**Module – 1**

Characteristic equation – Eigen values of a real matrix – Eigen vectors of a real matrix – Properties of Eigen values – Cayley-Hamilton theorem – Finding $A^{-1}$ using Cayley-Hamilton theorem – Finding higher powers of A using Cayley-Hamilton theorem – Orthogonal reduction of a symmetric matrix to diagonal form – Reduction of quadratic form to canonical form by orthogonal transformations – Orthogonal matrices – Applications of Matrices in Engineering.

## BASIC CONCEPTS

### CHARACTERISTIC EQUATION

The characteristic equation of any square matrix $A$ is $|A-\lambda I|=0$.

For $2 \times 2$ matrix, the characteristic equation is $\lambda^2 - S_1\lambda + S_2 = 0$.

where $S_1$ = Sum of the main diagonal elements

$S_2$ = Determinant of the matrix

For $3 \times 3$ matrix, the characteristic equation is $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$.

where $S_1$ = Sum of the main diagonal elements

$S_2$ = Sum of the minors of the main diagonal elements.

$S_3$ = Determinant of the matrix

### EIGEN VALUES

The roots of the characteristic equation are called eigen values.

### EIGEN VECTOR

The eigen vector of the matrix $A$ is $(A-\lambda I)X = 0$, $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$, a non-zero column vector.

### SINGULAR & NON-SINGULAR MATRIX

A square matrix $A$ is said to be singular if $|A| = 0$, otherwise it is a non-singular matrix.

### PROPERTIES OF EIGEN VALUES

1. Sum of the eigen values is equal to sum of the main diagonal elements.
2. Product of the eigen values is equal to determinant of the matrix.
3. A square matrix A and its transpose have the same eigen values.
4. If the matrix is singular, then one of its eigen value is 0.
5. If the matrix is upper or lower triangular, then the eigen values are its main diagonal values.
6. If $\lambda$ is an eigen value of an orthogonal matrix, then $\frac{1}{\lambda}$ is also its eigen value.

7. If $\lambda_1, \lambda_2, \lambda_3$ are the eigen values of the matrix $A$, then $\lambda_1^k, \lambda_2^k, \lambda_3^k$ are the eigen values of $A^k$.
8. If $\lambda_1, \lambda_2, \lambda_3$ are the eigen values of the matrix $A$, then $k\lambda_1, k\lambda_2, k\lambda_3$ are eigen values of matrix $kA$.
9. If $\lambda_1, \lambda_2, \lambda_3$ are the eigen values of the matrix $A$, then $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \frac{1}{\lambda_3}$ are the eigen values of $A^{-1}$.
10. If $\lambda_1, \lambda_2, \lambda_3$ are the eigen values of the matrix $A$, then $\lambda_1 - K$, $\lambda_2 - K$, $\lambda_3 - K$ are the eigenvalues of matrix $A - KI$.
11. The eigen values of a real symmetric matrix are real numbers.

## PROPERTIES OF EIGEN VECTORS

1. If the eigen values of a matrix are distinct, then the corresponding eigen vectors are linearly independent. If $|A| \neq 0$, then the eigenvectors are *linearly independent*. If $|A| = 0$, then the eigenvectors are *linearly dependent*.
2. The eigen vectors corresponding to distinct eigen values of a real symmetric matrix are orthogonal.
3. The eigen vector corresponding to an eigenvalue is not unique.
4. If two or more eigen values are equal, then the eigen vectors may be linearly dependent or linearly independent.

## ORTHOGONAL MATRIX

A square matrix $A$ with real entries is said to be orthogonal if $AA^T = A^T A = I$, where $A^T$ is the transpose of the matrix $A$. (*i.e.*) $A^T = A^{-1}$ for an orthogonal matrix.

## PROPERTIES OF AN ORTHOGONAL MATRIX

1. If $\lambda$ is an eigen value of an orthogonal matrix, then $\frac{1}{\lambda}$ is also an eigen value.
2. If $A$ is an orthogonal matrix, then $|A| = \pm 1$.
3. The transpose of an orthogonal matrix is also orthogonal.
4. The inverse of an orthogonal matrix is also orthogonal.

## CONDITIONS FOR PAIRWISE ORTHOGONAL VECTORS

In a real symmetric matrix, the eigen vectors $X_1, X_2, X_3$ are said to be pair wise orthogonal, if $X_1 X_2^T = 0,\ X_2 X_3^T = 0,\ X_3 X_1^T = 0.$

# PROBLEMS

## EIGEN VALUES AND EIGEN VECTORS

1. **Find the Eigen values and Eigen vectors of the matrix** $A = \begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix}$**.**

**Solution:**

Let $A = \begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix}$.

Step 1: To find the characteristic equation

The Characteristic equation of A is $|A - \lambda I| = 0$

(i.e.) $\lambda^2 - S_1\lambda + S_2 = 0$ where

$S_1$ = Sum of the main diagonal elements = (1) + ( –1) = 0

$S_2 = |A| = \begin{vmatrix} 1 & 1 \\ 3 & -1 \end{vmatrix} = -1 - 3 = -4$

Hence the characteristic equation is

$$\lambda^2 - (0)\lambda + (-4) = 0$$

$$\lambda^2 - 4 = 0$$

Step 2: To solve the characteristic equation

$$\lambda^2 - 4 = 0$$

$$\lambda^2 = 4$$

$$\lambda = \pm 2$$

Hence, the Eigen values are – 2, 2.

Step 3: To find the Eigenvectors:

To find the Eigenvectors, solve $(A - \lambda I)X = 0$

$$\left(\begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{pmatrix} 1-\lambda & 1 \\ 3 & -1-\lambda \end{pmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{———————} (1)$$

**Case 1.** If $\lambda = -2$, then Eqn. (1) becomes

$$\begin{pmatrix} 3 & 1 \\ 3 & 1 \end{pmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$3x_1 + x_2 = 0$$

$$3x_1 + x_2 = 0$$

i.e., we get, only one equation $3x_1 + x_2 = 0$

$i.e., 3x_1 = -x_2$

$$\frac{x_1}{1} = \frac{x_2}{-3}$$

Hence the corresponding Eigenvector is $X_1 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$

**Case 2.** If $\lambda = 2$, then equation (1) becomes

$$\begin{pmatrix} -1 & 1 \\ 3 & -3 \end{pmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-x_1 + x_2 = 0$$

$$3x_1 - 3x_2 = 0$$

i.e., we get, only one equation $x_1 - x_2 = 0$

*i.e.,* $x_1 = x_2$

$$\frac{x_1}{1} = \frac{x_2}{1}$$

Hence the corresponding Eigenvector is $X_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

**TYPE – 1 NON-SYMMETRIC MATRIX WITH NON-REPEATED EIGENVALUES**

2. **Find the eigen values and eigen vectors of the matrix** $A = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{pmatrix}$ **.**

**Solution:**

The characteristic equation of $A$ is $|A - \lambda I| = 0$

⇨ The characteristic equation is $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

where $S_1$ = Sum of the diagonal elements=1+2+3 = 6

$S_2$ = Sum of the minors of the diagonal elements.

$$= \begin{vmatrix} 2 & 1 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 1 & -1 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix} = 4 + 5 + 2 = 11$$

$$S_3 = |A| = \begin{vmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{vmatrix} = 6$$

Therefore, the Characteristic equation is: $\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$

$$(\lambda - 1)(\lambda^2 - 5\lambda + 6) = 0$$

$$\lambda = 1 \text{ and } \lambda^2 - 5\lambda + 6 = 0$$

$$(\lambda - 2)(\lambda - 3) = 0$$

$$\lambda = 1, 2, 3$$

<u>To find the Eigen Vectors:</u>

To get the Eigenvectors, solve $(A - \lambda I)X = 0$

$$\begin{pmatrix} 1-\lambda & 0 & -1 \\ 1 & 2-\lambda & 1 \\ 2 & 2 & 3-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \rightarrow \text{(A)}$$

$(1 – \lambda) x_1 + 0 x_2 – x_3 = 0$
$x_1 + (2 – \lambda) x_2 + x_3 = 0$
$2 x_1 + 2 x_2 + (3 – \lambda) x_3 = 0$

**Case (1) :** $\lambda = 1$ Then Equation (A) becomes

$$\begin{pmatrix} 0 & 0 & -1 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$0 x_1 + 0 x_2 – x_3 = 0$
$x_1 + x_2 + x_3 = 0$
$2 x_1 + 2 x_2 + 2 x_3 = 0$

Solve (1) and (2), using cross multiplication rule, we get

$$\frac{x_1}{\begin{vmatrix} 0 & -1 \\ 1 & 1 \end{vmatrix}} = \frac{x_2}{\begin{vmatrix} -1 & 0 \\ 1 & 1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 0 & 0 \\ 1 & 1 \end{vmatrix}}$$

$$\frac{x_1}{1} = \frac{x_2}{-1} = \frac{x_3}{0}$$

Hence, the corresponding Eigen vector is $X_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$.

**Case (2) :** $\lambda = 2$ Then Equation (A) becomes

$$\begin{pmatrix} -1 & 0 & -1 \\ 1 & 0 & 1 \\ 2 & 2 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$–1 x_1 + 0 x_2 – x_3 = 0$
$x_1 + 0 x_2 + x_3 = 0$
$2 x_1 + 2 x_2 + x_3 = 0$

Solve (2) and (3),using cross multiplication rule, we get

$$\frac{x_1}{\begin{vmatrix} 0 & 1 \\ 2 & 1 \end{vmatrix}} = \frac{x_2}{\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 1 & 0 \\ 2 & 2 \end{vmatrix}}$$

$$\frac{x_1}{-2} = \frac{x_2}{1} = \frac{x_3}{2}$$

Hence, the corresponding Eigen vector is $X_2 = \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix}$.

**Case (3) :** $\lambda = 3$ Then Equation (A) becomes

$$\begin{pmatrix} -2 & 0 & -1 \\ 1 & -1 & 1 \\ 2 & 2 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$-2\, x_1 + 0\, x_2 - x_3 = 0$
$x_1 - x_2 + x_3 = 0$
$2\, x_1 + 2\, x_2 + 0\, x_3 = 0$

Solve (1) and (2), using cross multiplication rule, we get

$$\frac{x_1}{\begin{vmatrix} 0 & -1 \\ -1 & 1 \end{vmatrix}} = \frac{x_2}{\begin{vmatrix} -1 & -2 \\ 1 & 1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -2 & 0 \\ 1 & -1 \end{vmatrix}}$$

$$\frac{x_1}{-1} = \frac{x_2}{1} = \frac{x_3}{2}$$

Hence, the corresponding Eigen vector is $X_3 = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$.

## Type – 2 NON-SYMMETRIC MATRIX WITH REPEATED EIGENVALUES

3. **Find the eigen values and eigen vectors of the matrix** $A = \begin{pmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{pmatrix}$**.**

**Solution:**

The characteristic equation of $A$ is $|A - \lambda I| = 0$

⇨ The characteristic equation is $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

where $S_1$ = Sum of the main diagonal elements = – 2 +1+0 = – 1

$S_2$ = Sum of the minors of the main diagonal elements.

$$= \begin{vmatrix} 1 & -6 \\ -2 & 0 \end{vmatrix} + \begin{vmatrix} -2 & -3 \\ -1 & 0 \end{vmatrix} + \begin{vmatrix} -2 & 2 \\ 2 & 1 \end{vmatrix} = -12 - 3 - 6 = -21$$

$$S_3 = |A| = \begin{vmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{vmatrix} = 45$$

Therefore, the Characteristic equation is $\lambda^3 + \lambda^2 - 21\,\lambda - 45 = 0$.

To solve the Characteristic equation

$\lambda^3 + \lambda^2 - 21\lambda - 45 = 0$

If $\lambda = -3$, then $\lambda^3 + \lambda^2 - 21\lambda - 45 = 0$.

Therefore, $\lambda = -3$ is a root.

By Synthetic division

$$\begin{array}{r|rrrr|l} -3 & 1 & 1 & -21 & -45 \\ & 0 & -3 & 6 & 45 \\ \hline & 1 & -2 & -15 & 0 \end{array}$$

Other roots are given by $\lambda^2 - 2\lambda - 15 = 0$.

$$(\lambda - 5)(\lambda + 3) = 0$$

Hence , the Eigen values of $A$ are $\lambda = 5, -3, -3$.

To find the Eigen Vectors:

To get the Eigenvectors, solve $(A - \lambda I)X = 0$

$$\begin{pmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & 0-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad \rightarrow \text{(A)}$$

**Case (1) : $\lambda = 5$** Then equation (A) becomes

$$\begin{pmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$-7x_1 + 2x_2 - 3x_3 = 0$

$2x_1 - 4x_2 - 6x_3 = 0$

$-x_1 - 2x_2 - 5x_3 = 0$

Solving (1) & (2) by rule of cross multiplication, we get

$$\frac{x_1}{\begin{vmatrix} 2 & -3 \\ -2 & -5 \end{vmatrix}} = \frac{x_2}{\begin{vmatrix} -3 & -7 \\ -5 & -1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -7 & 2 \\ -1 & -2 \end{vmatrix}}$$

$$\frac{x_1}{-16} = \frac{x_2}{-32} = \frac{x_3}{16}$$

$$\frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{-1}$$

Hence, the corresponding Eigen vector is $X_1 = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$.

**Case (2) : λ = –3** Then equation (A) becomes

$$\begin{pmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$x_1 + 2x_2 - 3x_3 = 0$
$2x_1 + 4x_2 - 6x_3 = 0$
$-x_1 - 2x_2 + 3x_3 = 0$
All are same equations.
Put $x_1 = 0$.
$2x_2 - 3x_3 = 0$
$2x_2 = 3x_3$

$$\frac{x_2}{3} = \frac{x_3}{2}$$

Hence, the corresponding Eigen vector is $X_2 = \begin{pmatrix} 0 \\ 3 \\ 2 \end{pmatrix}$.

Put $x_2 = 0$.
$x_1 - 3x_3 = 0$
$x_1 = 3x_3$

$$\frac{x_1}{3} = \frac{x_3}{1}$$

Hence, the corresponding Eigen vector is $X_3 = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}$.

4. **Find the eigen values and eigen vectors of the matrix** $A = \begin{pmatrix} 6 & -6 & 5 \\ 14 & -13 & 10 \\ 7 & -6 & 4 \end{pmatrix}$**.**

**Solution:**

The characteristic equation of $A$ is $|A - \lambda I| = 0$

⇨ The characteristic equation is $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

Where $S_1$ = Sum of the diagonal elements = 6 – 13 + 4 = – 3

$S_2$ = Sum of the minors of the diagonal elements.

$$= \begin{vmatrix} -13 & 10 \\ -6 & 4 \end{vmatrix} + \begin{vmatrix} 6 & 5 \\ 7 & 4 \end{vmatrix} + \begin{vmatrix} 6 & -6 \\ 14 & -13 \end{vmatrix} = 8 - 11 + 6 = 3$$

=4 – 0 + 4 – 0 + 4 – 0 = 12

$$S_3 = |A| = \begin{vmatrix} 6 & -6 & 5 \\ 14 & -13 & 10 \\ 7 & -6 & 4 \end{vmatrix} = -1$$

Therefore, The Characteristic equation is $\lambda^3 + 3\lambda^2 + 3\lambda + 1 = 0$.

<u>To Solve the Characteristic equation</u>

$\lambda^3 + 3\lambda^2 + 3\lambda + 1 = 0$.

$(\lambda + 1)^3 = 0$

Hence , the Eigen values are $\lambda = -1, -1, -1$.

<u>To find the Eigen Vectors:</u>

To get the Eigenvectors, solve $(A - \lambda I)X = 0$

$$\begin{pmatrix} 6-\lambda & -6 & 5 \\ 14 & -13-\lambda & 10 \\ 7 & -6 & 4-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \rightarrow (A)$$

**Case (1) : $\lambda = -1$** Equation (A) becomes

$$\begin{pmatrix} 7 & -6 & 5 \\ 14 & -12 & 10 \\ 7 & -6 & 5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$7x_1 - 6x_2 + 5x_3 = 0$

$14x_1 - 12x_2 + 10x_3 = 0$

$7x_1 - 6x_2 + 5x_3 = 0$

All are same equations.

Put $x_1 = 0$.

$-6x_2 + 5x_3 = 0$

$5x_3 = 6x_2$

$$\frac{x_2}{5} = \frac{x_3}{6}$$

Hence, the corresponding Eigen vector is $X_1 = \begin{pmatrix} 0 \\ 5 \\ 6 \end{pmatrix}$.

Put $x_2 = 0$.

$7x_1 + 5x_3 = 0$

$7x_1 = -5x_3$

$$\frac{x_1}{-5} = \frac{x_3}{7}$$

Hence, the corresponding Eigen vector is $X_2 = \begin{pmatrix} -5 \\ 0 \\ 7 \end{pmatrix}$.

Put $x_3 = 0$.

$7x_1 - 6x_2 + 5x_3 = 0$

$7x_1 = 6x_2$

$\frac{x_1}{6} = \frac{x_2}{7}$

Hence, the corresponding Eigen vector is $X_3 = \begin{pmatrix} 6 \\ 7 \\ 0 \end{pmatrix}$.

**TYPE – 3 SYMMETRIC MATRIX WITH NON-REPEATED EIGEN VALUES**

5. **Find the eigen values and eigen vectors of the matrix** $A = \begin{pmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{pmatrix}$**.**

**Solution:**

The characteristic equation of $A$ is $|A - \lambda I| = 0$

⇨ The characteristic equation is $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

where $S_1$ = Sum of the main diagonal elements=3 + 5 + 3 = 11

$S_2$ = Sum of the minors of the main diagonal elements.

$= \begin{vmatrix} 3 & -1 \\ -1 & 5 \end{vmatrix} + \begin{vmatrix} 5 & -1 \\ -1 & 3 \end{vmatrix} + \begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix}$

= (15 – 1) + (15 – 1) + (9 – 1) = 36

$S_3 = |A| = \begin{vmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{vmatrix}$

$= 3(15-1) + 1(-3+1) + 1(1-5) = 42 - 2 - 4 = 36$

Therefore, The Characteristic equation is $\lambda^3 - 11\lambda^2 + 36\lambda - 36 = 0$.

<u>To Solve the Characteristic equation</u>

$\lambda^3 - 11\lambda^2 + 36\lambda - 36 = 0$

If $\lambda = 2$, *then* $\lambda^3 - 11\lambda^2 + 36\lambda - 36 = 0$

Therefore, $\lambda = 2$ is a root.

By Synthetic division

$$\begin{array}{r|rrrr} 2 & 1 & -11 & 36 & -36 \\ & 0 & 2 & -18 & 36 \\ \hline & 1 & -9 & 18 & \lfloor 0 \end{array}$$

Other roots are given by $\lambda^2 - 9\lambda + 18 = 0$

$(\lambda - 3)(\lambda - 6) = 0$

*i.e.*, $\lambda = 3$, $\lambda = 6$

Hence , the Eigen values are $\lambda = 2, 3, 6$

To find the Eigen Vectors:

To get the Eigenvectors, solve $(A-\lambda I)X = 0$

$$\begin{pmatrix} 3-\lambda & -1 & 1 \\ -1 & 5-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_(A)}$$

**Case (1) : $\lambda$ = 2** Then equation (A) becomes

$$\begin{pmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \qquad \begin{aligned} x_1 - x_2 + x_3 &= 0 \quad \to (1) \\ -x_1 + 3x_2 - x_3 &= 0 \quad \to (2) \\ x_1 - x_2 + x_3 &= 0 \quad \to (3) \end{aligned}$$

Solving (1) & (2) by rule of cross multiplication, we get

$$\frac{x_1}{1-3} = \frac{x_2}{-1+1} = \frac{x_3}{3-1}$$

$$\frac{x_1}{-2} = \frac{x_2}{0} = \frac{x_3}{2}$$

*i.e.,* $\frac{x_1}{1} = \frac{x_2}{0} = \frac{x_3}{-1}$

Hence, the corresponding Eigen vector is $X_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$.

**Case (2) : $\lambda$ = 3** Then equation (A) becomes

$$\begin{pmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{aligned} 0x_1 - x_2 + x_3 &= 0 \quad \to (4) \\ -x_1 + 2x_2 - x_3 &= 0 \quad \to (5) \\ x_1 - x_2 + 0x_3 &= 0 \quad \to (6) \end{aligned}$$

Solving (4) & (5) by rule of cross multiplication, we get

$$\frac{x_1}{1-2} = \frac{x_2}{-1-0} = \frac{x_3}{0-1}$$

$$\frac{x_1}{-1} = \frac{x_2}{-1} = \frac{x_3}{-1}$$

*i.e.,* $\frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{1}$

Hence, the corresponding Eigen vector is $X_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

**Case (3) : λ = 6** then equation (A) becomes

$$\begin{pmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$-3x_1 - x_2 + x_3 = 0 \qquad \rightarrow (7)$$

$$-x_1 - x_2 - x_3 = 0 \qquad \rightarrow (8)$$

$$x_1 - x_2 - 3x_3 = 0 \qquad \rightarrow (9)$$

Solving (7) & (8) by rule of cross multiplication, we get

$$\frac{x_1}{1+1} = \frac{x_2}{-1-3} = \frac{x_3}{3-1}$$

$$\frac{x_1}{1} = \frac{x_2}{-2} = \frac{x_3}{1}$$

Hence, the corresponding Eigen vector is $X_3 = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$.

**TYPE – 4 SYMMETRIC MATRIX WITH REPEATED EIGEN VALUES**

**6. Find the eigen values and eigen vectors of the matrix** $A = \begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix}$**.**

**Solution:**

The characteristic equation of $A$ is $|A - \lambda I| = 0$

⇨ The characteristic equation is $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

where $S_1$ = Sum of the main diagonal elements=6 + 3 + 3 = 12

$S_2$ = Sum of the minors of the main diagonal elements.

$$= \begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix} + \begin{vmatrix} 6 & 2 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 6 & -2 \\ -2 & 3 \end{vmatrix}$$

=( 9 – 1) + (18 – 4 ) + ( 18 – 4 ) = 8+ 14 + 14 = 36

$$S_3 = |A| = \begin{vmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{vmatrix}$$

$$= 6(9-1) + 2(-6+2) + 2(2-6)$$

$$= 6(8) + 2(-4) + 2(-4) = 48 - 8 - 8 = 32$$

Therefore, the characteristic equation is $\lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0$.

To Solve the Characteristic equation

$\lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0.$

If $\lambda = 2$, *then* $\lambda^3 - 12\lambda^2 + 36\lambda - 32 = 8 - 42 + 72 - 32 = 0$

Therefore, $\lambda = 2$ is a root.

By Synthetic division

$$\begin{array}{c|cccc} & 1 & -12 & 36 & -32 \\ 2 & 0 & 2 & -20 & 32 \\ \hline & 1 & -10 & 16 & \lfloor 0 \end{array}$$

The other roots are given by $\lambda^2 - 10\lambda + 16 = 0$

$(\lambda - 8)(\lambda - 2) = 0$

*i.e.,* $\lambda = 8,\ \lambda = 2$

Hence , the Eigen values are $\lambda = 8, 2, 2$

To find the Eigen Vectors:

To get the Eigenvectors, solve $(A - \lambda I)X = 0$

$$\begin{pmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \to (A)$$

**Case (1) : $\lambda = 8$** Equation (A) becomes

$$\begin{pmatrix} -2 & -2 & 2 \\ -2 & -5 & -1 \\ 2 & -1 & -5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$-2x_1 - 2x_2 + 2x_3 = 0 \quad \to (1)$

$-2x_1 - 5x_2 - x_3 = 0 \quad \to (2)$

$2x_1 - x_2 - 5x_3 = 0 \quad \to (3)$

Solving (1) & (2) by rule of cross multiplication, we get

$$\frac{x_1}{2+10} = \frac{x_2}{-4-2} = \frac{x_3}{10-4}$$

$$\frac{x_1}{12} = \frac{x_2}{-6} = \frac{x_3}{6}$$

*i.e.,* $\dfrac{x_1}{2} = \dfrac{x_2}{-1} = \dfrac{x_3}{1}$

Hence, the corresponding Eigen vector is $X_1 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$

**Case (2) : λ = 2** Equation (A) becomes

$$\begin{pmatrix} 4 & -2 & 2 \\ -2 & 1 & -1 \\ 2 & -1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$4x_1 - 2x_2 + 2x_3 = 0 \quad \rightarrow (4)$

$-2x_1 + x_2 - x_3 = 0 \quad \rightarrow (5)$

$2x_1 - x_2 + x_3 = 0 \quad \rightarrow (6)$

(4) , (5) & (6) are same as

$2x_1 - x_2 + x_3 = 0$

If

$x_1 = 0$ , *we get* $-x_2 + x_3 = 0$

$$-x_2 = -x_3$$

$$x_2 = x_3$$

$$i.e., \frac{x_2}{1} = \frac{x_3}{1}$$

Hence, the corresponding Eigen vector $X_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

**Case (3) : λ = 2**

Let the third eigen vector be $X_3 = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ and $X_3$ should be orthogonal with $X_1$ and $X_2$

$X_1^T X_3 = 0 \Rightarrow 2a - b + c = 0 \quad \rightarrow (7)$

$X_2^T X_3 = 0 \Rightarrow 0a + b + c = 0 \quad \rightarrow (8)$

Solving (7) & (8) by rule of cross multiplication, we get

$$\frac{a}{-1-1} = \frac{b}{0-2} = \frac{c}{2-0}$$

$$\frac{a}{-2} = \frac{b}{-2} = \frac{c}{2}$$

$$i.e., \frac{a}{1} = \frac{b}{1} = \frac{c}{-1}$$

Hence, the corresponding Eigen vector $X_3 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$.

**PROBLEMS BASED ON PROPERTIES**

7. **Find the sum and product of all the eigen values of A =** $\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$ **.**

**Solution :** Sum of the eigen values = Sum of the main diagonal elements = 8+7+3 = 18

Product of the eigen values = $|A| = 8(5)+6(-10)+2(10) = 0$

8. **Find the eigen values of the inverse of the matrix** $\mathbf{A} = \begin{bmatrix} \mathbf{2} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{3} & \mathbf{4} \\ \mathbf{0} & \mathbf{0} & \mathbf{4} \end{bmatrix}$**.**

**Solution :** In a triangular matrix, the main diagonal values are the eigen values of the matrix.

$\therefore$ 2, 3, 4 are the eigen values of A. Hence the eigen values of $A^{-1} = \frac{1}{2}, \frac{1}{3}, \frac{1}{4}$.

9. **Find the eigen vector of** $A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$ **corresponding to the eigen value 2.**

**Solution :** Let X = $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ be the eigen vector of the matrix corresponding to the eigen value$\lambda$.

The eigen vectors are obtained from the equation $(A - \lambda I)X = 0$

$$\Rightarrow \begin{pmatrix} 2-\lambda & 1 & 0 \\ 0 & 2-\lambda & 1 \\ 0 & 0 & 2-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

When $\lambda = 2$, $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow x_2 = 0, x_3 = 0$ and $x_1$ takes any value, say $k \neq 0$.

Therefore the eigenvector is $X_1 = \begin{pmatrix} k \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.

10. **If 3 and 5 are two eigen values of the matrix** $A = \begin{pmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{pmatrix}$ **then find its third eigen**

**value and hence find $|A|$.**

**Solution:**

Let the third eigen value of A be $\lambda_3$

Sum of the eigen values = 3 + 5 + $\lambda_3$=8 + $\lambda_3$=trace of $A$

$$8 + \lambda_3 = 8 + 7 + 3$$

$$\lambda_3 = 18 - 8 = 10, \qquad \lambda_3 = 10.$$

Hence, $|A| =$ product of the eigen values of $A$

$$|A| = 3 \times 5 \times 10 = 150.$$

11. **If the eigen values of the matrix A of order 3 × 3 matrix are 2,3 and 1, then find the eigen values of adjoint of A.**

**Solution:** We know that, adjoint of A = $A^{-1}|A|$.

$|A|$ = product of the eigen values = (2)(3)(1) = 6.

Eigen values of $A^{-1}$= $\frac{1}{2}, \frac{1}{3}, 1.$

$\therefore$ Eigen values of adj A = $\frac{1}{2}(6), \frac{1}{3}(6), (1)(6) = 3, 2, 6$

12. **If 1 and 2 are the two eigen values of $A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$, find $|A|$ without expanding the determinant.**

**Solution:** Let λ be the third eigen value of the given matrix.

We know that, sum of the eigen values = sum of the main diagonal elements.

i.e. 1 + 2 + λ = 2+2+2 $\Rightarrow$ λ = 3

Now, $|A|$ = product of all eigen values = (1)(2)(3) = 6

13. **One of the eigen values of $\begin{bmatrix} \mathbf{7} & \mathbf{4} & \mathbf{4} \\ \mathbf{4} & \mathbf{-8} & \mathbf{-1} \\ \mathbf{4} & \mathbf{-1} & \mathbf{-8} \end{bmatrix}$ is –9, find the other two eigen values.**

**Solution:** Let $\lambda_1$, $\lambda_2$ be the other two eigen values.

We know that, sum of the eigen values = sum of the main diagonal elements

i.e. $\lambda_1 + \lambda_2 - 9 = 7 - 8 - 8 = -9$

$\lambda_1 + \lambda_2 = 0 \Rightarrow \lambda_1 = -\lambda_2$ …(1)

We know that, product of the eigen values = $|A|$

$-9\lambda_1\lambda_2 = |A| = 441$

$\lambda_1\lambda_2 = -49 \Rightarrow \lambda_1 = \frac{-49}{\lambda_2}$ … (2)

substitute in (1) we get, $-\lambda_2 = \frac{-49}{\lambda_2}$

$\lambda_2{}^2 = 49 \Rightarrow \lambda_2 = \pm 7$

(1) $\Rightarrow \lambda_1 = \mp 7$. Hence the other two eigen values are 7 and -7.

14. **If $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ is an eigen vector of $\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$, find the corresponding eigen value.**

**Solution:** (A - λI)X = 0 $\Rightarrow \begin{pmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$\Rightarrow$ (–2–λ)(1)+2(2)+(-3)(-1) = 0 $\Rightarrow$ λ = 5.

15. **Find the constants 'a' & 'c' such that the matrix $\begin{pmatrix} a & 4 \\ 1 & c \end{pmatrix}$ has 3 & –2 as its eigen values.**

**Solution:**
Sum of the eigen values = sum of the main diagonals $\Rightarrow$ a + c = 3–2 = 1-----(1)
product of the eigen values = $|A| \Rightarrow$ (3)(-2) = ac – 4

i.e. –6 = ac – 4 $\Rightarrow$ ac = -2

$\therefore$ c = -2/a
sub c in (1) a + c = 1 $\Rightarrow$ a + (-2/a) =1 $\Rightarrow$ $a^2$-2 = a i.e. $a^2$-a-2 =0
solving a = -1, 2 $\Rightarrow$ c = 2,-1

16. **If λ is the eigen value of the matrix A, then prove that $\lambda^2$ is the eigen value of $A^2$.**
**Solution:**
Let X be the eigen vector of the matrix A corresponding to the eigen value λ, then AX = λ X.
Multiply by A $\Rightarrow$ $A^2$ X = A (λX)
= λ(AX)
= λ(λ X)
= $\lambda^2$X
Hence, $\lambda^2$ is the eigen value of $A^2$.

17. **If 2,-1,-3 are the eigen values of the matrix A, find the eigen values of the matrix $A^2 - 2I$.**
**Solution :** The eigen values of $A^2$ are $2^2$, $(-1)^2$, $(-3)^2$ = 4, 1, 9.
The eigen values of $A^2$-2I are 4 – 2,1–2,9 –2 = 2,–1,7

18. **If 2,3 are the two eigen values of $\begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ b & 0 & 2 \end{pmatrix}$,then find the value of b.**

**Solution:** Let λ be the third eigen value of the given matrix.
Sum of the eigen values = sum of the main diagonals
i.e. 2+3+λ = 6 $\Rightarrow$ λ = 1.
product of the eigen values = $|A|$
(1)(2)(3) = 2(4) +1(-2b) $\Rightarrow$ 6 = 8-2b $\Rightarrow$ b=1.

## DIAGONALIZATION

The process of transforming a square matrix $A$ in to a diagonal matrix $D$ is called diagonalization. A real symmetric matrix $A$ is said to be orthogonal diagonalizable, if there exists an orthogonal matrix $N$ such that $D = N^{-1} A N = N^T A N$, where $N$ is the modal matrix and $N^T$ is the transpose of the modal matrix. Diagonisation by orthogonal transformation is possible only for a real symmetric matrix.

19. **Diagonalize the matrix A=** $\begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}$ **by means of an orthogonal transformation.**

**Solution:**

The characteristic equation of $A$ is $|A-\lambda I| = 0$

⇨ The characteristic equation is: $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

where $S_1$ = Sum of the main diagonal elements=2 + 2 + 2 = 6

$S_2$ = Sum of the minors of the main diagonal elements.

$$= \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix}$$

=( 4-1) + (4 - 1 ) + ( 4 - 1 ) = 3+3+3 = 9

$$S_3 = |A| = \begin{vmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{vmatrix}$$

$= 2(4-1) + 1(-2+1) + 1(1-2) = = 6 - 1 - 1 = 4$

Therefore, The Characteristic equation is $\lambda^3 - 6\lambda^2 + 9\lambda - 4 = 0$.

<u>To Solve the Characteristic equation</u>

$\lambda^3 - 6\lambda^2 + 9\lambda - 4 = 0$.

If $\lambda = 1$, *then* $\lambda^3 - 6\lambda^2 + 9\lambda - 4 = 1 - 6 + 9 - 4 = 0$

Therefore, $\lambda = 1$ is a root.

By Synthetic division

$$\begin{array}{r|rrrr} 1 & 1 & -6 & 9 & -4 \\ & 0 & 1 & -5 & 4 \\ \hline & 1 & -5 & 4 & \underline{|0} \end{array}$$

Other roots are given by $\lambda^2 - 5\lambda + 4 = 0$

$(\lambda-1)(\lambda-4)=0$

*i.e.,* $\lambda=1,\ \lambda=4$

Hence , the Eigen values are λ = 1, 1, 4

<u>To find the Eigen Vectors:</u>

To get the Eigenvectors, solve $(A-\lambda I)X = 0$

$$\begin{pmatrix} 2-\lambda & -1 & 1 \\ -1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \rightarrow (A)$$

**Case (1) : λ = 4** Equation (A) becomes

$$\begin{pmatrix} -2 & -1 & 1 \\ -1 & -2 & -1 \\ 1 & -1 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$-2x_1-x_2+x_3=0 \quad \rightarrow (1)$

$-x_1-2x_2-x_3=0 \quad \rightarrow (2)$

$x_1-x_2-2x_3=0 \quad \rightarrow (3)$

Solving (1) & (2) by rule of cross multiplication, we get

$$\frac{x_1}{1+2}=\frac{x_2}{-1-2}=\frac{x_3}{4-1}$$

$$\frac{x_1}{3}=\frac{x_2}{-3}=\frac{x_3}{3}$$

*i.e.,* $\dfrac{x_1}{1}=\dfrac{x_2}{-1}=\dfrac{x_3}{1}$

Hence, the corresponding Eigen vector is $X_1=\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$

**Case (2) : λ= 1** Equation (A) becomes

$$\begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$4x_1-2x_2+2x_3=0 \quad \rightarrow (4)$

$-2x_1+x_2-x_3=0 \quad \rightarrow (5)$

$2x_1-x_2+x_3=0 \quad \rightarrow (6)$

(4) , (5) & (6) are same as

$x_1-x_2+x_3=0$

If

$x_3=0$ , $x_1=1$ *we get* $x_2=1$

Hence, the corresponding Eigen vector $X_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

**Case (3) : λ = 1**

Let the third eigen vector be $X_3 = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ and $X_3$ should be orthogonal with $X_1$ and $X_2$ .

$X_3^T X_1 = 0 \Rightarrow a - b + c = 0 \quad \rightarrow (7)$

$X_3^T X_2 = 0 \Rightarrow a + b + 0c = 0 \quad \rightarrow (8)$

Solving (7) & (8) by rule of cross multiplication, we get

$$\frac{a}{0-1} = \frac{b}{1-0} = \frac{c}{1+1}$$

$$\frac{a}{-1} = \frac{b}{1} = \frac{c}{2}$$

Hence, the corresponding Eigen vector $X_3 = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$

| Eigenvector | Normalized form |
|---|---|
| $X_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ | $\begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{-1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix}$ |
| $X_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ | $\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ \frac{0}{\sqrt{2}} \end{pmatrix}$ |
| $X_3 = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$ | $\begin{pmatrix} \frac{-1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \end{pmatrix}$ |

Normalized modal matrix $N=\begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} \\ \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{0}{\sqrt{2}} & \frac{2}{\sqrt{6}} \end{bmatrix}$, $N^T=\begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{0}{\sqrt{2}} \\ \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} \end{bmatrix}$

**To find AN:**

$$AN=\begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}\begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} \\ \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \end{bmatrix}$$

$$=\begin{bmatrix} \frac{2+1+1}{\sqrt{3}} & \frac{2-1+0}{\sqrt{2}} & \frac{-2-1+2}{\sqrt{6}} \\ \frac{-1-2-1}{\sqrt{3}} & \frac{-1+2+0}{\sqrt{2}} & \frac{1+2-2}{\sqrt{6}} \\ \frac{1+1+2}{\sqrt{3}} & \frac{1-1+0}{\sqrt{2}} & \frac{-1-1+4}{\sqrt{6}} \end{bmatrix}=\begin{bmatrix} \frac{4}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} \\ \frac{-4}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{4}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \end{bmatrix}$$

$\underline{D=N^T AN}$:

$$D=N^T AN=\begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{0}{\sqrt{2}} \\ \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} \end{bmatrix}\begin{bmatrix} \frac{4}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} \\ \frac{-4}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{4}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \end{bmatrix}$$

$$=\begin{bmatrix} \frac{4+4+4}{3} & \frac{1-1+0}{\sqrt{6}} & \frac{-1-1+2}{\sqrt{18}} \\ \frac{4-4+0}{\sqrt{6}} & \frac{1+1+0}{2} & \frac{-1+1+0}{\sqrt{12}} \\ \frac{-4-4+8}{\sqrt{18}} & \frac{1-1+0}{\sqrt{12}} & \frac{1+1+4}{6} \end{bmatrix}=\begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}=D\,(4,1,1)$$

## QUADRATIC FORM

A homogeneous polynomial of second degree in any number of variables is called quadratic form. Every quadratic form can be expressed as $XAX^T$, where $A$ is a symmetric matrix of the

$$\text{form } A=\begin{pmatrix} \text{coeff.}x^2 & \frac{1}{2}\text{coeff.}xy & \frac{1}{2}\text{coeff.}xz \\ \frac{1}{2}\text{coeff.}yx & \text{coeff.}y^2 & \frac{1}{2}\text{coeff.}yz \\ \frac{1}{2}\text{coeff.}zx & \frac{1}{2}\text{coeff.}zy & \text{coeff.}z^2 \end{pmatrix}.$$

**CANONICAL FORM**

Canonical form is equal to the sum or difference of squares of any number of variables.

**Matrix form of the Canonical Form**: Every canonical form can be expressed as $Y^{T}DY$ where $D$ is a diagonal matrix.

**ORTHOGONAL REDUCTION**

The orthogonal transformation $X=NY$ reduces the quadratic form to canonical form provided $N^TAN = D$ where N is normalized modal matrix.

$$\text{Quadratic form } = X^TAX = (NY)^TA(NY) = (Y^TN^T)A(NY) = Y^T(N^TAN)Y$$

$$= Y^T(D)Y$$

$$= y_1^2\lambda_1 + y_2^2\lambda_2 + y_3^2\lambda_3$$

**RANK OF THE QUADRATIC FORM (*r*)**

The number of nonzero terms in the canonical form is called rank of the quadratic form.

**INDEX OF THE QUADRATIC FORM (*p*)**

The number of positive terms in the canonical form is called index of the quadratic form.

**SIGNATURE OF THE QUADRATIC FORM (*s*)**

The difference between positive and negative terms in the canonical form is called signature.

**NATURE OF THE QUADRATIC FORM**

| Nature | If the eigen values are known | If the eigen values are unknown |
|---|---|---|
| Positive definite | All the eigen values are positive | $D_1,D_2,D_3$ are positive |
| Negative definite | All the eigen values are negative | $D_1$, $D_3$ are negative $D_2$ is positive |

| | | |
|---|---|---|
| Positive semi definite | All the eigen values are positive and atleast one is zero | $D_1 \geq 0$ , $D_2 \geq 0$, $D_3 \geq 0$ and atleast one is zero |
| Negative semi definite | All the eigen values are negative and atleast one is zero | $D_1 \leq 0$ , $D_2 \leq 0$, $D_3 \leq 0$ and atleast one is zero |
| Indefinite | eigen values are positive and negative | All the other cases |

Where, $D_1 = |a_{11}|$, $D_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$, $D_3 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$.

## REDUCTION OF QUADRATIC FORM TO CANONICAL FORM

20. **Reduce the quadratic form $x_1^2 + x_2^2 + 2x_1x_2$ into a canonical form by using orthogonal transformation. Hence find its rank, index, signature and nature.**

**Solution:**

The symmetric matrix $A = \begin{bmatrix} \text{coeff}.x^2 & \frac{1}{2}\text{coeff}.xy \\ \frac{1}{2}\text{coeff}.yx & \text{coeff}.y^2 \end{bmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

The characteristic equation of $A$ is $|A - \lambda I| = 0$

⇨ The characteristic equation is $\lambda^2 + S_1\lambda - S_2 = 0$

where $S_1$ = Sum of the main diagonal elements = 1+1 = 2

$S_2 = |A| = \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 0$

Therefore, The Characteristic equation is $\lambda^2 - 2\lambda = 0$.

<u>To solve the characteristic equation</u>

$\lambda^2 - 2\lambda = 0$.

$\lambda(\lambda - 2) = 0$

$\lambda(\lambda - 2) = 0$

*i.e.*, $\lambda = 0$, $\lambda = 2$

Hence , the Eigen values are $\lambda = 0, 2$.

<u>To find the Eigen Vectors:</u>

To get the Eigenvectors, solve $(A - \lambda I)X = 0$

$$\begin{pmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow (A)$$

**Case (1) : $\lambda = 0$** Equation (A) becomes

$$\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}$$

$$x_1+x_2=0 \quad \rightarrow (1)$$

$$x_1+x_2=0 \quad \rightarrow (2)$$

$$\Rightarrow \quad x_1=-x_2, \quad \Rightarrow \quad \frac{x_1}{-1}=\frac{x_2}{1}$$

Hence, the corresponding Eigen vector is $X_1=\begin{pmatrix}-1\\ 1\end{pmatrix}$

**Case (2) : λ = 2** Then equation (A) becomes

$$\begin{pmatrix}-1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}, \qquad \begin{aligned}-x_1+x_2&=0 \quad \rightarrow (3)\\ x_1-x_2&=0 \quad \rightarrow (4)\end{aligned}$$

Solving (4) & (5) by rule of cross multiplication, we get

$$\Rightarrow -x_1=-x_2$$

$$\frac{x_1}{1}=\frac{x_2}{1}$$

Hence, the corresponding Eigen vector is $X_2=\begin{pmatrix}1\\ 1\end{pmatrix}$

**Orthogonal Condition:**

$$X_1X_2^{T}=\begin{pmatrix}-1\\ 1\end{pmatrix}\begin{pmatrix}1 & 1\end{pmatrix}=0$$

They are pairwise Orthogonal.

| Eigenvector | Normalised form | Eigen vector | Normalized form |
|---|---|---|---|
| $X_1=\begin{pmatrix}-1\\ 1\end{pmatrix}$ | $\begin{pmatrix}-\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{pmatrix}$ | $X_2=\begin{pmatrix}1\\ 1\end{pmatrix}$ | $\begin{pmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{pmatrix}$ |

Normalized modal matrix N= $\begin{bmatrix}-\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{bmatrix}$ , $N^{T}=\begin{bmatrix}-\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{bmatrix}$

**<u>To find AN:</u>**

$$AN = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 0 & \sqrt{2} \\ 0 & \sqrt{2} \end{bmatrix}$$

$\underline{D = N^T AN}$ :

$$D = N^T AN = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 0 & \sqrt{2} \\ 0 & \sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -1+1 \\ 0 & 1+1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} = D\ (0,2)$$

Canonical Form is $Y^T(N^T AN)Y = Y^T DY = \begin{bmatrix} y_1 & y_2 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$

$$= 0y_1^{\,2} + 2y_2^{\,2}$$

Rank ($r$) = 1 ( No. of non zero terms in the canonical form)

Index ($p$) = 1 ( No. of Positive terms in the canonical form )

Signature (s) = $2p - r$ = 2(1) – 1= 1

Nature: Positive Semi-definite.

21. **Reduce the quadratic form** $8x_1^{\,2} + 7x_2^{\,2} + 3x_3^{\,2} - 12x_1x_2 - 8x_2x_3 + 4x_3x_1$ **into a canonical form by using orthogonal transformation. Hence find its rank, index, signature and nature.**

**Solution:**

The symmetric matrix A = $\begin{pmatrix} \text{coeff.}x^2 & \frac{1}{2}\text{coeff.}xy & \frac{1}{2}\text{coeff.}xz \\ \frac{1}{2}\text{coeff.}yx & \text{coeff.}y^2 & \frac{1}{2}\text{coeff.}yz \\ \frac{1}{2}\text{coeff.}zx & \frac{1}{2}\text{coeff.}zy & \text{coeff.}z^2 \end{pmatrix} = \begin{pmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{pmatrix}$

The characteristic equation of $A$ is $|A - \lambda I| = 0$

⇨ The characteristic equation is $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

where $S_1$ = Sum of the main diagonal elements=8+7+3 = 18

$S_2$ = Sum of the minors of the main diagonal elements.

$$= \begin{vmatrix} 8 & -6 \\ -6 & 7 \end{vmatrix} + \begin{vmatrix} 7 & -4 \\ -4 & 3 \end{vmatrix} + \begin{vmatrix} 8 & 2 \\ 2 & 3 \end{vmatrix}$$

=(56 – 36)+(21 – 16)+(24 – 4)=20+5+20 = 45

$$S_3 = |A| = \begin{vmatrix} 2 & 1 & -1 \\ 1 & 1 & -2 \\ -1 & -2 & 1 \end{vmatrix}$$

$$= 8(21-16) + 6(-18+8) + 2(24-14) = 40 - 60 + 20 = 0$$

Therefore, the characteristic equation is $\lambda^3 - 18\lambda^2 + 45\lambda = 0$.

<u>To solve the characteristic equation</u>

$\lambda^3 - 18\lambda^2 + 45\lambda = 0$.

$$\lambda\left(\lambda^2 - 18\lambda - 45\right) = 0$$

$$\lambda(\lambda-3)(\lambda-15) = 0$$

*i.e.*, $\lambda = 0,\ \lambda = 3,\ \lambda = 15$

Hence , the eigen values of **A** are $\lambda = 0, 3, 15$

<u>To find the Eigen Vectors:</u>

To get the Eigenvectors, solve $(A - \lambda I)X = 0$

$$\begin{pmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \to (A)$$

**Case (1) : $\lambda = 0$** Equation (A) becomes

$$\begin{pmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$8x_1 - 6x_2 + 2x_3 = 0 \quad \to (1)$$

$$-6x_1 + 7x_2 - 4x_3 = 0 \quad \to (2)$$

$$2x_1 - 4x_2 + 3x_3 = 0 \quad \to (3)$$

Solving (1) & (2) by rule of cross multiplication, we get

$$\frac{x_1}{24-14} = \frac{x_2}{-12+32} = \frac{x_3}{56-36}$$

$$\frac{x_1}{10} = \frac{x_2}{20} = \frac{x_3}{20}$$

*i.e.*, $\dfrac{x_1}{1} = \dfrac{x_2}{2} = \dfrac{x_3}{2}$

Hence, the corresponding Eigen vector is $X_1 = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$

**Case (2) : λ = 3** Equation (A) becomes

$$\begin{pmatrix} 5 & -6 & 2 \\ -6 & 4 & -4 \\ 2 & -4 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$5x_1 - 6x_2 + 2x_3 = 0 \quad \rightarrow (4)$$
$$-6x_1 + 4x_2 - 4x_3 = 0 \quad \rightarrow (5)$$
$$2x_1 - 4x_2 + 0x_3 = 0 \quad \rightarrow (6)$$

Solving (4) & (5) by rule of cross multiplication, we get

$$\frac{x_1}{24-8} = \frac{x_2}{-12+20} = \frac{x_3}{20-36}$$

$$\frac{x_1}{16} = \frac{x_2}{8} = \frac{x_3}{-16}$$

*i.e.,* $\frac{x_1}{2} = \frac{x_2}{1} = \frac{x_3}{-2}$

Hence, the corresponding Eigen vector is $X_2 = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$

**Case (3) : λ = 15** Equation (A) becomes

$$\begin{pmatrix} -7 & -6 & 2 \\ -6 & -8 & -4 \\ 2 & -4 & -12 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$-7x_1 - 6x_2 + 2x_3 = 0 \quad \rightarrow (7)$$
$$-6x_1 - 8x_2 - 4x_3 = 0 \quad \rightarrow (8)$$
$$2x_1 - 4x_2 - 12x_3 = 0 \quad \rightarrow (9)$$

Solving (7) & (8) by rule of cross multiplication, we get

$$\frac{x_1}{24+16} = \frac{x_2}{-12-28} = \frac{x_3}{56-36}$$

$$\frac{x_1}{40} = \frac{x_2}{-40} = \frac{x_3}{20}$$

*i.e.,* $\frac{x_1}{2} = \frac{x_2}{-2} = \frac{x_3}{1}$

Hence, the corresponding Eigen vector is $X_3 = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$

| Eigenvector | Normalised form |
|---|---|
| $X_1 = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$ | $\begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \\ \frac{2}{3} \end{pmatrix}$ |
| $X_2 = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$ | $\begin{pmatrix} \frac{2}{3} \\ \frac{1}{3} \\ \frac{-2}{3} \end{pmatrix}$ |
| $X_3 = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$ | $\begin{pmatrix} \frac{2}{3} \\ \frac{-2}{3} \\ \frac{1}{3} \end{pmatrix}$ |

Normalized modal matrix N= $\begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{-2}{3} \\ \frac{2}{3} & \frac{-2}{3} & \frac{1}{3} \end{bmatrix}$ , $N^T = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{-2}{3} \\ \frac{2}{3} & \frac{-2}{3} & \frac{1}{3} \end{bmatrix}$

**To find AN:**

$$\mathrm{AN} = \begin{pmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{pmatrix} \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{-2}{3} \\ \frac{2}{3} & \frac{-2}{3} & \frac{1}{3} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{8-12+4}{3} & \frac{16-6-4}{3} & \frac{16+12+2}{3} \\ \frac{-6+14-8}{3} & \frac{-12+7+8}{3} & \frac{-12-14-4}{3} \\ \frac{2-8+6}{3} & \frac{4-4-6}{3} & \frac{4+8+3}{3} \end{bmatrix} = \begin{bmatrix} 0 & 2 & 10 \\ 0 & 1 & -10 \\ 0 & -2 & 5 \end{bmatrix}$$

$\underline{D = N^T AN}$ :

$$\mathrm{D} = \mathrm{N}^{\mathrm{T}}\mathrm{AN} = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{-2}{3} \\ \frac{2}{3} & \frac{-2}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 0 & 2 & 10 \\ 0 & 1 & -10 \\ 0 & -2 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & \frac{2+2-4}{3} & \frac{10-20+10}{3} \\ 0 & \frac{4+1+4}{3} & \frac{20-10-10}{3} \\ 0 & \frac{4-2-2}{3} & \frac{20+20+5}{3} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 15 \end{bmatrix} = \mathrm{D}\ (0,3,15)$$

Canonical Form is $\mathrm{Y}^{\mathrm{T}}(\mathrm{N}^{\mathrm{T}}\mathrm{AN})\mathrm{Y} = \mathrm{Y}^{\mathrm{T}}\mathrm{DY} = \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 15 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

$= 0\mathrm{y}_1{}^2 + 3\mathrm{y}_2{}^2 + 15\mathrm{y}_3{}^2$

Rank ($r$) = 2 (No. of non zero terms in the canonical form)

Index ($p$) = 2 ( No. of positive terms in the canonical form )

Signature (s) = $2p - r$ = 2(2) – 2= 2

Nature: Positive Semi-definite.

22. **Reduce the quadratic form $6x^2 + 3y^2 + 3z^2 - 4xy - 2yz + 4xz$ into a canonical form by using orthogonal transformation. Hence find its rank, index, signature and nature.**

**Solution:**

The symmetric matrix $A=\begin{pmatrix} \text{coeff.}x^2 & \frac{1}{2}\text{coeff.}xy & \frac{1}{2}\text{coeff.}xz \\ \frac{1}{2}\text{coeff.}yx & \text{coeff.}y^2 & \frac{1}{2}\text{coeff.}yz \\ \frac{1}{2}\text{coeff.}zx & \frac{1}{2}\text{coeff.}zy & \text{coeff.}z^2 \end{pmatrix} = \begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix}$

The characteristic equation of $A$ is $|A-\lambda I|=0$

⇨ The characteristic equation is $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

where $S_1$ = Sum of the main diagonal elements=6 + 3 + 3 = 12

$S_2$ = Sum of the minors of the main diagonal elements.

$= \begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix} + \begin{vmatrix} 6 & 2 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 6 & -2 \\ -2 & 3 \end{vmatrix}$

=( 9 – 1) + (18 – 4 ) + ( 18 – 4 ) = 8+ 14 + 14 = 36

$S_3 = |A| = \begin{vmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{vmatrix}$

$= 6(9-1) + 2(-6+2) + 2(2-6)$

$= 6(8) + 2(-4) + 2(-4) = 48 - 8 - 8 = 32$

Therefore, the characteristic equation is $\lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0$.

<u>To Solve the Characteristic equation</u>

$\lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0$.

If $\lambda = 2$, *then* $\lambda^3 - 12\lambda^2 + 36\lambda - 32 = 8 - 42 + 72 - 32 = 0$

Therefore, $\lambda = 2$ is a root.

By Synthetic division

$$\begin{array}{r|rrrr} 2 & 1 & -12 & 36 & -32 \\ & 0 & 2 & -20 & 32 \\ \hline & 1 & -10 & 16 & \underline{|0} \end{array}$$

The other roots are given by $\lambda^2 - 10\lambda + 16 = 0$

$(\lambda - 8)(\lambda - 2) = 0$

*i.e.*, $\lambda = 8, \lambda = 2$

Hence , the Eigen values are $\lambda$ = 8, 2, 2

<u>To find the Eigen Vectors:</u>

To get the Eigenvectors, solve $(A - \lambda I)X = 0$

$$\begin{pmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad \to (A)$$

**Case (1) : $\lambda = 8$** Equation (A) becomes

$$\begin{pmatrix} -2 & -2 & 2 \\ -2 & -5 & -1 \\ 2 & -1 & -5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$-2x_1-2x_2+2x_3=0 \quad \to (1)$$
$$-2x_1-5x_2-x_3=0 \quad \to (2)$$
$$2x_1-x_2-5x_3=0 \quad \to (3)$$

Solving (1) & (2) by rule of cross multiplication, we get

$$\frac{x_1}{2+10}=\frac{x_2}{-4-2}=\frac{x_3}{10-4}$$

$$\frac{x_1}{12}=\frac{x_2}{-6}=\frac{x_3}{6}$$

*i.e.,* $\frac{x_1}{2}=\frac{x_2}{-1}=\frac{x_3}{1}$

Hence, the corresponding Eigen vector is $X_1=\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$

**Case (2) : $\lambda = 2$** Equation (A) becomes

$$\begin{pmatrix} 4 & -2 & 2 \\ -2 & 1 & -1 \\ 2 & -1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$4x_1-2x_2+2x_3=0 \quad \to (4)$$
$$-2x_1+x_2-x_3=0 \quad \to (5)$$
$$2x_1-x_2+x_3=0 \quad \to (6)$$

(4) , (5) & (6) are same as

$$2x_1-x_2+x_3=0$$

If

$x_1=0$ , *we get* $-x_2+x_3=0$

$$-x_2=-x_3$$
$$x_2=x_3$$

*i.e.,* $\frac{x_2}{1}=\frac{x_3}{1}$

Hence, the corresponding Eigen vector $X_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

**Case (3) : λ = 2**

Let the third eigen vector be $X_3 = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ and $X_3$ should be orthogonal with $X_1$ and $X_2$

$X_1^T X_3 = 0 \Rightarrow 2a - b + c = 0 \quad \rightarrow (7)$

$X_2^T X_3 = 0 \Rightarrow 0a + b + c = 0 \quad \rightarrow (8)$

Solving (7) & (8) by rule of cross multiplication, we get

$$\frac{a}{-1-1} = \frac{b}{0-2} = \frac{c}{2-0}$$

$$\frac{a}{-2} = \frac{b}{-2} = \frac{c}{2}$$

*i.e.*, $\frac{a}{1} = \frac{b}{1} = \frac{c}{-1}$

Hence, the corresponding Eigen vector $X_3 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$

| Eigenvector | Normalised form |
|---|---|
| $X_1 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$ | $\begin{pmatrix} \frac{2}{\sqrt{6}} \\ \frac{-1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{pmatrix}$ |
| $X_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ | $\begin{pmatrix} \frac{0}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$ |

| | |
|---|---|
| $X_3 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$ | $\begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{-1}{\sqrt{3}} \end{pmatrix}$ |

Normalized modal matrix $N = \begin{bmatrix} \frac{2}{\sqrt{6}} & 0 & \frac{1}{\sqrt{3}} \\ \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{3}} \end{bmatrix}$, $N^T = \begin{bmatrix} \frac{2}{\sqrt{6}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} \end{bmatrix}$

<u>To find AN:</u>

$$AN = \begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix} \begin{bmatrix} \frac{2}{\sqrt{6}} & 0 & \frac{1}{\sqrt{3}} \\ \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{3}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{12+2+2}{\sqrt{6}} & \frac{0-2+2}{\sqrt{2}} & \frac{6-2-2}{\sqrt{3}} \\ \frac{-4-3-1}{\sqrt{6}} & \frac{0+3-1}{\sqrt{2}} & \frac{-2+3+1}{\sqrt{3}} \\ \frac{4+1+3}{\sqrt{6}} & \frac{0-1+3}{\sqrt{2}} & \frac{2-1-3}{\sqrt{3}} \end{bmatrix} = \begin{bmatrix} \frac{16}{\sqrt{6}} & 0 & \frac{2}{\sqrt{3}} \\ \frac{-8}{\sqrt{6}} & \frac{2}{\sqrt{2}} & \frac{2}{\sqrt{3}} \\ \frac{8}{\sqrt{6}} & \frac{2}{\sqrt{2}} & \frac{-2}{\sqrt{3}} \end{bmatrix}$$

<u>$D = N^T AN$</u>:

$$D = N^TAN = \begin{bmatrix} \frac{2}{\sqrt{6}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} \frac{16}{\sqrt{6}} & 0 & \frac{2}{\sqrt{3}} \\ \frac{-8}{\sqrt{6}} & \frac{2}{\sqrt{2}} & \frac{2}{\sqrt{3}} \\ \frac{8}{\sqrt{6}} & \frac{2}{\sqrt{2}} & \frac{-2}{\sqrt{3}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{32+8+8}{6} & \frac{0-2+2}{\sqrt{12}} & \frac{4-2-2}{\sqrt{18}} \\ \frac{0-8+8}{\sqrt{12}} & \frac{0+2+2}{2} & \frac{0+2-2}{\sqrt{6}} \\ \frac{16-8-8}{\sqrt{18}} & \frac{0+2-2}{\sqrt{6}} & \frac{2+2+2}{3} \end{bmatrix} = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = D\,(8,\,2,\,2)$$

Canonical Form is $Y^T(N^TAN)Y = Y^TDY = \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix} \begin{bmatrix} 8 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

$$= 8y_1^2 + 2y_2^2 + 2y_3^2$$

Rank $(r)$ = 3 (No. of non zero terms in canonical form)

Index $(p)$ = 3 ( No. of Positive terms in canonical form)

Signature $(s) = 2p - r = 2(3) - 3 = 3$

Nature: Positive definite.

23. **Reduce the quadratic form $2x_1^2 + x_2^2 + x_3^2 + 2x_1x_2 - 4x_2x_3 - 2x_3x_1$ into a canonical form by using orthogonal transformation. Hence find its rank, index, signature and nature.**

**Solution:**

The symmetric matrix $A = \begin{pmatrix} \text{coeff.}x^2 & \frac{1}{2}\text{coeff.}xy & \frac{1}{2}\text{coeff.}xz \\ \frac{1}{2}\text{coeff.}yx & \text{coeff.}y^2 & \frac{1}{2}\text{coeff.}yz \\ \frac{1}{2}\text{coeff.}zx & \frac{1}{2}\text{coeff.}zy & \text{coeff.}z^2 \end{pmatrix} = \begin{pmatrix} 2 & 1 & -1 \\ 1 & 1 & -2 \\ -1 & -2 & 1 \end{pmatrix}$

The characteristic equation of $A$ is $|A - \lambda I| = 0$

⇨ The characteristic equation is $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

where $S_1$ = Sum of the main diagonal elements=2 + 1 + 1 = 4

$S_2$ = Sum of the minors of the main diagonal elements.

$$= \begin{vmatrix} 1 & -2 \\ -2 & 1 \end{vmatrix} + \begin{vmatrix} 2 & -1 \\ -1 & 1 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix}$$

=1 – 4 + 2 - 1 + 2 – 1 = -3 +1 +1 = - 1

$$S_3 = |A| = \begin{vmatrix} 2 & 1 & -1 \\ 1 & 1 & -2 \\ -1 & -2 & 1 \end{vmatrix}$$

$$= 2(1-4) - 1(1-2) - 1(-2+1)$$

$$= = -6 + 1 + 1 = -4$$

Therefore, The Characteristic equation is $\lambda^3 - 4\lambda^2 - \lambda + 4 = 0$.

<u>To Solve the Characteristic equation</u>

$\lambda^3 - 4\lambda^2 - \lambda + 4 = 0$

If $\lambda = 1$, *then* $\lambda^3 - 12\lambda^2 + 36\lambda - 32 = 1 - 4 - 1 + 4 = 0$

Therefore, $\lambda = 1$ is a root.

By Synthetic division

$$\begin{array}{c|cccc} & 1 & -4 & -1 & 4 \\ 1 & 0 & 1 & -3 & -4 \\ \hline & 1 & -3 & -4 & \underline{|0} \end{array}$$

Other roots are given by $\lambda^2 - 3\lambda - 4 = 0$

$(\lambda - 4)(\lambda + 1) = 0$

*i.e.*, $\lambda = 4,\ \lambda = -1$

Hence , the Eigen values are $\lambda = -1$ , 1, 4

<u>To find the Eigen Vectors:</u>

To get the Eigenvectors, solve $(A - \lambda I)X = 0$

$$\begin{pmatrix} 2-\lambda & 1 & -1 \\ 1 & 1-\lambda & -2 \\ -1 & -2 & 1-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \rightarrow (A)$$

**Case (1) : $\lambda = -1$** Equation (A) becomes

$$\begin{pmatrix} 3 & 1 & -1 \\ 1 & 2 & -2 \\ -1 & -2 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$3x_1 + x_2 - x_3 = 0 \quad \rightarrow (1)$$

$$x_1 + 2x_2 - 2x_3 = 0 \quad \rightarrow (2)$$

$$-x_1 - 2x_2 + 2x_3 = 0 \quad \rightarrow (3)$$

Solving (1) & (2) by rule of cross multiplication, we get

$$\frac{x_1}{-2+2}=\frac{x_2}{-1+6}=\frac{x_3}{6-1}$$

$$\frac{x_1}{0}=\frac{x_2}{5}=\frac{x_3}{5}$$

*i.e.,* $\frac{x_1}{0}=\frac{x_2}{1}=\frac{x_3}{1}$

Hence, the corresponding Eigen vector is $X_1=\begin{pmatrix}0\\1\\1\end{pmatrix}$

**Case (2) : λ =** 1 Then equation (A) becomes

$$\begin{pmatrix}1 & 1 & -1\\ 1 & 0 & -2\\ -1 & -2 & 0\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$x_1+x_2-x_3=0 \quad \rightarrow (4)$$

$$x_1+0x_2-2x_3=0 \quad \rightarrow (5)$$

$$-x_1-2x_2+0x_3=0 \quad \rightarrow (6)$$

Solving (4) & (5) by rule of cross multiplication, we get

$$\frac{x_1}{-2+0}=\frac{x_2}{-1+2}=\frac{x_3}{0-1}$$

$$\frac{x_1}{-2}=\frac{x_2}{1}=\frac{x_3}{-1}$$

*i.e.,* $\frac{x_1}{2}=\frac{x_2}{-1}=\frac{x_3}{1}$

Hence, the corresponding Eigen vector is $X_2=\begin{pmatrix}2\\-1\\1\end{pmatrix}$

**Case (3) : λ =** 4 Then equation (A) becomes

$$\begin{pmatrix}-2 & 1 & -1\\ 1 & -3 & -2\\ -1 & -2 & -3\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$-2x_1+x_2-x_3=0 \quad \rightarrow (7)$$

$$x_1-3x_2-2x_3=0 \quad \rightarrow (8)$$

$$-x_1-2x_2-3x_3=0 \quad \rightarrow (9)$$

Solving (7) & (8) by rule of cross multiplication, we get

$$\frac{x_1}{-2-3}=\frac{x_2}{-1-4}=\frac{x_3}{6-1}$$

$$\frac{x_1}{-5}=\frac{x_2}{-5}=\frac{x_3}{5}$$

*i.e.,* $\frac{x_1}{1}=\frac{x_2}{1}=\frac{x_3}{-1}$

Hence, the corresponding Eigen vector is $X_3=\begin{pmatrix}1\\1\\-1\end{pmatrix}$

| Eigenvector | Normalised form |
|---|---|
| $X_1=\begin{pmatrix}0\\1\\1\end{pmatrix}$ | $\begin{pmatrix}\frac{0}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\end{pmatrix}$ |
| $X_2=\begin{pmatrix}2\\-1\\1\end{pmatrix}$ | $\begin{pmatrix}\frac{2}{\sqrt{6}}\\\frac{-1}{\sqrt{6}}\\\frac{1}{\sqrt{6}}\end{pmatrix}$ |
| $X_3=\begin{pmatrix}1\\1\\-1\end{pmatrix}$ | $\begin{pmatrix}\frac{1}{\sqrt{3}}\\\frac{1}{\sqrt{3}}\\\frac{-1}{\sqrt{3}}\end{pmatrix}$ |

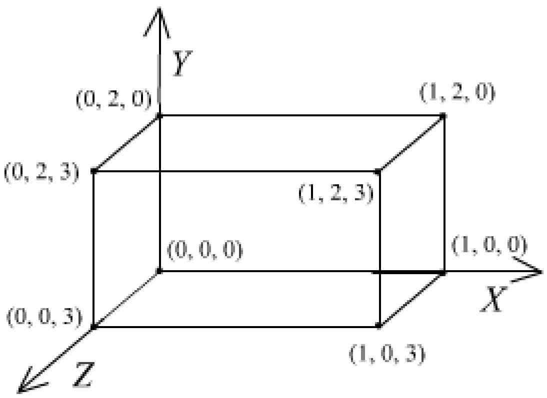
Normalized modal matrix N= $\begin{bmatrix}0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}}\\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{3}}\\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{3}}\end{bmatrix}$ , $N^T=\begin{bmatrix}0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ \frac{2}{\sqrt{6}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{6}}\\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{3}}\end{bmatrix}$

**To find AN:**

$$AN = \begin{pmatrix} 2 & 1 & -1 \\ 1 & 1 & -2 \\ -1 & -2 & 1 \end{pmatrix} \begin{bmatrix} 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{3}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{0+1-1}{\sqrt{2}} & \frac{4-1-1}{\sqrt{6}} & \frac{2+1+1}{\sqrt{3}} \\ \frac{0+1-2}{\sqrt{2}} & \frac{2-1-2}{\sqrt{6}} & \frac{1+1+2}{\sqrt{3}} \\ \frac{0-2+1}{\sqrt{2}} & \frac{-2+2+1}{\sqrt{6}} & \frac{-1-2-1}{\sqrt{3}} \end{bmatrix} = \begin{bmatrix} 0 & \frac{2}{\sqrt{6}} & \frac{4}{\sqrt{3}} \\ \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} & \frac{4}{\sqrt{3}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{-4}{\sqrt{3}} \end{bmatrix}$$

$\underline{D = N^T AN}$ :




Canonical Form is $\quad Y^T(N^TAN)Y = Y^TDY = \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

$$= -1y_1^2 + y_2^2 + 4y_3^2$$

Rank $(r)$ = 3 (No. of non zero terms in the canonical form)
Index $(p)$ = 2 ( No. of positive terms in the canonical form )
Signature $(s)$ = 2p – r = 2(2) – 3= 1
Nature: indefinite.

24. **Reduce the quadratic form** $2x^2 + 5y^2 + 3z^2 + 4xy$ **to canonical form through orthogonal transformation. Find also its nature.**

**Solution:**

The symmetric matrix $A=\begin{pmatrix} \text{coeff}.x^2 & \frac{1}{2}\text{coeff}.xy & \frac{1}{2}\text{coeff}.xz \\ \frac{1}{2}\text{coeff}.yx & \text{coeff}.y^2 & \frac{1}{2}\text{coeff}.yz \\ \frac{1}{2}\text{coeff}.zx & \frac{1}{2}\text{coeff}.zy & \text{coeff}.z^2 \end{pmatrix} = \begin{pmatrix} 2 & 2 & 0 \\ 2 & 5 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

The characteristic equation of $A$ is $|A-\lambda I|=0$

⇨ The characteristic equation is $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

where $S_1$ = Sum of the main diagonal elements=2 + 5 + 3 = 10

$S_2$ = Sum of the minors of the main diagonal elements.

$$=\begin{vmatrix} 5 & 0 \\ 0 & 3 \end{vmatrix} + \begin{vmatrix} 2 & 0 \\ 0 & 3 \end{vmatrix} + \begin{vmatrix} 2 & 2 \\ 2 & 5 \end{vmatrix} = 15+6+(10-4)=27$$

$$S_3 = |A| = \begin{vmatrix} 2 & 2 & 0 \\ 2 & 5 & 0 \\ 0 & 0 & 3 \end{vmatrix} = 2(15-0)-2(6-0)+0=18$$

Therefore, the Characteristic equation is $\lambda^3 - 10\lambda^2 + 27\lambda - 18 = 0$.

<u>To Solve the Characteristic equation</u>

$\lambda^3 - 10\lambda^2 - 27\lambda + 18 = 0$

If $\lambda = 1$, *then* $\lambda^3 - 10\lambda^2 - 27\lambda + 18 = 1 - 10 + 27 - 18 = 0$

Therefore, $\lambda = 1$ is a root.

By Synthetic division

$$\begin{array}{r|rrrr} 1 & 1 & -10 & 27 & -18 \\ & 0 & 1 & -9 & 18 \\ \hline & 1 & -9 & 18 & \underline{|0} \end{array}$$

The other roots are given by $\lambda^2 - 9\lambda + 18 = 0$

$(\lambda-6)(\lambda-3)=0$

*i.e.,* $\lambda = 6,\ \lambda = 3$

Hence , the Eigen values are $\lambda$ = 1, 3, 6

<u>To find the Eigen Vectors:</u>

To get the Eigenvectors, solve $(A-\lambda I)X = 0$

$$\begin{pmatrix} 2-\lambda & 2 & 0 \\ 2 & 5-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \rightarrow (A)$$

**Case (1) : $\lambda = 1$** Then equation (A) becomes

$$\begin{pmatrix} 1 & 2 & 0 \\ 2 & 4 & 0 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \rightarrow (A)$$

$$x_1 + 2x_2 + 0x_3 = 0 \quad \rightarrow (1)$$

$$2x_1 + 4x_2 + 0x_3 = 0 \quad \rightarrow (2)$$

$$0x_1 + 0x_2 + 2x_3 = 0 \quad \rightarrow (3)$$

Solving (1) & (2) by rule of cross multiplication, we get

$$\frac{x_1}{\begin{vmatrix} 4 & 0 \\ 0 & 2 \end{vmatrix}} = \frac{x_2}{\begin{vmatrix} 0 & 2 \\ 2 & 0 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 2 & 4 \\ 0 & 0 \end{vmatrix}}$$

$$\frac{x_1}{8} = \frac{x_2}{-4} = \frac{x_3}{0}$$

*i.e.,* $\frac{x_1}{2} = \frac{x_2}{-1} = \frac{x_3}{0}$

Hence, the corresponding Eigen vector is $X_1 = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$

**Case (2) : $\lambda = 3$** Then equation (A) becomes

$$\begin{pmatrix} -1 & 2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \rightarrow (A)$$

$$-x_1 + 2x_2 + 0x_3 = 0 \quad \rightarrow (4)$$

$$2x_1 + 2x_2 + 0x_3 = 0 \quad \rightarrow (5)$$

$$0x_1 + 0x_2 + 0x_3 = 0 \quad \rightarrow (6)$$

Solving (1) & (2) by rule of cross multiplication, we get

$$\frac{x_1}{\begin{vmatrix} 2 & 0 \\ 2 & 0 \end{vmatrix}} = \frac{x_2}{\begin{vmatrix} 0 & -1 \\ 0 & 2 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -1 & 2 \\ 2 & 2 \end{vmatrix}}$$

$$\frac{x_1}{0} = \frac{x_2}{0} = \frac{x_3}{-6}$$

*i.e.,* $\frac{x_1}{0} = \frac{x_2}{0} = \frac{x_3}{-1}$

Hence, the corresponding Eigen vector is $X_2 = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} \Rightarrow X_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

**Case (3) : λ = 6** Then equation (A) becomes

$$\begin{pmatrix} -4 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \rightarrow (A)$$

$$-4x_1 + 2x_2 = 0 \quad \rightarrow (4)$$

$$2x_1 - x_2 = 0 \quad \rightarrow (5)$$

$$0x_1 + 0x_2 - 3x_3 = 0 \quad \rightarrow (6)$$

Solving (1) & (2) by rule of cross multiplication, we get

$$\frac{x_1}{\begin{vmatrix} -1 & 3 \\ 0 & -3 \end{vmatrix}} = \frac{x_2}{\begin{vmatrix} 3 & 2 \\ -3 & 0 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 2 & -1 \\ 0 & 0 \end{vmatrix}}$$

$$\frac{x_1}{3} = \frac{x_2}{6} = \frac{x_3}{0}$$

*i.e.,* $\frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{0}$

Hence, the corresponding Eigen vector is $X_2 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$

| Eigenvector | Normalised form |
|---|---|
| $X_1 = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$ | $\begin{pmatrix} \frac{2}{\sqrt{5}} \\ \frac{-1}{\sqrt{5}} \\ 0 \end{pmatrix}$ |

| | |
|---|---|
| $X_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ | $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ |
| $X_2 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$ | $\begin{pmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \\ 0 \end{pmatrix}$ |

Normalized modal matrix

$$\mathrm{N} = \begin{bmatrix} \frac{2}{\sqrt{5}} & 0 & \frac{1}{\sqrt{5}} \\ \frac{-1}{\sqrt{5}} & 0 & \frac{2}{\sqrt{5}} \\ 0 & 1 & 0 \end{bmatrix}, \qquad \mathrm{N}^{\mathrm{T}} = \begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{-1}{\sqrt{5}} & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} & 0 \end{bmatrix}$$

To find AN:

$$\mathrm{AN} = \begin{pmatrix} 2 & 2 & 0 \\ 2 & 5 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{bmatrix} \frac{2}{\sqrt{5}} & 0 & \frac{1}{\sqrt{5}} \\ \frac{-1}{\sqrt{5}} & 0 & \frac{2}{\sqrt{5}} \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} \frac{4-2+0}{\sqrt{5}} & 0 & \frac{2+4+0}{\sqrt{5}} \\ \frac{4-5}{\sqrt{5}} & 0 & \frac{2+10+0}{\sqrt{5}} \\ 0 & 3 & 0 \end{bmatrix} = \begin{bmatrix} \frac{2}{\sqrt{5}} & 0 & \frac{6}{\sqrt{5}} \\ \frac{-1}{\sqrt{5}} & 0 & \frac{12}{\sqrt{5}} \\ 0 & 3 & 0 \end{bmatrix}$$

$\underline{D = N^T AN}$:

$$\mathrm{D} = \mathrm{N}^{\mathrm{T}}\mathrm{AN} = \begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{-1}{\sqrt{5}} & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} & 0 \end{bmatrix} \times \begin{bmatrix} \frac{2}{\sqrt{5}} & 0 & \frac{6}{\sqrt{5}} \\ \frac{-1}{\sqrt{5}} & 0 & \frac{12}{\sqrt{5}} \\ 0 & 3 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix} = \mathrm{D}$$

Thus $A$ has been diagonalized by $N$ *through the orthogonal transformation.*

Canonical Form is $Y^T(N^TAN)Y = Y^TDY = \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

$$= y_1^2 + 3y_2^2 + 6y_3^2$$

Nature: Positive definite.

25. **The Eigen vectors of a 3×3 real symmetric matrix A corresponding to the eigen values 2,3,6 are $(1,0,-1)^T$, $(1,1,1)^T$ and $(1,-2,1)^T$ respectively. Find the matrix A.**

**Solution:**

We know that the eigen vectors of a real symmetric matrix

$$\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \text{ are pairwise orthogonal} .$$

The normalized modal matrix

$$N = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & \frac{-2}{\sqrt{6}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{bmatrix}$$

Also $D = N^T A N$

Now $N D N^T = N (N^T A N) N^T$
$= (N N^T) A (N N^T)$
$= I A I$

Hence $\mathbf{N D N^T = A}$**,** since N is an orthogonal matrix and $N N^T = I$.

$$\therefore A = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & \frac{-2}{\sqrt{6}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{bmatrix}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{-2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2+0+0}{\sqrt{2}} & \frac{0+3+0}{\sqrt{3}} & \frac{0+0+6}{\sqrt{6}} \\ \frac{0+0+0}{\sqrt{2}} & \frac{0+3+0}{\sqrt{3}} & \frac{0+0-12}{\sqrt{6}} \\ \frac{-2+0+0}{\sqrt{2}} & \frac{0+3+0}{\sqrt{3}} & \frac{0+0+6}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{-2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2}{\sqrt{2}} & \frac{3}{\sqrt{3}} & \frac{6}{\sqrt{6}} \\ 0 & \frac{3}{\sqrt{3}} & \frac{-12}{\sqrt{6}} \\ \frac{-2}{\sqrt{2}} & \frac{3}{\sqrt{3}} & \frac{6}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{-2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2}{2}+\frac{3}{3}+\frac{6}{6} & 0+\frac{3}{3}-\frac{12}{6} & \frac{-2}{2}+\frac{3}{3}+\frac{6}{6} \\ 0+\frac{3}{3}-\frac{12}{6} & 0+\frac{3}{3}+\frac{24}{6} & 0+\frac{3}{3}-\frac{12}{6} \\ \frac{-2}{2}+\frac{3}{3}+\frac{6}{6} & 0+\frac{3}{3}-\frac{12}{6} & \frac{2}{2}+\frac{3}{3}+\frac{6}{6} \end{bmatrix} = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$$

Therefore, the matrix $A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$

26. **Identify the nature, index and signature of the quadratic form** $2x_1x_2 + 2x_2x_3 + 2x_3x_1$.

**Solution:**

The matrix of the quadratic form is given by $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$.

The characteristics equation is $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$.

$S_1$ = Sum of the main diagonal elements = 0

$S_2$ = Sum of the minors of the main diagonal element $=(0-1)+(0-1)+(0-1)=-3$;

$S_3 = |A| = -1(0-1) + 1(1-0) = 2$

The characteristics equation is $\lambda^3 - 3\lambda - 2 = 0$.

$$(\lambda + 1)^2(\lambda - 2) = 0 \Rightarrow \text{The eigen values of A are } \lambda = -1, -1, 2.$$

Nature: indefinite

Rank ($r$) = Number of non-zero eigen values = 3

Index ($p$) = Number of positive eigen values = 1

Signature ($s$) = $2p - r$ = 2(1) – 3 = –1.

27. **Find the rank, index and signature of the Quadratic form whose Canonical form is $x_1^2 + 2x_2^2 - 3x_3^2$.**

**Solution :**

Rank ($r$) = Number of non-zero terms in the C.F = 3
Index ($p$) = Number of Positive terms in the C.F = 2
Signature ($s$) = $2p - r$ = 1

28. **Write down the matrix of the quadratic form $2x^2 + 8z^2 + 4xy + 10xz - 2yz$.**

**Solution :**

The matrix of the quadratic form is given by

$a_{11}$ = coeff of $x^2$ = 2 , $a_{22}$ = coeff of $y^2$ = 0 , $a_{33}$ = coeff of $z^2$ = 8

$a_{12} = a_{21} = \frac{1}{2}$(coeff of xy) = $\frac{4}{2}$ =2, $a_{13} = a_{31} = \frac{1}{2}$(coeff of xz) = $\frac{10}{2}$ = 5

$a_{23} = a_{32} = \frac{1}{2}$(coeff of yz) = $\frac{-2}{2}$ =-1

$$\Rightarrow A = \begin{bmatrix} 2 & 2 & 5 \\ 2 & 0 & -1 \\ 5 & -1 & 8 \end{bmatrix}$$

29. **Determine $\lambda$ so that $\lambda(x^2 + y^2 + z^2) + 2xy - 2xz + 2zy$ is positive definite.**

**Solution:** The matrix of the given quadratic form is $A = \begin{pmatrix} \lambda & 1 & -1 \\ 1 & \lambda & 1 \\ -1 & 1 & \lambda \end{pmatrix}$

The principal sub determinants are given by

$D_1 = \lambda$, $D_2 = \begin{vmatrix} \lambda & 1 \\ 1 & \lambda \end{vmatrix} = \lambda^2 - 1 = (\lambda+1)(\lambda-1)$ & $D_3 = |A| = (\lambda+1)^2(\lambda - 2)$

The Quadratic form is +ve definite if $D_1$, $D_2$& $D_3 > 0 \Rightarrow \lambda > 2$.

30. **What is the nature of the quadratic form $x^2 + y^2 + z^2$ in four variables?**

**Solution:** The matrix of the given quadratic form is $A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

Since the matrix is the diagonal matrix, its main diagonal elements are its eigen values.
$\therefore$ The eigen values are 1,1,1,0. Hence the nature is positive semi definite.

31. **Write down the quadratic form corresponding to the matrix $A = \begin{bmatrix} 0 & 5 & -1 \\ 5 & 1 & 6 \\ -1 & 6 & 2 \end{bmatrix}$.**

**Solution :**

Quadratic form of A is given by $X^TAX$ = $\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}\begin{bmatrix} 0 & 5 & -1 \\ 5 & 1 & 6 \\ -1 & 6 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$= 0x_1^2 + x_2^2 + 2x_3^2 + 10x_1x_2 + 12x_2x_3 - 2x_3x_1$.

## CAYLEY-HAMILTON THEOREM

**Statement:** Every square matrix satisfies its own characteristic equation.

32. **Verify that A=$\begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}$ satisfies its own characteristic equation and hence find $A^4$.**

**Solution:**

Let $A = \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}$

The Characteristic equation of A is: $|A - \lambda I| = 0$

i.e., $\lambda^2 - S_1\lambda + S_2 = 0$

where, $S_1$ = Sum of the diagonal elements = (1) + ( –1) = 0

$S_2 = |A| = \begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix} = -1 - 4 = -5$

Hence the characteristic equation is

$\lambda^2 - (0)\lambda + (-5) = 0$

$\lambda^2 - 5 = 0$

Cayley-Hamilton theorem states that " Every Square matrix satisfies its own characteristic equation "

$\Rightarrow A^2 - 5I = 0$ ________________________________________ (1)

<u>Verification:</u> Find $A^2$ as $A^2 = A*A = \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} 1+4 & 2-2 \\ 2-2 & 4+1 \end{pmatrix} = \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}$

and then prove $A^2 - 5I = 0$

$\begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix} - 5\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

Hence, Cayley-Hamilton theorem is verified.

<u>To find $A^4$:</u> Pre multiply $A^2$ on both sides of Eqn. (1) and get $A^4 - 5A^2 = 0$

$A^4 = 5A^2$

$= 5\begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}$

$\Rightarrow A^4 = \begin{pmatrix} 25 & 0 \\ 0 & 25 \end{pmatrix}$

33. **Given A$=\begin{bmatrix} \mathbf{1} & \mathbf{2} \\ \mathbf{4} & \mathbf{3} \end{bmatrix}$, Find $A^{-1}$ using Cayley – Hamilton theorem.**

**Solution :** The characteristic equation of A is $\lambda^2$ - $S_1$ $\lambda$ + $S_2$ $= 0$,

Here, $S_1 = 4$ and $S_2 = -5$ $\Rightarrow \lambda^2 - 4\lambda - 5 = 0$.

By Cayley – Hamilton theorem $A^2 - 4A - 5I = 0$.

Multiply by $A^{-1}$, we get $A - 4I - 5A^{-1} = 0$ $\therefore$ $A^{-1} = \frac{1}{5}[A - 4I] = \begin{bmatrix} \frac{-3}{5} & \frac{2}{5} \\ \frac{4}{5} & \frac{-1}{5} \end{bmatrix}$

34. **Verify Cayley- Hamilton theorem for the matrix A=** $\begin{pmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{pmatrix}$ **and hence find $A^4$ and $A^{-1}$.**

**Solution:**

The characteristic equation of A is $|A - \lambda I| = 0$

⇨ The characteristic equation is $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

Where $S_1$ = Sum of the diagonal elements=1+2+1=4

$S_2$ = Sum of the minors of the diagonal elements.

$$= \begin{vmatrix} 2 & 3 \\ 2 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 7 \\ 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 3 \\ 4 & 2 \end{vmatrix} = (2-6)+(1-7)+(2-12) = -4-6-10 = -20$$

$$S_3 = |A| = \begin{vmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{vmatrix} = 1(2-6) - 3(4-3) + 7(8-2) = -4 - 3 + 42 = 35$$

Therefore, the Characteristic equation is: $\lambda^3 - 4\lambda^2 - 20\lambda - 35 = 0$ __________(1)

Verification:

Replace $\lambda$ by $A$ in Eqn. (1)

$A^3 - 4A^2 - 20A - 35I = 0$ ______________________________(2).

To find $A^2$

$$A^2 = A \times A = \begin{pmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1+12+7 & 3+6+14 & 7+9+7 \\ 4+8+3 & 12+4+6 & 28+6+3 \\ 1+8+1 & 3+4+2 & 7+6+1 \end{pmatrix}$$

$$= \begin{pmatrix} 20 & 23 & 23 \\ 15 & 22 & 37 \\ 10 & 9 & 14 \end{pmatrix}$$

$$A^3 = A^2 \times A = \begin{pmatrix} 20 & 23 & 23 \\ 15 & 22 & 37 \\ 10 & 9 & 14 \end{pmatrix}\begin{pmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 20+92+23 & 60+46+46 & 140+69+23 \\ 15+88+37 & 45+44+74 & 105+66+37 \\ 10+36+14 & 30+18+28 & 70+27+14 \end{pmatrix}$$

$$= \begin{pmatrix} 135 & 152 & 232 \\ 140 & 163 & 208 \\ 60 & 76 & 111 \end{pmatrix}$$

Substituting $A^3$ , $A^2$& A in (2)

$A^3 - 4A^2 - 20A - 35I = 0$

$$\begin{pmatrix} 135 & 152 & 232 \\ 140 & 163 & 208 \\ 60 & 76 & 111 \end{pmatrix} -4 \begin{pmatrix} 20 & 23 & 23 \\ 15 & 22 & 37 \\ 10 & 9 & 14 \end{pmatrix} -20 \begin{pmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{pmatrix} -35 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 135 & 152 & 232 \\ 140 & 163 & 208 \\ 60 & 76 & 111 \end{pmatrix} - \begin{pmatrix} 80 & 92 & 92 \\ 60 & 88 & 148 \\ 40 & 36 & 56 \end{pmatrix} - \begin{pmatrix} 20 & 60 & 140 \\ 80 & 40 & 60 \\ 20 & 40 & 20 \end{pmatrix} - \begin{pmatrix} 35 & 0 & 0 \\ 0 & 35 & 0 \\ 0 & 0 & 35 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Therefore, L. H. S = R. H. S

Hence, Cayley-Hamilton theorem is verified.

<u>To find $A^4$</u>

$A^3 - 4A^2 - 20A - 35I = 0$ ______________________________(3)

Pre-Multiply 'A' in Eqn. (3)

$A^4 - 4A^3 - 20A^2 - 35A = 0$

$A^4 = 4A^3 + 20A^2 + 35A$

$$= 4\begin{pmatrix} 135 & 152 & 232 \\ 140 & 163 & 208 \\ 60 & 76 & 111 \end{pmatrix} + 20\begin{pmatrix} 20 & 23 & 23 \\ 15 & 22 & 37 \\ 10 & 9 & 14 \end{pmatrix} + 35\begin{pmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 540 & 608 & 928 \\ 560 & 652 & 832 \\ 240 & 304 & 444 \end{pmatrix} + \begin{pmatrix} 400 & 460 & 460 \\ 300 & 440 & 740 \\ 200 & 180 & 280 \end{pmatrix} + \begin{pmatrix} 35 & 105 & 245 \\ 140 & 70 & 105 \\ 35 & 70 & 35 \end{pmatrix}$$

$$= \begin{pmatrix} 975 & 1173 & 1633 \\ 1000 & 1162 & 1677 \\ 475 & 554 & 759 \end{pmatrix}$$

<u>To find $A^{-1}$</u>

From Eqn. (3), $A^3 - 4A^2 - 20A - 35I = 0$ ______________________________(4)

Pre-Multiply '$A^{-1}$' in Eqn. (4)

$A^2 - 4A - 20I - 35A^{-1} = 0$

$$A^{-1} = \frac{1}{35}\left(A^2 - 4A - 20I\right)$$

$$= \frac{1}{35}\left[\begin{pmatrix} 20 & 23 & 23 \\ 15 & 22 & 37 \\ 10 & 9 & 14 \end{pmatrix} - 4\begin{pmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{pmatrix} - 20\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right]$$

$$= \frac{1}{35}\left[\begin{pmatrix} 20 & 23 & 23 \\ 15 & 22 & 37 \\ 10 & 9 & 14 \end{pmatrix} - \begin{pmatrix} 4 & 12 & 28 \\ 16 & 8 & 12 \\ 4 & 8 & 4 \end{pmatrix} - \begin{pmatrix} 20 & 0 & 0 \\ 0 & 20 & 0 \\ 0 & 0 & 20 \end{pmatrix}\right]$$

$$A^{-1} = \frac{1}{35}\begin{pmatrix} -4 & 11 & -5 \\ -1 & -6 & 25 \\ 6 & 1 & -10 \end{pmatrix}$$

35. **Using Cayley- Hamilton theorem for the matrix A=** $\begin{pmatrix} 1 & 2 & -2 \\ 2 & 5 & -4 \\ 3 & 7 & -5 \end{pmatrix}$ **and hence find $A^4$ and $A^{-1}$.**

**Solution:**

The characteristic equation of A is $|A - \lambda I| = 0$

⇨ The characteristic equation is: $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

Where, $S_1$ = Sum of the diagonal elements: 1+5-5=1

$S_2$ = Sum of the minors of the main diagonal elements.

$$= \begin{vmatrix} 5 & -4 \\ 7 & -5 \end{vmatrix} + \begin{vmatrix} 1 & -2 \\ 3 & -5 \end{vmatrix} + \begin{vmatrix} 1 & 2 \\ 2 & 5 \end{vmatrix} = (-25+28)+(-5+6)+(5-4)=3+1+1=5$$

$$S_3 = |A| = \begin{vmatrix} 1 & 2 & -2 \\ 2 & 5 & -4 \\ 3 & 7 & -5 \end{vmatrix} = 1(-25+28) - 2(-10+12) - 2(14-15) = 3 - 2(2) - 2(-1) = 3 - 4 + 2 = 1$$

Therefore, The Characteristic equation is $\lambda^3 - \lambda^2 + 5\lambda - 1 = 0$ ___________(1)

Replace $\lambda$ by A in Eqn. (1)

$$A^3 - A^2 + 5A - I = 0 \quad \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_} \quad (2)$$

<u>To find $A^4$</u>

$$A^3 - A^2 + 5A - I = 0 \quad \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_} \quad (3)$$

Pre-Multiply 'A' in Eqn. (3)

$$A^4 - A^3 + 5A^2 - A = 0$$

$$A^4 = A^3 - 5A^2 + A \quad \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_} \quad (4)$$

<u>To find $A^2$</u>

$$A^2 = A \times A = \begin{pmatrix} 1 & 2 & -2 \\ 2 & 5 & -4 \\ 3 & 7 & -5 \end{pmatrix}\begin{pmatrix} 1 & 2 & -2 \\ 2 & 5 & -4 \\ 3 & 7 & -5 \end{pmatrix}$$

$$= \begin{pmatrix} 1+4-6 & 2+10-14 & -2-8+10 \\ 2+10-12 & 4+25-28 & -4-20+20 \\ 3+14-15 & 6+35-35 & -6-28+25 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & -2 & 0 \\ 0 & 1 & -4 \\ 2 & 6 & -9 \end{pmatrix}$$

Substituting $A^3$ , $A^2$& A in Eqn. (4)

$$A^3 = A^2 \times A = \begin{pmatrix} -1 & -2 & 0 \\ 0 & 1 & -4 \\ 2 & 6 & -9 \end{pmatrix}\begin{pmatrix} 1 & 2 & -2 \\ 2 & 5 & -4 \\ 3 & 7 & -5 \end{pmatrix}$$

$$A^4 = A \qquad = \begin{pmatrix} -1-4+0 & -2-10+0 & 2+8+0 \\ 0+2-12 & 0+5-28 & 0-4+20 \\ 2+12-27 & 4+30-63 & -4-24+45 \end{pmatrix}^3 -5 A^2+A$$

$$= \begin{pmatrix} -5 & -12 & 10 \\ -10 & -23 & 16 \\ -13 & -29 & 17 \end{pmatrix}$$

$$= \begin{pmatrix} -5 & -12 & 10 \\ -10 & -23 & 16 \\ -13 & -29 & 17 \end{pmatrix} -5 \begin{pmatrix} -1 & -2 & 0 \\ 0 & 1 & -4 \\ 2 & 6 & -9 \end{pmatrix} + \begin{pmatrix} 1 & 2 & -2 \\ 2 & 5 & -4 \\ 3 & 7 & -5 \end{pmatrix}$$

$$= \begin{pmatrix} -5 & -12 & 10 \\ -10 & -23 & 16 \\ -13 & -29 & 17 \end{pmatrix} - \begin{pmatrix} -5 & -10 & 0 \\ 0 & 5 & -20 \\ 10 & 30 & -45 \end{pmatrix} + \begin{pmatrix} 1 & 2 & -2 \\ 2 & 5 & -4 \\ 3 & 7 & -5 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 8 \\ -8 & -23 & 32 \\ -20 & -52 & 57 \end{pmatrix}$$

<u>To find $A^{-1}$</u>

From Eqn. (3), $A^3 - A^2 + 5A - I = 0$

Pre-Multiply ‘$A^{-1}$’ in Eqn. (3)

$A^2 - A + 5I - A^{-1} = 0$

$A^{-1} = A^2 - A + 5I$

$$= \begin{pmatrix} -1 & -2 & 0 \\ 0 & 1 & -4 \\ 2 & 6 & -9 \end{pmatrix} - \begin{pmatrix} 1 & 2 & -2 \\ 2 & 5 & -4 \\ 3 & 7 & -5 \end{pmatrix} + \begin{pmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{pmatrix}$$

$$A^{-1} = \begin{pmatrix} 3 & -4 & 2 \\ -2 & 1 & 0 \\ -1 & -1 & 1 \end{pmatrix}$$

36. **Using Cayley-Hamilton theorem find the inverse of the given matrix** $A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 3 \end{pmatrix}$.

**Solution:**

The characteristic equation of $A$ is $|A - \lambda I| = 0$

⇨ The characteristic equation is $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

Where $S_1$ = Sum of the diagonal elements=1+2+3=6

$S_2$ = Sum of the minors of the diagonal elements.

$$= \begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix} + \begin{vmatrix} 1 & 2 \\ 2 & 2 \end{vmatrix} = (6-1) + (3-1) + (2-4) = 5 + 2 - 2 = 5$$

$$S_3 = |A| = \begin{vmatrix} 1 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 3 \end{vmatrix} = 1(6-1) - 2(6-1) + 1(2-2) = 5 - 2(5) + 0 = -5$$

Therefore, the Characteristic equation is: $\lambda^3 - 6\lambda^2 + 5\lambda + 5 = 0$

Using Cayley-Hamilton theorem, [Every square matrix satisfies its own characteristic equation]

$$A^3 - 6A^2 + 5A + 5I = 0$$

Multiply by $A^{-1}$ we get

$$A^{-1}\left(A^3 - 6A^2 + 5A + 5I\right) = 0$$

$$A^{-1}A^3 - 6A^{-1}A^2 + 5A^{-1}A + 5A^{-1}I = 0$$

$$IA^2 - 6IA + 5I + 5A^{-1} = 0$$

$$A^2 - 6A + 5I + 5A^{-1} = 0$$

$$A^{-1} = \frac{1}{5}\left(6A - A^2 - 5I\right)$$

**To find** $A^{-1}$:

$$A^2 = A \times A$$

$$= \begin{pmatrix} 1 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 3 \end{pmatrix} \times \begin{pmatrix} 1 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 6 & 7 & 6 \\ 7 & 9 & 7 \\ 6 & 7 & 11 \end{pmatrix}$$

$$6A = 6I \times A$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 6 & 12 & 6 \\ 12 & 12 & 6 \\ 6 & 6 & 18 \end{pmatrix}$$

$$6A - A^2 = \begin{pmatrix} 6 & 12 & 6 \\ 12 & 12 & 6 \\ 6 & 6 & 18 \end{pmatrix} - \begin{pmatrix} 6 & 7 & 6 \\ 7 & 9 & 7 \\ 6 & 7 & 11 \end{pmatrix} = \begin{pmatrix} 0 & 5 & 0 \\ 5 & 3 & -1 \\ 0 & -1 & 7 \end{pmatrix}$$

$$6A - A^2 - 5I = \begin{pmatrix} 0 & 5 & 0 \\ 5 & 3 & -1 \\ 0 & -1 & 7 \end{pmatrix} - \begin{pmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{pmatrix} = \begin{pmatrix} -5 & 5 & 0 \\ 5 & -2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

$$A^{-1} = \frac{1}{5}\begin{pmatrix} -5 & 5 & 0 \\ 5 & -2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

37. **Using Cayley-Hamilton theorem, find the matrix represented by** $A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 - 8A^2 + 2A - I$ **when** $A = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{pmatrix}$**.**

**Solution:**

The characteristic equation of A is $|A - \lambda I| = 0$

⇨ The characteristic equation is $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

Where $S_1$ = Sum of the diagonal elements=2+1+2=5

$S_2$ = Sum of the minors of the diagonal elements.

$$= \begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix}$$

=(2-0)+(4-1)+(2-0)=2+3+2=7

$$S_3 = |A| = \begin{vmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{vmatrix}$$

$$= 2(2-0) - 1(0-0) + 1(0-1) = 4 - 1 = 3$$

Therefore, The Characteristic equation is $\lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0$ ______________(1)

Replace $\lambda$ by A in (1)

$A^3 - 5A^2 + 7A - 3I = 0$ ______________________________(2)

Cayley-Hamilton theorem states that " Every Square matrix satisfies its own characteristic equation "

$\Rightarrow A^3 - 5A^2 + 7A - 3I = 0$

$A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 - 8A^2 + 2A - I$

$= A^5\left(A^3 - 5A^2 + 7A - 3I\right) + A\left(A^3 - 5A^2 + 7A - 3I\right) - 15A^2 + 5A - I$ [from Eqn. (2)]

$= -15A^2 + 5A - I$

<u>To find $A^2$</u>

$$A^2 = A * A = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{pmatrix}\begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 4+0+1 & 2+1+1 & 2+0+2 \\ 0+0+1 & 0+1+0 & 0+0+0 \\ 2+0+2 & 1+1+2 & 1+0+4 \end{pmatrix}$$

$$= \begin{pmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{pmatrix}$$

$$-15A^2 + 5A - I = -15\begin{pmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{pmatrix} + 5\begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -75 & -60 & -60 \\ 0 & -15 & 0 \\ -60 & -60 & -75 \end{pmatrix} + \begin{pmatrix} 10 & 5 & 5 \\ 0 & 5 & 0 \\ 5 & 5 & 10 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -66 & -55 & -55 \\ 0 & -11 & 0 \\ -55 & -55 & -66 \end{pmatrix}$$

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