B.Tech/M.Tech (Integrated) Degree Examination, January 2023

21MAB101T - Calculus and Linear Algebra

Answer key

Part - A (20×1=20M)

1) (B) Index
2) (A) 1, 4, 25
3) (B) ±1
4) (D) 1, 10
5) (A) $4x+5y$
6) (A) $-\dfrac{(\partial f/\partial x)}{(\partial f/\partial y)}$
7) (C) $2\pi+e$
8) (A) zero
9) (C) $y=A\cos 2x+B\sin 2x$
10) (A) xe^{ax}
11) (C) $y=Ae^{-x}+Be^{-4x}$
12) (D) $-\frac{1}{8}\cos 2x$
13) (C) 0
14) (B) $B^2-4AC=0$
15) (A) $\rho=(r^2+r_1^2)^{3/2}/r^2-rr''+2r_1^2$
16) (B) $(x-\bar{x})^2+(y-\bar{y})^2=\rho^2$
17) (B) $p>1$
18) (C) Leibnitz test
19) (B) 1
20) (B) Absolutely Convergent

Part - B (5×8=40M)

21) a)

$$A=\begin{bmatrix}10 & -2 & -5\\ -2 & 2 & 3\\ -5 & 3 & 5\end{bmatrix}$$

$S_1=17,\ S_2=42,\ S_3=0$

$\lambda^3-17\lambda^2+42\lambda=0 \rightarrow 1M$

$\lambda=0, 3, 14 \rightarrow 1M$

Case (i): $\lambda=0$

$$\begin{bmatrix}10 & -2 & -5\\ -2 & 2 & 3\\ -5 & 3 & 5\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$X_1=\begin{bmatrix}4\\-20\\16\end{bmatrix}$ or $\begin{bmatrix}1\\-5\\4\end{bmatrix} \rightarrow 2M$

Case (ii): $\lambda=3$

$$\begin{bmatrix}7 & -2 & -5\\ -2 & -1 & 3\\ -5 & 3 & 2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$X_2=\begin{bmatrix}-11\\-11\\-11\end{bmatrix}$ or $\begin{bmatrix}1\\1\\1\end{bmatrix} \rightarrow 2M$

Case (iii): $\lambda=14$

$$\begin{bmatrix}-4 & -2 & -5\\ -2 & -12 & 3\\ -5 & 3 & -9\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$X_3=\begin{bmatrix}-66\\22\\44\end{bmatrix}$ or $\begin{bmatrix}-3\\1\\2\end{bmatrix} \rightarrow 2M$

(OR)

21) b)

$$A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} \rightarrow 2M \qquad A^3 = \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix} \rightarrow 2M$$

$S_1 = 6, \quad S_2 = 9, \quad S_3 = 4$

$\therefore \lambda^3 - 6\lambda^2 + 9\lambda - 4 = 0 \rightarrow 1M$

To Verify : $A^3 - 6A^2 + 9A - 4I = 0$

$$A^3 - 6A^2 + 9A - 4I = \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix} - 6\begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} + 9\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} - 4\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow 3M$$

22) a)

$$\left.\begin{aligned} f(x,y) &= e^x \cos y \\ f_x(x,y) &= e^x \cos y \\ f_y(x,y) &= -e^x \sin y \\ f_{xx}(x,y) &= e^x \cos y \\ f_{xy}(x,y) &= -e^x \sin y \\ f_{yy}(x,y) &= -e^x \cos y \end{aligned}\right\} \rightarrow 2M \qquad \left.\begin{aligned} f(1, \pi/4) &= \frac{e}{\sqrt{2}} \\ f_x(1, \pi/4) &= e/\sqrt{2} \\ f_y(1, \pi/4) &= -e/\sqrt{2} \\ f_{xx}(1, \pi/4) &= \frac{e}{\sqrt{2}} \\ f_{xy}(1, \pi/4) &= -e/\sqrt{2} \\ f_{yy}(1, \pi/4) &= -\frac{e}{\sqrt{2}} \end{aligned}\right\} \rightarrow 2M$$

$$f(x,y) = f(a,b) + \frac{1}{1!}\left[(x-a)f_x(a,b) + (y-b)f_y(a,b)\right]$$
$$+ \frac{1}{2!}\left[(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b)f_{xy}(a,b) + (y-b)^2 f_{yy}(a,b)\right] \rightarrow 2M$$

$$= \frac{e}{\sqrt{2}}\left\{1 + (x-1) - (y-\pi/4) + \frac{1}{2!}\left[(x-1)^2 - 2(x-1)(y-\pi/4) - (y-\tfrac{\pi}{4})^2\right]\right\} \rightarrow 2M$$

(OR)

22) b)

$x = e^r \cos\theta, \quad y = e^r \sin\theta$

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial r} \rightarrow 1M$$

$$= \frac{\partial u}{\partial x}(e^r\cos\theta) + \frac{\partial u}{\partial y}(e^r\sin\theta) \rightarrow 2M$$

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial \theta} \rightarrow 1M$$

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}(-e^r \sin\theta) + \frac{\partial u}{\partial y}(e^r \cos\theta) \longrightarrow (2) \rightarrow 2M$$

Adding (1) and (2), we get

$$\left.\begin{aligned} &\left(\frac{\partial u}{\partial r}\right)^2 + \left(\frac{\partial u}{\partial \theta}\right)^2 = \left(\frac{\partial u}{\partial x}\right)^2 e^{2r} + \left(\frac{\partial u}{\partial y}\right)^2 e^{2r} \\ &\Rightarrow \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = e^{-2r}\left[\left(\frac{\partial u}{\partial r}\right)^2 + \left(\frac{\partial u}{\partial \theta}\right)^2\right] \end{aligned}\right\} \rightarrow 2M$$

23) a) $(D^2+2D+1)y = e^{3x} + \sin 2x$

The auxiliary equation is $m^2+2m+1=0 \rightarrow 1M$

$$\Rightarrow m = -1, -1 \rightarrow 1M$$

$$CF = (A+Bx)e^{-x} \rightarrow 1M$$

$$PI_1 = \frac{1}{D^2+2D+1} e^{3x} = \frac{1}{16} e^{3x} \rightarrow 2M$$

$$PI_2 = \frac{1}{D^2+2D+1} \sin 2x = \frac{1}{2D-3} \sin 2x$$

$$= \frac{4\cos 2x + 3\sin 2x}{-25} \rightarrow 2M$$

$$y = (A+Bx)e^{-x} + \frac{1}{16}e^{3x} - \frac{1}{25}[4\cos 2x + 3\sin 2x] \rightarrow 1M$$

(OR)

23) b) The auxiliary equation is $m^2+1=0$

$$\Rightarrow m = \pm i \rightarrow 1M$$

$CF = C_1 f_1 + C_2 f_2$ where $f_1 = \cos x$, $f_2 = \sin x \rightarrow 1M$

$f_1 f_2' - f_2 f_1' = 1$, $PI = Pf_1 + Qf_2$

$$\left.\begin{aligned} P &= -\int \frac{f_2 F(x)\,dx}{f_1 f_2' - f_2 f_1'} = -\int \sin x \tan x\,dx = -\int \frac{\sin^2 x}{\cos x}dx \\ &= -\int(\sec x - \cos x)\,dx = -\log(\sec x + \tan x) + \sin x \end{aligned}\right\} \rightarrow 3M$$

$$Q = \int \frac{f_1 F(x)\,dx}{f_1 f_2' - f_1' f_2} = \int \cos x \tan x\,dx = -\cos x \rightarrow 2M$$

$$y = C_1\cos x + C_2 \cos x - \log(\sec x + \tan x)\cos x \rightarrow 1M$$

24) a)

$$r = a + a\cos\theta \qquad (r)_{\pi/2} = a$$
$$r_1 = -a\sin\theta \rightarrow 1M \qquad (r_1)_{\pi/2} = -a$$
$$r_2 = -a\cos\theta \rightarrow 1M \qquad (r_2)_{\pi/2} = 0$$
$\rightarrow 2M$

$$\rho = \frac{(r^2 + r_1^2)^{3/2}}{r^2 + 2r_1^2 - r r_2} \rightarrow 1M$$

$$= \frac{(a^2+a^2)^{3/2}}{a^2+2a^2} = \frac{(2a^2)^{3/2}}{3a^2} = \frac{2\sqrt{2}a}{3} \rightarrow 3M$$

(OR)

24) b)

$$x = at^2, \quad y = 2at \qquad \rightarrow 1M$$

$$\frac{dx}{dt} = 2at, \qquad \frac{dy}{dt} = 2a$$

$$y_1 = \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{1}{t} \rightarrow 1M$$

$$y_2 = \frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dx}\right)\frac{dt}{dx} = \frac{d}{dt}\left(\frac{1}{t}\right)\cdot\frac{1}{2at} = \frac{-1}{2at^3} \rightarrow 1M$$

$$\bar{x} = x - \frac{y_1(1+y_1^2)}{y_2} = at^2 - \frac{(1/t)(1+1/t^2)}{-1/2at^3} = 3at^2 + 2a \rightarrow 2M$$

$$\Rightarrow \quad t^2 = \frac{(\bar{x} - 2a)}{3a} \longrightarrow (1)$$

$$\bar{y} = y + \frac{(1+y_1^2)}{y_2} = 2at + \frac{(1+1/t^2)}{-1/2at^3} = -2at^3 \rightarrow 2M$$

$$\Rightarrow \quad t^3 = -\frac{\bar{y}}{2a} \longrightarrow (2)$$

From (1) and (2), we get $\frac{(\bar{x}-2a)^3}{27a^3} = \frac{\bar{y}^2}{4a^2}$

$\therefore$ Evolute is $27ay^2 = 4(x-2a)^3 \rightarrow 1M$

25) a)

$$u_n = \sqrt{n^4+1} - \sqrt{n^4-1}$$

$$= \frac{(\sqrt{n^4+1} - \sqrt{n^4-1})(\sqrt{n^4+1} + \sqrt{n^4-1})}{(\sqrt{n^4+1} + \sqrt{n^4-1})} = \frac{2}{\sqrt{n^4+1} + \sqrt{n^4-1}} \rightarrow 2M$$

Take $V_n = \frac{1}{n^2} \rightarrow 1M$

$$\frac{u_n}{v_n} = \frac{2n^2}{\sqrt{n^4+1}+\sqrt{n^4-1}} = \frac{1}{\sqrt{1+\frac{1}{n^4}}+\sqrt{1-\frac{1}{n^4}}}$$

$$\lim_{n\to\infty} \frac{u_n}{v_n} = 1 \neq 0 \quad \rightarrow 4M$$

$\therefore \Sigma u_n$ is convergent since Σv_n is convergent. $\rightarrow 1M$

(OR)

25) b) Let $u_n = \frac{2\cdot 4\cdot 6\cdot 8 \cdots (2n)}{3\cdot 5\cdot 7 \cdots (2n+1)} \cdot \frac{1}{(2n+2)}$ $\rightarrow 1M$

$$u_{n+1} = \frac{2\cdot 4\cdot 6\cdot 8 \cdots (2n)(2n+2)}{3\cdot 5\cdot 7 \cdots (2n+1)(2n+3)} \cdot \frac{1}{(2n+4)} \rightarrow 1M$$

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \frac{(2n+2)}{(2n+3)} \cdot \frac{(2n+2)}{(2n+4)} = 1 \rightarrow 1M$$

Hence Ratio test fails.

Using Rabee's test, $n\left[\frac{u_n}{u_{n+1}} - 1\right] = n\left[\frac{6n+8}{4n^2+8n+4}\right]$ $\rightarrow 1M$

$$\lim_{n\to\infty}\left\{n\left[\frac{u_n}{u_{n+1}}-1\right]\right\} = \frac{3}{2} > 1 \rightarrow 3M$$

$\therefore \Sigma u_n$ is convergent $\rightarrow 1M$

Part-c $(1\times 15 = 15M)$

26) $A = \begin{bmatrix} 3 & -2 & 4 \\ -2 & -2 & 6 \\ 4 & 6 & -1 \end{bmatrix}$ $\rightarrow 1M$

$S_1 = 0, \quad S_2 = -63, \quad S_3 = -162$

$\lambda^3 - 63\lambda + 162 = 0 \rightarrow 1M$

$\therefore \lambda = 3, 6, -9 \rightarrow 2M$

Case (i): $\lambda = 3$

$$\begin{bmatrix} 0 & -2 & 4 \\ -2 & -5 & 6 \\ 4 & 6 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$X_1 = \begin{bmatrix} 8 \\ -8 \\ -4 \end{bmatrix} \text{ or } \begin{bmatrix} 2 \\ -2 \\ -1 \end{bmatrix} \rightarrow 2M$$

Case (ii): $\lambda = 6$

$$\begin{bmatrix} -3 & -2 & 4 \\ -2 & -8 & 6 \\ 4 & 6 & -7 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$X_2 = \begin{bmatrix} 20 \\ 10 \\ 20 \end{bmatrix} \text{ or } \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \rightarrow 2M$$

Case (iii): $\lambda = -9$

$$\begin{bmatrix} 12 & -2 & 4 \\ -2 & 7 & 6 \\ 4 & 6 & 8 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$X_3 = \begin{bmatrix} -40 \\ -80 \\ 80 \end{bmatrix} \text{ or } \begin{bmatrix} -1 \\ -2 \\ 2 \end{bmatrix} \rightarrow 2M$$

$$M = \begin{bmatrix} 2 & 2 & -1 \\ -2 & 1 & -2 \\ -1 & 2 & 2 \end{bmatrix}, \quad N = \begin{bmatrix} 2/3 & 2/3 & -1/3 \\ -2/3 & 1/3 & -2/3 \\ -1/3 & 2/3 & 2/3 \end{bmatrix} \rightarrow 1M \quad N^T = \begin{bmatrix} 2/3 & -2/3 & -1/3 \\ 2/3 & 1/3 & 2/3 \\ -1/3 & -2/3 & 2/3 \end{bmatrix}$$

$$AN = \begin{pmatrix} 2 & 4 & 3 \\ -2 & 2 & 6 \\ -1 & 4 & -6 \end{pmatrix} \text{ or } N^T A = \begin{pmatrix} 2 & -2 & -1 \\ 4 & 2 & 4 \\ 3 & 6 & -6 \end{pmatrix} \rightarrow 1M$$

$$D = N^T A N = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & -9 \end{pmatrix} \rightarrow 1M$$

Canonical form $= YDY' \quad \because X = NY$

$$= 3y_1^2 + 6y_2^2 - 9y_3^2 \rightarrow 1M$$

Nature = indefinite
Rank = 3
index = 2
signature = 1
$\rightarrow 1M$

27) $f(x, y, z) = xyz \rightarrow 1M$, $\phi(x, y, z) = xy + 2yz + 2zx - 432 \rightarrow$ (I) $\rightarrow 1M$

$F(x, y, z) = f(x, y, z) + \lambda g(x, y, z) = xyz + \lambda(xy + 2yz + 2zx - 432)$

$\frac{\partial F}{\partial x} = 0 \Rightarrow yz + \lambda(y + 2z) = 0 \rightarrow$ (1)

$\frac{\partial F}{\partial y} = 0 \Rightarrow xz + \lambda(x + 2z) = 0 \rightarrow$ (2)

$\frac{\partial F}{\partial z} = 0 \Rightarrow xy + \lambda(2y + 2x) = 0 \rightarrow$ (3)

$\rightarrow 5M$

From (1) & (2), $\Rightarrow x = y \rightarrow 2M$

From (2) & (3) $\Rightarrow y = 2z \rightarrow 2M$

Sub' $x = y$ and $y = 2z$ in (I), we get

$$x^2 = 144 \Rightarrow x = \pm 12 \rightarrow 2M$$

$\therefore y = 12, \; z = 6 \rightarrow 2M$

Prepared & Verified by

10/1/2023