1&-1&2\end{bmatrix}\begin{bmatrix}7&-6&9\\-5&6&-6\\5&-5&7\end{bmatrix}=\begin{bmatrix}29&-28&38\\-22&23&-28\\22&-22&29\end{bmatrix}$$

Substituting these values in (2),

$$\text{L.S.}=\begin{bmatrix}29&-28&38\\-22&23&-28\\22&-22&29\end{bmatrix}-\begin{bmatrix}42&-36&54\\-30&36&-36\\30&-30&42\end{bmatrix}+\begin{bmatrix}16&-8&16\\-8&16&-8\\8&-8&16\end{bmatrix}-\begin{bmatrix}3&0&0\\0&3&0\\0&0&3\end{bmatrix}$$

$$=\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}=\text{R.S.}$$

Thus A satisfies its characteristic equation.
Multiplying both sides of (2) by A, we have,

$$A^4-6A^3+8A^2-3A=0$$

$\therefore$ $$A^4=6A^3-8A^2+3A \quad (3)$$

$$=6(6A^2-8A+3I)-8A^2+3A, \text{ using (2)}$$

$$=28A^2-45A+18I \quad (4)$$

A^4 can be computed by using either (3) or (4).
From (4),

$$A^4=\begin{bmatrix}196&-168&252\\-140&168&-168\\140&-140&196\end{bmatrix}-\begin{bmatrix}90&-45&90\\-45&90&-45\\45&-45&90\end{bmatrix}+\begin{bmatrix}18&0&0\\0&18&0\\0&0&18\end{bmatrix}$$

$$=\begin{bmatrix}124&-123&162\\-95&96&-123\\95&-95&124\end{bmatrix}$$

Example 1.3 Use Cayley-Hamilton theorem to find the value of the matrix given by $(A^8-5A^7+7A^6-3A^5+8A^4-5A^3+8A^2-2A+I)$, if the matrix

$$A=\begin{bmatrix}2&1&1\\0&1&0\\1&1&2\end{bmatrix}.$$

The characteristic equation of A is

$$\begin{vmatrix} 2-\lambda & 1 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 1 & 2-\lambda \end{vmatrix} = 0$$

i.e. $(2-\lambda)(\lambda^2 - 3\lambda + 2) + \lambda - 1 = 0$

i.e. $\lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0$

$\therefore$ $A^3 - 5A^2 + 7A - 3I = 0$, by Cayley-Hamilton theorem (1)

Now the given polynomial in A

$= A^5(A^3 - 5A^2 + 7A - 3I) + A(A^3 - 5A^2 + 8A - 2I) + I$

$= 0 + A(A^3 - 5A^2 + 7A - 3I) + A^2 + A + I$, by (1)

$= A^2 + A + I$, again using (1) (2)

Now $$A^2 = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix}$$

Substituting in (2), the given polynomial

$$= \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix} + \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 5 & 5 \\ 0 & 3 & 0 \\ 5 & 5 & 8 \end{bmatrix}$$

Example 1.4 Find the eigenvalues of A and hence find A^n (n is a positive integer), given that $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$.

The characteristic equation of A is

$$\begin{vmatrix} 1-\lambda & 2 \\ 4 & 3-\lambda \end{vmatrix} = 0$$

i.e. $\lambda^2 - 4\lambda - 5 = 0$

$\therefore$ The eigenvalues of A are $\lambda = -1, 5$

When λ^n is divided by $(\lambda^2 - 4\lambda - 5)$, let the quotient be $Q(\lambda)$ and the remainder be $(a\lambda + b)$.

Then $\lambda^n \equiv (\lambda^2 - 4\lambda - 5)\, Q(\lambda) + (a\lambda + b)$ (1)

Put $\lambda = -1$ in (1). $-a + b = (-1)^n$ (2)

Put $\lambda = 5$ in (l). $5a + b = 5^n$ (3)

Solving (2) and (3), we get

$$a = \frac{5^n - (-1)^n}{6} \quad \text{and} \quad b = \frac{5^n + 5(-1)^n}{6}$$

Replacing λ by the matrix A in (1), we have

$$\begin{aligned} A^n &= (A^2 - 4A - 5I)\,Q(A) + aA + bI \\ &= 0 \times Q(A) + aA + bI \text{ (by Cayley-Hamilton theorem)} \\ &= \left\{\frac{5^n - (-1)^n}{6}\right\}\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} + \left\{\frac{5^n + 5(-1)^n}{6}\right\}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

For example, when $n = 3$,

$$\begin{aligned} A^3 &= \left(\frac{125+1}{6}\right)\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} + \left(\frac{125-5}{6}\right)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 21 & 42 \\ 84 & 63 \end{bmatrix} + \begin{bmatrix} 20 & 0 \\ 0 & 20 \end{bmatrix} \\ &= \begin{bmatrix} 41 & 42 \\ 84 & 83 \end{bmatrix} \end{aligned}$$

Example 1.5 Diagonalise the matrix $A = \begin{bmatrix} 2 & 2 & -7 \\ 2 & 1 & 2 \\ 0 & 1 & -3 \end{bmatrix}$ by similarity transformation and hence find A^4.

The characteristic equation of A is

$$\begin{vmatrix} 2-\lambda & 2 & -7 \\ 2 & 1-\lambda & 2 \\ 0 & 1 & -3-\lambda \end{vmatrix} = 0$$

i.e. $(2 - \lambda)(\lambda^2 + 2\lambda - 5) - 2(-6 - 2\lambda + 7) = 0$

i.e. $\lambda^3 - 13\lambda + 12 = 0$

i.e. $(\lambda - 1)(\lambda - 3)(\lambda + 4) = 0$

∴ Eigenvalues of A are $\lambda = 1, 3, -4$.

Case (i) $\lambda = 1$.

The eigenvector is given by $\begin{bmatrix} 1 & 2 & -7 \\ 2 & 0 & 2 \\ 0 & 1 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

$$\therefore \quad \frac{x_1}{-2} = \frac{x_2}{8} = \frac{x_3}{2}$$

$$\therefore \quad X_1 = \begin{bmatrix} 1 \\ -4 \\ -1 \end{bmatrix}$$

Case (ii) $\lambda = 3$.

The eigenvector is given by $\begin{bmatrix} -1 & 2 & -7 \\ 2 & -2 & 2 \\ 0 & 1 & -6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

$$\therefore \qquad \frac{x_1}{10} = \frac{x_2}{12} = \frac{x_3}{2}$$

$$\therefore \qquad X_2 = \begin{bmatrix} 5 \\ 6 \\ 1 \end{bmatrix}$$

Case (iii) $\lambda = -4$.

The eigenvector is given by $\begin{bmatrix} 6 & 2 & -7 \\ 2 & 5 & 2 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

$$\therefore \qquad \frac{x_1}{3} = \frac{x_2}{-2} = \frac{x_3}{2}$$

$$\therefore \qquad X_3 = \begin{bmatrix} 3 \\ -2 \\ 2 \end{bmatrix}$$

Hence the modal matrix is $M = \begin{bmatrix} 1 & 5 & 3 \\ -4 & 6 & -2 \\ -1 & 1 & 2 \end{bmatrix}$

Let $$M \equiv \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Then the co-factors are given by

$$A_{11} = 14, \quad A_{12} = 10, \quad A_{13} = 2, \quad A_{21} = -7, \quad A_{22} = 5, \quad A_{23} = -6,$$
$$A_{31} = -28, \quad A_{32} = -10, \quad A_{33} = 26.$$

and $$|M| = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} = 70.$$

$$\therefore \qquad M^{-1} = \frac{1}{70}\begin{bmatrix} 14 & -7 & -28 \\ 10 & 5 & -10 \\ 2 & -6 & 26 \end{bmatrix}$$

The required similarity transformation is

$$M^{-1} A\, M = D(1, 3, -4) \tag{1}$$

which is verified as follows:

$$AM = \begin{bmatrix} 2 & 2 & -7 \\ 2 & 1 & 2 \\ 0 & 1 & -3 \end{bmatrix}\begin{bmatrix} 1 & 5 & 3 \\ -4 & 6 & -2 \\ -1 & 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 15 & -12 \\ -4 & 18 & 8 \\ -1 & 3 & -8 \end{bmatrix}$$

$$M^{-1}\ A\ M = \frac{1}{70}\begin{bmatrix} 14 & -7 & -28 \\ 10 & 5 & -10 \\ 2 & -6 & 26 \end{bmatrix}\begin{bmatrix} 1 & 15 & -12 \\ -4 & 18 & 8 \\ -1 & 3 & -8 \end{bmatrix}$$

$$= \frac{1}{70}\begin{bmatrix} 70 & 0 & 0 \\ 0 & 210 & 0 \\ 0 & 0 & -280 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -4 \end{bmatrix}$$

A^4 is given by $$A^4 = M\,D^4\,M^{-1} \tag{2}$$

$$D^4\ M^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 81 & 0 \\ 0 & 0 & 256 \end{bmatrix} \times \frac{1}{70}\begin{bmatrix} 14 & -7 & -28 \\ 10 & 5 & -10 \\ 2 & -6 & 26 \end{bmatrix}$$

$$= \frac{1}{70}\begin{bmatrix} 14 & -7 & -28 \\ 810 & 405 & -810 \\ 512 & -1536 & 6656 \end{bmatrix}$$

$$M\,D^4\ M^{-1} = \begin{bmatrix} 1 & 5 & 3 \\ -4 & 6 & -2 \\ -1 & 1 & 2 \end{bmatrix} \times \frac{1}{70}\begin{bmatrix} 14 & -7 & -28 \\ 810 & 405 & -810 \\ 512 & -1536 & 6656 \end{bmatrix}$$

$$= \frac{1}{70}\begin{bmatrix} 5600 & -2590 & 15890 \\ 3780 & 5530 & -18060 \\ 1820 & -2660 & 12530 \end{bmatrix}$$

i.e. $$A^4 = \begin{bmatrix} 80 & -37 & 227 \\ 54 & 79 & -258 \\ 26 & -38 & 179 \end{bmatrix}$$

Example 1.6 Find the matrix M that diagonalises the matrix $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$ by means of a similarity transformation. Verify your answer. The characteristic equation of A is

$$\begin{vmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{vmatrix} = 0$$

i.e. $$(2-\lambda)(\lambda^2 - 5\lambda + 4) - 2(1-\lambda) + (\lambda - 1) = 0$$

i.e. $$\lambda^3 - 7\lambda^2 + 11\lambda - 5 = 0$$

i.e. $$(\lambda - 1)^2 (\lambda - 5) = 0$$

$\therefore$ The eigenvalues of A are $\lambda = 5, 1, 1$.

Case (i) $\lambda = 5$.

The eigenvector is given by $\begin{bmatrix} -3 & 2 & 1 \\ 1 & -2 & 1 \\ 1 & 2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

$\therefore$ $$\frac{x_1}{4} = \frac{x_2}{4} = \frac{x_3}{4}$$

$\therefore$ $$X_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Case (ii) $\lambda = 1$.

The eigenvector is given by $\begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

All the three equations are one and the same, namely, $x_1 + 2x_2 + x_3 = 0$

Two independent solutions are obtained as follows:

Putting $x_2 = -1$ and $x_3 = 0$, we get $x_1 = 2$

Putting $x_2 = 0$ and $x_3 = -1$, we get $x_1 = 1$

$\therefore$ $$X_2 = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} \quad \text{and} \quad X_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

Hence the modal matrix is

$$M = \begin{bmatrix} 1 & 2 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix} \equiv \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Then the co-factors are given by

$A_{11} = 1$, $A_{12} = 1$, $A_{13} = 1$, $A_{21} = 2$, $A_{22} = -2$, $A_{23} = 2$

$A_{31} = 1$, $A_{32} = 1$, $A_{33} = -3$ and

$$|M| = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} = 4$$

$$\therefore \qquad M^{-1} = \frac{1}{4}\begin{bmatrix} 1 & 2 & 1 \\ 1 & -2 & 1 \\ 1 & 2 & -3 \end{bmatrix}$$

The required similarity transformation is

$$M^{-1}\,A\,M = D(5,1,1) \tag{1}$$

We shall now verify (1).

$$AM = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}\begin{bmatrix} 1 & 2 & 1 \\ 1 & -2 & 1 \\ 1 & 2 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 2 & 1 \\ 5 & -2 & 1 \\ 5 & 2 & -3 \end{bmatrix}$$

$$M^{-1}\,A\,M = \frac{1}{4}\begin{bmatrix} 1 & 2 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}\begin{bmatrix} 5 & 2 & 1 \\ 5 & -2 & 1 \\ 5 & 2 & -3 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 20 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= D(5,1,1).$$

Example 1.7 Diagonalise the matrix $A = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 1 & -2 \\ -1 & -2 & 1 \end{bmatrix}$ by means of an orthogonal transformation. The characteristic equation of A is

$$\begin{vmatrix} 2-\lambda & 1 & -1 \\ 1 & 1-\lambda & -2 \\ -1 & -2 & 1-\lambda \end{vmatrix} = 0$$

i.e. $$(2-\lambda)(\lambda^2 - 2\lambda - 3) - (-\lambda - 1) - (-\lambda - 1) = 0$$

i.e. $$\lambda^3 - 4\lambda^2 - \lambda + 4 = 0$$

i.e. $$(\lambda + 1)(\lambda - 1)(\lambda - 4) = 0$$

$\therefore$ The eigenvalues of A are 1 = –1, 1, 4.

Case (i) $\lambda = -1$.

The eigenvector is given by $\begin{bmatrix} 3 & 1 & -1 \\ 1 & 2 & -2 \\ -1 & -2 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

$$\therefore \qquad \frac{x_1}{0} = \frac{x_2}{5} = \frac{x_3}{5}$$

$$\therefore \qquad X_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Case (ii) $\lambda = 1$.

The eigenvector is given by $\begin{bmatrix} 1 & 1 & -1 \\ 1 & 0 & -2 \\ -1 & -2 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

$$\therefore \qquad \frac{x_1}{-4} = \frac{x_2}{2} = \frac{x_3}{-2}$$

$$\therefore \qquad X_2 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

Case (iii) $\lambda = 4$.

The eigenvector is given by $\begin{bmatrix} -2 & 1 & -1 \\ 1 & -3 & -2 \\ -1 & -2 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

$$\therefore \qquad \frac{x_1}{5} = \frac{x_2}{5} = \frac{x_3}{-5}$$

$$\therefore \qquad X_3 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

Hence the modal matrix $M = \begin{bmatrix} 0 & 2 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$

Normalising each column vector of M, i.e. dividing each element of the first column by $\sqrt{2}$, that of the second column by $\sqrt{6}$ and that of the third column by $\sqrt{3}$, we get the normalised modal matrix N.

Thus
$$N=\begin{bmatrix} 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} \end{bmatrix}$$

The required orthogonal transformation that diagonalises A is

$$N^T A\, N = D\,(-1, 1, 4) \qquad (1)$$

which is verified below:

$$A\,N=\begin{bmatrix} 2 & 1 & -1 \\ 1 & 1 & -2 \\ -1 & -2 & 1 \end{bmatrix}\begin{bmatrix} 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} \end{bmatrix}$$

$$=\begin{bmatrix} 0 & \frac{2}{\sqrt{6}} & \frac{4}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{4}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & -\frac{4}{\sqrt{3}} \end{bmatrix}$$

$$N^T\ A\,N=\begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \end{bmatrix}\begin{bmatrix} 0 & \frac{2}{\sqrt{6}} & \frac{4}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{4}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & -\frac{4}{\sqrt{3}} \end{bmatrix}$$

$$=\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$=D\left(-1,1,4\right).$$

Example 1.8 Diagonalise the matrix $A=\begin{bmatrix} 2 & 0 & 4 \\ 0 & 6 & 0 \\ 4 & 0 & 2 \end{bmatrix}$ by means of an orthogonal transformation.

The characteristic equation of A is

$$\begin{vmatrix} 2-\lambda & 0 & 4 \\ 0 & 6-\lambda & 0 \\ 4 & 0 & 2-\lambda \end{vmatrix} = 0$$

i.e. $$(2-\lambda)(6-\lambda)(2-\lambda) - 16(6-\lambda) = 0$$

i.e. $$(6-\lambda)(\lambda^2 - 4\lambda - 12) = 0$$

i.e. $$(6-\lambda)(\lambda-6)(\lambda+2) = 0$$

$\therefore$ The eigenvalues of A are $\lambda = -2, 6, 6$.

Case (i) $\lambda = -2$.

The eigenvector is given by $\begin{bmatrix} 4 & 0 & 4 \\ 0 & 8 & 0 \\ 4 & 0 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

$\therefore$ $$\frac{x_1}{32} = \frac{x_2}{0} = \frac{x_3}{-32}$$

$\therefore$ $$X_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

Case (ii) $\lambda = 6$.

The eigenvector is given by $\begin{bmatrix} -4 & 0 & 4 \\ 0 & 0 & 0 \\ 4 & 0 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

We get only one equation,

i.e. $$x_1 - x_3 = 0 \qquad (1)$$

From this we get, $x_1 = x_3$ and x_2 is arbitrary.

x_2 must be so chosen that X_2 and X_3 are orthogonal among themselves and also each is orthogonal with X_1.

Let us choose X_2 arbitrarily as $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

Note ☑ This assumption of X_2 satisfies (1) and x_2 is taken as 0.

Let $X_3 = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

X_3 is orthogonal to X_1

$\therefore$ $$a - c = 0 \qquad (2)$$

X_3 is orthogonal to X_2

$\therefore$ $$a + c = 0 \qquad (3)$$

Solving (2) and (3), we get $a = c = 0$ and b is arbitrary.

Taking $b = 1,\ X_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

Note ☑ Had we assumed X_2 in a different form, we should have got a different X_3.

For example, if $X_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$, then $X_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

The modal matrix is $M = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}$

The normalised model matrix is

$$N = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix}$$

The required orthogonal transformation that diagonalises A is

$$N^T AN = D(-2, 6, 6) \tag{1}$$

which is verified below:

$$AN = \begin{bmatrix} 2 & 0 & 4 \\ 0 & 6 & 0 \\ 4 & 0 & 2 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{2}{\sqrt{2}} & \frac{6}{\sqrt{2}} & 0 \\ 0 & 0 & 6 \\ \frac{2}{\sqrt{2}} & \frac{6}{\sqrt{2}} & 0 \end{bmatrix}$$

$$N^T AN = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} -\frac{2}{\sqrt{2}} & \frac{6}{\sqrt{2}} & 0 \\ 0 & 0 & 6 \\ \frac{2}{\sqrt{2}} & \frac{6}{\sqrt{2}} & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

$$= D(-2, 6, 6)$$

Note ☑ From the above problem, it is clear that diagonalisation of a real symmetric matrix is possible by orthogonal transformation, even if two or more eigenvalues are equal.

EXERCISE 1(c)

Part A

(Short Answer Questions)

1. State Cayley-Hamilton theorem.
2. Give two uses of Cayley-Hamilton theorem.
3. When are two matrices said to be similar? Give a property of similar matrices.
4. What do you mean by diagonalising a matrix?
5. Explain how you will find A^k, using the similarity transformation $M^{-1} AM = D$.
6. What is the difference between diagonalisation of a matrix by similarity and orthogonal transformations?
7. What type of matrices can be diagonalised using (i) similarity transformation and (ii) orthogonal transformation?
8. Verify Cayley-Hamilton theorem for the matrix $A = \begin{bmatrix} 5 & 3 \\ 1 & 3 \end{bmatrix}$.
9. Use Cayley-Hamilton theorem to find the inverse of $A = \begin{bmatrix} 7 & 3 \\ 2 & 6 \end{bmatrix}$.
10. Use Cayley-Hamilton theorem to find A^3, given that $A = \begin{bmatrix} -1 & 3 \\ 2 & 4 \end{bmatrix}$.

11. Use Cayley-Hamilton theorem to find $(A^4 - 4A^3 - 5A^2 + A + 2I)$, when $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$.

12. Find the modal matrix that will diagonalise the matrix $A = \begin{bmatrix} 5 & 3 \\ 1 & 3 \end{bmatrix}$.

Part B

13. Show that the matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ satisfies its own characteristic equation and hence find A^{-1}.

14. Verify Cayley-Hamilton theorem for the matrix $A = \begin{bmatrix} 7 & 2 & -2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{bmatrix}$ and hence find A^{-1}

15. Verify Cayley-Hamilton theorem for the matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}$ and hence find A^{-1}.

16. Verify that the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{bmatrix}$ satisfies its own characteristic equation and hence find A^4.

17. Verify that the matrix $A = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$ satisfies its a own characteristic equation and hence find A^4.

18. Find A^n, using Cayley-Hamilton theorem, when $A = \begin{bmatrix} 5 & 3 \\ 1 & 3 \end{bmatrix}$. Hence find A^4.

19. Find A^n, using Cayley-Hamilton theorem, when $A = \begin{bmatrix} 7 & 3 \\ 2 & 6 \end{bmatrix}$. Hence find A^3.

20. Given that $A = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$, compute the value of $(A^6 - 5A^5 + 8A^4 - 2A^3 - 9A^2 + 31A - 36I)$, using Caylay-Hamilton theorem.

Diagonalise the following matrices by similarity transformation:

21. $\begin{bmatrix} 2 & 2 & 0 \\ 2 & 1 & 1 \\ -7 & 2 & -3 \end{bmatrix}$

22. $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ -4 & 4 & 3 \end{bmatrix}$; find also the fourth power of this matrix.

23. $\begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$ 24. $\begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix}$ 25. $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$

26. $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

Diagonalise the following matrices by orthogonal transformation:

27. $\begin{bmatrix} 10 & -2 & -5 \\ -2 & 2 & 3 \\ -5 & 3 & 5 \end{bmatrix}$ 28. $\begin{bmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{bmatrix}$ 29. $\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$

30. $\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$

1.11 QUADRATIC FORMS

A homogeneous polynomial of the second degree in any number of variables is called a quadratic form.

For example, $x_1^2 + 2x_2^2 - 3x_3^2 + 5x_1x_2 - 6x_1x_3 + 4x_2x_3$ is a quadratic form in three variables.

The general form of a quadratic form, denoted by Q in n variables is

$$\begin{aligned} Q = {} & c_{11}x_1^2 + c_{12}x_1x_2 + \cdots + c_{1n}x_1x_n \\ & + c_{21}x_2x_1 + c_{22}x_2^2 + \cdots + c_{2n}x_2x_n \\ & + c_{31}x_3x_1 + c_{32}x_3x_2 + \cdots + c_{3n}x_3x_n \\ & + (\text{-----------------------}) \\ & + c_{n1}x_nx_1 + c_{n2}x_nx_2 + \cdots + c_{nn}x_n^2 \end{aligned}$$

i.e.
$$Q = \sum_{j=1}^{n}\sum_{i=1}^{n} c_{ij}x_ix_j$$

In general, $c_{ij} \neq c_{ji}$. The coefficient of $x_i x_j = c_{ij} + c_{ji}$.

Now if we define $a_{ij} = \frac{1}{2}\left(c_{ij} + c_{ji}\right)$, for all i and j, then $a_{ii} = c_{ii}$, $a_{ij} = a_{ji}$ and $a_{ij} + a_{ji} = 2a_{ij} = c_{ij} + c_{ji}$.

$\therefore Q = \sum_{j=1}^{n}\sum_{i=1}^{n} a_{ij}x_ix_j$, where $a_{ij} = a_{ji}$ and hence the matrix $A = [a_{ij}]$ is a symmetric matrix. In matrix notation, the quadratic form Q can be represented as $Q = X^T AX$, where

$$A = [a_{ij}],\ X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \text{and} \quad X^T = [x_1, x_2, \ldots, x_n].$$

The symmetric matrix $A = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$ is called *the matrix of the quadratic form Q*.

Note ☑ To find the symmetric matrix A of a quadratic form, the coefficient of x_i^2 is placed in the a_{ii} position and $\left(\frac{1}{2} \times \text{coefficient } x_i\, x_j\right)$ is placed in each of the a_{ij} and a_{ji} positions.

For example, (i) if $Q = 2x_1^2 - 3x_1x_2 + 4x_2^2$, then

$$A = \begin{bmatrix} 2 & -\frac{3}{2} \\ -\frac{3}{2} & 4 \end{bmatrix}$$

(ii) if $Q = x_1^2 + 3x_2^2 + 6x_3^2 - 2x_1x_2 + 6x_1x_3 + 5x_2x_3$,

then $A = \begin{bmatrix} 1 & -1 & 3 \\ -1 & 3 & \frac{5}{2} \\ 3 & \frac{5}{2} & 6 \end{bmatrix}$

Conversely, the quadratic form whose matrix is

$$\begin{bmatrix} 3 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 6 \\ 0 & 6 & -7 \end{bmatrix} \text{ is } Q = 3x_1^2 - 7x_3^2 + x_1x_2 + 12x_2x_3$$

1.11.1 Definitions

If A is the matrix of a quadratic form Q, $|A|$ is called *the determinant* or *modulus of Q*.

The rank r of the matrix A is called *the rank of the quadratic form.*

If $r < n$ (the order of A) or $|A| = 0$ or A is singular, the quadratic form is called *singular*. Otherwise it is non-singular.

1.11.2 Linear Transformation of a Quadratic Form

Let $Q = X^T AX$ be a quadratic form in the n variables $x_1, x_2, \ldots, x_n$.

Consider the transformation $X = PY$, that transforms the variable set $X = [x_1, x_2, \ldots, x_n]^T$ to a new variable set $Y = [y_1, y_2, \ldots, y_n]^T$, where P is a non-singular matrix.

We can easily verify that the transformation $X = PY$ expresses each of the variables $x_1, x_2, \ldots, x_n$ as homogeneous linear expressions in $y_1, y_2, \ldots, y_n$. Hence $X = PY$ is called a non-singular linear trans formation.

By this transformation, $Q = X^T AX$ is transformed to

$$\begin{aligned} Q &= (PY)^T A(PY) \\ &= Y^T \left(P^T AP\right)Y \\ &= Y^T BY, \text{ where } B = P^T AP \end{aligned}$$

Now
$$\begin{aligned} B^T &= \left(P^T AP\right)^T = P^T A^T P \\ &= P^T AP \qquad (\because A \text{ is sysmmetric}) \\ &= B \end{aligned}$$

$\therefore$ B is also a symmetric matrix.

Hence B is the matrix of the quadratic form $Y^T BY$ in the variables $y_1, y_2, \ldots, y_n$. Thus $Y^T B\ Y$ is the linear transform of the quadratic form $X^T AX$ under the linear transformation $X = PY$, where $B = P^T AP$.

1.11.3 Canonical Form of a Quadratic Form

In the linear transformation $X = PY$, if P is chosen such that $B = P^T A P$ is a diagonal

matrix of the form $\begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$, then the quadratic form Q gets reduced as

$$\begin{aligned} Q &= Y^T BY \\ &= [y_1, y_2, \ldots, y_n] \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \\ &= \lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots + \lambda_n y_n^2 \end{aligned}$$

This form of Q is called *the sum of the squares form of Q* or *the canonical form of Q*.

1.11.4 Orthogonal Reduction of a Quadratic Form to the Canonical Form

If, in the transformation $X = PY$, P is an orthogonal matrix and if $X = PY$ transforms the quadratic form Q to the canonical form then Q is said to be reduced to the canonical form by an orthogonal transformation.

We recall that if A is a real symmetric matrix and N is the normalised modal matrix of A, then N is an orthogonal matrix such that $N^T AN = D$, where D is a diagonal matrix with the eigenvalues of A as diagonal elements.

Hence, to reduce a quadratic form $Q = X^T AX$ to the canonical form by an orthogonal transformation, we may use the linear transformation $X = NY$, where N is the normalised modal matrix of A. By this orthogonal transformation, Q gets transformed into $Y^T DY$, where D is the diagonal matrix with the eigenvalues of A as diagonal elements.

1.11.5 Nature of Quadratic Forms

When the quadratic form $X^T AX$ is reduced to the canonical form, it will contain only r terms, if the rank of A is r.

The terms in the canonical form may be positive, zero or negative.

The number of positive terms in the canonical form is called *the index* (p) of the quadratic form.

The excess of the number of positive terms over the number of negative terms in the canonical form i.e. $p - (r - p) = 2p - r$ is called the *signature(s)* of the quadratic form i.e. $s = 2p - r$.

The quadratic form $Q = X^T A X$ in n variables is said to be

(i) positive definite, if $r = n$ and $p = n$ or if all the eigenvalues of A are positive.
(ii) negative definite, if $r = n$ and $p = 0$ or if all the eigenvalues of A are negative.
(iii) positive semidefinite, if $r < n$ and $p = r$ or if all the eigenvalues of $A \geq 0$ and at least one eigenvalue is zero.
(iv) negative semidefinite, if $r < n$ and $p = 0$ or if all the eigenvalues of $A \leq 0$ and at least one eigenvalue is zero.
(v) indefinite in all other cases or if A has positive as well as negative eigenvalues.

WORKED EXAMPLE 1(d)

Example 1.1 Reduce the quadratic form $2x_1^2 + x_2^2 + x_3^2 + 2x_1x_2 - 2x_1x_3 - 4x_2x_3$ to canonical form by an orthogonal transformation. Also find the rank, index, signature and nature of the quadratic form.

$$\text{Matrix of the Q.F. is } A = \begin{bmatrix} 2 & 1 & -1 \\ 1 & -1 & -2 \\ -1 & -2 & -1 \end{bmatrix}$$

Refer to the worked example (7) in section 1(c).
The eigenvalues of A are $-1, 1, 4$.
The corresponding eigenvectors are $[0, 1, 1]^T$ $[2, -1, 1]^T$ and $[1, 1, -1]^T$ respectively.

The modal matrix $M = \begin{bmatrix} 0 & 2 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$

The normalised modal matrix $N = \begin{bmatrix} 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} \end{bmatrix}$

Hence $N^T AN = D\ (-1, 1, 4)$, where D is a diagonal matrix with -1, 1, 4 as the principal diagonal elements.

$\therefore$ The orthogonal transformation $X = NY$ will reduce the Q.F. to the canonical form $-y_1^2 + y_2^2 + 4y_3^2$

Rank of the Q.F. = 3.

Index = 2

Signature = 1

Q.F. is indefinite in nature, as the canonical form contains both positive and negative terms.

Example 1.2 Reduce the quadratic form $2x_1^2 + 6x_2^2 + 2x_3^2 + 8x_1x_3$ to canonical form by orthogonal reduction. Find also the nature of the quadratic form.

Matrix of the Q.F. is $A = \begin{bmatrix} 2 & 0 & 4 \\ 0 & 6 & 0 \\ 4 & 0 & 2 \end{bmatrix}$

Refer to worked example (8) in section 1(c).

The eigenvalues of A are -2, 6, 6.

The corresponding eigenvectors are $[1, 0, -1]^T$, $[1, 0, 1]^T$ and $[0, 1, 0]^T$ respectively.

Note ☑ Though two of the eigenvalues are equal, the eigenvectors have been so chosen that all the three eigenvectors are pairwise orthogonal.

The modal matrix $M = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}$

The normalised modal matrix is given by

$$N = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix}$$

Hence $N^T AN = \text{Diag}\,(-2, 6, 6)$

$\therefore$ The orthogonal transformation $X = NY$

i.e.
$$x_1 = \frac{1}{\sqrt{2}}\, y_1 + \frac{1}{\sqrt{2}}\, y_2$$
$$x_2 = y_2$$
$$x_3 = -\frac{1}{\sqrt{2}}\, y_1 + \frac{1}{\sqrt{2}}\, y_2$$

will reduce the given Q.F. to the canonical form $-2y_1^2 + 6y_2^2 + 6y_3^2$.

The Q.F. is indefinite in nature, as the canonical form contains both positive and negative terms.

Example 1.3 Reduce the quadratic form $x_1^2 + 2x_2^2 + x_3^2 - 2x_1x_2 + 2x_2x_3$ to the canonical form through an orthogonal transformation and hence show that it is positive semidefinite. Give also a non-zero set of values (x_1, x_2, x_3) which makes this quadratic form zero.

Matrix of the Q.F. is $A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

The characteristic equation of A is $\begin{vmatrix} 1-\lambda & -1 & 0 \\ -1 & 2-\lambda & 1 \\ 0 & 1 & 1-\lambda \end{vmatrix} = 0$

i.e.
$$(1-\lambda)\{(2-\lambda)(1-\lambda)-1\}-(1-\lambda)=0$$

i.e.
$$(1-\lambda)(\lambda^2 - 3\lambda) = 0$$

$\therefore$ The eigenvalues of A are $\lambda = 0, 1, 3$.

When $\lambda = 0$, the elements of the eigenvector are given by $x_1 - x_2 = 0$, $-x_1 + 2x_2 + x_3 = 0$ and $x_2 + x_3 = 0$.

Solving these equations, $x_1 = 1, x_2 = 1, x_3 = -1$

$\therefore$ The eigenvector corresponding to $\lambda = 0$ is

$$[1, 1, -1]^T$$

When $\lambda = 1$, the elements of the eigenvector are given by $-x_2 = 0$, $-x_1 + x_2 + x_3 = 0$ and $x_2 = 0$.

Solving these equations, $x_1 = 1, x_2 = 0, x_3 = 1$.

$\therefore$ When $\lambda = 1$, the eigenvector is

$$[1, 0, 1]^T$$

When $\lambda = 3$, the elements of the eigenvector are given by $-2x_1 - x_2 = 0$, $-x_1 - x_2 + x_3 = 0$ and $x_2 - 2x_3 = 0$

Solving these equation, $x_1 = -1, x_2 = 2, x_3 = 1$.

$\therefore$ When $\lambda = 3$, the eigenvector is $[-1, 2, 1]^T$.

Now the modal matrix is $M = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 0 & 2 \\ -1 & 1 & 1 \end{bmatrix}$

The normalised modal matrix is

$$N = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \\ -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{bmatrix}$$

Hence $N^T AN = \text{Diag}\ (0, 1, 3)$

$\therefore$ The orthogonal transformation $X = NY$.

i.e.

$$x_1 = \frac{1}{\sqrt{3}} y_1 + \frac{1}{\sqrt{2}} y_2 - \frac{1}{\sqrt{6}} y_3$$

$$x_2 = \frac{1}{\sqrt{3}} y_1 + \frac{2}{\sqrt{6}} y_3$$

$$x_3 = -\frac{1}{\sqrt{3}} y_1 + \frac{1}{\sqrt{2}} y_2 + \frac{1}{\sqrt{6}} y_3$$

will reduce the given Q.F. to the canonical form $0 \cdot y_1^2 + y_2^2 + 3y_3^2 = y_2^2 + 3y_3^2$.

As the canonical form contains only two terms, both of which are positive, the Q.F. is positive semi-definite.

The canonical form of the Q.F. is zero, when $y_2 = 0$, $y_3 = 0$ and y_1 is arbitrary.

Taking $y_1 = \sqrt{3}$, $y_2 = 0$ and $y_3 = 0$, we get $x_1 = 1$, $x_2 = 1$ and $x_3 = -1$.

These values of x_1, x_2, x_3 make the Q.F. zero.

Example 1.4 Determine the nature of the following quadratic forms without reducing them to canonical forms:

$$\text{(i)}\ x_1^2 + 3x_2^2 + 6x_3^2 + 2x_1x_2 + 2x_2x_3 + 4x_3x_1$$

$$\text{(ii)}\ 5x_1^2 + 5x_2^2 + 14x_3^2 + 2x_1x_2 - 16x_2x_3 - 8x_3\ x_1$$

$$\text{(iii)}\ 2x_1^2 + x_2^2 - 3x_3^2 + 12x_1\ x_2 - 8x_2\ x_3 - 4x_3\ x_1.$$

Note ☑ We can find the nature of a Q.F. without reducing it to canonical form. The alternative method uses the principal sub-determinants of the matrix of the Q.F., as explained below:

Let $A = (a_{ij})_{n\times n}$ be the matrix of the Q.F.

Let $D_1 = |a_{11}| = a_{11}$, $D_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$,

$$D_3 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \text{ etc. and } D_n = |A|$$

$D_1, D_2, D_3, \ldots D_n$ are called the principal sub-determinants or principal minors of A.

(i) The Q.F. is positive definite, if $D_1, D_2, \ldots, D_n$ are all positive i.e. $D_n > 0$ for all n.

(ii) The Q.F. is negative definite, if $D_1, D_3, D_5, \ldots$ are all negative and $D_2, D_4, D_6, \ldots$ are all positive i.e. $(-1)^n D_n > 0$ for all n.

(iii) The Q.F. is positive semidefinite, if $D_n \geq 0$ and least one $D_i = 0$.

(iv) The Q.F. is negative semidefinite, if $(-1)^n\ D_n \geq 0$ and at least one $D_i = 0$.

(v) The Q.F. is indefinite in all other cases.

(i) $Q = x_1^2 + 3x_2^2 + 6x_3^2 + 2x_1x_2 + 2x_2x_3 + 4x_3x_1$

Matrix of the Q.F. is $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 3 & 1 \\ 2 & 1 & 6 \end{bmatrix}$

Now $\quad D_1 = |1| = 1; \; D_2 = \begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix} = 2$;

$$D_3 = 1 \cdot (18 - 1) - 1 \cdot (6 - 2) + 2(1 - 6) = 3.$$

D_1, D_2, D_3 are all positive.

$\therefore$ The Q.F. is positive definite.

(ii) $Q = 5x_1^2 + 5x_2^2 + 14x_3^2 + 2x_1x_2 - 16x_2x_3 - 8x_3x_1$.

$$A = \begin{bmatrix} 5 & 1 & -4 \\ 1 & 5 & -8 \\ -4 & -8 & 14 \end{bmatrix}$$

Now $D_1 = 5; \; D_2 = \begin{vmatrix} 5 & 1 \\ 1 & 5 \end{vmatrix} = 24;$

$$D_3 = |A| = 5 \cdot (70 - 64) - 1 \cdot (14 - 32) - 4 \cdot (-8 + 20)$$
$$= 30 + 18 - 48 = 0$$

D_1 and D_2 are > 0, but $D_3 = 0$

$\therefore$ The Q.F. is positive semidefinite.

(iii) $Q = 2x_1^2 + x_2^2 - 3x_3^2 + 12x_1\ x_2 - 8x_2\ x_3 - 4x_3\ x_1$

$$A = \begin{bmatrix} 2 & 6 & -2 \\ 6 & 1 & -4 \\ -2 & -4 & -3 \end{bmatrix}$$

Now $D_1 = |2| = 2; \; D_2 = \begin{vmatrix} 2 & 6 \\ 6 & 1 \end{vmatrix} = -34;$

$$D_3 = |A| = 2\cdot(-3-16) - 6\cdot(-18-8) - 2\,(-24+2)$$
$$= -38 + 156 + 44 = 162$$

$\therefore$ The Q.F. is indefinite.

Example 1.5 Reduce the quadratic forms $6x_1^2 + 3x_2^2 + 14x_3^2 + 4x_1x_2 + 4x_2x_3 + 18x_3x_1$ and $2x_1^2 + 5x_2^2 + 4x_1x_2 + 2x_3x_1$ simultaneously to canonical forms by a real non-singular transformation.

Note ☑ We can reduce two quadratic forms $X^T AX$ and $X^T BX$ to canonical forms simultaneously by the same linear transformation using the following theorem, (stated without proof):

If A and B are two symmetric matrices such that the roots of $|A - \lambda B| = 0$ are all distinct, then there exists a matrix P such that $P^T AP$ and $P^T BP$ are both diagonal matrices.

The procedure to reduce two quadratic forms simultaneously to canonical forms is given below:

(1) Let A and B be the matrices of the two given quadratic forms.
(2) Form the characteristic equation $|A - \lambda B| = 0$ and solve it. Let the eigenvalues (roots of this equation) be $\lambda_1, \lambda_2, \ldots, \lambda_n$.
(3) Find the eigenvectors X_i ($i = 1, 2, \ldots, n$) corresponding to the eigenvalues λ_i, using the equation $(A - \lambda_i B)\, X_i = 0$.
(4) Construct the matrix P whose column vectors are $X_1, X_2, \ldots, X_n$. Then $X = PY$ is the required linear transformation.
(5) Find $P^T AP$ and $P^T BP$, which will be diagonal matrices.
(6) The quadratic forms corresponding to these diagonal matrices are the required canonical forms.

The matrix of the first quadratic form is

$$A = \begin{bmatrix} 6 & 2 & 9 \\ 2 & 3 & 2 \\ 9 & 2 & 14 \end{bmatrix}$$

The matrix of the second quadratic form is

$$B = \begin{bmatrix} 2 & 2 & 1 \\ 2 & 5 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

The characteristic equation is $|A - \lambda B| = 0$

i.e. $$\begin{vmatrix} 6-2\lambda & 2-2\lambda & 9-\lambda \\ 2-2\lambda & 3-5\lambda & 2 \\ 9-\lambda & 2 & 14 \end{vmatrix} = 0$$

Simplifying, $$5\lambda^3 - \lambda^2 - 5\lambda + 1 = 0$$
i.e. $$(\lambda - 1)(5\lambda - 1)(\lambda + 1) = 0$$

$$\therefore\ \lambda = -1, \frac{1}{5}, 1$$

When $\lambda = -1$, $(A - \lambda B)X = 0$ gives the equations.
$8x_1 + 4x_2 + 10x_3 = 0$; $4x_1 + 8x_2 + 2x_3 = 0$; $10x_1 + 2x_2 + 14x_3 = 0$.

Solving these equations, $\dfrac{x_1}{-72} = \dfrac{x_2}{24} = \dfrac{x_3}{48}$

$$\therefore \qquad X_1 = [-3, 1, 2]^T$$

When $\lambda = \dfrac{1}{5}, (A - \lambda B)X = 0$ gives the equations.

$28x_1 + 8x_2 + 44x_3 = 0$; $8x_1 + 10x_2 + 10x_3 = 0$; $44x_1 + 10x_2 + 70x_3 = 0$.

Solving these equations, $\dfrac{x_1}{-360} = \dfrac{x_2}{72} = \dfrac{x_3}{216}$

$$\therefore \qquad X_2 = [-5, 1, 3]^T$$

When $\lambda = 1, (A - \lambda B)X = 0$ gives the equations

$$4x_1 + 8x_3 = 0; \quad -2x_2 + 2x_3 = 0; \quad 8x_1 + 2x_2 + 14x_3 = 0$$

$$\therefore \qquad X_3 = [2, -1, -1]^T.$$

Now $P = [X_1, X_2, X_3] = \begin{bmatrix} -3 & -5 & 2 \\ 1 & 1 & -1 \\ 2 & 3 & -1 \end{bmatrix}$

Now $$P^T AP = \begin{bmatrix} -3 & 1 & 2 \\ -5 & 1 & 3 \\ 2 & -1 & -1 \end{bmatrix} \begin{bmatrix} 6 & 2 & 9 \\ 2 & 3 & 2 \\ 9 & 2 & 14 \end{bmatrix} \begin{bmatrix} -3 & -5 & 2 \\ 1 & 1 & -1 \\ 2 & 3 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 & 3 \\ -1 & -1 & -1 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} -3 & -5 & 2 \\ 1 & 1 & -1 \\ 2 & 3 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Hence the Q.F. $X^T AX$ is reduced to the canonical form $y_1^2 + y_2^2 + y_3^2$.

Now $$P^T BP = \begin{bmatrix} -3 & 1 & 2 \\ -5 & 1 & 3 \\ 2 & -1 & -1 \end{bmatrix} \begin{bmatrix} 2 & 2 & 1 \\ 2 & 5 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} -3 & -5 & 2 \\ 1 & 1 & -1 \\ 2 & 3 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & -1 & -3 \\ -5 & -5 & -5 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} -3 & -5 & 2 \\ 1 & 1 & -1 \\ 2 & 3 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Hence the Q.F. $X^T BX$ is reduced to the canonical form $-y_1^2 + 5y_2^2 + y_3^2$.
Thus the transformation $X = PY$ reduces both the Q.F.'s to canonical forms.

Note ☑ $X = PY$ is not an orthogonal transformation, but only a linear non-singular transformation.

EXERCISE 1(d)

Part A

(Short answer questions)

1. Define a quadratic form and give an example for the same in three variables:
2. Write down the matrix of the quadratic form $3x_1^2 + 5x_2^2 + 5x_3^2 - 2x_1x_2 + 2x_2x_3 + 6x_3x_1$.
3. Write down the quadratic form corresponding to the matrix $\begin{bmatrix} 2 & 1 & -2 \\ 1 & 2 & -2 \\ -2 & -2 & 3 \end{bmatrix}$.
4. When is a Q.F. said to be singular? What is its rank then?
5. If the Q.F. $X^T AX$ gets transformed to $Y^T BY$ under the transformation $X = PY$, prove that B is a symmetric matrix.
6. What do you mean by canonical form of a quadratic form? State the condition for $X = PY$ to reduce the Q.F. $X^T AX$ into the canonical form.
7. How will you find an orthogonal transformation to reduce a Q.F. $X^T AX$ to the canonical form?
8. Define index and signature of a quadratic form.
9. Find the index and signature of the Q.F. $x_1^2 + 2x_2^2 - 3x_3^2$.
10. State the conditions for a Q.F. to be positive definite and positive semidefinite.

Part B

11. Reduce the quadratic form $2x_1^2 + 5x_2^2 + 3x_3^2 + 4x_1x_2$ to canonical form by an orthogonal transformation. Also find the rank, index and signature of the Q.F.
12. Reduce the Q.F. $3x_1^2 - 3x_2^2 - 5x_3^2 - 2x_1x_2 - 6x_2x_3 - 6x_3x_1$ to canonical form by an orthogonal transformation. Also find the rank, index and signature of the Q.F.
13. Reduce the Q.F. $6x_1^2 + 3x_2^2 + 3x_3^2 - 4x_1x_2 - 2x_2x_3 + 4x_3x_1$ to canonical form by an orthogonal transformation. Also state its nature.
14. Obtain an orthogonal transformation which will transform the quadratic form $2x_1^2 + 2x_2^2 + 2x_3^2 - 2x_1x_2 - 2x_2x_3 + 2x_3x_1$ into sum of squares form and find also the reduced form.
15. Find an orthogonal transformation which will reduce the quadratic form $2x_1x_2 + 2x_2x_3 + 2x_3x_1$ into the canonical form and hence find its nature.
16. Reduce the quadratic form $8x_1^2 + 7x_2^2 + 3x_3^2 - 12x_1x_2 - 8x_2x_3 + 4x_3x_1$ to the canonical form through an orthogonal transformation and hence show that it is positive definite. Find also a non-zero set of values for x_1, x_2, x_3 that will make the Q.F. zero.
17. Reduce the quadratic form $10x_1^2 + 2x_2^2 + 5x_3^2 + 6x_2x_3 - 10x_3x_1 - 4x_1x_2$ to a canonical form by orthogonal reduction. Find also a set of non-zero values of x_1, x_2, x_3, which will make the Q.F. zero.
18. Reduce the quadratic form $5x_1^2 + 26x_2^2 + 10x_3^2 + 6x_1x_2 + 4x_2x_3 + 14x_3x_1$ to a canonical form by orthogonal reduction. Find also a set of non-zero values of x_1, x_2, x_3, which will make the Q.F. zero.

19. Determine the nature of the following quadratic forms without reducing them to canonical forms:

    (i) $6x_1^2 + 3x_2^2 + 14x_3^2 + 4x_2x_3 + 18x_1x_3 + 4x_1x_2$

    (ii) $x_1^2 - 2x_1x_2 + x_2^2 + x_3^2$

    (iii) $x_1^2 + 2x_2^2 + 3x_3^2 + 2x_1x_2 + 2x_2x_3 - 2x_3x_1$

20. Find the value of λ so that the quadratic form $\lambda(x_1^2 + x_2^2 + x_3^2) + 2x_1x_2 - 2x_2x_3 + 2x_3x_1$ may be positive definite.
21. Find real non-singular transformations that reduce the following pairs of quadratic forms simultaneously to the canonical forms.

    (i) $6x_1^2 + 2x_2^2 + 3x_3^2 - 4x_1x_2 + 8x_3x_1$ and $5x_1^2 + x_2^2 + 5x_3^2 - 2x_1x_2 + 8x_3x_1$.

    (ii) $3x_1^2 + 3x_2^2 - 3x_3^2 + 2x_1x_2 - 2x_2x_3 + 2x_3x_1$ and $4x_1x_2 + 2x_2x_3 - 2x_3x_1$.

    (iii) $2x_1^2 + 2x_2^2 + 3x_3^2 + 2x_1x_2 - 4x_2x_3 - 4x_3x_1$ and $2x_2x_3 - 2x_1x_2 - x_2^2$.

    (iv) $3x_1^2 + 6x_2^2 + 2x_3^2 + 8x_1x_2 - 4x_2x_3$ and $5x_1^2 + 5x_2^2 + x_3^2 - 8x_1x_2 - 2x_2x_3$.

ANSWERS

Exercise 1(a)

Part A

(6) $X_1 = -\frac{1}{2}X_2 + \frac{3}{2}X_3$
(8) $a = 8$.
(12) $x + 2y = 3$ and $2x - y = 1$; $x + 2y = 3$ and $2x + 4y = 5$.
(13) $x + 2y = 3$ and $2x + 4y = 6$.
(14) $a = -4$, $b = 6$.
(15) Have a unique solution.
(16) $\lambda \neq 5$.
(17) No unique solution for any value of λ.
(18) $\lambda \neq -1$ and μ = any value.
(19) $\lambda = 2$ and $\mu = 3$.
(20) $\lambda = 8$ and $\mu \neq 11$.
(21) No, as $|A| \neq 0$.
(22) $\lambda = 3$
(23) $x = k$, $y = 2k$, $z = 5k$.

Part B

(24) $-7X_1 + X_2 + X_3 + X_4 = 0$;
(25) $2X_1 - X_2 - X_3 + X_4 = 0$;
(26) $2X_1 + X_2 - X_3 = 0$;
(27) $X_1 - 2X_2 + X_3 = 0$;
(28) $X_1 - X_2 + X_3 - X_4 = 0$;
(29) Yes. $X_5 = 2X_1 + X_2 - 3X_3 + 0.X_4$.
(34) $R(A) = R[A, B] = 2$; Consistent with many solutions.

(35) $R(A) = 3$ and $R[A, B] = 4$; Inconsistent.
(36) $R(A) = 3$ and $R[A, B] = 4$; Inconsistent.
(37) $R(A) = 3$ and $R[A, B] = 4$; Inconsistent.
(38) Consistent; $x = -1, y = 1, z = 2$. (39) Consistent; $x = 3, y = 5, z = 6$.
(40) Consistent; $x = 1, y = 1, z = 1$. (41) Consistent; $x = 2, y = 1, z = -4$.
(42) Consistent; $x_1 = 2, x_2 = 1, x_3 = -1, x_4 = 3$.
(43) Consistent; $x = 2,\ y = \frac{1}{5},\ z = 0,\ w = \frac{4}{5}$.
(44) Consistent; $x = 2k - 1, y = 3 - 2k, z = k$.
(45) Consistent; $x = \frac{7-16k}{11},\ y = \frac{k+3}{11},\ z = k$.
(46) Consistent; $x = \frac{16}{3} - \frac{9}{5}k,\ y = \frac{16}{3} - \frac{6}{5}k,\ z = k$.
(47) Consistent; $x = 3 - 4k - k', y = 1 - 2k + k', z = k, w = k'$.
(48) Consistent; $x_1 = -2k + 5k' + 7, x_2 = k, x3 = -2k' - 2, x_4 = k'$.
(49) $k = 1, 2$: When $k = 1, x = 2\lambda + 1, y = -3\lambda, z = \lambda$.
When $k = 2, x = 2\mu, y = 1 - 3\mu, z = \mu$.
(50) $\lambda = 1, 8$: When $\lambda = 1, x = k + 2, y = 1 - 3k, z = 5k$.
When $\lambda = 8, x = \frac{1}{5}(k+52),\quad y = -\frac{1}{5}(3k+16), z = k$.

(51) a + $2b - c = 0$.
(52) No solution, when $k = 1$; one solution, when $k \neq 1$ and -2; Many solutions, when $k = -2$.
(53) No solution when $\lambda = 8$; and $\mu \neq 6$; unique solution, when $\lambda \neq 8$ and $\mu =$ any value; many solutions when $\lambda = 8$ and $\mu = 6$.
(54) If $a = 8$, $b \neq 11$ no solution, ; If $a \neq 8$ and $b =$ any value, unique solution; If $a = 8$ and $b = 11$, many solutions.
(55) $x = k, y = -2k, z = 3k$. (56) $x = -4k, y = 2k, z = -2k, w = k$.
(57) $\lambda = 1, -9$; When $\lambda = 1, x = k, y = -k, z = 2k$ and when $\lambda = -9, x = 3k, y = 9k, z = -2k$.
(58) $\lambda = 0, 1, 2$; When $\lambda = 0$, solution is (k, k, k); When $\lambda = 1$, solution is $(k, -k, 2k)$; When $\lambda = 2$, solution is $(2k, k, 2k)$.

Exercise 1(b)

(3) 2, 50. (5) −2, −1.
(6) 38. (7) 36.
(8) 5. (9) 0.
(10) $\begin{bmatrix}1\\0\end{bmatrix}$ and $\begin{bmatrix}1\\1\end{bmatrix}$. (11) $\frac{47}{60}$.
(12) 2.

(15) 1, 3 −4; $(-2, 1, 4)^T, (2, 1, -2)^T, (1, -3, 13)^T$
(16) $1, \sqrt{5}, -\sqrt{5}; (1, 0, -1)^T, (\sqrt{5} - 1, 1, -1)^T, (\sqrt{5} + 1,\ -1, 1)^T$.
(17) $1, 3, -4; (-1, 4, 1)^T, (5, 6, 1)^T, (3, -2, 2)^T$

(18) 5, −3, −3; $(1, 2, -1)^T$, $(2, -1, 0)^T$, $(3, 0, 1)^T$
(19) 5, 1, 1, ; $(1, 1, 1)^T$, $(2, -1, 0)^T$, $(1, 0, -1)^T$
(20) 8, 2, 2; $(2, -1, 1)^T$, $(1, 2, 0)^T$, $(1, 0, -2)^T$
(21) 3,.2, 2; $(1, 1, -2)^T$, $(5, 2, -5)^T$
(22) −2, 2, 2 ; $(4, 1, -7)^T$, $(0, 1, 1)^T$
(23) 2, 2, 2; $(1, 0, 0)^T$.
(24) 1, 1, 6, 6; $(0, 0, 1, 2)^T$, $(1, -2, 0, 0)^T$, $(0, 0, 2, -1)^T$ and $(2, 1, 0, 0)^T$
(25) 0, 3, 15; $(1, 2, 2)^T$, $(2, 1, -2)^T$, $(2, -2, 1)^T$; A is singular
(26) Eigenvalues are 5, −10, −20; Trace = −25; $|A|$ = 1000
(28) 1, 4, 4; $(1, -1, 1)^T$, $(2, -1, 0)^T$, $(1, 0, -1)^T$
(29) −1, 1, 4; $(0, 1, 1)^T$, $(2, -1, 1)^T$, $(1, 1, -1)^T$

Exercise 1(c)

(9) $\frac{1}{36}\begin{bmatrix} 6 & -3 \\ -2 & 7 \end{bmatrix}$

(10) $\begin{bmatrix} -19 & 57 \\ 38 & 76 \end{bmatrix}$

(11) $\begin{bmatrix} 3 & 2 \\ 4 & 5 \end{bmatrix}$

(12) $M = \begin{bmatrix} 1 & 3 \\ -1 & 1 \end{bmatrix}$

(13) $\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

(14) $\frac{1}{3}\begin{bmatrix} -3 & -2 & 2 \\ 6 & 5 & -2 \\ -6 & -2 & 5 \end{bmatrix}$

(15) $-\frac{1}{11}\begin{bmatrix} 3 & -4 & -5 \\ -9 & 1 & 4 \\ -5 & 3 & 1 \end{bmatrix}$

(16) $\begin{bmatrix} 248 & 101 & 218 \\ 272 & 109 & 50 \\ 104 & 98 & 204 \end{bmatrix}$

(17) $\begin{bmatrix} 7 & -30 & 42 \\ 18 & -13 & 46 \\ -6 & -14 & 17 \end{bmatrix}$

(18) $A^n = \left(\frac{6^n - 2^n}{4}\right)\cdot\begin{bmatrix} 5 & 3 \\ 1 & 3 \end{bmatrix} + \left(\frac{3.2^n - 6^n}{2}\right)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; \begin{bmatrix} 976 & 960 \\ 320 & 336 \end{bmatrix}$

(19) $A^n = \left(\frac{9^n - 4^n}{5}\right)\begin{bmatrix} 7 & 3 \\ 2 & 6 \end{bmatrix} + \left(\frac{9.4^n - 4.9^n}{5}\right)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; \begin{bmatrix} 463 & 399 \\ 266 & 330 \end{bmatrix}$

(20) $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

(21) $D(1, 3, -4);\quad M = \begin{bmatrix} 2 & 2 & 1 \\ -1 & 1 & -3 \\ -4 & -2 & 13 \end{bmatrix}$

(22) $D(1, 2, 3);\quad M = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ -2 & 0 & 1 \end{bmatrix};\quad A^4 = \begin{bmatrix} -99 & 115 & 65 \\ -100 & 116 & 65 \\ -160 & 160 & 81 \end{bmatrix}$

(23) $D(2, 3, 6)$; $M = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -2 \\ -1 & 1 & 1 \end{bmatrix}$

(24) $D(4, -2, -2)$; $M = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 2 & 0 & -1 \end{bmatrix}$

(25) $D(8, 2, 2)$; $M = \begin{bmatrix} 2 & 1 & 1 \\ -1 & 0 & 2 \\ 1 & -2 & 0 \end{bmatrix}$

(26) $D(2, -1, -1)$; $M = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}$

(27) $D(0, 3, 14)$; $N = \begin{bmatrix} \frac{1}{\sqrt{42}} & \frac{1}{\sqrt{3}} & -\frac{3}{\sqrt{14}} \\ \frac{-5}{\sqrt{42}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{14}} \\ \frac{4}{\sqrt{42}} & \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{14}} \end{bmatrix}$

(28) $D(1, 3, 4)$; $N = \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{2}{\sqrt{6}} & 0 & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{bmatrix}$

(29) $D(4, 1, 1)$; $N = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \end{bmatrix}$

(30) $D(5, -3, -3)$; $N = \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{30}} \\ \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{30}} \\ -\frac{1}{\sqrt{6}} & 0 & \frac{5}{\sqrt{30}} \end{bmatrix}$

Exercise 1(d)

(2) $\begin{bmatrix} 3 & -1 & 3 \\ -1 & 5 & 1 \\ 3 & 1 & 5 \end{bmatrix}$

(3) $2x_1^2 + 2x_2^2 + 3x_3^2 + 2x_1x_2 - 4x_2x_3 - 4x_3x_1$.

(4) Singular, when $|A| = 0$; Rank $r < n$.

(6) $P^T AP$ must be a diagonal matrix.

(9) index = 2 and signature = 1.

(11) $N = \begin{bmatrix} \frac{1}{\sqrt{5}} & 0 & -\frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & 0 & \frac{1}{\sqrt{5}} \\ 0 & 1 & 0 \end{bmatrix}; Q = y_1^2 + 3y_2^2 + 6y_3^2; r = 3; p = 3; s = 3$

(12) $N = \begin{bmatrix} -\frac{3}{\sqrt{10}} & \frac{1}{\sqrt{35}} & \frac{1}{\sqrt{14}} \\ 0 & -\frac{5}{\sqrt{35}} & \frac{2}{\sqrt{14}} \\ \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{35}} & \frac{3}{\sqrt{14}} \end{bmatrix}; Q = 4y_1^2 - y_2^2 - 8y_3^2; r = 3; p = 1; s = -1$

(13) $N = \begin{bmatrix} \frac{2}{\sqrt{6}} & 0 & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} \end{bmatrix}; Q = 8y_1^2 + 2y_2^2 + 2y_3^2$; Q is positive definite

(14) $N = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \end{bmatrix}; Q = 4y_1^2 + y_2^2 + y_3^2$

(15) $N = \begin{bmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{5}} \end{bmatrix}; Q = 2y_1^2 - y_2^2 - y_3^2$; Q is indefinite

(16) $N = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \end{bmatrix}$; $Q = 3y_2^2 + 15y_3^2$; $x_1 = 1$, $x_2 = 2$, $x_3 = 2$

(17) $N = \begin{bmatrix} \frac{1}{\sqrt{42}} & \frac{1}{\sqrt{3}} & -\frac{3}{\sqrt{14}} \\ -\frac{5}{\sqrt{42}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{14}} \\ \frac{4}{\sqrt{42}} & \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{14}} \end{bmatrix}$; $Q = 3y_2^2 + 14y_3^2$; $x_1 = 1$, $x_2 = -5$, $x_3 = 4$.

(18) $N = \begin{bmatrix} \frac{16}{\sqrt{378}} & -\frac{2}{\sqrt{14}} & \frac{1}{\sqrt{27}} \\ -\frac{1}{\sqrt{378}} & \frac{1}{\sqrt{14}} & \frac{5}{\sqrt{27}} \\ -\frac{11}{\sqrt{378}} & -\frac{3}{\sqrt{14}} & \frac{1}{\sqrt{27}} \end{bmatrix}$; $Q = 14y_2^2 + 27y_3^2$; $x_1 = 16$, $x_2 = -1$, $x_3 = -11$.

(19) (i) positive definite; (ii) positive semidefinite; (iii) indefinite.

(20) $\lambda > 2$.

(21) (i) $P = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ -1 & 0 & 0 \end{bmatrix}$; $Q_1 = -y_1^2 + 4y_2^2 + 2y_3^2$; $Q_2 = y_1^2 + 4y_2^2 + y_3^2$

(ii) $P = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & 1 \\ -2 & 0 & 0 \end{bmatrix}$; $Q_1 = -16y_1^2 + 4y_2^2 + 8y_3^2$; $Q_2 = 4y_1^2 - 4y_2^2 + 4y_3^2$

(iii) $P = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$; $Q_1 = y_1^2 + y_2^2 + y_3^2$; $Q_2 = y_2^2 - y_3^2$.

(iv) $P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$; $Q_1 = 2y_1^2 + 4y_2^2 - y_3^2$; $Q_2 = y_1^2 + 4y_2^2 + y_3^2$.