18MAB101T
Calculus and Linear Algebra
Unit-3

Prepared by
Dr. S. ATHITHAN
Assistant Professor
Department of of Mathematics
College of Engineering and Technology
SRM Institute of Science and Technology
Kattankulathur-603203, Chengalpattu District.

1. Indroduction

A differential equation is a mathematical equation which involves a function and its derivatives.
For example,

$$(i)\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = e^{-3x}$$

$$(ii)\frac{d^2y}{dx^2} + 3\left(\frac{dy}{dx}\right)^2 + 2y = \sin 3x$$

$$(iii)\left(\frac{d^2y}{dx^2}\right)^2 + 5\frac{dy}{dx} + 6y = 5x$$

$$(iv)\frac{d^3y}{dx^3} + \frac{dy}{dx} = e^{-x}$$

are few differential equations.

2. Linear Differential Equations of Second Order with Constant Co-efficients

An equation of the form

$$a_0\frac{d^n y}{dx^n} + a_1\frac{d^{n-1}y}{dx^{n-1}} + a_2\frac{d^{n-2}y}{dx^{n-2}} + \cdots + a_{n-1}\frac{dy}{dx} + a_n y = F(x)$$

where $a_0, a_1, \ldots, a_n$ are constants, is said to be a LINEAR DIFFERENTIAL EQUATION of degree 'n' with constant coefficients.

Let $\frac{d}{dx} = D, \frac{d^2}{dx^2} = D^2 + \cdots + \frac{d^n}{dx^n} = D^n$. Then the above equation can be written as

$$\begin{aligned}(a_0D^n + a_1D^{n-1} + a_2D^{n-2} + \cdots + a_{n-1}D + a_n)y &= F(x) \\ \phi(D)y = F(x) \qquad (2.1)\end{aligned}$$

The general or Complete solution of (2.1) consists of two parts namely the `Complementary Function` (C.F.) and `Particular Integral` (P.I.). *i.e.* The General Solution is

$$y = C.F. + P.I. = y_c + y_p$$

Complementary Function

Definition 2.0.1 (Complimentary Function)**.** *The general solution of $\phi(D) = 0$ is called as Complementary Function and it id denoted by y_c.*

Definition 2.0.2 (Auxiliary Equation). *An equation of the form $\phi(m) = 0$ is called as an Auxiliary Equation.*

Depends on the roots of the polynomial $\phi(m) = 0$, *i.e.* the roots of the auxiliary equation, we have the following cases to write the Complimentary Function.

Case 1:
The roots of the auxiliary equation are real and distinct, then we write the roots of $\phi(m) = 0$ as m_1 and m_2 and the C.F. becomes, $y_c = c_1 e^{m_1 x} + c_2 e^{m_2 x}$.

Generalized form of C.F. in this case: If $m_1, m_2, \ldots, m_n$ be the real and distinct roots of the auxiliary equation, then the C.F. becomes, $y_c = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \cdots + c_n e^{m_n x}$.

Case 2:
The roots of the auxiliary equation are real and equal, then we write the roots of $\phi(m) = 0$ as $m_1 = m_2 = m$ and the C.F. becomes, $y_c = (c_1 + c_2 x)e^{m_x}$.

Generalized form of C.F.in this case: If $m_1 = m_2 = \cdots = m_n (= m)$, then the C.F. becomes, $y_c = (c_1 + c_2 x + c_3 x^2 + \cdots + c_n x^{n-1})e^{mx}$.

Case 3:
The roots of the auxiliary equation are complex conjugates *i.e.* $m = \alpha \pm i\beta$, then the

C.F. becomes,

$$\begin{aligned} y_c &= e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x) \\ \text{(OR)}\ y_c &= c_1 e^{\alpha x} \cos(\beta x + c_2) \\ \text{(OR)}\ y_c &= c_1 e^{\alpha x} \sin(\beta x + c_2) \end{aligned}$$

Note: For repeated complex roots, say $m = \alpha \pm i\beta,\ \alpha \pm i\beta$, the C.F. becomes, $y_c = e^{\alpha x}[(c_1 + c_2 x) \cos \beta x + (c_3 + c_4 x) \sin \beta x]$.

Case 4:
The roots of the auxiliary equation are surds, *i.e.* $m = \alpha \pm \sqrt{\beta}$, then the C.F. becomes,

$$\begin{aligned} y_c &= e^{\alpha x}(c_1 \cosh \sqrt{\beta} x + c_2 \sinh \sqrt{\beta} x) \\ \text{(OR)}\ y_c &= c_1 e^{\alpha x} \cosh(\sqrt{\beta} x + c_2) \\ \text{(OR)}\ y_c &= c_1 e^{\alpha x} \sinh(\sqrt{\beta} x + c_2) \end{aligned}$$

Note: For repeated complex roots, say $m = \alpha \pm \sqrt{\beta},\ \alpha \pm \sqrt{\beta}$, the C.F. becomes, $y_c = e^{\alpha x}[(c_1 + c_2 x) \cos \sqrt{\beta} x + (c_3 + c_4 x) \sin \sqrt{\beta} x]$.

Particular Integral

Definition 2.0.3 (Particular Integral)**.** *The general solution of $\phi(D)y = F(x)$ is called as Particular Integral and it is denoted by y_p.*

Based on the function on the RHS of the equation $\phi(D)y = F(x)$, *i.e.* based on $F(x)$, the following cases can be considered while evaluating the Particular Integral (P.I.).
Let us see some short cut methods of evaluating P.I. when $F(x)$ is of the following form:

A. $F(x) = e^{ax}$

B. $F(x) = \sin ax$ or $\cos ax$

C. $F(x) = x^m$ (polynomial function)

D. $F(x) = e^{ax}\chi(x)$, where $\chi(x) = x^m$ or $\sin ax$ or $\cos ax$

E. $F(x) = x^m\chi(x)$, where $\chi(x) = \sin ax$ or $\cos ax$

CASE A: $F(x) = e^{ax}$

We know that

$$y_p = \frac{1}{\phi(D)}F(x)$$

Now,

$$\begin{aligned} y_p &= \frac{1}{\phi(D)}e^{ax} \\ &= \frac{1}{\phi(a)}e^{ax}, \text{ if } \phi(a) \neq 0 \quad \boxed{\text{Directly replace } D \text{ by } a} \end{aligned}$$

If $\phi(a) = 0$, then we rewrite $\phi(D)$ as the product of its factors and then we have

$$y_p = x\frac{1}{\phi'(D)\psi(D)}e^{ax}, \quad \text{with } \psi(a) \neq 0$$

$$(OR)$$

$$y_p = x^2\frac{1}{\phi''(D)}e^{ax}$$

CASE B: $F(x) = \sin ax$ or $\cos ax$

We know that

$$y_p = \frac{1}{\phi(D)}F(x)$$

Let us consider $\phi(D) = \psi(D^2)$ (*i.e.* considering only the quadratic part), then the above equation becomes

$$\begin{aligned} y_p &= \frac{1}{\psi(D^2)}\sin ax \text{ (or) } \frac{1}{\psi(D^2)}\cos ax \\ &= \frac{1}{\psi(-a^2)}\sin ax \text{ (or) } \frac{1}{\psi(-a^2)}\cos ax \quad \boxed{\text{Replace } D^2 \text{ by } -a^2 \text{ if } \psi(-a^2) \neq 0} \end{aligned}$$

$\psi(a^2) = 0$, *i.e.* when y_p is of the form $y_p = \dfrac{1}{D^2+a^2}\sin ax$ (or) $\dfrac{1}{D^2+a^2}\cos ax$, then

$$y_p = \frac{x}{2}\int \sin ax \text{ (or) } \frac{x}{2}\int \cos ax$$

CASE C: $F(x) = x^k,\ k \in \mathbb{Z}^+$

We know that

$$y_p = \frac{1}{\phi(D)} F(x) = \frac{1}{\phi(D)} x^k$$

Now, taking the lowest degree term (may be constant term) and write the the denominator in the form of $[1 + \phi(D)]$, then we have

$$y_p = \frac{1}{[1 + \phi(D)]} F(x) = [1 + \phi(D)]^{-1} x^k$$

Expanding this relation upto k^{th} derivative using BINOMIAL EXPANSION and hence we get the desired y_p.

Few important Binomial Expansions:

1. $(1 - D)^{-1} = 1 + D + D^2 + \ldots$
2. $(1 + D)^{-1} = 1 - D + D^2 - D^3 + \ldots$
3. $(1 - D)^{-2} = 1 + 2D + 3D^2 + \ldots$
4. $(1 + D)^{-2} = 1 - 2D + 3D^2 - 4D^3 + \ldots$
5. $(1 - D)^{-3} = 1 + 3D + 6D^2 + \ldots$

6. $(1+D)^{-3} = 1 - 3D + 6D^2 - \ldots$

CASE D: $F(x) = e^{ax}\chi(x), \;\; \chi(x) = \cos ax \; (or) \; \sin ax \; (or) \; x^k$

We know that

$$\begin{aligned} y_p &= \frac{1}{\phi(D)}F(x) \\ &= \frac{1}{\phi(D)}e^{ax}\chi(x) \\ &= e^{ax}\frac{1}{\phi(D+a)}\chi(x) \;\; \boxed{\text{Replace } D \text{ by } D+a} \end{aligned}$$

Now, depends upon $\chi(x) = \cos ax \; (or) \; \sin ax \; (or) \; x^k$, we proceed by using CASE a (or) b.

CASE E: $F(x) = x^k\chi(x), \;\; \chi(x) = \cos ax \; (or) \; \sin ax$

We know that

$$\begin{aligned} y_p &= \frac{1}{\phi(D)}F(x) \\ &= \frac{1}{\phi(D)}x^k\chi(x) \end{aligned}$$

Now we consider the following subcases.

SUBCASE E1:

Let $k = 1$, then

$$y_p = \left[x - \frac{\phi'(D)}{\phi(D)}\right]\frac{1}{\phi(D)}\chi(x)$$

SUBCASE E2:

Let $k \neq 1$ (may be $k = 1$ also considered in this case)

$$\begin{aligned} y_p &= \frac{1}{\phi(D)} x^k \chi(x) \\ y_p &= \frac{1}{\phi(D)} x^k \cos ax \ (or) \ \sin ax \end{aligned}$$

We know that $e^{i\theta} = \cos\theta + i\sin\theta$, then

$$\cos\theta = Re(e^{i\theta}), \ \sin\theta = Im(e^{i\theta})$$

We may also write the above description as

$$\cos\theta = R.P.(e^{i\theta}), \ \sin\theta = I.P.(e^{i\theta})$$

Now, if $\chi(x) = \cos ax$, then

$$\begin{aligned} y_p &= \frac{1}{\phi(D)} x^k \cos ax \\ &= \frac{1}{\phi(D)} x^k Re(e^{iax}) \\ &= Re\left[\frac{1}{\phi(D)} x^k e^{iax}\right] \\ &= Re\left[e^{iax} \frac{1}{\phi(D+ia)} x^k\right] \end{aligned}$$

Using the previous cases, we will proceed to solve the above y_p by substituting $e^{iax} = \cos ax + i \sin ax$ and then collecting the terms in the Real Part.
if $\chi(x) = \sin ax$, then

$$
\begin{aligned}
y_p &= \frac{1}{\phi(D)} x^k \sin ax \\
&= \frac{1}{\phi(D)} x^k Im(e^{iax}) \\
&= Im\left[\frac{1}{\phi(D)} x^k e^{iax}\right] \\
&= Im\left[e^{iax} \frac{1}{\phi(D+ia)} x^k\right]
\end{aligned}
$$

Using the previous cases, we will proceed to solve the above y_p by substituting $e^{iax} = \cos ax + i \sin ax$ and then collecting the terms in the Imaginary Part.

Example 2.1. *Solve:* $\dfrac{d^3y}{dx^3} - 7\dfrac{dy}{dx} - 6y = x^2 + \sin x + e^{4x}$

Hints/Solution:
Given equation is of the form $(D^3 - 7D - 6)y = x^2 + \sin x + e^{4x}$
The auxiliary equation is $m^3 - 7m - 6 = 0 \implies m = -1, -2, 3$.

$\therefore$ C.F. $y_c = c_1 e^{-x} + c_2 e^{-2x} + c_3 e^{3x}$

$$
\begin{aligned}
(P.I.)_1 &= \frac{1}{D^3 - 7D - 6} x^2 \\
&= -6\left[1 - \left(\frac{1}{6}D^3 - \frac{7}{6}D\right)\right]^{-1}(x^2) \\
&= -6\left[1 + \left(\frac{1}{6}D^3 - \frac{7}{6}D\right) + \left(\frac{1}{6}D^3 - \frac{7}{6}D\right)^2\right](x^2) \\
&= -6\left[1 - \frac{7}{6}D + \frac{49}{36}D^2\right](x^2) \\
&= -6\left[x^2 - \frac{7x}{3} + \frac{49}{18}\right] \\
&= \frac{7x}{3} - 6x^2 - \frac{49}{18}
\end{aligned}
$$

$$
\begin{aligned}
(P.I.)_2 &= \frac{1}{D^3 - 7D - 6}\sin x \\
&= \frac{1}{-D - 7D - 6}\sin x \\
&= -\frac{1}{8D + 6}\frac{8D - 6}{8D - 6}\sin x \\
&= -\frac{8D - 6}{64D^2 - 36}\sin x \\
&= -\frac{8\cos x - 6\sin x}{-64 - 36} \\
&= \frac{8\cos x - 6\sin x}{100}
\end{aligned}
$$

$$
\begin{aligned}
(P.I.)_3 &= \frac{1}{D^3 - 7D - 6}e^{4x} \\
&= \frac{1}{64 - 28 - 6}e^{4x} \\
&= \frac{1}{30}e^{4x}.
\end{aligned}
$$

Hence the complete solution is given by

$y = C.F. + P.I. = y_c + y_p = c_1e^{-x} + c_2e^{-2x} + c_3e^{3x} + \frac{7x}{3} - 6x^2 - \frac{49}{18} + \frac{8\cos x - 6\sin x}{100} + \frac{1}{30}e^{4x}.$

3. Linear ODE with variable co-efficients

Methodology to convert ODE with variable co-efficients as ODE with constant co-efficients

If the given ODE is of the form $[X^2D^2 + XD + 1]y = G(X)$

Let $X = e^z$ and $\theta = \dfrac{d}{dz}$.

$\therefore \quad z = \log X$ and $\dfrac{dy}{dX} = \dfrac{dy}{dz}\cdot\dfrac{dz}{dX} = \dfrac{dy}{dz}\cdot\dfrac{1}{X} \implies X\dfrac{dy}{dX} = \dfrac{dy}{dz}$

i.e. $\boxed{X\dfrac{dy}{dX} = XD = \dfrac{dy}{dz} = \theta y}$ and Differentiating $X\dfrac{dy}{dX} = \dfrac{dy}{dz}$ w.r.to X, we have

$$\begin{aligned} X\frac{d^2y}{dX^2} + \frac{dy}{dX} &= \frac{d^2y}{dz^2}\frac{dz}{dX} \\ \implies X^2\frac{d^2y}{dX^2} + X\frac{dy}{dX} &= \frac{d^2y}{dz^2} \\ \implies X^2\frac{d^2y}{dX^2} &= \frac{d^2y}{dz^2} - \frac{dy}{dz} = (\theta^2 - \theta)y \end{aligned}$$

i.e. $\boxed{X^2\dfrac{d^2y}{dX^2} = X^2D^2y = \theta(\theta - 1)y}$

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Similarly we have $X^3\dfrac{d^3y}{dX^3} = X^3D^3y = \theta(\theta-1)(\theta-2)y$,

$X^4\dfrac{d^4y}{dX^4} = X^4D^4y = \theta(\theta-1)(\theta-2)(\theta-3)y$ and so on.

Example 3.1. *Solve:* $(1+x)^2\dfrac{d^2y}{dx^2}+(1+x)\dfrac{dy}{dx}+y = \log(1+x)+\cos[\log(1+x)]$

Hints/Solution:

Given equation is of the form $[(1+x)^2D^2+(1+x)D+1]y = \log(1+x)+\cos[\log(1+x)]$

Let $X = 1+x$, then the ODE becomes $[X^2D^2+XD+1]y = \log X+\cos[\log X]$

Let $X = e^z$ and $\theta = \dfrac{d}{dz}$.

$$\therefore \quad z = \log X \text{ and } \frac{dy}{dX} = \frac{dy}{dz}\cdot\frac{dz}{dX} = \frac{dy}{dz}\cdot\frac{1}{X} \implies X\frac{dy}{dX} = \frac{dy}{dz}$$

i.e. $X\dfrac{dy}{dX} = XD = \dfrac{dy}{dz} = \theta y$ and $X^2\dfrac{d^2y}{dX^2} = X^2D^2y = \theta(\theta-1)y$

Now, the ODE takes the form

$$[\theta(\theta-1)+\theta+1]y = (\theta^2+1)y = z+\cos z$$

The auxiliary equation is $m^2 + 1 = 0 \implies m = \pm i$.

$\therefore$ C.F. $y_c = e^{0z}(c_1 \cos z + c_2 \sin z) = c_1 \cos z + c_2 \sin z$

$$\begin{aligned}
(P.I.)_1 &= \frac{1}{\theta^2 + 1} z \\
&= \left[1 + \theta^2\right]^{-1} z \\
&= \left[1 - \theta^2 + \theta^4 - \dots\right] z \\
&= z
\end{aligned}$$

$$\begin{aligned}
(P.I.)_2 &= \frac{1}{\theta^2 + 1} \cos z \\
&= \frac{1}{-1 + 1} \cos z \\
&= \frac{z}{2\theta} \frac{\theta}{\theta} \cos z \\
&= \frac{z}{2} \sin z
\end{aligned}$$

Hence the complete solution is given by

$$
\begin{aligned}
y = C.F. + P.I. = y_c + y_p &= c_1 \cos z + c_2 \sin z + z + \frac{z}{2} \sin z \\
&= c_1 \cos \log(1+x) + c_2 \sin \log(1+x) \\
&\quad + \log(1+x) + \frac{\log(1+x)}{2} \sin \log(1+x)
\end{aligned}
$$

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4. Method of Variation of Parameters

Finding the solution of the ODE in the form

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R \tag{4.1}$$

where P, Q and R are the functions of x or constants.
The homogeneous equation corresponding to (4.1) is

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = 0 \tag{4.2}$$

Method 1:
Let the general solution of (4.2) be

$$y = Ay_1 + By_2 \tag{4.3}$$

where A and B are constants and $y = y_1(x)$ and $y = y_2(x)$ are independent particular solutions of (4.2).
Now we consider A and B as functions of x and assume (4.3) to be the general solution of (4.1).
Differentiating (4.3) w.r.to x, we have

$$\frac{dy}{dx} = (Au' + Bv') + (A'u + B'v) \tag{4.4}$$

We select A and B in such a way that

$$(A'u + B'v) = 0 \tag{4.5}$$

$\therefore$ (4.4) becomes

$$\frac{dy}{dx} = (Au' + Bv') \tag{4.6}$$

Differentiating (??) w.r.to x, we have

$$\frac{d^2y}{dx^2} = (Au'' + Bv'') + (A'u' + B'v') \tag{4.7}$$

Since (4.3) is the solution of (4.1), the equations (4.3),(4.6) and (4.7) satisfies (4.1). *i.e.*

$$\begin{aligned} (Au'' + Bv'' + A'u' + B'v') + P(Au' + Bv') + Q(Au + Bv) &= R \\ A(u'' + Pu' + Qu) + B(v'' + Pv' + Qv) + (A'u' + B'v') &= R \end{aligned} \tag{4.8}$$

Since u and v are the solution of equation (4.2), we have

$$\begin{aligned} (u'' + Pu' + Qu) &= 0 \\ (v'' + Pv' + Qv) &= 0 \end{aligned}$$

Substituting these in (4.8), we get

$$A'u' + B'v' = R \tag{4.9}$$

Solving (4.5) and (4.9), we get the values of A' and B' and then integrating, we get the values of A and B in terms of x. Substituting A and B in (4.3), we get the required general solution of (4.1).

Method 2:

Let the general solution of (4.2) be

$$y_c = Ay_1(x) + By_2(x) \tag{4.10}$$

which is the C.F. of (4.1) with constants A and B.
The P.I. is given by

$$y_p = C(x)y_1(x) + D(x)y_2(x) \tag{4.11}$$

where

$$\begin{aligned} C(x) &= -\int \frac{R \cdot y_2}{W}\,dx \\ D(x) &= \int \frac{R \cdot y_1}{W}\,dx \end{aligned}$$

where W is called the Wronskian which is given by

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1y_2' - y_1'y_2$$

Example 4.1. *Solve:* $\dfrac{d^2y}{dx^2} + y = cosec\, x$

Hints/Solution:
Given equation is of the form $(D^2 + 1)y = \text{cosec}\, x$
The auxiliary equation is $m^2 + 1 = 0 \implies m = -\pm i.$
$\therefore$ C.F. $y_c = c_1 \cos x + c_2 \sin x$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1y_2' - y_1'y_2 = 1$$

The P.I. is given by

$$y_p = C(x)y_1(x) + D(x)y_2(x) \tag{4.12}$$

where

$$C(x) = -\int \frac{R \cdot y_2}{W}\, dx = -x$$
$$D(x) = \int \frac{R \cdot y_1}{W}\, dx = \log(\sin x)$$

$P.I. = -x\cos x + \log(\sin x)$
Hence the complete solution is given by
$y = C.F. + P.I. = y_c + y_p = c_1\cos x + c_2\sin x + -x\cos x + \log(\sin x).$

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5. Miscellaneous Solved Problems

Example 5.1. *Solve the differential equation* $(D^2 + 4)y = \sin 2x$

Hints/Solution:

Given equation is of the form $(D^3 + 4)y = \sin 2x$

The auxiliary equation is $m^2 + 4 = 0 \implies m = \pm 2i$.

$\therefore C.F. = c_1 \cos 2x + c_2 \sin 2x$.

$$\begin{aligned} (P.I.) &= \frac{1}{D^2+4}\sin 2x \\ &= \frac{x}{4}\cos 2x. \\ y = C.F. + P.I. &= c_1 \cos 2x + c_2 \sin 2x + \frac{x}{4}\cos 2x \end{aligned}$$

Example 5.2. *Solve:* $(5+2x)^2 \dfrac{d^2y}{dx^2} - 6(5+2x)\dfrac{dy}{dx} + 8y = 0$

Hints/Solution:

Let $2x + 5 = e^z$ and $\theta = \dfrac{d}{dz}$.

$\therefore \quad z = \log(2x+5)$ and $(2x+5)\dfrac{dy}{dx} = \dfrac{dy}{dz} = 2\theta y,$

$$(2x+5)^2 \frac{d^2y}{dx^2} = 2^2\theta(\theta-1)y.$$

Now, the ODE takes the form $[2^2\theta(\theta-1)+12\theta+8]y=(4\theta^2-16\theta+8)y=0$

The A.E. is $m^2-4m+2=0 \implies m=2\pm\sqrt{2}. \therefore$

$C.F. = c_1e^{(2+\sqrt{2})z}+c_2e^{(2-\sqrt{2})z}=c_1(2x+5)^{(2+\sqrt{2})}+c_2(2x+5)^{(2+\sqrt{2})}$

Example 5.3. *Solve the differential equation* $(D^2+4)y=4\tan 2x$

Hints/Solution:
Given equation is of the form $(D^+4)y=4\tan 2x$

The auxiliary equation is $m^2+4=0 \implies m=\pm 2i$.

$\therefore C.F. = c_1\cos 2x+c_2\sin 2x$

$$W=\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}=y_1y_2'-y_1'y_2=2$$

The P.I. is given by
$y_p=C(x)y_1(x)+D(x)y_2(x)$ where

$$C(x)=-\int\frac{F(x)\cdot y_2}{W}\,dx=-\int\frac{4\tan 2x\cdot\sin 2x}{2}\,dx=-\log(\sec 2x+\tan 2x)+\sin 2x$$

and

$$D(x)=\int\frac{F(x)\cdot y_1}{W}\,dx=\int\frac{4\tan 2x\cdot\cos 2x}{2}\,dx=-\cos 2x$$

$$\therefore \quad P.I. = -\cos 2x \log(\sec 2x + \tan 2x)$$

Hence the complete solution is given by
$y = C.F. + P.I. = y_c + y_p = c_1 \cos 2x + c_2 \sin 2x - \cos 2x \log(\sec 2x + \tan 2x)$.

Example 5.4. *Solve:* $\dfrac{dx}{dt} + 2y = \sin 2t; \ \dfrac{dy}{dt} - 2x = \cos 2t$

Hints/Solution:
Eliminating $x(t)$ from the given equations, we get

$$\left(\frac{d^2}{dt} + 4\right) y = 0$$
$$\implies y = c_1 \cos 2t + c_2 \sin 2t$$

and hence

$$\frac{dy}{dt} = -2c_1 \sin 2t + 2c_2 \cos 2t.$$

Substituting these in the given equations, we get

$$x(t) = -c_1 \sin 2t + c_2 \cos 2t - \frac{1}{2} \cos 2t$$

Example 5.5. *Solve the differential equations (a)* $(D^2 - 2D + 1)y = e^{2x}$ *(b)* $(D^2 - 5D + 6)y = x^2 + 3$

Hints/Solution:

(a) Given equation is of the form $(D^2 - 2D + 1)y = (D - 2)^2 y = e^{2x}$

The auxiliary equation is $(m - 2)^2 = 0 \implies m = 2, 2.$

$\therefore C.F. = (c_1 x + c_2)e^{2x}.$

$$(P.I.) = \frac{1}{(D-2)^2} e^{2x} = \frac{x^2}{2} e^{2x}.$$

$$y = C.F. + P.I. = (c_1 x + c_2)e^{2x} + \frac{x^2}{2} e^{2x}$$

(b) Given equation is of the form $(D^2 - 5D + 6)y = x^2 + 3$

The auxiliary equation is $m - 5m + 6 = 0 \implies m = 2, 3.$

$\therefore C.F. = c_1 e^{2x} + c_2 e^{3x}.$

$$(P.I.) = \frac{1}{D^2 - 5D + 6} x^2 + 3e^0$$

$$= \frac{1}{6}\left[x^2 + \frac{5}{3}x + \frac{19}{18} + 3\right]$$

$$= \frac{1}{108}\left[18x^2 + 30x + 73\right]$$

$$y = C.F. + P.I. = c_1 e^{2x} + c_2 e^{3x} + \frac{1}{108}\left[18x^2 + 30x + 73\right]$$

Example 5.6. *Solve the differential equations (a)* $(D^2 + 4)y = \sec 2x$ *(b)* $x^2 \dfrac{d^2y}{dx^2} + 4x\dfrac{dy}{dx} + 2y = x$.

Hints/Solution:

(a) Given equation is of the form $(D^2 + 4)y = \sec 2x$

The auxiliary equation is

$m^2 + 4 = 0 \implies m = \pm 2i.$

$\therefore \quad C.F. = c_1 \cos 2x + c_2 \sin 2x$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_1' y_2 = 2$$

The P.I. is assumed as $y_p = C(x)y_1(x) + D(x)y_2(x)$

where

$$\begin{aligned} C(x) &= -\int \frac{F(x)\cdot y_2}{W}\,dx = -\int \frac{\sec 2x \cdot \sin 2x}{2}\,dx \\ &= \frac{1}{4}\log(\cos 2x) \text{ and} \\ D(x) &= \int \frac{F(x)\cdot y_1}{W}\,dx = \int \frac{\sec 2x \cdot \cos 2x}{2}\,dx = \frac{1}{2}x \end{aligned}$$

$\therefore\ P.I. = \frac{1}{4}\log(\cos 2x)\cos 2x + \frac{1}{2}x\sin 2x$
Hence the complete solution is given by
$y = C.F. + P.I. = y_c + y_p = c_1\cos 2x + c_2\sin 2x + \frac{1}{4}\log(\cos 2x)\cos 2x + \frac{1}{2}x\sin 2x.$

(b) Given equation is of the form $[x^2D^2 + 4xD + 2]y = x$

Let $x = e^z$ and $\theta = \frac{d}{dz}$.

$\therefore\ z = \log x$ and $x\frac{dy}{dx} = \frac{dy}{dz} = \theta y,$

$\boxed{x^2\frac{d^2y}{dx^2} = \theta(\theta-1)y.}$

Now, the ODE takes the form $[\theta(\theta-1)+4\theta+2]y=(\theta^2+3\theta+2)y=e^z$

The A.E. is $m^2+3m+2=0 \implies m=-1,-2.$

$\therefore \boxed{C.F. = c_1e^{-z}+c_2e^{-2z}=c_1/x+c_2/x^2}$

$$(P.I. = \frac{1}{\theta^2+3\theta+2}e^z = \frac{x}{[6]}$$

$$y = C.F. + P.I. = c_1/x + c_2/x^2 + \frac{x}{6}$$

Example 5.7. *Solve the differential equations (a)* $(D^2-2D+1)y=e^x\sin x$ *(b)* $(x+2)^2\frac{d^2y}{dx^2}-(x+2)\frac{dy}{dx}+y=0.$

Hints/Solution:

(a) Given equation is of the form $(D^2-2D+1)y=(D-1)^2=e^x\sin x$

The auxiliary equation is $(m-1)^2=0 \implies m=1,1.$

$\therefore C.F. = (c_1+c_2x)e^x.$

$$(P.I.) = \frac{1}{(D-1)^2}e^x\sin x = -e^x\sin x$$

Hence the complete solution is given by

$$y = C.F. + P.I. = y_c + y_p = (c_1 + c_2 x)e^x - e^x \sin x.$$

(b) Given equation is of the form $[(2+x)^2 D^2 - (2+x)D + 1]y = 0$

Let $x + 2 = e^z$ and $\theta = \dfrac{d}{dz}$.

$\therefore \quad z = \log(x+2)$ and $(x+2)\dfrac{dy}{dx} = \dfrac{dy}{dz} = \theta y,$

$$(x+2)^2 \frac{d^2y}{dx^2} = \theta(\theta - 1)y.$$

Now, the ODE takes the form $[\theta(\theta-1) - \theta + 1]y = (\theta^2 - 2\theta + 1)y = 0.$

The A.E. is $m^2 - 2m + 1 = 0 \implies m = 1, 1.$

$\therefore C.F. = (c_1 x + c_2)e^z = (c_1 \log(x+2) + c_2)(x+2)$

Example 5.8. *Solve the differential equations (a)* $(D^2+1)y = cosec\, x$ *(b)* $\dfrac{dx}{dt} + y = e^t;\ \ x - \dfrac{dy}{dt} = t.$

Hints/Solution:

(a) Given equation is of the form $(D^2 + 1)y = \operatorname{cosec} x$

The auxiliary equation is $m^2 + 1 = 0 \implies m = \pm i.$

$\therefore \quad C.F. = c_1 \cos x + c_2 \sin x$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_1' y_2 = 1$$

The P.I. is given by $y_p = C(x)y_1(x) + D(x)y_2(x)$ where

$$\begin{aligned} C(x) &= -\int \frac{F(x) \cdot y_2}{W}\, dx \\ &= -\int \frac{cosec\, x \cdot \sin x}{1}\, dx \\ &= -x \end{aligned}$$

$$\begin{aligned} D(x) &= \int \frac{F(x) \cdot y_1}{W}\, dx \\ &= \int \frac{cosec\, x \cdot \cos x}{1}\, dx \\ &= \log(\sin x) \end{aligned}$$

$\therefore \quad P.I. = -x \cos x + \log(\sin x) \sin x$

Hence the complete solution is given by
$y = C.F. + P.I. = y_c + y_p = c_1 \cos x + c_2 \sin x - x \cos x + \log(\sin x) \sin x.$

(b) Eliminating $x(t)$ from the given equations, we get

$$\left(\frac{d^2}{dt}+1\right)y=e^t-1$$
$$\implies y=c_1\cos t+c_2\sin t+\frac{1}{2}e^t+1$$

and hence

$$\frac{dy}{dt}=-c_1\sin t+2c_2\cos t+\frac{1}{2}e^t.$$

Substituting these in the given equations, we get

$$x(t)=-c_1\sin t+c_2\cos t-\frac{1}{2}e^t+t$$

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6. Exercise/Practice/Assignment Problems

In all the following problems D represents the operator $\dfrac{d}{dx}$ and y' represents $\dfrac{dy}{dx}$.

1. Solve the following differential equations

   (a) $(D^2 - 4D + 3)y = \sin 3x + x^2$

   Ans: $y = c_1 e^x + c_2 e^{3x} + \dfrac{1}{30}(2\cos 3x - \sin 3x) + + \dfrac{1}{3}\left(x^2 + \dfrac{8}{3}x + \dfrac{26}{9}\right)$

   (b) $y'' - 6y' + 8y = e^{-2x} + 4$

   (c) $y'' + 4y = x^4 + \cos^2 x$

   Ans: $y = c_1 \cos 2x + c_2 \sin 2x + \dfrac{1}{8}\left(4 - 6x^2 + +2x^4 + x \sin 2x\right)$

   (d) $(D^2 + 4D + 3)y = e^{2x} + 5$

2. Solve the following differential equations

   (a) $(D^2 + 5D + 6)y = \cos(-3x) + 6$

   (b) $(D^2 - 6D + 8)y = \cos 5x + e^{4x} + \sin 4x$

   (c) $(D^2 - 6D + 9)y = e^{3x} + \sin 2x$

   (d) $(D^2 + 4D + 3)y = \cos^2 2x$

3. Solve the following differential equations

   (a) $(D^2 + 5D + 6)y = x^3 + 2x^2$

   (b) $(D^2 - 6D + 8)y = x^4$

(c) $(D^2 - 6D + 9)y = x^2 + 1$

(d) $(D^2 + 4D + 3)y = x^2 - 4x$

4. Solve the following differential equations

(a) $(D^2 + 5D + 6)y = \cos^3 3x + e^{-2x} \sin 2x$

(b) $(D^2 - 6D + 8)y = e^{-2x} \cos 4x$

(c) $(D^2 - 6D + 9)y = e^{3x} x^2$

(d) $(D^2 + 4D + 3)y = x e^{2x}$

5. Solve the following differential equations

(a) $(D^2 + 5D + 6)y = x e^{-3x} \cos 2x$

(b) $(D^2 - 6D + 8)y = x^2 \sin 3x$

(c) $(D^2 - 6D + 9)y = x^2 e^{2x} \sin 4x$

(d) $(D^2 + 4D + 3)y = x e^{2x} \sin 5x$

6. Solve the following differential equations

(a) $(x^2 D^2 - 3xD + 5)y = x^2 \sin(\log x)$

(b) $(x^2 D^2 + 4xD + 2)y = \sin(\log x)$

(c) $(x^2 D^2 - 3xD)y = x + 1$

(d) $(D^2 + \frac{1}{x} D + 5)y = \frac{12 \log x}{x^2}$

(e) $((1 + x)^2 D^2 + (1 + x)D + 1)y = 2 \sin(\log(x + 1))$

(f) $((3x+2)^2D^2+3(3x+2)D-36)y=3x^2+4x+1$

Ans: $y=c_1e^{2z}+c_2e^{-2z}+\frac{1}{108}\left(ze^{2z}+1\right)$ with $z=\log(3x+2)$

(g) $((2x+1)^2D^2-2(2x+1)D-12)y=6x$

Ans: $y=c_1e^{3z}+c_2e^{-z}-\frac{3}{16}\left(e^{z}\right)-\frac{9}{12}$ with $z=\log(2x+1)$

(h) $((2x+5)^2D^2-6(2x+5)D-8)y=6x$

Ans: $y=e^{2z}\left[c_1e^{\sqrt{2}z}+c_2e^{-\sqrt{2}z}\right]-\frac{3}{4}\left(e^{z}\right)-\frac{15}{8}$ with $z=\log(2x+5)$

(i) $((1+2x)^2D^2+3(1+2x)D+1)y=8(1+2x)^2$

7. Solve the following differential equations using the method of Variation of Parameters

(a) $(D^2+1)y=\tan x$

(b) $(D^2+a^2)y=\sec ax$

(c) $(D^2-1)y=e^x\sin x$

(d) $(D^2+9)y=3\sin 3t$

(e) $(D^2-2D+1)y=\frac{e^x}{x^2+1}$

(f) $(D^2+2D+1)y=e^{-x}\cos x$

8. Solve the following system of differential equations

(a) $\dfrac{dx}{dt} + y = e^t;\ \dfrac{dy}{dt} = t$

(b) $\dfrac{dx}{dt} + 2x + 3y = 2e^{2t};\ \dfrac{dy}{dt} + 3x + 2y = 0$

(c) $\dfrac{dx}{dt} + 2y = -\sin t;\ \dfrac{dy}{dt} - 2x = \cos t$

(d) $\dfrac{dx}{dt} - \left(\dfrac{dx}{dt} - 2\right) y = \cos 2t;\ \left(\dfrac{dx}{dt} - 2\right) x + \dfrac{dy}{dt} = \sin 2t$

(e) $\dfrac{dx}{dt} + \dfrac{dy}{dt} + y = 1;\ \dfrac{dx}{dt} - \dfrac{dz}{dt} + 2x + z = 1;\ \dfrac{dy}{dt} + \dfrac{dz}{dt} + y + 2z = 2$

(f) $2\dfrac{dx}{dt} + \dfrac{dy}{dt} - 4x - y = e^t;\ \dfrac{dx}{dt} + 3x + y = 0$

(g) $\dfrac{d^2x}{dt^2} - 5x + 3y = \sin t;\ \dfrac{d^2y}{dt^2} - 3x + 5y = t$

(h) $\dfrac{d^2x}{dt^2} - 3x - 4y = 0;\ \dfrac{d^2y}{dt^2} + x + y = 0$

(i) $D^2x - 2x - Dy = 2t;\ Dx + 4Dy - 3y = 0$