Analytic Functions.

Analytic functions.

- Properties.
- Construction of analytic function (Milne-Thompson method)
- Conformal mapping
- Bilinear Transformation.

* Analytic function: (Regular function / Holomorphic function)

A Complex function $f(z)$ is said to be analytic at a point z_0, if $f(z)$ is differentiable at z_0 and at every point of some neighbourhood of z_0.

A function is analytic in a domain D if it is analytic at each point of D.

* Necessary & Sufficient Conditions for $f(z)$ to be analytic.

The necessary & sufficient conditions for the function $f(z) = u(x,y) + iv(x,y)$ to be analytic in a domain D are

i) u_x, u_y, V_x, V_y are continuous functions of x & y in the domain D.

ii) $$\left.\begin{array}{l} u_x = V_y \\ \text{and } u_y = -V_x \end{array}\right\} \text{ cauchy-Riemann equns (C-R equns) in cartesian form}$$

* C-R equns in polar form:

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$$

and $$\frac{\partial v}{\partial r} = \frac{-1}{r}\frac{\partial u}{\partial \theta}$$

Pbs:

① Test whether the following functions are analytic or not,

i) $f(z) = z^2$

Soln:

Given: $f(z) = z^2$

$$\Rightarrow u + iv = (x+iy)^2$$
$$= x^2 + (iy)^2 + 2ixy$$
$$u + iv = x^2 - y^2 + i(2xy)$$

Equating real & imaginary parts, we get

$u = x^2 - y^2$	$v = 2xy$
$u_x = 2x$	$v_x = 2y$
$u_y = -2y$	$v_y = 2x$.

$\therefore$ $\left.\begin{array}{l} u_x = v_y \\ u_y = -v_x \end{array}\right\}$ C-R equns are satisfied.

$\therefore$ $f(z)$ is analytic.

ii) $w = \sin z$.

Soln: Given: $f(z) = \sin z$

$$u + iv = \sin(x+iy)$$
$$= \sin x \cos(iy) + \cos x \sin(iy)$$
$$= \sin x \cosh y + i \cos x \sinh y.$$

$\therefore \quad u = \sin x \cosh y \qquad v = \cos x \sinh y$

$u_x = \cos x \cosh y \qquad v_x = -\sin x \sinh y$

$u_y = \sin x \sinh y \qquad v_y = \cos x \cosh y$

$\therefore \quad \left.\begin{matrix} u_x = v_y \\ u_y = -v_x \end{matrix}\right\}$ C-R equns are satisfied.

$\therefore \quad f(z) = \sin z$ is analytic.

iii) $f(z) = \bar{z}$

Soln: Given: $f(z) = \bar{z}$

$u + iv = x - iy$

$\therefore \quad u = x \qquad v = -y$

$u_x = 1 \qquad v_x = 0$

$u_y = 0 \qquad v_y = -1$

But $\left.\begin{matrix} u_x \neq v_y \\ u_y = -v_x \end{matrix}\right\}$ C-R equns are not satisfied.

$\therefore \quad f(z) = \bar{z}$ is not analytic anywhere.

iv). $f(z) = |z|^2$.

Soln: Given: $f(z) = |z|^2$

$= z\bar{z}$

$= (x+iy)(x-iy)$

$u + iv = x^2 + y^2$

$\therefore \quad u = x^2 + y^2 \qquad v = 0$

$u_x = 2x \qquad v_x = 0$

$u_y = 2y \qquad v_y = 0$

Here, $\left.\begin{array}{l} u_x \neq v_y \\ u_y \neq -v_x \end{array}\right\}$. C-R equns are not satisfied

But at $(0,0)$

$\left.\begin{array}{l} u_x = v_y \\ u_y = -v_x \end{array}\right\}$ C-R equns are satisfied at $x=0,\ y=0$.

$\therefore$ $f(z)$ is not analytic for all z including $z=0$.

v) $f(z) = 2xy + i(x^2-y^2)$

Soln:

Given: $f(z) = 2xy + i(x^2-y^2)$

$u + iv = 2xy + i(x^2-y^2)$

Here, $u = 2xy \qquad v = x^2 - y^2$

$u_x = 2y \qquad v_x = 2x$

$u_y = 2x \qquad v_y = -2y$

$\therefore \left.\begin{array}{l} u_x \neq v_y \\ u_y \neq -v_x \end{array}\right\}$ C-R equns are not satisfied.

$\therefore$ $f(z)$ is not analytic.

(2) check for the analyticity of $\log z$.

Soln: Let $f(z) = \log z$. Put $z = re^{i\theta}$, $r > 0$.

$\therefore f(z) = \log(re^{i\theta})$

$= \log_e r + \log_e e^{i\theta}$

$= \log_e r + i\theta$ ($\because \log r$ is defined for $r > 0$)

If $f(z) = u(r,\theta) + iv(r,\theta)$

then $u = \log_e r$ $\quad$ $v = \theta$

$$\frac{\partial u}{\partial r} = \frac{1}{r} \qquad \frac{\partial v}{\partial r} = 0$$

$$\frac{\partial u}{\partial \theta} = 0 \qquad \frac{\partial v}{\partial \theta} = 1.$$

$$\therefore \left.\begin{aligned} \frac{\partial u}{\partial r} &= \frac{1}{r}\frac{\partial v}{\partial \theta} \\ \frac{\partial v}{\partial r} &= -\frac{1}{r}\frac{\partial u}{\partial \theta} \end{aligned}\right\} \text{C-R equns are satisfied for } r \neq 0.$$

$\therefore f(z)$ is analytic ~~for~~ except for $r = 0$ (or) $z = 0$.

$\therefore \log z$ is analytic for all $z \neq 0$..

(3). Prove That $f(z) = z^n$ is analytic, where n is positive integer.

Soln: Let $f(z) = z^n$.

$u + iv = (re^{i\theta})^n = r^n e^{in\theta}$

$= r^n \cos n\theta + i r^n \sin n\theta$.

$\therefore \quad u = r^n \cos n\theta \qquad\qquad v = r^n \sin n\theta$

$$\frac{\partial u}{\partial r} = n \cdot r^{n-1} \cos n\theta \qquad \frac{\partial v}{\partial r} = n \cdot r^{n-1} \sin n\theta$$

$$\frac{\partial u}{\partial \theta} = r^n (-n \sin n\theta) \qquad \frac{\partial v}{\partial \theta} = r^n (n \cos n\theta)$$

Here, $\left.\begin{array}{l} \dfrac{\partial u}{\partial r} = \dfrac{1}{r} \dfrac{\partial v}{\partial \theta} \\ \dfrac{\partial v}{\partial r} = \dfrac{-1}{r} \dfrac{\partial u}{\partial \theta} \end{array}\right\}$ C-R equns are satisfied.

$\therefore \quad f(z) = z^n$ is analytic.

(4). Test for analyticity of $f(z) = e^x (\cos y + i \sin y)$

Soln:

Let $f(z) = e^x (\cos y + i \sin y)$

$u + iv = e^x \cos y + i (e^x \sin y)$

$\therefore \quad u = e^x \cos y \qquad\qquad v = e^x \sin y$

$$u_x = e^x \cos y \qquad v_x = e^x \sin y$$

$$u_y = e^x (-\sin y) \qquad v_y = e^x \cos y.$$

Here, $\left.\begin{array}{l} u_x = v_y \\ u_y = -v_x \end{array}\right\}$ C-R equns are satisfied.

$\therefore \quad f(z)$ is analytic.

* Harmonic function. $\boxed{u_{xx}+u_{yy}=0}$

Any function which has continuous 2^{nd} order partial derivatives and which satisfies the Laplace eqn is called harmonic function. (ie) $u_{xx}+u_{yy}=0$.

Note:

1) Both real & imaginary parts of an analytic function satisfies the Laplace eqn.

$w=f(z)=\phi+i\psi$
$\phi \to$ Potential function
$\psi \to$ Steam function.

2) $w=f(z)=u+iv$ is analytic.

v is harmonic conjugate of u.
(u) " (v)

Properties:

P1: If $f(z)$ is an analytic function of z,

S.T $\nabla^2|f(z)|^2=4|f'(z)|^2$.

(OR)

If $f(z)=u+iv$ is an analytic function of z,

S.T $$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^2=4|f'(z)|^2.$$

(OR)

If $f(z)$ is a regular function of z,

P.T $$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^2=4|f'(z)|^2.$$

5. Find the constants a, b, c if $f(z) = x + ay + i(bx + cy)$ is analytic.

Soln:

Let $u + iv = x + ay + i(bx + cy)$

$\therefore$

$u = x + ay$	$v = bx + cy$
$u_x = 1$	$v_x = b$
$u_y = a$	$v_y = c$

Using C-R eqns,

$u_x = v_y \Rightarrow \boxed{1 = c}$

$u_y = -v_x \Rightarrow \boxed{a = -b}$

6. S.T an analytic function with constant real part is constant.

Soln:

Let $f(z) = u + iv$ be an analytic function.

Given: u = constant $= C$.

$\therefore$ $u_x = 0$

$u_y = 0$.

By C-R eqns,

$u_x = v_y \Rightarrow v_y = 0$

$u_y = -v_x \Rightarrow v_x = 0.$ $\Big\} \Rightarrow v$ is a constant.

$\therefore$ $f(z) = u + iv$ = constant. //

Proof:

Let $f(z) = u + iv$

$\therefore \ |f(z)|^2 = u^2 + v^2.$

$$\frac{\partial}{\partial x} |f(z)|^2 = 2u \cdot \frac{\partial u}{\partial x} + 2v \frac{\partial v}{\partial x}$$

and
$$\frac{\partial^2}{\partial x^2} |f(z)|^2 = 2\left\{u \cdot \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} \cdot \frac{\partial u}{\partial x}\right\} + 2\left\{v \cdot \frac{\partial^2 v}{\partial x^2} + \frac{\partial v}{\partial x} \cdot \frac{\partial v}{\partial x}\right\}$$

$$\frac{\partial^2}{\partial x^2} |f(z)|^2 = 2\left\{u \frac{\partial^2 u}{\partial x^2} + \left(\frac{\partial u}{\partial x}\right)^2\right\} + 2\left\{v \frac{\partial^2 v}{\partial x^2} + \left(\frac{\partial v}{\partial x}\right)^2\right\}$$

|||ly,
$$\frac{\partial^2}{\partial y^2} |f(z)|^2 = 2\left\{u \frac{\partial^2 u}{\partial y^2} + \left(\frac{\partial u}{\partial y}\right)^2\right\} + 2\left\{v \frac{\partial^2 v}{\partial y^2} + \left(\frac{\partial v}{\partial y}\right)^2\right\}$$

Adding,
$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) |f(z)|^2 = 2\left\{u\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right\} + 2\left\{v\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right\} \longrightarrow (1)$$

Using C-R equns,

$u_x = v_y$ & $u_{xx} + u_{yy} = 0$

$u_y = -v_x$ $v_{xx} + v_{yy} = 0$, (1) becomes,

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) |f(z)|^2 = 2\left\{u(0) + \left(\frac{\partial u}{\partial x}\right)^2 + \left(-\frac{\partial v}{\partial x}\right)^2\right\} + 2\left\{v(0) + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial x}\right)^2\right\}$$

$$= 4\left\{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right\} \longrightarrow (2)$$

WKT $f(z) = u+iv$

$f'(z) = u_x + i v_x$

$|f'(z)|^2 = u_x^2 + v_x^2$

$= \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2$

$\therefore$ (2) $\Rightarrow \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) |f(z)|^2 = 4\, |f'(z)|^2$

(ie) $\boxed{\nabla^2 |f(z)|^2 = 4\,|f'(z)|^2}$ //.

P2: P.T an analytic function with constant modulus is constant.

Proof: let $f(z) = u+iv$

$|f(z)| = \sqrt{u^2+v^2}$ = constant.

Let $u^2+v^2 = k$ (constant) $\rightarrow$ (*)

Diff (*) w.r.to x Partially, $2u.u_x + 2v.v_x = 0$

$\Rightarrow u.u_x + v.v_x = 0 \rightarrow (1)$

Diff (*) w.r.to y Partially, $2u.u_y + 2v.v_y = 0$

$\Rightarrow u.u_y + v.v_y = 0$

Using C-R eqns,

$\left.\begin{array}{l} u_x = v_y \\ u_y = -v_x \end{array}\right\} \Rightarrow u.(-v_x) + v.(u_x) = 0 \rightarrow (2)$

Solving ① & ②, $u \cdot u_x + v \cdot v_x = 0$

$v \cdot u_x - u \cdot v_x = 0$

Eliminating v_x, we get $(u^2+v^2)\,u_x = 0$

$\Rightarrow \boxed{u_x = 0}$

Eliminating u_x, we get $\boxed{v_x = 0}$

∴ By C-R equns,

$$\left.\begin{array}{l} u_y = 0 \\ v_y = 0 \end{array}\right\} \Rightarrow \begin{array}{l} u = \text{constant} \\ v = \text{constant}. \end{array}$$

∴ $f(z)$ is constant //.

P3: If $f(z) = u+iv$ be an analytic function, then the family of curves $u(x,y) = c_1$ and $v(x,y) = c_2$ cut each other orthogonally.

((ie) The real & imaginary parts of an analytic function form an orthogonal system.)

Proof: Let $u(x,y) = c_1$

Diff partially w.r.to x, we get

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{-\left(\frac{\partial u}{\partial x}\right)}{\left(\frac{\partial u}{\partial y}\right)} = m_1 \rightarrow ①.$$

Again, $v(x,y) = c_2$

Diff partially w.r. to x, we get $\dfrac{dy}{dx} = \dfrac{-\left(\frac{\partial v}{\partial x}\right)}{\left(\frac{\partial v}{\partial y}\right)}$

Using C-R equns,

$$\left.\begin{array}{l} u_x = v_y \\ u_y = -v_x \end{array}\right\} \Rightarrow \frac{dy}{dx} = \frac{\left(\frac{\partial u}{\partial y}\right)}{\left(\frac{\partial u}{\partial x}\right)} = m_2 \rightarrow ②.$$

① × ② $\Rightarrow$ $m_1 m_2 = \dfrac{-\left(\frac{\partial u}{\partial x}\right)}{\left(\frac{\partial u}{\partial y}\right)} \times \dfrac{\left(\frac{\partial u}{\partial y}\right)}{\left(\frac{\partial u}{\partial x}\right)}$

$$\boxed{m_1 m_2 = -1}$$

$\therefore$ The curves cut each other orthogonally. //

HW

P4: If $f(z)$ is analytic, then $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \log|f(z)| = 0$

(ie) $\boxed{\nabla^2 \log|f(z)| = 0}$

Construction of an analytic function

Milne - Thomson method to find $f(z) = u + iv$

1) If real part $u(x,y)$ is given

$$f(z) = \int u_x(z,0)\,dz - i\int u_y(z,0)\,dz + c.$$

2) If Imaginary part $v(x,y)$ is given

$$f(z) = \int v_y(z,0)\,dz + i\int v_x(z,0)\,dz + c.$$

Pbs:

① S.T the function $u(x,y) = 3x^2y + 2x^2 - y^3 - 2y^2$ is harmonic.

Soln:

$$u(x,y) = 3x^2y + 2x^2 - y^3 - 2y^2$$

$u_x = 6xy + 4x$ | $u_y = 3x^2 - 3y^2 - 4y$

$u_{xx} = 6y + 4$ | $u_{yy} = -6y - 4$

$$\therefore u_{xx} + u_{yy} = 6y + 4 - 6y - 4$$
$$= 0.$$

$\therefore$ $u(x,y)$ is harmonic.

②. S.T $u = 3x^2y - y^3$ is harmonic.

Soln:

$$u = 3x^2y - y^3$$

$u_x = 6xy$ | $u_y = 3x^2 - 3y^2$

$u_{xx} = 6y$ | $u_{yy} = -6y$

$$\therefore u_{xx} + u_{yy} = 6y - 6y$$
$$= 0.$$

$\therefore$ u is harmonic.

③. S.T $V(x,y) = \dfrac{-y}{x^2+y^2}$ is harmonic.

Soln:

$$V_x = -y \left(\frac{-1}{(x^2+y^2)^2}\right) \times 2x$$

$$V_x = \frac{2xy}{(x^2+y^2)^2}$$

Again, $$V_{xx} = \frac{(x^2+y^2)^2 (2y) - 2xy\left[2(x^2+y^2)\right] 2x}{(x^2+y^2)^4}$$

$$= \frac{(x^2+y^2)\,2y - 2xy(2)(2x)}{(x^2+y^2)^3}$$

$$= \frac{2x^2y + 2y^3 - 8x^2y}{(x^2+y^2)^3}$$

$$\boxed{V_{xx} = \frac{2y^3 - 6x^2y}{(x^2+y^2)^3}} \longrightarrow ①.$$

Now

$$V_y = -\left[\frac{(x^2+y^2)(1) - y(2y)}{(x^2+y^2)^2}\right]$$

$$= -\left[\frac{x^2+y^2-2y^2}{(x^2+y^2)^2}\right]$$

$$V_y = \frac{y^2-x^2}{(x^2+y^2)^2}$$

Again, $$V_{yy} = \frac{(x^2+y^2)^2(2y) - (y^2-x^2)\,2(x^2+y^2)(2y)}{(x^2+y^2)^4}$$

$$= \frac{(x^2+y^2)\,2y - (y^2-x^2)\,4y}{(x^2+y^2)^3}$$

$$V_{yy} = \frac{2x^2y + 2y^3 - 4y^3 + 4x^2y}{(x^2+y^2)^3}$$

$$\boxed{V_{yy} = \frac{6x^2y - 2y^3}{(x^2+y^2)^3}} \rightarrow (2).$$

$$\therefore V_{xx} + V_{yy} = \left(\frac{2y^3 - 6x^2y}{(x^2+y^2)^3}\right) + \left(\frac{6x^2y - 2y^3}{(x^2+y^2)^3}\right)$$

$$= 0$$

$\therefore$ V is harmonic.

4. S.T $u = x^2 - y^2$ satisfies (harmonic) Laplace's equation.

Soln:

$$u = x^2 - y^2$$

$u_x = 2x$	$u_y = -2y$
$u_{xx} = 2$	$u_{yy} = -2$

$$\therefore u_{xx} + u_{yy} = 2 - 2$$
$$= 0.$$

$\therefore$ u is harmonic.

HW

5. S.T the function $u = 2xy + 3y$ is harmonic.

⑥. S.T $u = x^3 - 3xy^2 + 3x^2 - 3y^2$ is harmonic and determine its harmonic conjugate. Also find $f(z)$

Soln:

Given: $u = x^3 - 3xy^2 + 3x^2 - 3y^2$

$u_x = 3x^2 - 3y^2 + 6x$ | $u_y = -6xy - 6y$

$u_{xx} = 6x + 6$ | $u_{yy} = -6x - 6$

$\therefore u_{xx} + u_{yy} = 6x + 6 - 6x - 6$
$= 0.$

$\therefore u$ is harmonic.

To find $f(z) = u + iv$

Here, u is given. So, by Milne-Thomson method,

$$f(z) = \int u_x(z,0)\,dz - i\int u_y(z,0)\,dz$$

$$= \int (3z^2 + 6z)\,dz - i\int (0)\,dz$$

$$= \cancel{3}\cdot\frac{z^3}{\cancel{3}} + \frac{\cancel{6}^{3} z^2}{\cancel{2}} + c$$

$$\boxed{f(z) = z^3 + 3z^2 + c}$$

To find harmonic conjugate V

$f(z) = z^3 + 3z^2 +$ $(z = x + iy)$

$= (x+iy)^3 + 3(x+iy)^2 + c$

$= (x^3 - iy^3 + 3yix^2 - 3xy^2) + 3(x^2 - y^2 + 2ixy) + c$

Comparing imaginary parts,

$$\boxed{V = y^3 + 3x^2y + 6xy}$$ //

7. Verify whether $u = \frac{1}{2}\log(x^2+y^2)$ is harmonic. Find the harmonic conjugate. Also find $f(z)$

Soln:

Given: $u = \frac{1}{2}\log(x^2+y^2)$

$$u_x = \frac{1}{2}\cdot\frac{1}{x^2+y^2} \times 2x$$

$$= \frac{x}{x^2+y^2}$$

$$u_{xx} = \frac{(x^2+y^2).1 - x(2x)}{(x^2+y^2)^2}$$

$$u_{xx} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$u_y = \frac{1}{2}\cdot\frac{1}{x^2+y^2} \times 2y$$

$$u_y = \frac{y}{x^2+y^2}$$

$$u_{yy} = \frac{(x^2+y^2)(1)-y(2y)}{(x^2+y^2)^2}$$

$$= \frac{x^2-y^2}{(x^2+y^2)^2}$$

$$\therefore u_{xx}+u_{yy} = \frac{y^2-x^2+x^2-y^2}{(x^2+y^2)^2}$$

$$= 0.$$

$\therefore$ u is harmonic.

To find $f(z)$: If u is given, By Milne-Thomson method,

$$f(z) = \int u_x(z,0)\,dz - i\int u_y(z,0)\,dz + c$$

$$= \int \frac{z}{z^2}\,dz - i(0) + c$$

$$= \int \frac{1}{z}\,dz + c$$

$$\boxed{f(z) = \log z + c}$$

To find harmonic conjugate v:

$$f(z) = \log(x+iy) \qquad (z = x+iy)$$

$$= \log\sqrt{x^2+y^2} + i\tan^{-1}(y/x)$$

$$\therefore \boxed{v = \tan^{-1}(y/x) + c}$$

//

⑧. Find the analytic function $w = u + iv$

if $u = e^x(x\sin y + y\cos y)$.

Soln:

Given: $u = e^x(x\sin y + y\cos y)$

$$u_x = e^x[\sin y] + [x\sin y + y\cos y]\cdot e^x$$

$$u_x(z,0) = 0 + 0 = 0$$

&, $$u_y = e^x[x\cos y + [y\cdot(-\sin y) + \cos y\cdot 1]]$$

$$u_y(z,0) = e^z[z\cdot 1 + [0+1]]$$

$$= ze^z + e^z.$$

∴ By Milne-Thomson method, (u is given)

$$f(z) = \int u_x(z,0)\,dz - i\int u_y(z,0)\,dz + c$$

$$= \int (0)\,dz - i\int (ze^z + e^z) + c$$

$$= -i\int(ze^z + e^z) + c$$

$$= -i[z\cdot e^z - \cancel{e^z} + \cancel{e^z}] + c$$

$$\boxed{f(z) = -ize^z + c}$$

⑨. If $f(z) = u + iv$ is analytic, find $f(z)$ and v

if $u = \dfrac{\sin 2x}{\cosh 2y + \cos 2x}$.

Soln:

Given: $u = \dfrac{\sin 2x}{\cosh 2y + \cos 2x}$.

$$u_x = \frac{(\cosh 2y + \cos 2x)\cdot 2\cos 2x - \sin 2x\,(-2\sin 2x)}{(\cosh 2y + \cos 2x)^2}$$

$$= \frac{2\cosh 2y\cdot\cos 2x + 2\cos^2 2x + 2\sin^2 2x}{(\cosh 2y + \cos 2x)^2}$$

$$= \frac{2\cosh 2y\cdot\cos 2x + 2\left[\cos^2 2x + \sin^2 2x\right]}{(\cosh 2y + \cos 2x)^2}$$

$$= \frac{2\cosh 2y\cos 2x + 2(1)}{(\cosh 2y + \cos 2x)^2}$$

$$u_x(z,0) = \frac{2\cosh 0\cos 2z + 2}{(\cosh 0 + \cos 2z)^2} = \frac{2\cos 2z + 2}{(1+\cos 2z)^2}$$

$$= \frac{2(1+\cancel{\cos 2z})}{(1+\cos 2z)^{\cancel{2}}} = \frac{2}{1+\cos 2z}$$

$$= \frac{2}{2\cos^2 z} = \frac{1}{\cos^2 z} \qquad \left[\cos^2\theta = \frac{1+\cos 2\theta}{2}\right]$$

$$\boxed{u_x(z,0) = \sec^2 z}$$

Again,

$$u_y = \frac{(\cosh 2y + \cos 2x)(0) - \sin 2x\,(2\sinh 2y)}{(\cosh 2y + \cos 2x)^2}$$

$$u_y(z,0) = \frac{0-0}{(\ )^2} = 0.$$

$\therefore$ By Milne-Thomson method (If u is given)

$$f(z) = \int u_x(z,0)\,dz - i\int u_y(z,0)\,dz$$

$$= \int \sec^2 z\,dz - i\int (0)$$

$$\boxed{f(z) = \tan z + c}$$

To find v :

$$f(z) = \tan(x+iy) \qquad (z = x+iy)$$

$$= \frac{2\sin(x+iy)}{2\cos(x+iy)} \cdot \frac{\cos(x-iy)}{\cos(x-iy)}$$

$$= \frac{\sin 2x + i\sinh 2y}{\cos 2x + \cosh 2y}$$

$$\boxed{v = \frac{\sinh 2y}{\cos 2x + \cosh 2y}}$$ //.

(HW)

(10). S.T the function $u = 2xy + 3y$ is harmonic & find the corresponding analytic function.

(11) If $f(z) = u+iv$ is analytic, find $f(z)$ given that $u - v = \dfrac{\sin 2x}{\cosh 2y - \cos 2x}$.

Soln:

Let $f(z) = u + iv$

$i\,f(z) = iu - v$

(+)

$$(1+i)\,f(z) = (u-v) + i(u+v)$$

(i.e) $\boxed{F(z) = U + iV}$

where $U = u - v = \dfrac{\sin 2x}{\cosh 2y - \cos 2x}$ (real part)

$$\therefore \quad U_x = \frac{(\cosh 2y - \cos 2x)(2\cos 2x) - \sin 2x\,(+2\sin 2x)}{(\cosh 2y - \cos 2x)^2}$$

$$U_x(z,0) = \frac{(\cosh 0 - \cos 2z)\,2\cos 2z - \sin 2z\,(2\sin 2z)}{(\cosh 0 - \cos 2z)^2}$$

$$= \frac{(1-\cos 2z)\,2\cos 2z - 2\sin^2 2z}{(1-\cos 2z)^2}$$

$$= \frac{2\cos 2z - 2\cos^2 2z - 2\sin^2 2z}{(1-\cos 2z)^2}$$

$$= \frac{2\cos 2z - 2(\cos^2 2z + \sin^2 2z)}{(1-\cos 2z)^2}$$

$$= \frac{2\cos 2z - 2(1)}{(1-\cos 2z)^2} = \frac{-2(1-\cos 2z)}{(1-\cos 2z)^2}$$

$$= \frac{-2}{1-\cos 2z} = \frac{-\cancel{2}}{\cancel{2}\sin^2 z}$$

$$\boxed{U_x(z,0) = -\operatorname{cosec}^2 z} \qquad \left(\frac{1-\cos 2\theta}{2} = \sin^2\theta\right)$$

Again,

$$U_y = \frac{(\cosh 2y - \cos 2x)(0) - \sin 2x\,(2\sinh 2y)}{(\cosh 2y - \cos 2x)^2}$$

$$U_y(z,0) = \frac{0-0}{(\ )^2} = 0.$$

$\therefore$ By Milne-Thomson method, (U is given)

$$f(z) = \int U_x(z,0)\,dz - i\int U_y(z,0)\,dz$$

$$= \int -\operatorname{cosec}^2 z\,dz - i\,(0)$$

$$\boxed{F(z) = \cot z + C}$$

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(ie) $(1+i)\, f(z) = \cot z + c$

$$\boxed{f(z) = \frac{\cot z}{1+i} + C}$$ //.

(12). Find the analytic function $f(z) = u+iv$ where $u-v = e^x (\cos y - \sin y)$.

Soln:

$$\text{Let } f(z) = u + iv$$

$$i f(z) = iu - v$$

(+)

$$(1+i)\, f(z) = (u-v) + i(u+v)$$

(ie) $\boxed{F(z) = U + iV}$

where $U = u-v = e^x (\cos y - \sin y)$ (real part)

$$\therefore \quad U_x = e^x (\cos y - \sin y)$$

$$U_x(z,0) = e^z (\cos 0 - \sin 0)$$

$$= e^z$$

Again, $\quad U_y = e^x [-\sin y - \cos y]$

$$U_y(z,0) = e^z[-\sin 0 - \cos 0]$$

$$= -e^z.$$

$\therefore$ By Milne-Thomson method, (U is given)

$$F(z) = \int U_x(z,0)\, dz - i \int U_y(z,0)\, dz$$

$$= \int e^z dz - i \int (-e^z)\, dz$$

$$= e^z + ie^z + c$$

$$F(z) = (1+i)\, e^z + c.$$

(ie) $(1+i) f(z) = (1+i) e^{z} + c$

$$\boxed{f(z) = e^{z} + c}$$ //.

(13). Find the analytic function $f(z)$ interms of z if $u+v = (x-y)(x^2+4xy+y^2)$

Soln:

Let $f(z) = u + iv$

$$i f(z) = iu - v$$

(+)

$$(1+i) f(z) = (u-v) + i(u+v)$$

(ie) $$\boxed{F(z) = U + iV}$$

where $V = u+v = (x-y)(x^2+4xy+y^2)$ Imaginary part.

$\therefore$ $V_x = (x-y)(2x+4y) + (x^2+4xy+y^2)(1-0)$

$V_x(z,0) = (z-0)(2z+0) + (z^2+0+0)(1)$

$= 2z^2 + z^2$

$= 3z^2$.

Again,

$V_y = (x-y)(4x+2y) + (x^2+4xy+y^2)(0-1)$

$V_y(z,0) = (z-0)(4z+0) + (z^2+0+0)(-1)$

$= 4z^2 - z^2$

$= 3z^2$.

$\therefore$ By Milne-Thomson method (V is given)

$F(z) = \int V_y(z,0)\, dz + i \int V_x(z,0)\, dz$

$= \int 3z^2\, dz + i \int 3z^2\, dz$

$= \cancel{3} \cdot \frac{z^3}{\cancel{3}} + i\, \cancel{3} \cdot \frac{z^3}{\cancel{3}} + c$

$$F(z) = z^3 + i z^3 + c$$

$$F(z) = (1+i)\, z^3 + c$$

(ie) $$(1+i)\, f(z) = (1+i)\, z^3 + c$$

$$\boxed{f(z) = z^3 + C}$$

//.

(14). Determine the analytic function $f(z) = u + iv$ given that $3u + 2v = y^2 - x^2 + 16xy$.

Soln:

let $f(z) = u + iv$.

$$2f(z) = 2u + i2v$$

$$3if(z) = 3iu - 3v$$

(+)

$$(2+3i)\, f(z) = (2u - 3v) + i\,(3u + 2v)$$

(ie) $$\boxed{F(z) = U + iV}$$

where $V = 3u + 2v = y^2 - x^2 + 16xy$ (Imaginary part).

$$\therefore V_x = -2x + 16y$$

$$V_x(z,0) = -2z + 0$$

$$= -2z.$$

Again,

$$V_y = 2y + 16x$$

$$V_y(z,0) = 0 + 16z$$

$$= 16z.$$

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$\therefore$ By Milne-Thomson method (V is given)

$$F(z) = \int V_y(z,0)\,dz + i\int V_x(z,0)\,dz$$

$$= \int 16z\,dz + i\int(-2z)\,dz$$

$$= \cancel{16}^{8}\cdot\frac{z^2}{\cancel{2}} + i\left(-\cancel{2}\cdot\frac{z^2}{\cancel{2}}\right) + c$$

$$= 8z^2 - iz^2 + c.$$

$$F(z) = (8-i)z^2 + c$$

(ie) $(2+3i)\,f(z) = (8-i)z^2 + c$

$$f(z) = \frac{(8-i)\,z^2}{(2+3i)} + \frac{c}{2+3i}.$$

$$= \frac{(8-i)(2-3i)}{(2+3i)(2-3i)}z^2 + C$$

$$= \frac{16 - 24i - 2i + 3i^2}{4 - 9i^2}z^2 + C$$

$$= \frac{-26i + 13}{13}z^2 + C$$

$$\boxed{f(z) = (1-2i)z^2 + C}$$ (where $C = \frac{c}{2+3i}$)

(HW)

(15) Find the analytic function $f(z) = u+iv$, given that $2u+v = e^x(\cos y - \sin y)$.

Soln:

$$f(z) = u + iv$$

$$2if(z) = 2iu - 2v$$

(+)

$$(1+2i)f(z) = (u-2v) + i(2u+v)$$

(ie) $\boxed{F(z) = U + iV}$

where $V = 2u+v = e^x(\cos y - \sin y)$ (Imaginary part)

↓ solving,

Ans: $f(z) = \left(\frac{1+3i}{5}\right)e^z + C.$

(HW)

(16) Find the analytic function $f(z) = u+iv$, given that $u-2v = e^x(\cos y - \sin y)$

Soln:

$$f(z) = u + iv$$

here,

$$-2f(z) = -2u + i(-2v)$$

$$if(z) = iu - v$$

(+)

$$(-2+i)f(z) = (-2u-v) + i(u-2v)$$

(ie) $\boxed{F(z) = U + iV.}$

where $V = u-2v = e^x(\cos y - \sin y)$

↓ solving,

$$f(z) = \frac{3-i}{5}e^z + C.$$

Conformal mapping

* A transformation that preserves angles between every pair of curves through a point, both in magnitude and sense is said to be conformal at that point.

* A transformation under which angles between every pair of curves through a point are preserved in magnitude, but altered in sense is said to be isogonal at that point.

I translation:

$$w = z + b \quad , \quad b \to \text{complex constant.}$$

$$u + iv = (x + iy) + (b_1 + ib_2)$$

$$= (x + b_1) + i(y + b_2)$$

Under this transformation, Size & shape remain same.

(i.e) Circle into circles,
Squares into Squares etc.

pbs:

1. Find the image of the region bounded by $x=0, y=0, x=1, y=2$ under the map $w = z + 2 - i$.

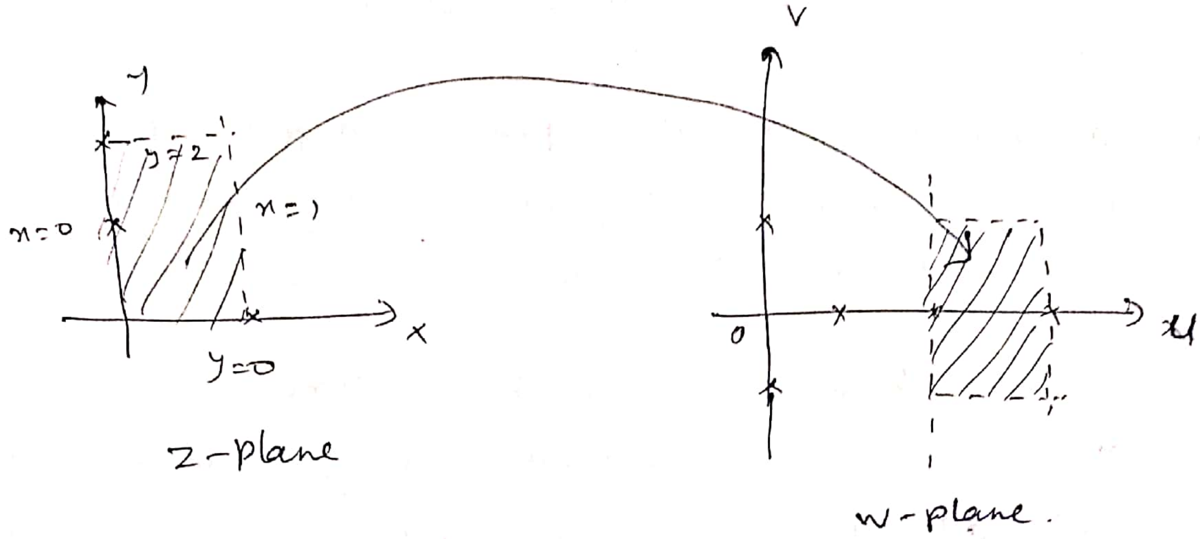
Soln:

Given: $w = z + 2 - i$

$$u + iv = x + iy + 2 - i$$

$$= (x+2) + i(y-1)$$

	$\therefore u = x + 2$	$v = y - 1$
when	$x = 0 \Rightarrow u = 2$	$y = 0 \Rightarrow v = -1$
	$x = 1 \Rightarrow u = 3$	$y = 2 \Rightarrow v = 1.$



Hence the lines $x=0, x=1, y=0, y=2$ in the z-plane are mapped onto $u=2, u=3, v=-1, v=1$ in the w-plane, and they form a rectangle.

②. Find the image of the circle $|z|=1$ under the map $w=z+(2+2i)$

Soln:

Given: $w = z+2+2i$,

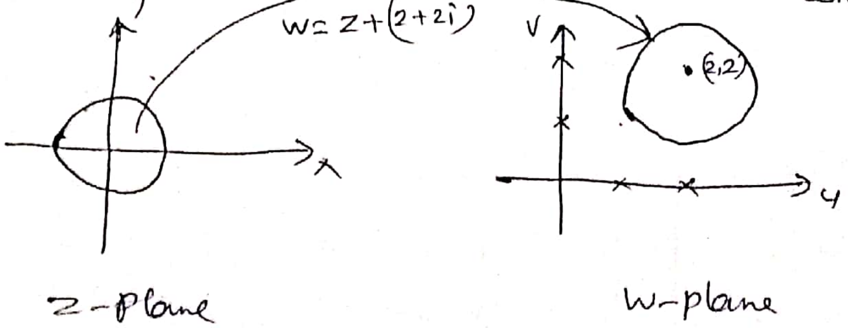
$$u+iv = x+iy+2+2i$$

$$= (x+2)+i(y+2)$$

$\therefore$ $u = x+2$ $\Rightarrow x = u-2$ | $v = y+2$ $\Rightarrow y = v-2$.

$\therefore$ $|z|=1 \Rightarrow x^2+y^2=1$

$\Rightarrow (u-2)^2+(v-2)^2=1$ is a circle with centre $(2,2)$ & radius $=1$.



(II). Magnification & Rotation:

i) $w = az$, $a \to$ real constant $\longrightarrow$ Magnification.

ii) $w = az$, $a \to$ complex constant $\longrightarrow$ Magnification & Rotation.

Here, circles transformed into circles.

Pbs:

(1). Find the image of the rectangular region in the z-plane bounded by the lines $x=0$, $y=0$, $x=2$ and $y=1$ under the transformation $w = 2z$.

Soln:

The transformation is $w = 2z$.

$$\Rightarrow u + iv = 2(x + iy)$$

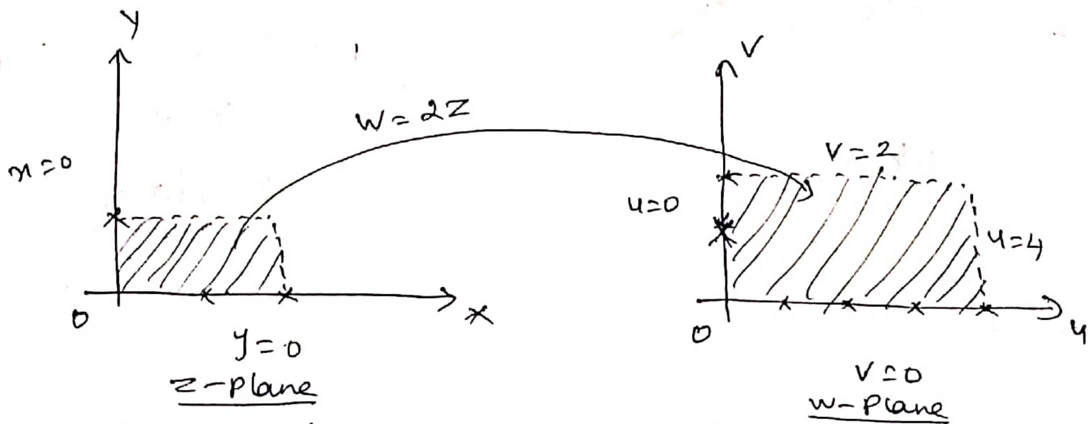
$$\Rightarrow u + iv = 2x + i(2y)$$

$\therefore$ $u = 2x$ | $v = 2y$.

when $x = 0 \Rightarrow u = 0$ | $y = 0 \Rightarrow v = 0$

$x = 2 \Rightarrow u = 4$ | $y = 1 \Rightarrow v = 2$.

$\therefore$



In this transformation, rectangle in the z-plane is mapped into w-plane but it is magnified twice.

(2). Determine the region D of the w-plane into which the triangular region D enclosed by the lines $x=0, y=0, x+y=1$ is transformed under the transform $w = 2z$.

Soln:

The given transformation is $w = 2z$

$$\Rightarrow u+iv = 2(x+iy)$$

$$\Rightarrow u+iv = 2x+i(2y)$$

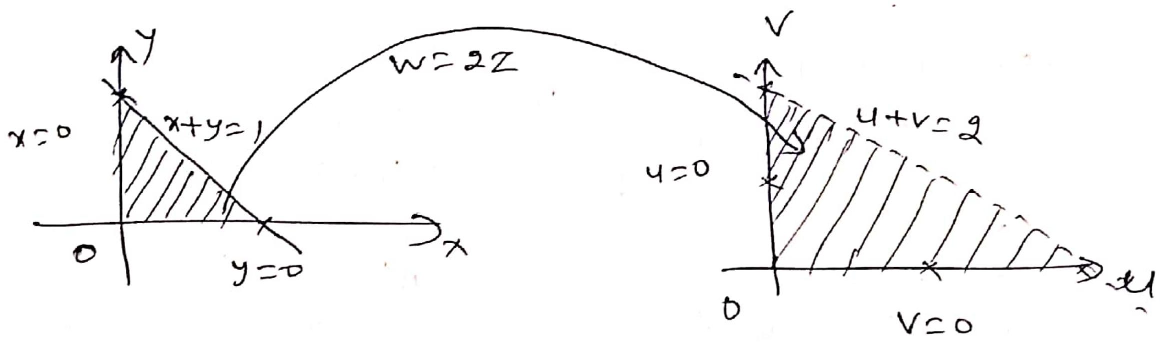
$\therefore$ $u = 2x$ and $v = 2y$

when $x=0 \Rightarrow u=0$

$y=0 \Rightarrow v=0$

and $x+y=1 \Rightarrow \frac{u}{2}+\frac{v}{2}=1$

$\Rightarrow u+v=2$.



z-plane

w-plane.

III Inversion and Reflection $w = \frac{1}{z}$

$$w = \frac{1}{z}$$

$$\Rightarrow \quad z = \frac{1}{w} = \frac{1}{u+iv}$$

$$\Rightarrow \quad x+iy = \frac{1}{u+iv} \times \frac{u-iv}{u-iv}$$

$$= \frac{u-iv}{u^2+v^2}$$

$$\Rightarrow \quad x+iy = \left(\frac{u}{u^2+v^2}\right) + i\left(\frac{-v}{u^2+v^2}\right)$$

Equating real & imaginary parts, we get

$$x = \frac{u}{u^2+v^2}$$

$$y = \frac{-v}{u^2+v^2}$$

Pbs:

① Find the image of $|z-2i| = 2$ under the transformation $w = \frac{1}{z}$.

Soln: Given transformation is $w = \frac{1}{z}$.

$$\Rightarrow \quad z = \frac{1}{w}$$

$$x+iy = \frac{1}{u+iv}$$

$$= \frac{1}{u+iv} \times \frac{u-iv}{u-iv}$$

$$= \frac{u-iv}{u^2+v^2}$$

$$x+iy = \left(\frac{u}{u^2+v^2}\right) + i\left(\frac{-v}{u^2+v^2}\right)$$

Equating real & imaginary parts, we get

$$x = \frac{u}{u^2+v^2}, \quad y = \frac{-v}{u^2+v^2}.$$

Now, Given circle : $|z-2i| = 2$

$$\Rightarrow |x+iy-2i| = 2$$

$$\Rightarrow |x+i(y-2)| = 2$$

$$\Rightarrow x^2+(y-2)^2 = 2^2$$ is a circle with $c(0,2)$ & $r=2$.

$$\Rightarrow x^2+y^2+4-4y = 4$$

$$\Rightarrow \boxed{x^2+y^2-4y=0} \longrightarrow (1)$$

Put $x = \frac{u}{u^2+v^2}$, $y = \frac{-v}{u^2+v^2}$ in (1), we get

$$\left(\frac{u}{u^2+v^2}\right)^2 + \left(\frac{-v}{u^2+v^2}\right)^2 - 4\left(\frac{-v}{u^2+v^2}\right) = 0$$

$$\frac{u^2}{(u^2+v^2)^2} + \frac{v^2}{(u^2+v^2)^2} + \frac{4v}{u^2+v^2} = 0$$

$$\frac{u^2+v^2+4v(u^2+v^2)}{(u^2+v^2)^2} = 0$$

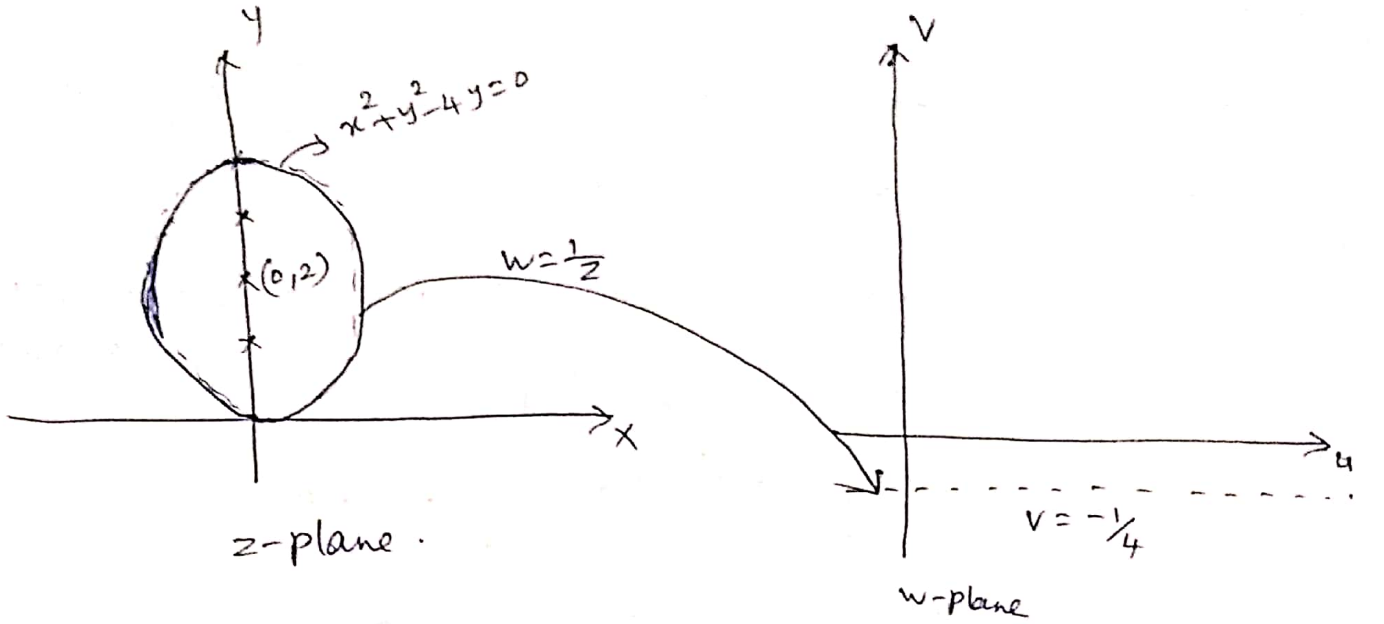
$$\Rightarrow (u^2+v^2)+4v(u^2+v^2) = 0$$

$$\Rightarrow (u^2+v^2)(1+4v) = 0$$

$$\Rightarrow 1+4v = 0$$

$$\Rightarrow \boxed{v = -\frac{1}{4}}$$ is a straight line.

Rough Sketch :



$\therefore$ The given circle $|z-2i|=2$ in the z-plane is transformed into a straight line $v=\frac{-1}{4}$ in the w-plane under the transformation $w=\frac{1}{z}$.

② . Find the image of the circle $|z-1|=1$ under the transformation $w=\frac{1}{z}$.

Soln: Given transformation is $w=\frac{1}{z}$

$$\Rightarrow \quad z=\frac{1}{w}$$

$$\Rightarrow \quad x+iy=\frac{1}{u+iv}$$

$$=\frac{1}{u+iv}\times\frac{u-iv}{u-iv}$$

$$=\frac{u-iv}{u^2+v^2}$$

$$x+iy=\left(\frac{u}{u^2+v^2}\right)+i\left(\frac{-v}{u^2+v^2}\right)$$

Equating real & imaginary parts, we get

$$x=\frac{u}{u^2+v^2}, \quad y=\frac{-v}{u^2+v^2}$$

Now, Given circle : $|z-1| = 1$

$$\Rightarrow |x+iy-1| = 1$$

$$\Rightarrow |(x-1)+iy| = 1$$

$$\Rightarrow (x-1)^2 + y^2 = 1^2$$ is a circle with $c(1,0)$ & $r=1$.

$$\Rightarrow x^2+1-2x+y^2 = 1$$

$$\Rightarrow \boxed{x^2+y^2-2x=0} \longrightarrow (1).$$

Put $x = \frac{u}{u^2+v^2}$, $y = \frac{-v}{u^2+v^2}$ in (1), we get

$$\left(\frac{u}{u^2+v^2}\right)^2 + \left(\frac{-v}{u^2+v^2}\right)^2 - 2\left(\frac{u}{u^2+v^2}\right) = 0$$

$$\frac{u^2}{(u^2+v^2)^2} + \frac{v^2}{(u^2+v^2)^2} - \frac{2u}{u^2+v^2} = 0$$

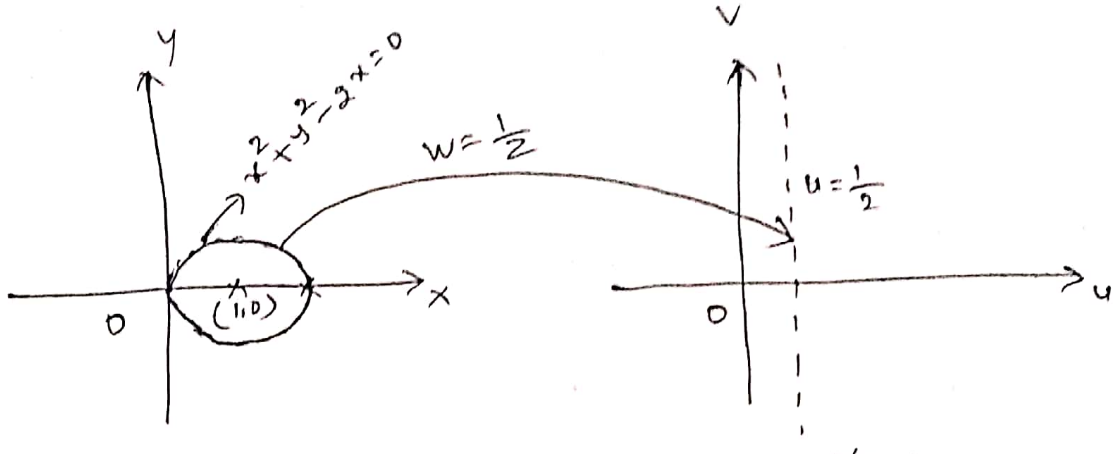
$$\frac{u^2+v^2-2u(u^2+v^2)}{(u^2+v^2)^2} = 0$$

$$\Rightarrow u^2+v^2-2u(u^2+v^2) = 0$$

$$\Rightarrow (u^2+v^2)(1-2u) = 0$$

$$\Rightarrow 1-2u = 0$$

$$\Rightarrow \boxed{u = \frac{1}{2}}$$ is a straight line.



z-plane w-plane.

$\therefore$ The given circle $|z-1|=1$ in z-plane is transformed into a straight line $u=\frac{1}{2}$ in w-plane under the transformation $w=\frac{1}{z}$.

(3). Find the image of the infinite strips

(i) $\frac{1}{4} < y < \frac{1}{2}$

(ii) $0 < y < \frac{1}{2}$ under the transformation $w=\frac{1}{z}$.

Soln:

Given transformation is $w=\frac{1}{z}$

$$\Rightarrow z=\frac{1}{w}$$

$$\Rightarrow x+iy=\frac{1}{u+iv}$$

$$=\frac{1}{u+iv}\times\frac{u-iv}{u-iv}$$

$$=\frac{u-iv}{u^2+v^2}$$

$$\Rightarrow x+iy=\left(\frac{u}{u^2+v^2}\right)+i\left(\frac{-v}{u^2+v^2}\right)$$

Equating real & imaginary parts, we get

$$x = \frac{u}{u^2+v^2}, \quad y = \frac{-v}{u^2+v^2}.$$

Now (i). Given: $\frac{1}{4} < y < \frac{1}{2}$

(ie) $\frac{1}{4} < \frac{-v}{u^2+v^2} < \frac{1}{2}$.

$$\frac{1}{4} < \frac{-v}{u^2+v^2}$$

$$u^2+v^2 < -4v$$

$$u^2+v^2+4v < 0$$

$$\Rightarrow \boxed{u^2+(v+2)^2 < 4}$$

is a circle with c(0,−2) & r = 2 (interior portion)

$$\frac{-v}{u^2+v^2} < \frac{1}{2}$$

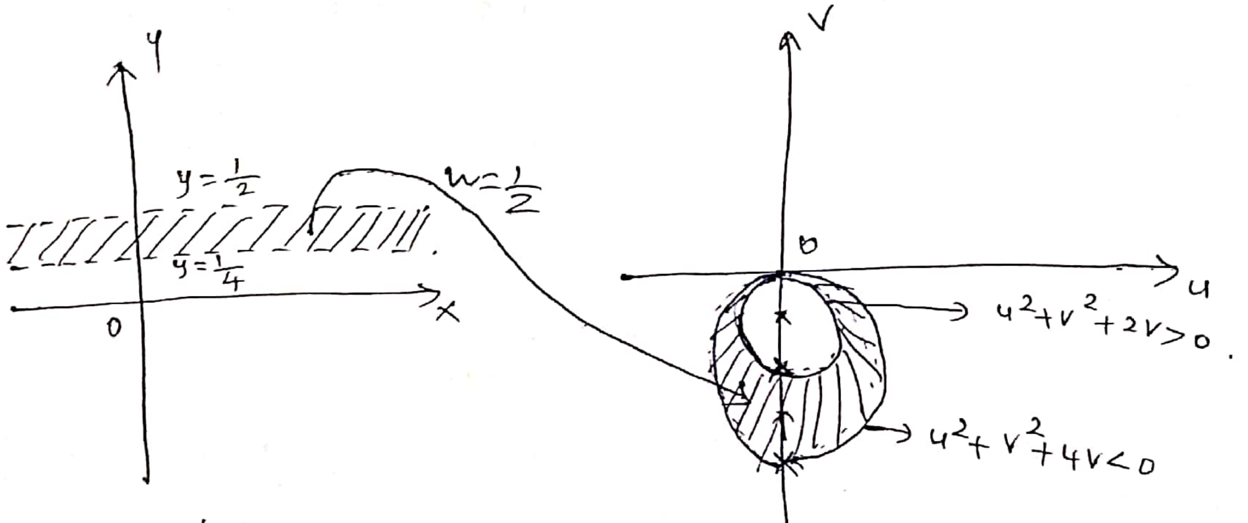
$$-2v < u^2+v^2$$

(ie) $u^2+v^2 > -2v$

$$u^2+v^2+2v > 0$$

$$\Rightarrow \boxed{u^2+(v+1)^2 > 1}$$

is a circle with c(0,−1) & r = 1 (exterior portion)



z-plane w-plane

Now (ii) Given: $0 < y < \frac{1}{2}$.

(ie) $0 < \frac{-v}{u^2+v^2} < \frac{1}{2}$

$0 < \frac{-v}{u^2+v^2}$

$0 < -v$

(ie) $-v > 0$

$\Rightarrow \boxed{v < 0}$

$\frac{-v}{u^2+v^2} < \frac{1}{2}$

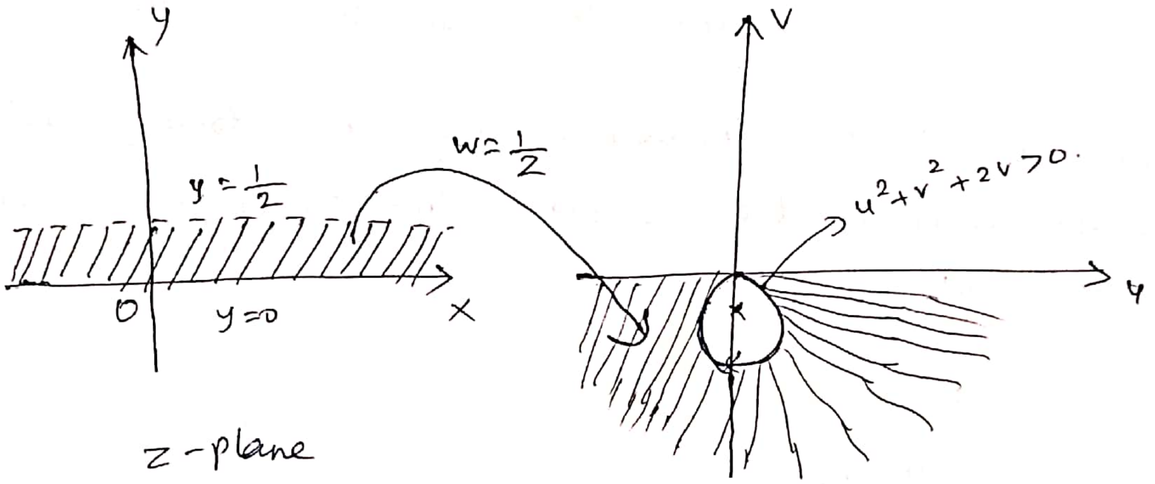
$-2v < u^2+v^2$

(ie) $u^2+v^2 > -2v$

$u^2+v^2+2v > 0$

$\Rightarrow \boxed{u^2+(v+1)^2 > 1}$

is a circle with $c(0,-1)$ & radius $= 1$ (exterior portion)



z-plane

w-plane.

(4). Discuss the transformation $w = \frac{1}{z}$.

Soln: Let $z = re^{i\theta}$ & $w = Re^{i\phi}$

$\therefore$ The transformation $w = \frac{1}{z}$ gives

$$Re^{i\phi} = \frac{1}{re^{i\theta}} = \frac{1}{r}e^{-i\theta}.$$

Hence $R = \frac{1}{r}$ $\longrightarrow$ (1)

& $\phi = -\theta$ $\longrightarrow$ (2).

From ①.

* ① shows that circles in the z-plane with centre at the origin map onto circles in the w-plane with centre at the origin of the plane.

* Unit circle of the z-plane maps onto the unit circle of the w-plane.

* The interior of the unit circle of the z-plane maps onto the exterior of the unit circle of the w-plane & vice versa.

From ②,

* ② shows that rays through the origin of the z-plane maps onto their reflections in the real axis. //

Note:

① $w=\frac{1}{z}$ is not conformal at $z=0$.

② $w=\frac{1}{z}$ is a particular case of a Bilinear transformation.

(HCO)

⑤ Show that the transformation $w=\frac{1}{z}$ transforms circles and straight lines in the z-plane into circles or straight lines in the w-plane.

Soln:

$$w=\frac{1}{z}$$

$$u+iv=\frac{1}{x+iy}$$

$$\Rightarrow x+iy=\frac{1}{u+iv}$$

$$=\frac{1}{u+iv}\times\frac{u-iv}{u-iv}$$

$$=\frac{u-iv}{u^2+v^2}$$

$$\Rightarrow x+iy=\left(\frac{u}{u^2+v^2}\right)+i\left(\frac{-v}{u^2+v^2}\right)$$

Equating real & imaginary parts, we get

$$\boxed{x = \frac{u}{u^2+v^2}}, \quad \boxed{y = \frac{-v}{u^2+v^2}}$$

& $$x^2+y^2 = \left(\frac{u}{u^2+v^2}\right)^2 + \left(\frac{-v}{u^2+v^2}\right)^2$$

$$= \frac{u^2}{(u^2+v^2)^2} + \frac{v^2}{(u^2+v^2)^2}$$

$$= \frac{u^2+v^2}{(u^2+v^2)^2}$$

$$\boxed{x^2+y^2 = \frac{1}{u^2+v^2}}$$

Consider the general eqn of circle in the z-plane is

$$\boxed{a(x^2+y^2) + 2gx + 2fy + c = 0} \longrightarrow (1)$$

Put x, y, x^2+y^2 values, we get

$$a\left(\frac{1}{u^2+v^2}\right) + 2g\frac{u}{u^2+v^2} - \frac{2fv}{u^2+v^2} + c = 0$$

$$\Rightarrow \boxed{c(u^2+v^2) + 2gu - 2fv + a = 0} \longrightarrow (2).$$

case (i): $a \neq 0, c \neq 0$

(1) represents a circle not passing through the origin (z-plane)

(2) represents a circle not passing through the origin (w-plane).

$\therefore$ All circles of the z-plane (not passing through the origin) are mapped onto circles of the w-plane (not passing through the origin).

case (ii): $a \neq 0, c = 0$

(1) represents a circle through the origin (z-plane)

(2) represents a straight line not through the origin (w-plane)

$\therefore$ All circles of the z-plane (through origin) mapped onto straight lines in w-plane (not through origin).

case (iii) : $a=0, c\neq 0$

① represents Straight Lines not through the origin (z-plane)

② represents circles through the origin (w-plane)

$\therefore$ All Straight lines of the z-plane (not through the origin) are mapped onto circles in w-plane (through the origin)

case (iv) : $a=0, c=0$

All Straight lines through the origin of the z-plane are mapped onto Straight Lines through the origin of the w-plane.

Hence, In general, circles & Straight lines of the z-plane are mapped onto circles (or) straight lines of the w-plane. //

Bilinear Transformation.

(Mobius' Transformation / Linear fractional Transformation / Linear Transformation.)

* The transformation

$$\boxed{w = \frac{az+b}{cz+d}} \quad , \quad a,b,c,d \to \text{Complex constants.}$$

such that ~~a~~ $ad - bc \neq 0$, is called a bilinear Transformation.

* **Critical points:**

The critical points of $w = f(z)$ occur at $\boxed{\frac{dw}{dz} = 0}$

and $\boxed{\frac{dz}{dw} = 0.}$

* **Fixed points (or) Invariant points:**

To find the fixed points, of $w = f(z)$,

solve $\boxed{z = f(z)}$.

(A bilinear Transformation has atmost two fixed points).

* **Cross Ratio:** (Normal forms (or) Canonical forms of Bilinear Transformation)

The cross ratio of 4 points z_1, z_2, z_3, z_4 is defined as

$$\frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_4)(z_3 - z_2)}$$ and it is denoted by (z_1, z_2, z_3, z_4)

* The bilinear Transformation that maps the points z_1, z_2, z_3 of the z-plane onto the points w_1, w_2, w_3 of the w-plane is given by

$$\frac{(w - w_1)(w_2 - w_3)}{(w - w_3)(w_2 - w_1)} = \frac{(z - z_1)(z_2 - z_3)}{(z - z_3)(z_2 - z_1)}$$

Pbs:

1. Find the critical points for the transformation $w^2 = (z-\alpha)(z-\beta)$

Soln:

Given: $w^2 = (z-\alpha)(z-\beta)$

Hence, $2w \cdot \frac{dw}{dz} = (z-\alpha)(1) + (z-\beta)(1)$

$= z-\alpha+z-\beta$

$2w\frac{dw}{dz} = 2z-(\alpha+\beta)$

$w\frac{dw}{dz} = z - \frac{1}{2}(\alpha+\beta) \longrightarrow (1)$

critical points occur at $\frac{dw}{dz} = 0$.

(ie) $z - \frac{1}{2}(\alpha+\beta) = 0 \Rightarrow \boxed{z = \frac{1}{2}(\alpha+\beta)}$

Again, (1) $\Rightarrow$ $\frac{dz}{dw} = \frac{w}{z-\frac{1}{2}(\alpha+\beta)}$

Critical points occur at $\frac{dz}{dw} = 0$ also.

(ie) $\frac{w}{z-\frac{1}{2}(\alpha+\beta)} = 0$

$\Rightarrow w = 0$

$\Rightarrow w^2 = 0$

$\Rightarrow (z-\alpha)(z-\beta) = 0$

$\Rightarrow \boxed{z = \alpha, \beta}$

$\therefore$ critical points $= \alpha, \beta, \frac{1}{2}(\alpha+\beta)$. //

(2). Find the fixed points (or) invariant points of the transformation $w = \frac{2z+6}{z+7}$.

Soln:

The fixed points are given by $w = z$

$$\Rightarrow \quad z = \frac{2z+6}{z+7}$$

$$\Rightarrow \quad z^2 + 7z - 2z - 6 = 0$$

$$\Rightarrow \quad z^2 + 5z - 6 = 0$$

$$\Rightarrow \quad (z+6)(z-1) = 0$$

$$\Rightarrow \quad \boxed{z = 1, -6}$$

(3) Find the invariant points (fixed points) of the transformation $w = \frac{2z+4i}{1+iz}$

Soln:

The fixed points are given by $w = z$

$$\Rightarrow \quad z = \frac{2z+4i}{1+iz}$$

$$\Rightarrow \quad z + iz^2 - 2z - 4i = 0$$

$$\Rightarrow \quad iz^2 - z - 4i = 0$$

$$\therefore \quad z = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(i)(-4i)}}{2i}$$

$$= \frac{1 \pm \sqrt{1-16}}{2i}$$

$$= \frac{1 \pm \sqrt{-15}}{2i}$$

$$\boxed{z = \frac{1 \pm i\sqrt{15}}{2i}}$$

④. Find the fixed points of $w = \frac{3z-5}{z+1}$.

Soln:

The fixed points are given by $w = z$

$$\Rightarrow \quad z = \frac{3z-5}{z+1}$$

$$\Rightarrow \quad z^2 + z - 3z + 5 = 0$$

$$\Rightarrow \quad z^2 - 2z + 5 = 0$$

$$\Rightarrow \quad z = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(5)}}{2}$$

$$= \frac{2 \pm \sqrt{4 - 20}}{2}$$

$$= \frac{2 \pm \sqrt{-16}}{2}$$

$$= \frac{2 \pm 4i}{2}$$

$$\boxed{z = 1 \pm 2i} \; //.$$

⑤. Find the bilinear Transformation which maps the Point $z_1 = \infty$, $z_2 = i$, $z_3 = 0$ onto the points $w_1 = 0$, $w_2 = i$, $w_3 = \infty$ respectively.

Soln: Given:

$z_1 = \infty$	$w_1 = 0$
$z_2 = i$	$w_2 = i$
$z_3 = 0$	$w_3 = \infty$.

The bilinear Transformation is

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\frac{(w-0)\cancel{(w_2-w_3)}}{\cancel{(w-w_3)}(i-0)} = \frac{\cancel{(z-z_1)}(i-0)}{(z-0)\cancel{(z_2-z_1)}}$$

Since $\dfrac{w_2-w_3}{w-w_3} = \dfrac{z-z_1}{z_2-z_1}$, here $w_3 = \infty$, $z_1 = \infty$.

$$\frac{\cancel{w_3}\left[\frac{w_2}{w_3}-1\right]}{\cancel{w_3}\left[\frac{w}{w_3}-1\right]} = \frac{\cancel{z_1}\left[\frac{z}{z_1}-1\right]}{\cancel{z_1}\left[\frac{z_2}{z_1}-1\right]}$$

$$\frac{\left[\frac{w_2}{\infty}-1\right]}{\left[\frac{w}{\infty}-1\right]} = \frac{\left(\frac{z}{\infty}-1\right)}{\left(\frac{z_2}{\infty}-1\right)}$$

$$\frac{0-1}{0-1} = \frac{0-1}{0-1}$$

$$1 = 1$$

$\therefore$

$$\frac{w}{i} = \frac{i}{z}$$

$$w = \frac{i^2}{z}$$

$\boxed{w = \dfrac{-1}{z}}$ is the bilinear transformation.

6. Find the bilinear transformation which maps the point $z_1 = \infty, z_2 = 0, z_3 = i$ onto the points $w_1 = 0, w_2 = \infty, w_3 = -i$ respectively.

Soln:

Given:

$z_1 = \infty$	$w_1 = 0$
$z_2 = 0$	$w_2 = \infty$
$z_3 = i$	$w_3 = -i$

$\therefore$ The bilinear transformation is

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\frac{(w-0)(\cancel{w_2}-\cancel{w_3})}{(w+i)(\cancel{w_2}-\cancel{w_1})} = \frac{(z-\cancel{z_1})(0-i)}{(z-i)(z_2-\cancel{z_1})}, \quad \text{here } z_1 = \infty, \; w_2 = \infty.$$

$$\Rightarrow \quad \frac{w}{w+i} = \frac{-i}{z-i}$$

$$w(z-i) = -i(w+i)$$

$$wz - iw = -iw - i^2$$

$$wz - \cancel{iw} + \cancel{iw} - 1 = 0$$

$$wz = 1$$

$\boxed{w = \frac{1}{z}}$ is the bilinear transformation.

(7). Find the bilinear transformation mapping the points $z = 1, i, -1$ into the points $w = 2, i, -2$.

Soln: Given:

$z_1 = 1$	$w_1 = 2$
$z_2 = i$	$w_2 = i$
$z_3 = -1$	$w_3 = -2$

$\therefore$ The bilinear transformation is

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\frac{(w-2)(i+2)}{(w+2)(i-2)} = \frac{(z-1)(i+1)}{(z+1)(i-1)}$$

$$\frac{w-2}{w+2} = \frac{(z-1)(i+1)(i-2)}{(z+1)(i-1)(i+2)}$$

$$= \frac{(z-1)(i^2-2i+i-2)}{(z+1)(i^2+2i-i-2)}$$

$$= \frac{(z-1)(-3-i)}{(z+1)(-3+i)}$$

$$\frac{w-2}{w+2} = \frac{(z-1)(3+i)}{(z+1)(3-i)}$$

Using Componendo and dividendo rule,

$\frac{a}{b} = \frac{c}{d}$ then $\frac{a+b}{a-b} = \frac{c+d}{c-d}$, we get

$$\frac{w-2+w+2}{w-2-(w+2)} = \frac{(z-1)(3+i)+(z+1)(3-i)}{(z-1)(3+i)-[(z+1)(3-i)]}$$

$$\Rightarrow \frac{2w}{-4} = \frac{3z+\cancel{iz}-\cancel{3}-i+3z-\cancel{iz}+\cancel{3}-i}{\cancel{3z}+iz-3-\cancel{i}-\cancel{3z}+iz-3+\cancel{i}}$$

$$\Rightarrow \frac{w}{-2} = \frac{6z-2i}{2iz-6}$$

$$\Rightarrow w = \frac{-2(6z-2i)}{2iz-6}$$

$$\boxed{w = \frac{-(6z-2i)}{iz-3}}$$ is the bilinear transformation.

8. Find the bilinear transformation which maps $z = 0, 1, \infty$ onto $w = i, -1, -i$.

Soln:

Given:

$z_1 = 0$	$w_1 = i$
$z_2 = 1$	$w_2 = -1$
$z_3 = \infty$	$w_3 = -i$

$\therefore$ The bilinear transformation is

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\frac{(w-i)(-1+i)}{(w+i)(-1-i)} = \frac{(z-0)(z_2-\cancel{z_3})}{(z-\cancel{z_3})(1-0)} \qquad \because z_3 = \infty$$

$$\frac{w-i}{w+i} = \frac{z(-1-i)}{(-1+i)}$$

$$= z\left[\frac{-1-i}{-1+i} \times \frac{-1-i}{-1-i}\right]$$

$$= z\left(\frac{1+i+i+i^2}{-1-i^2}\right)$$

$$= z\left(\frac{\cancel{1}+2i-\cancel{1}}{-1-(-1)}\right)$$

$$= z\left(\frac{\cancel{2}i}{\cancel{2}}\right)$$

$$\frac{w-i}{w+i} = \frac{iz}{1}$$

Using componendo and dividendo rule,

(ie) $\frac{a}{b} = \frac{c}{d} \Rightarrow \frac{a+b}{a-b} = \frac{c+d}{c-d}$, we get

$$\frac{w-\cancel{i}+w+\cancel{i}}{\cancel{w}-i-(\cancel{w}+i)} = \frac{iz+1}{iz-1}$$

$$\frac{\cancel{2}w}{-\cancel{2}i} = \frac{iz+1}{iz-1}$$

$$\boxed{w = \frac{-i(iz+1)}{iz-1}}$$ is the bilinear transformation.

(Q). Find the bilinear map which maps the points $z=1, i, -1$ onto the points $w = i, 0, -i$ & hence find the image of $|z|<1$.

Soln: Given:

$z_1 = 1$	$w_1 = i$
$z_2 = i$	$w_2 = 0$
$z_3 = -1$	$w_3 = -i$

$\therefore$ The bilinear transformation is

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\frac{(w-i)(0+i)}{(w+i)(0-i)} = \frac{(z-1)(i+1)}{(z+1)(i-1)}$$

$$\frac{(w-i)(\cancel{i})}{(w+i)(\cancel{+i})} = \frac{(z-1)(1+i)}{\cancel{-}(z+1)(1-i)}$$

$$\Rightarrow \quad \frac{w-i}{w+i} = \frac{(z-1)(1+i)}{(z+1)(1-i)}$$

Using componendo and dividendo rule,

(ie) $\dfrac{a}{b} = \dfrac{c}{d} \Rightarrow \dfrac{a+b}{a-b} = \dfrac{c+d}{c-d}$, we get

$$\frac{w-\cancel{i}+w+\cancel{i}}{\cancel{w}-i-(w+i)} = \frac{(z-1)(1+i)+(z+1)(1-i)}{(z-1)(1+i)-[(z+1)(1-i)]}$$

$$\frac{\cancel{2}w}{-\cancel{2}i} = \frac{z+\cancel{iz}-\cancel{1}-i+z-\cancel{iz}+\cancel{1}-i}{\cancel{z}+iz-1-\cancel{i}-\cancel{z}+iz-1+\cancel{i}}$$

$$\frac{w}{-i} = \frac{\cancel{2}z-\cancel{2}i}{\cancel{2}iz-\cancel{2}}$$

$$w = \frac{-i(z-i)}{iz-1}$$

$$= \frac{-iz+i^2}{iz-1}$$

$$= \frac{-iz-1}{iz-1}$$

$$\boxed{w = \frac{1+iz}{1-iz}}$$ is the bilinear transformation.

Again, $w(1-iz) = 1+iz$

$\Rightarrow \quad w - iwz - 1 - iz = 0$

$\Rightarrow \quad w-1 = iwz + iz$

$w-1 = (iw+i)z$

$\dfrac{w-1}{i(w+1)} = z \quad \Rightarrow \quad \boxed{z = \dfrac{-i(w-1)}{w+1}}$

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NOW, $|z|<1 \Rightarrow \left|\dfrac{w-1}{w+1}\right|<1$

$$\Rightarrow |w-1| < |w+1|$$

$$\Rightarrow |(u+iv)-1| < |(u+iv)+1|$$

$$\Rightarrow |(u-1)+iv| < |(u+1)+iv|$$

$$\Rightarrow (u-1)^2+v^2 < (u+1)^2+v^2$$

$$\Rightarrow u^2+1-2u+v^2 < u^2+1+2u+v^2$$

$$\Rightarrow u^2+2u+v^2+1 > u^2+1-2u+v^2$$

$$\Rightarrow 2u+2u > 0$$

$$\Rightarrow 4u > 0$$

$$\Rightarrow u > 0.$$

$\therefore$ $|z|<1$ is mapped onto $u>0$.