Engineering Mathematics-1

About the Author

T Veerarajan is the retired Dean, Department of Mathematics, Velammal College of Engineering and Technology, Viraganoor, Madurai, Tamil Nadu. A Gold Medalist from Madras University, he has had a brilliant academic career all through. He has 50 years of teaching experience at undergraduate and postgraduate levels in various established Engineering Colleges in Tamil Nadu including Anna University, Chennai.

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T Veerarajan

Dean, Department of Mathematics, (Retd.)
Velammal College of Engineering & Technology
Viraganoor, Madurai,
Tamil Nadu

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Chapter 2

Sequences and Series

2.1 DEFINITIONS

If $u_1, u_2, u_3, \ldots u_n \ldots$ be an ordered set of quantities formed according to a certain law (called a *sequence*), then $u_1 + u_2 + u_3 + \ldots u_n + \ldots$ is called a *series*. If the number of terms in a series is limited, then it is called a *finite series*. If the series consists of an infinite number of terms, then it is called an *infinite series*.
For example

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots \text{ to } \infty \text{ and}$$

$$\frac{1}{2}x^2 + \frac{2}{3}x^3 + \frac{3}{4}x^4 + \ldots \text{ to } \infty$$

are infinite series.

The terms of an infinite series may be constants or variables. The infinite series $u_1 + u_2 + \cdots + u_n + \cdots$ to ∞ is denoted by $\sum_{n=1}^{\infty} u_n$ or simply $\sum u_n$ .The sum of its first n terms, namely, $(u_1 + u_2 + \cdots + u_n)$ is called the n^{th} partial sum and is denoted by s_n.

If s_n tends to a finite limit s as n tends to infinity, then the series $\sum u_n$ is said to be *convergent* and s is called the sum to infinity (or simply the sum) of the series. If $s_n \to \pm\infty$ as $n \to \infty$, then the series $\sum u_n$ is said to be *divergent*.

If s_n neither tends to a finite limit nor to $\pm\infty$ as $n \to \infty$, then the series $\sum u_n$ is said to be *oscillatory*. When $\sum u_n$ oscillates, s_n may tend to more than one limit as $n \to \infty$.

To understand the ideas of convergence, divergence and oscillation of infinite series, let us consider the familiar geometric series

$$a + ar + ar^2 + \cdots + ar^{n-1} + \cdots \text{ to } \infty \tag{1}$$

For the geometric series, s_n is given by.

$$s_n = a + ar + ar^2 + \ldots + ar^{n-1}$$

$$= \frac{a(1-r^n)}{1-r} \quad \text{or} \quad \frac{a(r^n - 1)}{r-1}$$

Case (i) Let $|r| < 1$ or $-1 < r < 1$.

$$s_n = \frac{a}{1-r} - \frac{a}{1-r}\cdot r^n$$

$$\therefore \quad \lim_{n\to\infty}(s_n) = \frac{a}{1-r} - \frac{a}{1-r}\cdot \lim_{n\to\infty}(r^n)$$

$$= \frac{a}{1-r} - \frac{a}{1-r}\times 0, \text{ since } |r|<1$$

$$= \frac{a}{1-r} = a \text{ finite quantity.}$$

$\therefore$ The geometric series (1) converges and its sum is $\dfrac{a}{1-r}$.

Case (ii) Let $r > 1$.

$$s_n = \frac{a(r^n-1)}{r-1} = \frac{a}{r-1}.r^n - \frac{a}{r-1}$$

$$\therefore \quad \lim_{n\to\infty}(s_n) = \frac{a}{r-1}\times \lim_{n\to\infty}(r^n) - \frac{a}{r-1}$$

$$= \frac{a}{r-1}\times\infty$$

$= \pm\infty$, according as a is positive or negative.

$\therefore$ Series (1) is divergent.

Case (iii) Let $r = 1$.

Then $\quad s_n = a + a + a + \cdots + a$ (n terms)

$= na$

$$\therefore \quad \lim_{n\to\infty}(s_n) = a\lim_{n\to\infty}(n)$$

$= \pm\infty$, according as a is positive or negative.

$\therefore$ Series (1) is divergent.

Case (iv) Let $r < -1$ and put $r = -k$

Then $k > 1$

$$s_n = \frac{a}{1-r} - \frac{ar^n}{1-r}$$

$$= \frac{a}{1+k} - a\frac{(-k)^n}{1+k}$$

$$= \frac{a}{1+k} + a\frac{(-1)^{n+1}\cdot k^n}{1+k}$$

Now $\lim\limits_{n\to\infty}(k^n) = \infty$, since $k > 1$

$\therefore$ $\lim\limits_{n\to\infty}(s_n) = \infty$, if n is odd and $= -\infty$, if n is even.

i.e. s_n oscillates between $-\infty$ and $+\infty$.
$\therefore$ Series (1) is oscillatory, oscillating between $-\infty$ and ∞.

Case (v) Let $r = -1$
Then
$$s_n = a - a + a - a + \ldots \text{ to } n \text{ terms}$$
$$= a \text{ or } 0, \text{ according as } n \text{ is odd or even.}$$

$\therefore$ Series (1) oscillates between a and 0.

Thus the geometric series $a + ar + ar^2 + \cdots + ar^{n-1} + \cdots$ is convergent, if $|r| < 1$, divergent if $|r| \geq 1$ and oscillatory if $|r| \leq -1$.

2.2 GENERAL PROPERTIES OF SERIES

1. If a finite number of terms are added to or deleted from a series, the convergence or divergence or oscillation of the series is unchanged.
2. The convergence or divergence of an infinite series is not affected when each of its terms is multiplied by a finite quantity.
3. If the two series $\sum u_n$ and $\sum v_n$ are convergent to s and s', then $\sum(u_n + v_n)$ is also convergent and its sum is $(s + s')$.

Note ☑ Form the geometric series example, it is clear that, to find the convergence or divergence of a series, we have to find s_n and its limit. In many situations, it may not be possible to find s_n and hence the definition of convergence cannot be applied directly in such cases. Tests have been devised to determine whether a given series is convergent or not, without finding s_n. Some important tests of convergence of series of positive terms are described below without proof.

2.2.1 Necessary Condition for Convergence

If a series of positive terms $\sum u_n$ is convergent, then $\lim\limits_{n\to\infty}(u_n) = 0$.

Since $\sum u_n$ is convergent, $\lim\limits_{n\to\infty}(s_n) = s$, where $s_n = u_1 + u_2 + \cdots + u_n$.

Now
$$\lim_{n\to\infty}(s_{n-1}) = \lim_{n\to\infty}(u_1 + u_2 + \cdots + u_{n-1})$$
$$= \lim_{m\to\infty}(u_1 + u_2 + \cdots + u_m), \text{ putting } m = n - 1$$
$$= \lim_{m\to\infty} s_m$$
$$= s$$

$$\therefore \qquad \lim_{n\to\infty}(u_n) = \lim_{n\to\infty}(s_n - s_{n-1})$$
$$= \lim_{n\to\infty}(s_n) - \lim_{n\to\infty}(s_{n-1})$$
$$= s - s$$
$$= 0$$

Note ☑ The condition is only necessary but not sufficient, i.e. $\lim_{n\to\infty}(u_n) = 0$ does not imply that $\sum u_n$ is convergent.

For example, if $u_n = \dfrac{1}{n}$, then $\lim_{n\to\infty}\left(\dfrac{1}{n}\right) = 0$, but $\sum u_n$ is known to be divergent.

2.2.2 A Simple Test for Divergence

If $\lim_{n\to\infty}(u_n) \neq 0$, then $\sum u_n$ is not convergent. Since a series of positive terms either converges or diverges, we conclude that $\sum u_n$ is divergent, when $\lim_{n\to\infty}(u_n) \neq 0$.

2.2.3 Simplified Notation

When a series is convergent, it is written as **Series is (C)**.
When a series is divergent, it is written as **series is (D)**.

2.2.4 Comparison Test (Form I)

1. If $\sum u_n$ and $\sum v_n$ are two series of positive terms such that $u_n \leq v_n$ for all n (= 1, 2, 3, …) and if $\sum v_n$ is (C), then $\sum u_n$ is also (C).
2. If $\sum u_n$ and $\sum v_n$ are two series of positive terms such that $u_n \geq v_n$ for all n and if $\sum v_n$ (D), then $\sum u_n$ is also (D).

2.2.5 Comparison Test (Form II or Limit Form)

If $\sum u_n$ and $\sum v_n$ are two series of positive terms such that $\lim_{n\to\infty}\left(\dfrac{u_n}{v_n}\right) = l$, a finite number ≠ 0, then $\sum u_n$ and $\sum v_n$ converge together or diverge together.

Note ☑ Using comparison test, we can test the convergence of $\sum u_n$, provided we know another series $\sum v_n$ (known as auxiliary series) whose convergence or divergence is known beforehand.

In most situations, one of the following series is chosen as the auxiliary series for the application of comparison test.

1. The geometric series $1 + r + r^2 + \cdots$, which is (C), when $|r| < 1$ and (D), when $r \geq 1$.

2. The factorial series $\sum_{n=1}^{\infty} \frac{1}{n!} = \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots$ which is (C) as discussed below.

3. The p-series $\sum_{n=1}^{\infty} \frac{1}{n^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots$ which is (C), when $p > 1$ and is (D), when $p \le 1$.

2.2.6 Convergence of the Series $\sum_{n=1}^{\infty} \frac{1}{n!}$

Let $$\sum u_n = \sum \frac{1}{n!} = \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots \text{ to } \infty$$

Consider $$\sum v_n = \frac{1}{1} + \frac{1}{1 \cdot 2} + \frac{1}{1 \cdot 2 \cdot 2} + \frac{1}{1 \cdot 2 \cdot 2 \cdot 2} + \cdots \text{ to } \infty$$

We note that $u_1 = v_1$ and $u_2 = v_2$

Since $3! > 1 \cdot 2 \cdot 2$, $\frac{1}{3!} < \frac{1}{1 \cdot 2 \cdot 2}$, i.e., $u_3 < v_3$

Similarly $u_4 < v_4$ and so on.

Thus each term of $\sum u_n$ after the second is less then the corresponding term of $\sum v_n$.

But $\sum v_n = 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \ldots$ is a geometric series in which $|r| = \frac{1}{2} < 1$.

Hence $\sum v_n$ is (C).

$\therefore$ By the comparison test, $\sum u_n$ is also (C).

2.2.7 Convergence of the *p*-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$

Let $$\sum u_n = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \ldots \infty$$

Case (i) Let $p > 1$.

$\sum u_n$ can be rewritten as

$$\sum u_n = \left(\frac{1}{1^p}\right) + \left(\frac{1}{2^p} + \frac{1}{3^p}\right) + \left(\frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p}\right) + \ldots,$$

such that the n^{th} group contains 2^{n-1} terms of $\sum u_n$.

Consider the auxiliary series

$$\sum v_n = \left(\frac{1}{1^p}\right) + \left(\frac{1}{2^p} + \frac{1}{2^p}\right) + \left(\frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p}\right) + \cdots$$

We note that $u_1 = v_1$, $u_2 = v_2$, $u_4 = v_4$, $u_8 = v_8$ and so on.
Since $p > 1$, $3^p > 2^p$

$\therefore \qquad \dfrac{1}{3^p} < \dfrac{1}{2^p}$, i.e. $u_3 < v_3$

Similarly $\dfrac{1}{5^p} < \dfrac{1}{4^p}$, i.e. $u_5 < v_5$, $u_6 < v_6$, $u_7 < v_7$ and so on.

Hence in the two series $\sum u_n$ and $\sum v_n$, $u_n \le v_n$ for all n.

$\therefore$ By comparison test, $\sum u_n$ is (C), provided $\sum v_n$ is (C).

Now
$$\sum v_n = \frac{1}{1^p} + \frac{2}{2^p} + \frac{4}{4^p} + \cdots$$
$$= \frac{1}{1^p} + \frac{1}{2^{p-1}} + \frac{1}{4^{p-1}} + \frac{1}{8^{p-1}} + \cdots$$
$$= 1 + \frac{1}{2^{p-1}} + \left(\frac{1}{2^{p-1}}\right)^2 + \left(\frac{1}{2^{p-1}}\right)^3 + \cdots$$

This series is a geometric series with $r = \dfrac{1}{2^{p-1}}$

Since $\quad p > 1, p - 1 > 0$

$\therefore \qquad 2^{p-1} > 1$ and so $\dfrac{1}{2^{p-1}} < 1$

$\therefore \qquad \sum v_n$ is (C).

Hence $\sum u_n$ is also (C).

Case (ii) $p = 1$

Now $\sum u_n = 1 + \dfrac{1}{2} + \dfrac{1}{3} + \dfrac{1}{4} + \cdots + \text{to } \infty$ (This series is called the *harmonic series*).

$\sum u_n$ can be rewritten as

$$\sum u_n = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \left(\frac{1}{9} + \frac{1}{10} + \cdots + \frac{1}{16}\right) + \cdots \text{ to } \infty$$

Consider the auxiliary series

$$\sum v_n = 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) + \cdots \text{ to } \infty$$

We note that $u_1 = v_1$, $u_2 = v_2$, $u_4 = v_4$, $u_8 = v_8$, $u_{16} = v_{16}$ and so on.

Also since $\frac{1}{3} > \frac{1}{4}$, $u_3 > v_3$; since $\frac{1}{5} > \frac{1}{8}$, $u_5 > v_5$;

Similarly $u_6 > v_6$, $u_7 > v_7$ and so on.

$\therefore$ In the two series $\sum u_n$ and $\sum v_n$, $u_n \geq v_n$ for all n.

$\therefore$ By comparison test, $\sum u_n$ is (D), provided $\sum v_n$ is (D).

Now
$$\sum v_n = 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots \infty$$

$$s_n = n^{\text{th}} \text{ partial sum of } \sum v_n = 1 + \frac{n-1}{2} = \frac{n+1}{2}$$

$$\lim_{n\to\infty} (s_n) = \infty$$

$\therefore$ $\sum v_n$ is (D)

$\therefore$ $\sum u_n$ is also (D)

Case (iii) Let $p < 1$.

$$\sum u_n = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots \text{ to } \infty$$

Consider the auxiliary series

$$\sum v_n = 1 + \frac{1}{2} + \frac{1}{3} \cdots \text{ to } \infty$$

Since $p < 1$, $n^p < n$ (except when $n = 1$)

$\therefore$
$$\frac{1}{n^p} \geq \frac{1}{n}, \text{ for all values of } n$$

i.e.
$$u_n \geq v_n, \text{ for all values of } n$$

But, by case (ii), $\sum v_n$ is (D).

$\therefore$ By comparison test, $\sum u_n$ is also (D).

Cauchy's Root Test

If $\sum u_n$ is a series of positive terms such that $\lim_{n\to\infty} \left(u_n^{1/n}\right) = l$, then the series $\sum u_n$ is (C), when $l < 1$ and (D) when $l > 1$. When $l = 1$, the test fails.

WORKED EXAMPLE 2(a)

Example 2.1 Test the convergence of the series

(i) $\sum_{n=1}^{\infty} \frac{1}{1+3^n}$; (ii) $\sum_{n=1}^{\infty} \frac{\cos^2 n}{2^n}$.

(i) Let $$\sum u_n = \sum \frac{1}{1+3^n}$$

Let $$\sum v_n = \sum \frac{1}{3^n}$$

Now $1 + 3^n > 3^n$ for all n

$\therefore$ $\frac{1}{1+3^n} < \frac{1}{3^n}$ for all n.

i.e. $u_n < v_n$ for all n

$\therefore$ $\sum u_n$ is (C), if $\sum v_n$ is (C).

Now $\sum v_n = \sum \frac{1}{3^n} = \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \cdots$ is a geometric series with $r = \frac{1}{3} < 1$.

$\therefore$ $\sum v_n$ is (C)

Hence $\sum u_n$ is also (C).

(ii) Let $$\sum u_n = \sum \frac{\cos^2 n}{2^n}$$

Let $$\sum v_n = \sum \frac{1}{2^n}$$

Now $|\cos n| < 1$ or $-1 < \cos n < 1$

$\therefore$ $\cos^2 n < 1$ for all n

Hence $\frac{\cos^2 n}{2^n} < \frac{1}{2^n}$ for all n,

i.e. $u_n < v_n$ for all n

$\therefore$ $\sum u_n$ is (C), if $\sum v_n$ is (C).

Now $\sum v_n = \sum \frac{1}{2^n} = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots$ is a geometric series with $r = 1/2 < 1$.

$\therefore$ $\sum v_n$ is (C)

Hence $\sum u_n$ is also (C).

Example 2.2 Test the convergence of the following series

(i) $\dfrac{1}{1\cdot 2}+\dfrac{2}{3\cdot 4}+\dfrac{3}{5\cdot 6}+\cdots$ to ∞

(ii) $\dfrac{1\cdot 2}{3^2\cdot 4^2}+\dfrac{3\cdot 4}{5^2\cdot 6^2}+\dfrac{5\cdot 6}{7^2\cdot 8^2}+\cdots$ to ∞

(iii) $1+\dfrac{1}{2^2}+\dfrac{2^2}{3^3}+\dfrac{3^3}{4^4}+\cdots$ to ∞.

(i) The given series is $\sum u_n = \sum \dfrac{n}{(2n-1)\cdot 2n}$

$$= \frac{1}{2}\sum \frac{1}{2n-1}$$

Let $$\sum v_n = \sum \frac{1}{n}.$$

Note ☑ If the numerator and denominator of u_n are expressions of degree p and q in n, then we choose $v_n = \dfrac{n^p}{n^q} = \dfrac{1}{n^{q-p}}$

Then $$\frac{u_n}{v_n} = \frac{1}{2}\cdot\frac{\left(\frac{1}{2n-1}\right)}{\left(\frac{1}{n}\right)} = \frac{1}{2}\cdot\frac{n}{2n-1}$$

$$= \frac{1}{2}\cdot\frac{1}{2-\frac{1}{n}}$$

$\therefore$ $$\lim_{n\to\infty}\left(\frac{u_n}{v_n}\right) = \frac{1}{2}\cdot\lim_{n\to\infty}\left(\frac{1}{2-\frac{1}{n}}\right)$$

$$= \frac{1}{4} \neq 0.$$

$\therefore$ By the limit form of comparison test, $\sum u_n$ and $\sum v_n$ converge or diverge together.

Now $\sum v_n = \sum \dfrac{1}{n}$, which is the harmonic series is (D).

$\therefore$ $\sum u_n$ is also (D).

(ii) The given series is $\sum u_n = \dfrac{1\cdot 2}{3^2\cdot 4^2} + \dfrac{3\cdot 4}{5^2\cdot 6^2} + \ldots$ to ∞

i.e. $$\sum u_n = \sum \frac{(2n-1)\,(2n)}{(2n+1)^2\,(2n+2)^2}$$

Let $\sum v_n = \sum \dfrac{n^2}{n^4}$ ($\therefore$ the numerator is degree 2 and the denominator is of degree 4)

$$= \sum \frac{1}{n^2}$$

Then $$\frac{u_n}{v_n} = \frac{(2n-1)(2n)\cdot n^2}{(2n+1)^2\,(2n+2)^2}$$

$$= \frac{\left(2-\frac{1}{n}\right)\cdot 2}{\left(2+\frac{1}{n}\right)^2\left(2+\frac{2}{n}\right)^2}$$

$\therefore$ $$\lim_{n\to\infty}\left(\frac{u_n}{v_n}\right) = \frac{2^2}{2^2\cdot 2^2} = \frac{1}{4} \neq 0$$

$\therefore$ By comparison test, $\sum u_n$ and $\sum v_n$ converge or diverge together.

Now $\sum v_n = \sum \dfrac{1}{n^2}$ is (C) $\left[\because \sum \dfrac{1}{n^p}\text{ is (C), when } p>1\right]$

$\therefore$ $\sum u_n$ is also (C).

(iii) The given series is $\sum u_n = \dfrac{1}{2^2} + \dfrac{2^2}{3^3} + \dfrac{3^3}{4^4} + \ldots$ to ∞ (omitting the first term)

i.e. $$\sum u_n = \sum \frac{n^n}{(n+1)^{n+1}}$$

Let $$\sum v_n = \sum \frac{n^n}{n^{n+1}} \text{ or } \sum \frac{1}{n}$$

Then $$\frac{u_n}{v_n} = \frac{n^n}{(n+1)^{n+1}}\cdot n$$

$$= \left(\frac{n}{n+1}\right)^{n+1} = \left(\frac{1}{1+\frac{1}{n}}\right)^{n+1}$$

$$= \frac{1}{\left(1+\frac{1}{n}\right)^n} \cdot \left(\frac{1}{1+\frac{1}{n}}\right)$$

$$\therefore \qquad \lim_{x\to\infty}\left(\frac{u_n}{v_n}\right) = \frac{1}{\lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^n} \cdot \frac{1}{\lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)}$$

$$= \frac{1}{e} \neq 0.$$

$\therefore$ By comparison test, $\sum u_n$ and $\sum v_n$ converge or diverge together.

Now $\qquad \sum v_n = \sum \frac{1}{n}$ is (D)

$\therefore \qquad \sum u_n$ is also (D).

Note ☑ Omission of the first term (= 1) of the given series does not alter the convergence or divergence of the series.

Example 2.3 Examine the convergence of the following series:

(i) $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt[3]{2n^2-1}}{\sqrt[4]{3n^3+2n+5}}$; (ii) $\displaystyle\sum_{n=1}^{\infty}\left(\frac{3^n+4^n}{4^n+5^n}\right)$;

(iii) $\displaystyle\sum_{n=1}^{\infty} n\sin^2\left(\frac{1}{n}\right)$; (iv) $\displaystyle\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}\tan\left(\frac{1}{n}\right)$;

(i) $$\sum u_n = \sum \frac{(2n^2-1)^{1/3}}{(3n^3+2n+5)^{1/4}}$$

Let $$\sum v_n = \sum \frac{n^{2/3}}{n^{3/4}} \text{ or } \sum \frac{1}{n^{1/12}}$$

Then $$\frac{u_n}{v_n} = \frac{(2n^2-1)^{1/3}}{(3n^3+2n+5)^{1/4}} \times n^{1/12}$$

$$= \frac{(2n^2-1)^{1/3}}{(3n^3+2n+5)^{1/4}} \times \frac{n^{3/4}}{n^{2/3}}$$

$$= \frac{\left(\frac{2n^2-1}{n^2}\right)^{1/3}}{\left(\frac{3n^3+2n+5}{n^3}\right)^{1/4}}$$

$$= \frac{\left(2 - \frac{1}{n^2}\right)^{1/3}}{\left(3 + \frac{2}{n^2} + \frac{5}{n^3}\right)^{1/4}}$$

$$\therefore \qquad \lim_{n\to\infty}\left(\frac{u_n}{v_n}\right) = \frac{2^{1/3}}{3^{1/4}} \neq 0.$$

$\therefore$ By comparison test, $\sum u_n$ and $\sum v_n$ converge or diverge together.

Now $\qquad \sum v_n = \sum \frac{1}{n^{1/12}}$ is (D) $\qquad \left[\because \sum \frac{1}{n^p} \text{ is (D) when } p<1\right]$

$\therefore\ \sum u_n$ is (D).

(ii) $\qquad \sum u_n = \sum\left(\frac{3^n + 4^n}{4^n + 5^n}\right)$

Let $\qquad \sum v_n = \sum\left(\frac{4}{5}\right)^n$

Then $\qquad \frac{u_n}{v_n} = \frac{3^n + 4^n}{4^n + 5^n} \times \frac{5^n}{4^n}$

$$= \frac{\left(\frac{3}{4}\right)^n + 1}{\left(\frac{4}{5}\right)^n + 1}$$

$$\therefore \qquad \lim_{n\to\infty}\left(\frac{u_n}{v_n}\right) = \frac{0+1}{0+1} = 1 \neq 0.$$

$\therefore\ \sum u_n$ and $\sum v_n$ converge or diverge together, by comparison test.

Now $\sum v_n = \sum\left(\frac{4}{5}\right)^n$ is a geometric series with $r = \frac{4}{5} < 1$ and hence is convergent.

Hence $\sum u_n$ is also (C).

(iii) $\qquad \sum u_n = \sum_{n=1}^{\infty} n \sin^2\left(\frac{1}{n}\right)$

Let $\qquad \sum v_n = \sum \frac{1}{n}$

Then $$\frac{u_n}{v_n} = n^2 \sin^2\left(\frac{1}{n}\right)$$

$$= \left\{\frac{\sin\left(\frac{1}{n}\right)}{\left(\frac{1}{n}\right)}\right\}^2$$

$$\therefore \quad \lim_{n\to\infty}\left(\frac{u_n}{v_n}\right) = \left\{\lim_{n\to\infty}\left[\frac{\sin\left(\frac{1}{n}\right)}{\left(\frac{1}{n}\right)}\right]\right\}^2$$

$$= 1\left[\because \lim_{\theta\to 0}\left(\frac{\sin\theta}{\theta}\right) = 1\right]$$

$$\neq 0$$

$\therefore$ By comparison test, $\sum u_n$ and $\sum v_n$ converge or diverge together.

$$\sum v_n = \sum \frac{1}{n} \text{ is (D).}$$

$\therefore \quad \sum u_n$ is also (D).

(iv) $$\sum u_n = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \tan\left(\frac{1}{n}\right)$$

Let $$\sum v_n = \sum \frac{1}{n^{3/2}}$$

Then $$\frac{u_n}{v_n} = \frac{1}{\sqrt{n}} \tan\left(\frac{1}{n}\right) \times n^{3/2}$$

$$= \frac{\tan\left(\frac{1}{n}\right)}{\left(\frac{1}{n}\right)}$$

$$\therefore \quad \lim_{n\to\infty}\left(\frac{u_n}{v_n}\right) = \lim_{n\to\infty}\left[\frac{\tan\left(\frac{1}{n}\right)}{\left(\frac{1}{n}\right)}\right]$$

$$= 1\left[\because \lim_{\theta\to\infty}\left(\frac{\tan\theta}{\theta}\right) = 1\right]$$

$$\neq 0$$

$\therefore$ By comparison test, $\sum u_n$ and $\sum v_n$ converge or diverge together.

$$\sum v_n = \sum \frac{1}{n^{3/2}} \text{ is (C)} \qquad \left[\because \sum \frac{1}{n^p} \text{ is (C), when } p > 1\right]$$

Hence $\sum u_n$ is also (C).

Example 2.4 Determine whether the following series are (C) or (D).

(i) $\sum\left\{\sqrt{n^4+1}-\sqrt{n^4-1}\right\}$; (ii) $\sum\left(\dfrac{\sqrt{n+1}-\sqrt{n}}{n^\alpha}\right)$;

(iii) $\sum\left(\sqrt[3]{n^3+1}-n\right)$; (iv) $\sum\left(\dfrac{1}{\sqrt{n}+\sqrt{n+1}}\right)$.

(i)
$$\sum u_n = \sum\left(\sqrt{n^4+1}-\sqrt{n^4-1}\right);$$

$$= \sum\left\{\frac{\left(\sqrt{n^4+1}-\sqrt{n^4-1}\right)\left(\sqrt{n^4+1}+\sqrt{n^4-1}\right)}{\sqrt{n^4+1}+\sqrt{n^4-1}}\right\}$$

$$= \sum\left(\frac{2}{\sqrt{n^4+1}+\sqrt{n^4-1}}\right)$$

Let
$$\sum v_n = \sum \frac{1}{n^2}$$

Then
$$\frac{u_n}{v_n} = \frac{2n^2}{\sqrt{n^4+1}+\sqrt{n^4-1}}$$

$$= \frac{2}{\sqrt{1+\dfrac{1}{n^4}}+\sqrt{1-\dfrac{1}{n^4}}}$$

$\therefore$
$$\lim_{n\to\infty}\left(\frac{u_n}{v_n}\right) = \frac{2}{1+1} = 1 \neq 0$$

$\therefore$ By comparison test, $\sum u_n$ and $\sum v_n$ converge or diverge together.

$$\sum v_n = \sum \frac{1}{n^2} \text{ is (C).}$$

Hence $\sum u_n$ is also (C).

(ii)

$$\sum u_n = \sum \left(\frac{\sqrt{n+1} - \sqrt{n}}{n^\alpha} \right)$$

$$= \sum \frac{\left(\sqrt{n+1} - \sqrt{n}\right)\left(\sqrt{n+1} + \sqrt{n}\right)}{n^\alpha \left(\sqrt{n+1} + \sqrt{n}\right)}$$

$$= \sum \frac{1}{n^\alpha \left(\sqrt{n+1} + \sqrt{n}\right)} \qquad (1)$$

Let

$$\sum v_n = \sum \frac{1}{n^{\alpha + \frac{1}{2}}}$$

Then

$$\frac{u_n}{v_n} = \frac{n^{\alpha + \frac{1}{2}}}{n^\alpha \left(\sqrt{n+1} + \sqrt{n}\right)}$$

$$= \frac{1}{\sqrt{1 + \frac{1}{n}} + 1}$$

$\therefore$

$$\lim_{n \to \infty} \left(\frac{u_n}{v_n} \right) = \frac{1}{2} \neq 0.$$

$\therefore$ $\sum u_n$ and $\sum v_n$ converge or diverge together.

$\sum v_n = \sum \frac{1}{n^{\alpha + \frac{1}{2}}}$ is (C) when $\alpha + \frac{1}{2} > 1$ or $\alpha > \frac{1}{2}$, and it is (D) when $\alpha + \frac{1}{2} \leq 1$ or $\alpha \leq \frac{1}{2}$.

$\therefore$ $\sum u_n$ is (C) when $\alpha > \frac{1}{2}$ and (D) when $\alpha \leq \frac{1}{2}$.

Note ☑ Keeping $u_n = \frac{\sqrt{n+1} - \sqrt{n}}{n^\alpha}$, if we choose $v_n = \frac{n^{1/2}}{n^\alpha} = \frac{1}{n^{\alpha - 1/2}}$ we will get $\lim_{n \to \infty} \left(\frac{u_n}{v_n} \right) = 0$ and so comparison test fails.

(iii)

$$u_n = (n^3 + 1)^{1/3} - (n^3)^{1/3}$$

$$= \frac{(n^3 + 1) - n^3}{(n^3 + 1)^{2/3} + (n^3 + 1)^{1/3} n + (n^3)^{2/3}} \quad \left[\because a - b = \frac{a^3 - b^3}{a^2 + ab + b^2} \right]$$

$$= \frac{1}{(n^3+1)^{2/3} + n(n^3+1)^{1/3} + n^2}$$

Let $\quad v_n = \dfrac{1}{n^2}$

Then $\quad \dfrac{u_n}{v_n} = \dfrac{n^2}{(n^3+1)^{2/3} + n(n^3+1)^{1/3} + n^2}$

$$= \frac{1}{\left(1+\frac{1}{n^3}\right)^{2/3} + \left(1+\frac{1}{n^3}\right)^{1/3} + 1}$$

$\therefore \quad \lim_{n\to\infty}\left(\dfrac{u_n}{v_n}\right) = \dfrac{1}{3} \neq 0$

$\therefore$ $\sum u_n$ and $\sum v_n$ converge or diverge together.

$$\sum v_n = \sum \frac{1}{n^2} \text{ is (C)}$$

Hence $\sum u_n$ is also (C).

(iv) $\quad \sum u_n = \sum \dfrac{1}{\sqrt{n} + \sqrt{n+1}}$

Let $\quad \sum v_n = \sum \dfrac{1}{\sqrt{n}}$

$$\frac{u_n}{v_n} = \frac{\sqrt{n}}{\sqrt{n} + \sqrt{n+1}} = \frac{1}{1 + \sqrt{1+\frac{1}{n}}}$$

$\therefore \quad \lim_{n\to\infty}\left(\dfrac{u_n}{v_n}\right) = \dfrac{1}{2} \neq 0$

$\therefore \quad \sum u_n$ and $\sum v_n$ converge or diverge together.

$$\sum v_n = \sum \frac{1}{n^{1/2}} \text{ is (D)} \left[\because \text{In } \sum \frac{1}{n^p},\ p = \frac{1}{2} < 1\right]$$

$\therefore \quad \sum u_n$ is also (D).

Example 2.5 Examine the convergence of the following series:

(i) $\left(\dfrac{1}{4}\right) + \left(\dfrac{2}{7}\right)^2 + \left(\dfrac{3}{10}\right)^3 + \ldots \text{to } \infty$

(ii) $\frac{1}{2}+\frac{2}{3}x+\left(\frac{3}{4}\right)^2 x^2+\left(\frac{4}{5}\right)^3 x^3+\ldots$ to ∞;

(iii) $\left(\frac{2^2}{1^2}-\frac{2}{1}\right)^{-1}+\left(\frac{3^3}{2^3}-\frac{3}{2}\right)^{-2}+\left(\frac{4^4}{3^4}-\frac{4}{3}\right)^{-3}+\ldots$ to ∞;

(iv) $a+b+a^2+b^2+a^3+b^3+\ldots$ to ∞, given $a>0$, $b>0$.

(i) Given series is $\frac{1}{4}+\left(\frac{2}{7}\right)^2+\left(\frac{3}{10}\right)^3+\ldots$ to ∞

$$\therefore \qquad u_n = n^{\text{th}} \text{ term of the given series}$$

$$= \left(\frac{n}{3n+1}\right)^n$$

$$\therefore \qquad \lim_{n\to\infty}(u_n)^{1/n} = \lim_{n\to\infty}\left(\frac{n}{3n+1}\right)$$

$$= \lim_{n\to\infty}\left(\frac{1}{3+\frac{1}{n}}\right)$$

$$= \frac{1}{3} < 1.$$

$\therefore$ By Cauchy's root test, $\sum u_n$ is (C).

(ii) $\sum u_n = \frac{2}{3}x+\left(\frac{3}{4}\right)^2 x^2+\left(\frac{4}{5}\right)^3 x^3+\ldots$ (omitting the first term)

$$u_n = \left(\frac{n+1}{n+2}\right)^n x^n$$

Then $$u_n^{1/n} = \left(\frac{n+1}{n+2}\right)x \text{ or } \left(\frac{1+\frac{1}{n}}{1+\frac{2}{n}}\right)x$$

$$\therefore \qquad \lim_{n\to\infty}(u_n)^{1/n} = x$$

$\therefore$ By Cauchy's root test,

$\sum u_n$ is (C) if $x<1$ and it is (D) if $x>1$

If $x = 1$, Cauchy's root test fails.

In this case, $$\lim_{n\to\infty}(u_n) = \lim_{n\to\infty}\left(\frac{n+1}{n+2}\right)^n$$

$$= \lim_{n\to\infty} \frac{\left(1+\frac{1}{n}\right)^n}{\left(1+\frac{2}{n}\right)^n}$$

$$= \frac{e}{e^2} \text{ or } \frac{1}{e} \neq 0$$

i.e. the necessary condition for convergence of $\sum u_n$ is not satisfied.

$\therefore$ $\sum u_n$ is (D)

(iii) $$\sum u_n = \left(\frac{2^2}{1^2} - \frac{2}{1}\right)^{-1} + \left(\frac{3^3}{2^3} - \frac{3}{2}\right)^{-2} + \left(\frac{4^4}{3^4} - \frac{4}{3}\right)^{-3} + \cdots$$

$\therefore$ $$u_n = \left[\left(\frac{n+1}{n}\right)^{n+1} - \left(\frac{n+1}{n}\right)\right]^{-n}$$

$\therefore$ $$(u_n)^{1/n} = \left[\left(\frac{n+1}{n}\right)^{n+1} - \left(\frac{n+1}{n}\right)\right]^{-1}$$

$$= \left(\frac{n+1}{n}\right)^{-1}\left[\left(\frac{n+1}{n}\right)^{n} - 1\right]^{-1}$$

$$= \left(1+\frac{1}{n}\right)^{-1}\left[\left(1+\frac{1}{n}\right)^{n} - 1\right]^{-1}$$

Then $$\lim_{n\to\infty} (u_n)^{1/n} = (e-1)^{-1} = \frac{1}{e-1} < 1$$

$\therefore$ By Cauchy's root test, $\sum u_n$ is (C)

(iv) $$\sum u_n = a + b + a^2 + b^2 + a^3 + b^3 + \cdots$$

Then $$u_n = a^{\frac{n+1}{2}}, \text{ if } n \text{ odd}$$

$$= b^{n/2}, \text{ if } n \text{ is even}$$

$\therefore$ $$u_n^{1/n} = a^{\frac{1}{2}+\frac{1}{2n}}, \text{ if } n \text{ is odd}$$

$$= b^{1/2}, \text{ if } n \text{ is even}$$

$$\lim_{n\to\infty} (u_n^{1/n}) = \sqrt{a}, \text{ if } n \text{ is odd}$$

$$= \sqrt{b}, \text{ if } n \text{ is even}$$

$\therefore$ Whether n is odd or even, $\sum u_n$ is (C) if $a < 1$ and $b < 1$, and (D) if $a > 1$ and $b > 1$, by Cauchy's root test.

When $a = 1 = b$, the series becomes

$1 + 1 + 1 \cdots$ to ∞, which is (D).

Example 2.6 Test the convergence of the following series:

(i) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{(\log n)^n}$; (ii) $\displaystyle\sum \left(\frac{n}{n+1}\right)^{n^2}$;

(iii) $\displaystyle\sum \left(1+\frac{1}{\sqrt{n}}\right)^{n^{3/2}}$; (iv) $\displaystyle\sum \frac{[(n+1)x]^n}{n^{n+1}}$, $x > 0$.

(i) $$\sum u_n = \sum \frac{1}{(\log n)^n} \therefore u_n = \frac{1}{(\log n)^n},\ n \geq 2$$

$$\lim_{n\to\infty} (u_n^{1/n}) = \lim_{n\to\infty} \left(\frac{1}{\log n}\right)$$

$$= 0 < 1$$

$\therefore$ By Cauchy's root test, $\sum u_n$ is (C).

(ii) $$\sum u_n = \sum \left(\frac{n}{n+1}\right)^{n^2}$$

$\therefore$ $$u_n = \frac{1}{\left(1+\frac{1}{n}\right)^{n^2}} \quad \text{and so} \quad u_n^{1/n} = \frac{1}{\left(1+\frac{1}{n}\right)^{n}}$$

$\therefore$ $$\lim_{n\to\infty} (u_n)^{1/n} = \frac{1}{\lim\limits_{n\to\infty} \left(1+\frac{1}{n}\right)^{n}} = \frac{1}{e} < 1$$

$\therefore$ By Cauchy's root test, $\sum u_n$ is (C).

(iii) $$\sum u_n = \sum \left(1+\frac{1}{\sqrt{n}}\right)^{n^{3/2}}$$

$\therefore$ $$u_n = \left(1+\frac{1}{\sqrt{n}}\right)^{n^{3/2}} \quad \text{and so} \quad u_n^{1/n} = \left(1+\frac{1}{\sqrt{n}}\right)^{\sqrt{n}}$$

Hence $$\lim_{n\to\infty} \left(u_n^{\frac{1}{n}}\right) = \lim_{n\to\infty} \left(1+\frac{1}{\sqrt{n}}\right)^{\sqrt{n}} = e > 1$$

$\therefore$ By Cauchy's root test, $\sum u_n$ is (D).

(iv) $$\sum u_n = \sum \frac{[(n+1)x]^n}{n^{n+1}}$$

$\therefore$ $$u_n = \frac{[(n+1)x]^n}{n^{n+1}} \quad \text{and so} \quad u_n^{1/n} = \frac{(n+1)x}{n^{\frac{n+1}{n}}}$$

$$= \frac{(n+1)x}{n^{1+1/n}}$$

$$= \left(\frac{n+1}{n}\right) \cdot \frac{x}{n^{1/n}}$$

$\therefore$ $$\lim_{n\to\infty} (u_n^{1/n}) = \lim_{n\to\infty} \left(1+\frac{1}{n}\right) \cdot \frac{x}{\lim\limits_{n\to\infty} (n^{1/n})}$$

$$= \frac{x}{\lim\limits_{n\to\infty} (n^{1/n})} \tag{1}$$

Now let $v = n^{1/n}$ $\therefore$ $\log v = \frac{1}{n} \log n$

$$\lim_{n\to\infty} (\log v) = \lim_{n\to\infty} \left(\frac{\log n}{n}\right)$$

$$= \lim_{n\to\infty} \left(\frac{1}{n}\right), \text{by L'Hospital's rule}$$

i.e. $$\log\left[\lim_{n\to\infty} (v)\right] = 0 \quad \text{or} \quad \lim_{n\to\infty} v = e^0 = 1 \tag{2}$$

Using (2) in (1), we have

$$\lim_{n\to\infty} (u_n^{1/n}) = x$$

$\therefore$ By Cauchy's root test, $\sum u_n$ is (C) if $x < 1$, and (D) if $x > 1$

If $x = 1$, the series becomes $u_n = \sum \frac{(n+1)^n}{n^{n+1}}$

Let $$\sum v_n = \sum \frac{1}{n}$$

Then $$\frac{u_n}{v_n} = \frac{(n+1)^n}{n^{n+1}} \times n = \left(\frac{n+1}{n}\right)^n = \left(1+\frac{1}{n}\right)^n$$

$$\therefore \qquad \lim_{n\to\infty}\left(\frac{u_n}{v_n}\right) = e \neq 0$$

$\therefore$ By comparison test, $\sum u_n$ and $\sum v_n$ converge or diverge together.

$\sum v_n$ is (D). Hence $\sum u_n$ is (D) if $x = 1$.

EXERCISE 2(a)

Part A

(Short Answer Questions)

1. Define convergence of an infinite series with an example.
2. Define divergence of an infinite series with an example.
3. Show that the series $1 + x + x^2 + \cdots$ to ∞ oscillates when $x = -1$.
4. Show that the series $1 + x + x^2 + \cdots$ to ∞ oscillates between $-\infty$ and ∞, when $x < -1$.
5. Give an example to shown that $\sum u_n$ is not (C), even though $\lim\limits_{n\to\infty} (u_n) = 0$.
6. Prove that the series $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots$ to ∞ is (D).
7. State two forms of comparison test for the convergence of $\sum u_n$.
8. State two forms of comparison test for the divergence of $\sum u_n$.
9. State Cauchy's root test.
10. Test the convergence of the series $1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \ldots$ to ∞.
11. Test the convergence of the series $\sum \sin\left(\frac{1}{n}\right)$.
12. Test the convergence of the series $\sum e^{-n^2}$.

Part B

Examine the convergence of the following series:

13. $\displaystyle\sum_{n=1}^{\infty} \frac{\sin^2 n}{3^n}$
14. $\displaystyle\sum_{1}^{\infty} \frac{1}{1+4^n}$
15. $\dfrac{1\cdot 2}{3\cdot 4\cdot 5} + \dfrac{2\cdot 3}{4\cdot 5\cdot 6} + \dfrac{3\cdot 4}{5\cdot 6\cdot 7} + \cdots$
16. $\dfrac{1}{1+\sqrt{2}} + \dfrac{2}{1+2\sqrt{3}} + \dfrac{3}{1+3\sqrt{4}} + \cdots$
17. $\dfrac{1}{\sqrt{3}} + \dfrac{1}{\sqrt{5}} + \dfrac{1}{\sqrt{7}} + \cdots$
18. $\dfrac{1}{1\cdot 2\cdot 3} + \dfrac{1}{2\cdot 3\cdot 4} + \dfrac{1}{3\cdot 4\cdot 5} + \cdots$

19. $\dfrac{1}{1\cdot 2\cdot 3}+\dfrac{3}{2\cdot 3\cdot 4}+\dfrac{5}{3\cdot 4\cdot 5}+\cdots$

20. $\sum_{n=1}^{\infty}\dfrac{2^n+3^n}{4^n+5^n}$

21. $\sum\left(\dfrac{n^3-4n+2}{n^5+2n-3}\right)^{\frac{4}{3}}$

22. $\left(\dfrac{n^3-5n^2+7}{n^5+4n^4-n}\right)^{\frac{1}{3}}$

23. $\sum\sin^2\left(\dfrac{1}{n}\right)$

24. $\sum\sqrt{n\tan^{-1}\left(\dfrac{1}{n^3}\right)}$ $\left[Hint\text{: } \lim_{x\to 0}\left(\dfrac{\tan^{-1}x}{x}\right)=1\right]$

25. $\sum\sin\left(\dfrac{n}{n^2+1}\right)$

26. $\sum\sqrt{\dfrac{2^n-1}{3^n-1}}$

27. $\sum\dfrac{(n+1)^3}{n^k+(n+2)^k}$

28. $\sum\dfrac{(n+1)}{n^p}$

29. $\sum\dfrac{1}{n^{1+1/n}}$ $\left[Hint\text{: Choose } v_n=\dfrac{1}{n} \text{ and } \lim_{n\to\infty} n^{1/n}=1\right]$

30. $\sum\left(\sqrt{n^2+1}-n^2\right)$

31. $\sum\left(\sqrt{n^3+1}-\sqrt{n^3}\right)$

32. $\sum\left(\sqrt{n^4+1}-n^2\right)$

33. $\sum\left(\dfrac{\sqrt{n+1}-\sqrt{n}}{n}\right)$

34. $\dfrac{1}{3}+\left(\dfrac{2}{5}\right)^2+\left(\dfrac{3}{7}\right)^3+\cdots$

35. $\dfrac{3x}{4}+\left(\dfrac{4}{5}\right)^2x^2+\left(\dfrac{5}{6}\right)^3x^3+\cdots$

36. $1+\dfrac{x}{2}+\dfrac{x^2}{3^2}+\dfrac{x^3}{4^3}+\cdots\ (x>0)$

37. $x+\dfrac{x^2}{2}+\dfrac{x^3}{3}+\dfrac{x^4}{4}+\cdots\ (x>0)$

38. $1+2\left(\dfrac{1}{3}\right)+\left(\dfrac{1}{3}\right)^2+2\left(\dfrac{1}{3}\right)^3+\left(\dfrac{1}{3}\right)^4+2\left(\dfrac{1}{3}\right)^5+\cdots$

39. $\sum\left(1+\dfrac{1}{n}\right)^{-n^2}$

40. $\sum\left(1+\dfrac{1}{\sqrt{n}}\right)^{n^{-3/2}}$

D'Alambert's Ratio Test

If $\sum u_n$ is a series of positive terms such that $\lim_{n\to\infty}\left(\dfrac{u_{n+1}}{u_n}\right)=l$, then the series $\sum u_n$ is (C) when $l<1$ and is (D) when $l>1$. When $l=1$, the test fails.

Raabe's Test

If $\sum u_n$ is a series of positive terms such that $\lim_{n\to\infty}\left[n\left(\frac{u_n}{u_{n+1}}-1\right)\right]=l$, then the series $\sum u_n$ is (C) when $l > 1$ and is (D) when $l < 1$. When $l = 1$, the test fails.

WORKED EXAMPLE 2(b)

Example 2.1 Test the convergence of the following series:

(i) $\frac{2}{1}+\frac{2^2}{2}+\frac{2^3}{3}+\cdots$ to ∞;

(ii) $\frac{3}{4}+\frac{3\cdot4}{4\cdot6}+\frac{3\cdot4\cdot5}{4\cdot6\cdot8}+\cdots$ to ∞;

(iii) $\frac{1}{1+3}+\frac{2}{1+3^2}+\frac{3}{1+3^3}+\cdots$ to ∞;

(iv) $1+\frac{2^p}{2!}+\frac{3^p}{3!}+\frac{4^p}{4!}+\cdots\infty.$

(i) $$\sum u_n = \frac{2}{1}+\frac{2^2}{2}+\frac{2^2}{3}+\cdots+\frac{2^n}{n}+\cdots$$

$\therefore$ $$u_n = \frac{2^n}{n} \quad\text{and}\quad u_{n+1} = \frac{2^{n+1}}{n+1}$$

$$\frac{u_{n+1}}{u_n}=\frac{2^{n+1}}{n+1}\times\frac{n}{2^n}=\frac{1}{\left(1+\frac{1}{n}\right)}\times 2$$

$\therefore$ $$\lim_{n\to\infty}\left(\frac{u_{n+1}}{u_n}\right)=2>1$$

$\therefore$ By ratio test, $\sum u_n$ is (D).

(ii) $$\sum u_n = \frac{3}{4}+\frac{3\cdot4}{4\cdot6}+\frac{3\cdot4\cdot5}{4\cdot6\cdot8}+\cdots$$

$\therefore$ $$u_n = \frac{3\cdot4\cdot5\ldots(n+2)}{4\cdot6\cdot8\ldots(2n+2)}$$

[*Note* ☑ There are n factors each in the numerator and denominator of the n^{th} term. The factors are in A.P.]

$$u_{n+1} = \frac{3\cdot 4\cdot 5 \ldots (n+2)\,(n+3)}{4\cdot 6\cdot 8 \ldots (2n+2)\,(2n+4)}$$

$$\therefore \quad \frac{u_{n+1}}{u_n} = \frac{n+3}{2n+4} = \frac{1+\frac{3}{n}}{2+\frac{4}{n}}$$

$$\lim_{n\to\infty}\left(\frac{u_{n+1}}{u_n}\right) = \frac{1}{2} < 1.$$

$\therefore$ By ratio test, $\sum u_n$ is (C).

(iii)
$$\sum u_n = \frac{1}{1+3} + \frac{2}{1+3^2} + \frac{3}{1+3^3} + \cdots$$

$$\therefore \quad u_n = \frac{n}{1+3^n} \quad \text{and} \quad u_{n+1} = \frac{n+1}{1+3^{n+1}}$$

$$\frac{u_{n+1}}{u_n} = \frac{n+1}{1+3^{n+1}} \times \frac{1+3^n}{n}$$

$$= \left(1+\frac{1}{n}\right)\cdot\left(\frac{\frac{1}{3^n}+1}{\frac{1}{3^n}+3}\right)$$

$$\therefore \quad \lim_{n\to\infty}\left(\frac{u_{n+1}}{u_n}\right) = \frac{1}{3} < 1.$$

$\therefore$ By ratio test, $\sum u_n$ is (C).

(iv)
$$\sum u_n = 1 + \frac{2^p}{2!} + \frac{3^p}{3!} + \frac{4^p}{4!}$$

$$\therefore \quad u_n = \frac{n^p}{n!} \quad \text{and} \quad u_{n+1} = \frac{(n+1)^p}{(n+1)!}$$

$$\therefore \quad \frac{u_{n+1}}{u_n} = \frac{(n+1)^p}{(n+1)!} \times \frac{n!}{n^p}$$

$$= \left(\frac{n+1}{n}\right)^p \cdot \frac{1}{n+1}$$

$$= \left(1+\frac{1}{n}\right)^p \cdot \frac{1}{n+1}$$

$$\lim_{n\to\infty}\left(\frac{u_{n+1}}{u_n}\right) = 0 < 1.$$

$\therefore$ By ratio test, $\sum u_n$ is (C).

Example 2.2 Test the convergence of the following series:

(i) $\displaystyle\sum_{n+1}^{\infty}\frac{(a+1)\,(2a+1)\,(3a+1)\ldots(na+1)}{(b+1)(2b+1)\,(3b+1)\ldots(nb+1)}; a, b, > 0.$

(ii) $\displaystyle\sum\frac{3^n\; n!}{n^n};$

(iii) $\displaystyle\sum\frac{a^n\; x^n}{1+n^2}(x>0);$

(iv) $\displaystyle\sum\frac{x^n}{1+x^{2n}}(x>0);$

(i) $$\sum u_n = \sum\frac{(a+1)\,(2a+1)\ldots(na+1)}{(b+1)\,(2b+1)\ldots(nb+1)}$$

$$\frac{u_{n+1}}{u_n} = \frac{(a+1)\,(2a+1)\ldots(na+1)\,(\overline{n+1}\cdot a+1)}{(b+1)\,(2b+1)\ldots(nb+1)\,(\overline{n+1}\cdot b+1)}\times\frac{(b+1)\,(2b+1)\ldots(nb+1)}{(a+1)\,(2a+1)\ldots(na+1)}$$

$$= \frac{(n+1)\,a+1}{(n+1)b+1} = \frac{a+\dfrac{1}{n+1}}{b+\dfrac{1}{n+1}}$$

$$\therefore \quad \lim_{n\to\infty}\left(\frac{u_{n+1}}{u_n}\right) = \frac{a}{b}$$

$\therefore$ By Ratio test, $\sum u_n$ is (C) if $\dfrac{a}{b}<1$ or $a<b$, and $\sum u_n$ is (D) if $\dfrac{a}{b}>1$ or $a>b$.

If $a=b$, the ratio test fails.
But in this case, the series becomes $1+1+1+\cdots$ to ∞, which is (D).
Thus $\sum u_n$ is (C) when $0<a<b$, and it is (D) when $0<b\le a$.

(ii) $$\sum u_n = \sum\frac{3^n\; n!}{n^n}$$

$$\therefore \quad u_n = \frac{3^n\cdot n!}{n^n} \quad\text{and}\quad y_{n+1} + \frac{3^{n+1}\cdot(n+1)!}{(n+1)^{n+1}}$$

$$\frac{u_{n+1}}{u_n} = \frac{3^{n+1}(n+1)!}{(n+1)^{n+1}} \times \frac{n^n}{3^n \cdot n!}$$

$$= 3\cdot\left(\frac{n}{n+1}\right)^n \text{ or } 3\left(\frac{1}{1+\frac{1}{n}}\right)^n$$

$\therefore$ $$\lim_{n\to\infty}\left(\frac{u_{n+1}}{u_n}\right) = \frac{3}{e} > 1 \quad (\because e = 2.71828 \cdots)$$

$\therefore$ By ratio test, $\sum u_n$ is (D).

(iii) $$\sum u_n = \sum \frac{a^n\ x^n}{1+n^2}$$

$\therefore$ $$u_n = \frac{a^n\ x^n}{1+n^2} \quad \text{and} \quad u_{n+1} = \frac{a^{n+1}\ x^{n+1}}{1+(n+1)^2}$$

$$\frac{u_{n+1}}{u_n} = \frac{a^{n+1}\ x^{n+1}}{1+(n+1)^2} \times \frac{1+n^2}{a^n\ x^n}$$

$$= ax\cdot\frac{(1+n^2)}{2+2n+n^2}$$

$$= ax\cdot\left(\frac{\frac{1}{n^2}+1}{\frac{2}{n^2}+\frac{2}{n}+1}\right)$$

$\therefore$ $$\lim_{n\to\infty}\left(\frac{u_{n+1}}{u_n}\right) = ax$$

$\therefore$ By ratio test, $\sum u_n$ is (C), if $ax < 1$ or $x > \frac{1}{a}$.

$\sum u_n$ = is (D), if $ax > 1$ or $x < \frac{1}{a}$

Ratio test fails, when $x = \frac{1}{a}$.

But when $x = \frac{1}{a}$, $\sum u_n = \sum \frac{1}{1+n^2}$

By choosing $\sum v_n = \frac{1}{n^2}$ and using comparison test, we can prove that $\sum u_n$ is (C).

Thus $\sum u_n$ is (C) when $x \le \frac{1}{a}$ and (D) when $x > \frac{1}{a}$.

(iv) $$\sum u_n = \sum \frac{x^n}{1+x^{2n}}$$

$$u_n = \frac{x^n}{1+x^{2n}} \quad \text{and} \quad u_{n+1} = \frac{x^{n+1}}{1+x^{2n+2}}$$

$$\frac{u_{n+1}}{u_n} = \frac{x^{n+1}}{1+x^{2n+2}} \times \frac{1+x^{2n}}{x^n}$$

$$= \frac{x+x^{2n+1}}{1+x^{2n+2}}$$

$$\lim_{n\to\infty}\left(\frac{u_{n+1}}{u_n}\right) = x, \text{ if } x<1 \left[\because \lim_{n\to\infty}(x^{2n+1}) = 0 = \lim_{n\to\infty}(x^{2n+2})\right]$$

$$\frac{u_{n+1}}{u_n} = \left(\frac{\frac{1}{x^{2n+1}}+\frac{1}{x}}{\frac{1}{x^{2n+2}}+1}\right)$$

$$\lim_{n\to\infty}\left(\frac{u_{n+1}}{u_n}\right) = \frac{1}{x}, \text{ if } x>1$$

Thus when $x < 1$ and $x > 1$, $\lim_{n\to\infty}\left(\frac{u_{n+1}}{u_n}\right) < 1$ and hence $\sum u_n$ is (C).

But when $x = 1$, the ratio test fails.
In this case, the series becomes

$\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\ldots$ to ∞, which is (D).

Example 2.3 Test the convergence of the following series:

(i) $\sqrt{\frac{1}{2}}x+\sqrt{\frac{2}{5}}x^2+\sqrt{\frac{3}{10}}x^3+\ldots+\sqrt{\frac{n}{n^2+1}}x^n+\ldots(x>0)$

(ii) $\frac{1}{2\sqrt{1}}+\frac{x^2}{3\sqrt{2}}+\frac{x^4}{4\sqrt{3}}+\frac{x^6}{5\sqrt{4}}+\ldots(x>0)$

(iii) $\frac{x}{1+x}+\frac{x^2}{1+x^2}+\frac{x^3}{1+x^3}+\ldots(x>0)$

(iv) $x+\frac{2^2x^2}{2!}+\frac{3^3x^3}{3!}+\frac{4^4x^4}{4!}+\ldots(x>0)$

(i) $$\sum u_n = \sqrt{\frac{1}{2}}x + \sqrt{\frac{2}{5}}x^2 + \sqrt{\frac{3}{10}}x^3 + \ldots$$

$\therefore$ $$u_n = \sqrt{\frac{n}{n^2+1}}x^n \text{ and } u_{n+1} = \sqrt{\frac{n+1}{(n+1)^2+1}}x^{n+1}$$

$$\frac{u_{n+1}}{u_n} = \sqrt{\left(\frac{n+1}{n}\right).\left\{\frac{n^2+1}{(n+1)^2+1}\right\}}x$$

$$= \sqrt{\left(1+\frac{1}{n}\right)\left\{\frac{1+\frac{1}{n^2}}{1+\frac{2}{n}+\frac{2}{n^2}}\right\}}x$$

$\therefore$ $$\lim_{n\to\infty}\left(\frac{u_{n+1}}{u_n}\right) = x.$$

$\therefore$ By ratio test, $\sum u_n$ is (C) if $x < 1$ and it is (D) if $x > 1$.

When $x = 1$, ratio test fails.

In this case, the series becomes $\sum u_n = \sum\sqrt{\frac{n}{n^2+1}}$.

Choosing $\sum v_n = \frac{1}{\sum n^{1/2}}$ and using comparison test, we can prove that $\sum u_n$ is (D).

Thus $\sum u_n$ is (C) when $x < 1$ and (D) when $x \geq 1$.

(ii) $$\sum u_n = \frac{1}{2\sqrt{1}} + \frac{x^2}{3\sqrt{2}} + \frac{x^4}{4\sqrt{3}} + \ldots$$

$\therefore$ $$u_n = \frac{x^{2n-2}}{(n+1)\sqrt{n}} \text{ and } u_{n+1} = \frac{x^{2n}}{(n+2)\sqrt{n+1}}$$

$$\frac{u_{n+1}}{u_n} = \frac{x^{2n}}{x^{2n-2}}\cdot\frac{(n+1)\sqrt{n}}{(n+2)\sqrt{n+1}}$$

$$= \frac{\left(1+\frac{1}{n}\right)}{\left(1+\frac{2}{n}\right)\sqrt{\left(1+\frac{1}{n}\right)}}\cdot x^2$$

$\therefore \qquad \lim_{n\to\infty}\left(\frac{u_{n+1}}{u_n}\right) = x^2$

$\therefore \quad$ By ratio test $\sum u_n$ is (C) if $x^2 < 1$ and (D) if $x^2 > 1$.

When $x^2 = 1$, $\sum u_n = \frac{1}{2\sqrt{1}} + \frac{1}{3\sqrt{2}} + \frac{1}{4\sqrt{3}} + \ldots + \frac{1}{(n+1)\sqrt{n}}$. Choosing

$\sum v_n = \frac{1}{\sum n^{3/2}}$ and using comparison test, we can prove that $\sum u_n$ is (C).

Thus $\sum u_n$ is (C) when $x^2 \le 1$ and it is (D) when $x^2 > 1$.

(iii) $$\sum u_n = \sum \frac{x^n}{1+x^n}$$

$\therefore$ $$\frac{u_{n+1}}{u_n} = \frac{x^{n+1}}{1+x^{n+1}} \times \frac{1+x^n}{x^n}$$

$$= \frac{x+x^{n+1}}{1+x^{n+1}}$$

$$\lim_{n\to\infty}\left(\frac{u_{n+1}}{u_n}\right) = x, \text{ if } x<1 \ \left[\because x^{n+1} \to 0 \text{ as } n\to\infty\right]$$

Also $$\frac{u_{n+1}}{u_n} = \frac{\frac{x}{x^{n+1}}+1}{\frac{1}{x^{n+1}}+1}$$

$$\lim_{n\to\infty}\left(\frac{u_n+1}{u_n}\right) = 1, \text{ if } x > 1$$

$\therefore$ By ratio test, $\sum u_n$ is (C). If $x < 1$ and ratio test fails if $x > 1$.

Also when $x = 1$, the ratio test fails.

In this case, $\sum u_n = \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \ldots$, which is (D).

(iv) $$\sum u_n = x + \frac{2^2 x^2}{2!} + \frac{3^3 x^3}{3!} + \ldots + \frac{n^n x^n}{n!} + \ldots$$

$\therefore$ $$\frac{u_{n+1}}{u_n} = \frac{(n+1)^{n+1} x^{n+1}}{(n+1)!} \times \frac{n!}{n^n x^n}$$

$$= \left(\frac{n+1}{n}\right)^n x \text{ or } \left(1+\frac{1}{n}\right)^n \cdot x$$

$$\lim_{n\to\infty}\left(\frac{u_{n+1}}{u_n}\right) = e.x$$

$\therefore$ By ratio test, $\sum u_n$ is (C) if $ex < 1$ or $x < \frac{1}{e}$ and it is (D) if $ex > 1$ or $x > \frac{1}{e}$.

When $x = \frac{1}{e}$, ratio test fails.

In this case, $\sum u_n = \sum \frac{\left(\frac{n}{e}\right)^n}{n!}$.

$$\lim_{n\to\infty}(u_n) = \lim_{n\to\infty}\left[\frac{\left(\frac{n}{e}\right)^n}{n!}\right] \neq 0. \quad \therefore \sum u_n \text{ is (D), when } x = \frac{1}{e}.$$

Example 2.4 Test the convergence of the following series:

(i) $1 + \frac{1}{2} + \frac{1\cdot 3}{2\cdot 4} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6} + \ldots$

(ii) $\frac{1^2}{4^2} + \frac{1^2\cdot 5^2}{4^2\cdot 8^2} + \frac{1^2\cdot 5^2\cdot 9^2}{4^2\cdot 8^2\cdot 12^2} + \ldots$

(iii) $\frac{3}{7}x + \frac{3\cdot 6}{7\cdot 10}x^2 + \frac{3\cdot 6\cdot 9}{7\cdot 10\cdot 13}x^3 + \ldots (x > 0)$

(iv) $1 + \frac{x}{2^2} + \frac{x^2}{3^2} + \frac{x^3}{4^2} + \ldots (x > 0)$.

(i) $$\sum u_n = \frac{1}{2} + \frac{1\cdot 3}{2\cdot 4} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6} + \ldots \text{ (omitting the first term)}$$

$\therefore$ $$u_n = \frac{1\cdot 3\cdot 5 \ldots (2n-1)}{2\cdot 4\cdot 6 \ldots 2n} \text{ and}$$

$$u_{n+1} = \frac{1\cdot 3\cdot 5 \ldots (2n-1)(2n+1)}{2\cdot 4\cdot 6 \ldots 2n\cdot (2n+2)}$$

$$\frac{u_{n+1}}{u_n} = \frac{2n+1}{2n+2} \text{ or } \frac{2+\frac{1}{n}}{2+\frac{2}{n}}$$

$\therefore$ $$\lim_{n\to\infty}\left(\frac{u_{n+1}}{u_n}\right) = 1.$$

Hence ratio test fails.
Let us try now Raabe's test.

$$\lim_{n\to\infty}\left\{n\left(\frac{u_n}{u_{n+1}}-1\right)\right\} = \lim_{n\to\infty}\left[n\left(\frac{2n+2}{2n+1}-1\right)\right]$$

$$= \lim_{n\to\infty}\left[\frac{n}{2n+1}\right]$$

$$= \lim_{n\to\infty}\left[\frac{1}{2+\frac{1}{n}}\right] = \frac{1}{2} < 1$$

$\therefore$ By Rabbe's test $\sum u_n$ is (D).

(ii) $$\sum u_n = \frac{1^2}{4^2} + \frac{1^2\cdot 5^2}{4^2\cdot 8^2} + \frac{1^2\cdot 5^2\cdot 9^2}{7^2\cdot 8^2\cdot 12^2} + \ldots$$

$\therefore$ $$u_n = \frac{1^2\cdot 5^2\cdot 9^2\ldots(4n-3)^2}{4^2\cdot 8^2\cdot 12^2\ldots(4n)^2}$$

and $$u_{n+1} = \frac{1^2\cdot 5^2\ldots(4n-3)^2\,(4n+1)^2}{4^2\cdot 8^2\ldots(4n)^2\,(4n+4)^2}$$

$$\frac{u_{n+1}}{u_n} = \frac{(4n+1)^2}{(4n+4)^2} = \frac{\left(4+\frac{1}{n}\right)^2}{\left(4+\frac{4}{n}\right)^2}$$

$\therefore$ $$\lim_{n\to\infty}\left(\frac{u_{n+1}}{u_n}\right) = 1.$$

Hence ratio test fails.

Now $$n\left(\frac{u_n}{u_{n+1}}-1\right) = n\left\{\frac{(4n+4)^2}{(4n+1)^2}-1\right\}$$

$$= n\left\{\frac{(8n+5)\cdot 3}{(4n+1)^2}\right\}$$

$\therefore$ $$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}}-1\right) = \lim_{n\to\infty} 3\left\{\frac{8+\frac{5}{n}}{\left(4+\frac{1}{n}\right)^2}\right\}$$

$$= \frac{3\times 8}{4^2} = \frac{3}{2} > 1.$$

$\therefore$ By Raabe's test $\sum u_n$ is (C).

(iii) $$\sum u_n = \frac{3}{7}x + \frac{3\cdot 6}{7\cdot 10}x^2 + \frac{3\cdot 6\cdot 9}{7\cdot 10\cdot 13}x^3 + \ldots$$

$\therefore$ $$u_n = \frac{3\cdot 6\cdot 9\ldots(3n)}{7\cdot 10\cdot 13\ldots(3n+4)}x^n$$

and $$u_{n+1} = \frac{3\cdot 6\cdot 9\ldots(3n)(3n+3)}{7\cdot 10\cdot 13\ldots(3n+4)(3n+7)}x^{n+1}$$

$$\frac{u_{n+1}}{u_n} = \frac{3n+3}{3n+7}x = \left(\frac{3+\frac{3}{n}}{3+\frac{7}{n}}\right).x$$

$\therefore$ $$\lim_{n\to\infty}\left(\frac{u_{n+1}}{u_n}\right) = x$$

$\therefore$ By ratio test, $\sum u_n$ is (C) if $x < 1$ and $\sum u_n$ is (D) if $x > 1$.
When $x = 1$, ratio test fails.

In the case, $$\frac{u_{n+1}}{u_n} = \frac{3n+3}{3n+7}$$

$\therefore$ $$n\left(\frac{u_n}{u_{n+1}} - 1\right) = n\left(\frac{3n+7}{3n+3} - 1\right) = \frac{4n}{3n+3}$$

$\therefore$ $$\lim_{n\to\infty}\left[n\left(\frac{u_n}{u_{n+1}} - 1\right)\right] = \lim_{n\to\infty}\left(\frac{4}{3+\frac{3}{n}}\right)$$

$$= \frac{4}{3} > 1.$$

$\therefore$ By Raabe's test, $\sum u_n$ is (C), when $x = 1$.

Thus $\sum u_n$ is (C) if $x \le 1$ and it is (D) if $x > 1$.

(iv) $$\sum u_n = \frac{x}{2^2} + \frac{x^2}{3^2} + \frac{x^3}{4^2} + \ldots \text{ (omitting the first term)}$$

$\therefore$ $$u_n = \frac{x^n}{(n+1)^2} \text{ and } u_{n+1} = \frac{x^{n+1}}{(n+2)^2}$$

$$\frac{u_{n+1}}{u_n} = \frac{x^{n+1}}{(n+2)^2} \times \frac{(n+1)^2}{x^n} = \frac{\left(1+\frac{1}{n}\right)^2}{\left(1+\frac{2}{n}\right)^2} \cdot x$$

$$\therefore \quad \lim_{n\to\infty} \left(\frac{u_{n+1}}{u_n}\right) = x.$$

$\therefore$ By ratio test, $\sum u_n$ is (C) if $x < 1$ and is (D) if $x > 1$.

Ratio test fails, when $x = 1$.

When $x = 1$, $\dfrac{u_n}{u_{n+1}} = \dfrac{(n+2)^2}{(n+1)^2}$

$$\therefore \quad \lim_{n\to\infty} \left[n\left(\frac{u_n}{u_{n+1}} - 1\right)\right] = \lim_{n\to\infty} \left[\frac{n(2n+3)}{(n+1)^2}\right]$$

$$= \lim_{n\to\infty} \left[\frac{2+\frac{3}{n}}{\left(1+\frac{1}{n}\right)^2}\right]$$

$$= 2 > 1$$

$\therefore$ By Raabe's test, $\sum u_n$ is (C).

Note ☑ The convergence of $\sum u_n$ can be proved, when $x = 1$, by comparison test also.

$\therefore$ $\sum u_n$ is (C) if $x \le 1$ and is (D) if $x > 1$.

Example 2.5 Examine the convergence of the following series:

(i) $\sum \dfrac{a(a+1)(a+2)\dots(a+\overline{n-1})}{b(b+1)(b+2)\dots(b+\overline{n-1})}$; (ii) $\sum \dfrac{x^{n-1}}{n \cdot 3^n}$;

(iii) $\sum \dfrac{(2n)!}{(n!)^2} x^n$.

(i) $$\sum u_n = \sum \frac{a(a+1)(a+2)\dots(a+\overline{n-1})}{b(b+1)(b+2)\dots(b+\overline{n-1})}$$

$$\therefore \quad \frac{u_{n+1}}{u_n} = \frac{a+n}{b+n}$$

$$\lim_{n\to\infty}\left(\frac{u_{n+1}}{u_n}\right) = \lim_{n\to\infty}\left(\frac{\frac{a}{n}+1}{\frac{b}{n}+1}\right) = 1$$

$\therefore$ Ratio test fails

Now $$n\left(\frac{u_n}{u_{n+1}}-1\right) = n\left(\frac{b+n}{a+n}-1\right)$$

$$= \frac{n(b-a)}{a+n} \text{ or } \frac{(b-a)}{\frac{a}{n}+1}$$

$$\lim_{n\to\infty}\left[n\left(\frac{u_n}{u_{n+1}}-1\right)\right] = b-a$$

$\therefore$ By Raabe's test, $\sum u_n$ (C) if $b-a>1$ or $b>a+1$ and it is (D) if $b-a<1$ or $b<a+1$.

If $b = a + 1$, Raabe's test fails.

In this case,

$$u_n = \frac{a\,(a+1)\,(a+2)\ldots(a+\overline{n-1})}{(a+1)\,(a+2)\,(a+3)\ldots(a+\overline{n-1})(a+n)}$$

$$= \frac{a}{a+n}$$

Let $$\sum v_n = \sum \frac{1}{n}$$

$$\frac{u_n}{v_n} = \frac{na}{a+n} = \frac{a}{\frac{a}{n}+1}$$

$\therefore$ $$\lim_{n\to\infty}\left(\frac{u_n}{v_n}\right) = a \neq 0.$$

$\therefore$ By comparison test, $\sum u_n$ and $\sum v_n$ converge or diverge together.

$\sum v_n = \sum \frac{1}{n}$ is (D).

$\therefore$ $\sum u_n$ is also (D).

Thus $\sum u_n$ is (C) if $b > (a + 1)$ and is (D) if $b \leq (a + 1)$.

(ii) $$\sum u_n = \sum \frac{x^{n-1}}{n \cdot 3^n}$$

$$\therefore \quad \frac{u_{n+1}}{u_n} = \frac{x^n}{(n+1)\, 3^{n+1}} \times \frac{n \cdot 3^n}{x^{n-1}}$$

$$= \frac{x}{3} \cdot \frac{1}{1+\frac{1}{n}}$$

$$\therefore \quad \lim_{n\to\infty}\left(\frac{u_{n+1}}{u_n}\right) = \frac{x}{3}$$

$\therefore$ By ratio test, $\sum u_n$ is (C) if $\frac{x}{3} < 1$ or $x > 3$ and it is (D) if $\frac{x}{3} > 1$ or $x > 3$.

If $x = 3$, ratio test fails.

When $x = 3$, $\sum u_n = \sum \frac{1}{3n}$ or $\frac{1}{3}\sum \frac{1}{n}$ is (D)

$\therefore$ $\sum u_n$ is (C) if $x < 3$ and it is (D), if $x \geq 3$.

(iii) $$\sum u_n = \sum \frac{(2n)!}{(n!)^2}\, x^n$$

$$\therefore \quad \frac{u_{n+1}}{u_n} = \frac{(2n+2)!}{[(n+1)!]^2}\, x^{n+1} \times \frac{(n!)^2}{(2n)!\, x^n}$$

$$= \frac{(2n+1)\,(2n+2)}{(n+1)^2}\, x$$

$$= \frac{\left(2+\frac{1}{n}\right)\left(2+\frac{2}{n}\right)}{\left(1+\frac{1}{n}\right)^2} \cdot x$$

$$\therefore \quad \lim_{n\to\infty}\left(\frac{u_{n+1}}{u_n}\right) = 4x$$

$\therefore$ By ratio test, $\sum u_n$ is (C) if $4x < 1$ or $x < \frac{1}{4}$ and is (D) if $4x > 1$ or $x < \frac{1}{4}$.

If $x = \frac{1}{4}$, ratio test fails.

When $x = 1/4$,

$$n\left(\frac{u_n}{u_{n+1}}-1\right) = n\left[\frac{4(n+1)^2}{(2n+1)\,(2n+2)}-1\right]$$

$$= \frac{n}{2n+1} \quad \text{or} \quad \frac{1}{2+\dfrac{1}{n}}$$

$$\therefore \quad \lim_{n\to\infty}\left[n\left(\frac{u_n}{u_{n+1}}-1\right)\right] = \frac{1}{2}<1$$

$\therefore$ By Raabe's test, $\sum u_n$ is (D).

Thus $\sum u_n$ is (C), if $x < 1/4$ and is (D) if $x \geq 1/4$.

EXERCISE 2(b)

Part A

(Short Answer Questions)

1. State D'Alembert's ratio test.
2. State Raabe's test.
3. For the series $\sum u_n = \sum \frac{1}{n}$, show that both the ratio test and Raabe's test fail.
4. Use Raabe's test to establish the convergence of $\sum \frac{1}{n^2}$.
5. Prove that series $\sum (n+1)\,x^n$ is (C) if $0 < x < 1$.

Part B

Examine the convergence or divergence of the following series:

6. $\dfrac{3}{2^4}+\dfrac{3^2}{2^5}+\dfrac{3^3}{2^6}+\cdots$ to ∞.
7. $1+\dfrac{1\cdot 2}{1\cdot 3}+\dfrac{1\cdot 2\cdot 3}{1\cdot 3\cdot 5}+\cdots$ to ∞.
8. $\dfrac{1}{1+2}+\dfrac{2}{1+2^2}+\dfrac{3}{1+2^3}+\cdots$ to ∞.
9. $\dfrac{1!}{2}+\dfrac{2!}{4}+\dfrac{3!}{8}+\dfrac{4!}{16}+\cdots$ to ∞.

10. $\dfrac{3^2}{2\cdot 2}+\dfrac{3^3}{3\cdot 2^2}+\dfrac{3^4}{4\cdot 2^3}+\cdots$ to ∞.

11. $\displaystyle\sum_{n=1}^{\infty}\frac{n^3+k}{2^n+k}$.

12. $\displaystyle\sum\frac{n!}{n^n}$.

13. $\sum n^4\, e^{-n^2}$.

14. $\sum (2n+1)\, x^n \cdot (x>0)$.

15. $\displaystyle\sum\frac{x^{2n-2}}{(n+1)\sqrt{n}}\ (x>0)$.

16. $\dfrac{x}{1\cdot 2}+\dfrac{x^2}{3\cdot 4}+\dfrac{x^3}{5\cdot 6}+\cdots$ to ∞. $(x>0)$

17. $2+\dfrac{3}{2}x+\dfrac{4}{3}x^2+\dfrac{5}{4}x^3+\cdots$ to ∞. $(x>0)$

18. $1+2^2x+3^2x^2+4^2x^3+\cdots$ to ∞. $(x>0)$

19. $\dfrac{2}{1\cdot 3}x+\dfrac{3}{2\cdot 4}x^2+\dfrac{4}{3\cdot 5}x^3+\cdots$ to ∞. $(x>0)$

20. $\dfrac{2x}{1}+\dfrac{3x^2}{8}+\dfrac{4x^3}{27}+\cdots$ to ∞. $(x>0)$

21. $\dfrac{1}{2}\cdot\dfrac{1}{4}+\dfrac{1\cdot 3}{2\cdot 4}\cdot\dfrac{1}{6}+\dfrac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cdot\dfrac{1}{8}+\cdots$ to ∞.

22. $\dfrac{3^2}{5^2}+\dfrac{3^2\cdot 5^2}{5^2\cdot 7^2}+\dfrac{3^2\cdot 5^2\cdot 7^2}{5^2\cdot 7^2\cdot 9^2}+\cdots$ to ∞.

23. $x+\dfrac{1}{2}\cdot\dfrac{x^3}{3}+\dfrac{1\cdot 3}{2\cdot 4}\cdot\dfrac{x^5}{5}+\dfrac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cdot\dfrac{x^7}{7}+\cdots$ to ∞. $(x>0)$

24. $x+\dfrac{2^2\cdot x^2}{2!}+\dfrac{3^3\cdot x^3}{3!}+\dfrac{4^4\cdot x^4}{4!}+\cdots$ to ∞. $(x>0)$

25. $\dfrac{1}{3}x+\dfrac{1\cdot 2}{3\cdot 5}x^2+\dfrac{1\cdot 2\cdot 3}{3\cdot 5\cdot 7}x^3+\cdots$ to ∞. $(x>0)$

26. $\displaystyle\sum\frac{(n!)^2}{(n+1)!}x^n\ (x>0)$.

27. $\sum \dfrac{4\cdot 7 \ldots (3n+1)}{n!} x^n$

28. $\sum \dfrac{2\cdot 5\cdot 8 \ldots (3n-1)}{7\cdot 10\cdot 13 \ldots (3n+4)}$.

29. $\sum \left\{\dfrac{1\cdot 4\cdot 7 \ldots (3n-2)}{3\cdot 6\cdot 9 \ldots 3n}\right\}^2$.

30. $\sum \left\{\dfrac{2^2\cdot 4^2\cdot 6^2 \ldots (2n)^2}{3\cdot 4\cdot 5 \ldots (2n+2)}\right\}$.

2.3 ALTERNATING SERIES

A series in which the terms are alternately positive and negative is called an alternating series.

An alternating series is of the form

$$u_1 - u_2 + u_3 - u_4 + \cdots + (-1)^{n-1} u_n + \cdots = \sum_{n=1}^{\infty} (-1)^{n-1} u_{n_1}$$

where all the u's are positive.

2.3.1 Leibnitz Test for Convergence of an Alternating Series

The alternating series $u_1 - u_2 + u_3 - u_4 + \cdots\ (-1)^{n-1} u_n + \cdots,$ in which $u_1, u_2, u_3, \ldots$ are all positive, is convergent if (i) each term is numerically less than the preceding term, i.e. $u_{n+1} < u_n$, for all n and (ii) $\lim\limits_{n\to\infty} (u_n) = 0$.

Note ☑ If $\lim\limits_{n\to\infty} (u_n) \neq 0$, then $\sum (-1)^{n-1} u_n$ is not convergent, but oscillating.

For example, let us consider the series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + (-1)^{n-1} \cdot \frac{1}{n} + \cdots \text{ to } \infty.$$

Here $u_n = \dfrac{1}{n}$ and $u_{n+1} = \dfrac{1}{n+1}$

Since $n+1 > n, \dfrac{1}{n+1} < \dfrac{1}{n}$

i.e. $u_{n+1} < u_n$ for all n.

Also $\lim\limits_{n\to\infty} (u_n) = \lim\limits_{n\to\infty} \left(\dfrac{1}{n}\right) = 0$

$\therefore$ By Leibnitz test, $\sum (-1)^{n-1} \cdot \dfrac{1}{n}$ is (C).

2.3.2 Absolute and Conditional Convergence

A series $\sum u_n$, in which any term is either positive or negative, is said to be *absolutely convergent* if the series $\sum |u_n|$ is convergent.

A series $\sum u_n$ consisting of positive and negative terms is said to be *conditionally convergent*, if $\sum u_n$ is (C), but $\sum |u_n|$ is (D). For example, let us consider the series.

$\sum u_n = 1 - \frac{1}{2} + \frac{1}{2^2} - \frac{1}{2^3} + \cdots$, which is a series of +ve and –ve terms. (In fact, it is an alternating series).

Now $\sum |u_n| = 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots$ is (C), since it is a geometric series with $r = 1/2 < 1$.

$\therefore$ The given series $\sum u_n$ is absolutely (C).

Let us now consider $\sum u_n = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ By Leibnitz test, we have proved that $\sum u_n$ is (C).

Now $\sum |u_n| = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$ is known to be divergent.

$\therefore$ $\sum u_n$ is conditionally (C).

Note ☑ 1. We can prove that an absolutely convergent series is (ordinarily) convergent. The converse of this result is not true i.e. series which is convergent need not be absolutely convergent, as in the case of the series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots \infty$.

2. To prove the absolute convergence of $\sum u_n$, we have to prove the convergence of $\sum |u_n|$. Since $\sum |u_n|$ is a series of positive terms, we may use any of the standard tests (comparison, Cauchy's root, Ratio and Raabie's tests) to prove its convergence.

2.3.3 Convergence of the Binomial Series

The series $1 + \frac{n}{1!}x + \frac{n(n-1)}{2!}x^2 + \cdots + \frac{n(n-1)\cdots(n-r+1)}{r!}x^r + \cdots$ is called the *Binomial series*. The sum to which this series converges is $(1+x)^n$.

Let us now find the values of x for which the binomial series is (C) for any n.

Omitting the first term 1 in the binomial series,

let $\sum u_r = \frac{n}{1!}x + \frac{n(n-1)}{2!}x^2 + \cdots + \frac{n(n-1)\dots(n-r+1)}{r!}x^r + \cdots$

$\therefore$ The general term $u_r = \frac{n(n-1)\dots(n-r+1)}{r!}x^r$

Note ☑ As 'n' is now a given constant occurring in the given series, the r^{th} term u_r is taken as the general term.

$$u_{r+1} = \frac{n(n-1)\dots(n-r+1)(n-r)}{(r+1)!}x^{r+1}$$

$$\therefore \qquad \frac{u_{r+1}}{u_r} = \frac{n-r}{r+1}\cdot x = \frac{\frac{n}{r}-1}{1+\frac{1}{r}}\cdot x$$

$$\lim_{r\to\infty}\left|\frac{u_{r+1}}{u_r}\right| = \lim_{r\to\infty}\frac{\left|\frac{n}{r}-1\right|}{1+\frac{1}{r}}|x|$$

$$= |-1|\cdot|x|$$

$$= |x|$$

$\therefore$ By ratio test, $\sum|u_r|$ is (C) if $|x|<1$ and it is (D) if $|x|>1$.

$\therefore$ $\sum u_r$, i.e. the given binomial series is absolutely convergent and hence (C) if $|x| < 1$ and not (C) if $|x| > 1$.

Note ☑ When $|x| = 1$, the convergence or divergence of $\sum u_r$ can be established with further analysis. If $x = -1$, $\sum u_r$ is (C) when $n > 0$ and is (D) when $n < 0$.

If $x = 1$, $\sum u_r$ is (C) when $n > -1$ and oscillatory when $n \le -1$.

2.3.4 Convergence of the Exponential Series

The series $1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots$ to ∞ is called the *Exponential series*. The sum to which the series converges is e^x.

Let us now consider the convergence of the series $\sum u_n = \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots$ (omitting the first term)

$$\frac{u_{n+1}}{u_n} = \frac{x^{n+1}}{(n+1)!} \times \frac{n!}{x^n}$$

$$= \frac{x}{n+1}$$

$$\therefore \qquad \lim_{n\to\infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n\to\infty} \frac{|x|}{n+1} = 0 < 1.$$

$\therefore$ By ratio test, $\sum |u_n|$ is (C), for all x.

$\therefore$ The given exponential series $\sum u_n$ is absolutely (C) and hence (C) for all values of x.

2.3.5 Convergence of the Logarithmic Series

The series $x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n-1} \frac{x^n}{n} + \cdots$ is called the *Logarithmic series*. The sum to which the series converges is log $(1 + x)$.

Let us now consider the convergence of the series

$$\sum u_n = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n-1} \frac{x^n}{n} + \cdots$$

$$u_n = \frac{(-1)^{n-1}}{n} x^n \quad \text{and} \quad u_{n+1} = \frac{(-1)^n \, x^{n+1}}{n+1}$$

$$\therefore \qquad \frac{u_{n+1}}{u_n} = -x \cdot \frac{n}{n+1} = -x \cdot \frac{1}{1 + \frac{1}{n}}$$

$$\lim_{n\to\infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n\to\infty} \frac{|-x|}{1 + \frac{1}{n}} = |x|$$

$\therefore$ By ratio test, $\sum |u_n|$ is (C) if $|x| < 1$ and is (D) if $|x| > 1$.

$\therefore$ The logarithmic series $\sum u_n$ is absolutely (C) and hence (C) if $|x| < 1$ and not (C) if $|x| > 1$.

If $x = 1$, the series becomes $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$. It is an alternating series which has been proved to be (C) by Leibnitz test.

If $x = -1$, the series becomes $-1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{4} - \cdots \infty$. which is (D).

Thus the logarithmic series is (C), if $-1 < x \leq 1$.

WORKED EXAMPLE 2(c)

Example 2.1 Examine the convergence of the series:

(i) $\frac{1}{2}-\frac{2}{5}+\frac{3}{10}-\cdots+(-1)^{n-1}\cdot\frac{n}{n^2+1}+\cdots$

(ii) $\frac{1}{1}-\frac{2}{3}+\frac{3}{5}-\frac{4}{7}+\cdots$

(i) The given series is $\sum(-1)^{n-1}\frac{n}{n^2+1}=\sum(-1)^{n-1}u_n$, say.

$$\therefore \qquad u_n=\frac{n}{n^2+1}=\frac{\frac{1}{n}}{1+\frac{1}{n^2}}$$

$$\therefore \qquad \lim_{n\to\infty}(u_n)=0$$

Now
$$u_n-u_{n+1}=\frac{n}{n^2+1}-\frac{n+1}{(n+1)^2+1}$$
$$=\frac{n\{(n+1)^2+1\}-(n+1)(n^2+1)}{(n^2+1)\{(n+1)^2+1\}}$$
$$=\frac{n(n^2+2n+2)-(n+1)(n^2+1)}{(n^2+1)(n^2+2n+2)}$$
$$=\frac{n^2+n-1}{(n^2+1)(n^2+2n+2)}$$
$$=\frac{n(n+1)-1}{(n^2+1)(n^2+2n+2)}$$
= positive, for $n\geq 1$

$\therefore$ $u_n+1<u_n$ for all n.

$\therefore$ By Leibnitz test, $\sum(-1)^{n-1}u_n$ is (C) .

(ii)
$$\sum(-1)^{n-1}u_n=\frac{1}{1}-\frac{2}{3}+\frac{3}{5}-\frac{4}{7}+\cdots$$

$$\therefore \qquad u_n=\frac{n}{2n-1}$$

$$u_n - u_{n+1} = \frac{n}{2n-1} - \frac{n+1}{2n+1}$$

$$= \frac{n(2n+1)-(n+1)(2n-1)}{(2n-1)(2n+1)}$$

$$= \frac{1}{4n^2-1} > 0, \text{ for all } n.$$

$\therefore$ $\qquad u_{n+1} < u_n$ for all n.

But $\qquad \lim_{n\to\infty}(u_n) = \lim_{n\to\infty}\left(\frac{n}{2n-1}\right)$

$$= \lim_{n\to\infty}\left(\frac{1}{2-\frac{1}{n}}\right)$$

$$= \frac{1}{2} \neq 0$$

$\therefore$ The given series is not (C). It is oscillating.

Example 2.2 Examine the convergence of the series:

(i) $1 - \frac{1}{2\sqrt{2}} + \frac{1}{3\sqrt{3}} - \frac{1}{4\sqrt{4}} + \cdots$ to ∞

(ii) $\sum_{n=1}^{\infty}(-1)^{n-1}\sqrt{n+1}-\sqrt{n}).$

(i) Let $\qquad \sum(-1)^{n-1}u_n = 1 - \frac{1}{2\sqrt{2}} + \frac{1}{3\sqrt{3}} - \frac{1}{4\sqrt{4}} + \cdots + (-1)^{n-1}\cdot\frac{1}{n\sqrt{n}} + \cdots$

$$u_n = \frac{1}{n\sqrt{n}}$$

$$\lim_{n\to\infty}(u_n) = \lim_{n\to\infty}\left(\frac{1}{n\sqrt{n}}\right) = 0$$

Now $\qquad u_n - u_{n+1} = \frac{1}{n\sqrt{n}} - \frac{1}{(n+1)\sqrt{n+1}}$

$$= \frac{(n+1)\sqrt{n+1} - n\sqrt{n}}{n(n+1)\sqrt{n(n+1}} > 0, \text{ for all } n \geq 1$$

$\therefore$ $\qquad u_{n+1} < u_n$, for all n.

$\therefore$ The given series is (C) by Leibnitz test.

(ii) Let $\sum(-1)^{n-1} u_n = \sum(-1)^{n-1} (\sqrt{n+1} - \sqrt{n})$

$$\therefore \quad u_n = \sqrt{n+1} - \sqrt{n} = \frac{1}{\sqrt{n+1} + \sqrt{n}}$$

$$\lim_{n\to\infty} (u_n) = \lim_{n\to\infty} \left\{\frac{1}{\sqrt{n+1} + \sqrt{n}}\right\}$$

$$= 0$$

Now
$$u_n - u_{n+1} = \frac{1}{\sqrt{n+1} + \sqrt{n}} - \frac{1}{\sqrt{n+2} + \sqrt{n+1}}$$

$$= \frac{(\sqrt{n+2} + \sqrt{n+1}) - (\sqrt{n+1} + \sqrt{n})}{(\sqrt{n+1} + \sqrt{n})\ (\sqrt{n+2} + \sqrt{n+1})}$$

$$= \frac{\sqrt{n+2} - \sqrt{n}}{(\sqrt{n+1} + \sqrt{n})(\sqrt{n+2} + \sqrt{n+1})} > 0, \text{ for all } n \geq 1.$$

$$\therefore \quad u_{n+1} < u_n \text{ for all } n.$$

$\therefore$ By Leibnitz test, the given series is (C).

Example 2.3 Examine the convergence of the series:

(i) $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$ to ∞;

(ii) $\frac{1}{1^2} - \frac{1}{4^2} + \frac{1}{7^2} - \frac{1}{10^2} + \cdots$ to ∞;

(i) Let $\sum(-1)^{n-1} u_n = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \cdots + (-1)^{n-1} \frac{1}{n^2} + \cdots$

$$\therefore \quad u_n = \frac{1}{n^2}$$

$$\lim_{n\to\infty} \left(\frac{1}{n^2}\right) = 0$$

Also
$$u_n - u_{n+1} = \frac{1}{n^2} - \frac{1}{(n+1)^2}$$

$$= \frac{(n+1)^2 - n^2}{n^2(n+1)^2}$$

$$= \frac{2n+1}{n^2(n+1)^2} > 0, \text{ for all } n \geq 1$$

$\therefore$ $u_{n+1} < u_n$, for all n.

$\therefore$ By Leibnitz test, the given series is (C).

(ii) Let $\sum(-1)^{n-1} u_n = \frac{1}{1^2} - \frac{1}{4^2} + \frac{1}{7^2} - \cdots + (-1)^{n-1} \cdot \frac{1}{(3n-2)^2} + \cdots$

$$\therefore \quad u_n = \frac{1}{(3n-2)^2}$$

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{(3n-2)^2} = 0$$

Also
$$u_n - u_{n+1} = \frac{1}{(3n-2)^2} - \frac{1}{(3n+1)^2}$$
$$= \frac{(3n+1)^2 - (3n-2)^2}{(3n-2)^2\,(3n+1)^2}$$
$$= \frac{3(6n-1)}{(3n-2)^2\,(3n+1)^2} > 0 \text{ for all } n \geq 1$$

i.e. $u_{n+1} < u_n$ for all n.

$\therefore$ The given series is (C), by Leibnitz test.

Example 2.4 Examine the convergence of the series:

(i) $\frac{1}{1\cdot 2} - \frac{1}{3\cdot 4} + \frac{1}{5\cdot 6} - \frac{1}{7\cdot 8} + \cdots;$

(ii) $\frac{1}{1\cdot 2\cdot 3} - \frac{1}{2\cdot 3\cdot 4} + \frac{1}{3\cdot 4\cdot 5} - \cdots;$

(i) Let $\sum(-1)^{n-1} u_n = \frac{1}{1\cdot 2} - \frac{1}{3\cdot 4} + \frac{1}{5\cdot 6} - \cdots$

$$\therefore \quad u_n = \frac{1}{(2n-1)\cdot 2n}$$

$$\lim_{n\to\infty} (u_n) = \lim_{n\to\infty} \frac{1}{(2n-1)\cdot 2n} = 0$$

Also
$$u_n - u_{n+1} = \frac{1}{(2n-1)2n} - \frac{1}{(2n+1)\,(2n+2)}$$
$$= \frac{(2n+1)(2n+2) - 2n(2n-1)}{(2n-1)\cdot 2n\cdot(2n+1)\,(2n+2)}$$
$$= \frac{8n+2}{(2n-1)\cdot 2n\cdot(2n+1)\,(2n+2)} > 0, \text{ for all } n \geq 1$$

$\therefore$ $u_{n+1} < u_n$ for all n.

$\therefore$ By Leibnitz test, the given series is (C).

(ii) Let
$$\sum(-1)^{n-1}\,u_n = \frac{1}{1\cdot 2\cdot 3} = \frac{1}{2\cdot 3\cdot 4} + \frac{1}{3\cdot 4\cdot 5} - \cdots$$

$$u_n = \frac{1}{n(n+1)(n+2)}$$

$$\lim_{n\to\infty}(u_n) = \lim_{n\to\infty}\left\{\frac{1}{n(n+1)\,(n+2)}\right\} = 0$$

Also
$$u_n - u_{n+1} = \frac{1}{n(n+1)(n+2)} - \frac{1}{(n+1)(n+2)\,(n+3)}$$

$$= \frac{(n+3)-n}{n(n+1)(n+2)(n+3)} = \frac{3}{n(n+1)(n+2)(n+3)}$$

$$u_n - u_{n+1} > 0, \text{ for all } n \geq 1$$

or $u_{n+1} < u_n$, for all n.

$\therefore$ By Leibnitz test, the given series is (C).

Example 2.5 Examine the convergence of the following series:

(i) $\dfrac{1}{2!} - \dfrac{2}{3!} + \dfrac{3}{4!} - \cdots;$

(ii) $\dfrac{x}{1+x} - \dfrac{x^2}{1+x^2} + \dfrac{x^3}{1+x^3} - \cdots\ (0 < x < 1).$

(i) Let
$$\sum(-1)^{n-1}\,u_n = \frac{1}{2!} - \frac{2}{3!} + \frac{3}{4!} - \cdots$$

$\therefore$
$$u_n = \frac{n}{(n+1)!}$$

$$\lim_{n\to\infty}(u_n) = \lim_{n\to\infty}\left[\frac{n}{1\cdot 2\cdot 3\cdots n\cdot(n+1)}\right]$$

$$= \lim_{n\to\infty}\left[\frac{1}{1\cdot 2\cdot 3\cdots(n-1)(n+1)}\right]$$

$$= 0$$

Also
$$u_n - u_{n+1} = \frac{n}{(n+1)!} - \frac{n+1}{(n+2)!}$$

$$= \frac{n(n+2)-(n+1)}{(n+2)!}$$

$$= \frac{n^2 + n - 1}{(n+2)!}$$

$$= \frac{n(n+1) - 1}{(n+2)!} > 0, \text{ for all } n \geq 1$$

$\therefore$ $u_{n+1} < u_n$, for all n.

$\therefore$ By Leibnitz test, the given series is (C).

(ii) Let $\sum (-1)^{n-1} u_n = \frac{x}{1+x} - \frac{x^2}{1+x^2} + \frac{x^3}{1+x^3} - \cdots (0 < x < 1).$

$$\therefore \quad u_n = \frac{x^n}{1+x^n}$$

$$\lim_{n\to\infty} (u_n) = \lim_{n\to\infty} \left(\frac{x^n}{1+x^n} \right)$$

$$= \lim_{n\to\infty} \left\{ \frac{1}{\left(\frac{1}{x}\right)^n + 1} \right\}$$

$$= 0, \text{ since } \left(\frac{1}{x}\right)^n \to \infty, \text{ as } \frac{1}{x} > 1$$

Also $$u_n - u_{n+1} = \frac{x^n}{1+x^n} - \frac{x^{n+1}}{1+x^{n+1}}$$

$$= \frac{x^n (1 + x^{n+1}) - x^{n+1} (1 + x^n)}{(1+x^n)(1+x^{n+1})}$$

$$= \frac{x^n (1-x)}{(1+x^n)(1+x^{n+1})} > 0, \text{ for all } n, \text{ since } 0 < x < 1$$

i.e. $u_{n+1} < u_n$ for all n.

$\therefore$ By Leibnitz test, the given series is (C).

EXERCISE 2(c)

Part A

(Short answer questions)

1. State Leibnitz test for the convergence of an alternating series.
2. Show that the series $\sum (-1)^{n-1} . \frac{1}{n}$ is (C).

3. What is meant by absolute convergence?
4. Give an example for a series which is absolutely convergent.
5. What do you mean by conditional convergence?
6. Give an example for a series that is conditionally convergent.
7. Give the values of x for which the binomial series and the logarithmic series are convergent.
8. Show that the series $\frac{1}{\log 2} - \frac{1}{\log 3} + \frac{1}{\log 4} - \frac{1}{\log 5} + \dots$ is convergent.

Part B

Examine the convergence of the following alternating series.

9. $1 - \frac{1}{5} + \frac{1}{9} - \frac{1}{13} + \dots$
10. $1 - \frac{3}{2} + \frac{5}{4} - \frac{7}{8} + \frac{9}{16} \dots$
11. $\frac{3}{4} - \frac{5}{7} + \frac{7}{10} - \frac{9}{13} + \dots$
12. $1 - \frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{3} - \frac{\sqrt{4}}{4} + \dots$
13. $\sum_{n=1}^{\infty} (-1)^{n-1} . \frac{(n^2+1)}{n^3+1}$
14. $\frac{1}{1^2} - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \dots$
15. $\frac{1}{1^2+1} - \frac{1}{2^2+1} + \frac{1}{3^2+1} - \frac{1}{4^2+1} + \dots$
16. $\frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - \dots$
17. $\frac{1}{3.4.5} - \frac{1}{4.5.6} + \frac{1}{5.6.7} - \dots$
18. $\frac{1}{1.1.3} - \frac{1}{2.3.5} + \frac{1}{3.5.7} - \frac{1}{4.7.9} + \dots$
19. $\frac{3}{2!} - \frac{5}{4!} + \frac{7}{6!} - \dots$
20. $\frac{1}{2\log 2} - \frac{1}{3\log 3} + \frac{1}{4\log 4} - \dots$

2.4 SEQUENCES AND SERIES

Definition

If to each positive integer n, a quantity a_n is assigned, then the quantities $a_1, a_2, \ldots,$ $a_n, \ldots$ are said to form an *infinite sequence* or simply *sequence*, denoted by $\{a_n\}$. The individual quantities a_n are called the *terms* of the sequence.

If the terms of a sequence are real, then it is called *a real sequence*.

Limit of a Sequence

A sequence $\{a_n\}$ is said to be *convergent* to the limit 'l', if there exists an integer N, such that

$|a_n - l| < \in$ for all $n > N$, where $\in$ is a positive real quantity, however small it may be, but not zero.

This is denoted as $\lim\limits_{n \to \infty} (a_n) = l$ or $a_n \to l$, as $n \to \infty$

Note ☑ If a sequence converges, the limit is unique.

Note ☑ For all $n > N$ 'in the definition means for infinitely many n'. A sequence, that is not *convergent*, is said to be *divergent.*

Examples

1. The sequence $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots$, viz., $\left\{\frac{n}{n+1}\right\}$ is convergent to the limit 1, as

   $$\left|\frac{n}{n+1} - 1\right| = \frac{1}{n+1} < \in, \text{ when } n+1 > \frac{1}{\in} \text{ or } n > \frac{1}{\in} - 1.$$

   We note that $n > 99$, if we chose $\varepsilon = 0.01$.

2. The sequence $\left|2 + \frac{3}{n}\right|$, viz., 5, $\frac{5}{7}$, 3, $\frac{11}{4}, \frac{13}{5}, \cdots$ converges to the limit 2, as

   $$\left|2 + \frac{3}{n} - 2\right| = \frac{3}{n} < 6, \text{ when } n > \frac{3}{6}$$

It is noted that $n > 300$, if we choose $\in = 0.01$.

Definitions

1. The sequence $\{a_n\}$ is said to be *bounded*, if there is a positive number K such that $|a_n| < K$, for all n. A sequence that is not bounded is said to be *unbounded*.
2. A real sequence $\{a_n\}$ is said to be monotonic increasing or monotonic decreasing, according as

   $$a_1 \le a_2 \le a_3 \le \ldots \quad \text{or} \quad a_1 \ge a_2 \ge a_3 \ge \ldots$$

A sequence that is either *monotonic increasing* or *monotonic decreasing* is called a *monotonic sequence*.

We give below three theorems regarding convergence of sequences without proof:

Theorems

1. A sequence $\{a_n\}$ is convergent, if and only if for every position number $\in$, we can find a number N (which may depend on $\in$) such that

$$|a_m - a_n| < \in, \text{ when } m > N \text{ and } n > N$$

2. Every convergent sequence is bounded. Hence if a sequence is unbounded, it diverges.

Example The sequences 1, 2, 3, 4,… and $\frac{1}{2}, 2, \frac{1}{3}, 3, \frac{1}{4}, 4, \ldots$ are unbounded and hence diverge.

Note ☑ Boundedness is *not sufficient* for convergence.

Example The sequence $\frac{1}{4}, \frac{3}{4}, \frac{1}{5}, \frac{4}{5}, \frac{1}{6}, \frac{5}{6}, \ldots$ is bounded since $|a_n| < 1$, but it is divergent, since $\lim\limits_{n \to a} (a_n) = 0$ or 1.

3. If a real sequence is bounded and monotonic, it is convergent.

Examples

1. Thought the sequence 1, 2, 3, … is monotonic increasing, it is divergent, as it is unbounded.
2. The sequence $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots$ is both monotonic increasing and bounded and hence it converges to the limit 1.

EXERCISE 2(d)

Test the convergence of the following sequences: If convergent find the limit also.

1. $\left\{\frac{2n-1}{n}\right\} \equiv 1, \frac{3}{2}, \frac{5}{3}, \frac{7}{4}, \ldots$
2. $\left\{\frac{n+2}{n}\right\} \equiv 3, 2, \frac{5}{3}, \frac{3}{2}, \ldots$
3. $\frac{1}{2}, 2, \frac{1}{3}, 3, \frac{1}{4}, 4, \ldots$
4. $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots$
5. $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$
6. $1, -2, 3, -4, \ldots$
7. $\left\{\frac{1}{n} \log_{e^n}\right\} \ldots$
8. $\{n^{1/n}\} \ldots$

9. $\left\{\frac{a^n}{n!}\right\}....(a<0)$

10. $a, 2a^2, 3a^3, ...(|a|<1)$

More Tests of Convergence for Series of Positive Terms

1. Cauchy's Integral Test

If $\sum_{n=1}^{\infty} u_n$ is a series of decreasing positive terms, so that *u(x)* is a decreasing function of *x*, for $x \geq 1$, then the given series is convergent, if $\int_1^{\infty} u(x)\,dx$ exists and divergent if $\int_1^{\infty} u(x)\,dx$ does not exists.

2. Cauchy's Condensation Test

If *f*(*n*) is a decreasing positive function of *n* and '*a*' is any positive integer > 1, then the two series $\sum f(n)$ and $\sum a^n . f(a^n)$ are both divergent.

3. Logarithmic Test

If $\sum_{n=1}^{\infty} u_n$ is a series of positive terms such that $\lim_{n\to\infty}\left(n\log\frac{u_n}{u_{n+1}}\right)=l$, then $\sum u_n$ is convergent if $l > 1$ and divergent if $l < 1$.

4. Gauss's test

If $\sum_{n=1}^{\infty} u_n$ is a series of positive terms such that $\frac{u_n}{u_{n+1}}=1+\frac{h}{n}+\frac{A(n)}{n^2}$, where $A(n)$ is a bounded function of *n* as $n \to \infty$, then the series is convergent if $h > 1$ and divergent if $h \leq 1$.

5. Kummer's Test

If $\sum_{n=1}^{\infty} u_n$ is a series of positive terms and $\{a_n\}$ is a sequence of positive terms such that $\left(a_n . \frac{u_n}{u_{n+1}} - a_{n+1}\right) \geq r > 0$, for $n \geq m$, then $\sum u_n$ is convergent. If $\left(a_n . \frac{u_n}{u_{n+1}} - a_{n+1}\right) \leq 0$, then $\sum u_n$ is divergent, provided that $\sum_{n=1}^{\infty}\frac{1}{a_n}$ is divergent.

Note ☑ Raabe's test is a particular case of Kummer's test corresponding to $a_n = n$.

WORKED EXAMPLE 2(d)

Example 2.1 Test the convergence of the p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$, by using the integral test.

$$\sum_{n=1}^{\infty} \frac{1}{np} = \sum_{n=1}^{\infty} u_n \qquad \therefore\ u(x) = \frac{1}{x^p}$$

$$\int_1^{\infty} \frac{1}{x^p}\, dx = \left[\frac{x^{-p+1}}{-p+1}\right]_1^{\infty} = -\frac{1}{p-1}\cdot\left(\frac{1}{x^{p-1}}\right)_1^{\infty} = \begin{cases} \frac{1}{p-1}, \text{if } p>1 \\ \infty, \text{if } p \le 1 \end{cases}$$

$\therefore\ \sum_{n=1}^{\infty} \frac{1}{np}$ is (c), when $p > 1$ and (D), when $p \le 1$, by Cauchy's integral test.

Example 2.2 Test the convergence of the series $\sum \frac{1}{n \log n}$.

Let $$\sum \frac{1}{n \log n} = \sum u_n \qquad \therefore\ u(x) = \frac{1}{x \log x}$$

Now $$\int_1^{\infty} \frac{1}{x \log x}\, dx = [\log \log x]_1^{\infty} = \infty$$

$\therefore$ By integral test, $\sum u_n$ is (D).

Example 2.3 Test the convergence of the series $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}$

Let $$f(n) = \frac{1}{n(\log n)^p}$$

$\therefore$ $$a^n f(a^n) = a^n \frac{1}{a^n (\log a^n)^p} = \frac{1}{(n \log a)^p} = \frac{1}{(\log a)^p} \cdot \frac{1}{n^p}$$

Now $$\sum \frac{1}{(\log a)^p . n^p} = \frac{1}{(\log a)^p} \sum \frac{1}{n^p}$$

is (C), if $p > 1$ and (D), if $p \le 1$, by Cauchy's condensation test,

Example 2.4 Test the convergence of the series $x + \frac{2^2 x^2}{2!} + \frac{3^3 x^3}{3!} + \frac{4^4 x^4}{4!} + \cdots \infty$.

$$\frac{u_n}{u_{n+1}} = \frac{n^n x^n}{n!} \times \frac{(n+1)!}{(n+1)^{n+1} x^{n+1}} = \left(\frac{n}{n+1}\right)^n \cdot \frac{1}{x}$$

$$\lim_{n\to\infty}\left(\frac{u_n}{u_{n+1}}\right) = \lim_{n\to\infty}\frac{1}{(1+1/n)^n}\cdot\frac{1}{x} = \frac{1}{ex}$$

$\therefore$ By Ratio test, $\sum u_n$ is (C), when $x < \dfrac{1}{e}$ and (D) when $x > \dfrac{1}{e}$.

When $x = \dfrac{1}{e}$, the ratio test fails.

$$\log\left(\frac{u_n}{u_{n+1}}\right) = \log\left\{\frac{e}{(1+1/n)^n}\right\} = 1 - n\log\left(1+\frac{1}{n}\right)$$

$$= 1 - n\left[\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} - \ldots\right] = \frac{1}{2n} - \frac{1}{3n^2} + \ldots$$

Now $\displaystyle\lim_{n\to\infty}\left\{n\log\frac{u_n}{u_{n+1}}\right\} = \frac{1}{2} < 1.$

$\therefore$ By the Logarithmic test, $\sum u_n$ is (D).

Example 2.5 Test the convergence of the series $\dfrac{1}{2}.\dfrac{1}{3} + \dfrac{1.3}{2.4}.\dfrac{1}{5} + \dfrac{1.3.5}{2.4.6}.\dfrac{1}{7} + \cdots\infty.$

$$u_n = \frac{1\cdot 3\cdot 5\ldots(2n-1)}{2\cdot 4\cdot 6\ldots(2n)}\cdot\frac{1}{2n+1}$$

$$\therefore \quad \frac{u_n}{u_{n+1}} = \frac{1\cdot 3\cdot 5\ldots(2n-1)}{2\cdot 4\cdot 6\ldots(2n)}\cdot\frac{1}{2n+1}\times \frac{2\cdot 4\cdot 6\ldots(2n)(2n+2)}{1\cdot 3\cdot 5\ldots(2n-1)(2n+1)}\cdot(2n+3)$$

$$= \frac{(2n+2)(2n+3)}{(2n+1)^2}$$

$$= \frac{(1+1/n)(1+3/2n)}{(1+1/2n)^2} = \left\{1 + \frac{(5/2)}{n} + \frac{(3/2)}{n^2}\right\}\left(1+\frac{1}{2n}\right)^{-2}$$

$$= \left\{1 + \frac{(5/2)}{n} + \frac{(3/2)}{n^2}\right\}\left\{1 - \frac{2}{2n} + \frac{3}{4n^2}\cdots\right\}^{-2}$$

$$= 1 + \frac{(3/2)}{n} + \text{terms containing}\frac{1}{n^2}\text{and higher power of }\frac{1}{n}$$

$$= 1 + \frac{(3/2)}{n} + 0\left(\frac{1}{n^2}\right) \equiv 1 + \frac{h}{n} + 0\left(\frac{1}{n^2}\right)$$

$\therefore \quad h > 1$ and so $\sum u_n$ is (C), by Gauss's test.

EXERCISE 2(e)

Test the convergence of the following series:

1. $\sum_{n=1}^{\infty} \frac{1}{1+n^2}$, using integral test.

2. $\sum_{n=1}^{\infty} \frac{1}{n}$, using integral test.

3. $\sum_{n=1}^{\infty} \frac{n}{1+n^4}$, using integral test.

4. $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}$, using integral test.

5. $\sum_{n=1}^{\infty} \frac{1}{n^p}$, using condensation test.

6. $\sum_{n=2}^{\infty} \frac{1}{n \log n}$, using condensation test.

7. $1 + \frac{2x}{2!} + \frac{3^2 x^2}{3!} + \frac{4^3 x^3}{4!} + \cdots \infty$, using logarithmic test.

8. $\frac{1}{2}x + \frac{1 \cdot 3}{2 \cdot 4}x^2 + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}x^3 + \cdots \infty$, using logarithmic test.

9. $\frac{1^2}{2^2} + \frac{1^2 \cdot 3^2}{2^2 \cdot 4^2} + \frac{1^2 \cdot 3^2 \cdot 5^2}{2^2 \cdot 4^2 \cdot 6^2} + \cdots \infty$, using Gauss's test.

10. $\frac{1}{3} + \frac{1 \cdot 2}{3 \cdot 4} + \frac{1 \cdot 2 \cdot 3}{3 \cdot 4 \cdot 5}x^3 + \cdots \infty$, using Gauss's test.

ANSWERS

Exercise 2(a)

(5) $\sum \frac{1}{n}$	(10) Dgt.	(11) Dgt.	(12) Cgt.	(13) Cgt.	(14) Cgt.
(15) Dgt.	(16) Dgt.	(17) Dgt.	(18) Cgt.	(19) Cgt.	(20) Cgt.
(21) Cgt.	(22) Dgt.	(23) Cgt.	(24) Dgt.	(25) Dgt.	(26) Cgt.

(27) Cgt. if $k > 4$ and dgt. if $k \geq 4$. (28) Cgt. if $p > 2$ and dgt. if $p \leq 2$
(29) Dgt. (30) Dgt. (31) Cgt. (32) Cgt. (33) Cgt. (34) Cgt.
(35) Cgt. if $x > 4$ and dgt. if $x \geq 1$. (36) Cgt.
(37) Cgt. if $x > 4$ and dgt. if $x \geq 1$. (38) Cgt. (39) Cgt. (40) Cgt.

Exercise 2(b)

(6) Dgt. (7) Cgt. (8) Cgt. (9) Dgt. (10) Dgt. (11) Cgt.
(12) Cgt. (13) Cgt. (14) Cgt. if $x < 1$ and dgt. if $x \geq 1$.
(15) Cgt. if $x^2 \leq 1$ and dgt. if $x^2 > 1$. (16) Cgt. if $x \leq 1$ and dgt. if $x > 1$.
(17) Cgt. if $x < 1$ and dgt. if $x \geq 1$. (18) Cgt. if $x < 1$ and dgt. if $x \geq 1$.
(19) Cgt. if $x < 1$ and dgt. if $x \geq 1$. (20) Cgt. if $x \leq 1$ and dgt. if $x > 1$.
(21) Cgt. (22) Cgt. (23) Cgt. if $x^2 \leq 1$ and dgt. if $x^2 > 1$.
(24) Cgt. if $x < \frac{i}{e}$ and dgt. if $x \geq \frac{i}{e}$. (25) Cgt. if $x < 2$ and dgt. if $x \geq 2$.
(26) Cgt. if $x < 4$ and dgt. if $x \geq 4$. (27) Cgt. if $x < \frac{1}{3}$ and dgt. if $x \geq \frac{1}{3}$.
(28) Cgt. (29) Cgt. (30) Cgt.

Exercise 2(c)

(4) $1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots$ (6) $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$ (9) Cgt.
(10) Cgt. (11) Oscillatory (12) Cgt. (13) Cgt. (14) Cgt.
(15) Cgt. (16) Cgt. (17) Cgt. (18) Cgt. (19) Cgt. (20) Cgt.

Exercise 2(d)

(1) Cgt. to 2 (2) Cgt. to 1 (3) Dgt. (4) Cgt. to 1 (5) Cgt. to 0 (6) Dgt.
(7) Cgt. to 0 (8) Cgt. to 1 (9) Cgt. to 0 (10) Cgt. to 0, when $0 \leq 1/2$

Exercise 2(e)

(1) Cgt. (2) Dgt. (3) Cgt. (4) Cgt. if $p > 1$ and dgt. if $p \leq 1$
(5) Cgt. if $p > 1$ and dgt, if $p \leq 1$. (6) Dgt.
(7) Cgt., if $x \leq \frac{1}{e}$ and dgt., if $x > \frac{1}{e}$ (8) Cgt., if $x < 1$ and dgt. If $x \geq 1$
(9) Dgt. (10) Cgt.