Unit - $\underline{V}$

Complex Integration

1) Using cauchy's integral, evaluate $\int_c \frac{\cos\pi z^2}{(z-1)(z-2)} dz$ where c is a curve : $|z| = \frac{3}{2}$

Note:

$z-1=0$ and $z-2=0$

↓ lies inside the curve ↓ lies outside the curve.

Sol: Given : $f(z) = \frac{\cos\pi z^2}{z-2}$

$\Rightarrow z=1$ lies inside the curve $|z| = \frac{3}{2}$

By cauchy's Integral formula:

$$\int_c \frac{f(z)}{z-a} dz = 2\pi i . f(z)$$

$$= \int_c \frac{\frac{\cos\pi z^2}{z-2}}{z-1} dz = 2\pi i . f(1) \quad — ①$$

$\because f(z) = \frac{\cos\pi z^2}{z-2}$

($\because$ Found that $z=1$)

$f(z) = \frac{\cos\pi z^2}{z-2}$ and

$f(1) = \frac{\cos\pi}{-1} = \frac{-1}{-1} = 1$ ⟶ subs in ①:

$$\int \frac{\frac{\cos\pi z^2}{z-2}}{z-1} dz = 2\pi i(1) = \underline{\underline{2\pi i}}.$$

2) Evaluate : $\int_c \frac{z^2}{(z-1)^2 z+2} dz$ where $|z| = 3$.

Sol: Reference: $(z-1)^2(z-2) = 0 \rightarrow z-1=0 \quad z=1$

$\searrow z+2=0 \quad z=-2$

$\Rightarrow |1| = 3$ and $|-2| = 3$

$|2| = 3$

$\Rightarrow$ Both of these points lie inside the curve.

Using partial fraction:

$$\frac{z^2}{(z-1)^2(z+2)} = \frac{A}{z-1} + \frac{B}{(z-1)^2} + \frac{C}{z+2}$$

$$z^2 = A(z-1)(z+2) + B(z+2) + C(z-1)^2$$

Put $z=1$:

$1 = 3B$

$\therefore B = \frac{1}{3}$

Put $z=-2$:

$4 = C(-3)^2$

$\therefore C = \frac{4}{9}$

Put $z=0$:

$0 = -2A + \frac{2}{3} + \frac{4}{9}$

$0 = \frac{6+4}{9} - 2A$

$\therefore A = \frac{5}{9}$

$$\int \frac{z^2}{(z-1)^2(z+2)} dz = \int \frac{5\,dz}{9(z-1)} + \int \frac{1\,dz}{3(z-1)^2} + \int \frac{4}{9(z+2)} dz$$

$z=1$ lies inside the curve $|z| = 3$

$z=-2$ lies inside the curve $|z| = 3$

$\therefore$ By Cauchy's Integral formula:

$$= \frac{5}{9} \int \frac{1}{z-1} dz + \frac{1}{3} \int \frac{dz}{(z-1)^2} + \frac{4}{9} \int \frac{dz}{(z+2)}$$

$$= \frac{5}{9} 2\pi i . f(a) + \frac{1}{3} 2\pi i f'(a) + \frac{4}{9} 2\pi i . f(a)$$

$[f(z) = 1]$

$$= \frac{5}{9} 2\pi i . (1) + \frac{1}{3} 2\pi i(0) + \frac{4}{9} 2\pi i(1)$$

$$= 2\pi i \left(\frac{5}{9} + \frac{4}{9}\right) = 2\pi i(1)$$

$$= \underline{\underline{2\pi i}}.$$

3) Expand $f(z) = \log(1+z)$ about $z=0$

Sol: Given: $f(z) = \log(1+z)$

	$f(z)$	At $z=0$
$f(z)$	$\log(1+z)$	$0 \longrightarrow f(a)$
$f'(z)$	$\frac{1}{1+z}$	1
$f''(z)$	$\frac{-1}{(1+z)^2}$	-1
$f'''(z)$	$\frac{-2}{(1+z)^3}$	2
$\vdots$	$\vdots$	$\vdots$

$$f(z) = f(0) + \frac{f'(0)}{1!} z + \frac{f''(0)}{2!} z^2 + \frac{f'''(0)}{3!} z^3 + \dots$$

$\downarrow$ (By Maclaurin series)

$$= 0 + \frac{1}{1!} z - \frac{1 z^2}{2!} + \frac{2}{3} z^3 + \dots$$

$$= z - \frac{z^2}{2} + \frac{z^3}{3} + \dots$$

4) Expand $f(z) = \cos z$ about $z = \pi/3$ in the Taylor series.

Sol: Given: $f(z) = \cos z$ and about $z = \pi/3$

Taylor series:

$$f(z) = f(a) + \frac{f'(a)}{1!}(z-a) + \frac{f''(a)}{2!}(z-a)^2 + \frac{f'''(a)}{3!}(z-a)^3 + \dots$$

	$f(z)$	At $z=\pi/3$
$f(z)$	$\cos z$	$1/2 \rightarrow$
$f'(z)$	$-\sin z$	$-\sqrt{3}/2$
$f''(z)$	$-\cos z$	$-1/2$
$f'''(z)$	$\sin z$	$\sqrt{3}/2$
$\vdots$	$\vdots$	$\vdots$

So, by Taylor series:

$$f(z) = \frac{1}{2} + \frac{-\frac{\sqrt{3}}{2}}{1!}\left(z-\frac{\pi}{3}\right) + \frac{\frac{-1}{2}}{2!}\left(z-\frac{\pi}{3}\right)^2 + \frac{\frac{\sqrt{3}}{2}}{3!}\left(z-\frac{\pi}{3}\right.$$

$$= \frac{1}{2} - \frac{\sqrt{3}}{2}\left(z-\frac{\pi}{3}\right) - \frac{1}{4}\left(z-\frac{\pi}{3}\right)^2 + \frac{\sqrt{3}}{2}\left(z-\frac{\pi}{3}\right)^3 + \dots$$

5) obtain the Laurent's expansion for $f(z) = \frac{1}{(z-1)(z-2)}$ valid in the region $1 < |z| < 2$ and $|z| > 2$ and $|z-1| < 1$

Sol: Given: $f(z) = \frac{1}{(z-1)(z-2)}$

considering: $\frac{1}{(z-1)(z-2)} = \frac{A}{(z-1)} + \frac{B}{(z-2)}$

$1 = A(z-2) + B(z-1)$

Put $z=1$:

$1 = A(-1)$

$\therefore A = -1$

Put $z=2$:

$1 = (2-1)B$

$\therefore B = 1$

$= \frac{-1}{z-1} + \frac{1}{z-2}$ Taking the condition:

$\underline{1<|z|<2}$: $1<|z|$ and $|z|<2$

$\left|\frac{1}{z}\right|<1$ and $\left|\frac{z}{2}\right|<1$

$f(z) = \frac{-1}{z\left(1-\frac{1}{z}\right)} + \frac{1}{2\left(\frac{z}{2}-1\right)}$ Taking z and 2 common in 1st and 2nd term respectively.

$= \frac{-1}{z} \cdot \frac{1}{\left(1-\frac{1}{z}\right)} + \frac{-1}{2} \cdot \frac{1}{\left(1-z/2\right)}$

$= \frac{-1}{z} \frac{1}{\left(1-\frac{1}{z}\right)} \frac{-1}{2} \frac{1}{\left(1-\frac{z}{2}\right)}$

$= \frac{-1}{z}\left(1-\frac{1}{z}\right)^{-1} - \frac{1}{2}\left(1-\frac{z}{2}\right)^{-1}$

$= \frac{-1}{z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^{n} - \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^{n}$ Formulas ① and ② are used

Note:

Formulae used:

$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} b_n (z-a)^{-n}$ — ①

Analytical ✓ Principal

$(1-x)^{-1} = 1+x+x^2+x^3+\ldots$ — ②

ii) $|z|>2$: $2<|z| \Rightarrow \left|\frac{2}{z}\right|<1$ and $\left|\frac{1}{z}\right|<1$

$f(z) = \frac{-1}{z\left(1-\frac{1}{z}\right)} + \frac{1}{z\left(1-\frac{2}{z}\right)} \Rightarrow \frac{-1}{z}\left(1-\frac{1}{z}\right)^{-1} + \frac{1}{z}\left(1-\frac{2}{z}\right)^{-1}$

$\Rightarrow \frac{-1}{z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^{n} + \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{2}{z}\right)^{n}$

iii) $|z-1|<1$: Let $u = z-1$, then $|u|<1$

$f(z) = \frac{-1}{z-1\boxed{+1-1}} + \frac{1}{z-2\boxed{+1-1}}$ Adding 1 and subtracting 1

$= \frac{-1}{z-1} + \frac{1}{(z-1)-1} \Rightarrow \frac{-1}{u} + \frac{1}{u-1} \Rightarrow \frac{-1}{u} - \frac{1}{1-u}$

$= \frac{-1}{u} - (1-u)^{-1} \Rightarrow \frac{-1}{u} - \sum_{n=0}^{\infty} u^n$

$= \frac{-1}{z-1} - \sum_{n=0}^{\infty}(z-1)^n$

6) $f(z) = \frac{7z-2}{z(z-2)(z+1)}$ with $1<|z+1|<3$

Sol: Consider $\frac{7z-2}{z(z-2)(z+1)} = \frac{A}{z} + \frac{B}{z-2} + \frac{C}{z+1}$

$7z-2 = A(z-2)(z+1) + B(z)(z+1) + C(z)(z-2)$

Put $z=0$:
$-2 = A(-2)(1)$
$\therefore A = 1$

Put $z=-1$:
$-9 = C(-1)(-1-2)$
$-9 = (-C)(-3)$
$\therefore C = -3$

Put $z=2$:
$12 = B(2)(3)$
$\therefore B = 2$

$f(z) = \frac{1}{z} + \frac{2}{z-2} - \frac{3}{z+1}$

Let $u = z+1$

condition: $1<|z+1|<3 \longrightarrow 1<u<3$

$\Rightarrow \left|\frac{1}{u}\right| < 1$ and $\left|\frac{u}{3}\right| < 1$

$$f(z) = \frac{1}{z+1-1} + \frac{2}{z-2+1-1} - \frac{3}{z+1+1-1}$$

$$= \frac{1}{u-1} + \frac{2}{u-3} - \frac{3}{u} \Rightarrow \frac{1}{u\left(1-\frac{1}{u}\right)} + \frac{2}{3\left(\frac{u}{3}-1\right)} - \frac{3}{u}$$

$$= \frac{1}{u}\left(1-\frac{1}{u}\right)^{-1} - \frac{2}{3}\left(1-\frac{u}{3}\right)^{-1} - \frac{3}{u}$$

$$= \frac{1}{u}\sum_{n=0}^{\infty}\left(\frac{1}{u}\right)^n - \frac{2}{3}\sum_{n=0}^{\infty}\left(\frac{u}{3}\right)^n - \frac{3}{u}$$

$$= \frac{1}{z+1}\sum_{n=0}^{\infty}\left(\frac{1}{z+1}\right)^n - \frac{2}{3}\sum_{n=0}^{\infty}\left(\frac{z+1}{3}\right)^n - \frac{3}{z+1} \quad (\because u = z+1)$$

7) $f(z) = \frac{z^2-1}{z^2+5z+6}$ valid in the region $2<|z|<3$, $|z|<2$, $|z|>3$.

(Note: Both Nr. and Dr. are of same degree)

Sol:

$$z^2+5z+6 \overset{1}{\overline{\big)\, z^2+0z-1}}$$
$$\underset{(-)}{z^2} \underset{(-)}{+5z} \underset{(-)}{+6}$$
$$-5z-7$$

$\Rightarrow 1 - \frac{(5z+7)}{z^2+5z+6}$

Taking $\frac{5z+7}{(z+2)(z+3)} = \frac{A}{(z+2)} + \frac{B}{(z+3)}$

$5z+7 = A(z+3) + B(z+2)$

Put $z=-2$:
$-10+7 = A$
$\therefore A = -3$.

Put $z=-3$:
$-15+7 = -B$
$\therefore B = 8$.

$$f(z) = \frac{-3}{z+2} + \frac{8}{z+3} \Rightarrow 1 - \left(\frac{-3}{z+2} + \frac{8}{z+3}\right)$$

$$= 1 + \frac{3}{z+2} - \frac{8}{z+3}$$

i) $2<|z|<3$: $2<|z| \rightarrow \left|\frac{2}{z}\right|<1$ and $|z|<3 \rightarrow \left|\frac{z}{3}\right|<1$

$$\Rightarrow 1 + \frac{3}{z}\left(\frac{1}{1+\frac{2}{z}}\right) - \frac{8}{3}\,\frac{1}{\left(\frac{z}{3}+1\right)}$$

$$= 1 + \frac{3}{z}\left(1+\frac{2}{z}\right)^{-1} - \frac{8}{3}\left(\frac{z}{3}+1\right)^{-1}$$

$$= 1 + \frac{3}{z}\sum_{n=0}^{\infty}(-1)^n\left(\frac{2}{z}\right)^n - \frac{8}{3}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z}{3}\right)^n$$

$\left[\because (1+x)^{-1} = 1 - x + x^2 - x^3 + \dots\right]$

ii) $|z|<2 \rightarrow \left|\frac{z}{2}\right|<1$
$|z|<3 \rightarrow \left|\frac{z}{3}\right|<1$ (Logic)

$$f(z) = 1 + \frac{3}{2}\left(\frac{1}{1+\frac{z}{2}}\right) - \frac{8}{3}\,\frac{1}{\left(1+\frac{z}{3}\right)}$$

$$= 1 + \frac{3}{2}\left(1+\frac{z}{2}\right)^{-1} - \frac{8}{3}\left(1+\frac{z}{3}\right)^{-1}$$

$$= 1 + \frac{3}{2}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z}{2}\right)^n - \frac{8}{3}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z}{3}\right)^n$$

iii) $|z|>3 \rightarrow \left|\frac{3}{2}\right|<1$ ultimately, $\left|\frac{2}{z}\right|<1$

$$f(z) = 1 + \frac{3}{z+2} - \frac{8}{z+3}$$

$$\frac{3}{2}\left(\frac{1}{1+\frac{2}{2}}\right) - \frac{8}{z}\frac{1}{\left(1+\frac{3}{z}\right)}$$

$$1+\frac{3}{z}\left(1+\frac{2}{z}\right)^{-1} - \frac{8}{z}\left(1+\frac{3}{z}\right)^{-1}$$

$$= 1+\frac{3}{z}\sum_{n=0}^{\infty}(-1)^n\left(\frac{2}{z}\right)^n - \frac{8}{z}\sum_{n=0}^{\infty}(-1)^n\left(\frac{3}{z}\right)^n.$$

3) Evaluate $\int_0^{2\pi}\frac{1}{a+b\cos\theta}d\theta$, $a>|b|$

sol:

$\because \cos\theta = \frac{z^2+1}{2z}$

Put $z=e^{i\theta}$

$dz = \underline{e^{i\theta}}.i.d\theta = zi\,d\theta \quad \therefore d\theta = \frac{dz}{zi}$

$$\int_0^{2\pi}\frac{1}{a+b\cos\theta}d\theta = \int_c \frac{dz/zi}{a+b\left(\frac{z^2+1}{2z}\right)} = \int_c \frac{1}{\frac{2az+bz^2+b}{2z}}\cdot\frac{dz}{iz}$$

$$\int_c \frac{2\cancel{z}}{2az+bz^2+b}\cdot\frac{dz}{i\cancel{z}} = \frac{2}{ib}\int\frac{dz}{z^2+\frac{2a}{b}z+1}$$

Consider: $z^2+\frac{2a}{b}z+1=0$

$\Rightarrow bz^2+2a\,z+b=0$

By Quadratic eq.:

$x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$

Here, $a=b$, $b=2a$, $c=b$

$$\Rightarrow \frac{-2a\pm\sqrt{4a^2-4b^2}}{2b} = \frac{-\cancel{2}a\pm\cancel{2}\sqrt{a^2-b^2}}{\cancel{2}b} = \frac{-a\pm\sqrt{a^2-b^2}}{b}$$

Form: $(z-\alpha)(z-\beta)$ $\therefore \alpha = \frac{-a+\sqrt{a^2-b^2}}{b}$ and

$\beta = \frac{-a-\sqrt{a^2-b^2}}{b}$

Given $a>|b|$ then $a=2$ and $b=1$:

$\alpha = \frac{-2+\sqrt{4-1}}{1} = -2+\sqrt{3} = -2+1.7361 = |-0.264| <$

$\therefore$ It lies inside the curve $|z|=1$

$\beta = \frac{-2-\sqrt{4-1}}{1} = -2-\sqrt{3} = -2-1.7361 = |-3.1736| > 1$

$\therefore$ It lies outside the curve $|z|=1$.

$$= \frac{2}{ib}\int_c \frac{dz}{(z-\alpha)(z-\beta)} = \frac{2}{\cancel{i}b}\times 2\pi\cancel{i}\times \text{sum of residues}$$

$= \frac{4\pi}{b}\times$ sum of residues. ∟①

Residue at $z=\alpha$:

$$[\operatorname{Res} f(z)]_{z=\alpha} = \lim_{z\to\alpha}(z-\alpha).f(z)$$

$$= \lim_{z\to\alpha}\cancel{(z-\alpha)}.\frac{1}{\cancel{(z-\alpha)}(z-\beta)} = \frac{1}{\alpha-\beta}$$ (Subs values of α and β)

$$= \frac{1}{\frac{-a+\sqrt{a^2-b^2}}{b}-\left(\frac{-a-\sqrt{a^2-b^2}}{b}\right)} = \frac{1}{\frac{-\cancel{a}}{\cancel{b}}+\frac{\sqrt{a^2-b^2}}{b}+\frac{\cancel{a}}{b}+\frac{\sqrt{a^2-b^2}}{b}}$$

$$= \frac{1}{\frac{2\sqrt{a^2-b^2}}{b}} = \frac{b}{2\sqrt{a^2-b^2}}$$ = sum of residues.

① becomes:

$$\int_c\frac{dz}{(z-\alpha)(z-\beta)} = \frac{\overset{2}{\cancel{4}}\pi}{\cancel{b}}\cdot\frac{\cancel{b}}{\cancel{2}\sqrt{a^2-b^2}} = \frac{2\pi}{\sqrt{a^2-b^2}}$$ and $a>|b|$

Remarks: $\int_0^{2\pi} \frac{1}{13+5\cos\theta} d\theta$

Sol: Put $z = e^{i\theta}$ and $\cos\theta = \frac{z^2+1}{2z}$

$dz = e^{i\theta} . i . d\theta$

$dz = zi\, d\theta$

$\therefore d\theta = \frac{dz}{zi}$ $\Rightarrow \int_0^{2\pi} \frac{d\theta}{13+5\cos\theta} = \int_c \frac{\frac{dz}{zi}}{13+5\left(\frac{z^2+1}{2z}\right)}$

$= \int_c \frac{1}{\frac{26z+5z^2+5}{2z}} \cdot \frac{dz}{zi} = \int_c \frac{2\cancel{z}dz}{5z^2+26z+5} \cdot \frac{1}{\cancel{z}i}$

$= \frac{2}{5i} \int_c \frac{dz}{z^2+\frac{26}{5}z+1}$ Consider: $z^2 + \frac{26}{5} z + 1$

$a = 1, b = \frac{26}{5}, c = 1$

By Quadratic eq:

$\Rightarrow \frac{\frac{-26}{5} \pm \sqrt{\left(\frac{26}{5}\right)^2 - 4}}{2} = \frac{\frac{-26}{5} \pm \sqrt{576}}{2} = \frac{-26 \pm 24}{10}$

$\Rightarrow \frac{-1}{5}$ and $-5 \longrightarrow \left|\frac{-1}{5}\right| < 1$ and $|-5| > 1$

$\Rightarrow z = \frac{-1}{5}$ is a simple pole which lies inside the curve $|z| = 1$ and

$z = -5$ is a simple pole which lies outside the curve $|z| = 1$.

$\Rightarrow \frac{2}{5i} \int \frac{dz}{z^2+\frac{26}{5}z+1} = \frac{2}{5\cancel{i}} \times 2\pi\cancel{i} \times$ sum of residues

Residue at $z = \frac{-1}{5}$:

$\left[\operatorname{Res}_{z=\frac{-1}{5}} f(z)\right] = \lim_{z \to \frac{-1}{5}} \cancel{\left(z+\frac{1}{5}\right)} \cdot \frac{1}{\cancel{\left(z+\frac{1}{5}\right)}(z+5)}$

$= \frac{1}{\left(\frac{-1}{5}+5\right)} = \frac{1}{\frac{24}{5}} = \frac{5}{24}$

$\left[\operatorname{Res}_{z=-5} f(z)\right] = 0$ $\left[\because z = -5 \text{ lies outside the curve } |z| = 1\right]$

$\Rightarrow \int \frac{dz}{z^2+\frac{26}{5}z+1} = 2\pi i \times \frac{5}{24}$

$\Rightarrow \frac{2}{5i} \int \frac{dz}{z^2+\frac{26}{5}z+1} = \frac{\cancel{2}}{\cancel{5}\cancel{i}} \times \cancel{2}\pi\cancel{i} \times \frac{\cancel{5}}{\cancel{24}_{\cancel{12}\,6}} = \frac{\pi}{6}$

9) $\int_{-\infty}^{\infty} \frac{x^2 dx}{(x^2+a^2)(x^2+b^2)}$ Evaluate

Sol: Let $\int_c f(z) dz = \int_{-\infty}^{\infty} \frac{z^2 dz}{(z^2+a^2)(z^2+b^2)}$

where c is the upper half of the semicircle
with the boundary region $[-R, R]$

Note: $\int_0^{\infty} f(x) dx = \frac{1}{2} \int_{-\infty}^{\infty} f(x) dx$

$$\int_C f(z)dz = \int_{-\infty}^{\infty} f(z)dz + \int_\Gamma f(z)dz$$

when $R \to \infty$ $\int f(z)dz$

$$\therefore \int_C f(z)dz = \int_{-\infty}^{\infty} f(z)dz$$

By cauchy's Residue theorem:

$= 2\pi i$ (Sum of residues) — ①

$\Rightarrow (z^2+a^2)(z^2+b^2)=0$

$[z^2-(ia)^2]\ [z^2-(ib)^2]=0$

$z^2=(ia)^2$ $\quad$ $z^2=(ib)^2$

$z=\pm ia$ & $\quad$ $z=\pm ib$

$\Rightarrow z=ia$ is a simple pole, lies inside the curve Γ

$z=-ia$ " " " " lies outside the curve Γ

$z=ib$ " " " " lies inside the curve Γ

$z=-ib$ " " " " lies outside the curve Γ

Residue at $z=ia$:

$$\left[\operatorname{Res}_{z=ia} f(z)\right] = \lim_{z\to ia}\left[\cancel{(z-ia)}\cdot\frac{z^2}{\cancel{(z-ia)}\underbrace{(z+ia)(z^2+b^2)}_{z^2+a^2}}\right.$$

$$=\frac{(ia)^2}{2ai\,(ia^2+b^2)} = \frac{-a^2}{2ai(-a^2+b^2)} = \frac{a}{2i(a^2-b^2)}$$

Residue at $z=ib$:

$$\left[\operatorname{Res}_{z=ib} f(z)\right] = \lim_{z\to ib}\left[\cancel{(z-ib)}\cdot\frac{z^2}{\cancel{(z-ib)}(z+ib)(z^2+a^2)}\right]$$

$$=\frac{(ib)^2}{2ib(a^2+ib^2)} = \frac{-b^{\cancel{2}}}{2i\cancel{b}(-b^2+a^2)} = \frac{-b}{2i(a^2-b^2)}$$

$\therefore$ ① becomes:

$$= 2\pi i\left[\frac{a}{2i(a^2-b^2)} - \frac{b}{2i(a^2-b^2)}\right] = \frac{\cancel{2}\pi \cancel{i}(a-b)}{\cancel{2}\cancel{i}(a^2-b^2)}$$

$$= \frac{\pi\,\cancel{(a-b)}}{(a+b)\cancel{(a-b)}} \Rightarrow \frac{\pi}{a+b}$$

$\left[\operatorname{Res}_{z=-ia} f(z)\right]$ and $\left[\operatorname{Res}_{z=-ib} f(z)\right] = 0$ ($\because$ They lie outside the curve).

10) Evaluate $\displaystyle\int_0^{\infty}\frac{dx}{(x^2+a^2)(x^2+b^2)}$

Sol:

change the limits by multiplying 2:

let $\displaystyle\int_C f(z)dz = 2\int_{-\infty}^{\infty}\frac{dz}{(z^2+a^2)(z^2+b^2)} = \frac{1}{2}\int_{-\infty}^{\infty}\frac{dz}{(z^2+a^2)(z^2+b^2)}$

where c is the upper half of the semicircle Γ with boundary region $[-R, R]$.

$$\therefore \int_C f(z)dz = \int_R f(z)dz + \int_\Gamma f(z)dz$$

$\therefore$ when $R\to\infty$ $\int f(z)dz = 0$

$\therefore \int f(z)dz = \int^{\infty} f(z)dz$

By Cauchy's Residue theorem:

$$= 2\pi i \times (\text{sum of Residues}) - ①$$

$$(z^2+a^2)(z^2+b^2) = 0 \qquad \therefore z = \pm ia \text{ and}$$

$$\left[z^2 - (ia)^2\right]\left[z^2 - (ib)^2\right] = 0 \qquad z = \pm ib.$$

$z = ia$ is a simple pole, lies inside the curve Γ
$z = -ia$ " " " " " outside the curve Γ
$z = ib$ " " " " " inside the curve Γ
$z = -ib$ " " " " " outside the curve Γ

Residue at $z = ia$:

$$\left[\underset{z=ia}{\text{Res}}\, f(z)\right] = \lim_{z\to ia} \cancel{(z-ia)} \left[\frac{1}{2\cancel{(z-ia)}(z+ia)(z^2+b^2)}\right]$$

$$= \frac{1}{2(ia)(-a^2+b^2)} = \frac{-1}{4(ia)(a^2-b^2)}$$

$$\left[\underset{z=ib}{\text{Res}}\, f(z)\right] = \lim_{z\to ib} \cancel{(z-ib)} \cdot \left[\frac{1}{2\cancel{(z-ib)}(z+ib)(z^2+a^2)}\right.$$

$$= \frac{1}{4ib(-b^2+a^2)} = \frac{1}{4(ib)(a^2-b^2)}$$

$\therefore$ ① becomes:

$$\int_C f(z)dz = 2\pi i \cdot \left[\frac{-1}{4(ia)(a^2-b^2)} + \frac{1}{4(ib)(a^2-b^2)}\right]$$

$$= \frac{2\pi i}{i(a^2-b^2)} \left[\frac{-1}{4a} + \frac{1}{4b}\right] = \frac{\cancel{2}\pi}{2\cancel{4}(a^2-b^2)} \cdot \left[\frac{-b+a}{ab}\right]$$

$$= \frac{\pi}{2(a+b)\cancel{(a-b)}} \times \frac{\cancel{a-b}}{ab} = \frac{\pi}{2ab(a+b)}$$