Problem based on Harmonic function

Q.1.) If $f(z) = u+iv$, is a regular function of z, in the domain d, then

$$\nabla^2 |f(z)|^2 = 4|f'(z)|^2$$

Solution :

Proof : $|f(z)| = \sqrt{u^2+v^2}$

$$|f(z)|^2 = u^2+v^2.$$

$$\nabla^2 |f(z)|^2 = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(u^2+v^2)$$

$$= \frac{\partial^2}{\partial x^2}(u^2+v^2) + \frac{\partial^2}{\partial y^2}(u^2+v^2)$$

$$= \frac{\partial^2 u^2}{\partial x^2} + \frac{\partial^2 v^2}{\partial x^2} + \frac{\partial^2 u^2}{\partial y^2} + \frac{\partial^2 v^2}{\partial y^2} \rightarrow (1)$$

Now, $\frac{\partial^2}{\partial x^2} u^2 = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x} \cdot u^2\right)$

$$= \frac{\partial}{\partial x}(2u \cdot u_x)$$

$$= 2[u \cdot u_{xx} + u_x u_x]$$

$$= 2uu_{xx} + 2u_x^2.$$

Similarly, $\frac{\partial^2 v^2}{\partial y^2} = 2vv_{yy} + 2v_y^2$, $\frac{\partial^2 u^2}{\partial y^2} = 2uu_{yy} + 2u_y^2$

$$\frac{\partial^2 v^2}{\partial x^2} = 2vv_{xx} + 2v_x^2$$

from (1), $\Rightarrow$ $2[uu_{xx} + u_x^2 + v_{yy} + u_y^2 + v_{xx} \cdot v + v_x^2 + uv_{yy} + v_y^2]$

$$= 2[u(u_{xx} + u_{yy}) + u_x^2 + u_y^2 + v(v_{xx} + v_{yy}) + v_x^2 + v_y^2]$$

$\left[\because u_{xx} + u_{yy} = 0 \text{ (i.e. } u \text{ is harmonic)} \quad v_{yy} + v_{xx} = 0 \text{ } (\because v \text{ is harmonic})\right]$

$\Rightarrow \nabla^2 |f(z)|^2 = 2[u_x^2 + u_y^2 + v_x^2 + v_y^2]$ $\quad$ [from CR eqn. $u_x = v_y$, $-v_x = u_y$]

$$= 2[u_x^2 + (-v_x)^2 + v_x^2 + u_x^2]$$

$$= 2[2u_x^2 + 2v_x^2]$$

$$= 4[u_x^2 + v_x^2]$$

$\Rightarrow \nabla^2 |f(z)|^2 = 4|f'(z)|^2$ $\quad$ $(\because f'(z) = u_x + v_x)$.

Hence, proved//

Q.2.) If $f(z) = u+iv$ is a regular function of z, in the domain D then, $\nabla^2 \log|f(z)| = 0$. If $f(z)$ and $f'(z)$ is not equal to zero i.e. $\log|f(z)|$ is a Harmonic in D.

Proof: $|f(z)| = \sqrt{u^2+v^2}$

$$\log|f(z)| = \frac{1}{2}\log(u^2+v^2)$$

$$\nabla^2 \log|f(z)| = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\frac{1}{2}\log(u^2+v^2)$$

$$= \frac{1}{2}\frac{\partial^2}{\partial x^2}\log(u^2+v^2) + \frac{1}{2}\frac{\partial^2}{\partial y^2}\log(u^2+v^2) \rightarrow (1)$$

Consider,

$$\frac{1}{2}\frac{\partial^2}{\partial x^2}\log(u^2+v^2) = \frac{1}{2}\times\frac{\partial}{\partial x}\left(\frac{1}{u^2+v^2}\cdot(2uu_x+2vv_x)\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{uu_x+vv_x}{u^2+v^2}\right)$$

Here, $u = uu_x + vv_x$ $\quad v = u^2+v^2$

$u' = uu_{xx} + u_x^2 + vv_x + v_x^2$ $\quad v' = 2uu_x + 2vv_x$

by $\frac{u}{v}$ method,

$$= \left(\frac{(u^2+v^2)(uu_{xx}+u_x^2+vv_x+v_x^2) - (uu_x+vv_x)(2uu_x+2vv_x)}{(u^2+v^2)^2}\right)$$

$$= \frac{(u^2+v^2)(uu_{xx}+u_x^2+v_{xx}v+v_x^2) - 2(uu_{xx}+vv_{xx})^2}{(u^2+v^2)^2} \rightarrow (2)$$

Similarly,

$$\frac{1}{2}\frac{\partial^2}{\partial y^2}\log(u^2+v^2) = \frac{(u^2+v^2)(uu_{yy}+u_y^2+v v_{yy}-v_y^2) - 2(uu_y+vv_y)^2}{(u^2+v^2)^2} \rightarrow (3)$$

adding (2) & (3),

$$\nabla^2\log|f(z)| = \frac{(u^2+v^2)\left[uu_{xx}+u_x^2+vv_{xx}+v_x^2+uu_{yy}+v_y^2+vv_{yy}+v_y^2 - 2\left[(uu_x+vv_x)^2+(uu_{yy}+vv_{yy})^2\right]\right]}{(u^2+v^2)^2}$$

$$= \frac{(u^2+v^2)\left[u(u_{xx}+u_{yy})+v(v_{xx}+v_{yy})+u_x^2+v_x^2+u_y^2+v_y^2\right] - 2\left(u^2u_x^2+v^2v_x^2+u^2u_{yy}^2+v^2v_{yy}^2+2uu_xv_xv+2uvu_yv_y\right)}{(u^2+v^2)^2}$$

$$= \frac{(u^2+v^2)\left[u_x^2+v_x^2+(-v_x)^2+u_x^2\right] - 2\left[u^2(u_x^2+u_y^2)+v^2(v_x^2+v_y^2)+2uv(u_xv_x+u_yv_y)\right]}{(u^2+v^2)^2}$$

$\left[u_{xx}+u_{yy}=0\right.$
u Harmonic
$v_{xx}+v_{yy}=0$
$\left.v\text{ harmonic}\right]$

$$= \frac{(u^2+v^2)\left[u_x^2+v_x^2 \cdot 2\right] - 2\left[u^2(u_x^2+v_x^2)+v^2(v_x^2+u_x^2)\right]}{(u^2+v^2)^2}$$

$$= \frac{\left[2(u^2+v^2)(u_x^2+v_x^2)\right] - 2\left[(u_x^2+v_x^2)(u^2+v^2)\right]}{(u^2+v^2)^2}$$

$$= 0.$$

Hence, $\nabla^2\log|f(z)| = 0$

Hence proved//

Milne's Thompson Method.

Q.3.) Show that $v = e^x(x\cos y - y\sin y)$ is a harmonic function. Find the analytic function for which v is a imaginary part.

Solution:

Given: $V = e^x (x\cos y - y\sin y)$

$= e^x x\cos y - e^x y\sin y$

$V_x = \cos y\,(e^x(1) + xe^x) - y\sin y\, e^x$

$\Rightarrow V_x = \cos y\, e^x + xe^x\cos y - y\sin y\, e^x \rightarrow (1)$

$V_y = -e^x \sin y\, x - e^x [y\cos y + \sin y(1)]$

$V_y = -e^x x\sin y - e^x y\cos y - e^x \sin y \rightarrow (2)$

$V_{xx} = e^x\cos y + \cos y\,[xe^x + e^x(1)] - y\sin y\, e^x$

$= \cos y\, e^x + \cos y \cdot xe^x + e^x\cos y - y\sin y\, e^x$

$V_{yy} = -e^x x\cos y - e^x[\sin y(-y) + \cos y] - e^x\cos y$

$= -xe^x\cos y + ye^x\sin y - e^x\cos y - e^x\cos y$

So, $V_{xx} + V_{yy} = \cancel{e^x\cos y} + \cancel{\cos y\, xe^x} + \cancel{e^x\cos y} - \cancel{y\sin y\, e^x} - \cancel{x e^x\cos y} + \cancel{ye^x\sin y} - \cancel{e^x\cos y} - \cancel{e^x\cos y}$

$\Rightarrow V_{xx} + V_{yy} = 0$

$\therefore$ V is harmonic.

$(1) \Rightarrow \phi_1(z,0) = V_x = e^z(1) + z\cdot e^z(1) - e^z y(0)$

$= e^z + ze^z.$

$(2) \Rightarrow \phi_2(z,0) = V_y = -e^z z(0) - e^z(0)\cos 0 - e^z(0)$

$= 0$

$(z-1)\,(i+1)$

By Milne's Thompson method,

$$\boxed{f(z) = \int \phi_1(z,0)\,dz - i\int \phi_2(z,0)\,dz}$$

$f(z) = \int (e^z + ze^z)\,dz - i\int 0\,dz$

$f(z) = \int e^z dz + \int z\cdot e^z dz$

$u = z \quad dv = e^z$

$u_1 = 1 \quad v = e^z$

$u_2 = 0 \quad v_2 = e^z$

$\Rightarrow f(z) = e^z + (ze^z - e^z)$

$= \cancel{e^z} + ze^z - \cancel{e^z}$

$\therefore f(z) = ze^z + c$

$u + iv = (x+iy)\cdot e^{x+iy} + c$

$u + iv = x\cdot e^{x+iy} + iye^{x+iy} + c$

$= xe^x\cdot e^{iy} + iy\cdot e^x\cdot e^{iy} + c$

(w.k.t. $e^{i\theta} = \cos\theta + i\sin\theta$)

$\Rightarrow u + iv = xe^x(\cos y + i\sin y) + iye^x(\cos y + i\sin y)$

$= xe^x\cos y + xe^x i\sin y + iye^x\cos y - ye^x\sin y$

$u + iv = (xe^x\cos y - ye^x\sin y) + i(xe^x\sin y + ye^x\cos y)$

$\therefore \boxed{u = e^x(x\cos y - y\sin y)}$

Q.4.) Determine the analytical function whose real part is $\dfrac{\sin 2x}{\cosh 2y - \cos 2x}$ and show that $f(z) = \cot z +$

Solution:

Given: $u = \dfrac{\sin 2x}{\cosh 2y - \cos 2x}$

$$u_x = \frac{(\cosh 2y - \cos 2x)(2\cos 2x) - \sin 2x[0 - (-\sin 2x \cdot 2)]}{(\cosh 2y - \cos 2x)^2}$$

$$= \frac{(\cosh 2y - \cos 2x)(2\cos 2x) - \sin 2x(2\sin 2x)}{(\cosh 2y - \cos 2x)^2}$$

$$= 2\cosh 2y \cos 2x - 2\cos^2 2x - 2\sin^2 2x$$

$$\Rightarrow u_x = \frac{2\cosh 2y - 2}{(\cosh 2y - \cos 2x)^2}$$

$$\phi_1(z,0) = u_x(z,0) = \frac{2\cos 2z - 2}{(1-\cos 2z)^2} \rightarrow (1)$$

$$u_y = \frac{(\cosh 2y - \cos 2x)(0) - \sin 2x(\sinh 2y \times 2)}{(\cosh 2y - \cos 2x)^2}$$

$$\phi_2(z,0) = u_y(z,0) = \frac{-\sin 2z(-2)\times 0}{(1-\cos 2z)^2} = 0.$$

$$\phi_2(z,0) = u_y(z,0) = 0 \rightarrow (2)$$

By Milne's thompson method,

$$\boxed{f(z) = \int \phi_1(z,0)\,dz - i\int \phi_2(z,0)\,dz}$$

$$= \int \frac{-2\cancel{(1-\cos 2z)}}{(1-\cos 2z)^{\cancel{2}}}\,dz - i\int 0\,dz.$$

$$= -2\int \frac{1}{1-\cos 2z}\,dz \qquad \left[\sin^2\theta = \frac{1-\cos 2\theta}{2}\right]$$

$$f(z) = -\cancel{2}\times\int \frac{1}{\cancel{2}\sin^2\theta}\,d\theta$$

$$= -\int \operatorname{cosec}^2\theta\,d\theta$$

$$\therefore f(z) = -\int \operatorname{cosec}^2 z\,dz$$

$$\Rightarrow f(z) = \cot z + C$$

Hence, the result.

Q.5.) Show that function, $u = \frac{1}{2}\log(x^2+y^2)$ is harmonic and determine its conjugate, also $f(z)$.
(Hint: Solve problem using Polar coordinates)

Solution: Given: $u = \frac{1}{2}\log(x^2+y^2)$

$$u_x = \frac{1}{\cancel{2}}\left[\frac{1}{x^2+y^2}\cdot \cancel{2}x\right] = \frac{x}{x^2+y^2} \rightarrow (1)$$

$$u_y = \frac{1}{\cancel{2}}\left[\frac{1}{x^2+y^2}\cdot \cancel{2}y\right] = \frac{y}{x^2+y^2} \rightarrow (2).$$

$$u_{xx} = \frac{(x^2+y^2)(1) - x(2x)}{(x^2+y^2)^2} = \frac{x^2+y^2-2x^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$u_{yy} = \frac{(x^2+y^2)(1) - y(2y)}{(x^2+y^2)^2} = \frac{x^2+y^2-2y^2}{(x^2+y^2)^2} = \frac{x^2-y^2}{(x^2+y^2)^2}$$

$$u_{xx} + u_{yy} = \frac{y^2-x^2}{(x^2+y^2)^2} + \frac{x^2-y^2}{(x^2+y^2)^2} = \frac{\cancel{y^2}-\cancel{x^2}+\cancel{x^2}-\cancel{y^2}}{(x^2+y^2)^2} = 0$$

$u_{xx} + u_{yy} = 0$

$\therefore$ u is Harmonic.

① $\Rightarrow$ $\phi_1(z,0) = u_x(z,0) = \dfrac{x}{x^2+y^2} = \dfrac{z}{z^2+0} = \dfrac{z}{z^2} = \dfrac{1}{z}$.

② $\Rightarrow$ $\phi_2(z,0) = u_y(z,0) = \dfrac{y}{x^2+y^2} = \dfrac{0}{z^2} = 0$

By Milne's thompson method,

$$\boxed{f(z) = \int \phi_1(z,0)\,dz - i\int \phi_2(z,0)\,dz}$$

$= \int \frac{1}{z} dz - i \int 0$

$= \int \frac{1}{z} dz$

$\Rightarrow f(z) = \log z + c$.

$\therefore$ Put $z = re^{i\theta}$.

$u + iv = \log(r \cdot e^{i\theta}) + c$

$= \log r + \log e^{i\theta} + c$

$\therefore$ $\boxed{v = \theta}$

$\Rightarrow$ $v = \tan^{-1}\left(\frac{y}{x}\right)$

Hence, the result.

$r^2 = x^2+y^2$

$\therefore \log e^{i\theta} = i\theta$.

$\therefore \tan\theta = \frac{y}{x}$

Q.6) Find the analytical function $f(z) = u + iv$ given that $u - 2v = e^x(\cos y - \sin y)$

Solution: Given: $(1+2i) f(z) = U + iV$.

Given: $U = u - 2v = e^x(\cos y - \sin y)$

$U = e^x(\cos y - \sin y)$

$U_x = e^x(\cos y - \sin y)$

$U_x(z,0) = \phi_1 = e^z$.

$U_y = e^x(-\sin y - \cos y)$

$U_y(z,0) = \phi_2 = e^z(0-1) = -e^z$.

By Milne's thompson method,

$$\boxed{F(z) = \int \phi_1(z,0)\,dz - i\int \phi_2(z,0)\,dz}.$$

$(1+2i) f(z) = \int e^z dz + i \int e^z dz$

$(1+2i) f(z) = e^z + ie^z + C$

$f(z) = \dfrac{1}{1+2i} e^z(1+i)$

$= \left(\dfrac{1}{1+2i} \times \dfrac{1-2i}{1+2i}\right)(1+i)e^z + c$

$= \dfrac{(1-2i)(1+i)}{1^2+2^2} e^z + c$

$= \dfrac{1(e^z) + 2e^z + ie^z - 2ie^z}{5}$

$= \dfrac{3e^z}{5} - \dfrac{e^z i}{5} + c$

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Problems based on $w = \frac{1}{z}$ - Band

Q.7) Find the image of $|z| - 2i = 2$ under the transformation $w = 1/z$.

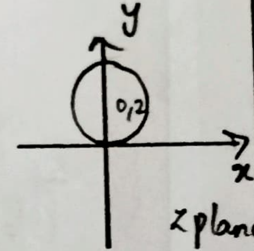
Solution: $|z - 2i| = 2$

$|(x+iy) - 2i| = 2$

$|x + i(y-2)| = 2$

$x^2 + (y-2)^2 = 2^2$ centre $(0, 2)$

radius : 2



Given: $z = \frac{1}{w}$.

$\left|\frac{1}{w} - 2i\right| = 2$

$\left|\frac{1 - 2iw}{w}\right| = 2^2$

$|1 - 2iw| = 4|w|$

$|1 - 2i(u+iv)| = 4|(u+iv)|$

$|1 - 2iu + 2v| = 4|u+iv|$

$|1 + 2v| - 2|iu| = 4|u+iv|$

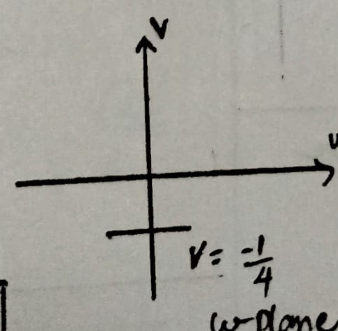
$(1+2v)^2 + (2u)^2 = 4(u^2+v^2)$

$= (1+2v)^2 + \cancel{4u^2} = \cancel{4u^2} + 4v^2$

$= 1 + \cancel{4v^2} + 4v = \cancel{4v^2}$

$\Rightarrow 1 + 4v = 0$

$\therefore \boxed{v = \frac{-1}{4}}$



Q.8) Show that the transformation $w = \frac{1}{z}$

(ii) transforms all the circle and straight line in z-plane into a circle or straight line in the w-plane.

Solution:

Given: $w = \frac{1}{z}$ (or) $z = \frac{1}{w}$

$(x+iy) = \frac{1}{u+iv} \times \frac{u-iv}{u-iv} \Rightarrow \frac{u-iv}{u^2+v^2}$

$\therefore x = \frac{u}{u^2+v^2}$; $y = \frac{-v}{u^2+v^2}$

W.K.T,

$a(x^2+y^2) + 2gx + 2fy + c = 0$.

$a\left(\frac{u^2}{(u^2+v^2)^2} + \frac{v^2}{(u^2+v^2)^2}\right) + 2\frac{gu}{u^2+v^2} + \left(-2f\left(\frac{v}{u^2+v^2}\right)\right) + c = 0$

$a\left(\frac{\cancel{u^2+v^2}}{(u^2+v^2)^{\cancel{2}}}\right) + \frac{2gu - 2fv}{u^2+v^2} + c = 0$

$\Rightarrow \frac{a + 2gu - 2fv + c(u^2+v^2)}{u^2+v^2} = 0$

$\Rightarrow a + 2gu - 2fv + c(u^2+v^2) = 0$

(If $a = 0$, this becomes a circle
If $c = 0$, this becomes a straight line)

Case(i) If $a \neq 0$, $c \neq 0$, circle does not pass through the origin, in the z-plane map into a circle not passes through the

the origin, in the w-plane.

Case (ii):

If $a \neq 0, c = 0$; then circle passes through the origin in z-plane maps into a straight line not through the origin in the w-plane.

Case (iii):

If $a = 0, c \neq 0$; then a straight line not pass through the origin in the z-plane maps onto a circle through the origin in the w plane.

Case (iv):

If $a = 0, c = 0$; A straight line through the origin z-plane onto a straight line through the origin on the w plane.

Bilinear Transformation:

Q.9.) Find the bilinear transformation which maps $z = 1, i, -1$ respectively onto $w = i, 0, -i$. Hence, find the fixed points.

Solution:

W.K.T

$$\boxed{\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}}$$

Given: $w_1 = i \quad w_2 = 0 \quad w_3 = -i$

$z_1 = 1 \quad z_2 = i \quad z_3 = -1$

$$= \frac{(w-i)(0+i)}{(w+i)(0-i)} = \frac{(z-1)(i+1)}{(z+1)(i-1)}$$

$$= \frac{\cancel{i}(w-i)}{-\cancel{i}(w+i)} = \frac{(z-1)(i+1)}{(z+1)(i-1)}$$

$$\Rightarrow \frac{w-i}{-(w+i)} = \frac{zi+z-i-1}{zi-z+i-1}$$

$$\Rightarrow w(zi-z+i-1)-i(zi-z+i-1) = -w(zi+z-i-1)-i(zi+z-i-1)$$

$$\Rightarrow wzi-wz+wi-w+z+zi+1+i = -wzi-wz+wi+w+z-zi+1+i$$

$$\Rightarrow wzi-\cancel{wz}+\cancel{wi}-w+wzi+\cancel{wz}-\cancel{wi}-w = \cancel{z}-zi-1+\cancel{i}-\cancel{z}-zi-1-\cancel{i}$$

$$\Rightarrow 2wzi-2w = -2zi-2$$

$$wzi - w = -zi-1$$

$$\Rightarrow w(zi-1) = -(zi+1)$$

$$\Rightarrow \boxed{w = \frac{-zi-1}{zi-1}}$$

Verification:-

when $z = 1$,

$$w = \frac{-i-1}{i-1} \times \frac{i+1}{i+1} = \frac{\cancel{1}-i-i-\cancel{1}}{-1-1} = \frac{-2i}{-2} = i$$

$w = i$

Hence verified.

Q.10) Find the bilinear transformation which
-X- maps the point $(1, i, -1)$ on to the point $(0, 1, \infty)$.
show that the transformation map the interior of a unit circle of z-plane onto the upper half of w-plane.

Given:

$z_1 = 1 \qquad z_2 = i \qquad z_3 = -1$

$w_1 = 0 \qquad w_2 = 1 \qquad w_3 = \infty$

Solution

$$\boxed{\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}}$$

$$\Rightarrow \frac{(w-w_1)}{(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\Rightarrow \frac{(w-1)}{1-0} = \frac{(z-1)(i+1)}{(z+1)(i-1)}$$

$$\frac{w}{1} = \frac{(z-1)(i+1)}{(z+1)(i-1)}$$

$$w = \frac{zi+z-i-1}{zi-z+i-1} = \frac{(z-1)(i+1)}{(z+1)(i-1)}$$

$$= \frac{(z-1)}{(z+1)}\frac{(i+1)}{(i-1)} \times \frac{(i+1)}{(i+1)}$$

$$= \frac{(z-1)(i+1)^2}{(z+1)(-1-1)} \Rightarrow \frac{z-1}{z+1} \times \left(\frac{-1+1+2i}{-2}\right)$$

$$= \frac{z-1}{} \cdot \left(\frac{i}{-1}\right)$$

$$\therefore w = \frac{zi-i}{-z-1} \Rightarrow -wz-w = zi-i$$

$$-wz-zi = w-i$$

$$-z(w+i) = w-i$$

$$\Rightarrow z = -\left(\frac{w-i}{w+i}\right)$$

$$z = \frac{-w+i}{w+i}$$

Given, $|z| < 1$

$$\Rightarrow \left|\frac{-w+i}{w+i}\right| < 1 \Rightarrow \left|\frac{-(u+iv)+i}{(u+iv)+i}\right| < 1$$

$$= \left|\frac{-u-iv+i}{u+iv+i}\right| < 1$$

$$= \left|\frac{-u-i(v-1)}{u+i(v-1)}\right| < 1$$

$$\Rightarrow |-u-i(v-1)| < |u+i(v+1)|$$

$$\cancel{u^2}+(v-1)^2 < \cancel{u^2}+(v+1)^2$$

$$(\cancel{v^2}-2v+\cancel{1}) < (\cancel{v^2}+2v+\cancel{1})$$

$$-2v < 2v$$

$$-4v < 0$$

$$-v < 0$$

$$\Rightarrow \boxed{v > 0}$$