→ MULTIPLE INTEGRALS ←

→ Evaluation of Double Integral using Cartesian and Polar Coordinates -

* Type 1 - Limits are constant
* Type 2 - Limits are variable.

1. evaluate $\int_0^1 \int_1^2 (x^2+y^2)\,dx\,dy$

$$\Rightarrow \int_0^1 \left(x^3/3 + y^2 x\right) dy \Rightarrow \int_0^1 (8/3 + 2y^2) - (1/3 + y^2)\,dy$$

$$\Rightarrow \int_0^1 \left(\frac{7}{3} + y^2\right) dy \Rightarrow \left[\frac{7y}{3} + \frac{y^3}{3}\right]_0^1 \Rightarrow \frac{7}{3} + \frac{1}{3} \Rightarrow \frac{8}{3}$$

NOTE - If the limits of integration are constant, then the order of integration is insignificant, i.e.,

$$\Rightarrow \int_Y \int_X f(x,y)\,dx\,dy = \int_X \int_Y f(x,y)\,dy\,dx$$

2. evaluate $\int_0^3 \int_0^2 xy(x+y)\,dy\,dx$

$$\Rightarrow \int_0^3 \int_0^2 (x^2 y + x y^2)\,dy\,dx \Rightarrow \int_0^3 \left[\frac{x^2 y^2}{2} + \frac{x y^3}{3}\right]_0^2 dx$$

$$\Rightarrow \int_0^3 \left(2x^2 + 8x/3\right) dx \Rightarrow \left[\frac{2x^3}{3} + \frac{8x^2}{6}\right]_0^3 \Rightarrow 18 + 12$$

$$= 30$$

3. evaluate $\int_2^a \int_2^b \frac{dx\,dy}{xy}$

$$\Rightarrow \int_2^a \int_2^b \frac{dx\,dy}{xy} \Rightarrow \int_2^a \left[\frac{\log x}{y}\right]_2^b dy \Rightarrow \int_2^a \frac{1}{y}(\log b - \log 2)\,dy$$

$$\Rightarrow \int_2^a \frac{1}{y} \times \log(b/2)\,dy \Rightarrow \log(b/2)\left(\log y\right)_2^a \Rightarrow \log(b/2)\,\log(a/2)$$

(2)

* evaluate $\int_0^1\int_x^{\sqrt{x}} xy(x+y)\,dy\,dx$

$$\Rightarrow \int_0^1\int_x^{\sqrt{x}} (x^2y+xy^2)\,dy\,dx \Rightarrow \int_0^1\left[\frac{x^2y^2}{2}+\frac{xy^3}{3}\right]_x^{\sqrt{x}} dx$$

$$\Rightarrow \int_0^1\left\{\left[\frac{x^3}{2}+\frac{x^2\sqrt{x}}{3}\right]-\left[\frac{x^4}{2}+\frac{x^4}{3}\right]\right\} dx$$

$$\Rightarrow \int_0^1\left[\frac{x^3}{2}+\frac{x^{5/2}}{3}-\frac{x^4}{2}-\frac{x^4}{3}\right]dx$$

$$\Rightarrow \int_0^1\left(\frac{x^3}{2}+\frac{x^{5/2}}{3}-\frac{5x^4}{6}\right)dx$$

$$\Rightarrow \left[\frac{x^4}{8}+\frac{2x^{7/2}}{21}-\frac{x^5}{6}\right]_0^1$$

$$\Rightarrow \frac{1}{8}+\frac{2}{21}-\frac{1}{6} \Rightarrow 3/56 \,//$$

HW * evaluate $\int_x^{\sqrt{x}}\int_0^1 xy(x+y)\,dx\,dy$

$$\Rightarrow \int_x^{\sqrt{x}}\int_0^1 (x^2y+xy^2)\,dx\,dy \Rightarrow \int_x^{\sqrt{x}}\left[\frac{x^3y}{3}+\frac{x^2y^2}{2}\right]_0^1 dy$$

$$\Rightarrow \int_x^{\sqrt{x}}\left[\frac{y}{3}+\frac{y^2}{2}\right]dy \Rightarrow \left[\frac{y^2}{6}+\frac{y^3}{6}\right]_x^{\sqrt{x}}$$

$$\Rightarrow \frac{x}{6}+\frac{x\sqrt{x}}{6}-\frac{x^2}{6}-\frac{x^3}{6} \Rightarrow \frac{x+x^{3/2}-x^3-x^2}{6}\,//$$

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* evaluate $\int_0^3\int_0^2 r\,dr\,d\theta$

$$\Rightarrow \int_0^3\left[\frac{r^2}{2}\right]_0^2 d\theta \Rightarrow \int_0^3 2\,d\theta \Rightarrow 2\times[\theta]_0^3\,d\theta \Rightarrow 2\times 3 = 6\,//$$

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* evaluate $\int_0^a\int_0^{\sqrt{a^2-x^2}} y\,dy\,dx$

$$\Rightarrow \int_0^a\left[\frac{y^2}{2}\right]_0^{\sqrt{a^2-x^2}} dx \Rightarrow \int_0^3\left(\frac{a^2-x^2}{2}\right)dx \Rightarrow \frac{1}{2}\times a^3\times\frac{2}{3} \Rightarrow a^3/3$$

* evaluate $\int_0^a \int_0^{\sqrt{ay}} xy\,dx\,dy$

$$\Rightarrow \int_0^a \left[\frac{x^2 y}{2}\right]_0^{\sqrt{ay}} dy \Rightarrow \int_0^a \frac{ay^2}{2} dy \Rightarrow \frac{a}{2}\left[\frac{y^3}{2}\right]_0^a \Rightarrow \frac{a^4}{6} \,||$$

* evaluate $\int_0^{\pi} \int_0^{a\cos\theta} r\,dr\,d\theta$

$$\Rightarrow \int_0^{\pi} \left(\frac{r^2}{2}\right)_0^{a\cos\theta} d\theta \Rightarrow \int_0^{\pi} \frac{a^2\cos^2\theta}{2} d\theta \Rightarrow \frac{a^2}{2}\int_0^{\pi} \cos^2\theta\,d\theta$$

$$\Rightarrow \frac{a^2}{2}\int_0^{\pi} \frac{1+\cos 2\theta}{2} d\theta \Rightarrow \frac{a^2}{4}\int_0^{\pi} (1+\cos 2\theta)\,d\theta$$

$$\Rightarrow \frac{a^2}{4}\left[\theta + \frac{\sin 2\theta}{4}\right]_0^{\pi} \Rightarrow \frac{a^2}{4}\left[\pi + \frac{\sin 2\pi}{2} + \frac{\sin 0}{2}\right]$$

$$\Rightarrow \frac{a^2}{4} \times (\pi + 0 + 0) \Rightarrow \frac{a^2\pi}{4} \,||$$

* evaluate $\int_0^1 \int_0^x e^{y/x}\,dy\,dx$

$$\Rightarrow \int_0^1 \int_0^x e^{y/x}\,dy\,dx \rightarrow \text{put } e^{y/x} = t$$

$$\Rightarrow dt = e^{y/x} \times \frac{1}{x} \times dy \Rightarrow \underbrace{dy = \frac{x\,dt}{e^{y/x}}}$$

$$y = 0 \rightarrow t = 1$$
$$y = x \rightarrow t = e$$

$$\Rightarrow \int_0^1 \int_1^e e^{y/x} \times \frac{x\,dt}{e^{y/x}} \times dx$$

$$\Rightarrow \int_0^1 [xt]_1^e\,dx \Rightarrow \int_0^1 (xe - x)\,dx$$

$$\Rightarrow (e-1)\int_0^1 x\,dx \Rightarrow (e-1)\left(\frac{x^2}{2}\right)_0^1 \Rightarrow \frac{(e-1)}{2} \,||$$

④ → Changing The Order Of Integration -

Step 1 - Go through the given limits & order of integration.

Step 2 - If the given order of integration is $dx\,dy$ (horizontal strip) then change it to $dy\,dx$ (vertical strip) & vice-versa.

Step 3 - Draw the diagram for the changed order of integration & hence find the limits.

Step 4 - Evaluate the integral.

→ Standard form of DI -

$$\int_{\beta_1(x)}^{\beta_2(x)} \int_{\alpha_1(y)}^{\alpha_2(y)} f(x,y)\,dx\,dy$$

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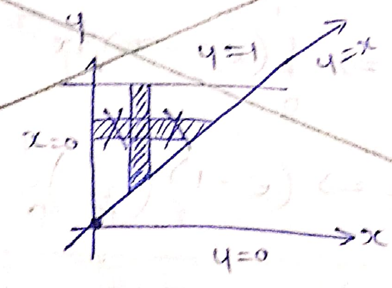
* change the order of integration (COI)

$$\int_0^1 \int_0^x dx\,dy \Rightarrow \int_0^1 \int_0^x dy\,dx \text{ (correct form)}$$

$y=0$ to $y=1$

$x=0$ to $\underbrace{x=x}_{y=x}$



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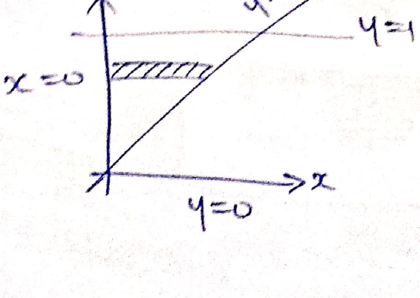
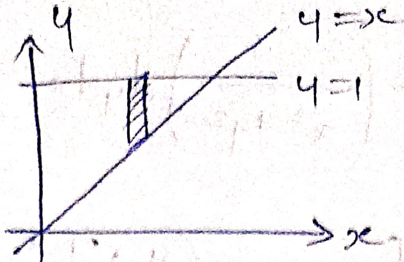
* change the order of integration (COI)

$$\int_0^1 \int_0^x dx\,dy$$



$$\int_0^1\int_x^1 dy\,dx$$



$$\Rightarrow \int_0^1 [y]_x^1\,dx$$

$$\Rightarrow \int_0^1 (1-x)\,dx \Rightarrow \left[x-\frac{x^2}{2}\right]_0^1 \Rightarrow 1-\frac{1}{2} \Rightarrow \frac{1}{2}$$

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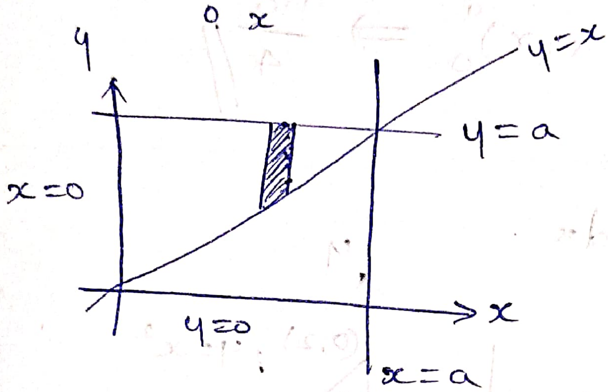
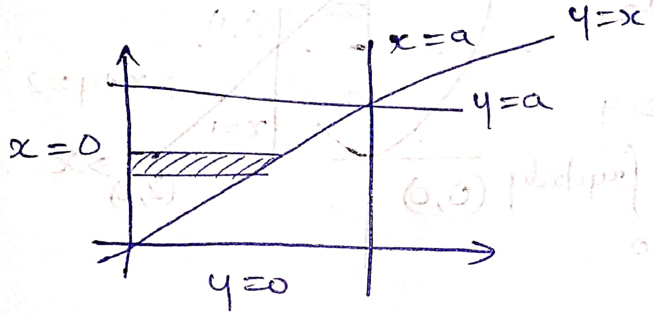
* COI $\longrightarrow$ $\displaystyle\int_0^a\int_x^a (x^2+y^2)\,dy\,dx$





$$\int_0^a\int_0^y (x^2+y^2)\,dx\,dy$$

$$\Rightarrow \int_0^a \left[\frac{x^3}{3}+y^2x\right]_0^y dy \Rightarrow \int_0^a \left[\frac{y^3}{3}+y^3\right]dy$$

$$\Rightarrow \int_0^a \left[\frac{4y^3}{3}\right]dy \Rightarrow \left[\frac{4\times y^4}{3\times 4}\right]_0^a \Rightarrow \frac{a^4}{3}$$

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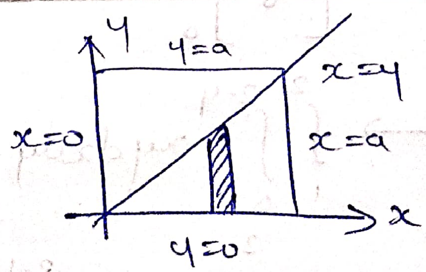
* COI $\longrightarrow$ $\displaystyle\int_0^a\int_y^a \frac{x}{x^2+y^2}\,dx\,dy$



$$\int_0^a\int_0^x \frac{x}{x^2+y^2}\,dy\,dx$$

⑥

$$\Rightarrow \int_0^a \int_0^x x\left(\frac{1}{x^2+y^2}\right) dy\,dx$$

$$\Rightarrow \int_0^a x\left[\frac{1}{x}\tan^{-1}\left(\frac{y}{x}\right)\right]_0^x dx$$

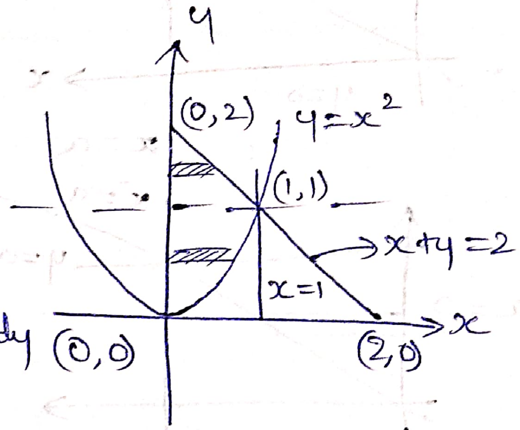
$$\Rightarrow \int_0^a \frac{x}{x}\left[\tan^{-1}(1) - \tan^{-1}(0)\right] dx$$

$$\Rightarrow \int_0^a \frac{\pi}{4} dx \Rightarrow \frac{\pi}{4} \times (x)_0^a \Rightarrow \frac{\pi a}{4} //$$

$*$ COI $\longrightarrow \int_0^1 \int_{x^2}^{2-x} xy\,dy\,dx$

$$\int_0^1 \int_{x^2}^{2-x} xy\,dx\,dy = I_1 + I_2$$

$$= \int_0^1 \int_0^{\sqrt{y}} xy\,dx\,dy + \int_1^2 \int_0^{2-y} xy\,dx\,dy$$



$\underline{\underline{I_1}} \rightarrow \int_0^1 \int_0^{\sqrt{y}} xy\,dx\,dy$

$$= \int_0^1 y\left[\frac{x^2}{2}\right]_0^{\sqrt{y}} dy \Rightarrow \int_0^1 y\left[\frac{y}{2}\right] dy \Rightarrow \int_0^1 \frac{y^2}{2} dy$$

$$= \left[\frac{y^3}{6}\right]_0^1 \Rightarrow 1/6 //$$

$\underline{\underline{I_2}} \rightarrow \int_1^2 \int_0^{2-y} xy\,dx\,dy$

$$\Rightarrow \int_1^2 y\left(\frac{x^2}{2}\right)_0^{2-y} dy$$

$$= \int_1^2 y\left[\frac{(2-y)^2}{2}\right] dy$$

$$= \frac{1}{2}\int_1^2 (4y - 4y^2 + y^3)\, dy$$

$$= \frac{1}{2}\left[\frac{4y^2}{2} - \frac{4y^3}{3} + \frac{y^4}{4}\right]_1^2$$

$$= \frac{1}{2}\left\{8 - \frac{32}{3} + 4 - 2 + \frac{4}{3} - \frac{1}{4}\right\}$$

$$= \frac{1}{2} \times \left[10 - \frac{28}{3} - \frac{1}{4}\right] = \frac{1}{2} \times \frac{5}{12}$$

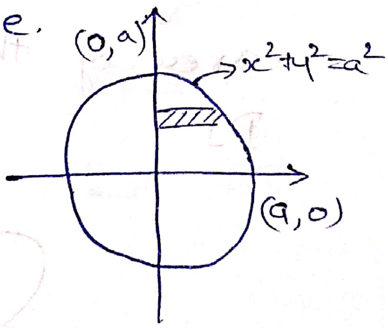
$$= \frac{5}{24}$$

Now, $I_1 + I_2 \Rightarrow \frac{1}{6} + \frac{5}{24} \Rightarrow \frac{9}{24} \Rightarrow \frac{3}{8}$

→ Area of bounded region using double integrat-

Type 1 → cartesian coordinates

* evaluate the area of the circle.



$$x^2 + y^2 = a^2$$

$$\Rightarrow \text{area} = \int_0^a \int_0^{\sqrt{a^2-y^2}} dx\,dy$$

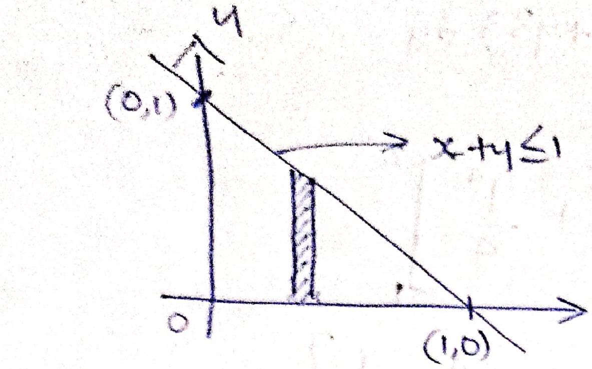
$$= \int_0^a (x)_0^{\sqrt{a^2-y^2}} dy \Rightarrow \int_0^a \sqrt{a^2-y^2}\, dy$$

$$= \left[\frac{y}{2}\sqrt{a^2-y^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{y}{a}\right)\right]_0^a \Rightarrow \frac{a}{2} \times 0 + \frac{a^2}{2} \times \frac{\pi}{2}$$

$$\Rightarrow \frac{a^2\pi}{4}$$

for 4 sides, $\Rightarrow 4 \times A \Rightarrow a^2\pi$

* evaluate $\iint_R (x^2+y^2)\,dy\,dx$ over the region R over which $x, y \geq 0$ and $x+y \leq 1$.



$$\int_0^1 \int_0^{1-x} (x^2+y^2)\,dy\,dx$$

$$\Rightarrow \int_0^1 \left[x^2 y + \frac{y^3}{3}\right]_0^{1-x} dx$$

$$\Rightarrow \int_0^1 \left[x^2(1-x) + \frac{(1-x)^3}{3}\right] dx$$

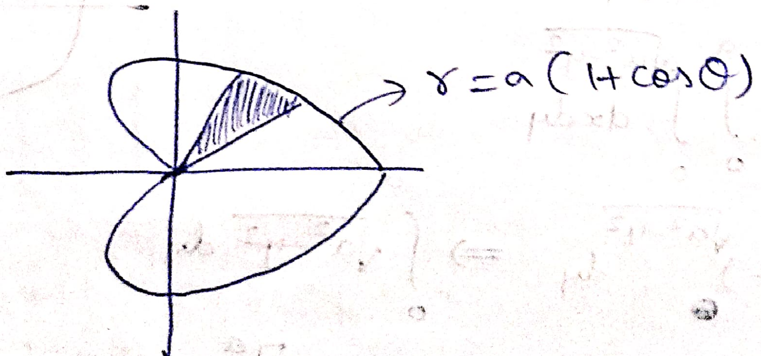
$$\Rightarrow \left[\frac{x^3}{3} - \frac{x^4}{4} - \frac{(1-x)^4}{12}\right]_0^1 = \frac{1}{3} - \frac{1}{4} + \frac{1}{12} = \frac{1}{6}$$

Type 2 - Polar coordinates - type

* $\underbrace{\text{evaluate}}$ $\left[\iint_R r\,dr\,d\theta\right]$

↓

area of the cardioid $r = a(1+\cos\theta)$ using DI.



$$\text{area} = 2\int_0^{\pi} \int_0^{a(1+\cos\theta)} r\,dr\,d\theta$$

$$\Rightarrow 2\int_0^{\pi}\left[\frac{r^2}{2}\right]_0^{a(1+\cos\theta)} d\theta$$

$$\Rightarrow \cancel{2}\int_0^{\pi} a^2(1+\cos\theta)^2 d\theta$$

$$\Rightarrow a^2\int_0^{\pi}(1+2\cos\theta+\cos^2\theta)d\theta$$

$$\Rightarrow a^2\int_0^{\pi}\left\{1+2\cos\theta+\left[\frac{1+\cos\theta}{2}\right]\right\}d\theta$$

$$\Rightarrow a^2\int_0^{\pi}\left\{\frac{3}{2}+2\cos\theta+\frac{1}{2}\cos 2\theta\right\}d\theta$$

$$\Rightarrow a^2\left[\frac{3}{2}\theta+2\sin\theta+\frac{1}{2}\times\sin 2\theta\times\frac{1}{2}\right]_0^{\pi}$$

$$\Rightarrow a^2\times\frac{3}{2}\times\pi \Rightarrow \frac{3\pi a^2}{2}$$ //

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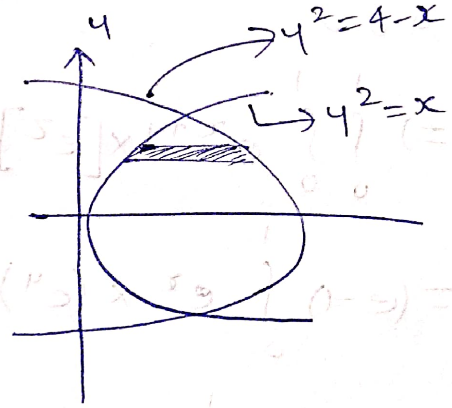
* find the area bounded by the parabola $y^2=4-x$ & $y^2=x$ by double integration.



$$\text{area} \Rightarrow 2\int_0^{\sqrt{2}}\int_{y^2}^{4-y^2} dx\,dy$$

$$\Rightarrow 2\int_0^{\sqrt{2}}(x)_{y^2}^{4-y^2} dy$$

$$\Rightarrow 2\int_0^{\sqrt{2}}(4-y^2-y^2)dy$$

$$\Rightarrow 2\int_0^{\sqrt{2}}(4-2y^2)dy \Rightarrow 2\left[4y-\frac{2y^3}{3}\right]_0^{\sqrt{2}}$$

$$\Rightarrow 2\left[4\sqrt{2}-\frac{2(\sqrt{2})^3}{3}\right] \Rightarrow 2\left[4\sqrt{2}-\frac{4\sqrt{2}}{3}\right]$$

$$\Rightarrow 4\sqrt{2}\times 2\times\frac{2}{3} \Rightarrow 8\sqrt{2}\times 2/3 \Rightarrow \frac{16}{3}\sqrt{2}$$ //

→ TRIPLE INTEGRAL -

* $\int_0^1\int_0^2\int_0^3 xyz\,dx\,dy\,dz$

$\Rightarrow \int_0^1\int_0^2\int_0^3 yz\left[\frac{x^2}{2}\right]_0^3 dy\,dz$

$\Rightarrow \int_0^1\int_0^2 yz \times \frac{9}{2}\,dy\,dz$

$\Rightarrow \frac{9}{2}\int_0^1 z\left[\frac{y^2}{2}\right]_0^2 dz$

$\Rightarrow \frac{9}{2}\int_0^1 2z\,dz \Rightarrow 9\int_0^1 z\,dz$

$\Rightarrow 9 \times \left[\frac{z^2}{2}\right]_0^1 \Rightarrow 9 \times 1/2 \Rightarrow 9/2$ //

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* $\int_0^1\int_0^1\int_0^1 e^{x+y+z}\,dz\,dy\,dx$

$\Rightarrow \int_0^1\int_0^1 e^{x+y} \times [e^z]_0^1\,dy\,dx \Rightarrow \int_0^1\int_0^1 e^{x+y}(e-1)\,dy\,dx$

$= (e-1)\int_0^1 e^x \times (e^y)_0^1\,dx \Rightarrow (e-1)\int_0^1 e^x(e-1)\,dx$

$\Rightarrow (e-1)^2\int_0^1 e^x\,dx \Rightarrow (e-1)^2(e^x)_0^1 \Rightarrow (e-1)^3$ //

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* evaluate $\int_0^4\int_0^x\int_0^{\sqrt{x+y}} z\,dz\,dy\,dx$

$\Rightarrow \int_0^4\int_0^x \left[\frac{z^2}{2}\right]_0^{\sqrt{x+y}} dy\,dx$

$$\Rightarrow \int_0^4 \int_0^x \left(\frac{x+y}{2}\right) dy\,dx \Rightarrow \frac{1}{2}\int_0^4\int_0^x (x+y)\,dy\,dx$$

$$\Rightarrow \frac{1}{2}\int_0^4 \left[xy + \frac{y^2}{2}\right]_0^x dx \Rightarrow \frac{1}{2}\int_0^4 \left[x^2 + \frac{x^2}{2}\right] dx$$

$$\Rightarrow \frac{1}{2}\int_0^4 \frac{3}{2} \times x^2\,dx \Rightarrow \frac{3}{4} \times \left[\frac{x^3}{3}\right]_0^4 = 16 //$$

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*evaluate $\int_0^{\log 2}\int_0^x\int_0^{x+y} e^{x+y+z}\,dz\,dy\,dx$

$$\Rightarrow \int_0^{\log 2}\int_0^x e^{x+y} \times (e^z)_0^{x+y}\,dy\,dx$$

$$\Rightarrow \int_0^{\log 2}\int_0^x e^{x+y} \times (e^{x+y}-1)\,dy\,dx$$

$$\Rightarrow \int_0^{\log 2}\int_0^x \left[e^{2x}\frac{e^{2y}}{2} - e^x e^y\right]_0^x dx$$

$$\Rightarrow \int_0^{\log 2}\left\{\left[e^{2x} \times \frac{e^{2x}}{2} - e^x \times e^x\right] - \left[\frac{e^{2x}}{2} - e^x\right]\right\} dx$$

$$\Rightarrow \int_0^{\log 2} \left(\frac{e^{4x}}{2} - e^{2x}\right) - \left(\frac{e^{2x}}{2} + e^x\right) dx$$

$$\Rightarrow \int_0^{\log 2} \left(\frac{e^{4x}}{2} - \frac{3}{2}e^{2x} + e^x\right) dx$$

$$\Rightarrow \left[\frac{e^{4x}}{8} + \frac{3}{2} \times \frac{e^{2x}}{2} + e^x\right]_0^{\log 2}$$

$$\Rightarrow \left[\frac{e^{4\log 2}}{8} + \frac{3}{4}e^{2\log 2} + e^{\log 2}\right] - \left(\frac{1}{8} + \frac{3}{4} + 1\right) = \frac{5}{8} //$$

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* evaluate the volume of the sphere $x^2+y^2+z^2=a^2$.

$\Rightarrow$ z varies from 0 to $\sqrt{a^2-x^2-y^2}$

$\Rightarrow$ y " " 0 " $\sqrt{a^2-x^2}$

$\Rightarrow$ x " " 0 " a

$$\text{vol} \Rightarrow 8 \times \int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} dz\,dy\,dx$$

$$= 8 \times \int_0^a \int_0^{\sqrt{a^2-x^2}} (z)_0^{\sqrt{a^2-x^2-y^2}} dy\,dx$$

$$= 8 \times \int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{a^2-x^2-y^2}\,dy\,dx$$

$$= 8 \times \int_0^a \left\{ \frac{y\sqrt{a^2-x^2-y^2}}{2} + \frac{a^2-x^2}{2} \sin^{-1} \frac{y}{\sqrt{a^2-x^2}} \right\}_0^{\sqrt{a^2-x^2}} dx$$

$$= 8 \times \int_0^a \left[\frac{a^2-x^2}{2} \sin^{-1}(1) - \frac{a^2-x^2}{2} \sin^{-1}(0) \right] dx$$

$$= \frac{8}{2} \times \int_0^a a^2-x^2 \times \frac{\pi}{2}\,dx$$

$$= 4 \int_0^a (a^2-x^2) \times \frac{\pi}{2} dx = 2\pi \left[a^2x - \frac{x^3}{3} \right]_0^a$$

$$= 2\pi \left[a^3 - \frac{a^3}{3} \right] = 2\pi \times \frac{2a^3}{3} = \frac{4\pi a^3}{3}$$

→ Conversion from Cartesian to Polar Coord

$$(x,y) \longrightarrow (r,\theta)$$

$$\left.\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ dx\,dy &= r\,dr\,d\theta \end{aligned}\right\} \quad \iint_R f(x,y)\,dx\,dy \;\Downarrow\; \iint_R f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$

* evaluate $\displaystyle\int_0^2 \int_0^{\sqrt{2x-x^2}} \frac{x}{\sqrt{x^2+y^2}}\,dy\,dx$ by changing to polar coordinates.

→ $x = r\cos\theta$, $y = r\sin\theta$, $dx\,dy = r\,dr\,d\theta$

→ $x \Rightarrow 0$ to 2

→ $y \Rightarrow 0$ to $\sqrt{2x-x^2}$

$y=0 \Rightarrow \underbrace{r\sin\theta = 0}$

$r=0;\ \theta=0$

$x=0 \Rightarrow \underbrace{r\cos\theta=0}$

$r=0;\ \theta=\pi/2$

$y=\sqrt{2x-x^2} \rightarrow y^2 = 2x-x^2$

$x^2+y^2 = 2x \Rightarrow x^2+y^2-2x=0$

$\Rightarrow r^2 - 2r\cos\theta = 0$

$\Rightarrow r(r-2\cos\theta)=0$

$\Rightarrow r=0,\ r=2\cos\theta$

$$\Rightarrow \int_0^{\pi/2}\int_0^{2\cos\theta} \frac{r\cos\theta}{r^2}\,r\,dr\,d\theta \Rightarrow \int_0^{\pi/2}\int_0^{2\cos\theta} r\cos\theta\,dr\,d\theta$$

$$\Rightarrow \int_0^{\pi/2} \cos\theta\left(\frac{r^2}{2}\right)_0^{2\cos\theta} d\theta \Rightarrow \frac{1}{2}\int_0^{\pi/2} 4\cos^3\theta\,d\theta \Rightarrow \int_0^{\pi/2}\cos^3\theta\,d\theta \times 2$$

$$\Rightarrow 2 \times \frac{2}{3} \times 1 = \underline{4/3}$$

NOTE -

$$\int_0^{\pi/2} \sin^n \theta \, d\theta = \int_0^{\pi/2} \cos^n \theta \, d\theta = \begin{cases} \frac{(n-1)(n-3)(n-5)\cdots 2}{n(n-2)(n-4)\cdots 3} \times 1 & \text{if } n - \text{odd} \\ \frac{(n-1)(n-3)(n-5)\cdots 1}{n(n-2)(n-4)\cdots 2} \times \frac{\pi}{2} & \text{if } n - \text{even} \end{cases}$$

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* evaluate $\int_0^a \int_y^a \frac{x^2}{\sqrt{x^2+y^2}} \, dx \, dy$ by changing to polar coordinates.

$x \Rightarrow y \to a$

$y \Rightarrow 0 \to a$

$x = y \Rightarrow \underbrace{r\cos\theta = r\sin\theta}_{\theta = \pi/4}$

$x = a \Rightarrow \underbrace{r\cos\theta = a}_{r = \frac{a}{\cos\theta}}$

$y = 0 \Rightarrow \underbrace{r\sin\theta = 0}_{r = 0,\ \theta = 0}$

$$\Rightarrow \int_0^{\pi/4} \int_0^{a/\cos\theta} \frac{r^2\cos^2\theta}{r} \times r \, dr \, d\theta$$

$$\Rightarrow \int_0^{\pi/4} \int_0^{a/\cos\theta} r^2 \cos^2\theta \, dr \, d\theta \Rightarrow \int_0^{\pi/4} \cos^2\theta \times \left[\frac{r^3}{3}\right]_0^{a\cos\theta} d\theta$$

$$\Rightarrow \frac{1}{3} \int_0^{\pi/4} \cos^2\theta \left[\frac{a^3}{\cos^3\theta}\right] d\theta \Rightarrow \frac{a^3}{3} \int_0^{\pi/4} \sec\theta \, d\theta$$

$$\Rightarrow \frac{a^3}{3} \left[\log(\sec\theta + \tan\theta)\right]_0^{\pi/4} \Rightarrow \frac{a^3}{3} \log\left[\sec\frac{\pi}{4} + \tan\frac{\pi}{4}\right]$$

$$\Rightarrow \frac{a^3}{3} \left[\log(\sqrt{2}+1)\right] //$$