Module – 2

Functions of two variables – Partial derivatives – Total differential – Taylor's expansion with two variables upto second order terms – Taylor's expansion with two variables upto third order terms – Maxima and Minima – Constrained Maxima and Minima by Lagrangian Multiplier method – Jacobians of two variables – Jacobians of three variables – Properties of Jacobians and problems – Applications of Taylor's series, Maxima and Minima, Jacobians in Engineering.

FUNCTIONS OF TWO VARIABLES

PARTIAL DERIVATIVES

Let $z = f(x, y)$ be a function. Then

(i) First order partial derivatives : $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$

(ii) Second order partial derivatives : $\frac{\partial^2 z}{\partial x^2}, \frac{\partial^2 z}{\partial y^2}, \frac{\partial^2 z}{\partial x \partial y}$

(iii) Third order partial derivatives : $\frac{\partial^3 z}{\partial x^3}, \frac{\partial^3 z}{\partial y^3}, \frac{\partial^3 z}{\partial x^2 \partial y}, \frac{\partial^3 z}{\partial x \partial y^2}$

1. **If $u = (x-y)(y-z)(z-x)$, show that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$.**

Solution:

Given $u = (x-y)(y-z)(z-x)$, then

$$\frac{\partial u}{\partial x} = (y-z)\left[(x-y)(-1) + (z-x)(1)\right] = (y-z)(z-x) - (y-z)(x-y) --- (1)$$

$$\frac{\partial u}{\partial x} = (z-x)\left[(x-y)(1) + (y-z)(-1)\right] = (x-y)(z-x) - (y-z)(z-x) --- (2)$$

$$\frac{\partial u}{\partial z} = (x-y)\left[(y-z)(1) + (z-x)(-1)\right] = (x-y)(y-z) - (x-y)(z-x) --- (3)$$

Adding (1),(2) and (3) we get $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$

2. **If $u = x^y$, then find $(i)\ u_{xy}$ $(ii)\ u_{xyx}$.**

Solution

Given $u = x^y$ --------(1) then

(i) Differentiating (1) w.r.t 'y' , we get

$u_y = x^y \log x$

Again differentiating w.r.t 'x' we get

$$u_{xy} = yx^{y-1}\left[\log x\right] + x^{y-1} = x^{y-1}\left(1 + y\log x\right)$$

(ii) Differentiating (1) w.r.t ' x', we get

$u_x = yx^{y-1}$

Again differentiating w.r.t 'y' we get

$u_{yx} = yx^{y-1} \log x + x^{y-1}$

Again differentiating w.r.t 'x' we get

$$u_{xyx} = x^{y-1}\left(\frac{y}{x}\right) + (1+y\log x)(y-1)x^{y-2} \;= yx^{y-2} + (1+y\log_e x)(y-1)x^{y-2}$$

3. **If** $z = x^2\tan^{-1}\left(\frac{y}{x}\right) - y^2\tan^{-1}\left(\frac{x}{y}\right)$**, prove that** $z_{xy} = \frac{x^2-y^2}{x^2+y^2}$.

Solution

Given $z = x^2\tan^{-1}\left(\frac{y}{x}\right) - y^2\tan^{-1}\left(\frac{x}{y}\right)$

$$z_x = 2x\tan^{-1}\left(\frac{y}{x}\right) + x^2\frac{1}{1+\left(\frac{y^2}{x^2}\right)}\left(\frac{-y}{x^2}\right) - y^2\frac{1}{1+\left(\frac{x^2}{y^2}\right)}\left(\frac{1}{y}\right)$$

Differentiating w.r.t 'x' we get

$$= 2x\tan^{-1}\left(\frac{y}{x}\right) + \frac{-x^2y}{x^2+y^2} - \frac{y^3}{x^2+y^2}$$

$$= 2x\tan^{-1}\left(\frac{y}{x}\right) - y$$

Again differentiating w.r.t 'y' we get

$$z_{yx} = z_{xy} = 2x\frac{1}{1+\left(\frac{y^2}{x^2}\right)}\left(\frac{1}{x}\right) - 1 = \frac{2x^2}{x^2+y^2} - 1 = \frac{x^2-y^2}{x^2+y^2}$$

4. **If** $u = \log(x^3+y^3+z^3-3xyz)$**, then prove that**

(i) $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3}{x+y+z}$ (ii) $\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = \frac{-9}{(x+y+z)^2}$.

Solution

Given $u = \log(x^3+y^3+z^3-3xyz)$

Then $\frac{\partial u}{\partial x} = \frac{3(x^2-zy)}{x^3+y^3+z^3-3xyz}$; $\frac{\partial u}{\partial y} = \frac{3(y^2-zx)}{x^3+y^3+z^3-3xyz}$; $\frac{\partial u}{\partial z} = \frac{3(z^2-xy)}{x^3+y^3+z^3-3xyz}$

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3(x^2+y^2+z^2-yz-zx-xy)}{x^3+y^3+z^3-3xyz}$$

$$= \frac{3(x^2+y^2+z^2-yz-zx-xy)}{(x+y+z)(x^2+y^2+z^2-yz-zx-xy)} = \frac{3}{(x+y+z)}$$

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)u = \frac{3}{x+y+z} \quad \text{--------------(1)}$$

(ii) Operating $\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)$ on both sides of (1), we get

$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u=\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{3}{x+y+z}\right)$$

$$=\frac{\partial}{\partial x}\left(\frac{3}{x+y+z}\right)+\frac{\partial}{\partial y}\left(\frac{3}{x+y+z}\right)+\frac{\partial}{\partial z}\left(\frac{3}{x+y+z}\right)$$

$$=\frac{-3}{(x+y+z)^2}+\frac{-3}{(x+y+z)^2}+\frac{-3}{(x+y+z)^2}$$

$$=\frac{-9}{(x+y+z)^2}$$

5. **If x = rcos θ, y = rsin θ, prove that** (i) $\frac{\partial^2 r}{\partial x^2}+\frac{\partial^2 r}{\partial y^2}=\frac{1}{r}\left[\left(\frac{\partial r}{\partial x}\right)^2+\left(\frac{\partial r}{\partial y}\right)^2\right]$**.**

Solution:

$x = r\cos\theta,\ y = r\sin\theta.$

$\therefore\quad x^2 + y^2 = r^2$ and $\tan\theta = y/x$

Differentiating $r^2 = x^2 + y^2$ partially w.r.t x, we get

$$2r.\frac{\partial r}{\partial x}=2x \qquad \text{i.e., } \frac{\partial r}{\partial x}=\frac{x}{r} \tag{1}$$

Differentiating $r^2 = x^2 + y^2$ partially w.r.t y, we get

$$2r.\frac{\partial r}{\partial y}=2y \qquad \text{i.e., } \frac{\partial r}{\partial y}=\frac{y}{r} \tag{2}$$

$$\therefore\ \frac{1}{r}\left[\left(\frac{\partial r}{\partial x}\right)^2+\left(\frac{\partial r}{\partial y}\right)^2\right]=\frac{1}{r}\left[\frac{x^2}{r^2}+\frac{y^2}{r^2}\right]$$

$$=\frac{1}{r}.\frac{1}{r^2}(x^2+y^2)$$

$$=\frac{1}{r} \tag{3}$$

Differentiating (1) partially w.r.t x, we get

$$\frac{\partial^2 r}{\partial x^2}=x\left(\frac{-1}{r^2}\right)\frac{\partial r}{\partial x}+1.\frac{1}{r}$$

$$=\left(\frac{-x}{r^2}\right).\frac{x}{r}+\frac{1}{r}$$

Similarly from (2), we get,

$$\frac{\partial^2 r}{\partial y^2} = y\left(\frac{-1}{r^2}\right)\frac{\partial r}{\partial y} + 1.\frac{1}{r}$$

$$= \left(\frac{-y}{r^2}\right).\frac{y}{r} + \frac{1}{r}$$

$$\therefore \quad \frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2} = -\frac{1}{r^3}(x^2 + y^2) + \frac{2}{r}$$

$$= -\frac{1}{r} + \frac{2}{r} = \frac{1}{r} \qquad (4)$$

From (3) and (4), we get, $\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2} = \frac{1}{r}\left[\left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2\right]$

Total Differential

- If u = u (x, y), then $du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$ is called the **total differential** of u.
- If u = u (x, y) and y is a function of x, then $\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx}$.
- **Differentiation of Composite Functions**

  If u = u (x, y) and both x and y are functions of t, then $\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}$.
- **Chain Rule**

  If u = u (x, y) and both x and y are functions of v and w, then

  $\frac{\partial u}{\partial v} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial v}$ and $\frac{\partial u}{\partial w} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial w} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial w}$.

6. **Find $\frac{du}{dx}$ if $u = x^2y$ and $x^2 + xy + y^2 = 1$.**

We have $\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}.\frac{dy}{dx}$

$$= 2xy + x^2.\frac{dy}{dx} \qquad (1)$$

Let $f(x, y) = x^2 + xy + y^2 - 1$.

Then $\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y} = \frac{-(2x+y)}{2y+x}$

$\therefore$ From (1) $\frac{du}{dx} = 2xy + x^2.\frac{-(2x+y)}{2y+x}$

$$= \frac{4xy^2 + 2x^2y - 2x^3 - x^2y}{x+2y} = \frac{x(4y^2 + xy - 2x^2)}{x+2y}$$

Change of Variables

7. **If** $u = f(x-y, y-z, z-x)$, then show that $\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=0$ **.**

Solution :

Given $u = f(x-y, y-z, z-x) = f(r,s,t)$ $where\, r = x-y;\ s = y-z; t = z-x$

$$\frac{\partial r}{\partial x}=1, \frac{\partial r}{\partial y}=-1, \frac{\partial r}{\partial z}=0\ ; \frac{\partial s}{\partial x}=0, \frac{\partial s}{\partial y}=1, \frac{\partial s}{\partial z}=-1\ ;\ \frac{\partial t}{\partial x}=-1, \frac{\partial t}{\partial y}=0, \frac{\partial t}{\partial z}=1$$

$$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial r}\frac{\partial r}{\partial x}+\frac{\partial u}{\partial s}\frac{\partial s}{\partial x}+\frac{\partial u}{\partial t}\frac{\partial t}{\partial x}= \frac{\partial u}{\partial r}(1)+\frac{\partial u}{\partial s}(0)+\frac{\partial u}{\partial t}(-1)=\frac{\partial u}{\partial r}-\frac{\partial u}{\partial t}$$

$$\frac{\partial u}{\partial y}=\frac{\partial u}{\partial r}\frac{\partial r}{\partial y}+\frac{\partial u}{\partial s}\frac{\partial s}{\partial y}+\frac{\partial u}{\partial t}\frac{\partial t}{\partial y}= \frac{\partial u}{\partial r}(-1)+\frac{\partial u}{\partial s}(1)+\frac{\partial u}{\partial t}(0)=-\frac{\partial u}{\partial r}+\frac{\partial u}{\partial s}$$

$$\frac{\partial u}{\partial z}=\frac{\partial u}{\partial r}\frac{\partial r}{\partial z}+\frac{\partial u}{\partial s}\frac{\partial s}{\partial z}+\frac{\partial u}{\partial t}\frac{\partial t}{\partial z}= \frac{\partial u}{\partial r}(0)+\frac{\partial u}{\partial s}(-1)+\frac{\partial u}{\partial t}(1)=-\frac{\partial u}{\partial s}+\frac{\partial u}{\partial t}$$

Now $\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=\frac{\partial u}{\partial r}-\frac{\partial u}{\partial t}-\frac{\partial u}{\partial r}+\frac{\partial u}{\partial s}-\frac{\partial u}{\partial s}+\frac{\partial u}{\partial t}=0$

8. **If** $u = f\left(\frac{x}{y},\frac{y}{z},\frac{z}{x}\right)$, then show that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=0$ **.**

Solution : Given $u = f\left(\frac{x}{y},\frac{y}{z},\frac{z}{x}\right)= f(r,s,t)$ $where\, r=\frac{x}{y};\ s=\frac{y}{z}; t=\frac{z}{x}$

$$\frac{\partial r}{\partial x}=\frac{1}{y}, \frac{\partial r}{\partial y}=-\frac{x}{y^2}, \frac{\partial r}{\partial z}=0\ ; \frac{\partial s}{\partial x}=0, \frac{\partial s}{\partial y}=\frac{1}{z}, \frac{\partial s}{\partial z}=-\frac{y}{z^2}\ ;\ \frac{\partial t}{\partial x}=-\frac{z}{x^2}, \frac{\partial t}{\partial y}=0, \frac{\partial t}{\partial z}=\frac{1}{x}$$

$$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial r}\frac{\partial r}{\partial x}+\frac{\partial u}{\partial s}\frac{\partial s}{\partial x}+\frac{\partial u}{\partial t}\frac{\partial t}{\partial x}= \frac{\partial u}{\partial r}\left(\frac{1}{y}\right)+\frac{\partial u}{\partial s}(0)+\frac{\partial u}{\partial t}\left(-\frac{z}{x^2}\right)=\frac{\partial u}{\partial r}\left(\frac{1}{y}\right)-\frac{\partial u}{\partial t}\left(-\frac{z}{x^2}\right)$$

$$\frac{\partial u}{\partial y}=\frac{\partial u}{\partial r}\frac{\partial r}{\partial y}+\frac{\partial u}{\partial s}\frac{\partial s}{\partial y}+\frac{\partial u}{\partial t}\frac{\partial t}{\partial y}= \frac{\partial u}{\partial r}\left(-\frac{x}{y^2}\right)+\frac{\partial u}{\partial s}\left(\frac{1}{z}\right)+\frac{\partial u}{\partial t}(0)=\frac{\partial u}{\partial r}\left(-\frac{x}{y^2}\right)+\frac{\partial u}{\partial s}\left(\frac{1}{z}\right)$$

$$\frac{\partial u}{\partial z}=\frac{\partial u}{\partial r}\frac{\partial r}{\partial z}+\frac{\partial u}{\partial s}\frac{\partial s}{\partial z}+\frac{\partial u}{\partial t}\frac{\partial t}{\partial z}= \frac{\partial u}{\partial r}(0)+\frac{\partial u}{\partial s}\left(-\frac{y}{z^2}\right)+\frac{\partial u}{\partial t}\left(\frac{1}{z}\right)=\frac{\partial u}{\partial s}\left(-\frac{y}{z^2}\right)+\frac{\partial u}{\partial t}\left(\frac{1}{x}\right)$$

Now $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=\frac{\partial u}{\partial r}\left(\frac{x}{y}\right)+\frac{\partial u}{\partial t}\left(-\frac{z}{x}\right)+\frac{\partial u}{\partial r}\left(-\frac{x}{y}\right)+\frac{\partial u}{\partial s}\left(\frac{y}{z}\right)+\frac{\partial u}{\partial s}\left(-\frac{y}{z}\right)+\frac{\partial u}{\partial t}\left(\frac{z}{x}\right)=0$

9. **If** $u = f\left(\frac{x-y}{xy},\frac{y-z}{yz},\frac{z-x}{xz}\right)$, then show that $x^2\frac{\partial u}{\partial x}+y^2\frac{\partial u}{\partial y}+z^2\frac{\partial u}{\partial z}=0$**.**

Solution : Given $u = f\left(\frac{x-y}{xy},\frac{y-z}{yz},\frac{z-x}{xz}\right)= f(r,s,t)$ $where\, r=\frac{x-y}{xy};\ s=\frac{y-z}{yz}; t=\frac{z-x}{xz}$

$$\frac{\partial r}{\partial x}=\frac{1}{x^2},\frac{\partial r}{\partial y}=-\frac{1}{y^2},\frac{\partial r}{\partial z}=0\ ;\frac{\partial s}{\partial x}=0,\frac{\partial s}{\partial y}=\frac{1}{y^2},\frac{\partial s}{\partial z}=-\frac{1}{z^2}\ ;\ \frac{\partial t}{\partial x}=-\frac{1}{x^2},\frac{\partial t}{\partial y}=0,\frac{\partial t}{\partial z}=\frac{1}{z^z}$$

$$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial r}\frac{\partial r}{\partial x}+\frac{\partial u}{\partial s}\frac{\partial s}{\partial x}+\frac{\partial u}{\partial t}\frac{\partial t}{\partial x}=\ \frac{\partial u}{\partial r}\left(\frac{1}{x^2}\right)+\frac{\partial u}{\partial s}(0)+\frac{\partial u}{\partial t}\left(-\frac{1}{x^2}\right)=\frac{\partial u}{\partial r}\left(\frac{1}{x^2}\right)+\frac{\partial u}{\partial t}\left(-\frac{1}{x^2}\right)$$

$$\frac{\partial u}{\partial y}=\frac{\partial u}{\partial r}\frac{\partial r}{\partial y}+\frac{\partial u}{\partial s}\frac{\partial s}{\partial y}+\frac{\partial u}{\partial t}\frac{\partial t}{\partial y}=\ \frac{\partial u}{\partial r}\left(-\frac{1}{y^2}\right)+\frac{\partial u}{\partial s}\left(\frac{1}{y^2}\right)+\frac{\partial u}{\partial t}(0)=\frac{\partial u}{\partial r}\left(-\frac{1}{y^2}\right)+\frac{\partial u}{\partial s}\left(\frac{1}{y^2}\right)$$

$$\frac{\partial u}{\partial z}=\frac{\partial u}{\partial r}\frac{\partial r}{\partial z}+\frac{\partial u}{\partial s}\frac{\partial s}{\partial z}+\frac{\partial u}{\partial t}\frac{\partial t}{\partial z}=\ \frac{\partial u}{\partial r}(0)+\frac{\partial u}{\partial s}\left(-\frac{1}{z^2}\right)+\frac{\partial u}{\partial t}\left(\frac{1}{z^2}\right)=\frac{\partial u}{\partial s}\left(-\frac{1}{z^2}\right)+\frac{\partial u}{\partial t}\left(\frac{1}{z^2}\right)$$

Now $x^2\frac{\partial u}{\partial x}+y^2\frac{\partial u}{\partial y}+z^2\frac{\partial u}{\partial z}=\frac{\partial u}{\partial r}-\frac{\partial u}{\partial t}-\frac{\partial u}{\partial r}+\frac{\partial u}{\partial s}-\frac{\partial u}{\partial s}+\frac{\partial u}{\partial t}=0$

10. **If** $u=f(x,y)$ *and* $x=rcos\theta,\ y=rsin\theta,$ **prove that** $\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2=\left(\frac{\partial u}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2$

Solution:

$x=r\cos\theta\ \Rightarrow \frac{\partial x}{\partial r}=\cos\theta\,;\ \frac{\partial x}{\partial \theta}=-r\sin\theta$

$y=r\sin\theta \Rightarrow \frac{\partial y}{\partial r}=\sin\theta \qquad \frac{\partial y}{\partial \theta}=r\cos\theta$

We have $\frac{\partial u}{\partial r}=\frac{\partial u}{\partial x}.\frac{\partial x}{\partial r}+\frac{\partial u}{\partial y}.\frac{\partial y}{\partial r}=\frac{\partial u}{\partial x}.\cos\theta+\frac{\partial u}{\partial y}.\sin\theta$ ------(1)

Also we have $\frac{\partial u}{\partial \theta}=\frac{\partial u}{\partial x}.\frac{\partial x}{\partial \theta}+\frac{\partial u}{\partial y}.\frac{\partial y}{\partial \theta}=\frac{\partial u}{\partial x}.(-r\sin\theta)+\frac{\partial u}{\partial y}.r\cos\theta$

$\therefore \quad \frac{1}{r}\frac{\partial u}{\partial \theta}=-\frac{\partial u}{\partial x}.\sin\theta+\frac{\partial u}{\partial y}.\cos\theta$ --------------(2)

Squaring and adding (1) and (2), we get,

$$\left(\frac{\partial u}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2=\left(\frac{\partial u}{\partial x}\right)^2\left(\cos^2\theta+\sin^2\theta\right)+\left(\frac{\partial u}{\partial y}\right)^2\left(\sin^2\theta+\cos^2\theta\right)$$

$$=\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2$$

$$\therefore \quad \left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2=\left(\frac{\partial u}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2$$

11. **If** $z = f(x, y)$ *and* $x = u^2 - v^2,\ y = 2uv,$ **prove that** $4(u^2+v^2)\left(z_{xx}+z_{yy}\right) = (z_{uu}+z_{vv})$.

Solution:

$$x = u^2 - v^2 \Rightarrow \frac{\partial x}{\partial u} = 2u \ , \frac{\partial x}{\partial v} = -2v \text{ and } y = 2uv \Rightarrow \frac{\partial y}{\partial u} = 2v \ , \frac{\partial y}{\partial v} = 2u$$

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} = \frac{\partial z}{\partial x}(2u) + \frac{\partial z}{\partial y}(2v)$$

$$\frac{\partial^2 z}{\partial u^2} = \frac{\partial z}{\partial x}(2) + (2u)\left(\frac{\partial^2 z}{\partial x^2}.\frac{\partial x}{\partial u}\right) + \frac{\partial z}{\partial y}(0) + (2v)\left(\frac{\partial^2 z}{\partial y^2}.\frac{\partial y}{\partial u}\right) \ (uv \text{ formula})$$

$$\frac{\partial^2 z}{\partial u^2} = \frac{\partial z}{\partial x}(2) + (2u)\left(\frac{\partial^2 z}{\partial x^2}.2u\right) + \frac{\partial z}{\partial y}(0) + (2v)\left(\frac{\partial^2 z}{\partial y^2}.2v\right)$$

$$\frac{\partial^2 z}{\partial u^2} = \frac{\partial z}{\partial x}(2) + 4u^2\frac{\partial^2 z}{\partial x^2} + 4v^2\frac{\partial^2 z}{\partial y^2} \qquad -----------------------(1)$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v} = \frac{\partial z}{\partial x}(-2v) + \frac{\partial z}{\partial y}(2u)$$

$$\frac{\partial^2 z}{\partial v^2} = \frac{\partial z}{\partial x}(-2) + (-2v)\left(\frac{\partial^2 z}{\partial x^2}.\frac{\partial x}{\partial v}\right) + \frac{\partial z}{\partial y}(0) + (2u)\left(\frac{\partial^2 z}{\partial y^2}.\frac{\partial y}{\partial v}\right) \ (uv \text{ formula})$$

$$\frac{\partial^2 z}{\partial v^2} = \frac{\partial z}{\partial x}(-2) + (-2v)\left(\frac{\partial^2 z}{\partial x^2}.(-2v)\right) + \frac{\partial z}{\partial y}(0) + (2u)\left(\frac{\partial^2 z}{\partial y^2}.(2u)\right)$$

$$\frac{\partial^2 z}{\partial v^2} = \frac{\partial z}{\partial x}(-2) + 4v^2\frac{\partial^2 z}{\partial x^2} + 4u^2\frac{\partial^2 z}{\partial y^2} \qquad ---------------------(2)$$

Adding (1) and (2)

$$\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2} = 4u^2\left(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}\right) + 4v^2\left(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}\right)$$

$$z_{uu} + z_{vv} = 4(u^2+v^2)\left(z_{xx}+z_{yy}\right)$$

12. **If** $z = f(x, y)$ *and* $x = e^u \sin v,\ y = e^u \cos v,$

prove that $z_{xx} + z_{yy} = (x^2+y^2)(z_{uu}+z_{vv})$

Solution:

$$x = e^u \sin v \Rightarrow \frac{\partial x}{\partial u} = e^u \sin v \ , \frac{\partial x}{\partial v} = e^u \cos v \text{ and } y = e^u \cos v \Rightarrow \frac{\partial y}{\partial u} = e^u \cos v \ , \frac{\partial y}{\partial v} = -e^u \sin v$$

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} = \frac{\partial z}{\partial x}e^u \sin v + \frac{\partial z}{\partial y}e^u \cos v$$

$$\frac{\partial^2 z}{\partial u^2} = (e^u \sin v)\frac{\partial^2 z}{\partial x^2}\frac{\partial x}{\partial u} + \frac{\partial^2 z}{\partial y \partial x}(e^u \sin v)\frac{\partial y}{\partial u} + \frac{\partial z}{\partial x}e^u \sin v$$

$$+ \frac{\partial^2 z}{\partial x \partial y}(e^u \cos v)\frac{\partial x}{\partial u} + (e^u \cos v)\frac{\partial^2 z}{\partial y^2}\frac{\partial y}{\partial v} + \frac{\partial z}{\partial y}e^u \cos v$$

$$\frac{\partial^2 z}{\partial u^2} = e^{2u}\sin^2 u\frac{\partial^2 z}{\partial x^2} + 2\frac{\partial^2 z}{\partial y \partial x}(e^{2u}\sin v \cos v) + e^u\left(\sin v\frac{\partial z}{\partial x} + \cos v\frac{\partial z}{\partial y}\right) + (e^{2u}\cos^2 v)\frac{\partial^2 z}{\partial y^2} \quad ..(1)$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v} = \frac{\partial z}{\partial x}e^u \cos v - \frac{\partial z}{\partial y}e^u \sin v$$

$$\frac{\partial^2 z}{\partial v^2} = (e^u \cos v)\frac{\partial^2 z}{\partial x^2}\frac{\partial x}{\partial v} + \frac{\partial^2 z}{\partial y \partial x}(e^u \cos v)\frac{\partial y}{\partial v} + \frac{\partial z}{\partial x}(-e^u \sin v)$$

$$+ \frac{\partial^2 z}{\partial x \partial y}(-e^u \sin v)\frac{\partial x}{\partial v} + (-e^u \sin v)\frac{\partial^2 z}{\partial y^2}\frac{\partial y}{\partial v} + \frac{\partial z}{\partial y}(-e^u \cos v)$$

$$\frac{\partial^2 z}{\partial v^2} = e^{2u}\cos^2 u\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial^2 z}{\partial y \partial x}(e^{2u}\sin v \cos v) + e^u\left(-\sin v\frac{\partial z}{\partial x} - \cos v\frac{\partial z}{\partial y}\right) + (e^{2u}\sin^2 v)\frac{\partial^2 z}{\partial y^2} \quad ..(2)$$

Adding (1) and (2)

$$\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2} = \left(\frac{\partial^2 z}{\partial x^2}e^{2u} + \frac{\partial^2 z}{\partial y^2}e^{2u}\right) = e^{2u}(z_{xx} + z_{yy})$$

$$(z_{uu} + z_{vv}) = e^{2u}(z_{xx} + z_{yy})$$

$$(z_{uu} + z_{vv}) = (x^2 + y^2)(z_{xx} + z_{yy})$$

Homogeneous Function

A function f (x, y) is said to be homogeneous of degree n, if f (tx, ty) = t^n f (x, y).

Euler's Theorem

If f (x, y) is a homogenous function of degree n in x and y, then $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf$.

If f (x, y) is a homogenous function of degree n in x and y, then $x^2\frac{\partial^2 f}{\partial x^2} + 2xy\frac{\partial^2 f}{\partial x \partial y} + y^2\frac{\partial^2 f}{\partial y^2} = n\,(n-1)\,f$.

13. **Show that** $x.\frac{\partial u}{\partial x} + y.\frac{\partial u}{\partial y} + z.\frac{\partial u}{\partial z} = 2\tan u$ where $u = \sin^{-1}\left[\frac{x^3 + y^3 + z^3}{ax + by + cz}\right]$.

Solution: We have, $\sin u = \left[\frac{x^3 + y^3 + z^3}{ax + by + cz}\right]$

Let $f(x, y, z) = \frac{x^3 + y^3 + z^3}{ax + by + cz}$ (1)

$$f(tx, ty, tz) = \frac{t^3x^3 + t^3y^3 + t^3z^3}{atx + bty + ctz} = t^2 f(x, y, z)$$

$\therefore$ f(x, y, z) is a homogeneous function of degree 2.

$\therefore$ By Euler's theorem,

$$x.\frac{\partial f}{\partial x} + y.\frac{\partial f}{\partial y} + z.\frac{\partial f}{\partial z} = 2.f \qquad (2)$$

From (1), we have, f = sin u

$$\therefore \quad \frac{\partial f}{\partial x} = \cos u.\frac{\partial u}{\partial x} \qquad \frac{\partial f}{\partial y} = \cos u.\frac{\partial u}{\partial y} \quad \text{and} \quad \frac{\partial f}{\partial z} = \cos u.\frac{\partial u}{\partial z}$$

Substituting these in (2), we get,

$$x.\cos u.\frac{\partial u}{\partial x} + y.\cos u.\frac{\partial u}{\partial y} + z.\cos u.\frac{\partial u}{\partial z} = 2.\sin u$$

$$x.\frac{\partial u}{\partial x} + y.\frac{\partial u}{\partial y} + z.\frac{\partial u}{\partial z} = 2.\tan u$$

14. **If** $u = \tan^{-1}\left(\frac{x^3 + y^3}{x - y}\right)$, **prove that** $x^2u_{xx} + 2xyu_{xy} + y^2u_{yy} = 2\sin u \cos 3u$.

Solution:

Given $u = \tan^{-1}\left(\frac{x^3 + y^3}{x - y}\right)$

$\tan u = f(x, y) = \frac{x^3 + y^3}{x - y}$, a homogenous function of degree 2.

Therefore, by Euler's theorem $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf = 2f$

$$\Rightarrow \quad x\frac{\partial(\tan u)}{\partial x} + y\frac{\partial(\tan u)}{\partial y} = 2\tan u$$

$$\Rightarrow \quad \sec^2 u\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right) = 2\tan u$$

$$\Rightarrow \quad \left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right) = 2\frac{\sin u}{\cos u}\cos^2 u = \sin 2u \text{------------------}(1)$$

Differentiating (1) partially with respect to x and multiply with x, we get,

$$\Rightarrow \quad x\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} + y\frac{\partial^2 u}{\partial x \partial y} = 2\cos 2u\frac{\partial u}{\partial x}$$

$$\Rightarrow \quad x^2\frac{\partial^2 u}{\partial x^2} + x\frac{\partial u}{\partial x} + xy\frac{\partial^2 u}{\partial x \partial y} = 2x\cos 2u\frac{\partial u}{\partial x}$$

$$\Rightarrow \quad x^2\frac{\partial^2 u}{\partial x^2} + xy\frac{\partial^2 u}{\partial x \partial y} = (2\cos 2u - 1)x\frac{\partial u}{\partial x} \text{---------------------}(2)$$

Differentiating (1) partially with respect to y and multiply with y, we get,

$$\Rightarrow \quad y\frac{\partial^2 u}{\partial y^2}+\frac{\partial u}{\partial y}+x\frac{\partial^2 u}{\partial x\partial y}=2\cos 2u\frac{\partial u}{\partial y}$$

$$\Rightarrow \quad y^2\frac{\partial^2 u}{\partial y^2}+y\frac{\partial u}{\partial y}+xy\frac{\partial^2 u}{\partial x\partial y}=2y\cos 2u\frac{\partial u}{\partial y}$$

$$\Rightarrow \quad y^2\frac{\partial^2 u}{\partial y^2}+xy\frac{\partial^2 u}{\partial x\partial y}=\left(2\cos 2u-1\right)y\frac{\partial u}{\partial y}--------------(3)$$

Adding (2) and (3), we get

$$x^2u_{xx}+2xyu_{xy}+y^2u_{yy}=\left(2\cos 2u-1\right)\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)$$

$$=\left(2\cos 2u-1\right)\ \sin 2u\ =2\sin u\left[4\cos^3 u-3\cos u\right]\ \ =2\sin u\ \cos 3u$$

Formulas : $\cos 2\theta = 2\cos^2\theta-1$ and $\cos 3\theta=4\cos^3\theta-3\cos\theta$

TAYLOR'S SERIES

TAYLOR'S SERIES FORMULA

$$f(x,y)=f(a,b)+(x-a)f_x(a,b)+(y-b)f_y(a,b)$$
$$+\frac{1}{2!}\left[(x-a)^2 f_{xx}(a,b)+2(x-a)(y-b)f_{xy}(a,b)+(y-b)^2 f_{yy}(a,b)\right]$$
$$+\frac{1}{3!}\left[(x-a)^3 f_{xxx}(a,b)+3(x-a)^2(y-b)f_{xxy}(a,b)+3(x-a)(y-b)^2 f_{xyy}(a,b)+(y-b)^3 f_{yyy}(a,b)\right]+\ldots$$

When a = 0 and b = 0, the above series is called **Maclaurin's series.**

$$f(x,y)=f(0,0)+x\ f_x(0,0)+y\ f_y(0,0)$$
$$+\frac{1}{2!}\left[x^2\ f_{xx}(0,0)+2\,x\,y\ \ f_{xy}(0,0)+y^2\ f_{yy}(0,0)\right]+$$
$$+\frac{1}{3!}\left[x^3\ f_{xxx}(0,0)+3\,x^2\,y\ \ f_{xxy}(0,0)+3\,x\,y^2\ f_{xyy}(0,0)+y^3\ f_{yyy}(0,0)\right]+\ldots..$$

15. **Expand** $e^x\sin y$ **as Maclaurin's series**.

Solution:

Given $f(x,y)=e^x\sin y$ and here a = b = 0. We use Maclaurin's series formula

$$f(x,y)=f(0,0)+x\ f_x(0,0)+y\ f_y(0,0)$$
$$+\frac{1}{2!}\left[x^2\ f_{xx}(0,0)+2\,x\,y\ \ f_{xy}(0,0)+y^2\ f_{yy}(0,0)\right]+$$
$$+\frac{1}{3!}\left[x^3\ f_{xxx}(0,0)+3\,x^2\,y\ \ f_{xxy}(0,0)+3\,x\,y^2\ f_{xyy}(0,0)+y^3\ f_{yyy}(0,0)\right]+\ldots..$$

$f(x,y)=e^x \sin y$ $\quad$ $f(0,0)=e^0 \sin 0=0$

$f_x(x,y)=e^x \sin y$ $\quad$ $f_x(0,0)=e^0 \sin 0=0$

$f_{xx}(x,y)=e^x \sin y$ $\quad$ $f_{xx}(0,0)=e^0 \sin 0=0$

$f_{xxx}(x,y)=e^x \sin y$ $\quad$ $f_{xxx}(0,0)=e^0 \sin 0=0$

$f_y(x,y)=e^x \cos y$ $\quad$ $f_y(0,0)=e^0 \cos 0=1$

$f_{yy}(x,y)=-e^x \sin y$ $\quad$ $f_{yy}(0,0)=-e^0 \sin 0=0$

$f_{yyy}(x,y)=-e^x \cos y$ $\quad$ $f_{yyy}(0,0)=-e^0 \cos 0=-1$

$f_{xy}(x,y)=e^x \cos y$ $\quad$ $f_{xy}(0,0)=e^0 \cos 0=1$

$f_{xxy}(x,y)=e^x \cos y$ $\quad$ $f_{xxy}(0,0)=e^0 \cos 0=1$

$f_{xyy}(x,y)=-e^x \sin y$ $\quad$ $f_{xyy}(0,0)=-e^0 \sin 0=0$

$$e^x \sin y = 0 + x(0) + y(1) + \frac{1}{2!}\left[x^2(0) + 2xy(1) + y^2(0)\right] + \frac{1}{3!}\left[x^3(0) + 3x^2y(1) + 3xy^2(0) + y^3(-1)\right] + \ldots.$$

$$= y + xy + \frac{x^2 y}{2} - \frac{y^3}{6} + \ldots$$

16. **Expand e^{xy} in powers of x and y up to third degree.**

Solution:

Given $f(x,y)=e^{xy}$ and here a = b = 0. We use Maclaurin's series formula.

$$f(x,y) = f(0,0) + x\, f_x(0,0) + y\, f_y(0,0)$$
$$+ \frac{1}{2!}\left[x^2\, f_{xx}(0,0) + 2xy\, f_{xy}(0,0) + y^2\, f_{yy}(0,0)\right] +$$
$$+ \frac{1}{3!}\left[x^3\, f_{xxx}(0,0) + 3x^2y\, f_{xxy}(0,0) + 3xy^2\, f_{xyy}(0,0) + y^3\, f_{yyy}(0,0)\right] + \ldots..$$

$f(x,y)=e^{xy}$ $\quad$ $f(0,0)=e^0=1$

$f_x(x,y)=ye^{xy}$ $\quad$ $f_x(0,0)=0$

$f_{xx}(x,y)=y^2e^{xy}$ $\quad$ $f_{xx}(0,0)=0$

$f_{xxx}(x,y)=y^3e^{xy}$ $\quad$ $f_{xxx}(0,0)=0$

$f_y(x,y)=xe^{xy}$ $\quad$ $f_y(0,0)=0$

$f_{yy}(x,y)=x^2e^{xy}$ $\quad$ $f_{yy}(0,0)=0$

$f_{yyy}(x,y)=x^3e^{xy}$ $\quad$ $f_{yyy}(0,0)=0$

$f_{xy}(x,y)=e^{xy}+x^2e^{xy}$ $\qquad$ $f_{xy}(0,0)=1+0=1$

$f_{xxy}(x,y)=e^{xy}y+2xe^{xy}+x^2ye^{xy}$ $\qquad$ $f_{xxy}(0,0)=0$

$f_{xyy}(x,y)=e^{xy}2x+x^2e^{xy}y$ $\qquad$ $f_{xyy}(0,0)=0$

$$e^{xy}=1+x(0)+y(0)+\frac{1}{2!}\left[x^2(0)+2xy(1)+y^2(0)\right]$$

$$+\frac{1}{3!}\left[x^3(0)+3x^2y(0)+3xy^2(0)+y^3(0)\right]+...$$

$= 1+ x\,y +\ldots.$

17. **Expand $e^x\log(1+y)$ in powers of x and y up to third degree.**

Given $f(x,y)=e^x\log(1+y)$ and here a = b = 0. We use Maclaurin's series formula.

$$f(x,y)=f(0,0)+x\,f_x(0,0)+y\,f_y(0,0)$$

$$+\frac{1}{2!}\left[x^2\,f_{xx}(0,0)+2\,x\,y\,f_{xy}(0,0)+y^2\,f_{yy}(0,0)\right]+$$

$$+\frac{1}{3!}\left[x^3\,f_{xxx}(0,0)+3x^2y\,f_{xxy}(0,0)+3xy^2\,f_{xyy}(0,0)+y^3\,f_{yyy}(0,0)\right]+.....$$

$f_x(x,y)=e^x\log(1+y)$ $\qquad$ $f_x(0,0)=e^0(\log 1)=0$

$f_{xx}(x,y)=e^x\log(1+y)$ $\qquad$ $f_{xx}(0,0)=e^0(\log 1)=0$

$f_{xxx}(x,y)=e^x\log(1+y)$ $\qquad$ $f_{xxx}(0,0)=e^0(\log 1)=0$

$f_y(x,y)=\dfrac{e^x}{1+y}$ $\qquad$ $f_y(0,0)=\dfrac{e^0}{1+0}=1$

$f_{yy}(x,y)=-\dfrac{e^x}{(1+y)^2}$ $\qquad$ $f_{yy}(0,0)=-\dfrac{e^0}{(1+0)^2}=-1$

$f_{yyy}(x,y)=-\dfrac{2e^x}{(1+y)^3}$ $\qquad$ $f_{yyy}(0,0)=-\dfrac{2e^0}{(1+0)^3}=-2$

$f_{xy}(x,y)=\dfrac{e^x}{(1+y)}$ $\qquad$ $f_{xy}(0,0)=\dfrac{e^0}{1+0}=1$

$f_{xxy}(x,y)=\dfrac{e^x}{(1+y)}$ $\qquad$ $f_{xxy}(0,0)=\dfrac{e^0}{1+0}=1$

$f_{xyy}(x,y)=-\dfrac{e^x}{(1+y)^2}$ $\qquad$ $f_{xyy}(0,0)=-\dfrac{e^0}{(1+0)^2}=-1$

$$f(x,y) = f(0,0) + x\, f_x(0,0) + y\, f_y(0,0)$$
$$+\frac{1}{2!}\left[x^2\, f_{xx}(0,0) + 2\,x\,y\,\, f_{xy}(0,0) + y^2\, f_{yy}(0,0)\right] +$$
$$+\frac{1}{3!}\left[x^3\, f_{xxx}(0,0) + 3\,x^2\, y\,\, f_{xxy}(0,0) + 3\,x\,y^2\, f_{xyy}(0,0) + y^3\, f_{yyy}(0,0)\right] + ...$$
$$= y + x\,y - \frac{1}{2}y^2 + \frac{1}{2}x^2\, y - \frac{1}{2}x\,y^2 + \frac{1}{3}y^3 + ...$$

18. **Expand** $e^x \cos y$ **in powers of** $(x-1)\, and \left(y - \frac{\pi}{4}\right)$ **up to third degree.**

Given $f(x,y) = e^x \cos y$ and here $a = 1, b = \frac{\pi}{4}$.

$$f(x,y) = f(a,b) + (x-a)\, f_x(a,b) + (y-b)\, f_y(a,b)$$
$$+\frac{1}{2!}\left[(x-a)^2\, f_{xx}(a,b) + 2(x-a)(y-b)\, f_{xy}(a,b) + (y-b)^2\, f_{yy}(a,b)\right]$$
$$+\frac{1}{3!}\begin{bmatrix}(x-a)^3\, f_{xxx}(a,b) + 3(x-a)^2(y-b)\, f_{xxy}(a,b) \\ +3(x-a)(y-b)^2\, f_{xyy}(a,b) + (y-b)^3\, f_{yyy}(a,b)\end{bmatrix} + \ldots$$

$f_x(x,y) = e^x \cos y$ $\qquad f_x\left(1, \frac{\pi}{4}\right) = e^1 \cos\left(\frac{\pi}{4}\right) = \frac{e}{\sqrt{2}}$

$f_{xx}(x,y) = e^x \cos y$ $\qquad f_{xx}\left(1, \frac{\pi}{4}\right) = e^1 \cos\left(\frac{\pi}{4}\right) = \frac{e}{\sqrt{2}}$

$f_{xxx}(x,y) = e^x \cos y$ $\qquad f_{xxx}\left(1, \frac{\pi}{4}\right) = e^1 \cos\left(\frac{\pi}{4}\right) = \frac{e}{\sqrt{2}}$

$f_y(x,y) = -e^x \sin y$ $\qquad f_y\left(1, \frac{\pi}{4}\right) = -e^1 \sin\left(\frac{\pi}{4}\right) = -\frac{e}{\sqrt{2}}$

$f_{yy}(x,y) = -e^x \cos y$ $\qquad f_{yy}\left(1, \frac{\pi}{4}\right) = -e^1 \cos\left(\frac{\pi}{4}\right) = -\frac{e}{\sqrt{2}}$

$f_{yyy}(x,y) = e^x \sin y$ $\qquad f_{yyy}\left(1, \frac{\pi}{4}\right) = e^1 \sin\left(\frac{\pi}{4}\right) = \frac{e}{\sqrt{2}}$

$f_{xy}(x,y) = -xe^x \sin y$ $\qquad f_{xy}\left(1, \frac{\pi}{4}\right) = -1.e^1 \sin\left(\frac{\pi}{4}\right) = -\frac{e}{\sqrt{2}}$

$f_{xxy}(x,y) = -x^2 e^x \sin y$ $\qquad f_{xxy}\left(1, \frac{\pi}{4}\right) = -1.e^1 \sin\left(\frac{\pi}{4}\right) = -\frac{e}{\sqrt{2}}$

$f_{xyy}(x,y) = -xe^x \cos y \qquad f_{xyy}\left(1,\frac{\pi}{4}\right) = -1.e^1 \cos\left(\frac{\pi}{4}\right) = -\frac{e}{\sqrt{2}}$

$$f(x,y) = f\left(1,\frac{\pi}{4}\right) + (x-1) f_x\left(1,\frac{\pi}{4}\right) + \left(y-\frac{\pi}{4}\right) f_y\left(1,\frac{\pi}{4}\right)$$
$$+\frac{1}{2!}\left[(x-1)^2 f_{xx}\left(1,\frac{\pi}{4}\right) + 2(x-1)\left(y-\frac{\pi}{4}\right) f_{xy}\left(1,\frac{\pi}{4}\right) + \left(y-\frac{\pi}{4}\right)^2 f_{yy}\left(1,\frac{\pi}{4}\right)\right]$$
$$+\frac{1}{3!}\begin{bmatrix}(x-1)^3 f_{xxx}\left(1,\frac{\pi}{4}\right) + 3(x-1)^2\left(y-\frac{\pi}{4}\right) f_{xxy}\left(1,\frac{\pi}{4}\right) \\ +3(x-1)\left(y-\frac{\pi}{4}\right)^2 f_{xyy}\left(1,\frac{\pi}{4}\right) + \left(y-\frac{\pi}{4}\right)^3 f_{yyy}\left(1,\frac{\pi}{4}\right)\end{bmatrix} + \ldots$$

$$f(x,y) = \frac{e}{\sqrt{2}}\begin{pmatrix}1+(x-1)-\left(y-\frac{\pi}{4}\right)+\frac{1}{2!}\left[(x-1)^2 - 2(x-1)\left(y-\frac{\pi}{4}\right) - \left(y-\frac{\pi}{4}\right)^2\right] \\ +\frac{1}{3!}\left[(x-1)^3 - 3(x-1)^2\left(y-\frac{\pi}{4}\right) - 3(x-1)\left(y-\frac{\pi}{4}\right)^2 + \left(y-\frac{\pi}{4}\right)^3\right]\end{pmatrix} + \ldots$$

19. **Expand $x^2y + 3y - 2$ in powers of $(x-1)$ and $(y+2)$ upto 3rd degree by Taylor's theorem.**

Given $f(x,y) = x^2y + 3y - 2$ and here $a = 1, b = -2$.

$$f(x,y) = f(a,b) + (x-a) f_x(a,b) + (y-b) f_y(a,b)$$
$$+\frac{1}{2!}\left[(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b) f_{xy}(a,b) + (y-b)^2 f_{yy}(a,b)\right]$$
$$+\frac{1}{3!}\begin{bmatrix}(x-a)^3 f_{xxx}(a,b) + 3(x-a)^2(y-b) f_{xxy}(a,b) \\ +3(x-a)(y-b)^2 f_{xyy}(a,b) + (y-b)^3 f_{yyy}(a,b)\end{bmatrix} + \ldots$$

$f(x,y) = x^2y + 3y - 2 \qquad f(1,-2) = -2-6-2 = -10$

$f_x(x,y) = 2xy \qquad f_x(1,-2) = -4$

$f_{xx}(x,y) = 2y \qquad f_{xx}(1,-2) = -4$

$f_{xxx}(x,y) = 0 \qquad f_{xxx}(1,-2) = 0$

$f_y(x,y) = x^2 + 3 \qquad f_y(1,-2) = 4$

$f_{yy}(x,y) = 0 \qquad f_{yy}(1,-2) = 0$

$f_{yyy}(x,y) = 0 \qquad f_{yyy}(1,-2) = 0$

$f_{xy}(x,y) = 2x \qquad f_{xy}(1,-2) = 2$

$f_{xxy}(x,y) = 2 \qquad f_{xxy}(1,-2) = 2$

$f_{xyy}(x,y) = 0 \qquad f_{xyy}(0,0) = 0$

$$f(x,y)= f(1,-2)+(x-1)\,f_x(1,-2)+(y+2)\,f_y(1,-2)$$
$$+\frac{1}{2!}\left[(x-1)^2 f_{xx}(1,-2)+2(x-1)(y+2)\,f_{xy}(1,-2)+(y+2)^2 f_{yy}(1,-2)\right]$$
$$+\frac{1}{3!}\begin{bmatrix}(x-1)^3 f_{xxx}(1,-2)+3(x-1)^2(y+2)\,f_{xxy}(1,-2)\\ +3(x-1)(y+2)^2 f_{xyy}(1,-2)+(y+2)^3 f_{yyy}(1,-2)\end{bmatrix}+\ldots$$

$$x^2y+3y-2=-10-4(x-1)+4(y+2)+\frac{1}{2!}\left[(-4)(x-1)^2+4(x-1)(y+2)\right]+\frac{1}{3!}\left[6(x-1)^2(y+2)\right]+\ldots$$
$$=-10-4(x-1)+4(y+2)+\left[(-2)(x-1)^2+2(x-1)(y+2)\right]+\left[(x-1)^2(y+2)\right]+\ldots$$

20. **Expand** $x^2y^2+2x^2y+3xy^2$ **in powers of** $(x+2)$ **and** $(y-1)$ **using Taylor's theorem.**

Given $f(x,y)=x^2y^2+2x^2y+3xy^2$ and here $a=-2, b=1$.

$$f(x,y)= f(a,b)+(x-a)\,f_x(a,b)+(y-b)\,f_y(a,b)$$
$$+\frac{1}{2!}\left[(x-a)^2 f_{xx}(a,b)+2(x-a)(y-b)\,f_{xy}(a,b)+(y-b)^2 f_{yy}(a,b)\right]$$
$$+\frac{1}{3!}\begin{bmatrix}(x-a)^3 f_{xxx}(a,b)+3(x-a)^2(y-b)\,f_{xxy}(a,b)\\ +3(x-a)(y-b)^2 f_{xyy}(a,b)+(y-b)^3 f_{yyy}(a,b)\end{bmatrix}+\ldots$$

$f(x,y)=x^2y^2+2x^2y+3xy^2$	$f(-2,1)=4+8-6=6$
$f_x(x,y)=2xy^2+4xy+3y^2$	$f_x(-2,1)=-4-8+3=-9$
$f_{xx}(x,y)=2y^2+4y$	$f_{xx}(-2,1)=6$
$f_{xxx}(x,y)=0$	$f_{xxx}(-2,1)=0$
$f_y(x,y)=2x^2y+2x^2+6xy$	$f_y(-2,1)=4$
$f_{yy}(x,y)=2x^2+6x$	$f_{yy}(-2,1)=-4$
$f_{yyy}(x,y)=0$	$f_{yyy}(-2,1)=0$
$f_{xy}(x,y)=4xy+6y+4x$	$f_{xy}(-2,1)=-10$
$f_{xxy}(x,y)=4y+4$	$f_{xxy}(-2,1)=8$
$f_{xyy}(x,y)=4x+6$	$f_{xyy}(-2,1)=-2$

$$f(x,y)= f(2,-1)+(x+2)\,f_x(-2,1)+(y-1)\,f_y(-2,1)$$
$$+\frac{1}{2!}\left[(x+2)^2 f_{xx}(-2,1)+2(x+2)(y-1)\,f_{xy}(-2,1)+(y-1)^2 f_{yy}(-2,1)\right]$$
$$+\frac{1}{3!}\begin{bmatrix}(x+2)^3 f_{xxx}(-2,1)+3(x+2)^2(y-1)\,f_{xxy}(-2,1)\\ +3(x+2)(y-1)^2 f_{xyy}(-2,1)+(y-1)^3 f_{yyy}(-2,1)\end{bmatrix}+\ldots$$

$$f(x,y)=6+(x+2)(-9)+(y-1)(4)+\frac{1}{2!}\left[(x+2)^2(6)+2(x+2)(y-1)(-10)+(y-1)^2(-4)\right]$$

$$+\frac{1}{3!}\left[(x+2)^3(0)+3(x+2)^2(y-1)(8)+3(x+2)(y-1)^2(-2)+(y-1)^3(0)\right]+...$$

$$=6-9(x+2)+4(y-1)+\left[3(x+2)^2-10(x+2)(y-1)-2(y-1)^2\right]$$

$$+\left[(x+2)^2(y-1)(4)-3(x+2)(y-1)^2\right]$$

21. **Expand** $\tan^{-1}\left(\frac{y}{x}\right)$ **at the point (1,1) up to second degree.**

Given $f(x,y)=\tan^{-1}\left(\frac{y}{x}\right)$ and here $a=1, b=1$.

$$f(x,y)=f(a,b)+\left[(x-a)f_x(a,b)+(y-b)f_y(a,b)\right]$$

$$+\frac{1}{2!}\left[(x-a)^2 f_{xx}(a,b)+2(x-a)(y-b)f_{xy}(a,b)+(y-b)^2 f_{yy}(a,b)\right]$$

$$f(x,y)=\tan^{-1}\left(\frac{y}{x}\right) \qquad f(1,1)=\tan^{-1}(1)=\frac{\pi}{4}$$

$$f_x(x,y)=\frac{-y}{x^2+y^2} \qquad f_x(1,1)=-\frac{1}{2}$$

$$f_{xx}(x,y)=\frac{2xy}{(x^2+y^2)^2} \qquad f_{xx}(1,1)=\frac{1}{2}$$

$$f_y(x,y)=\frac{x}{x^2+y^2} \qquad f_y(1,1)=\frac{1}{2}$$

$$f_{yy}(x,y)=-\frac{2xy}{(x^2+y^2)^2} \qquad f_{yy}(1,1)=-\frac{1}{2}$$

$$f_{xy}(x,y)=\frac{y^2-x^2}{(x^2+y^2)^2} \qquad f_{xy}(1,1)=0$$

$$f(x,y)=f(1,1)+\left[(x-1)f_x(1,1)+(y-1)f_y(1,1)\right]$$

$$+\frac{1}{2!}\left[(x-1)^2 f_{xx}(1,1)+2(x-1)(y-1)f_{xy}(1,1)+(y-1)^2 f_{yy}(1,1)\right]$$

$$=\frac{\pi}{4}+\frac{1}{2}\left((y-1)-(x-1)\right)+\frac{1}{2}\left((x-1)^2-(y-1)^2\right)$$

Maxima and Minima of a function of two variables

Notation: $p = \frac{\partial f}{\partial x}$; $q = \frac{\partial f}{\partial y}$; $r = \frac{\partial^2 f}{\partial x^2}$; $s = \frac{\partial^2 f}{\partial x \partial y}$; $t = \frac{\partial^2 f}{\partial y^2}$

Working rule:

Let f(x, y) be the given function.

1. Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.
2. Solve $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$ simultaneously. Solution of the equations are stationary points.
3. Find the value of r, s, t and $rt - s^2$ at all the stationary points.

r *or* **t**	$\mathbf{rt - s^2}$	**Conclusion**
$r < 0$	$rt - s^2 > 0$	f(x, y) attains its maximum at that stationary point.
$r > 0$	$rt - s^2 > 0$	f(x, y) attains its minimum at that stationary point.
-	$rt - s^2 < 0$	Neither maximum nor minimum. The stationary point is saddle point.
-	$rt - s^2 = 0$	Further investigation is needed.

22. **Find the maximum and minimum value for the function** $f(x, y) = x^2 + y^2 + 6x + 12$.

Solution:

Let $f(x, y) = x^2 + y^2 + 6x + 12$

$$p = \frac{\partial f}{\partial x} = 2x + 6; \text{q} = \frac{\partial f}{\partial y} = 2y;\ r = \frac{\partial^2 f}{\partial x^2} = 2;\ \text{s} = \frac{\partial^2 f}{\partial x \partial y} = 0 \text{ and } t = \frac{\partial^2 f}{\partial y^2} = 2.$$

p = 0 and q = 0 implies x = –3 and y = 0.

Therefore the stationary point is (–3, 0).

At (–3, 0), r = 2 > 0 and $rt - s^2 = 4 > 0$.

Therefore f(x, y) obtains its minimum value at (–3, 0).

The minimum value is f(–3, 0) = 3.

23. Find the maximum and minimum of the function $f(x, y) = 3(x^2 - y^2) - x^3 + y^3$.

Let $f(x, y) = 3(x^2 - y^2) - x^3 + y^3$

$$p = \frac{\partial f}{\partial x} = 6x - 3x^2; \text{q} = \frac{\partial f}{\partial y} = -6y + 3y^2;$$

$$r = \frac{\partial^2 f}{\partial x^2} = 6 - 6\text{x};\ \ \text{s} = \frac{\partial^2 f}{\partial x \partial y} = 0 \text{ and } t = \frac{\partial^2 f}{\partial y^2} = -6 + 6y.$$

p = 0 implies x = 0 and x = 2.

and q = 0 implies y = 0 and y = 2

Therefore the stationary points are (0, 0), (0, 2), (2, 0) and (2, 2).

At stationary points	r = 6 – 6x	$rt - s^2$	Conclusion	Extreme value
(0, 0)	6	–36	Saddle point	–
(0, 2)	6	36	Minimum	f(0, 2) = – 4
(2, 0)	–6	36	Maximum	f(2, 0) = 4
(2, 2)	–6	–36	Saddle point	–

Thus f(x, y) obtains its maximum at (2, 0) and the maximum value is 4. Similarly, f(x, y) obtains its minimum at (0, 2) and the minimum value is – 4 .

24. **Find the maximum and minimum of the function** $f(x,y) = x^3 + y^3 - 12x - 3y + 20$.

Solution: Let $f(x,y) = x^3 + y^3 - 12x - 3y + 20$

$$p = \frac{\partial f}{\partial x} = 3x^2 - 12; \text{q} = \frac{\partial f}{\partial y} = 3y^2 - 3;$$

$$r = \frac{\partial^2 f}{\partial x^2} = 6\text{x}; \quad \text{s} = \frac{\partial^2 f}{\partial x \partial y} = 0 \text{ and } t = \frac{\partial^2 f}{\partial y^2} = 6y.$$

p = 0 implies x = –2 and x = 2.

and q = 0 implies y = –1 and y = 1

Therefore the stationary points are (–2, –1), (–2, 1), (2, –1) and (2, 1).

At stationary points	r = 6x	$rt - s^2$	Conclusion	Extreme value
(–2, –1)	–12	72	Maximum	f(–2,–1) = 38
(–2, 1)	–12	–72	Saddle point	–
(2, –1)	12	–72	Saddle point	–
(2, 1)	12	72	Minimum	f(2, 1) = 2

Thus f(x, y) obtains its maximum at (–2, –1) and the maximum value is 38. Similarly, f(x,y) obtains its minimum at (2, 1) and the minimum value is 2.

25. **Find the maximum and minimum values of f(x, y) = $x^3 + y^3 - 3axy$**.

Solution:

Let f(x, y) = $x^3 + y^3 - 3axy$

$p = f_x = 3x^2 - 3ay$; $q = f_y = 3y^2 - 3ax$;

$r = f_{xx} = 6x$; $s = f_{xy} = -3a$; $t = f_{yy} = 6y$.

p = 0 and q=0 implies $3x^2 - 3ay = 0$and $3y^2 - 3ax = 0$

i.e., $x^2 = ay$ and $y^2 = ax$

i.e., $x^4 = a^2y^2$

i.e., $x^4 = a^3x$

i.e., $x(x^3 - a^3) = 0$

i.e., x = 0 or x = a

When x = 0, we get, y = 0 and when x = a, we get, y = a.

$\therefore$ The stationary points are (0, 0) and (a, a).

At stationary points	r	$rt - s^2$	Conclusion	Extreme value
(0, 0)	0	$-9a^2 < 0$	Neither maximum nor minimum, Saddle point	–
(a, a)	6a	$27a^2$	If a > 0, then r > 0 and hence f(a, a) is a minimum value.	
			If a < 0, then r < 0 and hence f(a, a) is a maximum value.	

Thus the maximum or minimum value at (a, a) is $f(a, a) = -a^3$.

26. **Find the maxima or minima of $f(x, y) = 2(x - y)^2 - x^4 - y^4$.**

Solution:

Let $f(x,y) = 2(x - y)^2 - x^4 - y^4$

$p = f_x = 4(x - y) - 4x^3$; $q = f_y = -4(x - y) - 4y^3$;

$r = f_{xx} = 4 - 12x^2$; $s = f_{xy} = -4$; $t = f_{yy} = 4 - 12y^2$

solving p = 0 and q = 0 implies $x - y - x^3 = 0 \quad \rightarrow(1)$

and $-(x - y) - y^3 = 0 \quad \rightarrow(2)$

Adding (1) and (2) $x^3 + y^3 = 0$

i.e., $(x + y)(x^2 - xy + y^2) = 0$

$\therefore$ $x = -y$ or $x^2 - xy + y^2 = 0$ (Check: $x^2 - xy + y^2 > 0$, always)

Putting in (1) $x = -y$, we get,

$-2y + y^3 = 0$

i.e., $y(y^2 - 2) = 0$

i.e., $y = 0, \sqrt{2}, -\sqrt{2}$

The corresponding x values are $0, -\sqrt{2}, \sqrt{2}$

$\therefore$ The stationary points are (0, 0), $(\sqrt{2}, -\sqrt{2})$ and $(-\sqrt{2}, \sqrt{2})$.

At stationary points	$r = 4 - 12x^2$	$rt - s^2$	Conclusion	Extreme value
(0, 0)	4	0	Further investigation needed	–
$(\sqrt{2}, -\sqrt{2})$	–20	384	Maximum	$f(\sqrt{2}, -\sqrt{2}) = 8$
$(-\sqrt{2}, \sqrt{2})$	–20	384	Maximum	$f(-\sqrt{2}, \sqrt{2}) = 8$

Constrained Maximum and Minimum – Lagrange's Method of Undetermined Multipliers

Let f(x, y, z) = 0 be the function whose extreme values should be found subject to the condition (constraint) ϕ (x, y, z) = 0. We define $F(x, y, z) = f(x, y, z) + \lambda\phi(x, y, z)$, where λ is called Lagrange multiplier. For extreme values, solve $\frac{\partial F}{\partial x} = 0; \frac{\partial F}{\partial y} = 0; \frac{\partial F}{\partial z} = 0; \frac{\partial F}{\partial \lambda} = 0$.

27. **A rectangular box open at the top is to have a volume of 32 cubic feet. Find the dimensions of the box that requires the least material for its construction.**

Solution: Let x, y, z be the length, breadth and height of the box.

Then surface area of the box = xy + 2yz + 2zx, since the box is opened at the top.

Given, volume = 32. Therefore, xyz = 32 → xyz – 32 = 0

Thus F(x, y, z) = (xy +2yz + 2zx) + λ (xyz – 32) → (1)

$$\frac{\partial F}{\partial x} = y + 2z + \lambda(yz)$$

$$\frac{\partial F}{\partial y} = x + 2z + \lambda(zx)$$

$$\frac{\partial F}{\partial z} = 2y + 2x + \lambda(xy)$$

$$\frac{\partial F}{\partial \lambda} = xyz - 32$$

$$\frac{\partial F}{\partial x} = 0; \frac{\partial F}{\partial y} = 0; \frac{\partial F}{\partial z} = 0 \Rightarrow \lambda = \frac{-y-2z}{yz} = \frac{-x-2z}{xz} = \frac{-2y-2x}{xy}$$

$$\frac{-y-2z}{yz} = \frac{-x-2z}{xz} \Rightarrow x = y$$

$$\frac{-x-2z}{xz} = \frac{-2y-2x}{xy} \Rightarrow y = 2z$$

Thus x = y = 2z.

$$\frac{\partial F}{\partial \lambda} = 0 \Rightarrow xyz - 32 = 0$$

⇨ $x \times x \times \frac{x}{2} = 32$

⇨ x = 4

⇨ y = 4 and z = 2.

Thus the dimension of the box is (4, 4, 2).

28. **Find the dimensions of the rectangular box without top of maximum capacity whose surface area is 432 sq. cm.**

Solution: Let x, y, z be the length, breadth and height of the box.

Then surface area of the box = xy + 2yz + 2zx = 432, since the box is opened at the top.

Volume = xyz

Thus F(x, y, z) = xyz + λ (xy +2yz + 2zx – 432) → (1)

$$\frac{\partial F}{\partial x} = yz + \lambda(y + 2z)$$

$$\frac{\partial F}{\partial y} = xz + \lambda(x + 2z)$$

$$\frac{\partial F}{\partial z} = xy + \lambda(2y + 2x)$$

$$\frac{\partial F}{\partial \lambda} = xy + 2yz + 2zx - 432$$

$$\frac{\partial F}{\partial x} = \frac{\partial F}{\partial y} = \frac{\partial F}{\partial z} = 0 \Rightarrow \lambda = \frac{-yz}{y+2z} = \frac{-xz}{x+2z} = \frac{-xy}{2y+2x}$$

$$\frac{-yz}{y+2z} = \frac{-xz}{x+2z} \Rightarrow x = y$$

$$\frac{-xz}{x+2z} = \frac{-xy}{2y+2x} \Rightarrow y = 2z$$

Hence x = y = 2z.

$$\frac{\partial F}{\partial \lambda} = 0 \Rightarrow xy + 2y\ z + 2zx - 432 = 0$$

$$x^2 + x^2 + x^2 - 432 = 0$$

$$3x^2 = 432 \Rightarrow x^2 = 144 \Rightarrow x = \pm 12$$

Hence x = 12, y = 12, z = 6.

The dimension of the box is (12, 12, 6).

29. **Find the volume of the greatest rectangular parallelopiped that can be inscribed in the ellipsoid** $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$**.**

Solution: Let 2x, 2y, 2z be the dimension of the rectangular parallelepiped. We have to maximize 8xyz subject to $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

Therefore $F(x,\ y,\ z) = 8xyz + \lambda\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1\right)$

$$\frac{\partial F}{\partial x} = 8yz + \frac{2\lambda x}{a^2}; \quad \frac{\partial F}{\partial y} = 8xz + \frac{2\lambda y}{b^2}; \quad \frac{\partial F}{\partial z} = 8xy + \frac{2\lambda z}{c^2};$$

$$\frac{\partial F}{\partial x} = 0; \frac{\partial F}{\partial y} = 0; \frac{\partial F}{\partial z} = 0 \Rightarrow \lambda = \frac{a^2 yz}{x} = \frac{b^2 xz}{y} = \frac{c^2 xy}{z}$$

Choosing $\frac{a^2 yz}{x} = \frac{b^2 xz}{y} \Rightarrow \frac{x^2}{a^2} = \frac{y^2}{b^2}$

Choosing $\dfrac{b^2xz}{y}=\dfrac{c^2xy}{z}\Rightarrow\dfrac{y^2}{b^2}=\dfrac{z^2}{c^2}$

Thus $\dfrac{x^2}{a^2}=\dfrac{y^2}{b^2}=\dfrac{z^2}{c^2}$

$$\frac{\partial F}{\partial\lambda}=\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1\right)$$

$$\frac{\partial F}{\partial\lambda}=0\Rightarrow\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

$$\Rightarrow\frac{x^2}{a^2}+\frac{x^2}{a^2}+\frac{x^2}{a^2}=1$$

$3\dfrac{x^2}{a^2}=1\Rightarrow x=\dfrac{a}{\sqrt{3}}$ Similarly, we can prove $y=\dfrac{b}{\sqrt{3}}$ and $z=\dfrac{c}{\sqrt{3}}$

Thus the maximum volume is V = 8xyz = $\dfrac{8abc}{3\sqrt{3}}$.

30. **Find the minimum distance from the point (3, 4, 15) to the cone $x^2 + y^2 = 4z^2$.**

Solution:

Let (x, y, z) be any point on the cone $x^2 + y^2 = 4z^2$.

Then its distance from the point (3, 4, 15) is $d=\sqrt{(x-3)^2+(y-4)^2+(z-15)^2}$.

First we find the minimum value of d^2 subject to the condition $x^2 + y^2 = 4z^2$.

Let $F(x, y, z) = (x-3)^2+(y-4)^2+(z-15)^2+\lambda\,(x^2+y^2-4z^2)$

The stationary points are given by,

$$F_x = 2(x-3) + 2\lambda x = 0 \quad (1)$$

$$F_y = 2(y-4) + 2\lambda y = 0 \quad (2)$$

$$F_z = 2(z-15) - 8\lambda z = 0 \quad (3)$$

$$F_\lambda = x^2 + y^2 - 4z^2 = 0 \quad (4)$$

From (1), $x=\dfrac{3}{1+\lambda}$

From (2), $y=\dfrac{4}{1+\lambda}$

From (3), $z=\dfrac{15}{1-4\lambda}$

Substituting in (4), $\left(\dfrac{3}{1+\lambda}\right)^2+\left(\dfrac{4}{1+\lambda}\right)^2=4\left(\dfrac{15}{1-4\lambda}\right)^2$

i.e., $25(1-4\lambda)^2 = 4(225)(1+\lambda)^2$

i.e., $\dfrac{1-4\lambda}{1+\lambda}=\pm 6$

From $\dfrac{1-4\lambda}{1+\lambda} = 6$ we get $\lambda = -1/2$

From $\dfrac{1-4\lambda}{1+\lambda} = -6$ we get $\lambda = -7/2$

When $\lambda = -1/2$, we get x = 6, y = 8, z = 5.

When $\lambda = -7/2$, we get x = –6/5, y = –8/5, z = 1.

Thus the stationary points are (6, 8, 5) and (–6/5, –8/5, 1).

Distance of (6, 8, 5) from (3, 4, 15) is $d = \sqrt{(6-3)^2 + (8-4)^2 + (5-15)^2}$

$$= \sqrt{125} \quad = 5\sqrt{5}$$

Distance of (–6/5, –8/5, 1) from (3, 4, 15) is $d = \sqrt{(-6/5 - 3)^2 + (-8/5 - 4)^2 + (1-15)^2}$

$$= \sqrt{\frac{441}{25} + \frac{784}{25} + 196}$$

$$= \sqrt{\frac{6125}{25}} = \sqrt{245} = 7\sqrt{5}$$

$\therefore$ The minimum distance from the point (3, 4, 15) to the cone $x^2 + y^2 = 4z^2$ is $5\sqrt{5}$.

31. **Find the shortest and longest distance from (1, 2, –1) to the sphere $x^2 + y^2 + z^2 = 24$ using Lagrange's method of constrained maxima and minima.**

Solution: Similar to **Problem – 30**. The points are (x, y, z) = (2, 4, –2) and (–2, –4, 2).
Shortest distance = $\sqrt{6}$ and longest distance $= 3\sqrt{6}$.

32. **Find the maximum and minimum distance of the point (3, 4, 12) from the sphere $x^2 + y^2 + z^2 = 1$.**

Solution: Similar to **Problem – 30**. The points are $(x, y, z) = \left(\frac{3}{13}, \frac{4}{13}, \frac{12}{13}\right)$ and $(x, y, z) = \left(\frac{-3}{13}, \frac{-4}{13}, \frac{-12}{13}\right)$. Minimum distance = 12 units and maximum distance = 14 units.

33. **Find the length of the shortest line from the point $\left(0, 0, \frac{25}{9}\right)$ to the surface z = xy.**

Solution: Similar to **Problem – 30**. The points are $(x, y, z) = \left(\pm\frac{4}{3}, \pm\frac{4}{3}, \frac{16}{9}\right)$. The minimum and the maximum distance is $\frac{\sqrt{41}}{3}$.

34. **If u = $a^3x^2 + b^3y^2 + c^3z^2$, where $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1$, then show that the stationary value of u is given by $x = \frac{a+b+c}{a}$, $y = \frac{a+b+c}{b}$, $z = \frac{a+b+c}{c}$.**

Solution: Given u = $a^3x^2 + b^3y^2 + c^3z^2$ and $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=1$.

Let $F(x, y, z) = a^3x^2 + b^3y^2 + c^3z^2 + \lambda\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}-1\right)$

The stationary points are given by,

$$F_x = 2a^3x + \lambda.\left(\frac{-1}{x^2}\right) = 0 \qquad (1)$$

$$F_y = 2b^3y + \lambda.\left(\frac{-1}{y^2}\right) = 0 \qquad (2)$$

$$F_z = 2c^3z + \lambda.\left(\frac{-1}{z^2}\right) = 0 \qquad (3)$$

$$F_\lambda = \frac{1}{x}+\frac{1}{y}+\frac{1}{z}-1 = 0 \qquad (4)$$

From (1) $2a^3x^3 - \lambda = 0$

i.e., $x = \left(\frac{\lambda}{2}\right)^{1/3}\frac{1}{a}$

Similarly from (2) and (3) we get,

$$y = \left(\frac{\lambda}{2}\right)^{1/3}\frac{1}{b},\ z = \left(\frac{\lambda}{2}\right)^{1/3}\frac{1}{c}$$

substituting for x, y, z in (4) we get

$$\left(\frac{2}{\lambda}\right)^{1/3}(a+b+c)-1=0$$

i.e., $\left(\frac{2}{\lambda}\right)^{1/3} = \frac{1}{\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c}}$

i.e., $\left(\frac{\lambda}{2}\right)^{1/3} = \boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c}$

$\therefore$ $\boldsymbol{x} = \left(\frac{\lambda}{2}\right)^{1/3}\frac{1}{\boldsymbol{a}} = \frac{\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c}}{\boldsymbol{a}}$

Similarly $y = \frac{a+b+c}{b}$ and $z = \frac{a+b+c}{c}$.

Hence the stationary value of u is given by $x = \frac{a+b+c}{a}$, $y = \frac{a+b+c}{b}$, $z = \frac{a+b+c}{c}$.

35. **Find the minimum value of $x^2 + y^2 + z^2$ where $ax + by + cz = p$.**

Solution:

Let $f(x, y, z) = x^2 + y^2 + z^2$.

$\varphi(x, y, z) = ax + by + cz - p$ and $F(x, y, z) = f(x, y, z) + \lambda\varphi(x, y, z)$ where λ is the Lagrange multiplier.

Then $F(x, y, z) = x^2 + y^2 + z^2 + \lambda(ax + by + cz - p)$

The stationary points are obtained by solving

$$F_x = 2x + a\lambda = 0 \qquad (1)$$

$$F_y = 2y + b\lambda = 0 \qquad (2)$$

$$F_z = 2z + c\lambda = 0 \qquad (3)$$

and $F_\lambda = ax + by + cz - p \qquad (4)$

From (1), $x = -\dfrac{a\lambda}{2}$

From (2), $y = -\dfrac{b\lambda}{2}$

From (3), $z = -\dfrac{c\lambda}{2}$

From (4), $a.\left(-\dfrac{a\lambda}{2}\right) + b.\left(-\dfrac{b\lambda}{2}\right) + c.\left(-\dfrac{c\lambda}{2}\right) = p$

$$\lambda = \frac{-2p}{a^2 + b^2 + c^2}$$

$$\therefore \qquad x = \frac{ap}{a^2 + b^2 + c^2}, \quad y = \frac{bp}{a^2 + b^2 + c^2}, \quad z = \frac{cp}{a^2 + b^2 + c^2}$$

The only stationary point is $\left(\dfrac{ap}{a^2 + b^2 + c^2}, \dfrac{bp}{a^2 + b^2 + c^2}, \dfrac{cp}{a^2 + b^2 + c^2}\right)$.

The minimum value of $f(x, y, z) = \left(\dfrac{ap}{a^2 + b^2 + c^2}\right)^2 + \left(\dfrac{bp}{a^2 + b^2 + c^2}\right)^2 + \left(\dfrac{cp}{a^2 + b^2 + c^2}\right)^2$

$$= \frac{p^2(a^2 + b^2 + c^2)}{\left(a^2 + b^2 + c^2\right)^2} = \frac{p^2}{a^2 + b^2 + c^2}$$

36. **Find the maximum value of $x^m y^n z^p$ such that $x + y + z = a$.**

Solution: Given $f(x, y, z) = x^m y^n z^p$ and $\varphi(x, y, z) = x + y + z = a$

$F(x, y, z) = x^m y^n z^p + \lambda\left(x + y + z - a\right)$

$\dfrac{\partial F}{\partial x} = mx^{m-1} y^n z^p + \lambda$

$$\frac{\partial F}{\partial y} = nx^m y^{n-1} z^p + \lambda$$

$$\frac{\partial F}{\partial z} = px^m y^n z^{p-1} + \lambda$$

$$\frac{\partial F}{\partial x} = 0, \frac{\partial F}{\partial y} = 0, \frac{\partial F}{\partial z} = 0 \Rightarrow \lambda = mx^{m-1} y^n z^p = nx^m y^{n-1} z^p = px^m y^n z^{p-1}$$

$$\Rightarrow \frac{mx^m y^n z^p}{x} = \frac{nx^m y^n z^p}{y} = \frac{px^m y^n z^p}{z}$$

$$\Rightarrow \frac{m}{x} = \frac{n}{y} = \frac{p}{z} = \frac{m+n+p}{x+y+z} \text{(by property)} = \frac{m+n+p}{a}$$

$$\Rightarrow x = \frac{am}{m+n+p};\ y = \frac{an}{m+n+p};\ z = \frac{ap}{m+n+p}$$

Thus the maximum value of, $F(x, y, z) = \left(\frac{am}{m+n+p}\right)^m \left(\frac{an}{m+n+p}\right)^n \left(\frac{ap}{m+n+p}\right)^p$

$$= \frac{a^{m+n+p}\left(m^m n^n p^p\right)}{(m+n+p)^{m+n+p}}$$

JACOBIAN

If $u = u(x, y)$ and $v = v(x, y)$ are two functions of two independent variables x and y, then the Jacobian of u and v is denoted by $J\left(\frac{u, v}{x, y}\right)$ *or* $\frac{\partial(u, v)}{\partial(x, y)}$ and is defined by $\frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$.

Note: If u, v and w are functions of three independent variables x, y and z, then their Jacobian is

$$J\left(\frac{u, v, w}{x, y, z}\right) = \frac{\partial(u, v, w)}{\partial(x, y, z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}.$$

Property 1. If u and v are functions of x and y, then $\frac{\partial(u, v)}{\partial(x, y)} \text{X} \frac{\partial(x, y)}{\partial(u, v)} = 1$

Property 2. (Chain Rule or Jacobian of Composite Functions)

If u and v are functions of r and s, where r and s are functions of x and y, then

$$\frac{\partial(u, v)}{\partial(x, y)} = \frac{\partial(u, v)}{\partial(r, s)} X \frac{\partial(r, s)}{\partial(u, v)}$$

Property 3. If u, v, w are functionally dependent of a function x, y and z, then $\frac{\partial(u,v,w)}{\partial(x,y,z)} = 0.$

37. **If** $x = r\cos\theta$, $y = r\sin\theta$ **, then find** $\frac{\partial(x,y)}{\partial(r,\theta)}$**.**

Solution:

Given $x = r\cos\theta$, $y = r\sin\theta$

Then $\frac{\partial x}{\partial r} = \cos\theta$, $\frac{\partial x}{\partial \theta} = -r\sin\theta$, $\frac{\partial y}{\partial r} = \sin\theta$, $\frac{\partial y}{\partial \theta} = r\cos\theta$

Now $\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial v}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\left(\sin^2\theta + \cos^2\theta\right) = r(1) = r$

38. **If** $\boldsymbol{x = uv,\ y = \frac{u}{v}}$**, find** $\boldsymbol{\frac{\partial(x,y)}{\partial(u,v)}}$ **.**

Solution: Given $x = uv$, $y = \frac{u}{v}$

Then $\frac{\partial x}{\partial u} = v$, $\frac{\partial x}{\partial v} = u$, $\frac{\partial y}{\partial u} = \frac{1}{v}$, $\frac{\partial y}{\partial v} = -\frac{u}{v^2}$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} v & u \\ \frac{1}{v} & \frac{-u}{v^2} \end{vmatrix} = -\frac{2u}{v}$$

39. ***If*** $\boldsymbol{x = r\cos\theta}$ ***and*** $\boldsymbol{y = r\sin\theta}$***, then*** **find** $\boldsymbol{\frac{\partial r}{\partial x}}$**.**

Solution: Given $x = r\cos\theta$, $y = r\sin\theta$, then $r^2 = x^2 + y^2 \Rightarrow r = \sqrt{x^2+y^2}$

Now $\frac{\partial r}{\partial x} = \frac{1}{2\sqrt{x^2+y^2}}(2x) = \frac{x}{\sqrt{x^2+y^2}} = \frac{x}{r}$

40. **If** $x = r\cos\theta$, $y = r\sin\theta, z = z$ **, then find** $\frac{\partial(x,y,z)}{\partial(r,\theta,z)}$ **.**

Solution: Given $x = r\cos\theta$, $y = r\sin\theta$

Then $\frac{\partial x}{\partial r} = \cos\theta$, $\frac{\partial x}{\partial \theta} = -r\sin\theta$, $\frac{\partial x}{\partial z} = 0$, $\frac{\partial y}{\partial r} = \sin\theta$, $\frac{\partial y}{\partial \theta} = r\cos\theta$, $\frac{\partial y}{\partial z} = 0$, $\frac{\partial z}{\partial r} = 0$, $\frac{\partial z}{\partial \theta} = 0$, $\frac{\partial z}{\partial z} = 1$

Now $\dfrac{\partial(x,y,z)}{\partial(r,\theta,z)} = \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} & \dfrac{\partial x}{\partial z} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} & \dfrac{\partial y}{\partial z} \\ \dfrac{\partial z}{\partial r} & \dfrac{\partial z}{\partial \theta} & \dfrac{\partial z}{\partial z} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = \cos\theta(r\cos\theta) + r\sin\theta(\sin\theta) = r$

41. **If** $x = r\sin\theta\cos\phi, y = r\sin\theta\sin\phi, z = r\cos\theta$ **, then find** $\dfrac{\partial(x,y,z)}{\partial(r,\theta,z)}$ **.**

Solution : Given $x = r\sin\theta\cos\phi, y = r\sin\theta\sin\phi, z = r\cos\theta$

$\dfrac{\partial x}{\partial r} = \sin\theta\cos\phi \;, \dfrac{\partial x}{\partial \theta} = r\cos\theta\cos\phi \;, \dfrac{\partial x}{\partial z} = -r\sin\theta\sin\phi,$

$\dfrac{\partial y}{\partial r} = \sin\theta\sin\phi, \dfrac{\partial y}{\partial \theta} = r\cos\theta\sin\phi \;, \dfrac{\partial y}{\partial z} = r\sin\theta\cos\phi \;,$

$\dfrac{\partial z}{\partial r} = \cos\theta, \dfrac{\partial z}{\partial \theta} = -r\sin\theta, \dfrac{\partial z}{\partial z} = 0$

$\dfrac{\partial(x,y,z)}{\partial(r,\theta,z)} = \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$

$= \sin\theta\cos\phi(0 + r^2\sin^2\theta\cos\phi) - r\cos\theta\cos\phi(0 - (r\sin\theta\cos\phi)\cos\theta)$

$\quad - r\sin\theta\sin\phi(-r\sin^2\theta\sin\phi - r\cos^2\theta\sin\phi)$

$= r^2\sin^3\theta\cos^2\phi + r^2\sin\theta\cos^2\phi\cos^2\theta + r^2\sin^3\theta\sin^2\phi + r^2\sin\theta\cos^2\theta\sin^2\phi$

$= r^2\sin^3\theta(\cos^2\phi + \sin^2\phi) + r^2\sin\theta\cos^2\theta(\cos^2\phi + \sin^2\phi)$

$= r^2\sin\theta(\sin^2\theta + \cos^2\theta) = r^2\sin\theta$

42. **If** $u = x + y + z,\; uv = y + z,\; uvw = z,$ **show that** $\dfrac{\partial(\mathrm{x,y,z})}{\partial(\mathrm{u,v,w})} = \mathrm{u}^2\mathrm{v}$ **.**

Solution:

Given

$u = x + y + z \quad -(1)$

$uv = y + z \quad -(2)$

$uvw = z \quad -(3)$

Using (2) in (1), we get, $x = u - \left(y + z\right) = u - uv = u(1-v)$

Using (3) in (2) we get, $\mathrm{y} = uv - \mathrm{z} = \mathrm{uv} - uvw = uv(1-w)$

From (4) $\dfrac{\partial \mathrm{x}}{\partial \mathrm{u}} = 1 - \mathrm{v}, \qquad \dfrac{\partial \mathrm{x}}{\partial v} = -\mathrm{u}, \qquad \dfrac{\partial \mathrm{x}}{\partial w} = 0$

From (5) $\dfrac{\partial y}{\partial \mathrm{u}} = \mathrm{v}.(1-\mathrm{w}), \qquad \dfrac{\partial y}{\partial v} = \mathrm{u}.(1-\mathrm{w}), \qquad \dfrac{\partial \mathrm{y}}{\partial w} = -\mathrm{uv}$

From (3) $\frac{\partial z}{\partial u} = vw, \qquad \frac{\partial z}{\partial v} = uw, \qquad \frac{\partial z}{\partial w} = uv$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} 1-v & -u & 0 \\ v(1-w) & u(1-w) & -uv \\ vw & wu & uv \end{vmatrix}$$

$$= (1-v)\left[u^2v(1-w) + u^2vw\right] + u\left[uv^2(1-w) + uv^2w\right]$$

$$= (1-v)u^2v + u^2v^2 = u^2v$$

43. **Find the Jacobian of y_1, y_2, y_3 with respect to x_1, x_2, x_3 if $y_1 = \frac{x_2x_3}{x_1}, y_2 = \frac{x_1x_3}{x_2}, y_3 = \frac{x_1x_2}{x_3}$.**

Solution:

Given $y_1 = \frac{x_2x_3}{x_1}, y_2 = \frac{x_1x_3}{x_2}, y_3 = \frac{x_1x_2}{x_3}$

$\frac{\partial y_1}{\partial x_1} = -\frac{x_2x_3}{x_1^2}, \frac{\partial y_1}{\partial x_2} = \frac{x_3}{x_1}, \frac{\partial y_1}{\partial x_3} = \frac{x_2}{x_1}$; $\frac{\partial y_2}{\partial x_1} = \frac{x_3}{x_2}, \frac{\partial y_2}{\partial x_2} = -\frac{x_1x_3}{x_2^2}, \frac{\partial y_2}{\partial x_3} = \frac{x_1}{x_2}$ and

$\frac{\partial y_2}{\partial x_1} = \frac{x_2}{x_3}, \frac{\partial y_2}{\partial x_2} = \frac{x_1}{x_3}, \frac{\partial y_2}{\partial x_3} = -\frac{x_1x_2}{x_3^2}$

$$\frac{\partial(y_1,y_2,y_3)}{\partial(x_1,x_2,x_3)} = \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \frac{\partial y_1}{\partial x_3} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \frac{\partial y_2}{\partial x_3} \\ \frac{\partial y_3}{\partial x_1} & \frac{\partial y_3}{\partial x_2} & \frac{\partial y_3}{\partial x_3} \end{vmatrix} = \begin{vmatrix} -\frac{x_2x_3}{x_1^2} & \frac{x_3}{x_1} & \frac{x_2}{x_1} \\ \frac{x_3}{x_2} & -\frac{x_1x_3}{x_2^2} & \frac{x_1}{x_2} \\ \frac{x_2}{x_3} & \frac{x_1}{x_3} & -\frac{x_1x_2}{x_3^2} \end{vmatrix}$$

Taking $\frac{1}{x_1}$ from Row 1 , $\frac{1}{x_2}$ from Row 2 and $\frac{1}{x_3}$ from Row 3,we get

$$= \frac{1}{x_1x_2x_3}\begin{vmatrix} -\frac{x_2x_3}{x_1} & x_3 & x_2 \\ x_3 & -\frac{x_1x_3}{x_2} & x_1 \\ x_2 & x_1 & -\frac{x_1x_2}{x_3} \end{vmatrix}$$

$$= \frac{1}{x_1^2x_2^2x_3^2}\begin{vmatrix} -x_2x_3 & x_1x_3 & x_1x_2 \\ x_2x_3 & -x_1x_3 & x_1x_2 \\ x_2x_3 & x_1x_3 & -x_1x_2 \end{vmatrix} \text{ (multiply } R_1 \text{ by } x_1,\ R_2 \text{ by } x_2 \text{ and } R_3 \text{ by } x_3)$$

$$= \frac{x_1^2x_2^2x_3^2}{x_1^2x_2^2x_3^2}\begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix} = 1(1-1) - 1(-1-1) + 1(1+1) = 4$$

44. **If** $x = r\cos\theta$, $y = r\sin\theta$ **verify that** $\dfrac{\partial(x,y)}{\partial(r,\theta)} \text{X} \dfrac{\partial(r,\theta)}{\partial(x,y)} = 1$**.**

Given $x = r\cos\theta$, $y = r\sin\theta$

Then $\dfrac{\partial x}{\partial r} = \cos\theta$, $\dfrac{\partial x}{\partial\theta} = -r\sin\theta$, $\dfrac{\partial y}{\partial r} = \sin\theta$, $\dfrac{\partial y}{\partial\theta} = r\cos\theta$

Now $\dfrac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial\theta} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial v}{\partial\theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\left(\sin^2\theta + \cos^2\theta\right) = r(1) = r$

Now expressing r and θ in terms of x and y

$r^2 = x^2 + y^2 \Rightarrow r = \sqrt{x^2+y^2}$ and $\tan\theta = \dfrac{y}{x} \Rightarrow \theta = \tan^{-1}\left(\dfrac{y}{x}\right)$

$\dfrac{\partial r}{\partial x} = \dfrac{1}{2\sqrt{x^2+y^2}}(2x) = \dfrac{x}{\sqrt{x^2+y^2}} = \dfrac{x}{r}$; $\dfrac{\partial r}{\partial y} = \dfrac{1}{2\sqrt{x^2+y^2}}(2y) = \dfrac{y}{\sqrt{x^2+y^2}} = \dfrac{y}{r}$

$\dfrac{\partial\theta}{\partial x} = \dfrac{1}{1+\dfrac{y^2}{x^2}}\left(\dfrac{-y}{x^2}\right) = \dfrac{-y}{x^2+y^2} = \dfrac{-y}{r^2}$; $\dfrac{\partial\theta}{\partial x} = \dfrac{1}{1+\dfrac{y^2}{x^2}}\left(\dfrac{1}{x}\right) = \dfrac{x}{x^2+y^2} = \dfrac{x}{r^2}$

$$\frac{\partial(r,\theta)}{\partial(x,y)} = \begin{vmatrix} \dfrac{\partial r}{\partial x} & \dfrac{\partial r}{\partial y} \\ \dfrac{\partial\theta}{\partial x} & \dfrac{\partial\theta}{\partial y} \end{vmatrix} = \begin{vmatrix} \dfrac{x}{r} & \dfrac{y}{r} \\ \dfrac{-y}{r^2} & \dfrac{x}{r^2} \end{vmatrix} = \frac{1}{r^3}(x^2+y^2) = \frac{1}{r}$$

$\therefore \dfrac{\partial(x,y)}{\partial(r,\theta)} \text{X} \dfrac{\partial(r,\theta)}{\partial(x,y)} = r \,\text{X}\, \dfrac{1}{r} = 1$

45. **If** $x = e^r\sec\theta$, $y = e^r\tan\theta$ **verify that** $\dfrac{\partial(x,y)}{\partial(r,\theta)}\dfrac{\partial(r,\theta)}{\partial(x,y)} = 1$**.**

Given $x = e^r\sec\theta$, $y = e^r\tan\theta$, $x = r\cos\theta$, $y = r\sin\theta$

Then $\dfrac{\partial x}{\partial r} = e^r\sec\theta$, $\dfrac{\partial x}{\partial\theta} = e^r\sec\theta\tan\theta$, $\dfrac{\partial y}{\partial r} = e^r\tan\theta$, $\dfrac{\partial y}{\partial\theta} = e^r\sec^2\theta$

Now $\dfrac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial\theta} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial v}{\partial\theta} \end{vmatrix} = \begin{vmatrix} e^r\sec\theta & e^r\sec\theta\tan\theta \\ e^r\tan\theta & e^r\sec^2\theta \end{vmatrix} = e^{2r}\sec\theta\left(\sec^2\theta - \tan^2\theta\right)$

$= e^{2r}\sec\theta(1) = e^{2r}\sec\theta$

Now expressing r and θ in terms of x and y.

$x^2 - y^2 = e^{2r}(\sec^2\theta - \tan^2\theta) = e^{2r}$

$\Rightarrow$ r = $\frac{1}{2}$ log ($x^2 - y^2$)

Also $\frac{y}{x} = \frac{\tan\theta}{\sec\theta} = \sin\theta \Rightarrow \quad \theta = \sin^{-1}\left(\frac{y}{x}\right)$

$$\frac{\partial r}{\partial x} = \frac{1}{2(x^2-y^2)}(2x) = \frac{x}{(x^2-y^2)} =; \frac{\partial r}{\partial y} = \frac{1}{2(x^2-y^2)}(-2y) = \frac{-y}{(x^2-y^2)}$$

$$\frac{\partial \theta}{\partial x} = \frac{1}{\sqrt{1-\frac{y^2}{x^2}}}\left(\frac{-y}{x^2}\right) = \frac{-y}{x\sqrt{x^2-y^2}}; \quad \frac{\partial \theta}{\partial x} = \frac{1}{\sqrt{1-\frac{y^2}{x^2}}}\left(\frac{1}{x}\right) = \frac{1}{\sqrt{x^2-y^2}}$$

$$\frac{\partial(r,\theta)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{x}{(x^2-y^2)} & \frac{-y}{(x^2-y^2)} \\ \frac{-y}{x\sqrt{x^2-y^2}} & \frac{1}{\sqrt{x^2-y^2}} \end{vmatrix}$$

$$= \frac{x}{(x^2-y^2)^{3/2}} - \frac{y^2}{x(x^2-y^2)^{3/2}}$$

$$= \frac{x^2-y^2}{x(x^2-y^2)^{3/2}} = \frac{1}{x\sqrt{x^2-y^2}} = \frac{1}{e^r\sec\theta\sqrt{e^{2r}}} = \frac{1}{e^{2r}\sec\theta}$$

$$\therefore \frac{\partial(x,y)}{\partial(r,\theta)}\frac{\partial(r,\theta)}{\partial(x,y)} = e^{2r}\sec\theta\frac{1}{e^{2r}\sec\theta} = 1$$

46. **If** $u = 2xy, v = x^2 - y^2, x = r\cos\theta\ , y = r\sin\theta,$ compute $\frac{\partial(u,v)}{\partial(r,\theta)}$.

Solution : Given $u = 2xy, v = x^2 - y^2,$

$$\frac{\partial(u,u)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 2y & 2x \\ 2x & -2y \end{vmatrix} = -4y^2 - 4x^2 = -4(x^2+y^2)$$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial v}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\left(\sin^2\theta + \cos^2\theta\right) = r(1) = r$$

$$\therefore \frac{\partial(u,v)}{\partial(r,\theta)} = \frac{\partial(u,v)}{\partial(x,y)} \mathrm{X} \frac{\partial(x,y)}{\partial(r,\theta)} = -4r(x^2+y^2) = -4r^3 \quad (\sin ce\ x^2+y^2 = r^2)$$

47. **Prove that the functions** $u=\dfrac{x+y}{x-y}, v=\dfrac{xy}{(x-y)^2}$ **are functionally dependent.**

Solution: If u and v are functionally dependent, then their $\dfrac{\partial(u,u)}{\partial(x,y)}=0$.

Given $u=\dfrac{x+y}{x-y}, v=\dfrac{xy}{(x-y)^2}$

Then $\dfrac{\partial u}{\partial x}=\dfrac{(x-y)-(x+y)}{(x-y)^2}=\dfrac{-2y}{(x-y)^2}; \dfrac{\partial u}{\partial y}=\dfrac{(x-y)-(x+y)(-1)}{(x-y)^2}=\dfrac{2x}{(x-y)^2}$

$\dfrac{\partial v}{\partial x}=\dfrac{(x-y)^2-2xy(x-y)}{(x-y)^4}=\dfrac{-y(x+y)}{(x-y)^3}; \dfrac{\partial v}{\partial x}=\dfrac{(x-y)^2-2xy(x-y)(-1)}{(x-y)^4}=\dfrac{x(x+y)}{(x-y)^3}$

$$\frac{\partial(u,u)}{\partial(x,y)}=\begin{vmatrix}\dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y}\\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y}\end{vmatrix}=\begin{vmatrix}\dfrac{-2y}{(x-y)^2} & \dfrac{2x}{(x-y)^2}\\ \dfrac{-y(x+y)}{(x-y)^3} & \dfrac{x(x+y)}{(x-y)^3}\end{vmatrix}=-\frac{2xy(x+y)}{(x-y)^5}+\frac{2xy(x+y)}{(x-y)^5}=0$$

Therefore u and v are functionally dependent.
Also the relation between u and v is $u^2 - 4v = 1$.

48. **If** $u=xy+yz+zx, v=x^2+y^2+z^2, w=x+y+z$**, determine the functional relationship between u, v, w.**

Solution:

$u=xy+yz+zx \Rightarrow \dfrac{\partial u}{\partial x}=y+z,\ \dfrac{\partial u}{\partial y}=x+z,\ \dfrac{\partial u}{\partial z}=x+y$

$v=x^2+y^2+z^2 \Rightarrow \dfrac{\partial v}{\partial x}=2x,\ \dfrac{\partial v}{\partial y}=2y,\ \dfrac{\partial v}{\partial z}=2z$

$w=x+y+z,\ \Rightarrow \dfrac{\partial w}{\partial x}=1,\ \dfrac{\partial w}{\partial y}=1,\ \dfrac{\partial w}{\partial z}=1$

Hence, $\dfrac{\partial(u,v,w)}{\partial(x,y,z)}=\begin{vmatrix}y+z & x+z & x+y\\ 2x & 2y & 2z\\ 1 & 1 & 1\end{vmatrix}$

$$=2(y+z)(y-z)-2(x-z)2(x+z)+2(y+x)(y-x)=0$$

Therefore u , v and w are functionally dependent.
The relation is $w^2=(x+y+z)^2=x^2+y^2+z^2+2(xy+yz+zx)=v+2u.$

49. **If** $u=\sin^{-1}x+\sin^{-1}y, v=x\sqrt{1-y^2}+y\sqrt{1-x^2}$ **, determine the functional relationship between u and v.**

Solution : Given $u=\sin^{-1}x+\sin^{-1}y, v=x\sqrt{1-y^2}+y\sqrt{1-x^2}$

$$\frac{\partial u}{\partial x}=\frac{1}{\sqrt{1-x^2}}\ ;\ \frac{\partial u}{\partial y}=\frac{1}{\sqrt{1-y^2}}\ ;$$

$$\frac{\partial v}{\partial x}=\sqrt{1-y^2}+\frac{-xy}{\sqrt{1-x^2}};\frac{\partial v}{\partial x}=\sqrt{1-x^2}+\frac{-xy}{\sqrt{1-x^2}}$$

$$\frac{\partial(u,u)}{\partial(x,y)}=\begin{vmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}\end{vmatrix}=\begin{vmatrix}\frac{1}{\sqrt{1-x^2}} & \frac{1}{\sqrt{1-y^2}}\\ \sqrt{1-y^2}+\frac{-xy}{\sqrt{1-x^2}} & \sqrt{1-x^2}+\frac{-xy}{\sqrt{1-y^2}}\end{vmatrix}$$

$$=\left(1+\frac{-xy}{\left(\sqrt{1-y^2}\right)\left(\sqrt{1-x^2}\right)}\right)-\left(1-\frac{xy}{\left(\sqrt{1-y^2}\right)\left(\sqrt{1-x^2}\right)}\right)\ =0$$

Therefore u, v are functionally dependent.

Take $x=\sin\alpha, y=\sin\beta \Rightarrow \alpha=\sin^{-1}(x), \beta=\sin^{-1}(y)$

Now $u=\sin^{-1}x+\sin^{-1}y=\alpha+\beta$

$$v=x\sqrt{1-y^2}+y\sqrt{1-x^2}=\sin\alpha\sqrt{1-\sin^2\beta}+\sin\beta\sqrt{1-\sin^2\alpha}$$

$$=\sin\alpha\cos\beta+\sin\beta\cos\alpha=\sin(\alpha+\beta)\ =\sin u$$

The relation is v = sin u.

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