UNIT-4

ANALYTIC FUNCTIONS:-

Definition of Analytic functions - Cauchy - Riemann equations - properties of analytic functions - Determination of analytic function using Milne Thomson's method - conformal mappings - Magnification Relation, Inversion, Reflection, Bilinear Transformation - Cauchy's Integral Theorem (without Proof) - Cauchy's Integral theorem applications - Application of Bilinear transformation and Cauchy's Integral in Engineering.

Differentiable Function:-

A function $f(z)$ is said to be differentiable at a point z if $\lim\limits_{\Delta z \to 0} \dfrac{f(z+\Delta z)-f(z)}{\Delta z}$ exists. This limit is called the derivative of $f(z)$ at the point z and it is denoted by $f'(z)$

Analytic Function:-

A function $f(z)$ is said to be analytic at a point $z=a$ in a region $\mathbb{R}$ if i) $f(z)$ is differentiable at $z=a$ ii) $f(z)$ is differentiable at all points for some neighborhood of $z=a$

Regular Function:-

A function $f(z)$ is said to be regular function if $f(z)$ is analytic at all points in the region $\mathbb{R}$.

Necessary condition for $f(z) = u+iv$ is to be analytic.

The necessary conditions for a function $w = f(z) = u+iv$ to be analytic are $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ ie $u_x = v_y$ and $u_y = -v_x$. These two function are called Cauchy-Riemann Equations (C-R equation)

Sufficient condition for $f(z)$ to be analytic

The function $f(z) = u(x,y) + iv(x,y)$ is analytic in a domain D if i, $u(x,y)$ and $v(x,y)$ are differentiable in D and $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

ii, The partial derivative. $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$ $\frac{\partial v}{\partial x}$ & $\frac{\partial v}{\partial y}$ are all continuous in D.

Problems on Analytic function:-

1. prove that $f(z) = z^2$ in an analytic function and find its derivative.

Soln Let $f(z) = z^2$

$u+iv = (x+iy)^2 = x^2 + 2ixy + i^2 y^2$

$u+iv = x^2 + 2ixy - y^2$

$u+iv = x^2 - y^2 + 2ixy$

$u = x^2 + y^2$	$v = 2xy$
$\frac{\partial u}{\partial x} = 2x$	$\frac{\partial v}{\partial x} = 2y$
$\frac{\partial u}{\partial y} = -2y$	$\frac{\partial v}{\partial y} = 2x$

$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ & $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

$\therefore f(z)$ is analytic because it satisfies C-R equation.

To find derivative :

$\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are continuous $f(z) = z^2$ is analytic everywhere.

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$= 2x + i2y = 2(x+iy)$$

$$\boxed{f'(z) = 2z}$$

2. prove that $w = \frac{1}{z}$ is analytic except at the origin.

Soln

Given that $w = 1/z$

$$u+iv = \frac{1}{x+iy} \times \frac{x-iy}{x-iy} = \frac{x-iy}{x^2+y^2}$$

i) $u+iv = \frac{x-iy}{x^2+y^2}$

ie) $u = \frac{x}{x^2+y^2}$

$$\frac{\partial u}{\partial x} = \frac{(x^2+y^2)(1) - x(2x)}{(x^2+y^2)^2}$$

$$\frac{\partial u}{\partial x} = \frac{x^2+y^2-2x^2}{(x^2+y^2)^2}$$

$$\frac{\partial u}{\partial x} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\frac{\partial u}{\partial y} = \frac{-2xy}{(x^2+y^2)^2}$$

$v = \frac{-y}{x^2+y^2}$

$$\frac{\partial v}{\partial x} = \frac{2xy}{(x^2+y^2)^2}$$

$$\frac{\partial v}{\partial y} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

C-R eqn is satisfied

$\therefore f(z)$ is analytic except at $z=0$.

3. Prove that $w = \bar{z}$ is no where differentiable

Sol^n Given that $w = \bar{z}$

$u + iv = \overline{x+iy}$

$u + iv = x - iy$

$u = x$

$\frac{\partial u}{\partial x} = -1$ | $\frac{\partial v}{\partial y} = -1$

$\frac{\partial u}{\partial y} = 0$ | $\frac{\partial v}{\partial x} = 0$

$\frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y}$ & $\frac{\partial u}{\partial y} \neq -\frac{\partial v}{\partial x}$

$\therefore$ The C-R equations are not satisfied.

$\therefore$ $w = \bar{z}$ is not analytic $w = \bar{z}$ is no where differentiable.

4. For what values of a and b the function $f(z) = (2x+ay) + i(4x+by)$ is analytic.

Sol^n Given that

$f(z) = (2x+ay) + i(4x+by)$

$u + iv = (2x+ay) + i(4x+by)$

$u = 2x + ay$ | $v = 4x + by$

$\frac{\partial u}{\partial x} = 2$ | $\frac{\partial v}{\partial x} = 4$

$\frac{\partial u}{\partial y} = a$ | $\frac{\partial v}{\partial y} = b.$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \& \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$b=2$ & $a=4$ $\quad \therefore \; a=-4$ & $b=2$

Harmonic function:

An expression of the form $\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$ is called the Laplace Equation.

Any function having continuous second order partial derivatives which satisfies the laplace equation is called Harmonic function.

Any two harmonic functions u and v such that $f(z) = u + iv$ is analytic are called conjugate harmonic functions.

Note:

If u and v are conjugate harmonic functions then u is conjugate harmonic functions. to v and v is conjugate harmonic to u.

Properties of an Analytic function

Property: 1

Both real and imaginary part of an analytic functions satisfies the Laplace equations (or) Both real and imaginary part of an analytic functions are harmonic

Proof:

Let $f(z) = u + iv$ be an analytic function

Hence $f(z)$ satisfies its CR eqns

$$\partial u / \partial x = \partial v / \partial y \quad -(1) \quad ; \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad -(2)$$

To prove u (real part of f(z)) is harmonic:-

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \rightarrow (1)$$

Diff eqn (1) p.w.r.to 'x'

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} \rightarrow (3)$$

And

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \rightarrow (2)$$

Diff eqn (2) w.r.to 'y'

$$\frac{\partial^2 u}{\partial y^2} = \frac{-\partial^2 v}{\partial y \partial x} - (4)$$

(3) + (4) $\Rightarrow$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 v}{\partial y \partial x} = 0$$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$\therefore$ u satisfies Laplace equation

$\therefore$ u is a harmonic function.

To prove v harmonic

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} - (5)$$

Diff eqn (5) w.r.to 'x'

$$\frac{\partial^2 v}{\partial x^2} = \frac{-\partial^2 u}{\partial x \partial y} - (6)$$

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} - (7)$$

Diff eqn ⑥ w.r.to 'y'

$$\frac{\partial^2 v}{\partial y^2} = \frac{\partial^2 u}{\partial y \partial x} \quad — ⑧$$

⑥ + ⑧ $\Rightarrow$

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = -\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y \partial x} = 0$$

$$\therefore \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

v satisfies Laplace eqn

$\therefore$ v is harmonic

Note:-

Two curves cut orthogonally if product of their slopes $= -1$

Property: 2

If $f(z) = u+iv$ is analytic then prove that the family of curves $u(x,y) = c_1$ & $v(x,y) = c_2$ cut orthogonally.

proof: Given $f(z) = u+iv$ is analytic.

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad —① \quad \&$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \rightarrow ②$$

$u(x,y) = c_1$	$v(x,y) = c_2$
$du(x,y) = 0$	$dv(x,y) = 0$

$$\frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy = 0$$

$$\frac{\partial u}{\partial x} dy = -\frac{\partial u}{\partial x} dx$$

$$\frac{dy}{dx} = \frac{-(\partial u/\partial x)}{(\partial u/\partial y)} = m_1$$

$$\frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy = 0$$

$$\frac{\partial v}{\partial y} dy = -\frac{\partial v}{\partial x} dx$$

$$\frac{dy}{dx} = \frac{-(\partial v/\partial x)}{(\partial v/\partial y)} = m_2$$

$$m_1 m_2 = \frac{-(\partial u/\partial x)}{(\partial u/\partial y)} \cdot \frac{(\partial v/\partial x)}{(\partial v/\partial y)}$$

$$= \frac{-(\partial v/\partial y) \cdot (-\partial v/\partial x)}{-(\partial v/\partial x) \cdot (\partial v/\partial y)} = -1$$

$\therefore m_1 \cdot m_2 = -1$

$\therefore$ The curves cut orthogonally.

property: 3

If real part of an analytic function is constant, then prove that the function is constant.

proof:

Let $f(z) = u + iv$ be an analytic function

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad -① \quad \& \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad -②$$

Given that $u = c_1$ (say)

$$\frac{\partial u}{\partial x} = 0 \quad \& \quad \frac{\partial u}{\partial y} = 0$$

$$\therefore \frac{\partial v}{\partial y} = 0 \quad \& \quad \frac{\partial v}{\partial x} = 0$$

$\Rightarrow v$ is constant, let $v = c_2$ (say)

$c_1 + ic_2$

$f(z) = u + iv$ = constant

$\therefore f(z)$ is a constant.

property: 4

If modulus of an analytic function is constant, then the function is constant:-

proof: Let $f(z) = u + iv$ be an analytic function.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad - (1) \quad \& \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad - (2)$$

Given $|f(z)|$ is constant.

$$|f(z)| = |u+iv| = \sqrt{u^2+v^2} = \text{constant}.$$

$\Rightarrow u^2+v^2$ = constant

$\Rightarrow u^2+v^2 = c^2 \quad \rightarrow (3)$ c is any constant.

Diff eqn (3) p.w.r.to x,

$$2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x} = 0$$

$$\therefore u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x} = 0 \quad - (4)$$

Diff eqn (3) p.w.r.to y,

$$2u\frac{\partial u}{\partial y} + 2v\frac{\partial v}{\partial y} = 0$$

$$\therefore u\frac{\partial u}{\partial y} + v\frac{\partial v}{\partial y} = 0 \quad - (5)$$

From eqn 1 & 2 we have

$$u\left(-\frac{\partial v}{\partial x}\right) + v\left(\frac{\partial u}{\partial x}\right) = 0$$

$$\therefore -u\frac{\partial v}{\partial x} + v\frac{\partial u}{\partial x} = 0 \quad -(6)$$

$$(4) \times u \Rightarrow u^2\frac{\partial u}{\partial x} + uv\cancel{\frac{\partial v}{\partial x}} = 0$$

$$(6) \times v \Rightarrow -uv\cancel{\frac{\partial v}{\partial x}} + v^2\frac{\partial u}{\partial x} = 0$$

$$\therefore (u^2+v^2)\frac{\partial u}{\partial x} = 0$$

$$c^2\frac{\partial u}{\partial x} = 0 \Rightarrow \frac{\partial u}{\partial x} = 0$$

$$\therefore (4) \Rightarrow v\frac{\partial v}{\partial x} = 0 \Rightarrow \frac{\partial v}{\partial x} = 0$$

$$(1) \Rightarrow \frac{\partial v}{\partial y} = 0 \qquad (2) \Rightarrow \frac{\partial u}{\partial y} = 0$$

$\therefore u$ & v are constants

$\therefore f(z) = u+iv$ is a constant

Hence the proof//

property: 5

If $f(z)$ & $\overline{f(z)}$ are analytic, then prove that $f(z)$ is a constant function.

proof:

Given $f(z) = u+iv$ is an analytic

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad -(1) \quad \& \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad -(2)$$

Also given $\overline{f(z)} = \overline{u+iv} = u-iv$ is an analytic function.

$$\frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y} \quad -(3) \quad \& \quad \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} \quad -(4)$$

$(1)+(3) \Rightarrow \quad 2\frac{\partial u}{\partial x} = 0 \quad \Rightarrow \quad \frac{\partial u}{\partial x} = 0$

$(2)+(4) \Rightarrow \quad 2\frac{\partial u}{\partial y} = 0 \quad \Rightarrow \quad \frac{\partial u}{\partial y} = 0$

$\therefore$ u & v are constant. Hence it follows that $f(z)$ is constant.

Property: 6.

If $u+iv$ and $v+iu$ are analytic functions, analytic constants. then prove that u & v are

proof:

$u+iv$ is analytic function.

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad -(1)$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad -(2)$$

$v+iu$ is analytic function.

$$\therefore \frac{\partial v}{\partial x} = \frac{\partial u}{\partial y} \quad -(3)$$

$$\frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x} \quad -(4)$$

From (1) & (4)

$$\frac{\partial u}{\partial x} = -\frac{\partial u}{\partial x}$$

$$2\frac{\partial u}{\partial x} = 0$$

$$\partial u/\partial x = 0$$

From (2) & (3)

$$\frac{\partial v}{\partial x} = -\frac{\partial v}{\partial x}$$

$$2\,\partial v/\partial x = 0$$

$$\partial v/\partial x = 0.$$

$\frac{\partial v}{\partial y} = 0$ | $\frac{\partial u}{\partial y} = 0$

$\therefore$ u & v are constants.

Problems on properties:-

1. If $u = x^2 - y^2$ & $v = \frac{-y}{x^2+y^2}$; Prove that u and v satisfies Laplace equation but $u+iv$ is not an analytic function.

Soln Given $u = x^2 - y^2$

To prove u satisfies Laplace equation

$\frac{\partial u}{\partial x} = 2x$ | $\partial u/\partial y = -2y$

$\frac{\partial^2 u}{\partial x^2} = 2$ | $\frac{\partial^2 u}{\partial y^2} = -2$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2-2 = 0$$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$\therefore$ u satisfies Laplace equation.

To prove v satisfies Laplace equation.

$$\frac{\partial v}{\partial y} = \frac{(x^2+y^2)(-1) - (-y)(2y)}{(x^2+y^2)} = \frac{-x^2-y^2+2y^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\frac{\partial v}{\partial y} = y^2-x^2 / (x^2+y^2)^2$$

$$\frac{\partial^2 v}{\partial y^2} = \frac{(x^2+y^2)(2y) - (y^2-x^2)\,2(x^2+y^2)(2y)}{(x^2+y^2)^4}$$

$$= \frac{[x^2+y^2][2(x^2+y^2)y - 4y(y^2-x^2)]}{(x^2+y^2)^{\cancel{4}}}$$

$$= \frac{2x^2y + 2y^3 - 4y^3 + 4x^2y}{(x^2+y^2)^3}$$

$$\frac{\partial^2 v}{\partial y^2} = \frac{6x^2y - 2y^3}{(x^2+y^2)^3} \,/\!/$$

$$\frac{\partial v}{\partial x} = \frac{(x^2+y^2)(0) - (-y)(2x)}{(x^2+y^2)^2} = \frac{2xy}{(x^2+y^2)^2}$$

$$\frac{\partial^2 v}{\partial x^2} = \frac{(x^2+y^2)(2y) - (2xy)\,2(x^2+y^2)(2x)}{(x^2+y^2)^4}$$

$$= \frac{[x^2+y^2][2y(x^2+y^2) - 8x^2y]}{(x^2+y^2)^4}$$

$$= \frac{2yx^2 + 2y^3 - 8x^2y}{(x^2+y^2)^3} \Rightarrow \frac{-6x^2y + 2y^3}{(x^2+y^2)^3}$$

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = \frac{6x^2y - 2y^3}{(x^2+y^2)^3} + \frac{-6x^2y + 2y^3}{(x^2+y^2)^3}$$

$$= \frac{6x^2y - 2y^3 - 6x^2y + 2y^3}{(x^2+y^2)^3} = 0$$

$$\therefore \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

$\therefore$ v is a harmonic function.

To prove u+iv is not analytic function

we can see that,

$$\frac{\partial u}{\partial x} = 2x \quad ; \quad \frac{\partial v}{\partial y} = \frac{y^2 - x^2}{(x^2+y^2)^2}$$

$$\Rightarrow \frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y}$$

Also $\frac{\partial v}{\partial y} = -2y \quad ; \quad -\frac{\partial v}{\partial x} = \frac{-2xy}{(x^2+y^2)^2}$

$$\frac{\partial u}{\partial y} \neq -\frac{\partial v}{\partial x}$$

$\therefore$ u+iv is not an analytic function.

2. If u(x,y) and v(x,y) are harmonic functions in a region IR prove that the function $\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) + i\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$ is an analytic function of $z = x+iy$.

Proof: Given u and v are harmonic function.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad —(1)$$

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0 \quad —(2)$$

Let $u+iv = \left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) + i\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$

$\therefore U = \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}$ and $V = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}$

$$\frac{\partial U}{\partial x} = \frac{\partial^2 u}{\partial x \partial y} - \frac{\partial^2 v}{\partial x^2} \quad \Bigg| \quad \frac{\partial V}{\partial x} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial x \partial y}$$

$$\frac{\partial U}{\partial y} = \frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 v}{\partial y \partial x} \quad \Bigg| \quad \frac{\partial V}{\partial y} = \frac{\partial^2 u}{\partial y \partial x} + \frac{\partial^2 v}{\partial y^2}$$

$$\frac{\partial U}{\partial x} - \frac{\partial V}{\partial y} \Rightarrow \cancel{\frac{\partial^2 u}{\partial x \partial y}} - \frac{\partial^2 v}{\partial x^2} - \cancel{\frac{\partial^2 u}{\partial y \partial x}} - \frac{\partial^2 v}{\partial y^2}$$

$$= -\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)$$

$$\frac{\partial U}{\partial x} - \frac{\partial V}{\partial y} = 0$$

$$\therefore \frac{\partial U}{\partial x} = \frac{\partial V}{\partial y}$$

$$\frac{\partial U}{\partial y} + \frac{\partial V}{\partial x} = \frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 u}{\partial y \partial x} + \frac{\partial^2 v}{\partial y \partial x} + \frac{\partial^2 u}{\partial x^2}$$

$$\frac{\partial U}{\partial y} + \frac{\partial V}{\partial x} = 0$$

$$\therefore \frac{\partial U}{\partial x} = \frac{\partial V}{\partial y} \quad \& \quad \frac{\partial U}{\partial y} = -\frac{\partial V}{\partial x}$$

$\therefore U+iV$ satisfies C-R equation

$\therefore U+iV$ is analytic

$\therefore \left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) + i\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$ is analytic

function.

3. Prove that every analytic function $w = u(x,y) + iv(x,y)$ can be expressed as a function of z alone.

Proof:

Let $z = x+iy$ — ①

& $\bar{z} = x-iy$ — ②

① + ② $\Rightarrow z+\bar{z} = 2x$

$$x = \frac{z+\bar{z}}{2} \quad — ③$$

① - ② $\Rightarrow z-\bar{z} = 2iy$

$$y = \frac{z-\bar{z}}{2i} \quad — ④$$

From ③, $\frac{\partial x}{\partial \bar{z}} = 1/2$

From ④, $\frac{\partial y}{\partial \bar{z}} = -1/2i$

Given $w = u(x,y) + iv(x,y)$ w is a function of u and v. But u and v are function of z and $\bar{z}$.

$$\frac{\partial w}{\partial \bar{z}} = \frac{\partial}{\partial \bar{z}}(u+iv)$$

$$= \frac{\partial u}{\partial \bar{z}} + i\frac{\partial v}{\partial \bar{z}}$$

$$= \left(\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial \bar{z}} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \bar{z}}\right) + i\left(\frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial \bar{z}} + \frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial \bar{z}}\right)$$

$$= \left[\frac{\partial u}{\partial x}(1/2) + \frac{\partial u}{\partial y}(-1/2i)\right] + i\left[\frac{\partial v}{\partial x}(1/2) + \frac{\partial v}{\partial y}(-1/2i)\right]$$

$$= 1/2\,\frac{\partial u}{\partial x} - \frac{1}{2i}\frac{\partial u}{\partial y} + \frac{i}{2}\frac{\partial v}{\partial x} - 1/2\,\frac{\partial v}{\partial y}$$

$$= \frac{1}{2}\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) + \frac{i}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

$$= \frac{1}{2}\left(\frac{\partial v}{\partial y} - \frac{\partial v}{\partial y}\right) + \frac{i}{2}\left(-\frac{\partial v}{\partial x} + \frac{\partial v}{\partial x}\right)$$

$$= 0$$

$$\frac{\partial w}{\partial \bar{z}} = 0$$

$\Rightarrow$ w is independent of $\bar{z}$

$\therefore$ w ~~is~~ depends on z only

$\therefore$ w is a function of z alone

Hence the proof.

4. If $z = x+iy$, then prove that $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4\frac{\partial^2}{\partial z \partial \bar{z}}$

Proof:

Let $z = x+iy$ — ①

$\bar{z} = x-iy$ — ②

① + ② $\Rightarrow$ $x = \frac{z+\bar{z}}{2}$; ① − ② $\Rightarrow$ $y = \frac{z-\bar{z}}{2i}$

$\frac{\partial x}{\partial z} = 1/2$; $\frac{\partial x}{\partial \bar{z}} = 1/2$; $\frac{\partial y}{\partial z} = 1/2i$; $\frac{\partial y}{\partial \bar{z}} = -1/2i$

$$\frac{\partial}{\partial z} = \frac{\partial}{\partial x}\cdot\frac{\partial x}{\partial z} + \frac{\partial}{\partial y}\cdot\frac{\partial y}{\partial z}$$

$$= \frac{\partial}{\partial x}(1/2) + \partial/\partial y(1/2i)$$

$$= 1/2\left[\partial/\partial x - i\,\partial/\partial y\right]$$

$$\frac{\partial}{\partial \bar{z}} = \frac{\partial}{\partial x}\cdot\frac{\partial x}{\partial \bar{z}} + \frac{\partial}{\partial y}\cdot\frac{\partial y}{\partial \bar{z}}$$

$$= \partial/\partial x(1/2) + \partial/\partial y(-1/2i)$$

$$= 1/2(\partial/\partial x + i\,\partial/\partial y)$$

$$(\partial/\partial z)(\partial/\partial \bar{z}) = 1/2(\partial/\partial x - i\,\partial/\partial y) + 1/2(\partial/\partial x + i\,\partial/\partial y)$$

$$= 1/4\left(\frac{\partial^2}{\partial x^2} + i\frac{\partial^2}{\partial x \partial y} - i\frac{\partial^2}{\partial y \partial x} + \frac{\partial^2}{\partial y^2}\right)$$

$$= 1/4\left(\partial^2/\partial x^2 + \partial^2/\partial y^2\right)$$

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4 \frac{\partial^2}{\partial z \partial \bar{z}}$$

Hence the Proof //.

5. If $f(z) = u+iv$ is an analytic function, then prove that

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) |f(z)|^2 = 4|f'(z)|^2 \text{ (or)}$$

$$\nabla^2 |f(z)|^2 = 4|f'(z)|^2$$

Soln

Let $z = x+iy$ — ①

$\bar{z} = x-iy$ — ②

① + ② ⇒ $x = \frac{z+\bar{z}}{2}$; ① − ② = $y = \frac{z-\bar{z}}{2i}$

$\frac{\partial x}{\partial z} = 1/2$; $\frac{\partial y}{\partial z} = 1/2i$; $\frac{\partial x}{\partial \bar{z}} = 1/2$; $\frac{\partial y}{\partial \bar{z}} = -1/2i$

$$\frac{\partial}{\partial z} = \frac{\partial}{\partial x} \cdot \frac{\partial x}{\partial z} + \frac{\partial}{\partial y} \cdot \frac{\partial y}{\partial z}$$

$$= \partial/\partial x (1/2) + \partial/\partial y (1/2i)$$

$$= 1/2 (\partial/\partial x - i\, \partial/\partial y)$$

$$\frac{\partial}{\partial \bar{z}} = \partial/\partial x \cdot \frac{\partial x}{\partial \bar{z}} + \frac{\partial}{\partial y} \cdot \frac{\partial y}{\partial \bar{z}}$$

$$= \partial/\partial x (1/2) + \partial/\partial y (-1/2i)$$

$$= 1/2 (\partial/\partial x + i\, \partial/\partial y)$$

$$\partial/\partial z \cdot \partial/\partial \bar{z} = 1/4 (\partial/\partial x - i\, \partial/\partial y)(\partial/\partial x + i\, \partial/\partial y)$$

$$= 1/4 \left(\frac{\partial^2}{\partial x^2} - i \frac{\partial^2}{\partial y \partial x} + i \frac{\partial^2}{\partial x \partial y} + \partial^2/\partial y^2\right)$$

$$= 1/4 (\partial^2/\partial x^2 + \partial^2/\partial y^2)$$

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4 \frac{\partial^2}{\partial z \partial \bar{z}}$$

$f(z) = u+iv$ is analytic

WKT $f(\bar{z}) = \overline{f(z)}$

$|f(z)|^2 = f(z)\overline{f(z)} = f(z) f(\bar{z})$

$\nabla^2 |f(z)|^2 = 4 \frac{\partial}{\partial z} \frac{\partial}{\partial \bar{z}} \left(f(z) f(\bar{z}) \right)$

$= 4 f'(z) f'(\bar{z})$ [$\because$ $f(z)$ is independent of $\bar{z}$ and $\overline{f(z)}$ is independent of z]

$= 4 f'(z) \overline{f'(z)}$

$= 4 |f'(z)|^2$ [$\because z\bar{z} = |z|^2$]

$\therefore \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) |f(z)|^2 = 4 |f'(z)|^2$ (or)

$\nabla^2 |f(z)|^2 = 4 |f'(z)|^2$

Hence the proof.//

6. If $f(z) = u+iv$ is an analytic function prove that

$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) |f(z)|^p = p^2 |f(z)|^{p-2} |f'(z)|^2$ (or)

$\nabla^2 |f(z)|^p = p^2 |f(z)|^{p-2} |f'(z)|^2$

Proof: Let $z = x+iy$ — ①

$\bar{z} = x-iy$ — ②

① + ② $\Rightarrow x = \frac{z+\bar{z}}{2}$; ① - ② $\Rightarrow y = \frac{z-\bar{z}}{2i}$

$\frac{\partial x}{\partial z} = 1/2$; $\frac{\partial x}{\partial \bar{z}} = 1/2$; $\frac{\partial y}{\partial z} = 1/2i$; $\frac{\partial y}{\partial \bar{z}} = -1/2i$

$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4 \frac{\partial}{\partial z} \frac{\partial}{\partial \bar{z}}$ — ③

$f(z) = u+iv$ is analytic

$\overline{f(z)} = f(\bar{z})$

$$|f(z)|^2 = f(z)\,\overline{f(z)} = f(z)\,f(\bar{z})$$

$$f(z) = [f(z)\,f(\bar{z})]^{1/2}$$

$$|f(z)|^p = [f(z)\,f(\bar{z})]^{p/2}$$

$$|f(z)|^p = [f(z)]^{p/2} \cdot [f(\bar{z})]^{p/2} \quad \text{—(4)}$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^p$$

$$= 4\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}}[|f(z)|^p]$$

$$= 4\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}}[f(z)]^{p/2}\,[f(\bar{z})]^{p/2}$$

$$= 4\frac{\partial}{\partial z}[f(z)]^{p/2}\,\frac{\partial}{\partial \bar{z}}[f(\bar{z})]^{p/2}$$

$$= 4\,p/2\,[f(z)]^{p/2-1}\,f'(z)\,\frac{p}{2}\,[f(\bar{z})]^{p/2-1}\,f'(\bar{z})$$

$$= p^2\,[f(z)\,f(\bar{z})]^{p/2-1}\,f'(z)\,f'(\bar{z})$$

$$= p^2\,[|f(z)|^2]^{p/2-1}\,|f'(z)|^2$$

$$= p^2\,[|f(z)|]^{p-2}\,|f'(z)|^2$$

$$\therefore \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^p = p^2\,[|f(z)|]^{p-2}\,|f'(z)|^2$$

Hence the Proof.

7. If $f(z) = u+iv$ is an analytic function, then prove that

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\log|f'(z)| = 0 \quad \text{(or)}$$

$$\nabla^2 \log|f'(z)| = 0.$$

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4 \frac{\partial^2}{\partial z \partial \bar{z}}$$

$$\therefore \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \log |f'(z)| = 4 \frac{\partial^2}{\partial z \partial \bar{z}} \log |f'(z)|$$

$$= 4 \frac{\partial^2}{\partial z \partial \bar{z}} \cdot \tfrac{1}{2} \log |f'(z)|^2$$

$$= 2 \frac{\partial^2}{\partial z \partial \bar{z}} \log \left[f'(z) \cdot f'(\bar{z}) \right]$$

$$= 2 \frac{\partial^2}{\partial z \partial \bar{z}} \left\{ \log f'(z) + \log f'(\bar{z}) \right\}$$

$\because \log mn = \log m + \log n$

$$= 2 \frac{\partial^2}{\partial z} \left[0 + \frac{1}{f'(\bar{z})} \cdot f''(\bar{z}) \right]$$

$$= 0$$

Construction of Analytic function

$f(z) = u + iv$

Milne's Thompson method:-

Case1: If real part (u) is given

$$f(z) = \int [\phi_1(z,0) - i\phi_2(z,0)]\, dz$$

$$\phi_1(x,y) = \frac{\partial u}{\partial x} \text{ and } \phi_2(x,y) = \frac{\partial u}{\partial y} \quad \text{— (A)}$$

In eqn (A) replace $x = z$ and $y = 0$

1. Show that the function $u(x,y) = 3x^2y + 2x^2 - y^3 - 2y^2$ is harmonic. Find the conjugate harmonic function v and express $u+iv$ is an analytic function of z.

proof

To Prove u is harmonic:

$$u = 3x^2y + 2x^2 - y^3 - 2y^2$$

$$\frac{\partial u}{\partial x} = 6xy + 4x \qquad \frac{\partial u}{\partial y} = -3y^2 - 4y + 3x^2$$

$$\frac{\partial^2 u}{\partial y^2} = 6y + 4 \qquad \frac{\partial^2 u}{\partial y^2} = -6y - 4$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 6y + 4 - 6y - 4 = 0$$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{Hence u is harmonic}$$

To find the analytic function:-

$$\phi_1(x,y) = \frac{\partial u}{\partial x}$$

$$= \phi_1(x,y) = 6xy + 4x$$

put $x=z, y=0$

$$\phi_1(z,0) = 4z$$

$$\phi_2(x,y) = \partial u/\partial y$$

$$\Rightarrow \phi_2(x,y) = 3x^2 - 3y^2 - 4y$$

put $x=z,\ y=0$

$$\phi_2(z,0) = 3z^2$$

$$f(z) = \int [\phi_1(z,0) - i\phi_2(z,0)]dz$$

$$= \int (4z - i3z^2)dz$$

$$= 4\frac{z^2}{2} \cdot i\frac{3z^3}{3} + c$$

$$\boxed{f(z) = 2z^2 - iz^3 + c}$$

To find the harmonic conjugate of v:

$$f(z) = 2z^2 - iz^3 + c$$

WKT $z = x+iy$

$$f(z) = 2(x+iy)^2 - i(x+iy)^3 + c$$

$$= 2(x^2 + i2xy - y^2) - i(x^3 + 3x^2iy - 3xy^2 - iy^3) + c$$

$$= 2x^2 + i4xy - 2y^2 - x^3i + 3x^2y + 3ixy^2 - y^3 + c$$

$$= (x^2 + 3x^2y - 2y^2 - y^3) + i(4xy - x^3 + 3xy^2) + c$$

$$\therefore \boxed{V = 4xy - x^3 + 3xy^2}$$

Case ii, If Imaginary part (v) is given

$$f(z) = \int [\psi_1(z,0) + i\psi_2(z,0)]dz$$

$$\psi_1(x,y) = \frac{\partial v}{\partial y} \quad ; \quad \psi_2(x,y) = \frac{\partial v}{\partial x} \quad - (B)$$

put $x = z$ and $y = 0$ in (B)

1. Show that $v = e^x(x\cos y - y\sin y)$ is a harmonic function. Find the analytic function for which v is the imaginary part.

proof:

To prove that v is harmonic function:

$$v = xe^x\cos y - e^x y\sin y$$

$$\frac{\partial v}{\partial x} = e^x\cos y + xe^x\cos y - e^x y\sin y$$

$$\frac{\partial^2 v}{\partial x^2} = e^x\cos y + e^x\cos y + xe^x\cos y - e^x y\sin y$$

$$\frac{\partial v}{\partial y} = -xe^x\sin y - e^x\sin y - e^x y\cos y$$

$$\frac{\partial^2 v}{\partial y^2} = -xe^x\cos y - e^x\cos y - e^x\cos y + e^x y\sin y$$

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 2e^x\cos y + xe^x\cos y - e^x y\sin y - xe^x\cos y$$
$$- 2e^x\cos y + e^x y\sin y$$

$$= 0$$

$$\therefore \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

$\therefore v$ is harmonic function.

To find the analytic function:

$$\psi_1(x,y) = \frac{\partial v}{\partial y}$$

$\psi_1(x,y) = -xe^x \sin y - e^x \sin y - e^x y \cos y$

put $x = z$, $y = 0$

$\psi_1(z,0) = e^z(0) = 0$

$\psi_2(x,y) = \frac{\partial v}{\partial x}$

$\Rightarrow \psi_2(x,y) = e^x \cos y + xe^x \cos y - e^x y \sin y$

put $x = z$, $y = 0$

$\psi_2(z,0) = e^z + ze^z$

$f(z) = \int [\psi_1(z,0) + i\psi_2(z,0)] dz$

$= \int (0 + i(1+z)e^z) dz$

$= \int i (e^z + ze^z) dz$

$= i [\int e^z dz + \int ze^z dz]$

$dv = e^z dz$

$u = z$ $\quad v = e^z$

$u' = 1$ $\quad v_1 = e^z$

$= i [e^z + (ze^z - e^z) + c]$

$= i [ze^z] + c$

$\boxed{f(z) = ize^z + c}$

To find the conjugate of u:

$u + iv = i(x+iy) e^{(x+iy)} + c$

$= (xi - y) e^x . e^{iy} + c$

$= (xi - y) e^x (\cos y + i \sin y) + c$

$= e^x (xi - y)(\cos y + i\sin y) + c$

$= e^x [ix\cos y - x\sin y - y\cos y - iy\sin y] + c$

$= e^x [-x\sin y - y\cos y] + ie^x (x\cos y - y\sin y) + c$

$\boxed{\therefore u = -e^x (x\sin y + y\cos y)}$

Case iii)

Given $u+v$ or $u-v$ where u and v are real and imaginary parts of $f(z)$

$F(z) = U + iV$

$(1+i) f(z) = u - v + i(u+v)$

If $u + v = V$, is given, use case ii, to construct $f(z)$

If $u - v = U$ is given, use case i, to construct $f(z)$

1. construct $f(z) = u + iv$ if $u - v = e^x(\cos y - \sin y)$

Soln

$u - v = e^x (\cos y - \sin y)$

To find the analytic function:

$f(z) = u + iv$

$if(z) = iu - v$

$f(z) + if(z) = u - v + i(u+v)$

$(1+i) f(z) = (u-v) + i(u+v)$

$F(z) = U + iV$

$u - v = U$

$U = e^x(\cos y - \sin y)$

$$\frac{\partial u}{\partial x} = e^x(\cos y - \sin y) = \phi_1(x,y)$$

$$\frac{\partial u}{\partial y} = e^x(-\sin y - \cos y) = \phi_2(x,y)$$

Put $x = z$ $y = 0$

$$\phi_1(z,0) = e^z(\cos 0 - \sin 0) = e^z$$

$$\phi_2(z,0) = e^z(-\sin 0 - \cos 0) = -e^z$$

$$F(z) = \int [\phi_1(z,0) - i\,\phi_2(z,0)]\, dz$$

$$= \int e^z - i \cdot e^{+z}\, dz$$

$$(1+i) f(z) = \int (1+i) e^z dz$$

$$\boxed{f(z) = e^z + c}$$

2. Construct $f(z) = u + iv$ if $u + v = (x-y)(x^2 + 4xy + y^2)$

Soln $u + v = (x-y)(x^2 + 4xy + y^2)$

$f(z) = u + iv$; $\quad if(z) = iu - v$

$$f(z) + if(z) = u - v + i(u+v)$$

$$(1+i) f(z) = u - v + i(u+v)$$

$$F(z) = U + iV$$

$$V = x^3 + 4x^2y + xy^2 - x^2y - 4xy^2 - y^3$$

$$V = x^3 + 3x^2y - 3xy^2 - y^3$$

$$\frac{\partial V}{\partial x} = 3x^2 + 6xy - 3y^2 = \psi_2(x,y)$$

$$\frac{\partial V}{\partial y} = 3x^2 - 6xy - 3y^2 = \psi_1(x,y)$$

put $x = z, y = 0$.

$\psi_1(z,0) = 3z^2$; $\psi_2(z,0) = 3z^2$

$F(z) = \int \psi_1(z,0) + i\psi_2(z,0)\, dz$

$(1+i) f(z) = \int (3z^2 + i3z^2)\, dz$

$(1+i) f(z) = (1+i) \int 3z^2\, dz$

$f(z) = 3\left(\frac{z^3}{3}\right) + c$

$\boxed{f(z) = z^3 + c}$

CROSS RATIO:

The cross-Ratio of the four points z_1, z_2, z_3, z_4 in the z-plane is defined by

$$\frac{(z_1-z_2)(z_3-z_4)}{(z_1-z_4)(z_2-z_3)}$$

BILINEAR TRANSFORMATION:-

The transformation $w = \frac{az+b}{cz+d}$ where a, b, c, d are complex constants $ad-bc \neq 0$ is called a bilinear transformation.

Fixed points of BILINEAR TRANSFORMATION:-

$w = \frac{az+b}{cz+d}$, put $w = z$

$z = \frac{az+b}{cz+d}$

$$cz^2 + dz = az + b$$

$$cz^2 + (d-a)z - b = 0 \quad \text{--- (1)}$$

If $c \neq 0$ then eqn (1) is a quadratic in z, which has two points and so there are two fixed points.

If $c = 0$, $d \neq a$ then there is one fixed point.

1. Determine the billinear transformation that maps that point $-1, 0, 1$ in the z-plane on the points $0, i, 3i$, in the w-plane.

Soln

Let $z_1 = -1$; $z_2 = 0$; $z_3 = 1$

$w_1 = 0$; $w_2 = i$; $w_3 = 3i$

$$\frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w-w_3)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_1-z_2)}$$

$$\frac{(w-0)(i-3i)}{(w-3i)(0-i)} = \frac{(z+1)(0-1)}{(z-1)(-1-0)}$$

$$\frac{w \times 2\cancel{i}}{(w-3i)(-\cancel{i})} = \frac{(z+1)(\cancel{-1})}{(z-1)(\cancel{-1})}$$

$$\frac{2w}{w-3i} = \frac{z+1}{z-1}$$

$$2w(z-1) = (w-3i)(z+1)$$

$$2wz - 2w = wz + w - 3iz - 3i$$

$$2wz - wz - 2w - w = -3i(z+1)$$

$$wz - 3w = -3i(z+1)$$

$$w(z-3) = -3i(z+1)$$

$$\boxed{w = \frac{-3i(z+1)}{3-z}}$$

Theorem:-

A billinear transformation is invariant under cross ratios

i.e., $$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_1-w_2)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_1-z_2)}$$

2. Find the BLT that maps $z=0,-1,i$ onto the points $w=i, 0, \infty$

Soln Given $z_1=0$, $z_2=-1$, $z_3=i$

$w_1=i$, $w_2=0$, $w_3=\infty$, $\frac{1}{w_3}=0$

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_1-w_2)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_1-z_3)}$$

$$\frac{(w-w_1)\left[w_3\left(\frac{w_2}{w_3}-1\right)\right]}{w_3\left[\frac{w}{w_3}-1\right]\left[w_1-w_2\right]} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_1-z_2)}$$

$$\frac{(w-i)(0-1)}{(0-1)(i-0)} = \frac{(z-0)(-1-i)}{(z-i)(0+1)}$$

$$\frac{w-i}{i} = \frac{z(-1-i)}{(z-i)(+1)} = \frac{-z(1+i)}{(z-i)(+1)}$$

$$\frac{w-i}{i} = \frac{-z(1+i)}{z-i}$$

$(w-i)(z-i) = -zi(1+i)$

$wz - iz - iw - 1 = -zi + z$

$w(z-i) = -zi + z + zi + 1$

$$\boxed{w = \frac{z+1}{z-i}}$$

3. Find the BLT which maps the point $z = 0, 1, \infty$ into $w = 5, -1, 3$. respectively what are the invariant points of this transformation?

Soln

$z_1 = 0$; $z_2 = 1$; $z_3 = \infty$; $\frac{1}{z_3} = 0$

$w_1 = -5$; $w_2 = -1$, $w_3 = 3$

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_1-w_2)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_1-z_2)}$$

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_1-w_2)} = \frac{(z-z_1)\, z_3(z_2/z_3 - 1)}{z_3(\frac{z}{z_3}-1)(z_1-z_2)}$$

$$\frac{(w+5)(-1-3)}{(w-3)(-5+1)} = \frac{(z-0)(0-1)}{(0-1)(0-1)}$$

$$\frac{(w+5)(-4)}{(w-3)(-4)} = \frac{z}{-1}$$

$-(w+5) = (w-3)z$

$-w-5 = wz - 3z$

$w + 5 = 3z - wz$

$w + wz = 3z - 5$

$w(1+z) = 3z - 5$; $\boxed{w = \frac{3z-5}{1+z}}$

To find the invariant points:

$$w = \frac{3z-5}{1+z}$$

put $w = z$; $z = \frac{3z-5}{1+z}$

$z(1+z) = 3z-5$

$z + z^2 - 3z + 5 = 0 \Rightarrow z^2 - 2z + 5 = 0$

$a = 1$; $b = -2$; $c = 5$.

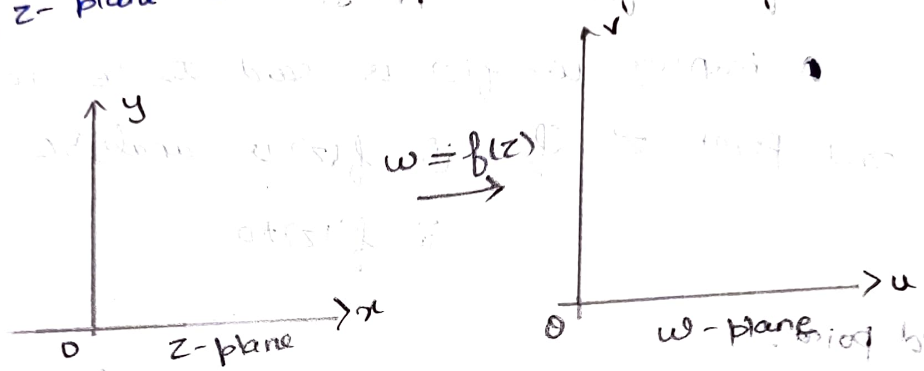
$$z = \frac{+2 \pm \sqrt{4 - 4 \cdot 1 \cdot 5}}{2 \cdot 1} = \frac{2 \pm \sqrt{4-20}}{2}$$

$$z = \frac{2 \pm 4i}{2} = \frac{\cancel{2}(1+2i)}{\cancel{2}} = 1 \pm 2i$$

invariant points are $1+2i$ & $1-2i$

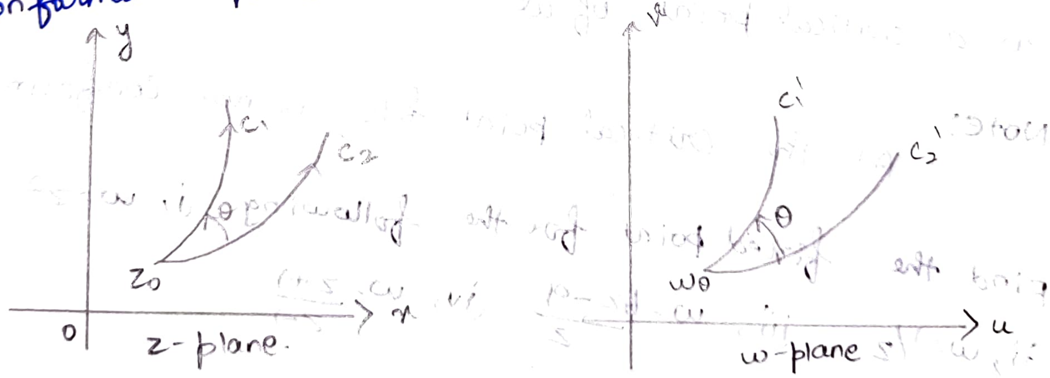
Transformation or Mappings:-

The point $z=x+iy$ is plotted in z-plane and the corresponding point $w=u+iv$ is plotted in w-plane. Thus the function $w=f(z)$ defines a relationship between the points of these two planes. Hence, the function $w=f(z)$ represents a transformation of z-plane into w-plane.



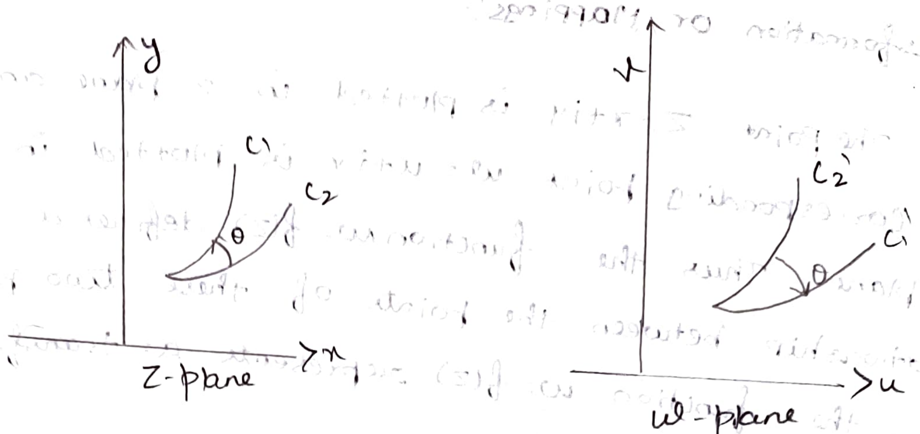
Conformal mapping:-

A mapping (transformation) which preserves the angles both in direction and in direction at the point of intersection of every pairs of curves is conformal mapping.



Isogonal mapping:-

A mapping (transformation) which preserves the angle in magnitude only not in direction at the point of intersection of every pairs of curves is called isogonal mapping.



Condition for conformal mapping:-

A mapping $w = f(z)$ is said to be conformal at each point 'z' if i, $f(z)$ is analytic

ii $f'(z) \neq 0$

Fixed point:

If $f(z) = 0$ (or) $w = z$, then z is fixed point of $f(z)$

Critical point:

If $f'(z) = 0$ (or) $\frac{dw}{dz} = 0$, then z is called as a critical point of w.

Note:

At the critical point $f(z)$ is not conformal.

1. Find the fixed point for the following i, $w = z^2$

ii, $w = 1/z$ iii, $w = \frac{6z-9}{z}$ iv, $w = \frac{z+1}{z-1}$

soln i, $w = z^2$

put $z = w$; $z = z^2$

$(z^2 - z) = 0$

$z(z-1) = 0$

$z = 0$ & $z = 1$

$\therefore$ The fixed points are 0 and 1

ii) $w = 1/z$

put $w = z$; $z = 1/z$

$z^2 - 1 = 0$

$z^2 = 1$

$z = \pm 1$

The fixed points are 1 and -1

iii) $w = \frac{6z - 9}{z}$

put $w = z$

$z = \frac{6z-9}{z} \Rightarrow z^2 - 6z + 9 = 0 \Rightarrow (z-3)^2 = 0$

$z = 3, 3$ Fixed point is 3.

iv) $w = \frac{z+1}{z-1}$

put $w = z$

$z(z-1) = z + 1$

$z^2 - z - z - 1 = 0$

$z^2 - 2z - 1 = 0$

$z = \frac{2 \pm \sqrt{4 - 4(1)(-1)}}{2} \Rightarrow \frac{2 \pm \sqrt{8}}{2} = \frac{2 \pm 2\sqrt{2}}{2} = 1 \pm \sqrt{2}$

The fixed points are $1+\sqrt{2}$ and $1-\sqrt{2}$

2. Find the cirtical points for i) $w = z^2$ ii) $w = z + \frac{k^2}{z}$
iii) $w^2 = (z-\alpha)(z-\beta)$

soln

i) $w = z^2$

$\frac{dw}{dz} = 2z$; $\frac{dw}{dz} = 0$

$2z = 0 \Rightarrow z = 0$ c.p is $z = 0$ //.

ii) $w = z + \frac{k^2}{z}$

$$\frac{dw}{dz} = 1 - \frac{k^2}{z^2}$$

$$\frac{dw}{dz} = 0 \Rightarrow 1 - \frac{k^2}{z^2} = 0$$

$$1 = \frac{k^2}{z^2} \Rightarrow z^2 - k^2 = 0 \Rightarrow \boxed{z = \pm k}$$

$\therefore$ The critical points are k and $-k$.

iii) $w^2 = (z-\alpha)(z-\beta)$

$$2w \cdot \frac{dw}{dz} = z - \alpha + z - \beta$$

$$\frac{dw}{dz} = 0 \Rightarrow z - \alpha + z - \beta = 0$$

$$2z - \alpha - \beta = 0 \Rightarrow 2z = \alpha + \beta$$

$$z = \frac{\alpha+\beta}{2}$$

$\therefore$ The critical point is $\frac{\alpha+\beta}{2}$

Mapping by Elementary function Translation:-

The mapping $w = z + b$ where b is a complex constant, is a translation by means of the vector representing b.

Let $z = x + iy$, $w = u + iv$, $b = b_1 + ib_2$

$$w = z + b$$

$$u + iv = x + iy + b_1 + ib_2$$

$$u + iv = (x + b_1) + i(y + b_2)$$

The image of any point in the z-plane is the point $(x+b_1, y+b_2)$ in the w-plane. If we assumed the w-plane is super on z-plane, the figure is shifted through a distance given by the vector b. But the size and shape remains same. So circles are transformed into circles, rectangles are transformed into rectangles etc.

Magnification:-

The transformation $w=az$, a is real constant, represents magnification. Let $w=u+iv$, $z=x+iy$

$w=az$

$u+iv = a(x+iy)$

$u+iv = ax+iay$

$\therefore$ The image of the point (x,y) is the point (ax, ay). Hence the size of any figure in the z-plane is magnified 'a' times but there is no change shape and orientation.

Rotation:-

The transformation $w=az$, where a is complex constant.

Let $z=re^{i\theta}$, $w=Re^{i\phi}$, $a=\rho e^{i\alpha}$

$w=az$

$Re^{i\phi} = \rho e^{i\alpha} re^{i\theta}$

$Re^{i\phi} = \rho r e^{i(\theta+\alpha)}$

$R=\rho r$ (or) $\phi=\theta+\alpha$

Thus the point (r,θ) in the z-plane is mapped onto $(\rho r, \theta+\alpha)$ in the w-plane.

1. Find the image of the circle $|z|=2$ under the transformation i, $w=2z$ ii, $w=(1+2i)z+3+4i$.

Soln Given $|z|=2$ — ①

$x^2+y^2=4$

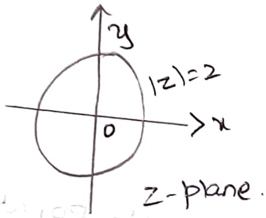
i, $w=2z$

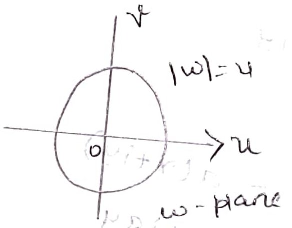
$|w| = 2|z| = 2\cdot 2 = |w| = 4$

$\Rightarrow \sqrt{u^2+v^2}=4$

$u^2+v^2=4^2$

$u^2+v^2=16$





ii, $w=(1+2i)z+(3+4i)$

Soln $w=(1+2i)z+(3+4i)$

$w-3-4i=(1+2i)z$

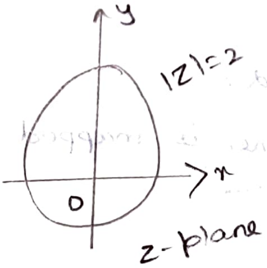
$|w-3-4i| = |1+2i|\,|z|$

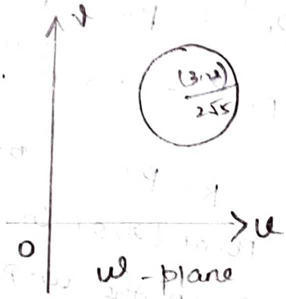
$|u+iv-3-4i| = |1+2i|\,2$

$|u-3+i(v-4)| = \sqrt{1+2^2}\cdot 2$

$\sqrt{(u-3)^2+(v-4)^2} = 2\sqrt{5}$

$(u-3)^2+(v-4)^2 = 4\cdot 5 = 20$





2. Find the image of the square region with vertices $(0,0)$, $(2,0)$, $(2,2)$, $(0,2)$ under the transformation $w = (1+i)z + (2+i)$

Soln

$$w = (1+i)z + (2+i)$$

$$\therefore w = (1+i)(x+iy) + (2+i)$$

At $A(0,0)$; $w = 2+i$; $A' = (2,1)$

At $B(2,0)$; $w = (1+i)(2+i0) + 2+i$

$$= 2+2i+2+i$$

$$= 4+3i$$

$B'(4,3)$

At $(2,2)$; $w = (1+i)(2+2i) + 2+i$

$$= 2(1+i)^2 + 2+i$$

$$= 2(1+2i-i^2) + 2+i$$

$$= 4i+2+i$$

$$= 2+5i$$

$$= C'(2,5)$$

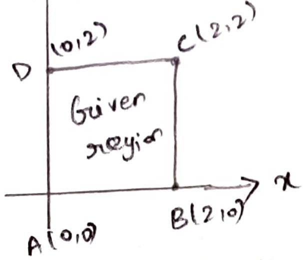
At $D(0,2)$; $w = (1+i)(0+2i) + (2+i)$

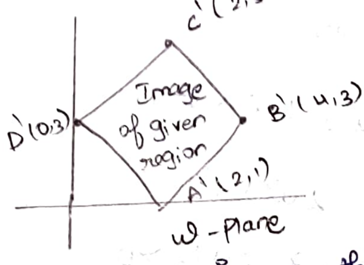
$$= (1+i)2i + 2+i$$

$$= 2i-2+2+i$$

$$= 3i$$

$D'(0,3)$





$\therefore$ Image of the square ABCD is another square A'B'C'D'

3. Find the images of the following under the map $w = 1/z$ i) $1 < x < 2$ ii) $1/4 < y < 1/2$. Also show the region graphically.

Sol: Given $w = 1/z$; $z = 1/w$

$$x + iy = \frac{1}{u+iv} \times \frac{u-iv}{u-iv}$$

$$= \frac{u-iv}{u^2+v^2}$$

$$\therefore x = \frac{u}{u^2+v^2} \quad ; \quad y = \frac{-v}{u^2+v^2}$$

i) To find the image of $1 < x < 2$

Consider $1 < x$; $\Rightarrow 1 < \frac{u}{u^2+v^2}$

$\therefore u^2+v^2 < u$

$u^2+v^2-u < 0$

$u^2 - 2 \cdot u \cdot 1/2 + (1/2)^2 - (1/2)^2 + v^2 < 0$

$(u-1/2)^2 - (1/2)^2 + v^2 < 0$

$(u-1/2)^2 + v^2 < (1/2)^2$

This region is the interior of the circle $(u-1/2)^2 + v^2 = (1/2)^2$ with center $(1/2, 0)$ and radius $1/2$

Consider $x < 2$

$$\frac{u}{u^2+v^2} < 2$$

$u < 2(u^2+v^2)$

$\frac{u}{2} < u^2+v^2$

$u^2+v^2 - u/2 > 0$

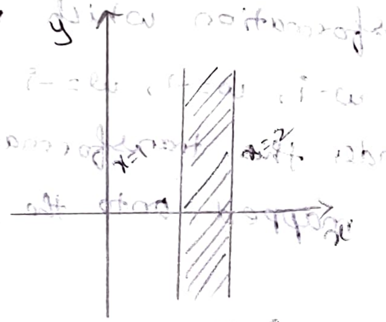
$$u^2 - 2\cdot u \, \tfrac{1}{4} + (\tfrac{1}{4})^2 - (\tfrac{1}{4})^2 + v^2 > 0$$

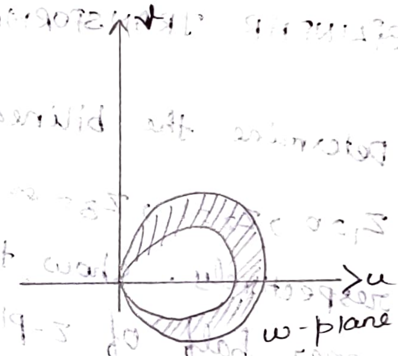
$$(u-\tfrac{1}{4})^2 - (\tfrac{1}{4})^2 + v^2 > 0$$

$$(u-\tfrac{1}{4})^2 + v^2 > (\tfrac{1}{4})^2$$

The region is the exterior of the circle $(u-\frac{1}{4})^2 + v^2 = (\frac{1}{4})^2$ with center $(\frac{1}{4}, 0)$ and radius $= \frac{1}{4}$

Therefore the image of the radius annulus $1 < x < 2$ is the region b/w the circles $(u-\frac{1}{2})^2 + v^2 = (\frac{1}{2})^2$ and $(u-\frac{1}{4})^2 + v^2 = (\frac{1}{4})^2$





(ii) To find image of $\frac{1}{4} < y < \frac{1}{2}$

$$\tfrac{1}{4} < \frac{-v}{u^2+v^2}$$

$$u^2+v^2 < -4v$$

$$u^2+v^2+4v < 0$$

$$u^2+v^2+4v+2^2-2^2 < 0$$

$$u^2+(v+2)^2-2^2 < 0$$

$$u^2+(v+2)^2 < 2^2$$

This is the region interior of the circle $u^2+(v+2)^2=2^2$ with the center $(0,-2)$ & radius $=2$

$$\tfrac{1}{2} > \frac{-v}{u^2+v^2}$$

$$-2v < u^2+v^2$$

$$0 < u^2+v^2+2v$$

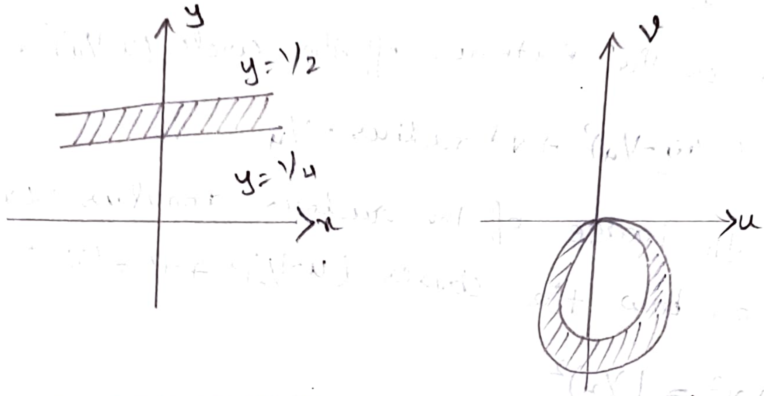
$$u^2+v^2+2v > 0$$

$$u^2+v^2+2v+1-1 > 0$$

$$u^2+(v+1)^2 > 1$$

This is the region exterior of the circle $u^2+(v+1)^2=1$ with center $(0,-1)$ and radius $r=1$

Thus the image of the annulus $1/4 < y < 1/2$ is the region b/w the circle is the image b/w the circles $u^2+(v+2)^2 = 2^2$ and $u^2+(v+1)^2 = 1$



BILINEAR TRANSFORMATION USING MAPPING.

1. Determine the bilinear transformation which maps $z_1 = 0, z_2 = 1, z_3 = \infty$ onto $w = i, w = -1, w = -i$ respectively. show that under this transformation the upper half of z-plane is mapped onto the interior of the unit circle.

Soln

Given $z_1 = 0$; $z_2 = 1$; $z_3 = \infty$; $1/z_3 = 0$

$w_1 = i$; $w_2 = -1$; $w_3 = -i$

$$\frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_1-z_2)} = \frac{(w-w_1)(w_2-w_3)}{(w-w_2)(w_1-w_2)}$$

$$\frac{(z-z_1)\, z_3(z_2/z_3-1)}{z_3(z/z_3-1)(z_1-z_2)} = \frac{(w-w_1)(w_2-w_3)}{(w-w_1)(w_1-w_2)}$$

$$\frac{(z-z_1)(0-1)}{(0-1)(z_1-z_2)} = \frac{(w-w_1)(w_2-w_3)}{(w-w_2)(w_1-w_2)}$$

$$\frac{(z-0)}{0-1} = \frac{(w-i)(-1+i)}{(w+i)(i+1)}$$

$$-z = \frac{(w-i)(-1+i)}{(w+i)(i+1)}$$

$$-z = \frac{(w-i)(-1+i)}{(w+i)(i+1)} \cdot \frac{(+i-1)}{(i-1)}$$

$$= \frac{(w-i)(i-1)^2}{(w+i)(i^2-1)}$$

$$= \frac{(w-i)}{(w+i)} \cdot \frac{(i-2i+1)}{-2}$$

$$-z = \frac{w-i}{w+i}\left(\frac{\cancel{-2}i}{\cancel{2}}\right)$$

$$z = \frac{i(w-i)}{w+i}$$

$$z(w+i) = iw+1$$

$$zw+zi = iw+1$$

$$zw+iw = zi+1$$

$$w(z+i) = zi+1$$

$$w = \frac{zi+1}{z+i} = \frac{zi-i^2}{z+i} = \frac{i(z-i)}{z+i}$$

$z=i$ is the point on the upper half of z-plane

$\therefore w=0$

$|w| = |0| = 0<1$

$|w|<1$

$\therefore$ The upper half of z-plane is mapped into the interior of the unit circle.

Cauchy's Integral theorem:-

Statement:

If $f(z)$ is analytic and $f'(z)$ is continuous on and inside a simple closed curve C, then

$$\oint_C f(z)\,dz = 0.$$

UNIT-4

Extra Questions:-

The theory of functions of a complex variable is most important in solving a large number of Engineering and science problems. Many complicated integrals of real functions are solved with the help of complex variable

$x+iy$ is a complex variable and its is denoted by z ie) $z=x+iy$ where $i=\sqrt{-1}$

i) single valued Function:-

If for each value of z in R there is correspondingly only one value of w, then w is called a single valued function of z.

eg: $w=z^2$; $w=1/z$

ii) Multiple valued Function:-

If there is more then one value of w corresponding to a given value of z, then w is called a multiple-valued function.

eg: $w=z^{1/2}$

Note: $C: |z-z_0|=\delta$; $C: |z-z_0|<\delta$

$C: |z-z_0|\leq\delta$; $C: |z-z_0|>\delta$

$C: |z|=1$

Limit of a Function:-

Let $f(z)$ be a single valued function defined at all points in some neighbourhood of point z_0

Then the limit of $f(z)$ as z approaches z_0 is w_0 ie $\lim_{z\to z_0} f(z) = w_0$

Continuity:

If $f(z)$ is said to be continuous at $z=z_0$

then $\lim_{z \to z_0} f(z) = f(z_0)$

Note:

If two functions are continuous at a point then their sum, product and quotient also continuous provided dr $\neq 0$ at any such point.

1. S.T the function $f(z) = \frac{z^2+4}{z-2i}$ is continuous at $z=2i$.

Sol.

Given $f(z) = \frac{z^2+4}{z-2i} = \frac{(z+2i)\cancel{(z-2i)}}{\cancel{(z-2i)}} = z+2i$

To prove $f(2i) = \lim_{z \to 2i} f(z)$

LHS: $\lim_{z \to 2i} f(z) = \lim_{z \to 2i} (z+2i) = 4i$

RHS: $f(2i) = 2i + 2i = 4i$

Hence $f(z)$ is continuous at $z=2i$

2. S.T $f(z) = \frac{\bar{z}}{z}$ is not continuous at $z=0$.

Sol:

Let $z = x+iy$; $\bar{z} = x-iy$

Suppose $z \to 0$ along x-axis, then we have $y=0$ (ie)

$z = x$; $\bar{z} = x$

$\therefore \lim_{z \to 0} \frac{\bar{z}}{z} = \frac{x}{x} = 1$

Suppose $z \to 0$ along y-axis, then $x=0$ (ie)

$z = iy$; $\bar{z} = -iy$

$\therefore \lim_{z \to 0} \frac{\bar{z}}{z} = \frac{-iy}{iy} = -1$

3. If Is the function $f(z) = \begin{cases} \frac{z^2 + 3iz - 2}{z + i} & \text{for } z \neq i \\ 5 & \text{for } z = -i \end{cases}$

Continuous? If not, can the function be refined to make it continuous at $z = -i$?

Soln Here $f(z) = \frac{g(z)}{h(z)}$ is continuous when $g(z)$ is & $h(z)$ are continuous except at $h(z) = 0$

So, $f(z)$ is continuous every where except at $z = -i$ since $g(z)$ and $h(z)$ are continuous.

Now, continuity at $z = -i$; $(x = 0 \; ; y = -1)$

$$\lim_{z \to -i} f(z) = \lim_{\substack{x \to 0 \\ y = -1}} \frac{(x+iy)^2 + 3i(x+iy) - 2}{(x + iy + i)}$$

$$= \frac{(-i)^2 + 3i(-i) - 2}{-i + i} = \lim_{\substack{x \to 0 \\ y \to -1}} \frac{2(x+iy) + 3i}{1}$$

$$= \frac{-1}{i} = i$$

But $f(-i) = 5 \neq i$

$\therefore f$ is not continuous at $z = -i$

Suppose we define $f(z)$ as $f(-i) = i$ instead of 5, then $f(z)$ is continuous at $z = i$ and is therefore continuous every where.

Differentiability at a point:

A function $f(z)$ is said to be differentiable at a point $z=z_0$ if the limit

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0+\Delta z) - f(z_0)}{\Delta z} \quad \text{exists.}$$

This limit is called the derivative of $f(z)$ at the point $z=z_0$

Note: If $f(z)$ is differentiable at z_0, then $f(z)$ is continuous at z_0 is called the necessary condition for differentiability. But the converse is not true.

Eg: 1 $\quad f(z) = xy + iy$

ie $u = xy$; $v = y$.

Analytic function: (Holomorphic or Regular)

A function is said to be analytic at a point if its derivative exists not only at that point but also in some neighbourhood of that point.

Note: i) The necessary condition for a complex function $f(z) = u(x,y) + iv(x,y)$ to be analytic $f(z) = u(x,y) + iv(x,y)$ to be analytic in a region R are

$u_x = v_y$ and $u_y = -v_x$ Called the C-R equation (Cauchy Reimann eqn)

ii) sufficient condition for $f(z)$ to be analytic is "If the partial derivatives u_x, u_y, v_x and v_y are all continuous in D and $u_x = v_y$ & $u_y = -v_x$ then the function $f(z)$ is ananalytic in a domain D.

Polar form of CR Equation:

In Polar Coordinates, $z = re^{i\theta}$ where r is the modulus and θ is the argument, then

$U_r = \frac{1}{r} V_\theta$; $u_\theta = -r V_r$.

① S.T $f(z) = xy + iy$ is continuous everywhere but is not analytic.

Soln

$$f(z) = xy + iy$$

$u = xy$ $\quad$ $v = y$

$u_x = y$ $\quad$ $v_x = 0$

$u_y = x$ $\quad$ $v_y = 1$

$u_x \neq v_y$ & $u_y \neq -v_x$ (ie) C-R eqn is not satisfied

$\Rightarrow f(z)$ is not analytic.

But $f(z) = xy + iy$ is continuous at all points of z since the limit of the function can exists at all points of z.

2. Check the analyticity of $|z|^2$.

Soln

Let $z = x + iy \Rightarrow \bar{z} = x - iy$

$$|z|^2 = z \cdot \bar{z} = x^2 + y^2$$
$$= (x^2 + y^2) + i(0)$$

$\Rightarrow u = x^2 + y^2$ $\quad$ $v = 0$

$u_x = 2x$ $\quad$ $v_x = 0$

$u_y = 2y$ $\quad$ $v_y = 0$

$\Rightarrow u_x \neq v_y$ & $v_y \neq -v_x$

(ie) C-R eqn is not satisfied everywhere except at $z = 0$.

But the function $f(z) = u+iv$ are continuous everywhere. (since u & v are continuous).

This shows that $f(z) = |z|^2 = x^2+y^2$ is "a real valued function and can not be analytic".

Result: $f(z) = |z|^2$ is continuous everywhere

$f(z) = |z|^2$ is differentiable only at $z=0$ $f(z) = |z|^2$ is not analytic.

3. ST $f(z) = z^3$ is analytic for all z.

soln.

$$\text{Let } f(z) = z^3$$

$$u+iv = (x+iy)^3$$

$$= x^3 - iy^3 + i3x^2y - 3xy^2$$

$$= (x^3 - 3xy^2) + i(3x^2y - y^3)$$

$\Rightarrow u = x^3 - 3xy^2$ | $v = 3x^2y - y^3$

$u_x = 3x^2 - 3y^2$ | $v_x = 6xy$

$u_y = -6xy$ | $v_y = 3x^2 - 3y^2$

Now, $u_x = v_y$ & $u_y = -v_x$

C-R eqn satisfied. Hence $f(z)$ is analytic

4) Check the analyticity of $\log z$.

soln

$$\text{Let } z = re^{i\theta}$$

$$f(z) = \log z = \log(re^{i\theta})$$

$$= \log r + \log(e^{i\theta})$$

$$u+iv = \log r + i\theta$$

$u = \log r$ | $v = \theta$

$u_r = 1/r$ | $v_r = 0$

$u_\theta = 0$ | $v_\theta = 1$

Now, $u_r = \frac{1}{r} v_\theta$; $u_\theta = -r v_r$

$\Rightarrow$ C-R eqn is satisfied and hence $f(z)$ is analytic.

Here $u_r = \frac{1}{r}$ $(r \neq 0)$ is continuous except at $r=0$.

So $f(z)$ is not defined at the origin ie, at $r=0$

$f(z) = -\infty + i0$

$$[\text{W.K.T } e^{-\infty} = 0$$

$$\log e^{-\infty} = \log 0$$

$$-\infty = \log 0]$$

and hence it is not differentiable at $z=0$

Result: $f(z) = \log z$ is analytic except at $z=0$

5. Check whether $w = \bar{z}$ is analytic everywhere.

Let $w = f(z) = \bar{z} = x - iy$

$u + iv = x - iy$

$u = x$ $\quad$ $v = -y$

$u_x = 1$ $\quad$ $v_x = 0$

$u_y = 0$ $\quad$ $v_y = -1$

Now, $u_x \neq v_y$ and $u_y = -v_x$

$\Rightarrow$ C-R eqn is not satisfied

Hence $f(z) = w = \bar{z}$ is not analytic.

6. Test the analyticity of $w = \sin z$.

Let $w = f(z) = \sin z$

$u + iv = \sin(x+iy)$

$= \sin x \cos iy + \cos x \sin iy$

$= \sin x \cosh y + i \cos x \sinh y.$

$u = \sin x \cosh y$ $\quad$ $v = \cos x \sinh y$

$u_x = \cos x \cosh y$ $\quad$ $v_x = -\sin x \sinh y$

$u_y = \sin x \sinh y$ $\quad$ $v_y = \cos x \cosh y$

$u_x = v_y$ and $u_y = -v_x$

C-R eqn is satisfied

Hence $f(z) = \sin z$ is analytic.

7. S.T $f(z) = 1/z$ is analytic everywhere except at $z=0$.

$f(z) = 1/z$ and $z = re^{i\theta}$

$f(z) = \frac{1}{re^{i\theta}} = \frac{1}{r} e^{-i\theta} = \frac{1}{r}(\cos\theta - i\sin\theta)$

at $r=0$; $f(z)$ is not defined. Hence $f(z)$ is not analytic at $z=0$.

Now, $f(z) = \frac{1}{r}\cos\theta - i\frac{1}{r}\sin\theta$

$u = \frac{1}{r}\cos\theta$	$v = \frac{-1}{r}\sin\theta$
$u_r = \frac{-1}{r^2}\cos\theta$	$v_r = \frac{1}{r^2}\sin\theta$
$u_\theta = \frac{-1}{r}\sin\theta$	$v_\theta = \frac{-1}{r}\cos\theta$
$u_r = \frac{1}{r}v_\theta$	$u_\theta = -rv_r$.

C-R eqn is satisfied. $\therefore$ $f(z)$ is analytic except at $z=0$.

8. Find the constants a, b, c if $f(z) = x + ay + i(bx + cy)$ is analytic.

Soln

G.T $f(z) = x + ay + i(bx + cy)$ is analytic

$\Rightarrow u + iv = (x + ay) + i(bx + cy)$ is analytic

$u = x + ay$; $v = bx + cy$ satisfies the C-R eqn ie, $u_x = v_y$ & $u_y = -v_x$

$\therefore$ $u_x = 1$; $u_y = a$ & $v_x = b$ $v_y = c$

$\therefore$ ~~$u_x = 1$~~ Now $c = 1$; $a = -b$

9. Test $f(z) = e^{-x}(\cos y - i\sin y)$ is analytic or not.

Soln

G.T $f(z) = e^{-x}(\cos y - i\sin y)$

$u + iv = e^{-x}\cos y - ie^{-x}\sin y$

$\Rightarrow u = e^{-x}\cos y \quad \& \quad v = -e^{-x}\sin y$

Now, $u_x = -e^{-x}\cos y \;;\quad u_y = -e^{-x}\sin y$

$v_x = e^{-x}\sin y \;;\quad v_y = -e^{-x}\cos y.$

$\Rightarrow u_x = v_y \quad \& \quad u_y = -v_x$

$\Rightarrow$ C-R eqn is satisfied, Hence $f(z)$ is analytic.

10. Test $f(z) = \frac{1}{2}\log(x^2+y^2) + i\tan^{-1}(y/x)$ is analytic or not.

G.T $f(z) = \frac{1}{2}\log(x^2+y^2) + i\tan^{-1}(y/x)$

$\Rightarrow u + iv = \frac{1}{2}\log(x^2+y^2) + i\tan^{-1}(y/x)$

ie $u = \frac{1}{2}\log(x^2+y^2) \;;\quad v = \tan^{-1}(y/x)$

$u_x = \frac{1}{2}\cdot\frac{1}{x^2+y^2}(2x) \;;\quad v_x = \frac{1}{1+y^2/x^2}(-y/x^2)$

$= \frac{x}{x^2+y^2} \qquad\qquad = \frac{-y}{x^2+y^2}$

$u_y = \frac{1}{2}\cdot\frac{1}{x^2+y^2}(2y) \qquad v_y = \frac{1}{1+y^2/x^2}(1/x)$

$= \frac{y}{x^2+y^2} \qquad\qquad = \frac{x}{x^2+y^2}$

Now, $u_x = v_y$ and $u_y = -v_x$

$\Rightarrow$ C-R eqn is satisfied

Hence $f(z)$ is analytic.

Properties of an analytic function:

Property: 1 An analytic function whose real part is constant must ~~itself~~ itself a constant.

Proof: G.T real part of an analytic function is constant (ie) $u =$ constant.

T.P $f(z)$ is constant

Now, $u = c$

$u_x = 0$; $u_y = 0$

By C-R eqn $u_x = 0 = v_y$

$u_y = 0 = v_x$

$\Rightarrow v$ is a constant

$\Rightarrow f(z) = u + iv$ is constant.

Property: 2

If $f(z)$ and $\overline{f(z)}$ are analytic in a region D. S.T $f(z)$ is constant in that region D.

proof: Let $f(z) = u(x,y) + iv(x,y)$ — ①

$\overline{f(z)} = u(x,y) - iv(x,y)$ — ②

Since $f(z)$ and $\overline{f(z)}$ are analytic in D

we get ① $\Rightarrow u_x = v_y$ & $u_y = -v_x$

② $\Rightarrow u_x = -v_y$ & $u_y = v_x$

Adding these, we get $2u_x = 0$ & $2u_y = 0$

$\Rightarrow u_x = 0$ & $u_y = 0$

$\Rightarrow u =$ Const & by C-R eqn $v_y = 0$

$\Rightarrow v =$ Const

$\therefore f(z)$ is constant in D.

Property: 3

An analytic function with constant modulus is constant.

Proof:

Let $f(z) = u + iv$ be analytic By C-R equation

$u_x = v_y$ & $u_y = -v_x$

G.T $|f(z)| =$ const

$\Rightarrow \sqrt{u^2+v^2} = c \neq 0$

$u^2+v^2 = c^2$

D.w.r.to x & y

$2uu_x + 2vv_x = 0$ & $2uu_y + 2vv_y = 0$

$uu_x + vv_x = 0$ — (1) & $uu_y + vv_y = 0$

$-uv_x + vu_x = 0$ — (2) By (C.R)

(1) $\times u$ + (2) $\times v \Rightarrow u^2u_x + uvv_x - u^2v_x + uvu_x = 0$

$u^2(u_x - v_x) = 0$.

$(u^2+v^2)\frac{\partial u}{\partial x} = 0$ $\Rightarrow u_x = 0$

$\Rightarrow u =$ Const

$|||^{ly}$ (1) $\times v$ - (2) $\times u \Rightarrow (u^2+v^2)\frac{\partial v}{\partial x}=0$

$\frac{\partial v}{\partial x}=0 \Rightarrow v = const$

$\therefore f(z)=u+iv$ is also constant.

Harmonic and conjugate Harmonic Functions:-

* Let $f(z)=u+iv$ is analytic, then u and v possesses continuous second order pp and that satisfies Laplace equations

(ie) $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$ & $\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=0$ is called a Harmonic function of u & v respectively

* If u and v are harmonic functions such that $u+iv$ is analytic, then each is called the conjugate harmonic function of the other

Verify that the families of curves $u=c_1$, and $v=c_2$ cut each other orthogonally when $w=z^3$.

Soln

Let $f(z)=z^3 \Rightarrow u+iv=(x+iy)^3 = x^3+3ix^2y-iy^3$

$\Rightarrow u=x^3-3xy^2=c_1$; $v=3x^2y-y^3=c_2$

Differentiate with respect to x, we have

$u=x^3-3xy^2$

$3x^2-3\left(y^2+2xy\frac{dy}{dx}\right)=0$

$\frac{dy}{dx}=\frac{3(y^2-x^2)}{6xy}=m_1$

$v=3x^2y-y^3$

$3\left(2xy+x^2\frac{dy}{dx}\right)-3y^2\frac{dy}{dx}=0$

$6xy+3x^2\frac{dy}{dx}-3y^2\frac{dy}{dx}=0$

$\frac{dy}{dx}=\frac{-6xy}{3(y^2-x^2)}=m_2$

$m_1m_2=\frac{y^2-x^2}{2xy}\times\frac{-2xy}{y^2-x^2}=-1$

$\therefore$ Hence the curve $u=c_1$ & $v=c_2$ cut each other orthogonally.

Construction of Analytic functions'

Method:1 Exact differential method

Suppose the harmonic function $u(x,y)$ is given

Now $dv=v_x\,dx+v_y\,dy$ is an exact differential where v_x and v_y are known from u by using C-R equations

$\therefore v=\int v_x\,dx+\int v_y\,dy = -\int u_y\,dx+\int u_x\,dy$

S.T the function $u = 2xy + 3y$ is harmonic and find the corresponding analytic function. Find its conjugate.

$$u = 2xy + 3y$$

$$\frac{\partial u}{\partial x} = 2y \qquad \frac{\partial u}{\partial y} = 2x+3$$

$$\frac{\partial^2 u}{\partial x^2} = 0 \qquad \frac{\partial^2 u}{\partial y^2} = 0$$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Hence u satisfies Laplace's equation. $\therefore$ u is harmonic

By Milne's Thomson method, we have

$$f'(z) = \phi_1(z,0) - i\phi_2(z,0) \quad — ①$$

where $\frac{\partial u(x,y)}{\partial x} = \phi_1(x,y) = 2y$

$$\therefore \phi_1(z,0) = 0 \quad — ②$$

& $\frac{\partial u(x,y)}{\partial y} = \phi_2(x,y) = 2x+3$

$$\therefore \phi_2(z,0) = 2z+3 \quad — ③$$

Sub ② & ③ in ① we get

$$f'(z) = 0 - i(2z+3)$$

Integrating we get

$$f(z) = -i\int(2z+3)dz + c$$

$$= -i(z^2+3z)+c$$

To find conjugate of u.

W.K.T $dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy$

$$\int dv = \int \frac{\partial v}{\partial x} dx + \int \frac{\partial v}{\partial y} dy$$

$$v = \int -\frac{\partial u}{\partial y} dx + \int \frac{\partial u}{\partial x} dy$$

$$v = \int -(2x+3)\, dx + \int 2y\, dy$$

$$= -x^2 + 3x + y^2 \Rightarrow y^2 - 3x - x^2 //.$$

Show that the function $u = \frac{1}{2} \log(x^2+y^2)$ is harmonic and determine its conjugate. Also find $f(z)$.

$$u = \frac{1}{2} \log(x^2+y^2)$$

$$\frac{\partial u}{\partial x} = \frac{x}{x^2+y^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{(x^2+y^2) - (2x^2)}{(x^2+y^2)^2}$$

$$= \frac{y^2 - x^2}{x^2+y^2}$$

$$\frac{\partial u}{\partial y} = \frac{y}{x^2+y^2}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{(x^2+y^2) - y \cdot 2y}{(x^2+y^2)^2}$$

$$= \frac{x^2 - y^2}{(x^2+y^2)^2}$$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{y^2 - x^2}{(x^2+y^2)^2} + \frac{x^2 - y^2}{(x^2+y^2)^2} = 0$$

Hence u satisfies Laplace equation.

$\therefore$ u is harmonic.

To find conjugate u ie (v)

W.K.T $dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy = -\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy$

[$\because$ using C.R equations]

W.K.

$$\Rightarrow \frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy = \frac{xdy - ydx}{x^2+y^2}$$

$$\Rightarrow \frac{xdy-ydx}{x^2} \cdot \frac{1}{1+(y/x)^2} \Rightarrow \frac{1}{1+(y/x)^2} \cdot d(y/x)$$

$$\int dv = \int \frac{d(y/x)}{1+(y/x)^2} \quad \left[\text{ie } d(u/v) = \frac{vdu - udv}{v^2}\right.$$

$$v = \tan^{-1}(y/x) //$$

$\therefore$ The required analytic function

$$f(z) = u+iv = \tfrac{1}{2}\log(x^2+y^2) + i\tan^{-1}(y/x)$$

$$= \log(x+iy)$$

$$= \log(z)$$

(or)

(ii) To find $f(z)$

By Milne's thomson method

$$f(z) = \int \phi_1(z,0)\,dz - i\int \phi_2(z,0)\,dz + c$$

$$\phi_1(z,0) = \left(\frac{\partial u}{\partial x}\right)_{(z,0)} = \left[\frac{x}{x^2+y^2}\right]_{(z,0)} = \frac{z}{z^2+0} = 1/z$$

$$\phi_2(z,0) = \left(\frac{\partial u}{\partial y}\right)_{(z,0)} = \left[\frac{y}{x^2+y^2}\right]_{(z,0)} = 0$$

$$f(z) = \int 1/z\,dz - i\int 0\,dz + c$$

$$= \log z + c //.$$

Find the analytic function $w = u + iv$ if $u = e^x(x\sin y + y\cos y)$ and hence find v.

Given

$u = e^x x\sin y + e^x y\cos y$.

$$\frac{\partial u}{\partial x} = \sin y(e^x + xe^x) + e^x(y\cos y).$$

$$\frac{\partial u}{\partial y} = xe^x\cos y + e^x(\cos y - y\sin y)$$

By milne' thomson method

$$f(z) = \int \phi_1(z,0)\,dz - i\int \phi_2(z,0)\,dz + c$$

$$\left(\frac{\partial u}{\partial x}\right)_{z,0} = 0 \quad \text{ie } \phi_1(z,0) = 0$$

$$\left(\frac{\partial u}{\partial y}\right)_{z,0} = ze^z + e^z \quad \text{ie } \phi_2(z,0) = ze^z + e^z$$

$$\Rightarrow \int 0\,dz - i\int \phi_2(z,0)\,dz + c$$

$$= -i\int ze^z + e^z\,dz$$

$$= -i\left[ze^z - e^z + e^z\right] + c$$

$$f(z) \Rightarrow -i(ze^z) + c$$

To find conjugate of u

$$-ize^z = -i\left[(x+iy)e^{x+iy}\right] = -i\left[(x+iy)e^x(\cos x + i\sin y)\right]$$

$$= -i\left[xe^x\cos y + ixe^x\sin y + iye^x\cos y - ye^x\sin y\right]$$

$$\Rightarrow -i\left[xe^x\cos y + ixe^x\sin y + ye^x\cos y + iye^x\sin y\right.$$

$$= -ixe^x\cos y + xe^x\sin y + ye^x\cos y) \,//.$$

$$= i(ye^x\sin y - xe^x\cos y)$$

Determine the analytic function whose real part is $\dfrac{\sin 2x}{\cosh 2y - \cos 2x}$.

Soln

$$u = \frac{\sin 2x}{\cosh 2y - \cos 2x}$$

$$u_x = \frac{(\cosh 2y - \cos 2x)[2\cos 2x] - \sin 2x\,[2\sin 2x]}{[\cosh 2y - \cos 2x]^2}$$

$$u_x(z,0) = \frac{(1-\cos 2z)(2\cos 2z) - 2\sin^2 2z}{(1-\cos 2z)^2}$$

$$= \frac{2\cos 2z - 2\cos^2 2z - 2\sin^2 2z}{(1-\cos 2z)^2}$$

$$\Rightarrow \frac{2\cos 2z - 2[\cos^2 2z + \sin^2 2z]}{(1-\cos 2z)^2}$$

$$= \frac{2\cos 2z - 2}{(1-\cos 2z)^2} = \frac{-2(1-\cos 2z)}{(1-\cos 2z)^2}$$

$$\Rightarrow \frac{-2}{(1-\cos 2z)} = \frac{-2}{2\sin^2 z}$$

$$= -\operatorname{cosec}^2 z$$

ie $\boxed{u_x(z,0) = -\operatorname{cosec}^2 z}$

ie $\phi_1(z,0) = -\operatorname{cosec}^2 z$

$$u_y = \frac{(\cosh 2y - \cos 2x.(0) - \sin 2x(2\sinh 2y)}{(\cosh 2y - \cos 2x)^2}$$

$\boxed{u_y(z,0) = 0}$ ie $\phi_2(z,0) = 0$

To find analytic function

$$f(z) = \int [\phi_1(z,0) - i\,\phi_2(z,0)]\,dz$$

$$= \int [-\operatorname{cosec}^2 z - i\,(0)]\,dz$$

$$= \cot z + c$$

Find the analytic function $f(z) = u + iv$. given that

$2u + v = e^x(\cos y - \sin y)$

Soln

$f(z) = u + iv$

$2f(z) = 2u + i\,2v$ — ①

and $i f(z) = iu - v$

$-i f(z) = v - iu$ — ②

① + ② $\Rightarrow$ $2f(z) - i f(z) = (2u+v) + i(2v-u)$

$(2-i) f(z) = (2u+v) + i(2v-u)$

$U + iV = F(z)$

Here $U = 2u + v = e^x(\cos y - \sin y)$

$$\frac{\partial U}{\partial x} = e^x(\cos y - \sin y)$$

$\phi_1(z,0) = e^z$

$$\frac{\partial U}{\partial y} = e^x(-\sin y - \cos y)$$

$\phi_2(z,0) = e^z(-1) \Rightarrow -e^z$

By Milne - Thomson method

$$F(z) = \int [\phi_1(z,0) - i\,\phi_2(z,0)]\,dz$$

$$(2-i)f(z) = \int [e^z - i(-e^z)]\,dz$$

$$= (e^z + ie^z) + c$$

$$(2-i)f(z) \Rightarrow e^z(1+i) + c$$

$$f(z) = \frac{e^z(1+i)}{(2-i)}$$

$$f(z) = \frac{e^z(1+i)(2+i)}{(2-i)(2+i)} = \frac{e^z[2+i+2i-1]}{(2)^2-(i)^2}$$

$$f(z) = \frac{e^z(1+3i)}{4+1} \Rightarrow \frac{e^z(1+3i)}{5}$$

Find the analytic function $f(z) = u+iv$ given that $u-2v = e^x(\cos y - \sin y)$

Soln

Let $f(z) = u+iv \quad -(1)$

$-if(z) = -iu+v \quad -(2)$

$(1) + (2) \times (-2)$

$$f(z) + 2if(z) = u+iv = 2iu - 2v$$

$$(1+2i)f(z) = (u-2v) + i(v+2u)$$

$$F(z) = U+iV$$

Here $U = u-2v = e^x(\cos y - \sin y)$

$\phi_1(x,y) = \frac{\partial u}{\partial x} = u_x = e^x(\cos y - \sin y)$

$\phi_1(z,0) = e^z$

$\phi_2(x,y) = \frac{\partial u}{\partial y} = u_y = e^x(-\sin y - \cos y)$

$\phi_2(z,0) = e^z(-1) = -e^z$

By Milne - Thomson method

$F(z) = \int [\phi_1(z,0) - i\,\phi_2(z,0)]\,dz$

$= \int e^z dz - i\int -e^z\,dz$

$u+iv \Rightarrow e^z + ie^z + c$

ie $(1+2i) f(z) = (1+i)e^z + c$

$f(z) = \frac{(1+i)e^z}{(1+2i)} + c \Rightarrow \frac{(1+i)e^z(1-2i)}{(1+2i)(1-2i)} + c$

$= \frac{e^z[1-2i+i+2]}{(1)^2-(2i)^2} + c \Rightarrow \frac{e^z[3-i]}{1+4} + c$

$f(z) = \frac{3-i}{5}e^z + c$ //.

Find the harmonic conjugate of $u = e^x\cos y$

To find the conjugate of u

$v = \int -\frac{\partial u}{\partial y}\,dx + \int \frac{\partial u}{\partial x}\,dy$

$\frac{\partial u}{\partial x} = e^x\cos y$; $\frac{\partial u}{\partial y} = e^x(-\sin y)$

$\frac{\partial^2 u}{\partial x^2} = e^x\cos y$; $\frac{\partial^2 u}{\partial y^2} = -e^x\cos y$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = e^x \cos y - e^x \cos y = 0$$

$\therefore$ u is harmonic

to find v

$$v = \int -\frac{\partial u}{\partial y}\, dx + \int \frac{\partial u}{\partial}\, dy$$

$$= -\int - e^x \sin y\, dx + \int e^x \cos y\, dy$$

$$\Rightarrow + e^x \sin y + e^x (+\sin y)$$

$$V = 2e^x \sin y + c$$

$$V = e^x \sin y + c$$