Module – 4

Radius of Curvature - Cartesian coordinates – Radius of curvature - Polar coordinates – Circle of curvature – Applications of Radius of curvature in Engineering – Centre of curvature – Evolute of a parabola – Evolute of an ellipse – Envelope of standard curves – Applications of curvature in Engineering – Beta Gamma functions – Beta Gamma functions and their properties

DIFFERENTIAL CALCULUS

Curvature of a curve

The rate of bending of a curve in an interval is called the curvature of the curve in that interval. It is denoted by *k*.

Radius of curvature

The reciprocal of the curvature of a curve at any point is called the radius of curvature at that point. It is denoted by ρ.

Hence $\rho = \dfrac{1}{k}$.

Radius of curvature in Cartesian form

$$\rho = \frac{\left[1+\left(\dfrac{dy}{dx}\right)^2\right]^{3/2}}{\dfrac{d^2y}{dx^2}}$$

If $\dfrac{dy}{dx} = \infty$, then $\rho = \dfrac{\left[1+\left(\dfrac{dx}{dy}\right)^2\right]^{3/2}}{\dfrac{d^2x}{dy^2}}$

Radius of curvature in Polar form

$\rho = \dfrac{(r^2 + r'^2)^{3/2}}{r^2 + 2r'^2 - r r''}$ where $r' = \dfrac{dr}{d\theta}, r'' = \dfrac{d^2 r}{d\theta^2}$

Radius of curvature in Parametric form

$$\rho = \frac{\left(x'^2 + y'^2\right)^{\frac{3}{2}}}{x' y'' - y' x''}$$

Note

1. Curvature of a straight line is zero.

2. Curvature of a circle is the reciprocal of its radius.

Formulae

$$\sinh x = \frac{e^x - e^{-x}}{2}, \quad \sinh 0 = 0, \ \frac{d}{dx}(\sinh x) = \cosh x$$

$$\cosh x = \frac{e^x + e^{-x}}{2}, \quad \cosh 0 = 1, \ \frac{d}{dx}(\cosh x) = \sinh x$$

Problems

1. **Find the radius of curvature at the point $(0,c)$ of the catenary $y=c\cosh\left(\frac{x}{c}\right)$.**

Solution:

$$y=c\cosh\left(\frac{x}{c}\right)$$

$$\frac{dy}{dx}=c\sinh\left(\frac{x}{c}\right)\times\frac{1}{c}=\sinh\left(\frac{x}{c}\right)$$

$$\frac{dy}{dx}\text{at}\,(0,c)=\sinh\,0=0$$

$$\frac{d^2y}{dx^2}=\cosh\left(\frac{x}{c}\right)\times\frac{1}{c}$$

$$\frac{d^2y}{dx^2}\text{at}\,(0,c)=\frac{1}{c}$$

$$\rho=\frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}$$

$$=c$$

2. **Find the radius of curvature at any point of the cycloid $x=a(\theta+\sin\theta)$, $y=a(1-\cos\theta)$.**

Solution:

$$x=a(\theta+\sin\theta)$$

$$\frac{dx}{d\theta}=a(1+\cos\theta)$$

$$y=a(1-\cos\theta)$$

$$\frac{dy}{d\theta}=a(\sin\theta)$$

$$\frac{dy}{dx}=\frac{a\sin\theta}{a(1+\cos\theta)}=\frac{2\sin(\theta/2)\cos(\theta/2)}{2\cos^2(\theta/2)}=\tan(\theta/2)$$

$$\frac{d^2y}{dx^2}=\sec^2(\theta/2)\times(1/2)\times\frac{d\theta}{dx}$$

$$=\frac{1}{\cos^2(\theta/2)}\times(1/2)\times\frac{1}{a(1+\cos\theta)}$$

$$=\frac{1}{\cos^2(\theta/2)}\times(1/2)\times\frac{1}{a\,2\cos^2(\theta/2)}=\frac{1}{4a\cos^4(\theta/2)}$$

$$\rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}$$

$$= \frac{\left[1+\tan^2\left(\frac{\theta}{2}\right)\right]^{3/2}}{\frac{1}{4a\cos^4\left(\frac{\theta}{2}\right)}}$$

$$= \left(\sec^2\left(\frac{\theta}{2}\right)\right)^{3/2} 4a\cos^4\left(\frac{\theta}{2}\right)$$

$$= \sec^3\left(\frac{\theta}{2}\right)4a\cos^4\left(\frac{\theta}{2}\right)$$

$$= 4a\cos\left(\frac{\theta}{2}\right)$$

3. **Show that the radius of curvature at the point θ on the curve** $x = 3a\cos\theta - a\cos 3\theta$**,** $y = 3a\sin\theta - a\sin 3\theta$ **is** $3a\sin\theta$**.**

4. **Find the radius of curvature at any point θ on the curve** $x = a(\cos t + t\sin t)$**,** $y = a(\sin t - t\cos t)$**. Ans** $\rho = at$

5. **Find the radius of curvature at the point** $\left(\frac{3a}{2},\frac{3a}{2}\right)$ **of the curve** $x^3 + y^3 = 3axy$**.**

Solution:

$x^3 + y^3 = 3axy$

Differentiate w.r.t. x

$$3x^2 + 3y^2\frac{dy}{dx} = 3a\left[x\frac{dy}{dx} + y.1\right]$$

$$3\left(y^2 - ax\right)\frac{dy}{dx} = 3\left(ay - x^2\right)$$

$$\frac{dy}{dx} = \frac{ay - x^2}{y^2 - ax}$$

$$\frac{dy}{dx} \text{ at } \left(\frac{3a}{2},\frac{3a}{2}\right) = -1$$

$$3\frac{d^2y}{dx^2} = \frac{(y^2 - ax)\left(a\frac{dy}{dx} - 2x\right) - (ay - x^2)\left(2y\frac{dy}{dx} - a\right)}{(y^2 - ax)^2}$$

$$\frac{d^2y}{dx^2} \text{ at } \left(\frac{3a}{2}, \frac{3a}{2}\right) = \frac{-32}{3a}$$

$$\rho = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}$$

$$= \frac{[1+1]^{3/2}}{\frac{-32}{3a}} = = -\frac{3\sqrt{2}\,a}{16}$$

$$|\rho| = \frac{3\sqrt{2}\,a}{16}$$

6. **Find the radius of curvature at the point $(a,0)$ of the curve $xy^2 = a^3 - x^3$.**

Solution:

$xy^2 = a^3 - x^3$

Differentiate w.r.t. x

$$x.2y\frac{dy}{dx} + y^2 = 0 - 3x^2$$

$$\frac{dy}{dx} = \frac{-(3x^2 + y^2)}{2xy}$$

$$\frac{dy}{dx} \text{ at } (\mathrm{a}, 0) = \infty$$

Here $\frac{dy}{dx} = \infty$

Differentiate w.r.t. y

$$x.2y + y^2 \frac{dx}{dy} = 0 - 3x^2 \frac{dx}{dy}$$

$$\frac{dx}{dy} = \frac{-2xy}{y^2 + 3x^2}$$

$$\frac{dx}{dy} \text{ at } (a,0) = 0$$

$$\frac{d^2x}{dy^2} = -2\left[\frac{(y^2+3x^2)\left(x + y\frac{dx}{dy}\right) - (xy)\left(2y + 6x\frac{dx}{dy}\right)}{(y^2+3x^2)^2}\right]$$

$$\frac{d^2x}{dy^2} \text{ at } (a,0) = \frac{-2}{3a}$$

$$\rho = \frac{\left[1 + \left(\frac{dx}{dy}\right)^2\right]^{3/2}}{\frac{d^2x}{dy^2}}$$

$$= \frac{[1+0]^{3/2}}{\frac{-2}{3a}} = -\frac{3a}{2}$$

$$|\rho| = \frac{3a}{2}$$

7. Find the radius of curvature at any point (r,θ) of the curve $r = a\cos\theta$.

Solution:

$$r = a\cos\theta$$

$$r' = -a\sin\theta$$

$$r'' = -a\cos\theta$$

$$\rho = \frac{(r^2 + r'^2)^{3/2}}{r^2 + 2r'^2 - rr''}$$

$$= \frac{(a^2 \cos^2\theta + a^2 \sin^2\theta)^{3/2}}{a^2\cos^2\theta + 2a^2\sin^2\theta + a^2\cos^2\theta} = \frac{a}{2}$$

8. Find the radius of curvature of the curve $r = a(1+\cos\theta)$ at the point $\theta = \frac{\pi}{2}$.

Solution:

$r = a(1+\cos\theta)$

$r' = -a\sin\theta$

$r'' = -a\cos\theta$

r at $\theta = \frac{\pi}{2} = a$

r' at $\theta = \frac{\pi}{2} = -a$

r'' at $\theta = \frac{\pi}{2} = 0$

$$\rho = \frac{(r^2 + r'^2)^{3/2}}{r^2 + 2r'^2 - r r''}$$

$$= \frac{(a^2 + a^2)^{3/2}}{a^2 + 2a^2} = \frac{2\sqrt{2}\,a}{3}$$

9. Find the radius of curvature at any point (r,θ) of the curve $r = e^{\theta}$. Ans $\rho = \sqrt{2}\,r$

Centre of curvature

Centre of curvature $= (\bar{x}, \bar{y})$ where $\bar{x} = x - \frac{y_1(1+y_1^2)}{y_2}, \bar{y} = y + \frac{1+y_1^2}{y_2}$.

Circle of curvature

Equation of circle of curvature is $(x-\bar{x})^2 + (y-\bar{y})^2 = \rho^2$.

10. Find the equation of the circle of curvature of the parabola $y^2 = 12x$ at (3, 6).

Solution:

$y^2 = 12x$

$2y\frac{dy}{dx} = 12$

$\frac{dy}{dx} = \frac{12}{2y} = \frac{6}{y}$

$\frac{dy}{dx} = \frac{12}{2y} = \frac{6}{y}$

$$\frac{dy}{dx} \text{ at } (3,6) = 1$$

$$\frac{d^2y}{dx^2} = 6\left(\frac{-1}{y^2}\right)\frac{dy}{dx}$$

$$\frac{d^2y}{dx^2} \text{ at } (3,6) = \frac{-1}{6}$$

$$\rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}$$

$$\rho = -6(2)^{3/2}$$

$$\bar{x} = x - \frac{y_1\,(1+y_1{}^2)}{y_2}$$

$$\bar{x} \text{ at } (3,6) = 3 - \frac{1(1+1)}{-1/6} = 15$$

$$\bar{y} = y + \frac{1+y_1{}^2}{y_2}$$

$$\bar{y} \text{ at } (3,6) = 6 + \frac{1+1}{-1/6} = -6$$

Centre of curvature $(\bar{x},\bar{y}) = (15,-6)$

Equation of circle of curvature is $(x-\bar{x})^2 + (y-\bar{y})^2 = \rho^2$

$$(x-15)^2 + (y+6)^2 = 288$$

11. Find the equation of the circle of curvature of the curve $\sqrt{x}+\sqrt{y}=\sqrt{a}$ at $\left(\frac{a}{4},\frac{a}{4}\right)$.

Solution:

$$\sqrt{x}+\sqrt{y}=\sqrt{a}$$

$$\frac{1}{2}x^{-1/2} + \frac{1}{2}y^{-1/2}\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-x^{-1/2}}{y^{-1/2}} = \frac{-\sqrt{y}}{\sqrt{x}}$$

$$\frac{dy}{dx} \text{ at } \left(\frac{a}{4},\frac{a}{4}\right) = -1$$

$$\frac{d^2y}{dx^2} = -\left[\frac{x^{1/2}\left(\frac{1}{2}\,y^{-1/2}\,\frac{dy}{dx}\right) - y^{1/2}\left(\frac{1}{2}\,x^{-1/2}\right)}{x}\right]$$

$$\frac{d^2y}{dx^2}\text{ at}\left(\frac{a}{4},\frac{a}{4}\right)=\frac{4}{a}$$

$$\rho=\frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}=\frac{a}{\sqrt{2}}$$

$$\bar{x}=x-\frac{y_1\,(1+y_1{}^2)}{y_2}$$

$$\bar{x}\text{ at}\left(\frac{a}{4},\frac{a}{4}\right)=\frac{a}{4}-\frac{-1(1+1)}{4/a}=\frac{3a}{4}$$

$$\bar{y}=y+\frac{1+y_1{}^2}{y_2}$$

$$\bar{y}\text{ at}\left(\frac{a}{4},\frac{a}{4}\right)=\frac{a}{4}+\frac{1+1}{4/a}=\frac{3a}{4}$$

Centre of curvature $(\bar{x},\bar{y})=\left(\frac{3a}{4},\frac{3a}{4}\right)$

Equation of circle of curvature is $(x-\bar{x})^2+(y-\bar{y})^2=\rho^2$

$$\left(x-\frac{3a}{4}\right)^2+\left(y-\frac{3a}{4}\right)^2=\frac{a^2}{2}$$

12. Find the equation of the circle of curvature of the curve $x\,y = c^2$ at (c, c).

Solution:

$$x\,y=c^2$$

$$x\frac{dy}{dx}+y\times 1=0$$

$$\frac{dy}{dx}=\frac{-y}{x}$$

$$\frac{dy}{dx}\text{ at}\,(c,c)=-1$$

$$\frac{d^2y}{dx^2}=-\left[\frac{x\frac{dy}{dx}-y.1}{x^2}\right]$$

$$\frac{d^2y}{dx^2}\text{ at}\,(c,c)=\frac{2}{c}$$

$$\rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}} = \sqrt{2}\,c$$

$$\bar{x} = x - \frac{y_1\,(1+y_1{}^2)}{y_2}$$

$$\bar{x}\text{ at }(c,c) = c - \frac{-1(1+1)}{2/c} = 2c$$

$$\bar{y} = y + \frac{1+y_1{}^2}{y_2}$$

$$\bar{y}\text{ at }(c,c) = c + \frac{1+1}{2/c} = 2c$$

Centre of curvature $(\bar{x},\bar{y}) = (2c,2c)$

Equation of circle of curvature is $(x-\bar{x})^2 + (y-\bar{y})^2 = \rho^2$

$$(x-2c)^2 + (y-2c)^2 = 2c^2$$

Evolutes and Involutes

Let C be the centre of curvature corresponding to a point P of the given curve. As the point P moves along the curve, C will trace out a locus, which is called **evolute** of the curve. (or) The locus of centre of curvature is called **evolute** of the curve.

If the curve C_1 is the evolute of a curve C_2, then C_2 is said to be an **involute** of C_1.

13. **Show that the evolute of the curve** $x = a(\cos\theta + \theta\sin\theta)$**,** $y = a(\sin\theta - \theta\cos\theta)$ **is a circle.**

Solution:

$$x = a(\cos\theta + \theta\sin\theta)$$

$$\frac{dx}{d\theta} = -a\sin\theta + a(\theta\cos\theta + \sin\theta.1) = a\theta\cos\theta$$

$$y = a(\sin\theta - \theta\cos\theta)$$

$$\frac{dy}{d\theta} = a\cos\theta - a(-\theta\sin\theta + \cos\theta.1) = a\theta\sin\theta$$

$$\frac{dy}{dx} = \frac{a\theta\sin\theta}{a\theta\cos\theta} = \tan\theta$$

$$\frac{d^2y}{dx^2} = \sec^2\theta \times \frac{d\theta}{dx}$$

$$\frac{d^2y}{dx^2} = \sec^2\theta \times \frac{1}{a\theta\sec\theta} = \frac{1}{a\theta\cos^3\theta}$$

$$\bar{x} = x - \frac{y_1\,(1+y_1{}^2)}{y_2}$$

$$= a(\cos\theta + \theta\sin\theta) - \frac{\tan\theta(1+\tan^2\theta)}{\frac{1}{a\theta\cos^3\theta}}$$

$$= a\cos\theta$$

$$\bar{y} = y + \frac{1+y_1^{\,2}}{y_2}$$

$$= a(\sin\theta - \theta\cos\theta) - \frac{(1+\tan^2\theta)}{\frac{1}{a\theta\cos^3\theta}}$$

$$= a\sin\theta$$

We know that $\cos^2\theta + \sin^2\theta = 1$

$$\left(\frac{\bar{x}}{a}\right)^2 + \left(\frac{\bar{y}}{a}\right)^2 = 1$$

$$\bar{x}^2 + \bar{y}^2 = a^2$$

Locus of $(\bar{x}, \bar{y})$ is $x^2 + y^2 = a^2$

14. Show that the evolute of the cycloid $x = a(\theta - \sin\theta)$, $y = a(1-\cos\theta)$ is another cycloid.

Solution:

$$x = a(\theta - \sin\theta)$$

$$\frac{dx}{d\theta} = a(1-\cos\theta)$$

$$y = a(1-\cos\theta)$$

$$\frac{dy}{d\theta} = a(\sin\theta)$$

$$\frac{dy}{dx} = \frac{a\sin\theta}{a(1-\cos\theta)} = \frac{2\sin(\theta/2)\cos(\theta/2)}{2\sin^2(\theta/2)} = \cot(\theta/2)$$

$$\frac{d^2y}{dx^2} = -\cos ec^2(\theta/2)\times(1/2)\times\frac{d\theta}{dx}$$

$$= -\frac{1}{2\sin^2(\theta/2)} \times \frac{1}{a(1-\cos\theta)}$$

$$= -\frac{1}{2a\sin^2(\theta/2)} \times \frac{1}{2\sin^2(\theta/2)} = -\frac{1}{4a\sin^4(\theta/2)}$$

$$\bar{x} = x - \frac{y_1(1+y_1^{\,2})}{y_2}$$

$$= a(\theta - \sin\theta) - \frac{\cot(\theta/2)(1+\cot^2(\theta/2))}{-\frac{1}{4a\sin^4(\theta/2)}}$$

$$= a(\theta - \sin\theta) + 4a\sin^4(\theta/2) \times \frac{\cos(\theta/2)}{\sin(\theta/2)} \times \frac{1}{\sin^2(\theta/2)}$$
$$= a(\theta - \sin\theta) + 4a\sin(\theta/2) \times \cos(\theta/2)$$
$$= a(\theta - \sin\theta) + 2a\,(2\sin(\theta/2) \times \cos(\theta/2))$$
$$= a(\theta - \sin\theta) + 2a\sin\theta$$
$$\bar{x} = a(\theta + \sin\theta)$$
$$\bar{y} = y + \frac{1 + {y_1}^2}{y_2}$$
$$= a(1 - \cos\theta) - \frac{1 + \cot^2(\theta/2)}{-\frac{1}{4a\sin^4(\theta/2)}}$$
$$\bar{y} = -a(1 - \cos\theta)$$

Locus of $(\bar{x}, \bar{y})$ is $x = a(\theta + \sin\theta),\ y = -a(1 - \cos\theta)$, which is another cycloid.

15. Find the evolute of the parabola $y^2 = 4ax$.

Solution:

Parametric coordinates: $x = at^2$, $y = 2at$

$$x = at^2$$
$$\frac{dx}{dt} = 2at$$
$$y = 2at$$
$$\frac{dy}{dt} = 2a$$
$$\frac{dy}{dx} = \frac{1}{t}$$
$$\frac{d^2y}{dx^2} = -\frac{1}{t^2} \times \frac{dt}{dx} = -\frac{1}{2at^3}$$
$$\bar{x} = x - \frac{y_1(1 + {y_1}^2)}{y_2}$$
$$= at^2 - \frac{\frac{1}{t}\left(1 + \frac{1}{t^2}\right)}{-\frac{1}{2at^3}}$$
$$= 3at^2 + 2a \qquad \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_(1)}$$
$$\bar{y} = y + \frac{1 + {y_1}^2}{y_2}$$

$$= 2at - \frac{1+\dfrac{1}{t^2}}{-\dfrac{1}{2at^3}}$$

$$= -2at^3 \quad \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_(2)}$$

Eliminate t from (1) and (2).

From (1)

$$t^2 = \frac{\bar{x}-2a}{3a}$$

$$(t^2)^3 = \left(\frac{\bar{x}-2a}{3a}\right)^3$$

From (2)

$$t^3 = \frac{-\bar{y}}{2a}$$

$$(t^3)^2 = \left(\frac{-\bar{y}}{2a}\right)^2$$

Hence $\left(\dfrac{\bar{x}-2a}{3a}\right)^3 = \dfrac{\bar{y}^2}{4a^2}$

$$4a^2(\bar{x}-2a)^3 = 27a^3\,\bar{y}^2$$

Locus of $(\bar{x},\bar{y})$ is $4\,(x-2a)^3 = 27\,a\,y^2$

Note: The parametric coordinates of the parabola $x^2 = 4ay$ are $x = 2at,\ \ y = at^2$**.**

16. Find the evolute of the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$.

Solution:

Parametric coordinates: $x = a\sec\theta$, $y = b\tan\theta$

$$x = a\sec\theta$$

$$\frac{dx}{d\theta} = a\sec\theta\tan\theta$$

$$y = b\tan\theta$$

$$\frac{dy}{d\theta} = b\sec^2\theta$$

$$\frac{dy}{dx} = \frac{b\sec^2\theta}{a\sec\theta\tan\theta} = \frac{b}{a\sin\theta} = \frac{b}{a}\operatorname{cosec}\theta$$

$$\frac{d^2y}{dx^2} = \frac{b}{a}(-\operatorname{cosec}\theta\times\cot\theta)\frac{d\theta}{dx}$$

$$= -\frac{b}{a}\cos ec\,\theta \times \cot\theta \times \frac{1}{a\sec\theta\tan\theta}$$

$$= -\frac{b}{a^2}\frac{\cos^3\theta}{\sin^3\theta}$$

$$\bar{x} = x - \frac{y_1(1+y_1{}^2)}{y_2}$$

$$= a\sec\theta - \frac{\frac{b}{a}\cos ec\,\theta\left(1+\frac{b^2}{a^2}\cos ec^2\theta\right)}{-\frac{b\cos^3\theta}{a^2\sin^3\theta}}$$

$$= \frac{a}{\cos\theta} + \frac{b}{a\sin\theta}\frac{a^2\sin^3\theta}{b\cos^3\theta}\left(1+\frac{b^2}{a^2\sin^2\theta}\right)$$

$$= \frac{a^2+b^2}{a}\sec^3\theta$$

$$\bar{y} = y + \frac{1+y_1{}^2}{y_2}$$

$$= b\tan\theta + \frac{1+\frac{b^2}{a^2}\cos ec^2\theta}{-\frac{b\cos^3\theta}{a^2\sin^3\theta}}$$

$$= \frac{b\sin\theta}{\cos\theta} - \frac{a^2\sin^3\theta}{b\cos^3\theta}\left(1+\frac{b^2}{a^2\sin^2\theta}\right)$$

$$= -\frac{(a^2+b^2)}{b}\tan^3\theta$$

We know that $1+\tan^2\theta = \sec^2\theta$

$$\sec^2\theta - \tan^2\theta = 1$$

$$\left(\frac{a\bar{x}}{a^2+b^2}\right)^{2/3} - \left(\frac{b\bar{y}}{a^2+b^2}\right)^{2/3} = 1$$

$$(a\bar{x})^{2/3} - (b\bar{y})^{2/3} = (a^2+b^2)^{2/3}$$

Locus of $(\bar{x},\bar{y})$ is $(a\,x)^{2/3} - (b\,y)^{2/3} = (a^2+b^2)^{2/3}$

17. Find the evolute of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Solution:

Parametric coordinates: $x = a\cos\theta$, $y = b\sin\theta$

$$x = a\cos\theta$$

$$\frac{dx}{d\theta} = -a\sin\theta$$

$$y = b\sin\theta$$

$$\frac{dy}{d\theta} = b\cos\theta$$

$$\frac{dy}{dx} = \frac{b\cos\theta}{-a\sin\theta} = -\frac{b}{a}\cot\theta$$

$$\frac{d^2y}{dx^2} = -\frac{b}{a}(-\operatorname{cosec}^2\theta)\frac{d\theta}{dx}$$

$$= \frac{b}{a}\operatorname{cosec}^2\theta \times \frac{-1}{a\sin\theta}$$

$$= -\frac{b}{a^2}\operatorname{cosec}^3\theta$$

$$\bar{x} = x - \frac{y_1(1+y_1^2)}{y_2}$$

$$= a\cos\theta - \frac{-\frac{b}{a}\cot\theta\left(1+\frac{b^2}{a^2}\cot^2\theta\right)}{-\frac{b}{a^2}\operatorname{cosec}^3\theta}$$

$$= a\cos\theta - a\sin^2\theta\cos\theta - \frac{b^2}{a}\cos^3\theta$$

$$= \frac{a^2-b^2}{a}\cos^3\theta$$

$$\bar{y} = y + \frac{1+y_1^2}{y_2}$$

$$= b\sin\theta + \frac{1+\frac{b^2}{a^2}\cot^2\theta}{-\frac{b}{a^2}\operatorname{cosec}^3\theta}$$

$$= b\sin\theta - b\cos^2\theta\sin\theta - \frac{a^2}{b}\sin^3\theta$$

$$= -\frac{(a^2-b^2)}{b}\sin^3\theta$$

We know that $\sin^2\theta + \cos^2\theta = 1$

$$\left(-\frac{b\bar{y}}{a^2-b^2}\right)^{2/3} + \left(\frac{a\bar{x}}{a^2-b^2}\right)^{2/3} = 1 \qquad \text{Value of } (-1)^{2/3} = 1$$

$$\left(\frac{a\bar{x}}{a^2-b^2}\right)^{2/3}+\left(\frac{b\bar{y}}{a^2-b^2}\right)^{2/3}=1$$

$$(a\bar{x})^{2/3}+(b\bar{y})^{2/3}=(a^2-b^2)^{2/3}$$

Locus of $(\bar{x},\bar{y})$ is $(ax)^{2/3}+(by)^{2/3}=(a^2-b^2)^{2/3}$

18. Find the evolute of the asteroid $x^{2/3}+y^{2/3}=a^{2/3}$**.**

Solution:

Parametric coordinates: $x = a\cos^3\theta$, $y = a\sin^3\theta$

$$x=a\cos^3\theta$$

$$\frac{dx}{d\theta}=3a\cos^2\theta(-\sin\theta)$$

$$y=a\sin^3\theta$$

$$\frac{dy}{d\theta}=3a\sin^2\theta(\cos\theta)$$

$$\frac{dy}{dx}=-\tan\theta$$

$$\frac{d^2y}{dx^2}=-\sec^2\theta\times\frac{d\theta}{dx}$$

$$\frac{d^2y}{dx^2}=-\sec^2\theta\times\frac{1}{3a\cos^2\theta(-\sin\theta)}=\frac{1}{3a\cos^4\theta\sin\theta}$$

$$\bar{x}=x-\frac{y_1\,(1+y_1{}^2)}{y_2}$$

$$=a\cos^3\theta-\frac{(-\tan\theta)\left(1+\tan^2\theta\right)}{\frac{1}{3a\cos^4\theta\sin\theta}}$$

$$=a\cos^3\theta+3a\cos\theta\sin^2\theta$$

$$\bar{y}=y+\frac{1+y_1{}^2}{y_2}$$

$$=a\sin^3\theta+\frac{1+\tan^2\theta}{\frac{1}{3a\cos^4\theta\sin\theta}}$$

$$=a\sin^3\theta+3a\sin\theta\cos^2\theta$$

Now $\bar{x}+\bar{y}=a(\cos^3\theta+3\cos^2\theta\sin\theta+3\cos\theta\sin^2\theta+\sin^3\theta)$

$$=a(\cos\theta+\sin\theta)^3$$

Now $\bar{x}-\bar{y}=a(\cos^3\theta+3\cos\theta\sin^2\theta-3\sin\theta\cos^2\theta-\sin^3\theta)$

$$=a(\cos\theta-\sin\theta)^3$$

Now $\left(\dfrac{\bar{x}+\bar{y}}{a}\right)^{2/3}+\left(\dfrac{\bar{x}-\bar{y}}{a}\right)^{2/3}=(\cos\theta+\sin\theta)^2+(\cos\theta-\sin\theta)^2=2$

Locus of $(\bar{x},\bar{y})$ is $(x+y)^{2/3}+(x-y)^{2/3}=2a^{2/3}$

Envelope

The envelope of a family of curves is the curve which touches each member of the family.

Example: All the straight lines of the family $x\cos\theta+y\sin\theta=1$, where θ is the parameter touches the circle $x^2+y^2=1$. (Refer Problem – 22)

Note

1. The envelope of the family of curves of the form $Am^2+Bm+C=0$ (quadratic form) is $B^2-4AC=0$.

2. Evolute of a curve is the envelope of the normals of the curve.

Type – 1 Envelope of SINGLE PARAMETER family of curves

19. Find the envelope of the family of straight lines $y=mx+\dfrac{a}{m}$, *m* being the parameter.

Solution:

$y=mx+\dfrac{a}{m}$

$y=\dfrac{m^2x+a}{m}$

$m^2x-my+a=0$

Here $A=x, B=-y, C=a$

Envelope is given by $B^2-4AC=0$.

$$y^2-4ax=0$$

20. Find the envelope of the family of straight lines $\dfrac{x}{t}+yt=2c$, *t* being the parameter.

Solution:

$\dfrac{x}{t}+yt=2c$

$t^2y-2ct+x=0$

Here $A=y, B=-2c, C=x$

Envelope is given by $B^2-4AC=0$.

$$xy=c^2$$

21. Find the envelope of the family of straight lines $x\cos\alpha+y\sin\alpha=a\sec\alpha$, α being the parameter.

Solution:

$x\cos\alpha+y\sin\alpha=a\sec\alpha$

Divide by cos α.

$x+y\tan\alpha=a\sec^2\alpha$

$x+y\tan\alpha=a(1+\tan^2\alpha)$

$a\tan^2\alpha-y\tan\alpha+(a-x)=0$

Here $A = a, B = -y, C = a - x$

Envelope is given by $B^2 - 4\,A\,C = 0$.

$$y^2 - 4\,a\,(a - x) = 0$$

22. **Find the envelope of the family of straight lines** $y = mx + \sqrt{a^2 m^2 + b^2}$ **, *m* being the parameter.**

Solution:

$y = mx + \sqrt{a^2 m^2 + b^2}$

Squaring

$(y - mx)^2 = a^2 m^2 + b^2$

$y^2 + m^2 x^2 - 2\,y\,m\,x = a^2 m^2 + b^2$

$m^2 (x^2 - a^2) - 2\,m\,x\,y + (y^2 - b^2) = 0$

Here $A = x^2 - a^2, B = -2xy, C = y^2 - b^2$

Envelope is given by $B^2 - 4\,A\,C = 0$.

$$4x^2y^2 - 4(x^2 - a^2)(y^2 - b^2) = 0$$

$$x^2b^2 + y^2a^2 = a^2b^2$$

Divide by a^2b^2, we get $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

23. **Find the envelope of the family of straight lines** $x\cos\theta + y\sin\theta = 1$**, θ being the parameter.**

Solution:

$x\cos\theta + y\sin\theta = 1$

Differentiate partially w.r.t. θ.

$x(-\sin\theta) + y\cos\theta = 1$

Squaring and adding

$(x\cos\theta + y\sin\theta)^2 + (-x\sin\theta + y\cos\theta)^2 = 1$

$$x^2 + y^2 = 1$$

Type – 2 Envelope of TWO PARAMETER family of curves

24. **Find the envelope of the family of straight lines** $\dfrac{x}{a} + \dfrac{y}{b} = 1$**, where *a* and *b* are the parameters connected by the relation** $a + b = c$**.**

Solution:

$\dfrac{x}{a} + \dfrac{y}{b} = 1$

Differentiate w.r.t. *b*

$$x\left(\frac{-1}{a^2}\right)\frac{da}{db} + y\left(\frac{-1}{b^2}\right) = 0$$

$$\frac{da}{db} = \frac{-a^2\,y}{b^2\,x} \qquad \text{\_\_\_\_\_\_\_\_\_\_(1)}$$

$a + b = c$

Differentiate w.r.t. *b*

$$\frac{da}{db}+1=0$$

$$\frac{da}{db}=-1 \qquad \text{\_\_\_\_\_\_\_\_(2)}$$

From (1) and (2)

$$\frac{-a^2 y}{b^2 x}=-1$$

$$a^2 y=b^2 x$$

$$\frac{x}{a^2}=\frac{y}{b^2}$$

$$\frac{x/a}{a}=\frac{y/b}{b}=\frac{\frac{x}{a}+\frac{y}{b}}{a+b}=\frac{1}{c}$$

Take first and last ratios.

$$\frac{x}{a^2}=\frac{1}{c}$$

$$a^2=cx$$

Take second and last two ratios.

$$\frac{y}{b^2}=\frac{1}{c}$$

$$b^2=cy$$

Given $:a+b=c$

$$(cx)^{1/2}+(cy)^{1/2}=c$$

$$x^{1/2}+y^{1/2}=c^{1/2}$$

25. Find the envelope of $\frac{x}{a}+\frac{y}{b}=1$, where $a^2+b^2=c^2$.

Solution:

$$\frac{x}{a}+\frac{y}{b}=1$$

Differentiate w.r.t. b

$$x\left(\frac{-1}{a^2}\right)\frac{da}{db}+y\left(\frac{-1}{b^2}\right)=0$$

$$\frac{da}{db}=\frac{-a^2 y}{b^2 x} \qquad \text{\_\_\_\_\_\_\_\_\_\_(1)}$$

$$a^2+b^2=c^2$$

Differentiate w.r.t. b

$$2a\frac{da}{db}+2b=0$$

$$\frac{da}{db} = -\frac{b}{a} \qquad \text{\_\_\_\_\_\_\_\_(2)}$$

From (1) and (2)

$$\frac{-a^2 y}{b^2 x} = -\frac{b}{a}$$

$$\frac{x}{a^3} = \frac{y}{b^3}$$

$$\frac{x/a}{a^2} = \frac{y/b}{b^2} = \frac{\frac{x}{a}+\frac{y}{b}}{a^2+b^2} = \frac{1}{c^2}$$

Take first and last ratios.

$$\frac{x}{a^3} = \frac{1}{c^2}$$

$$a^3 = c^2 x$$

Take second and last two ratios.

$$\frac{y}{b^3} = \frac{1}{c^2}$$

$$b^3 = c^2 y$$

Given $: a^2 + b^2 = c^2$

$$(c^2 x)^{2/3} + (c^2 y)^{2/3} = c^2$$

$$x^{2/3} + y^{2/3} = c^{2/3}$$

26. **Find the envelope of** $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$**, where** $a^2 + b^2 = c^2$**.**

Solution:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Differentiate w.r.t. b

$$x^2\left(\frac{-2}{a^3}\right)\frac{da}{db} + y^2\left(\frac{-2}{b^3}\right) = 0$$

$$\frac{da}{db} = \frac{-a^3 y^2}{b^3 x^2} \qquad \text{\_\_\_\_\_\_\_\_\_\_(1)}$$

$$a^2 + b^2 = c^2$$

Differentiate w.r.t. b

$$2a\frac{da}{db} + 2b = 0$$

$$\frac{da}{db} = -\frac{b}{a} \qquad \text{\_\_\_\_\_\_\_\_(2)}$$

From (1) and (2)

$$\frac{-a^3 y^2}{b^3 x^2} = -\frac{b}{a}$$

$$\frac{x^2}{a^4} = \frac{y^2}{b^4}$$

$$\frac{x^2/a^2}{a^2} = \frac{y^2/b^2}{b^2} = \frac{\frac{x^2}{a^2}+\frac{y^2}{b^2}}{a^2+b^2} = \frac{1}{c^2}$$

Take first and last ratios.

$$\frac{x^2}{a^4} = \frac{1}{c^2}$$

$$a^4 = c^2 x^2$$

Take second and last two ratios.

$$\frac{y^2}{b^4} = \frac{1}{c^2}$$

$$b^4 = c^2 y^2$$

Given $: a^2 + b^2 = c^2$

$$(c^2 x^2)^{1/2} + (c^2 y^2)^{1/2} = c^2$$

$$x + y = c$$

Beta and Gamma Functions

Gamma Function

$$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1}\, dx \text{ for } n > 0$$

Recurrence Formula

$$\Gamma(n+1) = n\Gamma(n) = n!$$

$$\Gamma(n) = (n-1)!$$

Note $\Gamma(1) = 0! = 1$

Beta Function

$$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\, dx \text{ where } m,\ n > 0$$

Property of Beta function

$$B(m,n) = B(n,m)$$

Other forms of Beta function

1. $B(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\, dx$ where $m,\ n > 0$

2. $B(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta\, d\theta$ where $m,\ n > 0$

Standard Result

$$B\left(\frac{m+1}{2},\frac{n+1}{2}\right)=2\int_0^{\pi/2}\sin^m\theta\cos^n\theta\,d\theta$$

Relation between Beta and Gamma functions

$$B(m,n)=\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

Note

1. $\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}$

2. $\displaystyle\int_0^{\pi/2}\sin^n\theta\,d\theta=\int_0^{\pi/2}\cos^n\theta\,d\theta=\frac{1}{2}\frac{\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{n+2}{2}\right)}$

27. Find $\Gamma(7/2)$**.**

Solution:

$$\Gamma(n+1)=n\Gamma(n)$$

$$\Gamma\left(\frac{7}{2}\right)=\frac{5}{2}\Gamma\left(\frac{5}{2}\right)$$

$$=\frac{5}{2}\Gamma\left(\frac{3}{2}+1\right)$$

$$=\frac{5}{2}\frac{3}{2}\Gamma\left(\frac{3}{2}\right)$$

$$=\frac{5}{2}\frac{3}{2}\Gamma\left(\frac{1}{2}+1\right)$$

$$=\frac{5}{2}\frac{3}{2}\frac{1}{2}\Gamma\left(\frac{1}{2}\right)$$

$$=\frac{15}{8}\sqrt{\pi}$$

28. Find $\displaystyle\int_0^{\pi/2}\sin^6\theta\cos^{10}\theta\,d\theta$**.**

Solution:

$m = 6$, $n = 10$

$$\int_0^{\pi/2}\sin^6\theta\cos^{10}\theta\,d\theta=\frac{1}{2}B\left(\frac{m+1}{2},\frac{n+1}{2}\right)$$

$$=\frac{1}{2}B\left(\frac{7}{2},\frac{11}{2}\right)$$

$$= \frac{1}{2} \frac{\Gamma\left(\frac{7}{2}\right)\Gamma\left(\frac{11}{2}\right)}{\Gamma\left(\frac{7}{2}+\frac{11}{2}\right)}$$

$$= \frac{1}{512} \frac{225 \times 63}{8!} \pi$$

29. **Find** $\int_0^{\pi/2} \cos^8 \theta \, d\theta$ **. Ans** $\frac{105}{768}\pi$

30. **Find** $\int_0^{\pi/2} \sin^5 \theta \, d\theta$ **. Ans** $\frac{8}{15}$

31. **Find** $\int_0^1 x^6 (1-x)^9 \, dx$ **.**

Solution:

$m = 7$, $n = 10$

$$B(m,n) = \int_0^1 x^{m-1} (1-x)^{n-1} \, dx$$

$$= \int_0^1 x^{7-1} (1-x)^{10-1} \, dx$$

$$= \frac{\Gamma(7)\Gamma(10)}{\Gamma(17)} = \frac{6!9!}{16!}$$

32. **Find** $\int_0^{\pi/2} \sqrt{\tan \theta} \, d\theta$ **.**

Solution:

$$\int_0^{\pi/2} \sqrt{\tan \theta} \, d\theta = \int_0^{\pi/2} \sqrt{\frac{\sin \theta}{\cos \theta}} \, d\theta = \int_0^{\pi/2} \sin^{1/2} \theta \cos^{-1/2} \theta \, d\theta$$

$$= \frac{1}{2} B\left(\frac{3/2}{2}, \frac{1/2}{2}\right)$$

$$= \frac{1}{2} B\left(\frac{3}{4}, \frac{1}{4}\right)$$

$$= \frac{1}{2} \frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right)}{\Gamma(1)}$$

Formula $\Gamma(n)\Gamma(1-n) = \frac{\pi}{\sin n\pi}$

$$= \frac{1}{2} \frac{\pi}{\left(1/\sqrt{2}\right)} = \frac{\pi}{\sqrt{2}}$$

33. **Find** $\int_0^{\pi/2} \sqrt{\cot\theta}\, d\theta$ **. Ans** $\frac{\pi}{\sqrt{2}}$

34. **Prove that** $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ **.**

Solution:

$$B(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta\, d\theta$$

Put $m = n = ½$

$$B\left(\frac{1}{2},\frac{1}{2}\right) = 2\int_0^{\pi/2} d\theta = \pi$$

$$\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(1)} = \pi$$

$$\left[\Gamma\left(\frac{1}{2}\right)\right]^2 = \pi\,\Gamma(1)$$ **Formula** $\Gamma(n) = (n-1)!$

$$= \pi\, 0!$$

$$= \pi$$

Hence $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

35. **Prove that** $\frac{B(m+1,n)}{B(m,n+1)} = \frac{m}{n}$ **.**

Solution:

$$\frac{B(m+1,n)}{B(m,n+1)} = \frac{\dfrac{\Gamma(m+1)\Gamma(n)}{\Gamma(m+n+1)}}{\dfrac{\Gamma(m)\Gamma(n+1)}{\Gamma(m+n+1)}} = \frac{m\Gamma(m)\Gamma(n)}{n\Gamma(m)\Gamma(n)} = \frac{m}{n}$$

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