27. Evaluate $\int_C [3x^2 \, dx + (2xy - y) \, dy - z \, dz]$ from $t = 0$ to $t = 1$ along the curve $C$ given by $x = 2t^2, y = t, z = 4t^3$.
28. Evaluate $\int xy \, ds$ along the arc of the curve given by the equations $x = a \tan \theta$, $y = a \cot \theta$, $z = \sqrt{2}\, a \log \tan \theta$ from the point $\theta = \frac{\pi}{4}$ to the point $\theta = \frac{\pi}{3}$.
29. Evaluate $\int_C (xy + z^2) ds$, where $C$ is the arc of the helix $x = \cos t$, $y = \sin t$, $z = t$ from $(1, 0, 0)$ to $(-1, 0, \pi)$.
30. Find the area of that part of the plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ that lies in the positive octant. $\left[\textit{Hint}: \text{Area of the surface} = \iint_S ds\right]$
31. Evaluate $\iint_S z \, dS$, where $S$ is the positive octant of the surface of the sphere $x^2 + y^2 + z^2 = a^2$.
32. Evaluate $\iint xy \, dS$, where $S$ is the curved surface of the cylinder $x^2 + y^2 = a^2$, $0 \le z \le k$, included in the positive octant.
33. Find the volume of the tetrahedron bounded by the planes $x = 0, y = 0, z = 0$, $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$.
34. Evaluate $\iiint_V z \, dx \, dy \, dz$, where $V$ is the region of space bounded by the sphere $x^2 + y^2 + z^2 = a^2$ above the $xoy$-plane.
35. Evaluate $\iiint_V (x^2 + y^2) \, dx \, dy \, dz$, where $V$ is the region of space inside the cylinder $x^2 + y^2 = a^2$ that is bounded by the planes $z = 0$ and $z = h$.

## 5.9 GAMMA AND BETA FUNCTIONS

***Definitions*** The definite integral $\int_0^\infty e^{-x} x^{n-1} \, dx$ exists only when $n > 0$ and when it exists, it is a function of $n$ and called *Gamma function* and denoted by $\Gamma(n)$ [read as "Gamma $n$"].

Thus
$$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} \, dx$$

The definite integral $\int_0^1 x^{m-1} (1 - x)^{n-1} \, dx$ exists only when $m > 0$ and $n > 0$ and when it exists, it is a function of $m$ and $n$ and called *Beta function* and denoted by $\beta (m, n)$ [read as "Beta $m, n$"].

Thus
$$\beta(m, n) = \int_0^1 x^{m-1} (1 - x)^{n-1} dx:$$

*Note* ☑ $$\Gamma(1) = \int_0^\infty e^{-x}\, dx = (-e^{-x})_0^\infty = 1\,.$$

$$\beta(1,1) = \int_0^1 dx = 1\,.$$

## 5.9.1 Recurrence Formula for Gamma Function

$$\Gamma(n) = \int_0^\infty e^{-x}\, x^{n-1}\, dx$$

$$= -(x^{n-1}\, e^{-x})_0^\infty + \int_0^\infty (n-1)\, e^{-x}\, x^{n-2}\, dx \quad \text{[integrating by parts]}$$

$$= (n-1)\,\Gamma(n-1), \text{ since } \lim_{n\to\infty}\left(\frac{x^{n-1}}{e^x}\right) = 0$$

This recurrence formula $\Gamma(n) = (n-1)\,\Gamma\,(n-1)$ is valid only when $n > 1$, as $\Gamma(n-1)$ exists only when $n > 1$.

### Cor.

$\Gamma(n+1) = n\,!$, where $n$ is a positive integer.

$$\begin{aligned}\Gamma(n+1) &= n\,\Gamma(n)\\ &= n\,(n-1)\,\Gamma(n-1)\\ &= n\,(n-1)\,(n-2)\,\Gamma\,(n-2)\\ &= \ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ &= n\,(n-1)\,(n-2)\,\ldots\,3.2.1\,\Gamma(1)\\ &= n\,!\ (\because\ \Gamma(1) = 1)\end{aligned}$$

*Note* ☑ 1. $\Gamma(n)$ does not exist (i.e. $= \infty$), when $n$ is 0 or a negative integer.

2. When $n$ is a negative fraction, $\Gamma(n)$ is defined by using the recurrence formula. i.e. when $n < 0$, but not an integer,

$$\Gamma(n) = \frac{1}{n}\,\Gamma(n+1)$$

For example, $$\Gamma(-3.5) = \frac{1}{(-3.5)}\,\Gamma(-2.5)$$

$$= \frac{1}{(-3.5)}\cdot\frac{1}{(-2.5)}\,\Gamma(-1.5)$$

$$= \frac{1}{(3.5)\,(2.5)\,(-1.5)}\,\Gamma(-.5)$$

$$= \frac{\Gamma\,(0.5)}{(3.5)\,(2.5)\,(1.5)\,(0.5)}$$

The value of $\Gamma(0.5)$ can be obtained from the table of Gamma functions, though its value can be found out mathematically as given below.

**Value of** $\Gamma\left(\frac{1}{2}\right)$

By definition, $\Gamma\left(\frac{1}{2}\right) = \int_0^\infty e^{-t} t^{-\frac{1}{2}} \, \mathrm{d}t$

$$= \int_0^\infty e^{-x^2} \cdot \frac{1}{x} \cdot 2x \, \mathrm{d}x \qquad \text{(on putting } t = x^2\text{)}$$

$$= 2\int_0^\infty e^{-x^2} \, \mathrm{d}x$$

Now $\left(\Gamma\left(\frac{1}{2}\right)\right)^2 = 2\int_0^\infty e^{-x^2} \, \mathrm{d}x \cdot 2\int_0^\infty e^{-y^2} \, \mathrm{d}y$ [∵ the variable in a definite integral is only a dummy variable]

$$= 4\int_0^\infty \int_0^\infty e^{-(x^2 + y^2)} \, \mathrm{d}x \, \mathrm{d}y \qquad (1)$$

[∵ the product of two definite integrals can be expressed as a double integral, when the limits are constants].

Now the region of the double integral in (1) is given by $0 \le x < \infty$ and $0 \le y < \infty$, i.e. the entire first quadrant of the $xy$-plane.

Let us change over to polar co-ordinates through the transformations

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta.$$

Then $\mathrm{d}x\,\mathrm{d}y = |J|\,\mathrm{d}r\,\mathrm{d}\theta = r\,\mathrm{d}r\,\mathrm{d}\theta$

The region of the double integration is now given by $0 \le r < \infty$ and $0 \le \theta \le \frac{\pi}{2}$.

Then, from (1), we have

$$\left\{\Gamma\left(\frac{1}{2}\right)\right\}^2 = 4\int_0^{\pi/2}\int_0^\infty e^{-r^2} \, r\,\mathrm{d}r\,\mathrm{d}\theta$$

$$= 4\int_0^{\pi/2} \mathrm{d}\theta \left(-\frac{1}{2}e^{-r^2}\right)_0^\infty$$

$$= 2\int_0^{\pi/2} \mathrm{d}\theta$$

$$= \pi$$

$$\therefore \qquad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

## 5.9.2 Symmetry of Beta Function

$$\beta(m, n) = \beta(n, m)$$

By definition, $\beta(m, n) = \int_0^1 x^{m-1}(1 - x)^{n-1} \, \mathrm{d}x \qquad (1)$

Using the property $\int_0^a f(x)\,\mathrm{d}x = \int_0^a f(a - x)\,\mathrm{d}x$ in (1),

$$\beta(m, n) = \int_0^1 (1-x)^{m-1} \{1-(1-x)^{n-1}\, dx$$
$$= \int_0^1 x^{n-1}(1-x)^{m-1}\, dx$$
$$= \beta(n, m).$$

## 5.9.3 Trigonometric Form of Beta Function

By definition, $\beta(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1}\, dx$

Put $x = \sin^2 \theta$ $\quad\therefore\ dx = 2 \sin\theta \cos\theta\, d\theta$

The limits for $\theta$ are 0 and $\frac{\pi}{2}$.

$$\therefore \qquad \beta(m, n) = \int_0^{\pi/2} \sin^{2m-2}\theta \cdot \cos^{2n-2}\theta \cdot 2 \sin\theta \cos\theta\, d\theta$$
$$= 2\int_0^{\pi/2} \sin^{2m-1}\theta \cdot \cos^{2n-1}\theta\, d\theta$$

**Note** ☑ $\int_0^{\pi/2} \sin^{2m-1}\theta \cdot \cos^{2n-1}\theta\, d\theta = \frac{1}{2}\beta(m, n)$

The first argument of the Beta function is obtained by adding 1 to the exponent of $\sin\theta$ and dividing the sum by 2. The second argument is obtained by adding 1 to the exponent of $\cos\theta$ and dividing the sum by 2.

Thus
$$\int_0^{\pi/2} \sin^p\theta \cos^q\theta\, d\theta = \frac{1}{2}\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$$

## 5.9.4 Relation Between Gamma and Beta Functions

$$\beta(m, n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$$

Consider
$$\Gamma(m)\,\Gamma(n) = \int_0^{\infty} e^{-t}\, t^{m-1}\, dt \cdot \int_0^{\infty} e^{-s} \cdot s^{n-1}\, ds$$

In the first integral, put $t = x^2$ and in the second, put $s = y^2$.

$$\therefore \qquad \Gamma(m)\cdot\Gamma(n) = 2\int_0^{\infty} e^{-x^2}\, x^{2m-1}\, dx \cdot 2\int_0^{\infty} e^{-y^2}\, y^{2n-1}\, dy$$

$$= 4 \int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\, x^{2m-1} \cdot y^{2n-1}\, dx\, dy$$

$$= 4 \int_0^{\pi/2} \int_0^\infty e^{-r^2}\, (r\cos\theta)^{2m-1} \cdot (r\sin\theta)^{2n-1}\, r\, dr\, d\theta$$

[changing over to polar co-ordinates]

$$= 4 \int_0^{\pi/2} \cos^{2m-1}\theta \sin^{2n-1}\theta\, d\theta \cdot \int_0^\infty e^{-r^2}\, r^{2m+2n-2} \cdot r\, dr$$

$$= \beta(m, n) \int_0^\infty e^{-r^2}\, r^{2(m+n-1)} \cdot 2r\, dr$$

$$= \beta(m, n) \cdot \int_0^\infty e^{-u} \cdot u^{m+n-1}\, du \qquad \text{[putting } r^2 = u\text{]}$$

$$= \beta(m, n) \cdot \Gamma(m, n)$$

$$\therefore \qquad \beta(m, n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}.$$

**Cor.**

Putting $m = n = \frac{1}{2}$ in the above relation, $\beta\left(\frac{1}{2}, \frac{1}{2}\right) = \dfrac{\left\{\Gamma\left(\frac{1}{2}\right)\right\}^2}{\Gamma(1)}$

$$\therefore \qquad \left\{\Gamma\left(\frac{1}{2}\right)\right\}^2 = \beta\left(\frac{1}{2}, \frac{1}{2}\right)$$

$$= 2\int_0^{\pi/2} \sin^0\theta \cdot \cos^0\theta\, d\theta$$

$$= \pi$$

$$\therefore \qquad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

## WORKED EXAMPLE 5(d)

**Example 5.1** Prove that $\int_0^\infty e^{-ax}\, x^{n-1}\, dx = \dfrac{\Gamma(n)}{a^n}$, where $a$ and $n$ are positive.

Hence find the value of $\int_0^1 x^{q-1} \left[\log\left(\frac{1}{x}\right)\right]^{p-1} dx$.

In $\int_0^\infty e^{-ax}\, x^{n-1}\, dx$, put $ax = t$, so that $dx = \dfrac{dt}{a}$

$$\therefore \qquad \int_0^\infty e^{-ax}\, x^{n-1}\, dx = \int_0^\infty e^{-t} \left(\frac{t}{a}\right)^{n-1} \cdot \frac{dt}{a}$$

$$= \frac{1}{a^n} \int_0^\infty e^{-t}\, t^{n-1}\, dt$$

$$= \frac{1}{a^n}\, \Gamma(n) \qquad (1)$$

In $I = \int_0^1 x^{q-1} \log\left(\frac{1}{x}\right)^{p-1} dx$,

put $\frac{1}{x} = e^y$

i.e. $x = e^{-y}$

Then $dx = -\, e^{-y}\, dy$

Also the limits for $y$ are $\infty$ and 0.

$$\therefore \qquad I = \int_\infty^0 e^{-(q-1)y} \cdot y^{p-1} \cdot (-e^{-y})\, dy$$

$$= \int_0^\infty e^{-qy}\, y^{p-1}\, dy$$

$$= \frac{1}{q^p} \cdot \Gamma(p) \text{ [by (1)]} .$$

**Example 5.2** Prove that $\beta(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx$.

Hence deduce that $\beta(m,n) = \int_0^1 \frac{x^{m-1} + x^{n-1}}{(1+x)^{m+n}} dx$ .

By definition, $\beta(m,n) = \int_0^1 t^{m-1} (1-t)^{n-1}\, dt$ (1)

In (1), put $t = \frac{x}{1+x}$. Then $dt = \frac{1}{(1+x)^2} dx$ $\left(\because x = \frac{t}{1-t}\right)$

When $t = 0$, $x = 0$; when $t = 1$, $x = \infty$

Then (1) becomes,

$$\beta(m,n) = \int_0^\infty \left(\frac{x}{1+x}\right)^{m-1} \cdot \left(\frac{1}{1+x}\right)^{n-1} \cdot \frac{1}{(1+x)^2} dx$$

$$= \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx \qquad (2)$$

$$= \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}} dx + \int_1^\infty \frac{x^{m-1}}{(1+x)^{m+n}} \qquad (3)$$

In $\int\limits_1^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}}\,dx$, put $x = \frac{1}{y}$. Then $dx = -\frac{1}{y^2}\,dy$

When $x = 1$, $y = 1$; when $x = \infty$, $y = 0$

$$\therefore \qquad \int\limits_1^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}}\,dx = \int\limits_1^{0} \frac{\frac{1}{y^{(m-1)}}}{\left(1+\frac{1}{y}\right)^{m+n}} \cdot \left(-\frac{1}{y^2}\right) dy$$

$$= \int\limits_0^{1} \frac{y^{m+n}}{(1+y)^{m+n} \cdot y^{m+1}}\,dy$$

$$= \int\limits_0^{1} \frac{y^{n-1}}{(1+y)^{m+n}}\,dy$$

$$= \int\limits_0^{1} \frac{x^{n-1}}{(1+x)^{m+n}}\,dx$$

[changing the dummy variable] (4)

Using (4) in (3), we have

$$\beta(m,n) = \int\limits_0^{1} \frac{x^{m-1} + x^{n-1}}{(1+x)^{m+n}}\,dx.$$

**Example 5.3** Evaluate $\int\limits_0^{1} x^m (1-x^n)^p\,dx$ in terms of Gamma functions and hence find $\int\limits_0^{1} \frac{dx}{\sqrt{1-x^n}}$.

In $$I = \int\limits_0^{1} x^m (1-x^n)^p\,dx,$$

put $x^n = t$;

then $nx^{n-1}\,dx = dt$

$$\therefore \qquad dx = \frac{1}{n} \cdot \frac{dt}{t^{1-\frac{1}{n}}}$$

When $x = 0$, $t = 0$; when $x = 1$, $t = 1$.

$$I = \int\limits_0^{1} t^{\frac{m}{n}} (1-t)^p \cdot \frac{1}{n} \cdot t^{\frac{1}{n}-1}\,dt$$

$$= \frac{1}{n} \int\limits_0^{1} t^{\frac{m+1}{n}-1} \cdot (1-t)^p\,dt$$

$$= \frac{1}{n} \beta\left(\frac{m+1}{n}, p+1\right)$$

$$= \frac{1}{n} \frac{\Gamma\left(\frac{m+1}{n}\right) \cdot \Gamma(p+1)}{\Gamma\left(\frac{m+1}{n} + p + 1\right)} \tag{1}$$

$$\int_0^1 \frac{\mathrm{d}x}{\sqrt{1-x^n}} = \int_0^1 x^0 (1-x^n)^{-\frac{1}{2}}\, \mathrm{d}x$$

Here $m = 0$, $n = n$, $p = -\frac{1}{2}$.

Using (1); we have

$$\int_0^1 \frac{\mathrm{d}x}{\sqrt{1-x^n}} = \frac{1}{n} \frac{\Gamma\left(\frac{1}{n}\right) \cdot \Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{n} + \frac{1}{2}\right)}$$

$$= \frac{\sqrt{\pi}}{n} \cdot \frac{\Gamma\left(\frac{1}{n}\right)}{\Gamma\left(\frac{1}{n} + \frac{1}{2}\right)}$$

**Example 5.4** Prove that $\beta(n,n) = \dfrac{\sqrt{\pi}\, \Gamma(n)}{2^{2n-1} \Gamma\left(n + \frac{1}{2}\right)}$

(or) $$\beta(n,n) = \frac{1}{2^{2n-1}} \cdot \beta\left(n, \frac{1}{2}\right)$$

$$\beta(n,n) = 2 \int_0^{\pi/2} \sin^{2n-1} \theta \cdot \cos^{2n-1} \theta\, \mathrm{d}\theta \quad \text{[using trigonometric form]}$$

$$= 2 \int_0^{\pi/2} (\sin \theta \cos \theta)^{2n-1} \mathrm{d}\theta$$

$$= 2 \int_0^{\pi/2} \left(\frac{\sin 2\theta}{2}\right)^{2n-1} \mathrm{d}\theta$$

$$= \frac{1}{2^{2n-2}} \int_0^{\pi/2} \sin^{2n-1} 2\theta\, \mathrm{d}\theta$$

$$= \frac{1}{2^{2n-2}} \int_0^{\pi} \sin^{2n-1} \phi \frac{\mathrm{d}\phi}{2}, \text{ putting } 2\theta = \phi$$

$$= \frac{1}{2^{2n-2}} \int_0^{\pi/2} \sin^{2n-1}\phi \, d\phi \quad \left[\because \int_0^{\pi} f(\sin\phi)\, d\phi = 2\int_0^{\pi/2} f(\sin\phi)\, d\phi\right]$$

$$= \frac{1}{2^{2n-2}} \cdot \frac{1}{2}\beta\left(n, \frac{1}{2}\right)$$

$$= \frac{1}{2^{2n-1}}\beta\left(n, \frac{1}{2}\right)$$

$$= \frac{1}{2^{2n-1}} \cdot \frac{\Gamma(n)\cdot\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(n+\frac{1}{2}\right)}$$

$$= \frac{\sqrt{\pi}\cdot\Gamma(n)}{2^{2n-1}\cdot\Gamma\left(n+\frac{1}{2}\right)}$$

**Example 5.5** Show that $\displaystyle\int_0^{\infty} x^n e^{-a^2x^2}\, dx = \frac{1}{2a^{n+1}}\Gamma\left(\frac{n+1}{2}\right).$

Deduce that $\displaystyle\int_0^{\infty} e^{-a^2x^2}\, dx = \frac{\sqrt{\pi}}{2a}$. Hence show that

$$\int_0^{\infty} \cos(x^2)\, dx = \int_0^{\infty} \sin(x^2)\, dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}$$

In $\displaystyle I = \int_0^{\infty} x^n e^{-a^2x^2}\, dx$, put $ax = \sqrt{t}$; then $dx = \dfrac{dt}{2a\sqrt{t}}$

When $x = 0$, $t = 0$; when $x = \infty$, $t = \infty$.

$$\therefore \qquad I = \int_0^{\infty} \left(\frac{\sqrt{t}}{a}\right)^n e^{-t} \frac{dt}{2a\sqrt{t}}$$

$$= \frac{1}{2a^{n+1}}\int_0^{\infty} t^{\frac{n-1}{2}}\cdot e^{-t}\, dt$$

$$= \frac{1}{2a^{n+1}}\Gamma\left(\frac{n+1}{2}\right) \qquad (1)$$

In (1), put $n = 0$.

$$\text{Then} \qquad \int_0^{\infty} e^{-a^2x^2}\, dx = \frac{\Gamma\left(\frac{1}{2}\right)}{2a} = \frac{\sqrt{\pi}}{2a} \qquad (2)$$

In (2), put $a = \dfrac{1-i}{\sqrt{2}}$; then $a^2 = -i$

$$\therefore \qquad \int_0^\infty e^{ix^2}\,dx = \frac{\sqrt{\pi}}{\sqrt{2}(1-i)}$$

$$= \frac{\sqrt{\pi}}{2\sqrt{2}}(1+i)$$

Equating the real parts on both sides,

$$\int_0^\infty \cos(x^2)\,dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}\,.$$

Equating the imaginary parts on both sides,

$$\int_0^\infty \sin(x^2)\,dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}\,.$$

**Example 5.6** Evaluate

(i) $\int_a^b (x-a)^{m-1}(b-x)^{n-1}\,dx$ and

(ii) $\int_{-a}^a (a+x)^{m-1}\cdot(a-x)^{n-1}\,dx$ in terms of Beta function.

(i) In $I_1 = \int_a^b (x-a)^{m-1}(b-x)^{n-1}\,dx$,

put $x - a = y$; then $dx = dy$

When $x = a$, $y = 0$; when $x = b$, $y = b - a$

$$\therefore \qquad I_1 = \int_0^{b-a} y^{m-1}\{(b-a)-y\}^{n-1}\,dy$$

$$= (b-a)^{n-1}\int_0^{b-a} y^{m-1}\left(1-\frac{y}{b-a}\right)^{n-1}dy \qquad (1)$$

In (1), put $\frac{y}{b-a} = t$ ; then $dy = (b - a)\,dt$

When $y = 0$, $t = 0$; when $y = b - a$, $t = 1$.

$$\therefore \qquad I_1 = (b-a)^{m+n-1}\int_0^1 t^{m-1}(1-t)^{n-1}\,dt$$

$$= (b-a)^{m+n-1}\,\beta(m,n)$$

(ii) In $I_2 = \int_{-a}^a (a+x)^{m-1}(a-x)^{n-1}\,dx$,

put $a + x = y$; then $dx = dy$

When $x = -a$, $y = 0$; when $x = a$, $y = 2a$.

$$\therefore \qquad I_2 = \int_0^{2a} y^{m-1}(2a-y)^{n-1}\,dy$$

$$= (2a)^{n-1} \int_0^{2a} y^{m-1} \left(1 - \frac{y}{2a}\right)^{n-1} dy \qquad (2)$$

In (2), put $\frac{y}{2a} = t$; then $dy = 2a\, dt$.

When $y = 0$, $t = 0$; when $y = 2a$, $t = 1$.

$$\therefore \qquad I_2 = (2a)^{m+n-1} \cdot \int_0^1 t^{m-1} (1-t)^{n-1}\, dt \cdot$$

$$= (2a)^{m+n-1} \beta\,(m,n)$$

**Example 5.7** Prove that $\displaystyle\int_0^{\infty} \frac{e^{-x^2}}{\sqrt{x}} dx \times \int_0^{\infty} x^2 e^{-x^4} dx = \frac{\pi}{4\sqrt{2}}$

In $I_1 = \displaystyle\int_0^{\infty} \frac{e^{-x^2}}{\sqrt{x}} dx$, put $x^2 = t$; then $dx = \dfrac{dt}{2x} = \dfrac{dt}{2\sqrt{t}}$

When $x = 0$, $t = 0$; when $x = \infty$, $t = \infty$

$$\therefore \qquad I_1 = \int_0^{\infty} \frac{e^{-t}}{t^{1/4}} \cdot \frac{dt}{2\sqrt{t}} = \frac{1}{2} \int_0^{\infty} e^{-t} \cdot t^{-\frac{3}{4}}\, dt$$

$$= \frac{1}{2} \Gamma\left(\frac{1}{4}\right)$$

In $I_2 = \displaystyle\int_0^{\infty} x^2\, e^{-x^4} dx$, put $x^4 = s$; then $dx = \dfrac{ds}{4x^3} = \dfrac{ds}{4s^{3/4}}$

When $x = 0$, $s = 0$; when $x = \infty$, $s = \infty$.

$$\therefore \qquad I_2 = \int_0^{\infty} \sqrt{s}\, e^{-s} \cdot \frac{ds}{4s^{3/4}} = \frac{1}{4} \int_0^{\infty} s^{-\frac{1}{4}}\, e^{-s} ds$$

$$= \frac{1}{4} \Gamma\left(\frac{3}{4}\right)$$

$$\therefore \qquad \int_0^{\infty} \frac{e^{-x^2}}{\sqrt{x}} dx \times \int_0^{\infty} x^2 e^{-x^4}\, dx = \frac{1}{8} \Gamma\left(\frac{1}{4}\right) \Gamma\left(\frac{3}{4}\right) \qquad (1)$$

From Example 5.4;

$$\beta\,(n, n) = \frac{1}{2^{\,2n-1}} \cdot \beta\left(n, \frac{1}{2}\right)$$

i.e.

$$\frac{\Gamma(n)\, \Gamma(n)}{\Gamma(2n)} = \frac{1}{2^{\,2n-1}} \cdot \frac{\Gamma(n) \cdot \sqrt{\pi}}{\Gamma\left(n + \frac{1}{2}\right)}$$

$$\therefore \qquad \Gamma(n) \cdot \Gamma\left(n + \frac{1}{2}\right) = \frac{\sqrt{\pi}\, \Gamma(2n)}{2^{2n-1}}$$

Putting $n=\frac{1}{4}$, we get

$$\Gamma\left(\frac{1}{4}\right)\cdot\Gamma\left(\frac{3}{4}\right)=\frac{\sqrt{\pi}\cdot\Gamma\left(\frac{1}{2}\right)}{2^{-\frac{1}{2}}}$$
$$=\pi\sqrt{2} \tag{2}$$

Using (2) in (1);

$$\int_0^\infty \frac{e^{-x^2}}{\sqrt{x}}\,dx\times\int_0^\infty x^2e^{-x^4}\,dx=\frac{\pi\sqrt{2}}{8}=\frac{\pi}{4\sqrt{2}}.$$

**Example 5.8** Evaluate $\int_0^\infty \frac{x^{m-1}}{(1+x^n)^p}\,dx$ and deduce that $\int_0^\infty \frac{x^{m-1}}{1+x^n}\,dx$ $=\frac{\pi}{n\sin\left(\frac{m\pi}{n}\right)}$. Hence show that $\int_0^\infty \frac{dx}{1+x^4}=\frac{\pi}{2\sqrt{2}}$.

In $I=\int_0^\infty \frac{x^{m-1}}{(1+x^n)^p}\,dx$, put $t=\frac{1}{1+x^n}$

Then $x^n=\frac{1-t}{t}$ $\quad\therefore\quad nx^{n-1}\,dx=-\frac{1}{t^2}\,dt$

When $x = 0$, $t = 1$; when $x = \infty$, $t = 0$

$$\therefore\qquad I=\int_0^1 \frac{t^{-\left(\frac{m-1}{n}\right)}\cdot(1-t)^{\frac{m-1}{n}}}{t^{-p}}\cdot\frac{dt}{nt^2\cdot t^{-\frac{(n-1)}{n}}(1-t)^{\frac{n-1}{n}}}$$
$$=\frac{1}{n}\int_0^1 t^{p-\frac{m}{n}-1}\cdot(1-t)^{\frac{m}{n}-1}\,dt$$
$$=\frac{1}{n}\beta\left(p-\frac{m}{n},\frac{m}{n}\right)$$
$$=\frac{1}{n}\frac{\Gamma\left(p-\frac{m}{n}\right)\cdot\Gamma\left(\frac{m}{n}\right)}{\Gamma(p)} \tag{1}$$

Putting $p = 1$ in (1), we get

$$\int_0^\infty \frac{x^{m-1}\,dx}{1+x^n}=\frac{1}{n}\Gamma\left(1-\frac{m}{n}\right)\Gamma\left(\frac{m}{n}\right)$$
$$=\frac{\pi}{n}\operatorname{cosec}\left(\frac{\pi m}{n}\right)\qquad\left[Hint: \text{Use}\,\Gamma(\alpha)\Gamma(1-\alpha)=\frac{\pi}{\sin\alpha\pi}\right] \tag{2}$$

Taking $m = 1$ and $n = 4$ in (2), we get

$$\int_0^\infty \frac{dx}{1+x^4} = \frac{\pi}{4}\cdot \operatorname{cosec}\left(\frac{\pi}{4}\right)$$

$$= \frac{\pi}{2\sqrt{2}}$$

**Example 5.9** Find the value of $\iint x^{m-1} y^{n-1}\, dx\, dy$, over the positive quadrant of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, in terms of Gamma functions.

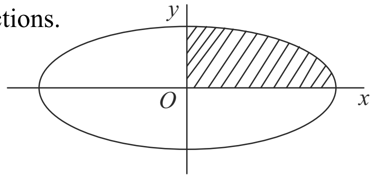


Fig. 5.53

Put $\frac{x}{a} = \sqrt{X}$ and $\frac{y}{b} = \sqrt{Y}$

Then $dx = \frac{a}{2\sqrt{X}}\, dX$ and $dy = \frac{b}{2\sqrt{Y}}\, dY$.

The region of double integration in the $xy$-plane is given by $x \geq 0$, $y \geq 0$ and

$\frac{x^2}{a^2}+\frac{y^2}{b^2} \leq 1$, shown in Fig. 5.53.

$\therefore$ The region of integration in the $XY$-plane is given by $X \geq 0$, $Y \geq 0$ and $X + Y \leq 1$, shown in Fig. 5.54.

The given integral

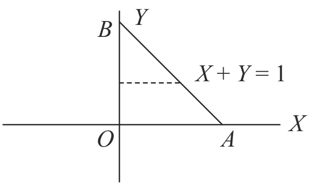


Fig. 5.54

$$I = \iint_{\Delta OAB} \left(a\sqrt{X}\right)^{m-1}\cdot\left(b\sqrt{Y}\right)^{n-1} \frac{ab}{4\sqrt{X}\sqrt{Y}}\, dX\, dY$$

$$= \frac{a^m b^n}{4} \iint_{\Delta OAB} X^{\frac{m}{2}-1}\, Y^{\frac{n}{2}-1}\, dX\, dY$$

$$= \frac{a^m b^n}{4} \int_0^1 \int_0^{1-Y} X^{\frac{m}{2}-1}\, Y^{\frac{n}{2}-1}\, dX\, dY$$

$$I = \frac{a^m b^n}{4} \int_0^1 Y^{\frac{n}{2}-1}\cdot\frac{2}{m}\left(X^{m/2}\right)_0^{1-Y} dY$$

$$= \frac{a^m b^n}{2m} \int_0^1 Y^{\frac{n}{2}-1}\cdot(1-Y)^{m/2}\, dY$$

$$= \frac{a^m b^n}{2m}\,\beta\cdot\left(\frac{n}{2}, \frac{m}{2}+1\right)$$

$$= \frac{a^m b^n}{2m}\cdot\frac{\Gamma\left(\frac{n}{2}\right)\cdot\Gamma\left(\frac{m}{2}+1\right)}{\Gamma\left(\frac{m}{2}+\frac{n}{2}+1\right)}$$