Complex Integration.

Complex integration.

- Cauchy's integral formula
- Taylor's Series, Laurent's Series
- Cauchy's Residue Theorem
- Contour Integration.

* Cauchy's Integral Theorem (or) Cauchy's Fundamental Theorem.

If a function $f(z)$ is analytic and its derivative $f'(z)$ is continuous at all points inside and on a simple closed curve C, then

$$\oint_C f(z)\,dz = 0.$$

* Singular points:

If a function 'f' fails to be a analytic at a point z_0, but is analytic at some point in every neighbourhood of z_0, then z_0 is a singular point of f.

Eg:

① $\int_C \frac{z}{z-2}\,dz$, here $z-2=0$
$\Rightarrow z=2$ is a singular pt.

② $\int \frac{\sin \pi z^2}{(z+1)(z-1)}\,dz$, here $(z+1)(z-1)=0$
$\Rightarrow z=-1, 1$ are singular points.

③ $\int \frac{1}{z}\,dz$, here $z=0$ is a singular pt.

* Cauchy's Integral formula :

Let $f(z)$ be an analytic function inside and on a simple closed contour c, taken in the positive sense. If 'a' is any point interior to c, then

$$\int_c \frac{f(z)}{z-a}\,dz = 2\pi i \cdot f(a)$$

* Cauchy's Integral formula for derivatives:

$$\int_c \frac{f(z)}{z-a}\,dz = 2\pi i\, f(a)$$

$$\int_c \frac{f(z)}{(z-a)^2}\,dz = \frac{2\pi i}{1!} f'(a)$$

$$\int_c \frac{f(z)}{(z-a)^3}\,dz = \frac{2\pi i}{2!} f''(a)$$

$$\therefore \int_c \frac{f(z)}{(z-a)^{n+1}}\,dz = \frac{2\pi i}{n!} f^{(n)}(a).$$

Pbs:

① Evaluate $\oint_c \frac{e^{-z}}{z+1}\,dz$, where c is a circle (i) $|z|=2$ (ii) $|z|=\frac{1}{2}$.

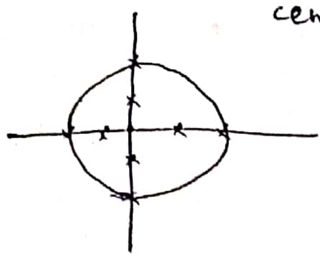
Soln:

Given; $I = \oint_c \frac{e^{-z}}{z+1}\,dz$

Here, $z+1=0$

$\Rightarrow z=-1$ is a singular point.

(i) Given: $c: |z|=2$



centre $(0,0)$
$r=2$.

Here $z=-1$ lies inside c.

$\therefore$ By Cauchy's Integral formula,

$$\int_C \frac{f(z)}{z-a}\, dz = 2\pi i\, f(a).$$

$$\therefore \int_C \frac{e^{-z}}{z+1}\, dz = 2\pi i\, f(-1)$$

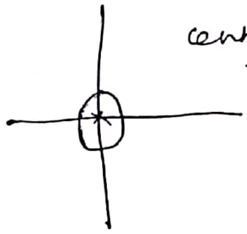
$$= 2\pi i\,(e).$$

here, $f(z) = e^{-z}$

$f(-1) = e^{-(-1)}$

$= e.$

(ii) Given: $c : |z| = \frac{1}{2}$.



centre $(0,0)$
$r = \frac{1}{2}$.

Here, $z = -1$ lies outside c,

$\therefore$ By cauchy's integral theorem,

$$\int_C \frac{e^{-z}}{z+1}\, dz = 0$$

② . Evaluate $\int_C \frac{\cos \pi z^2}{(z-1)(z-2)}\, dz$, where c is $|z| = \frac{3}{2}$.

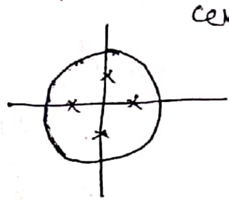
Soln: Given:

$$I = \int_C \frac{\cos \pi z^2}{(z-1)(z-2)}\, dz$$

Here, $(z-1)(z-2) = 0$

$\Rightarrow$ $z = 1, 2$ are Singular points.

Given: $c : |z| = \frac{3}{2}$.



centre $(0,0)$
$r = \frac{3}{2}$.

Here, $z = 1$ lies inside c.

$z = 2$ lies outside c.

$\therefore$ By Cauchy's integral formula,

$$\int_c \frac{f(z)}{z-a}\, dz = 2\pi i\, f(a)$$

$$\int_c \frac{\cos \pi z^2}{(z-1)(z-2)}\, dz = \int_c \frac{\left(\frac{\cos \pi z^2}{z-2}\right)}{z-1}\, dz$$

Here, $f(z) = \frac{\cos \pi z^2}{z-2}$

$$f(1) = \frac{\cos \pi (1^2)}{1-2} = \frac{-1}{-1} = 1.$$

$$= 2\pi i\, f(1)$$

$$= 2\pi i\, (1)$$

$$= 2\pi i$$

③. Using Cauchy's integral formula for derivatives,

evaluate $\int_c \frac{e^{2z}}{(z+1)^4}\, dz$, where c is the circle $|z| = 2$.

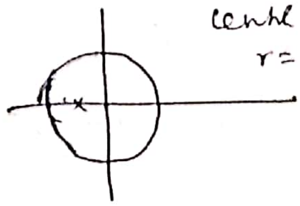
Soln:

Given: $I = \int_c \frac{e^{2z}}{(z+1)^4}\, dz$.

Here, $(z+1)^4 = 0$

$\Rightarrow$ $z = -1$ is a singular point.

Given: $c : |z| = 2$



centre $(0,0)$

$r = 2$.

$\therefore$ $z = -1$ lies inside c.

$\therefore$ By Cauchy's integral formula for derivatives,

$$\int_c \frac{f(z)}{(z-a)^{n+1}}\, dz = \frac{2\pi i}{n!} f^{(n)}(a)$$

$$\int_c \frac{e^{2z}}{(z+1)^4}\,dz = \frac{2\pi i}{3!} f'''(-1)$$

$$= \frac{\cancel{2}\pi i}{\cancel{6}_3}\left(8e^{-2}\right)$$

$$= \frac{8}{3}\pi i e^{-2} \;//.$$

Here, $a=-1$.

& $f(z) = e^{2z}$

$f'(z) = 2e^{2z}$

$f''(z) = 4e^{2z}$

$f'''(z) = 8e^{2z}$

$f'''(-1) = 8e^{-2}$.

(4). Evaluate $\oint_c \frac{ze^{2z}}{(z-1)^3}\,dz$, where c is the circle $|z+i|=2$.

soln:

Given: $I = \oint_c \frac{ze^{2z}}{(z-1)^3}\,dz$

Here, $(z-1)^3 = 0$

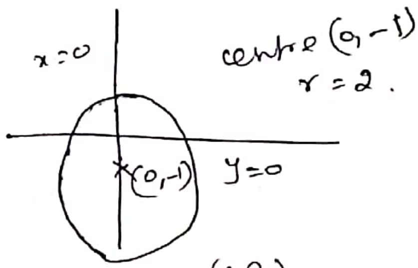
$\Rightarrow z=1$ is a singular point.

Given: $c : |z+i| = 2$

(ie) $|x+iy+i| = 2$

$\Rightarrow |x+i(y+1)| = 2$

$\Rightarrow (x-0)^2 + (y+1)^2 = 2^2$



$\therefore z=1$ lies inside c.

(OR)

($|z+i| = |1+i| = 2$

$= \sqrt{1^2+1^2} = 2$

$= \sqrt{2} < 2$.

$\therefore z=1$ lies inside c.)

$\therefore$ By Cauchy's Integral formula (for derivatives),

$$\int_c \frac{f(z)}{(z-a)^{n+1}}\,dz = \frac{2\pi i}{n!} f^{(n)}(a)$$

$$\int_c \frac{ze^{2z}}{(z-1)^3}\,dz = \frac{2\pi i}{2!} f''(a)$$

$$= \frac{2\pi i}{2!} f''(1)$$

$$= \frac{\cancel{2}\pi i}{\cancel{2}} (8e^2)$$

$$\int \frac{ze^{2z}}{(z-1)^3} dz = 8\pi i \, e^2 \quad //$$

Here, $a = 1$

& $f(z) = ze^{2z}$

$f'(z) = z \cdot 2e^{2z} + e^{2z} \cdot 1$

$f''(z) = 2[z \cdot 2e^{2z} + e^{2z} \cdot 1] + 2e^{2z}$

$f''(1) = 2[1 \cdot 2e^2 + e^2] + 2e^2$

$= 4e^2 + 2e^2 + 2e^2$

$= 8e^2$.

(5). Evaluate $\int_c \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} dz$ where c is $|z| = 3$.

Soln:

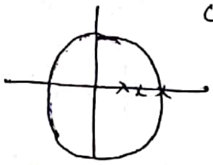
Given: $I = \int_c \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} dz$

Here, $(z-1)(z-2) = 0$

$\Rightarrow z = 1, 2$ are singular points.

Given: $c : |z| = 3$

centre $(0,0)$
$r = 3$.



$\therefore z = 1, 2$ both lies inside c.

$\therefore$ By Partial fraction method,

$$\frac{1}{(z-1)(z-2)} = \frac{A}{z-1} + \frac{B}{z-2}$$

$$1 = A(z-2) + B(z-1) \rightarrow (1)$$

For $z = 2 \Rightarrow 1 = A(0) + B(2-1)$

$\boxed{1 = B}$

For $z = 1 \Rightarrow 1 = A(1-2) + B(0)$

$\boxed{-1 = A}$

$$\therefore \frac{1}{(z-1)(z-2)} = \frac{-1}{z-1} + \frac{1}{z-2}.$$

$$(i) \int \frac{\sin\pi z^2 + \cos\pi z^2}{(z-1)(z-2)} dz = -\int \frac{\sin\pi z^2 + \cos\pi z^2}{z-1} dz + \int \frac{\sin\pi z^2 + \cos\pi z^2}{z-2} dz$$

$\therefore$ By Cauchy's Integral formula,

$$\int \frac{f(z)}{z-a} dz = 2\pi i \, f(a).$$

$$(i) \int \frac{\sin\pi z^2 + \cos\pi z^2}{(z-1)(z-2)} dz = -2\pi i \, f(1) + 2\pi i \, f(2)$$

$$= -2\pi i \left[\sin\pi(1^2) + \cos\pi(1^2)\right] + 2\pi i \left[\sin\pi(2^2) + \cos\pi(2^2)\right]$$

$$= -2\pi i \left[\sin\pi + \cos\pi\right] + 2\pi i \left[\sin 4\pi + \cos 4\pi\right]$$

$$= -2\pi i \left[0 - 1\right] + 2\pi i \left[0 + 1\right]$$

$$= 2\pi i + 2\pi i$$

$$= 4\pi i$$

6. Using Cauchy's Integral formula, Evaluate

$$\oint_c \frac{\sin\pi z^2 + \cos\pi z^2}{(z-2)(z-3)} dz, \text{ where } c: |z| = 4.$$

Soln:

Given: $c: |z| = 4$.

Here, $(z-2)(z-3) = 0$

$\Rightarrow z = 2, 3$ are singular points.

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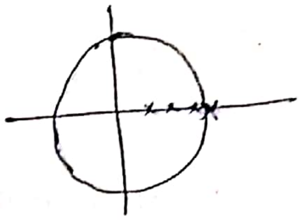
centre (0,0)
r=4.



Given: $c: |z| = 4$

Here, $z = 2, 3$ both lies inside c.

∴ By Partial fraction method,

$$\frac{1}{(z-2)(z-3)} = \frac{A}{z-2} + \frac{B}{z-3}$$

$$1 = A(z-3) + B(z-2) \longrightarrow ①.$$

For $z = 3 \Rightarrow 1 = A(0) + B(3-2)$

$$\boxed{1 = B}$$

For $z = 2 \Rightarrow 1 = A(2-3) + B(0)$

$$\boxed{-1 = A}.$$

$$\therefore \frac{1}{(z-2)(z-3)} = \frac{-1}{z-2} + \frac{1}{z-3}$$

$$\int_c \frac{\sin\pi z^2 + \cos\pi z^2}{(z-2)(z-3)}\,dz = -\int \frac{\sin\pi z^2 + \cos\pi z^2}{z-2}\,dz + \int \frac{\sin\pi z^2 + \cos\pi z^2}{z-3}\,dz$$

∴ By Cauchy's Integral formula,

$$\int_c \frac{f(z)}{z-a}\,dz = 2\pi i\, f(a)$$

(ie) $$\int_c \frac{\sin\pi z^2 + \cos\pi z^2}{(z-2)(z-3)}\,dz = -2\pi i\, f(2) + 2\pi i\, f(3)$$

$$= -2\pi i\left[\sin\pi(2^2) + \cos\pi(2^2)\right] + 2\pi i\left[\sin\pi(3^2) + \cos\pi(3^2)\right]$$

$$= -2\pi i\left[\sin 4\pi + \cos 4\pi\right] + 2\pi i\left[\sin 9\pi + \cos 9\pi\right]$$

$$= -2\pi i\,[0+1] + 2\pi i\,[0-1]$$

$$= -2\pi i - 2\pi i$$

$$= -4\pi i$$ //.

(7). Evaluate $\int_c \frac{z+4}{z^2+2z+5}\,dz$, where c is a circle $|z+1+i| = 2$ using Cauchy's integral formula.

Soln:

Given: $I = \int_c \frac{z+4}{z^2+2z+5}\,dz$.

Here, $z^2+2z+5 = 0$

$$z = \frac{-2 \pm \sqrt{2^2 - 4(1)(5)}}{2(1)}$$

$$= \frac{-2 \pm \sqrt{4-20}}{2}$$

$$= \frac{-2 \pm \sqrt{-16}}{2}$$

$$= \frac{-2 \pm 4i}{2}$$

$$= -1 \pm 2i$$

$\therefore$ The Singular points are $z_1 = -1+2i$

$z_2 = -1-2i$,

Given: $c: |z+1+i| = 2$.

For $z_1 = -1+2i$,

$|z+1+i| = |\cancel{-1}+2i+\cancel{1}+i| = |3i| = 3 > 2$.

$\therefore$ $z_1 = -1+2i$ lies outside c.

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For $\underline{z_2 = -1-2i}$,

$|z+1+i| = |\cancel{-1}-2i+\cancel{1}+i| = |-i| = 1 < 2$

$\therefore z_2 = -1-2i$ lies inside c.

$\therefore z^2+2z+5 = (z-z_1)(z-z_2)$

$= [z-(-1+2i)]\,[z-(-1-2i)]$.

$\therefore$ By Cauchy's integral formula,

$$\int_c \frac{f(z)}{z-a}\,dz = 2\pi i\, f(a)$$

(ii) $$\int_c \frac{z+4}{z^2+2z+5}\,dz = \int_c \frac{z+4}{[z-(-1+2i)][z-(-1-2i)]}\,dz$$

$$= \int \frac{\frac{z+4}{z-(-1+2i)}}{z-(-1-2i)}\,dz$$

$$= 2\pi i\, f(-1-2i)$$

$$= \cancel{2}\pi \cancel{i}\left(\frac{3-2i}{-\cancel{4}\cancel{i}_{\,2}}\right)$$

$$= \frac{\pi}{2}(2i-3) \;//.$$

here, $a = -1-2i$

& $f(z) = \dfrac{z+4}{z-(-1+2i)}$

$f(-1-2i) = \dfrac{-1-2i+4}{\cancel{-1}-2i+\cancel{1}-2i}$

$= \dfrac{3-2i}{-4i}$

HW.

(8) Evaluate $\oint_c \frac{\cos z}{z}\,dz$ where c is an ellipse $9x^2+4y^2=1$

Ans: $\dfrac{x^2}{1/9} + \dfrac{y^2}{1/4} = 1$ & $z=0$ is a singular pt lies inside c.

$\therefore 2\pi i\, f(a) = 2\pi i\, f(0)$

$= 2\pi i(1) = 2\pi i$ // $[\because \cos 0 = 1]$

Taylor's Series & Laurent's Series.

Taylor's Series:

A function $f(z)$, analytic inside a circle C with centre at 'a', can be expanded in the series

$$f(z) = f(a) + \frac{f'(a)}{1!}(z-a) + \frac{f''(a)}{2!}(z-a)^2 + \frac{f'''(a)}{3!}(z-a)^3 + \cdots$$

which is convergent at every point inside C.

Laurent's Series:

Let C_1, C_2 be two concentric circles $|z-a| = R_1$ and $|z-a| = R_2$ where $R_2 < R_1$. Let $f(z)$ be analytic on C_1 and C_2 and in the annular region R between them. Then, for any point z in R,

$$f(z) = \underbrace{\sum_{n=0}^{\infty} a_n (z-a)^n}_{\text{regular part}} + \underbrace{\sum_{n=1}^{\infty} \frac{b_n}{(z-a)^n}}_{\text{Principal part.}}$$

where

$$a_n = \frac{1}{2\pi i} \int_{C_1} \frac{f(z)}{(z-a)^{n+1}}\, dz$$

$$b_n = \frac{1}{2\pi i} \int_{C_2} \frac{f(z)}{(z-a)^{1-n}}\, dz,$$

the integrals being taken anticlockwise.

Formulae:

1) $(1-z)^{-1} = 1 + z + z^2 + z^3 + \cdots$

2) $(1+z)^{-1} = 1 - z + z^2 - z^3 + \cdots$

3) $(1-z)^{-2} = 1 + 2z + 3z^2 + 4z^3 + \cdots$

4) $(1+z)^{-2} = 1 - 2z + 3z^2 - 4z^3 + \cdots$

5) $\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots$

6) $\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \dots$

7) $\log(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \dots$

8) $e^z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots$

Pbs:

① Expand $f(z) = \sin z$ in a Taylor's series about $z = \frac{\pi}{4}$.

Soln:

$f(z) = \sin z$ is analytic in the infinite plane.

$\therefore f(z) = \sin z$	$f(\frac{\pi}{4}) = \sin \pi/4 = \frac{1}{\sqrt{2}}$
$f'(z) = \cos z$	$f'(\pi/4) = 1/\sqrt{2}$
$f''(z) = -\sin z$	$f''(\pi/4) = -1/\sqrt{2}$.

$\therefore$ By Taylor's series,

$$f(z) = f(a) + \frac{z-a}{1!} f'(a) + \frac{(z-a)^2}{2!} f''(a) + \dots$$

$$= \frac{1}{\sqrt{2}} + \frac{z-\pi/4}{1!}\left(\frac{1}{\sqrt{2}}\right) + \frac{(z-\pi/4)^2}{2!}\left(-\frac{1}{\sqrt{2}}\right) + \dots$$

② Expand $f(z) = e^z$ in a Taylor's series about $z=0$.

Soln:

Let $f(z) = e^z$.

$f'(z) = f''(z) = f'''(z) = e^z \dots$

(ie) $f(0) = f'(0) = f''(0) = \dots = e^0 = 1$

$\therefore$ By Taylor's series,

$$f(z) = f(a) + \frac{z-a}{1!} f'(a) + \frac{(z-a)^2}{2!} f''(a) + \dots$$

$$= 1 + \frac{z}{1!}(1) + \frac{z^2}{2!}(1) + \dots$$

$$= 1 + z + \frac{z^2}{2!} + \dots$$

(HW)

3. obtain the expansion of $\log(1+z)$ when $|z|<1$

Soln:

For $z \neq -1$, $f(z) = \log(1+z)$ is analytic.

Here, $f(z) = \log(1+z)$

$$f^{(n)}(z) = \frac{(-1)^{n-1}(n-1)!}{(1+z)^n}$$

$$f(0) = 0, \quad f^n(0) = (-1)^{n-1}(n-1)!$$

$\therefore$ Taylor's Series about $z=0$ (origin),

$$f(z) = f(0) + z f'(0) + \frac{z^2}{2!} f''(0) + \cdots$$

$$\log(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \cdots \text{ to } \infty.$$

4. Expand $f(z) = \dfrac{z^2-1}{(z+2)(z+3)}$ in a Laurent's series if

i) $|z|<2$

ii) $|z|>3$

iii) $2<|z|<3$.

Soln:

Let $f(z) = \dfrac{z^2-1}{(z+2)(z+3)}$

By partial fraction method,

$$\frac{z^2-1}{(z+2)(z+3)} = A + \frac{B}{z+2} + \frac{C}{z+3} \rightarrow (1)$$

$$\Rightarrow z^2-1 = A(z+2)(z+3) + B(z+3) + C(z+2) \rightarrow (2)$$

In (2), Put $z=-2$,

$$(-2)^2-1 = A(0) + B(-2+3) + C(0)$$

$$4-1 = B(1)$$

$$\boxed{3 = B}$$

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In ②, Put $z = -3$,

$$(-3)^2 - 1 = A(0) + B(0) + C(-3+2)$$

$$9 - 1 = C(-1)$$

$$\boxed{-8 = C}$$

Equating coefft of z^2 on both sides of ①,

$$\boxed{1 = A}$$

$$\therefore \frac{z^2-1}{(z+2)(z+3)} = A + \frac{B}{z+2} + \frac{C}{z+3}$$

$$\boxed{f(z) = 1 + \frac{3}{z+2} - \frac{8}{z+3}}$$

(i) Given region : $|z| < 2$.

(ie) $\left|\frac{z}{2}\right| < 1$ and $\left|\frac{z}{3}\right| < 1$.

$$\therefore f(z) = 1 + \frac{3}{2\left(1+\frac{z}{2}\right)} - \frac{8}{3\left(1+\frac{z}{3}\right)}$$

$$= 1 + \frac{3}{2}\left(1+\frac{z}{2}\right)^{-1} - \frac{8}{3}\left(1+\frac{z}{3}\right)^{-1}$$

$$f(z) = 1 + \frac{3}{2}\left[1 - \left(\frac{z}{2}\right) + \left(\frac{z}{2}\right)^2 - \cdots\right]$$

$$- \frac{8}{3}\left[1 - \left(\frac{z}{3}\right) + \left(\frac{z}{3}\right)^2 - \cdots\right]$$

//.

(ii) Given region : $|z| > 3$

(ie) $3 < |z| \Rightarrow \left|\frac{3}{z}\right| < 1$ and $\left|\frac{2}{z}\right| < 1$.

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$$\therefore\ f(z) = 1 + \frac{3}{z+2} - \frac{8}{z+3}$$

$$= 1 + \frac{3}{z\left(1+\frac{2}{z}\right)} - \frac{8}{z\left(1+\frac{3}{z}\right)}$$

$$= 1 + \frac{3}{z}\left[1+\frac{2}{z}\right]^{-1} - \frac{8}{z}\left[1+\frac{3}{z}\right]^{-1}$$

$$f(z) = 1 + \frac{3}{z}\left[1-\left(\frac{2}{z}\right)+\left(\frac{2}{z}\right)^2 - \cdots\right]$$
$$- \frac{8}{z}\left[1-\left(\frac{3}{z}\right)+\left(\frac{3}{z}\right)^2 - \cdots\right] \quad //.$$

(iii). <u>Given region:</u> $2 < |z| < 3$.

(ie) $2 < |z|$ and $|z| < 3$

(ie) $\left|\frac{2}{z}\right| < 1$ and $\left|\frac{z}{3}\right| < 1$

$$\therefore\ f(z) = 1 + \frac{3}{z+2} - \frac{8}{z+3}$$

$$= 1 + \frac{3}{z\left(1+\frac{2}{z}\right)} - \frac{8}{3\left(1+\frac{z}{3}\right)}$$

$$= 1 + \frac{3}{z}\left[1+\frac{2}{z}\right]^{-1} - \frac{8}{3}\left[1+\frac{z}{3}\right]^{-1}$$

$$f(z) = 1 + \frac{3}{z}\left[1-\left(\frac{2}{z}\right)+\left(\frac{2}{z}\right)^2 - \cdots\right]$$
$$- \frac{8}{3}\left[1-\left(\frac{z}{3}\right)+\left(\frac{z}{3}\right)^2 + \cdots\right] \quad //.$$

⑤. Expand $f(z) = \frac{1}{z^2-3z+2}$ in the region (i) $|z|<1$ (ii) $1<|z|<2$ (iii) $0<|z-1|<2$ (iv) $|z|>2$.

Soln:

Let $f(z) = \frac{1}{z^2-3z+2}$

$$f(z) = \frac{1}{(z-1)(z-2)}$$

By partial fraction method,

$$\frac{1}{(z-1)(z-2)} = \frac{A}{z-1} + \frac{B}{z-2} \longrightarrow ①$$

$$1 = A(z-2) + B(z-1) \longrightarrow ②$$

In ②, Put $z=2$,

$$1 = A(0) + B(2-1)$$

$$\boxed{1 = B}$$

In ②, Put $z=1$,

$$1 = A(1-2) + B(0)$$

$$\boxed{-1 = A}$$

$\therefore$ $$\frac{1}{(z-1)(z-2)} = \frac{A}{z-1} + \frac{B}{z-2}$$

$$\boxed{f(z) = \frac{-1}{z-1} + \frac{1}{z-2}}$$

(i) Given region: $|z|<1$

(ie) $\left|\frac{z}{1}\right|<1$ and $\left|\frac{z}{2}\right|<1$ (ie) $|z|<1$

$$\therefore f(z) = \frac{-1}{z-1} + \frac{1}{z-2}$$

$$= \frac{1}{1-z} - \frac{1}{2-z}$$

$$= (1-z)^{-1} - \frac{1}{2(1-\frac{z}{2})}$$

$$= (1-z)^{-1} - \frac{1}{2}\left(1-\frac{z}{2}\right)^{-1}$$

$$f(z) = \left[1+z+z^2+\cdots\right] - \frac{1}{2}\left[1+\left(\frac{z}{2}\right)+\left(\frac{z}{2}\right)^2+\cdots\right]$$ //.

(ii). Given region: $1 < |z| < 2$.

(ie) $1 < |z|$ and $|z| < 2$.

(ie) $\left|\frac{1}{z}\right| < 1$ and $\left|\frac{z}{2}\right| < 1$

$$\therefore f(z) = \frac{-1}{z-1} + \frac{1}{z-2}$$

$$= \frac{-1}{z(1-\frac{1}{z})} + \frac{1}{-2(1-\frac{z}{2})}$$

$$= \frac{-1}{z}\left[1-\frac{1}{z}\right]^{-1} - \frac{1}{2}\left[1-\frac{z}{2}\right]^{-1}$$

$$f(z) = \frac{-1}{z}\left[1+\left(\frac{1}{z}\right)+\left(\frac{1}{z}\right)^2+\cdots\right]$$

$$- \frac{1}{2}\left[1+\left(\frac{z}{2}\right)+\left(\frac{z}{2}\right)^2+\cdots\right]$$ //.

(iii). Given region: $0 < |z-1| < 2$.

Put $t = z-1$

$\therefore 0 < |t| < 2$.

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$\therefore \quad f(z) = \dfrac{-1}{z-1} + \dfrac{1}{z-2}$

$= \dfrac{-1}{t} + \dfrac{1}{t+1-2}$

$= \dfrac{-1}{t} + \dfrac{1}{t-1}$

$= \dfrac{-1}{t} - \dfrac{1}{1-t}$

$= -t^{-1} - [1-t]^{-1}$

$= -t^{-1} - [1+t+t^2+\cdots]$

$f(z) = -[z-1]^{-1} - [1+(z-1)+(z-1)^2+\cdots]$ //.

(iv). Given region: $|z| > 2$.

(ie) $2 < |z| \Rightarrow \left|\frac{2}{z}\right| < 1$ and $\left|\frac{1}{z}\right| < 1$

$\therefore \quad f(z) = \dfrac{-1}{z-1} + \dfrac{1}{z-2}$

$= \dfrac{-1}{z(1-\frac{1}{z})} + \dfrac{1}{z(1-\frac{2}{z})}$

$= \frac{-1}{z}[1-\frac{1}{z}]^{-1} + \frac{1}{z}[1-\frac{2}{z}]^{-1}$

$f(z) = \frac{-1}{z}\left[1+\left(\frac{1}{z}\right)+\left(\frac{1}{z}\right)^2+\cdots\right]$
$+ \frac{1}{z}\left[1+\left(\frac{2}{z}\right)+\left(\frac{2}{z}\right)^2+\cdots\right]$ //.

6. Expand $f(z) = \dfrac{z}{(z-1)(z-3)}$ as Laurent's series valid in the following regions:

(i) $1 < |z| < 3$

(ii) $0 < |z-1| < 2$

(iii) $|z| > 3$.

Soln:

Let $f(z) = \dfrac{z}{(z-1)(z-3)}$

By partial fraction method,

$$\frac{z}{(z-1)(z-3)} = \frac{A}{z-1} + \frac{B}{z-3} \longrightarrow (1)$$

$$z = A(z-3) + B(z-1) \longrightarrow (2)$$

In (2), Put $z=3$,

$$3 = A(0) + B(3-1)$$

$$3 = 2B$$

$$\boxed{\frac{3}{2} = B}$$

In (2), Put $z=1$,

$$1 = A(1-3) + B(0)$$

$$1 = A(-2)$$

$$\boxed{\frac{-1}{2} = A}$$

$$\therefore \frac{z}{(z-1)(z-3)} = \frac{A}{z-1} + \frac{B}{z-3}$$

$$\therefore \boxed{f(z) = \frac{-1/2}{z-1} + \frac{3/2}{z-3}}$$

(i) Given region: $1<|z|<3$

(ie) $1<|z|$ and $|z|<3$

(ie) $\left|\frac{1}{z}\right|<1$ and $\left|\frac{z}{3}\right|<1$

$$\therefore f(z) = \frac{-1/2}{z-1} + \frac{3/2}{z-3}$$

$$= \frac{-1/2}{z\left(1-\frac{1}{z}\right)} + \frac{\cancel{3}/2}{-\cancel{3}\left(1-\frac{z}{3}\right)}$$

$$= -\frac{1}{2z}\left[1-\frac{1}{z}\right]^{-1} - 2\left[1-\frac{z}{3}\right]^{-1}$$

$$f(z) = \frac{-1}{2z}\left[1+\left(\frac{1}{z}\right)+\left(\frac{1}{z}\right)^2+\cdots\right]$$

$$-2\left[1+\left(\frac{z}{3}\right)+\left(\frac{z}{3}\right)^2+\cdots\right] //$$

(ii) Given region: $0<|z-1|<2$.

Put $t = z-1$.

(ie) $0<|t|<2$.

$|t|<2$

$\left|\frac{t}{2}\right|<1$.

$$\therefore f(z) = \frac{-1/2}{z-1} + \frac{3/2}{z-3}$$

$$= \frac{-1/2}{t} + \frac{3/2}{t+1-3}$$

$$= \frac{-1}{2}t^{-1} + \frac{3/2}{t-2}$$

$$= \frac{-1}{2}t^{-1} + \frac{3/2}{-2\left(1-\frac{t}{2}\right)}$$

$$= \frac{-1}{2} t^{-1} - \frac{3}{4}\left(1-\frac{t}{2}\right)^{-1}$$

$$= \frac{-1}{2} t^{-1} - \frac{3}{4}\left[1+\left(\frac{t}{2}\right)+\left(\frac{t}{2}\right)^2+\cdots\right]$$

$$f(z) = -\frac{1}{2}(z-1)^{-1} - \frac{3}{4}\left[1+\frac{(z-1)}{2}+\frac{(z-1)^2}{4}+\cdots\right] //$$

(iii) Given region: $|z|>3$

(ie) $3<|z|$

$\Rightarrow \left|\frac{3}{z}\right|<1$ and $\left|\frac{1}{z}\right|<1$

$$\therefore f(z) = \frac{-1/2}{z-1} + \frac{3/2}{z-3}$$

$$= \frac{-1/2}{z\left(1-\frac{1}{z}\right)} + \frac{3/2}{z\left(1-\frac{3}{z}\right)}$$

$$= \frac{-1}{2z}\left[1-\frac{1}{z}\right]^{-1} + \frac{3}{2z}\left[1-\frac{3}{z}\right]^{-1}$$

$$f(z) = \frac{-1}{2z}\left[1+\left(\frac{1}{z}\right)+\left(\frac{1}{z}\right)^2+\cdots\right]$$

$$+ \frac{3}{2z}\left[1+\left(\frac{3}{z}\right)+\left(\frac{3}{z}\right)^2+\cdots\right] //$$

⑦ Obtain Taylor's (or) Laurent's series for $f(z) = \frac{1}{(z+2)(z^2+1)}$

in (i) $|z| < 1$

(ii) $1 < |z| < 2$

(iii) $|z| > 2$.

Soln:

Let $f(z) = \frac{1}{(z+2)(z^2+1)}$

By partial fraction method,

$$\frac{1}{(z+2)(z^2+1)} = \frac{A}{z+2} + \frac{Bz+C}{z^2+1} \longrightarrow (1)$$

$$1 = A(z^2+1) + (Bz+C)(z+2) \longrightarrow (2)$$

In (2), Put $z = -2$,

$$1 = A((-2)^2+1) + (Bz+C)(0)$$

$$1 = A(4+1)$$

$$1 = 5A$$

$$\boxed{1/5 = A}$$

In (2), Put $z = 0$,

$$1 = A(0+1) + [B(0)+C](0+2)$$

$$1 = A + 2C$$

$$1 = \frac{1}{5} + 2C$$

$$1 - \frac{1}{5} = 2C$$

$$\frac{4}{5} = 2C$$

$$\boxed{\frac{2}{5} = C}$$

In ②, Equating coeff of z^2 on both sides,

$$0 = A + B$$

$$\Rightarrow A = -B \Rightarrow B = -A.$$

$$\Rightarrow \boxed{\frac{-1}{5} = B}$$

$$\therefore \frac{1}{(z+2)(z^2+1)} = \frac{A}{z+2} + \frac{Bz+C}{z^2+1}$$

$$= \frac{1/5}{z+2} + \frac{-\frac{1}{5}z + \frac{2}{5}}{z^2+1}$$

$$\boxed{f(z) = \frac{1}{5}\left(\frac{1}{z+2} + \frac{2-z}{z^2+1}\right]}$$

(i) Given region: $|z|<1$.

(i.e) $\left|\frac{z}{1}\right|<1 \Rightarrow \left|\frac{z}{2}\right|<1$

$$\therefore f(z) = \frac{1}{5}\left[\frac{1}{z+2}\right] + \frac{1}{5}\left(\frac{2-z}{1+z^2}\right]$$

$$= \frac{1}{5\times 2}\left[\frac{1}{1+\frac{z}{2}}\right] + \frac{1}{5}(2-z)(1+z^2)^{-1}$$

$$= \frac{1}{10}\left[1+\frac{z}{2}\right]^{-1} + \frac{1}{5}(2-z)(1+z^2)^{-1}$$

$$f(z) = \frac{1}{10}\left[1-\left(\frac{z}{2}\right)+\left(\frac{z}{2}\right)^2 - \cdots\right] + \frac{1}{5}(2-z)\left[1-z^2+z^4 - \cdots\right]$$

(ii) Given region: $1<|z|<2$.

(i.e) $1<|z|$ and $|z|<2$.

(i.e) $\left|\frac{1}{z}\right|<1$ and $\left|\frac{z}{2}\right|<1$

$$\Rightarrow \left|\frac{1}{z^2}\right|<1$$

$$\therefore\ f(z) = \frac{1}{5}\left(\frac{1}{z+2}\right) + \frac{1}{5}\left(\frac{2-z}{z^2+1}\right)$$

$$= \frac{1}{5\times 2}\left(\frac{1}{1+\frac{z}{2}}\right) + \frac{1}{5\times z^2}\frac{(2-z)}{} \frac{1}{\left(1+\frac{1}{z^2}\right)}$$

$$= \frac{1}{10}\left[1+\frac{z}{2}\right]^{-1} + \frac{2-z}{5z^2}\left[1+\frac{1}{z^2}\right]^{-1}$$

$$f(z) = \frac{1}{10}\left[1-\left(\frac{z}{2}\right)+\left(\frac{z}{2}\right)^2-\cdots\right]$$

$$+ \frac{2-z}{5z^2}\left[1-\left(\frac{1}{z^2}\right)+\left(\frac{1}{z^2}\right)^2-\cdots\right] \quad //$$

(iii) <u>Given regim:</u> $|z| > 2$

(i.e) $2 < |z|$

(i.e) $\left|\frac{2}{z}\right| < 1 \Rightarrow \left|\frac{1}{z}\right| < 1$

$\Rightarrow \left|\frac{1}{z^2}\right| < 1$

$$\therefore\ f(z) = \frac{1}{5}\left(\frac{1}{z+2}\right) + \frac{1}{5}\left(\frac{2-z}{z^2+1}\right)$$

$$= \frac{1}{5z}\left(\frac{1}{1+\frac{2}{z}}\right) + \frac{1}{5}\cdot\frac{2-z}{z^2}\ \frac{1}{\left(1+\frac{1}{z^2}\right)}$$

$$= \frac{1}{5z}\left[1+\frac{2}{z}\right]^{-1} + \frac{2-z}{5z^2}\left[1+\frac{1}{z^2}\right]^{-1}$$

$$f(z) = \frac{1}{5z}\left[1-\left(\frac{2}{z}\right)+\left(\frac{2}{z}\right)^2-\cdots\right]$$

$$+ \frac{2-z}{5z^2}\left[1-\left(\frac{1}{z^2}\right)+\left(\frac{1}{z^2}\right)^2-\cdots\right] \quad //$$

Residues

* Zeros of analytic function:

Let $f(z)$ be analytic at $z = z_0$. If $f(z_0)=0$, then z_0 is called a zero of $f(z)$

* A point at which a complex function $f(z)$ is analytic is called a regular point (ordinary point) of $f(z)$

* Singular point:

A point $z=a$ at which a function $f(z)$ fails to be analytic is called a singular pt (or) singularity of $f(z)$

* Types of singularities:

(i) Isolated singularity:

If the singular point of $f(z)$ is such that there is no other singular point in its neighbourhood, we call such a singularity as an isolated singularity.

(ii) Poles:

An isolated singular point 'a' of $f(z)$ is said to be a pole of order m if the principal part of the Laurent's series of $f(z)$ about 'a' contains 'm' terms, ($m \to$ finite)

Laurent's series: $f(z) = \underbrace{\sum_{n=0}^{\infty} a_n (z-a)^n}_{\text{regular part}} + \underbrace{\sum_{n=1}^{\infty} b_n (z-a)^{-n}}_{\text{Principal part}}$

(iii) Essential singularity:

If the principal part of $f(z)$ in its Laurent's series does not terminate (ie) it possesses infinite no. of terms, then $z=a$ is called an essential singular point of $f(z)$.

(iv) Removable singularity:

The singular point $z=a$ is called a removable singularity of $f(z)$ if $\lim_{z \to a} f(z)$ exists.

(i.e) No principal part in Laurent's series)

* Entire function (Integral function):

A function $f(z)$ which is analytic everywhere in the finite plane (except at ∞) is called an entire function.

Eg: $e^z, \sin z, \cos z \rightarrow$ integral functions.

* Meromorphic function:

A function $f(z)$ which is analytic everywhere in the finite plane except at a finite no. of poles is called a meromorphic function.

Eg: $f(z) = \frac{z}{(z-1)(z+1)^2}$ is a meromorphic function.

(B/c Poles are $z = 1, -1$)

* Residues:

If $z=a$ is an isolated singular point of $f(z)$, then Laurent's series about $z=a$ is

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-a)^n}.$$

Here, The coefficient b_1 of $\frac{1}{z-a} \rightarrow$ Residue of $f(z)$ at $z=a$.

* Methods of finding Residues.

① If $z = a$ is a pole of order 'm', then

$$\boxed{R(a) = \frac{1}{(m-1)!} \lim_{z \to a} \frac{d^{m-1}}{dz^{m-1}} \left[(z-a)^m f(z)\right].}$$

For m=1,

$$R(a) = \lim_{z \to a} (z-a) f(z) \rightarrow \text{Simple pole}$$

For m=2,

$$R(a) = \lim_{z \to a} \frac{d}{dz} \left[(z-a)^2 f(z)\right] \rightarrow \text{Pole of order 2.}$$

② Residue of $f(z)$ at $z = a$ is $\frac{P(a)}{Q'(a)}$ if $f(z) = \frac{P(z)}{Q(z)}$.

* Cauchy's Residue Theorem:

If $f(z)$ be analytic at all points inside and on a simple closed curve c, except for a finite no. of isolated singularities $z_1, z_2, \ldots, z_n$ inside c, then

$$\int_c f(z)\, dz = 2\pi i \left[\text{Sum of the residues of } f(z)\right].$$

Pbs:

①. Classify the singularities of $f(z) = \frac{z+2}{(z-2)(z+1)^2}$ and find the residues of $f(z)$ at each singularity.

Soln:

The poles are given by $(z-2)(z+1)^2 = 0$

$\therefore$ $z = 2$ is a simple pole.

$z = -1$ is a pole of order 2.

i) Residue of $f(z)$ at $z=a$ $\}$ $= \underset{z\to a}{Lt} (z-a) f(z)$

$\therefore \quad R(2) = \underset{z\to 2}{Lt} (z-2) f(z)$

$$= \underset{z\to 2}{Lt} \cancel{(z-2)} \frac{z+2}{\cancel{(z-2)} (z+1)^2}$$

$$= \underset{z\to 2}{Lt} \frac{z+2}{(z+1)^2}$$

$$= \frac{2+2}{(2+1)^2} = \frac{4}{9} \,//$$

(ii) $\cancel{R(-1) = \underset{z\to -1}{Lt} (z+1) \, z+}$

$$\left.\begin{array}{l} R(a) \\ \text{of order 2} \end{array}\right\} = \underset{z\to a}{Lt} \frac{d}{dz} \left[(z-a)^2 f(z) \right]$$

$$\left.\begin{array}{l} R(-1) \\ \text{of order 2} \end{array}\right\} = \underset{z\to -1}{Lt} \frac{d}{dz} \left[\cancel{(z+1)^2} \frac{z+2}{(z-2)\cancel{(z+1)^2}} \right]$$

$$= \underset{z\to -1}{Lt} \frac{d}{dz} \left(\frac{z+2}{z-2} \right)$$

$$= \underset{z\to -1}{Lt} \frac{(z-2)(1) - (z+2)(1)}{(z-2)^2}$$

$$= \underset{z\to -1}{Lt} \frac{-4}{(z-2)^2}$$

$$= \frac{-4}{(-1-2)^2} = \frac{-4}{(-3)^2} = \frac{-4}{9} \,//$$

② . Determine the poles of $f(z) = \dfrac{z^2}{(z-1)^2 (z+2)}$ and the residues at each pole.

Soln:

Let $f(z) = \dfrac{z^2}{(z-1)^2 (z+2)}$

The poles are given by $(z-1)^2 (z+2) = 0$.

$\therefore$ $z=-2$ is a simple pole.

$z = 1$ is a pole of order 2.

(i) $R(a) = \lim\limits_{z\to a} (z-a) f(z)$

$$R(-2) = \lim_{z\to -2} \cancel{(z+2)} \frac{z^2}{(z-1)^2 \cancel{(z+2)}}$$

$$= \lim_{z\to -2} \frac{z^2}{(z-1)^2}$$

$$= \frac{(-2)^2}{(-2-1)^2} = \frac{4}{9} //$$

(ii) $\left.\begin{array}{l} R(a) \\ \text{of order } 2 \end{array}\right\} = \lim\limits_{z\to a} \dfrac{d}{dz} \left[(z-a)^2 f(z) \right]$

$$\left.\begin{array}{l} R(1) \\ \text{of order } 2 \end{array}\right\} = \lim_{z\to 1} \frac{d}{dz} \left[\cancel{(z-1)^2} \frac{z^2}{\cancel{(z-1)^2} (z+2)} \right]$$

$$= \lim_{z\to 1} \frac{d}{dz} \left(\frac{z^2}{z+2} \right)$$

$$= \lim_{z\to 1} \frac{(z+2)\, 2z - z^2 (1)}{(z+2)^2}$$

$$= \frac{(1+2)\, 2(1) - 1^2}{(1+2)^2} = \frac{6-1}{3^2}$$

$$= \frac{5}{9} //$$

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③. Evaluate $\int \frac{z\,dz}{(z-1)^2(z+1)}$ where c is (i) $|z| = \frac{1}{2}$

(ii) $|z| = 2$,

<u>Soln:</u>

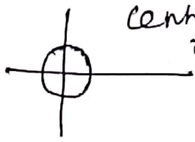
Let $f(z) = \frac{z}{(z-1)^2(z+1)}$

The poles are given by $(z-1)^2(z+1) = 0$

$\therefore$ $z = -1$ is a simple pole.

$z = 1$ is a pole of order 2.

(i) <u>Given</u>: $c : |z| = \frac{1}{2}$.



centre (0,0)
$r = 1/2$.

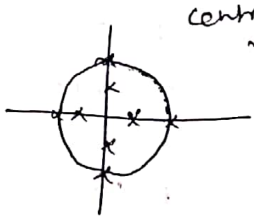
Here, both the poles lies outside c.

$\therefore$ By cauchy's integral theorem,

$$\int_c f(z)\,dz = 0$$

(ie) $$\int_c \frac{z}{(z-1)^2(z+1)}\,dz = 0.$$

(ii) <u>Given</u>: $c : |z| = 2$.



centre = (0,0)
$r = 2$.

Here, both the poles lies inside c.

$$R(a) = \underset{z \to a}{Lt}\,(z-a)\,f(z)$$

$$R(-1) = \underset{z \to -1}{Lt}\,\cancel{(z+1)}\,\frac{z}{(z-1)^2\cancel{(z+1)}}$$

$$= \underset{z \to -1}{Lt}\,\frac{z}{(z-1)^2} = \frac{-1}{(-1-1)^2} = \frac{-1}{4}$$

$$\boxed{R(-1) = \frac{-1}{4}}$$

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Now

$$\left.\begin{array}{l} R(a) \\ \text{of order } 2 \end{array}\right\} = \underset{z \to a}{Lt} \frac{d}{dz} (z-a)^2 f(z)$$

$$\left.\begin{array}{l} R(1) \\ \text{of order } 2 \end{array}\right\} = \underset{z \to 1}{Lt} \frac{d}{dz} \left[\cancel{(z-1)^2} \frac{z}{\cancel{(z-1)^2}(z+1)} \right]$$

$$= \underset{z \to 1}{Lt} \frac{d}{dz} \left(\frac{z}{z+1} \right)$$

$$= \underset{z \to 1}{Lt} \frac{(z+1)\cdot 1 - z(1)}{(z+1)^2}$$

$$= \underset{z \to 1}{Lt} \frac{1+\cancel{1}-\cancel{1}}{(1+1)^2} = \frac{1}{2^2}$$

$$\boxed{R(1) = \frac{1}{4}}$$

∴ By Cauchy's Residue Theorem,

$$\int_c f(z)\,dz = 2\pi i\,[\text{Sum of all the residues}]$$

$$= 2\pi i \left[\frac{-1}{4} + \frac{1}{4} \right]$$

$$= 0 \; //.$$

4. Evaluate $\oint_c \frac{\sin \pi z^2 + \cos \pi z^2}{z+z^2}\,dz$, where c is a circle $|z| = 2$.

Soln:

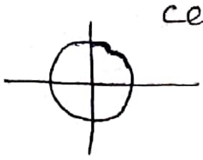
Let $f(z) = \frac{\sin \pi z^2 + \cos \pi z^2}{z+z^2}$.

The poles are given by $z+z^2 = 0$. (i.e) $z(z+1) = 0$

∴ $z = 0$ is a simple pole.

$z = -1$ is a simple pole.

Given: $c: |z| = 2$.



centre $(0,0)$

$r = 2$.

Here, $z = 0$ lies inside c.

& $z = -1$ lies inside c.

(i) $\therefore$ $R(a) = \underset{z \to a}{Lt} (z-a) f(z)$

$$R(0) = \underset{z \to 0}{Lt} (\cancel{z}-0) \frac{\sin\pi z^2 + \cos\pi z^2}{\cancel{z}(1+z)}$$

$$= \underset{z \to 0}{Lt} \left[\frac{\sin\pi z^2 + \cos\pi z^2}{1+z}\right]$$

$$= \frac{\sin 0 + \cos 0}{1+0} = \frac{0+1}{1}$$

$$\boxed{R(0) = 1}$$

(ii) $R(0) = \underset{z \to a}{Lt} (z-a) f(z)$

$$R(-1) = \underset{z \to -1}{Lt} (\cancel{z+1}) \frac{\sin\pi z^2 + \cos\pi z^2}{z(\cancel{1+z})}$$

$$= \underset{z \to -1}{Lt} \left[\frac{\sin\pi z^2 + \cos\pi z^2}{z}\right]$$

$$= \frac{\sin\pi(-1)^2 + \cos\pi(-1)^2}{-1}$$

$$= \frac{\sin\pi + \cos\pi}{-1} = \frac{0-1}{-1}$$

$$\boxed{R(-1) = 1}$$

$\therefore$ By Cauchy's Residue Theorem,

$$\int_C f(z)\,dz = 2\pi i \,[\text{Sum of all the residues}]$$

$$= 2\pi i \,[R(0) + R(-1)]$$

$$= 2\pi i \,[1+1]$$

$$= 4\pi i$$ //

(5). Evaluate $\int_c \frac{4z+1}{z(z-1)(z-3)}\,dz$, where c is the circle $|z|=2$.

Soln:

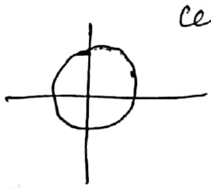
Let $f(z) = \frac{4z+1}{z(z-1)(z-3)}$

$\therefore$ The poles are given by $z(z-1)(z-3)=0$

Here, $z=0,1,3$ are the simple poles.

Given: $c: |z|=2$.

centre $(0,0)$
$r=2$.



Here, $z=0$ & $z=1$ } both lies inside c.

$z=3$ lies outside c.

$\therefore R(a) = \underset{z\to a}{Lt}\,(z-a)\,f(z)$

$$R(0) = \underset{z\to 0}{Lt}\,(\cancel{z}-0)\,\frac{4z+1}{\cancel{z}(z-1)(z-3)}$$

$$= \underset{z\to 0}{Lt}\left[\frac{4z+1}{(z-1)(z-3)}\right]$$

$$= \frac{4(0)+1}{(0-1)(0-3)}$$

$$\boxed{R(0) = \frac{1}{3}}$$

Now, $$R(1) = \underset{z\to 1}{Lt}\,(\cancel{z-1})\,\frac{4z+1}{z(\cancel{z-1})(z-3)}$$

$$= \underset{z\to 1}{Lt}\,\frac{4z+1}{z(z-3)}$$

$$= \frac{4(1)+1}{1(1-3)} = \frac{5}{-2}.$$

$$\boxed{R(1) = -\frac{5}{2}}$$

$\therefore$ By Cauchy's Residue Theorem,

$$\int_C f(z)\,dz = 2\pi i\,[\text{Sum of all the residues}]$$

$$= 2\pi i\,[R(0) + R(1)]$$

$$= 2\pi i\left[\frac{1}{3} - \frac{5}{2}\right]$$

$$= 2\pi i\left[\frac{2-15}{\cancel{6}_3}\right]$$

$$= -\frac{13}{3}\pi i \quad //.$$

(HW) (6). Evaluate $\oint_C \dfrac{e^{2z}}{\cos\pi z}\,dz$, where c is a circle $|z|=1$.

Soln:

Let $f(z) = \dfrac{e^{2z}}{\cos\pi z}$

centre $(0,0)$
$r=1$.

The poles are given by $\cos\pi z = 0$

(ie) $z = \pm\frac{1}{2}, \pm\frac{3}{2}, \ldots$

Here, $z = \pm\frac{1}{2}$ lies inside the circle $|z|=1$

$\therefore$ $R(a) = \lim\limits_{z\to a}(z-a)\,f(z)$

$$R\left(\tfrac{1}{2}\right) = \lim_{z\to\frac{1}{2}}\left(z-\tfrac{1}{2}\right)\frac{e^{2z}}{\cos\pi z} = \frac{0}{0} \text{ form.}$$

$\therefore$ Using L'hospital rule,

$$R\left(\tfrac{1}{2}\right) = \lim_{z\to\frac{1}{2}} \frac{\left(z-\frac{1}{2}\right)2e^{2z} + e^{2z}}{-\pi\sin\pi z} = \frac{-e}{\pi}$$

Again, $$R\left(-\tfrac{1}{2}\right) = \lim_{z\to -\frac{1}{2}} \frac{\left(z+\frac{1}{2}\right)2e^{2z} + e^{2z}}{-\pi\sin\pi z} = \frac{e^{-1}}{\pi}$$

$\therefore$ By Cauchy's residue theorem,

$$\int f(z)\,dz = 2\pi i\,[\text{Sum of all the residues}]$$

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$$= 2\pi i\left[R\left(\tfrac{1}{2}\right) + R\left(-\tfrac{1}{2}\right)\right]$$

$$= 2\pi i\left[\frac{-e}{\pi} + \frac{e^{-1}}{\pi}\right]$$

$$= 2\cancel{\pi} i\left[\frac{-e+e^{-1}}{\cancel{\pi}}\right]$$

$$= 2i\left[-e+e^{-1}\right] \quad //.$$

(7). Find the residues at the poles of $f(z) = \frac{z}{z^2+1}$.

Soln:

The poles are given by $z^2+1=0$

$$\Rightarrow z^2 = -1$$

$$\Rightarrow z = \pm i. \quad \text{(simple pole)}$$

$$\therefore \quad R(a) = \underset{z\to a}{Lt}\ (z-a)\, f(z)$$

$$R(i) = \underset{z\to i}{Lt}\ \cancel{(z-i)}\ \frac{z}{\cancel{(z-i)}(z+i)}$$

$$= \frac{i}{i+i} = \frac{i}{2i}$$

$$\boxed{R(i) = \frac{1}{2}} \quad //$$

Again,

$$R(-i) = \underset{z\to -i}{Lt}\ \cancel{(z+i)}\ \frac{z}{(z-i)\cancel{(z+i)}}$$

$$= \frac{-i}{-i-i} = \frac{-i}{-2i}$$

$$\boxed{R(-i) = \frac{1}{2}} \quad //$$

(HW) (8) Find the poles of $\cot z$, $\tan z$, $\operatorname{cosec} z$.

Soln:

(i) $f(z) = \cot z = \dfrac{\cos z}{\sin z}$.

$\therefore$ The poles are given by $\sin z = 0$.

$\Rightarrow z = n\pi$

$\Rightarrow z = 0, \pm\pi, \pm 2\pi, \ldots$ //.

(ii) $f(z) = \tan z = \dfrac{\sin z}{\cos z}$.

$\therefore$ The poles are given by $\cos z = 0$

$\Rightarrow z = (2n+1)\dfrac{\pi}{2} = n\pi + \dfrac{\pi}{2}$, $n = 0, \pm 1, \pm 2, \ldots$ //.

(iii) Let $f(z) = \operatorname{cosec} z = \dfrac{1}{\sin z}$.

$\therefore$ The poles are given by $\sin z = 0$

$\Rightarrow z = n\pi$

$\Rightarrow z = 0, \pm\pi, \pm 2\pi, \ldots$ //.

(HW) (9) Evaluate $\oint_C \dfrac{\cos \pi z^2}{(z-1)(z-2)}\, dz$ where $|z| = 3$.

Soln:

$R(1) = 1$

$R(2) = 1$

$\therefore$ By Cauchy's residue theorem $\int_C f(z)\,dz = 2\pi i(1+1)$

$= 4\pi i$ //.

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Contour Integration

Type I: $\int_0^{2\pi} F(\sin\theta, \cos\theta)\, d\theta$ (Real definite integrals)

Put $z = e^{i\theta}$ | $\frac{1}{z} = e^{-i\theta}$.

$\Rightarrow z = \cos\theta + i\sin\theta$ | $\frac{1}{z} = \cos\theta - i\sin\theta$.

$\therefore z + \frac{1}{z} = 2\cos\theta$ | $z - \frac{1}{z} = 2i\sin\theta$.

$\therefore \cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right)$ | $\frac{1}{2i}\left(z - \frac{1}{z}\right) = \sin\theta$

$\Rightarrow \boxed{\cos\theta = \frac{z^2+1}{2z}}$ $\boxed{\frac{z^2-1}{2iz} = \sin\theta}$

Since $F(\sin\theta, \cos\theta)$ is a rational function in $\sin\theta$ & $\cos\theta$, we get a rational function of z, (say $f(z)$).

$\therefore \frac{dz}{d\theta} = ie^{i\theta} \Rightarrow \frac{dz}{d\theta} = iz$.

(ie) $\boxed{d\theta = \frac{dz}{iz}}$

As θ varies from 0 to 2π, z moves on the unit circle $|z| = 1$ in the anticlockwise.

So, the contour c is the unit circle $|z| = 1$.

Pbs:

① Evaluate: $\int_0^{2\pi} \frac{d\theta}{13+5\sin\theta}$ by using contour integration.

Soln:

Let $I = \int_0^{2\pi} \frac{d\theta}{13+5\sin\theta}$.

Let $\sin\theta = \frac{z^2-1}{2iz}$ & $d\theta = \frac{dz}{iz}$.

$\cos\theta = \frac{z^2+1}{2z}$

$$\therefore \quad I = \int_0^{2\pi} \frac{\left(\frac{dz}{iz}\right)}{13+5\left(\frac{z^2-1}{2iz}\right)}$$

$$= \int_0^{2\pi} \frac{dz}{\cancel{iz}\left[\frac{26iz+5z^2-5}{2\cancel{iz}}\right]}$$

$$= 2\int_0^{2\pi} \frac{1}{5z^2+26iz-5}\, dz$$

$$\boxed{I = 2\int_0^{2\pi} f(z)\, dz} \longrightarrow (P)$$

where $f(z) = \frac{1}{5z^2+26iz-5}$ (Now, use Cauchy residue Theorem)

Here, the poles are given by $5z^2+26iz-5=0$.

$$\therefore z = \frac{-26i \pm \sqrt{(26i)^2 - 4(5)(-5)}}{2(5)}$$

$$= \frac{-26i \pm \sqrt{-676+100}}{10}$$

$$= \frac{-26i \pm \sqrt{-576}}{10}$$

$$= \frac{-26i \pm 24i}{10} = \frac{-13i \pm 12i}{5}$$

$$= \frac{-13i+12i}{5} \text{ and } \frac{-13i-12i}{5}$$

$$z = \frac{-i}{5} \text{ and } -5i.$$

are the Simple poles.

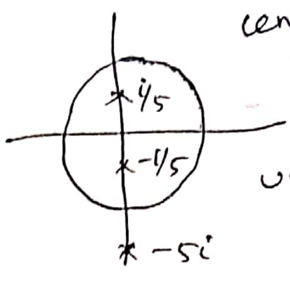
(ii) $\quad 5z^2+26iz-5 = 5\left(z+\frac{i}{5}\right)(z+5i)$

$\Big($ since $|z|<1$ $\qquad |z|<1$

$\left|-\frac{i}{5}\right|<1$ & $|-5i|<1$

But

$\Rightarrow \frac{1}{5} < 1$ $\qquad 5>1.$ $\Big)$



centre (0,0)

r = 1

unit circle.

$\therefore$ $z = \frac{-i}{5}$ is a simple pole which lies inside c.

$z = -5i$ is a simple pole which lies outside c.

$$\therefore R(a) = \underset{z \to a}{Lt} (z-a) f(z)$$

$$R\left(\frac{-i}{5}\right) = \underset{z \to \frac{-i}{5}}{Lt} \left(z+\frac{i}{5}\right) \frac{1}{5z^2+26iz-5}$$

$$= \underset{z \to \frac{-i}{5}}{Lt} \cancel{\left(z+\frac{i}{5}\right)} \frac{1}{5\cancel{\left(z+\frac{i}{5}\right)}(z+5i)}$$

$$= \underset{z \to \frac{-i}{5}}{Lt} \frac{1}{5(z+5i)}$$

$$= \frac{1}{5\left(-\frac{i}{5}+5i\right)} = \frac{1}{\cancel{5}\left(\frac{-i+25i}{\cancel{5}}\right)}$$

$$R\left(\frac{-i}{5}\right) = \frac{1}{24i}$$

$\therefore$ By Cauchy's Residue Theorem,

$$\int_c f(z)\,dz = 2\pi i\,[\text{Sum of all the residues}]$$

$$= 2\pi i \left[R\left(\frac{-i}{5}\right)\right]$$

$$= \cancel{2}\pi \cancel{i} \left[\frac{1}{\cancel{24}_{12} \cancel{i}}\right]$$

$$= \frac{\pi}{12}.$$

$\therefore (1) \Rightarrow$

$$I = 2\int_c f(z)\,dz$$

$$= 2\left(\frac{\pi}{12}\right)$$

$$\boxed{I = \frac{\pi}{6}}$$

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② Evaluate $\int_0^{2\pi} \frac{d\theta}{5+3\cos\theta}$ by method of residues. (Contour integration)

Soln:

Let $I = \int_0^{2\pi} \frac{d\theta}{5+3\cos\theta}$ with $|z|=1$

Let $\sin\theta = \frac{z^2-1}{2iz}$, $d\theta = \frac{dz}{iz}$.

$\cos\theta = \frac{z^2+1}{2z}$

$$\therefore I = \int_0^{2\pi} \frac{\left(\frac{dz}{iz}\right)}{5+3\left(\frac{z^2+1}{2z}\right)}$$

$$= \int_0^{2\pi} \frac{dz}{i\cancel{z}\left(\frac{10z+3z^2+3}{2\cancel{z}}\right)}$$

$$= \frac{2}{i}\int_0^{2\pi} \frac{dz}{3z^2+10z+3}$$

$$\boxed{I = \frac{2}{i}\int_C f(z)\,dz}$$

Where $f(z) = \frac{1}{3z^2+10z+3}$. (Use Cauchy's residue Theorem)

$\therefore$ The poles are given by $3z^2+10z+3=0$.

$$\therefore z = \frac{-10 \pm \sqrt{(10)^2 - 4(3)(3)}}{2(3)}$$

$$= \frac{-10 \pm \sqrt{100 - 36}}{6}$$

$$= \frac{-10 \pm \sqrt{64}}{6}$$

$$= \frac{-10 \pm 8}{6} \quad \# \quad \cancel{-10 \pm 8}$$

$$= \frac{-5 \pm 4}{3} = \frac{-5+4}{3} \text{ and } \frac{-5-4}{3}$$

$$z = \frac{-1}{3} \text{ and } -3$$

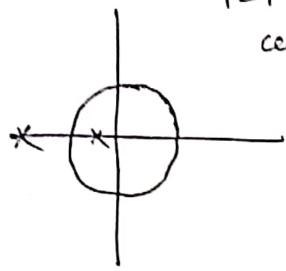
are the simple poles.

$|z| = 1$

centre $= (0,0)$

$r = 1$.

Here, $z = \frac{-1}{3}$ is a simple pole lies inside c.



$z = -3$ is a simple pole lies outside c.

$$\therefore \quad f(z) = 3\left(z + \frac{1}{3}\right)(z+3)$$

$$R(a) = \lim_{z \to a} (z-a) f(z)$$

$$R\left(-\frac{1}{3}\right) = \lim_{z \to -\frac{1}{3}} \cancel{\left(z + \frac{1}{3}\right)} \frac{1}{3\cancel{\left(z+\frac{1}{3}\right)}(z+3)}$$

$$= \lim_{z \to -\frac{1}{3}} \frac{1}{3(z+3)}$$

$$= \frac{1}{3\left(-\frac{1}{3}+3\right)} = \frac{1}{3\left(\frac{-1+9}{3}\right)}$$

$$= \frac{1}{\cancel{3}} \times \frac{\cancel{3}}{8}$$

$$R\left(-\frac{1}{3}\right) = \frac{1}{8}$$

$\therefore$ By Cauchy's residue Theorem,

$$\int_C f(z)\,dz = 2\pi i\,[\text{Sum of all the residues}]$$

$$= 2\pi i\left[R\left(-\frac{1}{3}\right)\right]$$

$$= \cancel{2}\pi i \cdot \left(\frac{1}{\cancel{8}_4}\right)$$

$$= \frac{\pi i}{4}.$$

$\therefore (1) \Rightarrow$ $$I = \frac{2}{i}\int_C f(z)\,dz$$

$$= \cancel{\frac{2\pi i}{4}} \quad \frac{\cancel{2}}{\cancel{i}} \times \frac{\pi \cancel{i}}{\cancel{4}_2}$$

$$\boxed{I = \frac{\pi}{2}}$$

(3). Evaluate $\int_0^{2\pi} \frac{d\theta}{2+\cos\theta}$ by Contour integration.

Soln:

Let $I = \int_0^{2\pi} \frac{d\theta}{2+\cos\theta}$ with $|z|=1$.

Let $\sin\theta = \frac{z^2-1}{2iz}$ & $d\theta = \frac{dz}{iz}$

$\cos\theta = \frac{z^2+1}{2z}$

$$\therefore I = \int_0^{2\pi} \frac{\left(\frac{dz}{iz}\right)}{2+\left(\frac{z^2+1}{2z}\right)}$$

$$= \int_0^{2\pi} \frac{dz}{i\cancel{z}\frac{(\cancel{4}z+z^2+1)}{2\cancel{z}}}$$

$$= \frac{2}{i}\int_0^{2\pi} \frac{1}{z^2+4z+1}\,dz$$

$$\boxed{I = \frac{2}{i}\int_C f(z)\,dz} \longrightarrow (1)$$

where $f(z) = \frac{1}{z^2+4z+1}$ (Use Cauchy's residue Theorem)

$\therefore$ The poles are given by $z^2+4z+1=0$

$$z = \frac{-4 \pm \sqrt{4^2-4(1)(1)}}{2(1)}$$

$$= \frac{-4 \pm \sqrt{16-4}}{2}$$

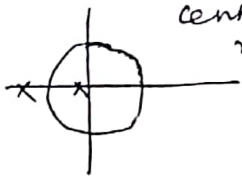
$$= \frac{-4 \pm \sqrt{12}}{2} = \frac{-4 \pm \sqrt{3 \times 4}}{2}$$

$$= \frac{-4 \pm 2\sqrt{3}}{2}$$

$z = -2 \pm \sqrt{3}$. are the simple poles.

(ie) $f(z) = [z-(-2+\sqrt{3})][z-(-2-\sqrt{3})]$.

$|z|=1$
centre $(0,0)$
$r=1$



Here, $z = -2+\sqrt{3}$ lies inside c.

$z = -2-\sqrt{3}$ lies outside c.

$$\therefore \quad R(a) = \underset{z \to a}{Lt} (z-a) f(z)$$

$$R(-2+\sqrt{3}) = \underset{z \to -2+\sqrt{3}}{Lt} [z-(-2+\sqrt{3})] \frac{1}{z^2+4z+1}$$

$$= \underset{z \to -2+\sqrt{3}}{Lt} \cancel{[z-(-2+\sqrt{3})]} \frac{1}{\cancel{[z-(-2+\sqrt{3})]}[z-(-2-\sqrt{3})]}$$

$$= \underset{z \to -2+\sqrt{3}}{Lt} \frac{1}{z+2+\sqrt{3}}$$

$$= \frac{1}{-\cancel{2}+\sqrt{3}+\cancel{2}+\sqrt{3}}$$

$$= \frac{1}{2\sqrt{3}}$$

$\therefore$ By Cauchy's Residue Theorem,

$$\int_c f(z)\,dz = 2\pi i \,[\text{Sum of all the residues}]$$

$$= 2\pi i \left[R(-2+\sqrt{3}) \right]$$

$$= \cancel{2}\pi i \times \frac{1}{\cancel{2}\sqrt{3}}$$

$$= \frac{\pi i}{\sqrt{3}}.$$

$\therefore$ (1) $\Rightarrow$ $$I = \frac{2}{i} \int_C f(z)\,dz$$

$$= \frac{2}{\cancel{i}} \left(\frac{\pi \cancel{i}}{\sqrt{3}} \right)$$

$$\boxed{I = \frac{2\pi}{\sqrt{3}}}.$$

HW

(4). Evaluate $I = \int_0^{2\pi} \frac{\cos 2\theta}{5+4\cos\theta}\,d\theta$, using Contour integration.

Sol^n:

Let $I = \int_0^{2\pi} \frac{\cos 2\theta}{5+4\cos\theta}\,d\theta$. with $|z|=1$.

Let $\boxed{\cos\theta = \frac{z^2+1}{2z}}$, $\boxed{d\theta = \frac{dz}{iz}}$

& ~~$z = e^{i\theta}$~~ $z = e^{i\theta} \Rightarrow z^2 = e^{i2\theta} = \cos 2\theta + i\sin 2\theta$

$\therefore$ $\cos 2\theta = \frac{z^2 + \frac{1}{z^2}}{2}$

OR

$\cos 2\theta = R.P$ of z^2.

$$\boxed{\cos 2\theta = \frac{z^4+1}{2z^2}}$$

$$\therefore \quad I = \int_0^{2\pi} \frac{\cos 2\theta}{5+4\cos\theta}\, d\theta .$$

$$= R.P \int_0^{2\pi} \frac{z^2}{5+\cancel{4}^{2}\left(\frac{z^2+1}{\cancel{2}z}\right)} \left(\frac{dz}{iz}\right)$$

$$= R.P \int_0^{2\pi} \frac{z^2}{i\cancel{z}\left(\frac{5z+2z^2+2}{\cancel{z}}\right)}\, dz$$

$$= R.P \; \frac{1}{i} \int_0^{2\pi} \frac{z^2}{2z^2+5z+2}\, dz$$

$$\boxed{I = R.P \; \frac{1}{i} \int_C f(z)\, dz} \longrightarrow (1)$$

(use Cauchy's residue Theorem)

where $\quad f(z) = \dfrac{z^2}{2z^2+5z+2}$

$\therefore$ The poles are given by $2z^2+5z+2=0$.

$$z = \frac{-5 \pm \sqrt{5^2-4(2)(2)}}{2(2)}$$

$$= \frac{-5 \pm \sqrt{25-16}}{4}$$

$$= \frac{-5 \pm \sqrt{9}}{4} = \frac{-5 \pm 3}{\cancel{4}}$$

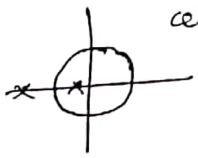
$$= \frac{-5+3}{4} \text{ and } \frac{-5-3}{4}$$

$$= \frac{-2}{4} \text{ and } \frac{-8}{4}$$

$z = \frac{-1}{2}$ and -2 are the simple poles.

(ii) $$f(z) = \frac{z^2}{2\left(z+\frac{1}{2}\right)(z+2)}$$

$|z|=1$
centre $(0,0)$
$r=1$



Here, $z = \frac{-1}{2}$ lies inside c

$z = -2$ lies outside c.

$$\therefore \quad R(a) = \underset{z \to a}{Lt} (z-a) f(z)$$

$$R\left(\frac{-1}{2}\right) = \underset{z \to -\frac{1}{2}}{Lt} \left(z+\frac{1}{2}\right) \frac{z^2}{2z^2+5z+2}$$

$$= \underset{z \to -\frac{1}{2}}{Lt} \cancel{\left(z+\frac{1}{2}\right)} \frac{z^2}{2\cancel{\left(z+\frac{1}{2}\right)}(z+2)}$$

$$= \underset{z \to -\frac{1}{2}}{Lt} \frac{z^2}{2(z+2)}$$

$$= \frac{\left(\frac{-1}{2}\right)^2}{2\left(-\frac{1}{2}+2\right)}$$

$$= \frac{1}{4} \cdot \frac{1}{\cancel{2}\left(\frac{3}{\cancel{2}}\right)}$$

$$\boxed{R\left(-\frac{1}{2}\right) = \frac{1}{12}}$$

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$\therefore$ By Cauchy's Residue Theorem,

$$\int_C f(z)\,dz = 2\pi i\,[\text{sum of all the residues}]$$

$$= 2\pi i\left[R\left(-\tfrac{1}{2}\right)\right]$$

$$= \cancel{2}\pi i\left[\frac{1}{\cancel{12}_{6}}\right]$$

$$= \frac{\pi i}{6}$$

$\therefore$ (1) $\Rightarrow$

$$I = R.P \text{ of } \frac{1}{i}\int_C f(z)\,dz$$

$$= R.P\ \frac{1}{\cancel{i}}\left(\frac{\pi \cancel{i}}{6}\right)$$

$$\boxed{I = \frac{\pi}{6}}$$ //.

(5). Evaluate: $\displaystyle\int_0^{2\pi} \frac{\cos 3\theta}{5-4\cos\theta}\,d\theta$, by Contour integration.

Soln:

Let $\displaystyle I = \int_0^{2\pi} \frac{\cos 3\theta}{5-4\cos\theta}\,d\theta.$ with $|z|=1$

Let $\cos\theta = \dfrac{z^2+1}{2z}$ & $d\theta = \dfrac{dz}{iz}$

$\cos 3\theta = R.P$ of z^3.

$$\therefore \quad I = \int_0^{2\pi} \frac{\cos 3\theta}{5-4\cos\theta}\, d\theta$$

$$= R.P \int_0^{2\pi} \frac{z^3}{5-\cancel{4}^{2}\left(\frac{z^2+1}{\cancel{2}z}\right)} \left(\frac{dz}{iz}\right)$$

$$= R.P \int_0^{2\pi} \frac{z^3}{i\cancel{z}\left(\frac{5z-2z^2-2}{\cancel{z}}\right)}\, dz$$

$$= R.P \left(\frac{-1}{i}\right) \int_0^{2\pi} \frac{z^3}{2z^2-5z+2}\, dz$$

$$\boxed{I = R.P \left(\frac{-1}{i}\right) \int_C f(z)\, dz} \longrightarrow (1)$$

where $f(z) = \dfrac{z^3}{2z^2-5z+2}$ (use Cauchy's Residue Theorem)

$\therefore$ The poles are given by $2z^2-5z+2=0$

$$z = \frac{-(-5) \pm \sqrt{(-5)^2-4(2)(2)}}{2(2)}$$

$$= \frac{5 \pm \sqrt{25-16}}{4}$$

$$= \frac{5 \pm \sqrt{9}}{4}$$

$$= \frac{5 \pm 3}{4}$$

$$= \frac{5+3}{4} \text{ and } \frac{5-3}{4}$$

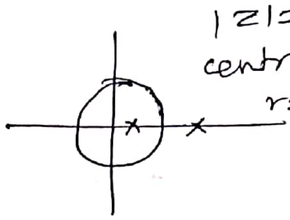
$$= \frac{8}{4} \text{ and } \frac{2}{4}$$

$z = 2$ and $\frac{1}{2}$ are the Simple poles.

$$\therefore \; 2z^2 - 5z + 2 = 2\left(z - \frac{1}{2}\right)(z-2).$$

Here, $z = \frac{1}{2}$ lies inside c.

$z = 2$ lies outside c.



$|z| = 1$
centre $(0,0)$
$r = 1$

$$\therefore \; R(a) = \underset{z \to a}{Lt} \; (z-a) \; f(z)$$

$$= \underset{z \to \frac{1}{2}}{Lt} \left(z - \frac{1}{2}\right) \frac{z^3}{2z^2 - 5z + 2}$$

$$= \underset{z \to \frac{1}{2}}{Lt} \cancel{\left(z - \frac{1}{2}\right)} \frac{z^3}{2\cancel{\left(z - \frac{1}{2}\right)}(z-2)}$$

$$= \underset{z \to \frac{1}{2}}{Lt} \frac{z^3}{2(z-2)}$$

$$= \frac{\left(\frac{1}{2}\right)^3}{2\left(\frac{1}{2} - 2\right)}$$

$$= \frac{1}{8} \cdot \frac{1}{\cancel{2}\left(\frac{-3}{\cancel{2}}\right)}$$

$$\boxed{R\left(\frac{1}{2}\right) = \frac{-1}{24}}$$

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$\therefore$ By Cauchy's residue Theorem,

$$\int_c f(z)\,dz = 2\pi i\,[\text{Sum of all the residues}]$$

$$= 2\pi i\,\left[R\left(\tfrac{1}{2}\right)\right]$$

$$= \cancel{2}\pi i\left(\frac{-1}{\cancel{24}\,12}\right)$$

$$= \frac{-\pi i}{12}$$

$\therefore$ (1) $\Rightarrow$

$$I = R.P\left(\frac{-1}{\cancel{i}}\right)\left(\frac{-\pi \cancel{i}}{12}\right)$$

$$\boxed{I = \frac{\pi}{12}}$$ //.

(HW)

(6) Evaluate: $\displaystyle\int_0^{2\pi} \frac{d\theta}{1-2a\cos\theta+a^2}$ if $|a|<1$ (or) $a^2<1$

by Contour integration.

Soln:

Let $$I = \int_0^{2\pi} \frac{d\theta}{1-2a\cos\theta+a^2}$$

Let $\cos\theta = \dfrac{z^2+1}{2z}$ & $d\theta = \dfrac{dz}{iz}$.

$$\therefore\quad I = \int_0^{2\pi} \frac{\left(\frac{dz}{iz}\right)}{1-\cancel{2}a\left(\frac{z^2+1}{\cancel{2}z}\right)+a^2}$$

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$$= \int_0^{2\pi} \frac{dz}{iz\left(\frac{z - az^2 - a + a^2 z}{z}\right)}$$

$$= \frac{1}{i} \int_0^{2\pi} \frac{dz}{-az^2 + (1+a^2)z - a}$$

$$\boxed{I = \frac{1}{i} \int_C f(z)\, dz} \longrightarrow (1)$$

(use Cauchy's residue Theorem)

where $f(z) = \dfrac{1}{-az^2 + (1+a^2)z - a}$

The poles are given by $-az^2 + (1+a^2)z - a = 0$

$$z = \frac{-(1+a^2) \pm \sqrt{(1+a^2)^2 - 4(-a)(-a)}}{2(-a)}$$

$$= \frac{-(1+a^2) \pm \sqrt{1 + a^4 + 2a^2 - 4a^2}}{-2a}$$

$$= \frac{-(1+a^2) \pm \sqrt{a^4 - 2a^2 + 1}}{-2a}$$

$$= \frac{-(1+a^2) \pm \sqrt{(a^2-1)^2}}{-2a}$$

$$= \frac{-(1+a^2) \pm (a^2-1)}{-2a}$$

$$= \frac{-(1+a^2)+(a^2-1)}{-2a} \quad \text{and} \quad \frac{-(1+a^2)-(a^2-1)}{-2a}$$

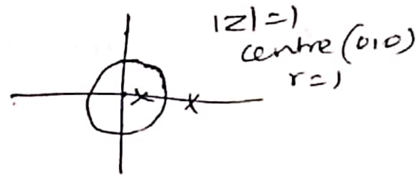
$$= \frac{-1-\cancel{a^2}+\cancel{a^2}-1}{-2a} \quad \text{and} \quad \frac{\cancel{-1}-a^2-a^2\cancel{+1}}{-2a}$$

$$= \frac{-\cancel{2}}{-\cancel{2}a} \quad \text{and} \quad \frac{-\cancel{2}a^2}{-\cancel{2}\cancel{a}}$$

$Z = \frac{1}{a}$ and a are the Simple poles.

$$\therefore -a z^2 + (1+a^2)z - a = a\left(z-\frac{1}{a}\right)(z-a)$$

Here, $z = \frac{1}{a}$ lies outside c



$z = a$ lies inside c.

(Since $|z| < 1$, $|a| < 1$ $\therefore$ lies inside | $|z| < 1$, $\left|\frac{1}{a}\right| < 1$, $|1| < |a|$, (ie) $|a| > 1$ $\therefore$ lies outside)

$$\therefore R(a) = \underset{z \to a}{Lt} (z-a) f(z)$$

$$= \underset{z \to a}{Lt} \cancel{(z-a)} \frac{1}{a\left(z-\frac{1}{a}\right)\cancel{(z-a)}}$$

$$= \frac{1}{a\left(a-\frac{1}{a}\right)} = \frac{1}{\cancel{a}} \frac{1}{\left(\frac{a^2-1}{\cancel{a}}\right)}$$

$$\boxed{R(a) = \frac{1}{a^2-1}}$$

$\therefore$ By Cauchy's Residue Theorem,

$$\int_C f(z)\,dz = 2\pi i\,[\text{Sum of all the residues}]$$

$$= 2\pi i\,[R(a)]$$

$$= 2\pi i\left(\frac{1}{a^2-1}\right).$$

$\therefore$ ① $\Rightarrow$

$$I = \frac{1}{i}\int_C f(z)\,dz$$

$$= \frac{1}{\cancel{i}}\,2\pi \cancel{i}\left(\frac{1}{a^2-1}\right)$$

$$\boxed{I = \frac{2\pi}{a^2-1}}$$

//.

<u>Type II :</u> $\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)}\,dx$ form. Upper semi-circle

Here, $P(x), Q(x) \longrightarrow$ Polynomials in x.

& degree of $Q(x)$ is atleast 2 more than the degree of $P(x)$

& $Q(x)$ does not vanish for any real x.

<u>To solve</u>, Let $f(z) = \frac{P(z)}{Q(z)}$

Then $\int_C f(z)\,dz = \int_{-R}^{R} f(x)\,dx + \int_\Gamma f(z)\,dz$. $\longrightarrow$ ①.

By Cauchy's residue Theorem, Solve $\int_C f(z)\,dz$ (as usual)

By Cauchy's Lemma,

$$\int_{\Gamma} f(z)\,dz \to 0 \quad \text{as } R \to \infty.$$

$\therefore$ ① $\Rightarrow$ $\lim\limits_{R \to \infty} \int_{-R}^{R} f(x)\,dx = \int_C f(z)\,dz.$

$\Downarrow$ residue method

(i.e) $\int_{-\infty}^{\infty} f(x)\,dx = 2\pi i$ [sum of all the residues] //

* Cauchy's Lemma:

If $f(z)$ is a continuous function such that $|z f(z)| \to 0$ as $|z| \to \infty$ on the upper semi-circle $\Gamma : |z| = R$ then $\int_{\Gamma} f(z)\,dz \to 0$ as $R \to \infty$.

Pbs:

① . Evaluate $\int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)(x^2+4)}\,dx$, using contour integration.

Soln:

Let $I = \int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)(x^2+4)}\,dx.$

(Here, Nr degree is $\nless$ Dr degree by atleast 2) & Dr does not vanish for real values.

$\therefore$ Consider $I = \int_C \frac{z^2}{(z^2+1)(z^2+4)}\,dz.$

(i.e) $\boxed{I = \int_C f(z)\,dz}$ $\to$ ① (upper semi-circle)

where $f(z) = \frac{z^2}{(z^2+1)(z^2+4)}$ (use Cauchy's residue Theorem)

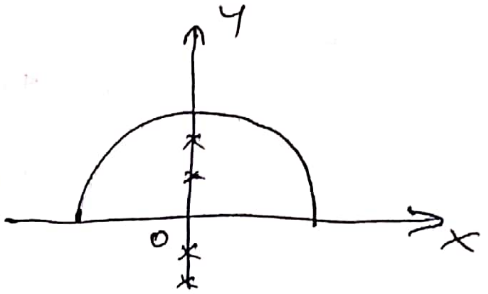
$\therefore$ The poles are given by $(z^2+1)(z^2+4) = 0.$

$\Rightarrow \quad z^2 = -1 \quad \text{and} \quad z^2 = -4$

$\Rightarrow \quad z = \pm i \quad \text{and} \quad z = \pm 2i.$ are the simple poles.

Here, $z = i, 2i$ lies inside c

$z = -i, -2i$ lies outside c.



(upper semi-circle)

$\therefore \; R(a) = \underset{z \to a}{Lt} \; (z-a) \, f(z)$

$$R(i) = \underset{z \to i}{Lt} \; (z-i) \frac{z^2}{(z^2+1)(z^2+4)}$$

$$= \underset{z \to i}{Lt} \; \cancel{(z-i)} \frac{z^2}{(z+i)\cancel{(z-i)}(z^2+4)}$$

$$= \underset{z \to i}{Lt} \; \frac{z^2}{(z+i)\;(z^2+4)}$$

$$= \frac{(i)^2}{[i+i]\,[(i)^2+4]} = \frac{-1}{2i\;(-1+4)}$$

$$\boxed{R(i) = \frac{-1}{6i}}$$

IIIly,

$$R(2i) = \underset{z \to 2i}{Lt} \; (z-2i) \frac{z^2}{(z^2+1)\;(z^2+4)}$$

$$= \underset{z \to 2i}{Lt} \; \cancel{(z-2i)} \frac{z^2}{(z^2+1)\;(z+2i)\cancel{(z-2i)}}$$

$$= \frac{(2i)^2}{[(2i)^2+1]\;(2i+2i)} = \frac{4i^2}{[4i^2+1]\;4i} = \frac{-\cancel{4}}{(-4+1)\cancel{4}i}$$

$$\boxed{R(2i) = +\frac{1}{3i}}$$

$\therefore$ By Cauchy's residue Theorem,

$$\int_C f(z)dz = 2\pi i \left[\text{Sum of all the residues}\right]$$

$$= 2\pi i \left[R(i) + R(2i)\right]$$

$$= 2\pi i \left[\frac{-1}{6i} + \frac{1}{3i}\right]$$

$$= \frac{2\pi \cancel{i}}{\cancel{i}} \left[\frac{-1}{6} + \frac{1}{3}\right]$$

$$= 2\pi \left[\frac{-1+2}{6}\right]$$

$$= \cancel{2}\pi \left[\frac{1}{\cancel{6}_3}\right]$$

$$\boxed{\int_C f(z)dz = \frac{\pi}{3}}$$

But, from ①,

$$\int_C f(z)\,dz = \int_{-R}^{R} f(x)\,dx + \int_{\Gamma} f(z)\,dz$$

By Cauchy's Lemma, as $R \to \infty$, $\int_{\Gamma} f(z)dz \to 0$.

$$\therefore \int_C f(z)\,dz = \int_{-\infty}^{\infty} f(x)\,dx$$

(ie) $$\int_{-\infty}^{\infty} f(x)\,dx = \frac{\pi}{3}$$

(ie) $$\boxed{\int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)(x^2+4)}\,dx = \frac{\pi}{3}}$$ //

(HW)

②. Evaluate $\int_0^\infty \frac{x^2}{(x^2+1)(x^2+4)}\,dx$, by contour integration.

(X)

Soln:

Here, $2\int_0^\infty = \int_{-\infty}^\infty$

$$\int_0^\infty = \frac{1}{2}\int_{-\infty}^\infty$$

(ie) $$\int_0^\infty \frac{x^2}{(x^2+1)(x^2+4)}\,dx = \frac{1}{2}\boxed{\int_{-\infty}^\infty \frac{x^2}{(x^2+1)(x^2+4)}\,dx}$$

$\downarrow$ solving,

$$= \frac{1}{2}\left(\frac{\pi}{3}\right)$$

$$= \frac{\pi}{6}$$ //

(HW).

③. Evaluate $\int_0^\infty \frac{x^2}{(x^2+9)(x^2+4)}\,dx$, by contour integration.

Soln:

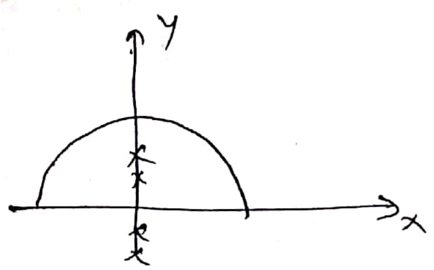
Let $$f(z) = \frac{z^2}{(z^2+9)(z^2+4)}$$

The poles are given by $(z^2+9)(z^2+4)=0$.

(ie) $z^2=-9$ and $z^2=-4$

(ie) $z=\pm 3i$ and $z=\pm 2i$.

Here, $z=2i, 3i$ are the simple poles which lies inside C.



$z=-2i,-3i$. are the simple poles which lies outside c.

$\therefore \; R(a) = \lim_{z\to a} (z-a)\, f(z)$

$$R(2i) = \lim_{z\to 2i} (z-2i) \frac{z^2}{(z^2+9)(z^2+4)}$$

$$= \lim_{z\to 2i} \cancel{(z-2i)} \frac{z^2}{(z^2+9)(z+2i)\cancel{(z-2i)}}$$

$$= \frac{(2i)^2}{[(2i)^2+9](2i+2i)} = \frac{4i^2}{(4i^2+9)(4i)}$$

$$= \frac{-\cancel{4}}{(-4+9)(\cancel{4}i)} = \frac{-1}{5i}$$

$$\boxed{R(2i) = \frac{-1}{5i}}$$

IIIly,

$$R(3i) = \lim_{z\to 3i} (z-3i) \frac{z^2}{(z^2+9)(z^2+4)}$$

$$= \lim_{z\to 3i} \cancel{(z-3i)} \frac{z^2}{(z+3i)\cancel{(z-3i)}(z^2+4)}$$

$$= \frac{(3i)^2}{(3i+3i)[(3i)^2+4]} = \frac{9i^2}{6i\,[9i^2+4]} = \frac{-\cancel{9}\,3}{\cancel{6}_{2}i(-9+4)}$$

$$= \frac{-3}{2i(-5)}$$

$$\boxed{R(3i) = \frac{3}{10i}}$$

$\therefore$ By Cauchy's Residue Theorem,

$$\int_C f(z)\,dz = 2\pi i \,[\text{Sum of all the residues}]$$

$$= 2\pi i \left[R(2i) + R(3i)\right]$$

$$= 2\pi i \left[\frac{-1}{5i} + \frac{3}{10i}\right]$$

$$= \frac{2\pi \cancel{i}}{\cancel{i}}\left[\frac{-1}{5} + \frac{3}{10}\right]$$

$$= 2\pi \left[\frac{-2+3}{10} \right]$$

$$= \cancel{2}\pi \times \frac{1}{\cancel{10}_{5}}$$

$$\boxed{\int_C f(z)dz = \frac{\pi}{5}}$$

But,

$$\int_C f(z)\,dz = \int_{-R}^{R} f(x)\,dx + \int_{\Gamma} f(z)\,dz$$

By Cauchy's Lemma, as $R \to \infty$, $\int_{\Gamma} f(z)dz \to 0$.

$$\therefore \int_C f(z)\,dz = \int_{-\infty}^{\infty} f(x)\,dx$$

(ie) $$\int_{-\infty}^{\infty} f(x)\,dx = \frac{\pi}{5}$$

$$2\int_{0}^{\infty} f(x)\,dx = \frac{\pi}{5}$$

$$\boxed{\int_{0}^{\infty} f(x)\,dx = \frac{\pi}{10}}$$ (or) $$\boxed{\int_{0}^{\infty} \frac{x^2}{(x^2+4)(x^2+9)}\,dx = \frac{\pi}{10}}$$ //

HW

(4) Evaluate $\int_{-\infty}^{\infty} \frac{x^2}{(x^2+a^2)(x^2+b^2)}\,dx$, $a>0$, $b>0$ using Contour integration.

Ans: $I = \frac{\pi}{a+b}$ (here, $a>0$, $b>0$ for upper semi-circle)