Module – 5

Sequences – Definition and Examples – Series - Types of convergence – Series of positive terms – Test of convergence – Comparison test – Integral test - D'Alemberts Ratio test, Raabe's root test – Convergent of Exponential Series – Cauchy's Root test – Log test – Alternating Series: Leibnitz test – Series of positive and Negative terms – Absolute Convergence – Conditional Convergence – Applications Convergence of series in Engineering

SEQUENCES

A *sequence* is defined as an arrangement of numbers in a definite order. A *sequence* is a set of numbers a_1, a_2, a_3, …, a_n,… such that to each positive integer n, there corresponds a number a_n of the set. It is denoted by $\{a_n\}$. Thus, a sequence and the set of natural numbers N have one to one correspondence.

Examples:

1. If $a_n = \frac{1}{n}$, the sequence is $1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots$

2. If $a_n = n^3$, the sequence is $1^3, 2^3, ,\ldots, n^3, \ldots$

3. If $a_n = k$, the sequence is $k, k, k, \ldots, k, \ldots$

4. If $a_n = (-1)^n$, the sequence is $-1, 1, -1, 1, \ldots, (-1)^n, \ldots$

Limit of a sequence (or) Convergence of a sequence

Let $\{a_n\}$ be a sequence and l, a real number. $\{a_n\}$ is said to *converge* to a limit l, if given any positive number ε, there exists a positive integer N such that $|a_n - l| < \varepsilon$ for all $n \geq N$.

We write $\lim\limits_{n\to\infty} a_n = l$.

Note:

$|a_n - l| < \varepsilon$ means $-\varepsilon < a_n - l < \varepsilon$, and hence $l - \varepsilon < a_n < l + \varepsilon$.

If $\lim\limits_{n\to\infty} a_n = l$ is finite and unique, then the sequence is said to be convergent.

If $\lim\limits_{n\to\infty} a_n$ is infinite, then the sequence is said to be divergent.

Bounded above sequence

A sequence $\{a_n\}$ is said to be *bounded above*, if there exists a real number M such that $a_n \leq M$ for all n.

Bounded below sequence

A sequence $\{a_n\}$ is said to be *bounded below*, if there exists a real number m such that $a_n \geq m$ for all n.

Bounded sequence

A sequence $\{a_n\}$ is said to be *bounded*, if it is bounded above and bounded below.

In other words, there exists two real numbers m and M such that $m \leq a_n \leq M$ for all n.

Divergent Sequence

A sequence $\{a_n\}$ is said to *diverge* to $+\infty$, if given any positive number M, there exists a positive integer N such that $a_n > M$ for all $n \geq N$.

We write $\lim_{n\to\infty} a_n = +\infty$.

A sequence $\{a_n\}$ is said to *diverge* to $-\infty$, if given any positive number M, there exists a positive integer N such that $a_n < -M$ for all $n \geq N$.

We write $\lim_{n\to\infty} a_n = -\infty$.

Examples: $\{n\}$, $\{n^2\}$, $\{n^3\}$, $\{3n\}$ are divergent sequences.

Monotonic increasing sequence

A sequence $\{a_n\}$ is said to be *monotonically increasing*, if $a_n \leq a_{n+1}$ for all n.

(*i.e.*) $a_1 \leq a_2 \leq a_3 \leq \ldots \leq a_n \leq a_{n+1} \leq \ldots.$

Examples: $\{n\}$, $\{n^2\}$, $\{2n + 7\}$ are monotonically increasing sequences.

Monotonic decreasing sequence

A sequence $\{a_n\}$ is said to be *monotonically decreasing*, if $a_n \geq a_{n+1}$ for all n.

(*i.e.*) $a_1 \geq a_2 \geq a_3 \geq \ldots \geq a_n \geq a_{n+1} \geq \ldots.$

Examples: $\left\{\frac{1}{n}\right\}, \left\{\frac{1}{2n+3}\right\}$ are monotonically decreasing sequences.

Monotonic sequence

A sequence which is either monotonically increasing or decreasing is called a *monotonic* sequence.

Oscillatory sequence

A sequence $\{a_n\}$ is said to be *oscillatory*, if it does not converge and does not diverge to $+\infty$ or $-\infty$.

Example: $\{\{-1\}^n\}$is an oscillating sequence.

Note:

A sequence is said to *oscillate finitely*, if it is bounded and is an oscillatory sequence.
A sequence is said to *oscillate infinitely*, if it is not bounded and is an oscillatory sequence.

Theorems (Without proof)

1. If $\{a_n\}$ and $\{b_n\}$are convergent sequences, then
   (i) $\lim (a_n + b_n) = \lim a_n + \lim b_n$
   (ii) $\lim (a_n - b_n) = \lim a_n - \lim b_n$
   (iii) $\lim (a_n \,.\, b_n) = (\lim a_n) \,.\, (\lim b_n)$
   (iv) $\lim \left(\frac{1}{a_n}\right) = \frac{1}{\lim a_n}$
2. If $\{a_n\}$ and $\{b_n\}$are convergent sequences, then $\{a_n + b_n\}$ is also a convergent sequence.
3. If $\{a_n\}$ and $\{b_n\}$are convergent sequences, then $\{a_n - b_n\}$ is also a convergent sequence.
4. A sequence cannot converge to two distinct limits.
5. If $\{a_n\}$ converges to a and $\{b_n\}$ converges to b, then $\{a_n b_n\}$ converges to ab.
6. If $\{a_n\}$ converges to a and $\{b_n\}$ converges to b, then $\{a_n / b_n\}$ converges to a / b.
7. A monotonic increasing sequence which is bounded above converges.
8. A monotonic increasing sequence which is not bounded above diverges $+\infty$.
9. A monotonic decreasing sequence which is bounded below converges.
10. A monotonic decreasing sequence which is not bounded below diverges $-\infty$.
11. Every convergent sequence is bounded.

Problems

1. **Show that** $\left\{\dfrac{n+1}{2n+7}\right\}$ **is convergent.**

Solution:

$$a_n = \frac{n+1}{2n+7} = \frac{n\left(1+\frac{1}{n}\right)}{n\left(2+\frac{7}{n}\right)}$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1+\frac{1}{n}}{2+\frac{7}{n}} = \frac{1}{2}$$

$\{a_n\}$ converges to $\frac{1}{2}$.

2. **Show that** $\left\{\dfrac{1}{n}\right\}$ **is convergent.**

Solution:

$$a_n = \frac{1}{n}$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n} = 0$$

$\{a_n\}$ converges to 0.

3. **Show that** $\left\{3+\dfrac{(-1)^n}{n}\right\}$ **is convergent.**

Solution:

$$a_n = 3+\frac{(-1)^n}{n}$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left[3+\frac{(-1)^n}{n}\right] = 3$$

$\{a_n\}$ converges to 3.

4. **Prove that** $\left\{\dfrac{n+1}{2n+3}\right\}$ **is monotonically increasing and convergent.**

Solution:

$$a_n = \frac{n+1}{2n+3}, \qquad a_{n+1} = \frac{n+2}{2n+5}$$

$$a_n - a_{n+1} = \frac{n+1}{2n+3} - \frac{n+2}{2n+5} = \frac{(n+1)(2n+5)-(n+2)(2n+3)}{(2n+5)(2n+3)}$$

$$= \frac{2n^2 + 5n + 2n + 5 - (2n^2 + 4n + 3n + 6)}{(2n+5)(2n+3)}$$

$$= \frac{-1}{(2n+5)(2n+3)} < 0$$

$a_n - a_{n+1} < 0$

$\therefore\ a_n < a_{n+1}$

$\therefore \{a_n\}$ is monotonically increasing.

$$a_n = \frac{n+1}{2n+3} = \frac{n\left(1+\frac{1}{n}\right)}{n\left(2+\frac{3}{n}\right)}$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1+\frac{1}{n}}{2+\frac{3}{n}} = \frac{1}{2}$$

$\therefore \{a_n\}$ converges to $\frac{1}{2}$.

Hence $\{a_n\}$ is monotonically increasing and convergent.

SERIES

A *series* is defined as the sum of the elements of a sequence.

INFINITE SERIES

If $\{a_n\}$ is a sequence, then $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$ is called an *infinite series*.

Convergence or divergence of a series

Let $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$ be a given series. Define $s_n = a_1 + a_2 + a_3 + \cdots + a_n$.

The *convergence* or *divergence* of the series $\Sigma\, a_n$ is defined in terms of the convergence or divergence of the sequence $\{s_n\}$.

Geometric Series

The *geometric series* $\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots$

(i) converges to $\frac{1}{1-x}$ if $|x| < 1$

(ii) diverges if $x \geq 1$

(iii) oscillates finitely if $x = -1$

(iv) oscillates infinitely if $x < -1$

SERIES OF POSITIVE TERMS

A series whose terms are all positive is called a *series of positive terms*.

Theorem:

1. A series of positive terms either converges or diverges. It cannot *oscillate*.
2. If Σu_n is convergent, then $\lim_{n\to\infty} u_n = 0$. (Necessary condition)
3. If $\lim_{n\to\infty} u_n = 0$, then $\sum u_n$ need not converge.

Theorem:

1. If Σu_n is convergent, then $\Sigma k u_n$ is convergent where k is a constant.
2. If Σu_n and Σv_n are two convergent series, then $\Sigma (u_n + v_n)$ is also a convergent series.

Theorem – Harmonic Series Test or *p*-series test

The series $\sum \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p} + \cdots$

(i) converges if $p > 1$ and

(ii) diverges if $p \leq 1$.

Note:

By Harmonic series test, $\sum \frac{1}{n}$ is divergent while $\sum \frac{1}{n^2}$ is convergent.

TEST OF CONVERGENCE OF SERIES OF POSITIVE TERMS

1. Comparison Test
2. D' Alembert's Ratio Test
3. Raabe's Test
4. Logarithmic Test
5. Cauchy's Root Test
6. Cauchy's Integral Test

1. COMPARISON TEST

Different Forms

(i) Let Σu_n and Σv_n be two series of positive terms with $u_n \leq v_n$ for all n.

If Σv_n converges, then Σu_n also converges.

(ii) Let Σu_n and Σv_n be two series of positive terms with $u_n \geq v_n$ for all n.

If Σv_n diverges, then Σu_n also diverges.

Other Form of Comparison Test (Limit Form)

Let $\Sigma\, u_n$ and $\Sigma\, v_n$ be two series of positive terms. If $\lim\limits_{n\to\infty} \dfrac{u_n}{v_n} = l\,(\text{a finite quantity and} \neq 0)$, then $\Sigma\, u_n$ and $\Sigma\, v_n$ both converge or diverge together.

Note:

Of the above forms, "Limit form" is the most useful.

5. Test the convergence of the series $\sum \dfrac{1}{\sqrt{n^2+1}}$.

Solution

$$u_n = \frac{1}{\sqrt{n^2\left(1+\frac{1}{n^2}\right)}} = \frac{1}{n}.\frac{1}{\sqrt{1+\frac{1}{n^2}}}$$

Let $v_n = \dfrac{1}{n}$

Now $\dfrac{u_n}{v_n} = \dfrac{1}{\sqrt{1+\frac{1}{n^2}}}$

$\lim\limits_{n\to\infty} \dfrac{u_n}{v_n} = 1$, which is finite and non – zero.

$\therefore$ Both $\Sigma\, u_n$ and $\Sigma\, v_n$ converge or diverge together.

$\sum v_n = \sum \dfrac{1}{n}$ is divergent.

Hence by comparison test, $\Sigma\, u_n$ is divergent.

6. Test the convergence of the series $\sum \dfrac{1}{\sqrt{n+1}}$.

Solution

$$u_n = \frac{1}{\sqrt{n+1}} = \frac{1}{\sqrt{n\left(1+\frac{1}{n}\right)}}$$

Let $v_n = \dfrac{1}{\sqrt{n}}$

Now $\dfrac{u_n}{v_n} = \dfrac{1}{\sqrt{1+\frac{1}{n}}}$

$\lim\limits_{n\to\infty}\dfrac{u_n}{v_n}=1$, which is finite and non – zero.

$\therefore$ Both $\Sigma\, u_n$ and $\Sigma\, v_n$ converge or diverge together.

$\sum v_n=\sum\dfrac{1}{\sqrt{n}}=\sum\dfrac{1}{n^{1/2}}$ is divergent.

Hence by comparison test, $\Sigma\, u_n$ is divergent.

7. Test the convergence of the series $\sum\dfrac{1}{(n+1)(2n+1)}$.

Solution

$$u_n=\frac{1}{(n+1)(2n+1)}=\frac{1}{n\left(1+\frac{1}{n}\right)n\left(2+\frac{1}{n}\right)}$$

Let $v_n=\dfrac{1}{n^2}$

Now $\dfrac{u_n}{v_n}=\dfrac{1}{\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)}$

$\lim\limits_{n\to\infty}\dfrac{u_n}{v_n}=\dfrac{1}{2}$, which is finite and non – zero.

$\therefore$ Both $\Sigma\, u_n$ and $\Sigma\, v_n$ converge or diverge together.

$\sum v_n=\sum\dfrac{1}{n^2}$ is convergent .

Hence by comparison test, $\Sigma\, u_n$ is convergent.

8. Test the convergence of the series $\sum\left(\sqrt{n^2+1}-n\right)$.

Solution

$$u_n=\sqrt{n^2+1}-n\times\frac{\sqrt{n^2+1}+n}{\sqrt{n^2+1}+n}$$

$$u_n=\frac{n^2+1-n^2}{\sqrt{n^2+1}+n}=\frac{1}{\sqrt{n^2\left(1+\frac{1}{n^2}\right)}+n}$$

$$u_n=\frac{1}{n\left[\sqrt{\left(1+\frac{1}{n^2}\right)}+1\right]}$$

Let $v_n=\dfrac{1}{n}$

Now $\dfrac{u_n}{v_n} = \dfrac{1}{\left[\sqrt{\left(1+\dfrac{1}{n^2}\right)}+1\right]}$

$\lim\limits_{n\to\infty} \dfrac{u_n}{v_n} = \dfrac{1}{2}$, which is finite and non – zero.

$\therefore$ Both $\Sigma\, u_n$ and $\Sigma\, v_n$ converge or diverge together.

$\sum v_n = \sum \dfrac{1}{n}$ is divergent.

Hence by comparison test, $\Sigma\, u_n$ is divergent.

9. Test the convergence of the series $\sum \sqrt{n^4+1} - \sqrt{n^4-1}$ **.**

Solution

$$u_n = \sqrt{n^4+1} - \sqrt{n^4-1} \times \frac{\sqrt{n^4+1}+\sqrt{n^4-1}}{\sqrt{n^4+1}+\sqrt{n^4-1}}$$

$$u_n = \frac{n^4+1-(n^4-1)}{\sqrt{n^4+1}+\sqrt{n^4-1}} = \frac{2}{\sqrt{n^4\left(1+\dfrac{1}{n^4}\right)}+\sqrt{n^4\left(1-\dfrac{1}{n^4}\right)}}$$

$$u_n = \frac{2}{n^2\left[\sqrt{\left(1+\dfrac{1}{n^4}\right)}+\sqrt{\left(1-\dfrac{1}{n^4}\right)}\right]}$$

Let $v_n = \dfrac{1}{n^2}$

Now $\dfrac{u_n}{v_n} = \dfrac{2}{\left[\sqrt{\left(1+\dfrac{1}{n^4}\right)}+\sqrt{\left(1-\dfrac{1}{n^4}\right)}\right]}$

$\lim\limits_{n\to\infty} \dfrac{u_n}{v_n} = \dfrac{2}{2} = 1$, which is finite and non – zero.

$\therefore$ Both $\Sigma\, u_n$ and $\Sigma\, v_n$ converge or diverge together.

$\sum v_n = \sum \dfrac{1}{n^2}$ is convergent .

Hence by comparison test, $\Sigma\, u_n$ is convergent.

10. Test the convergence of the series $\sum \dfrac{\sqrt{n+1}-\sqrt{n}}{n^p}$ **.**

Solution

$$u_n = \frac{\sqrt{n+1}-\sqrt{n}}{n^p} \times \frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}}$$

$$u_n = \frac{n+1-n}{n^p\left[\sqrt{n\left(1+\frac{1}{n}\right)}+\sqrt{n}\right]}$$

$$u_n = \frac{1}{n^{p+\frac{1}{2}}\left[\sqrt{1+\frac{1}{n}}+1\right]}$$

Let $v_n = \dfrac{1}{n^{p+\frac{1}{2}}}$

Now $\dfrac{u_n}{v_n} = \dfrac{1}{\sqrt{1+\frac{1}{n}}+1}$

$\lim\limits_{n\to\infty} \dfrac{u_n}{v_n} = \dfrac{1}{2}$, which is finite and non – zero.

$\therefore$ Both $\Sigma\, u_n$ and $\Sigma\, v_n$ converge or diverge together.

$\sum v_n = \sum \dfrac{1}{n^{p+\frac{1}{2}}}$ converges if $p+\dfrac{1}{2} > 1$ and diverges if $p+\dfrac{1}{2} \le 1$.

By Comparison test, $\sum u_n$ converges if $p > \dfrac{1}{2}$ and diverges if $p \le \dfrac{1}{2}$.

11. Test the convergence of the series $\sum \dfrac{n^p}{\sqrt{n+1}+\sqrt{n}}$.

Solution

$$u_n = \frac{n^p}{\sqrt{n+1}+\sqrt{n}} = \frac{n^p}{\sqrt{n}\left[\sqrt{1+\frac{1}{n}}+1\right]}$$

$$u_n = \frac{1}{n^{-p+\frac{1}{2}}\left[\sqrt{1+\frac{1}{n}}+1\right]}$$

Let $v_n = \dfrac{1}{n^{-p+\frac{1}{2}}}$

Now $\dfrac{u_n}{v_n} = \dfrac{1}{\sqrt{1+\dfrac{1}{n}}+1}$

$\lim\limits_{n\to\infty} \dfrac{u_n}{v_n} = \dfrac{1}{2}$, which is finite and non – zero.

$\therefore$ Both $\Sigma\, u_n$ and $\Sigma\, v_n$ converge or diverge together.

$\sum v_n = \sum \dfrac{1}{n^{-p+\frac{1}{2}}}$ converges if $-p+\dfrac{1}{2} > 1$ and diverges if $-p+\dfrac{1}{2} \le 1$.

By Comparison test, $\sum u_n$ converges if $-p > \dfrac{1}{2}$ and diverges if $-p \le \dfrac{1}{2}$.

$\sum u_n$ converges if $p < -\dfrac{1}{2}$ and diverges if $p \ge -\dfrac{1}{2}$.

12. Test the convergence of the series $\sum \dfrac{n^n}{(n+1)^{n+1}}$ **.**

Solution

$$u_n = \frac{n^n}{(n+1)^{n+1}} = \frac{n^n}{n^{n+1}\left(1+\dfrac{1}{n}\right)^{n+1}}$$

$$u_n = \frac{1}{n\left(1+\dfrac{1}{n}\right)^{n+1}}$$

Let $v_n = \dfrac{1}{n}$

Now $\dfrac{u_n}{v_n} = \dfrac{1}{\left(1+\dfrac{1}{n}\right)^{n+1}}$

$\lim\limits_{n\to\infty} \dfrac{u_n}{v_n} = \lim\limits_{n\to\infty} \dfrac{1}{\left(1+\dfrac{1}{n}\right)^{n}} \cdot \dfrac{1}{1+\dfrac{1}{n}} = \dfrac{1}{e}.1$, which is finite and non - zero. **Formula :** $\lim\limits_{n\to\infty} \dfrac{1}{\left(1+\dfrac{1}{n}\right)^{n}} = \dfrac{1}{e}$

$\therefore$ Both $\Sigma\, u_n$ and $\Sigma\, v_n$ converge or diverge together.

$\sum v_n = \sum \dfrac{1}{n}$ is divergent.

Hence by comparison test, $\Sigma\, u_n$ is divergent.

13. Show that the series $\dfrac{1}{1.2.3} + \dfrac{3}{2.3.4} + \dfrac{5}{3.4.5} + \cdots$ **is convergent.**

Solution

$$u_n = \frac{2n-1}{n(n+1)(n+2)} = \frac{n\left(2-\frac{1}{n}\right)}{n\,n\left(1+\frac{1}{n}\right)n\left(1+\frac{2}{n}\right)}$$

Let $v_n = \frac{1}{n^2}$

Now $\frac{u_n}{v_n} = \frac{2-\frac{1}{n}}{\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)}$

$\lim\limits_{n\to\infty} \frac{u_n}{v_n} = 2$, which is finite and non – zero.

$\therefore$ Both $\Sigma\, u_n$ and $\Sigma\, v_n$ converge or diverge together.

$\sum v_n = \sum \frac{1}{n^2}$ is convergent .

Hence by comparison test, $\Sigma\, u_n$ is convergent.

14. **Discuss the convergence of the series** $\frac{1^3}{1^p+2^p} + \frac{2^3}{2^p+3^p} + \frac{3^3}{3^p+4^p} + \cdots$.

Solution

$$u_n = \frac{n^3}{n^p+(n+1)^p} = \frac{1}{n^{p-3}\left[1+\left(1+\frac{1}{n}\right)^p\right]}$$

Let $v_n = \frac{1}{n^{p-3}}$

Now $\frac{u_n}{v_n} = \frac{1}{1+\left(1+\frac{1}{n}\right)^p}$

$\lim\limits_{n\to\infty} \frac{u_n}{v_n} = \frac{1}{2}$, which is finite and non – zero.

$\therefore$ Both $\Sigma\, u_n$ and $\Sigma\, v_n$ converge or diverge together.

$\sum v_n = \sum \frac{1}{n^{p-3}}$ converges if $p-3>1$ and diverges if $p-3\le 1$.

By Comparison test, $\sum u_n$ converges if $p>4$ and diverges if $p\le 4$.

15. **Discuss the convergence of the series** $\sum \frac{1}{n}\sin\left(\frac{1}{n}\right)$.

Solution

$$u_n = \frac{1}{n}\sin\left(\frac{1}{n}\right)$$

Let $v_n = \dfrac{1}{n^2}$

Now $\dfrac{u_n}{v_n} = \dfrac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}}$

$$\lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{\frac{1}{n}\to 0}\frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}} = 1, \text{which is finite and non} - \text{zero.}$$

Formula : $\lim_{x\to 0}\dfrac{\sin x}{x} = 1$

$\therefore$ Both $\Sigma\, u_n$ and $\Sigma\, v_n$ converge or diverge together.

$\sum v_n = \sum \dfrac{1}{n^2}$ is convergent .

Hence by comparison test, $\Sigma\, u_n$ is convergent.

16. Discuss the convergence of the series $\sum \dfrac{1}{\sqrt{n}}\tan\left(\dfrac{1}{n}\right)$**.**

Solution

$$u_n = \frac{1}{\sqrt{n}}\tan\left(\frac{1}{n}\right)$$

Let $v_n = \dfrac{1}{n^{3/2}}$

Now $\dfrac{u_n}{v_n} = \dfrac{\tan\left(\frac{1}{n}\right)}{\frac{1}{n}}$

$$\lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{\frac{1}{n}\to 0}\frac{\tan\left(\frac{1}{n}\right)}{\frac{1}{n}} = 1, \text{which is finite and non} - \text{zero.}$$

$\therefore$ Both $\Sigma\, u_n$ and $\Sigma\, v_n$ converge or diverge together.

$\sum v_n = \sum \dfrac{1}{n^{3/2}}$ is convergent .

Hence by comparison test, $\Sigma\, u_n$ is convergent.

2. D'ALEMBERT'S RATIO TEST

If $\Sigma\, u_n$ is a series of positive terms and $\lim\limits_{n\to\infty} \dfrac{u_{n+1}}{u_n} = l$, then

(i) $\Sigma\, u_n$ is convergent if $l < 1$ and

(ii) $\Sigma\, u_n$ is divergent if $l > 1$.

Note: If $l = 1$, then D' Alembert's ratio test fails.

Practical form of Ratio test

If $\Sigma\, u_n$ is a series of positive terms and $\lim\limits_{n\to\infty} \dfrac{u_n}{u_{n+1}} = l$, then

(i) $\Sigma\, u_n$ is convergent if $l > 1$ and

(ii) $\Sigma\, u_n$ is divergent if $l < 1$.

Note: If $l = 1$, then D' Alembert's ratio test fails.

Note: This test is applicable when the terms of the series involves powers of 'n' and factorials.

17. Test the convergence of the series $\sum \dfrac{n\,!}{n^n}$.

Solution

$$u_n = \frac{n\,!}{n^n}, \quad u_{n+1} = \frac{(n+1)\,!}{(n+1)^{n+1}}$$

$$\text{Now } \frac{u_n}{u_{n+1}} = \frac{n!}{n^n} \times \frac{(n+1)^{n+1}}{(n+1)\,!} = \frac{(n+1)^{n+1}}{n^n\,(n+1)} = \frac{n^n\left(1+\frac{1}{n}\right)^n}{n^n}$$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e = 2.718 > 1$$

Hence by Ratio test, $\Sigma\, u_n$ is convergent.

18. Test the convergence of the series $\sum \dfrac{n\,!\,3^n}{n^n}$.

Solution

$$u_n = \frac{n\,!\,3^n}{n^n}, \quad u_{n+1} = \frac{(n+1)\,!\,3^{n+1}}{(n+1)^{n+1}}$$

$$\text{Now } \frac{u_n}{u_{n+1}} = \frac{n!\,3^n}{n^n} \times \frac{(n+1)^{n+1}}{(n+1)\,!\,3^{n+1}} = \frac{3(n+1)\,n^n}{(n+1)^{n+1}} = \frac{n^n\left(1+\frac{1}{n}\right)^n}{n^n\,3}$$

$$\lim_{n\to\infty}\frac{u_n}{u_{n+1}}=\lim_{n\to\infty}\frac{\left(1+\frac{1}{n}\right)^n}{3}=\frac{e}{3}=0.906<1$$

Hence by Ratio test, $\Sigma\, u_n$ is divergent.

19. **Test the convergence of the series $\sum \frac{x^n}{n\,!}$ where $x > 0$.**

Solution

$$u_n=\frac{x^n}{n\,!},\quad u_{n+1}=\frac{x^{n+1}}{(n+1)\,!}$$

Now $\frac{u_n}{u_{n+1}}=\frac{x^n}{n!}\times\frac{(n+1)!}{x^{n+1}}=\frac{n+1}{x}=\frac{n\left(1+\frac{1}{n}\right)}{x}$

$$\lim_{n\to\infty}\frac{u_n}{u_{n+1}}=\lim_{n\to\infty}\frac{n\left(1+\frac{1}{n}\right)}{x}>1$$

Hence by Ratio test, $\Sigma\, u_n$ is convergent.

20. **Test the convergence of the series $\sum \frac{n^3+1}{2^n+1}$.**

Solution

$$u_n=\frac{n^3+1}{2^n+1},\quad u_{n+1}=\frac{(n+1)^3+1}{2^{n+1}+1}$$

Now $\frac{u_n}{u_{n+1}}=\frac{n^3+1}{2^n+1}\times\frac{2^{n+1}+1}{(n+1)^3+1}=\frac{n^3\left(1+\frac{1}{n^3}\right)}{2^n\left(1+\frac{1}{2^n}\right)}\times\frac{2^n\left(2+\frac{1}{2^n}\right)}{n^3\left[\left(1+\frac{1}{n}\right)^3+\frac{1}{n^3}\right]}$

$$\lim_{n\to\infty}\frac{u_n}{u_{n+1}}=2>1$$

Hence by Ratio test, $\Sigma\, u_n$ is convergent.

21. **Test the convergence of the series $\sum_{n=1}^{\infty}\sqrt{\frac{n}{n+1}}\;x^n$ where $x > 0$.**

Solution

$$u_n=\sqrt{\frac{n}{n+1}}\,x^n,\quad u_{n+1}=\sqrt{\frac{n+1}{n+2}}\,x^{n+1}$$

Now $\dfrac{u_n}{u_{n+1}} = \sqrt{\dfrac{n}{n+1}.\dfrac{n+2}{n+1}}\ \dfrac{1}{x} = \sqrt{\dfrac{n\,n\left(1+\dfrac{2}{n}\right)}{n\,n\left(1+\dfrac{1}{n}\right)^2}}.\ \dfrac{1}{x}$

$\lim\limits_{n\to\infty} \dfrac{u_n}{u_{n+1}} = \dfrac{1}{x}$

If $\dfrac{1}{x} > 1$, then the series is convergent .

If $\dfrac{1}{x} < 1$, then the series is divergent.

(*i.e.*) If $x < 1$, the series is convergent.
If $x > 1$, the series is divergent.

If $x = 1$, D' Alembert's test fails.

If $x = 1$, then $\lim\limits_{n\to\infty} u_n = \lim\limits_{n\to\infty} \sqrt{\dfrac{n}{n+1}} = \lim\limits_{n\to\infty} \sqrt{\dfrac{n}{n\left(1+\dfrac{1}{n}\right)}} = 1 \neq 0.$

$\therefore$ The series is divergent.

Hence by Ratio test, the series converges if $x < 1$ and diverges if $x \geq 1$.

3. RAABE'S TEST

If $\Sigma\, u_n$ is a series of positive terms and if $\lim\limits_{n\to\infty} n\left(\dfrac{u_n}{u_{n+1}} - 1\right) = l$, then

(i) $\Sigma\, u_n$ is convergent if $l > 1$ and
(ii) $\Sigma\, u_n$ is divergent if $l < 1$.

NOTE

1) If $l = 1$, then Raabe's test fails.

2) Raabe's test is used when D' Alembert's ratio test fails and when in the ratio test, $\dfrac{u_n}{u_{n+1}}$ does not involve e.

3) When $\dfrac{u_n}{u_{n+1}}$ involves e, we apply **Logarithmic test** after **Ratio test** and **NOT Raabe's test**.

22. **Discuss the convergence of the series** $\dfrac{2}{3.4} + \dfrac{2.4}{3.5.6} + \dfrac{2.4.6}{3.5.7.8} + \dfrac{2.4.6.8}{3.5.7.9.10} + \cdots\infty$**.**

Solution

$$u_n = \frac{2.4.6.8.\cdots(2n)}{3.5.7.9.\cdots(2n+1)}.\frac{1}{2n+2}$$

$$u_{n+1} = \frac{2.4.6.8.\cdots(2(n+1))}{3.5.7.9.\cdots(2(n+1)+1)}.\frac{1}{2(n+1)+2}$$

$$u_{n+1} = \frac{2.4.6.8.\cdots(2n+2)}{3.5.7.9.\cdots(2n+3)}.\frac{1}{2n+4}$$

$$\frac{u_n}{u_{n+1}} = \frac{2.4.6.8.\cdots(2n)}{3.5.7.9.\cdots(2n+1)}.\frac{1}{2n+2}.\frac{3.5.7.9.\cdots(2n+3)}{2.4.6.8.\cdots(2n+2)}.\frac{2n+4}{1}$$

$$\frac{u_n}{u_{n+1}} = \frac{(2n+3)}{(2n+2)}.\frac{2n+4}{(2n+2)} = \frac{n\left(2+\frac{3}{n}\right)n\left(2+\frac{4}{n}\right)}{n\left(2+\frac{2}{n}\right)n\left(2+\frac{2}{n}\right)}$$

$$\lim_{n\to\infty}\frac{u_n}{u_{n+1}} = \frac{4}{4} = 1$$

$\therefore$ **Ratio test fails. Apply Raabe's test.**

$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}}-1\right) = \lim_{n\to\infty} n\left(\frac{6n+8}{(2n+2)^2}\right) = \lim_{n\to\infty}\frac{n^2}{n^2}\left(\frac{6+\frac{8}{n}}{\left(2+\frac{2}{n}\right)^2}\right) = \frac{6}{4} = \frac{3}{2} > 1$$

Hence by Raabe's test, $\Sigma\, u_n$ is convergent.

23. **Test the convergence of the series** $\sum \frac{1.3.5.\cdots(2n-1)}{2.4.6.\cdots(2n)}.\frac{1}{n}$.

Solution

$$\frac{u_n}{u_{n+1}} = \frac{1.3.5.\cdots(2n-1)}{2.4.6.\cdots(2n)}.\frac{1}{n}.\frac{2.4.6.8.\cdots(2(n+1))}{1.3.5.7.\cdots(2(n+1)-1)}.\frac{n+1}{1}$$

$$\frac{u_n}{u_{n+1}} = \frac{1.3.5.\cdots(2n-1)}{2.4.6.\cdots(2n)}.\frac{1}{n}.\frac{2.4.6.8.\cdots(2n+2))}{1.3.5.7.\cdots(2n+1)}.\frac{n+1}{1}$$

$$\frac{u_n}{u_{n+1}} = \frac{(2n+2))}{(2n+1)}.\frac{n+1}{n}$$

$$\lim_{n\to\infty}\frac{u_n}{u_{n+1}} = \frac{n^2\left(2+\frac{2}{n}\right)\left(1+\frac{1}{n}\right)}{n^2\left(2+\frac{1}{n}\right)} = \frac{2}{2} = 1$$

$\therefore$ **Ratio test fails. Apply Raabe's test.**

$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}}-1\right)=\lim_{n\to\infty} n\left(\frac{(2n+2))}{(2n+1)}.\frac{n+1}{n}-1\right)=\lim_{n\to\infty}\frac{3n+2}{2n+1}=\lim_{n\to\infty}\frac{3+\frac{2}{n}}{2+\frac{1}{n}}=\frac{3}{2}>1$$

Hence by Raabe's test, $\Sigma\, u_n$ is convergent.

4. LOGARITHMIC TEST

If $\Sigma\, u_n$ is a series of positive terms and if $\lim_{n\to\infty} n\log\frac{u_n}{u_{n+1}}=l$, then

(i) $\Sigma\, u_n$ converges if $l>1$

(ii) $\Sigma\, u_n$ diverges if $l<1$.

NOTE

1) This test fails if $l=1$.

2) This test is applied after the failure of Ratio test and generally when $\frac{u_n}{u_{n+1}}$ involves e.

24. **Test the convergence of the series** $1+\frac{2}{2!}x+\frac{3^2}{3!}x^2+\frac{4^3}{4!}x^3+\cdots$.

Solution

$$u_n=\frac{n^{n-1}x^{n-1}}{n!}$$

$$u_{n+1}=\frac{(n+1)^n x^n}{(n+1)!}$$

$$\frac{u_n}{u_{n+1}}=\frac{n^{n-1}x^{n-1}}{n!}.\frac{(n+1)!}{(n+1)^n x^n}=\frac{n^{n-1}x^{-1}}{1}.\frac{(n+1)}{n^n\left(1+\frac{1}{n}\right)^n}=\frac{n^{-1}x^{-1}}{1}.\frac{n\left(1+\frac{1}{n}\right)}{\left(1+\frac{1}{n}\right)^n}=\frac{1}{x}.\frac{\left(1+\frac{1}{n}\right)}{\left(1+\frac{1}{n}\right)^n}$$

$$\lim_{n\to\infty}\frac{u_n}{u_{n+1}}=\frac{1}{x}.\frac{\left(1+\frac{1}{n}\right)}{\left(1+\frac{1}{n}\right)^n}=\frac{1}{e\,x}$$

Formula : $\lim_{n\to\infty}\frac{1}{\left(1+\frac{1}{n}\right)^n}=\frac{1}{e}$

Thus by Ratio test, the series converges if $\frac{1}{e\,x}>1$ and diverges if $\frac{1}{e\,x}<1$.

But the test fails if $\frac{1}{ex} = 1$.

(*i.e.*) The series converges if $x < \frac{1}{e}$ and diverges if $x > \frac{1}{e}$. But the test fails if $x = \frac{1}{e}$.

Since $\lim\limits_{n\to\infty} \frac{u_n}{u_{n+1}}$ involves e, apply Logarithmic test.

If $x = \frac{1}{e}$, then

$$\lim_{n\to\infty} n\log\frac{u_n}{u_{n+1}} = \lim_{n\to\infty} n\log\frac{1}{x}.\frac{\left(1+\frac{1}{n}\right)}{\left(1+\frac{1}{n}\right)^n} = \lim_{n\to\infty} n\log\left(e.\frac{\left(1+\frac{1}{n}\right)}{\left(1+\frac{1}{n}\right)^n}\right)$$

$$= \lim_{n\to\infty} n\left(\log e + \log\left(1+\frac{1}{n}\right) - \log\left(1+\frac{1}{n}\right)^n\right)$$

Formula : $\log(x+1) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots$ and $\log e = 1$

$$= \lim_{n\to\infty} n\left(1 + \left(\frac{1}{n} - \frac{(1/n)^2}{2} + \frac{(1/n)^3}{3} - \cdots\right) - n\left(\frac{1}{n} - \frac{(1/n)^2}{2} + \frac{(1/n)^3}{3} - \cdots\right)\right)$$

$$= \lim_{n\to\infty} \left(n + n\left(\frac{1}{n} - \frac{(1/n)^2}{2} + \frac{(1/n)^3}{3} - \cdots\right) - n^2\left(\frac{1}{n} - \frac{(1/n)^2}{2} + \frac{(1/n)^3}{3} - \cdots\right)\right)$$

$$= \lim_{n\to\infty} \left(n + \left(1 - \frac{1}{2n} + \frac{1}{3n^2} - \cdots\right) - \left(n - \frac{1}{2} + \frac{1}{3n} - \cdots\right)\right) = \frac{3}{2} > 1$$

By Logarithmic test, $\Sigma\, u_n$ is convergent.

Hence the series converges if $x \le \frac{1}{e}$ and diverges if $x > \frac{1}{e}$.

25. Test the convergence of the series $\frac{x}{1!} + \frac{2^2 x^2}{2!} + \frac{3^3 x^3}{3!} + \cdots$.

Solution

$$u_n = \frac{n^n x^n}{n!}$$

$$u_{n+1} = \frac{(n+1)^{n+1} x^{n+1}}{(n+1)!}$$

$$\frac{u_n}{u_{n+1}} = \frac{n^n\, x^n}{n!} \cdot \frac{(n+1)!}{(n+1)^{n+1}\, x^{n+1}} = \frac{n^n\, x^n}{1} \cdot \frac{(n+1)}{n^{n+1}\left(1+\frac{1}{n}\right)^{n+1} x^{n+1}} = \frac{n\left(1+\frac{1}{n}\right)}{n\left(1+\frac{1}{n}\right)^{n+1} .x} = \frac{1}{x} \cdot \frac{1}{\left(1+\frac{1}{n}\right)^{n}}$$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \frac{1}{e\,x} \qquad \textbf{Formula : } \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^{n}} = \frac{1}{e}$$

Thus by Ratio test, the series converges if $\frac{1}{e\,x} > 1$ and diverges if $\frac{1}{e\,x} < 1$.

But the test fails if $\frac{1}{e\,x} = 1$.

(*i.e.*) The series converges if $x < \frac{1}{e}$ and diverges if $x > \frac{1}{e}$. But the test fails if $x = \frac{1}{e}$.

Since $\lim_{n\to\infty} \frac{u_n}{u_{n+1}}$ involves e, apply Logarithmic test.

If $x = \frac{1}{e}$, then

$$\lim_{n\to\infty} n \log \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} n \log \frac{1}{x} \cdot \frac{1}{\left(1+\frac{1}{n}\right)^{n}} = \lim_{n\to\infty} n \log\left(\frac{e}{\left(1+\frac{1}{n}\right)^{n}}\right)$$

$$= \lim_{n\to\infty} n\left(\log e - \log\left(1+\frac{1}{n}\right)^{n}\right)$$

Formula : $\log(x+1) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots$ and $\log e = 1$

$$= \lim_{n\to\infty} n\left(1 - n\left(\frac{1}{n} - \frac{(1/n)^2}{2} + \frac{(1/n)^3}{3} - \cdots\right)\right)$$

$$= \lim_{n\to\infty} \left(n - n^2\left(\frac{1}{n} - \frac{(1/n)^2}{2} + \frac{(1/n)^3}{3} - \cdots\right)\right)$$

$$= \lim_{n\to\infty} \left(n - \left(n - \frac{1}{2} + \frac{1}{3n} - \cdots\right)\right) = \frac{1}{2} < 1$$

By Logarithmic test, $\Sigma\, u_n$ is divergent.

The series converges if $x < \frac{1}{e}$ and diverges if $x \geq \frac{1}{e}$.

5. CAUCHY'S ROOT TEST

If $\Sigma\, u_n$ is a series of positive terms and $\lim_{n\to\infty} (u_n)^{\frac{1}{n}} = l$, then

(i) $\Sigma\, u_n$ is converges if $l < 1$

(ii) $\Sigma\, u_n$ is diverges if $l > 1$.

NOTE: This test fails if $l = 1$.

26. Test the convergence of the series $\sum \frac{n^3}{3^n}$.

Solution

$$u_n = \frac{n^3}{3^n}$$

$$(u_n)^{1/n} = \frac{\left(n^{1/n}\right)^3}{3}$$

$$\lim_{n\to\infty} (u_n)^{1/n} = \frac{1}{3} < 1$$

By Cauchy's root test, $\Sigma\, u_n$ is convergent.

27. Test the convergence of the series $\sum \frac{1}{(\log n)^n}$.

Solution

$$u_n = \frac{1}{(\log n)^n}$$

$$(u_n)^{1/n} = \frac{1}{\log n}$$

$$\lim_{n\to\infty} (u_n)^{1/n} = \lim_{n\to\infty} \frac{1}{\log n} = 0 < 1$$

Formula : $\log \infty = \infty$

By Cauchy's root test, $\Sigma\, u_n$ is convergent.

28. Test the convergence of the series $\sum (\log n)^{-2n}$.

Solution

$$u_n = \frac{1}{(\log n)^{2n}}$$

$$(u_n)^{1/n} = \frac{1}{(\log n)^2}$$

$$\lim_{n\to\infty}(u_n)^{1/n} = \lim_{n\to\infty}\frac{1}{(\log n)^2} = 0 < 1$$ **Formula :** $\log\infty = \infty$

By Cauchy's root test, $\Sigma\, u_n$ is convergent.

29. **Test the convergence of the series** $\sum \frac{1}{\left(1+\frac{1}{n}\right)^{n^2}}$.

Solution

$$u_n = \frac{1}{\left(1+\frac{1}{n}\right)^{n^2}}$$

$$(u_n)^{1/n} = \frac{1}{\left(1+\frac{1}{n}\right)^{n}}$$

$$\lim_{n\to\infty}(u_n)^{1/n} = \lim_{n\to\infty}\frac{1}{\left(1+\frac{1}{n}\right)^{n}} = \frac{1}{e} < 1$$

By Cauchy's root test, $\Sigma\, u_n$ is convergent.

6. CAUCHY'S INTEGRAL TEST

If $\sum f(n) = f(1) + f(2) + f(3) + \ldots.. + f(n) + \ldots.$ is a series of positive terms and $f(n)$ decreases as n increases. (monotonic decreasing sequence)

Then (i) $\sum f(n)$ converges if $\int_1^\infty f(x)\,dx$ = finite

(ii) $\sum f(n)$ diverges if $\int_1^\infty f(x)\,dx$ = infinite

30. **Test the convergence of the series** $\sum \frac{1}{n^2+1}$.

Solution

$f(n) = \frac{1}{n^2+1}$ Clearly $f(n)$ is a monotonic decreasing sequence.

$$\int_1^\infty f(x)dx = \int_1^\infty \frac{1}{x^2+1}dx = \left(\tan^{-1} x\right)_1^\infty = \tan^{-1}(\infty) - \tan^{-1}(1) = \frac{\pi}{2} - \frac{\pi}{4} = \text{a finite quantity}$$

By Cauchy's integral test, the given series is convergent.

31. **Test the convergence of the series** $\sum \frac{1}{3n+1}$.

Solution

$f(n) = \frac{1}{3n+1}$ Clearly $f(n)$ is a monotonic decreasing sequence.

$$\int_1^{\infty} f(x)dx = \int_1^{\infty} \frac{1}{3x+1} dx = \frac{1}{3}\Big(\log(3x+1)\Big)_1^{\infty} = \frac{1}{3}\Big(\log\infty - \log 4\Big) = \infty$$

By Cauchy's integral test, the given series is divergent.

32. **Test the convergence of the series** $\sum_2^{\infty} \frac{1}{n \log n}$.

Solution

$f(n) = \frac{1}{n \log n}$ Clearly $f(n)$ is a monotonic decreasing sequence.

$$\int_2^{\infty} f(x)dx = \int_2^{\infty} \frac{1}{x \log x} dx = \int_2^{\infty} \frac{1/x}{\log x} dx = \Big(\log(\log(x)\Big)_2^{\infty} = \infty$$

By Cauchy's integral test, the given series is divergent.

SERIES OF POSITIVE AND NEGATIVE TERMS (ALTERNATING SERIES)

ALTERNATING SERIES

A series in which the terms are alternately positive and negative is called an alternating series.

TEST OF CONVERGENCE OF ALTERNATING SERIES

LEIBNITZ'S TEST

The alternating series $\sum_{n=1}^{\infty} (-1)^{n-1} u_n = u_1 - u_2 + u_3 - u_4 + \ldots$ converges if

(i) $u_n > u_{n+1}$ ((*i.e.*)$\{u_n\}$ is a monotonic decreasing sequence.)

(ii) $\lim_{n\to\infty} u_n = 0$

NOTE

If $\lim_{n\to\infty} u_n \neq 0$, then the alternating series oscillates.

ABSOLUTELY CONVERGENT

The alternating series $\Sigma\, u_n$ is *absolutely convergent*, if $\Sigma\, |\, u_n\,|$ is convergent.

CONDITIONALLY CONVERGENT

The alternating series $\Sigma\, u_n$ is *conditionally convergent or semi convergent,* if

(i) $\Sigma\, u_n$ is convergent while (ii) $\Sigma\, |\, u_n\,|$ is divergent.

Theorems (Statement Only)

1. Every absolutely convergent series is convergent. But converse is not true.

2. If the terms of an absolutely convergent series are rearranged, the series remains convergent and its sum is unaltered.

3. In a conditionally convergent series, a rearrangement of terms *may alter the sum of the series*.

33. **Test the convergence of the series** $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots$.

Solution

$u_n=\frac{1}{n}$ $\qquad$ $u_{n+1}=\frac{1}{n+1}$

(i) Clearly $u_n > u_{n+1}$.

(ii) $\lim\limits_{n\to\infty} u_n = \lim\limits_{n\to\infty}\frac{1}{n}=0$

$\therefore$ By Leibnitz's test, the given series is convergent.

34. **Test the convergence of the series** $1-\frac{\sqrt{2}}{2}+\frac{\sqrt{3}}{3}-\frac{\sqrt{4}}{4}+\cdots$ **(OR)** $1-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}-\frac{1}{\sqrt{4}}+\cdots$.

Solution

$u_n=\frac{1}{\sqrt{n}}$ $\qquad$ $u_{n+1}=\frac{1}{\sqrt{n+1}}$

(i) Clearly $u_n > u_{n+1}$.

(ii) $\lim\limits_{n\to\infty} u_n = \lim\limits_{n\to\infty}\frac{1}{\sqrt{n}}=0$

$\therefore$ By Leibnitz's test, the given series is convergent.

35. **Test the convergence of the series** $\frac{1}{1.2}-\frac{1}{3.4}+\frac{1}{5.6}-\cdots$.

Solution

$$u_n = \frac{1}{(2n-1)(2n)}$$

$$u_{n+1} = \frac{1}{(2n+1)(2n+2)}$$

(i) Since $\frac{1}{(2n-1)(2n)} > \frac{1}{(2n+1)(2n+2)}$ always, clearly $u_n > u_{n+1}$.

(ii) $\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{n\left(2-\frac{1}{n}\right)2n} = 0$

$\therefore$ By Leibnitz's test, the given series is convergent.

36. **Test the convergence of the series** $\sum (-1)^n \sin\left(\frac{1}{n}\right)$.

Solution

$$u_n = \sin\left(\frac{1}{n}\right)$$

$$u_{n+1} = \sin\left(\frac{1}{n+1}\right)$$

(i) Since $\sin\left(\frac{1}{n}\right) > \sin\left(\frac{1}{n+1}\right)$ always, clearly $u_n > u_{n+1}$.

(ii) $\lim_{n\to\infty} u_n = \lim_{n\to\infty} \sin\left(\frac{1}{n}\right) = 0$

$\therefore$ By Leibnitz's test, the given series is convergent.

37. **Test the convergence of the series** $(1+1)-\left(1+\frac{1}{2}\right)+\left(1+\frac{1}{3}\right)-\left(1+\frac{1}{4}\right)+\cdots$ **(OR)** $\sum (-1)^{n-1}\left(1+\frac{1}{n}\right)$.

Solution

$$u_n = 1+\frac{1}{n}$$

$$u_{n+1} = 1+\frac{1}{n+1}$$

(i) Since $1+\frac{1}{n} > 1+\frac{1}{n+1}$ always, clearly $u_n > u_{n+1}$.

(ii) $\lim_{n\to\infty} u_n = \lim_{n\to\infty} 1+\frac{1}{n} = 1 \neq 0$

$\therefore$ By Leibnitz's test, the given series is not convergent.

38. Test the convergence of the series $\sum \frac{(-1)^{n-1}}{\log(n+1)}$.

Solution

$u_n = \frac{1}{\log(n+1)}$

$u_{n+1} = \frac{1}{\log(n+2)}$

(i) Since $\frac{1}{\log(n+1)} > \frac{1}{\log(n+2)}$ always, clearly $u_n > u_{n+1}$.

(ii) $\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{\log(n+1)} = 0$ **Formula :** $\log \infty = \infty$

$\therefore$ By Leibnitz's test, the given series is convergent.

39. Test the convergence of the series $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$ (OR) $\sum \frac{(-1)^{n-1}}{n^2}$.

Solution

The series of absolute terms $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = \sum \frac{1}{n^2}$ is clearly *convergent* by Harmonic series test (or) p-series test.

$\therefore$ The series is absolutely convergent.

Since every absolutely convergent series is convergent, the given series is convergent.

40. Prove that the series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ is conditionally convergent.

Solution

$u_n = \frac{1}{n}$ $\qquad$ $u_{n+1} = \frac{1}{n+1}$

(i) Clearly $u_n > u_{n+1}$.

(ii) $\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{n} = 0$

$\therefore$ By Leibnitz's test, the given series is *convergent*.

Also $\Sigma\,|\,u_n\,| = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots = \sum \frac{1}{n}$ is *divergent* by Harmonic series test (or) p-series test.

Hence the given series is conditionally convergent.

41. Test the convergence of the series $\sum \frac{(-1)^{n-1}}{n^p}$ where $p > 0$.

Solution

$u_n = \dfrac{1}{n^p}$ $\qquad$ $u_{n+1} = \dfrac{1}{(n+1)^p}$

(i) Clearly $u_n > u_{n+1}$.

(ii) $\lim\limits_{n\to\infty} u_n = \lim\limits_{n\to\infty} \dfrac{1}{n^p} = 0, \text{if } \; p > 0.$

$\therefore$ By Leibnitz's test, the given series is *convergent*.

* * * * *