UNIT-5

Cauchy's Integral Formulae, Problems - Taylor's expansion with simple problems - Laurent's expansions with simple problems - singularities - Types of poles and Residues - Cauchy's Residue theorem (without proof) - contour Integration - unit circle, semi circular contour - Application of contour Integration in Engineering.

Cauchy's Integral Formula:

Statement:

Let $f(z)$ be an analytic function inside and on a simple closed contour C taken in the positive sense. If a is any point interior to C, then

$$f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)\,dz}{z-a}$$

Cauchy's Integral Formula for n^{th} Derivative:-

Statement:

Let $f(z)$ be an analytic function inside and on a simple closed contour C, taken in the positive sense. If a is any point interior to C, then

$$f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)\,dz}{z-a}$$

Cauchy's Integral Formula for n^{th} Derivative:-

Let $f(z)$ be analytic and on a simple closed curve C,

then $$\int_C \frac{f(z)}{(z-a)^{n+1}}\,dz = \frac{2\pi i}{n!} f^{n}(a)$$

$$\int \frac{f(z)}{(z-a)^2}\,dz = \frac{2\pi i}{1!} f'(a)$$

Problems:-

1. Evaluate $\int_C \frac{1-2z}{(z-2)(z-3)}\,dz$ where c is $|z|=1$

Soln

Let $(z-2)(z-3)=0$

$z=2,\ c$ is $|z|=1$

$|z|=2>1$

$z=2$ lies outside c

Take $z=3$, $|z|=|3|=3>1$

$z=3$ lies outside c

By cauchy's integral theorem $\int_C \frac{1-2z}{(z-2)(z-3)}=0$

$|z|=r$

$|x+iy|=r$

$\sqrt{x^2+y^2}=r$

$\Rightarrow x^2+y^2=r^2$

2. Evaluate using cauchy's integral formula $\int_C \frac{z}{z+1}\,dz$ where c is $|z+1|=1/2$

$a=-1$

$|x+iy+1|=1/2$

$|(x+1)+iy|=1/2$

$\sqrt{(x+1)^2+y^2}=1/2$

$(x+1)^2+y^2=(1/2)^2$

$c:(-1,0),\ r=1/2$

Soln

Let $z+1=0$

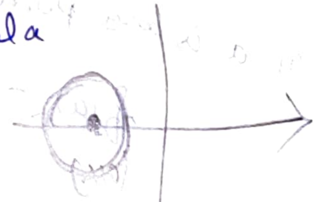
$z=-1$

$|z+1|=1/2$

$|z+1|=|-1+1|=0<1/2$

$z=-1$ lies inside c

By cauchy's Integral formula



$\int_C \frac{z}{z+1}\,dz$

$f(z)=z\,[f(z)]_{z=a}=f$

$\int \frac{f(z)}{z-a}\,dz=2\pi i\,f(a)$

$\int_C \frac{z}{z+1}\,dz = 2\pi i\,[f(z)]_{z=-1}$

$f(z)=z$

$f(a)=f(-1)=-1$

$=2\pi i\,[z]_{z=-1}$

$\int \frac{z}{z+1}\,dz=2\pi i(-1)$

$=-2\pi i$

$$\boxed{\int_C \frac{z}{z+1}\,dz=2\pi i}$$

3. Using Cauchy's Integral formula, evaluate

$$\int_C \frac{z}{(z-1)(z-2)}\,dz \quad \text{where } c \text{ is } |z-2| = 1/2.$$

Soln: Let $(z-1)(z-2)=0$

$z = 1, 2$ c is $|z-2| = 1/2$

At $z=1$

$|z-2| = 1/2$

$\Rightarrow |1-2| = |-1| = 1 > 1/2$

$\therefore z=1$ lies outside c

$$\int_C \frac{\left(z/z-1\right)}{z-2}\,dz = 2\pi i\, f(2)$$

$$f(z) = \frac{z}{z-1}$$

$$f(2) = \frac{2}{1} = 2$$

At $z=2$,

$|z-2| = 1/2$

$\Rightarrow |2-2| = 0 < 1/2$

$\therefore z=2$ lies inside c.

By Cauchy's integral formula,

$$\int_C \frac{f(z)\,dz}{z-a} = 2\pi i\,[f(z)]_{z=a}$$

$$\int_C \frac{f\left(\frac{z}{z-1}\right)dz}{z-2} = 2\pi i\left[\frac{z}{z-1}\right]_{z=2}$$

$$= 2\pi i\left[\frac{2}{2-1}\right]$$

$$\boxed{\int_C \frac{z\,dz}{(z-1)(z-2)} = 4\pi i}$$

4. Evaluate $\int_C \frac{\cos z^2}{(z-1)(z-2)}\, dz$ where c is $|z| = 3/2$ using Cauchy's integral formula.

Soln:

$(z-1)(z-2) = 0$

$\Rightarrow z = 1, 2$ c is $|z| = 3/2$

At $z=1$,

$|z| = |1| = 1 < 3/2$

$z=1$ lies inside c

At $z=2$,

$|z| = |2| = 2 > 3/2$

$z=2$ lies outside c

$$\int_C \frac{f(z)}{z-a}\, dz = 2\pi i \left[f(z)\right]_{z=a}.$$

$$\int_C \frac{\left(\frac{\cos z^2}{z-2}\right)}{z-1}\, dz = 2\pi i \left[\frac{\cos \pi z^2}{z-2}\right]_{z=1}$$

$$= 2\pi i \left[\frac{\cos \pi}{z-2}\right]_{z=1} \Rightarrow 2\pi i \left(\frac{-1}{-1}\right)$$

$$\boxed{\int_C \frac{\cos \pi z^2}{(z-1)(z-2)}\, dz = 2\pi i}$$

Problems using the formula,

$$\int_C \frac{f(z)\,dz}{(z-a)^n} = \frac{2\pi i}{(n-1)!} \frac{d^{n-1}}{dz^{n-1}} \left[f(z)\right]_{z=a}$$

$$\int_C \frac{f(z)\,dz}{(z-a)^2} = 2\pi i \frac{d}{dz}\left[f(z)\right]_{z=a}$$

1. Evaluate $\int_C \frac{z}{(z-1)(z-2)^2}\,dz$, where C is the circle $|z-2| = 1/2$, using cauchy's integral formula.

Soln

Let $(z-1)(z-2)^2 = 0$

$\Rightarrow z = 1, 2$ Given C is $|z-2| = 1/2$.

Consider $z=1$,

$|z-2| = |1-2| = |-1|$

$\Rightarrow 1 > 1/2$

$z=1$ lies outside

Consider $z=2$,

$|z-2| = |2-2| = 0 < 1/2$

$z=2$ lies inside c

$$\int_C \frac{f(z)\,dz}{(z-a)^2} = 2\pi i \frac{d}{dz}\left[f(z)\right]_{z=a}$$

$$\int_C \frac{z}{(z-1)(z-2)^2} = \int_C \frac{(z/z-1)}{(z-2)^2}\,dz$$

Here $f(z) = \frac{z}{z-1}$

$$\int_C \frac{z\,dz}{(z-1)(z-2)^2} = 2\pi i \left[\frac{d}{dz}\left(\frac{z}{z-1}\right)\right]_{z=2}$$

$$= 2\pi i \left[\frac{(z-1)(1) - z(1)}{(z-1)^2}\right]_{z=2}$$

$$= 2\pi i \left[\frac{z-1-z}{(z-1)^2}\right]_{z=2}$$

$$= 2\pi i \left[\frac{-1}{(z-1)^2}\right] = 2\pi i\,[-1]$$

$$\boxed{\int_C \frac{z\,dz}{(z-1)(z-2)^2} = -2\pi i}$$ //

2. Evaluate using Cauchy's integral formula

$$\int_c \frac{e^{2z}}{(z+1)^4} dz \quad \text{where } c \text{ is } |z|=2.$$

Soln

Let $(z+1)^4 = 0$

$z=-1$ Given $|z|=2$

Consider $z=-1$

$|z| = |-1| = 1 < 2$

$\therefore z=-1$ lies inside of c

$$\int_c \frac{f(z)}{(z-a)^n} dz = \frac{2\pi i}{n-1} \frac{d^{n-1}}{dz^{n-1}} \left[f(z)\right]_{z=a}$$

$$\int_c \frac{e^{2z} dz}{(z+1)^4} \quad \text{here } f(z) = e^{2z}, n=4$$

$$\Rightarrow \int_c \frac{e^{2z} dz}{(z+1)^4} = \frac{2\pi i}{(4-1)!} \left[\frac{d^3}{dz^3}(e^{2z})\right]_{z=-1}$$

$$= \frac{2\pi i}{3!} \left[\frac{d^2}{dz^2}\left(\frac{d}{dz}(e^{2z})\right)\right]_{z=-1}$$

$$= \frac{2\pi i}{3!} \left[\frac{d^2}{dz^2}(2e^{2z})\right]_{z=-1}$$

$$= \frac{2\pi i}{3!} \left[\frac{d}{dz} \frac{d}{dz}(2e^{2z})\right]_{z=-1}$$

$$= \frac{2\pi i}{3!} \left[\frac{d}{dz}(4e^{2z})\right]_{z=-1}$$

$$= \frac{2\pi i}{3!}\left[4 \cdot 2e^{2z}\right]_{z=-1}$$

$$= \frac{2\pi i}{36} 8 \cdot e^{-2} = \frac{8\pi i}{3} e^{-2} //.$$

Problems using partial Fraction:-

1) Evaluate $\int_c \frac{z}{(z-1)(z-2)} dz$ where c is $|z|=3$

Soln

Let $(z-1)(z-2)=0$

$\Rightarrow z=1,2$

Given : $|z|=3$

Consider $z=1$

$|z|=|3|=3<1$

$\therefore z=1$ lies inside c

Consider $z=2$,

$|z|=|2|=2<1$

$z=2$ lies inside circle

Consider $\frac{1}{(z-1)(z-2)} = \frac{A}{z-1} + \frac{B}{z-2}$

$$\frac{1}{(z-1)\cancel{(z-2)}} = \frac{A(z-2)+B(z-1)}{(z-1)\cancel{(z-2)}}$$

$1 = A(z-2) + B(z-1)$ — 1

Put $z=1$ in eqn 1

$1=-A$

$\boxed{A=-1}$

Put $z=2$ in eqn 1

$1=B$

$\boxed{B=1}$

$$\frac{1}{(z-1)(z-2)} = \frac{-1}{z-1} + \frac{1}{z-2}$$

$$\int_c \frac{z}{(z-1)(z-2)} dz = \int_c z\left(\frac{-1}{z-1} + \frac{1}{z-2}\right) dz$$

$$\int_C \left(\frac{-z}{z-1} + \frac{z}{z-2} \right) dz$$

$$= -\int_C \frac{z}{z-1} dz + \int \frac{z\,dz}{z-2}$$

$$= -2\pi i \left[z \right]_{z=1} + 2\pi i \left[z \right]_{z=2}$$

$$= -2\pi i (1) + 2\pi i (2)$$

$$= -2\pi i + 4\pi i = 2\pi i$$

$$\boxed{\int_C \frac{z\,dz}{(z-1)(z-2)} = 2\pi i}$$

2. Using Cauchy's Integral formula evaluate

$$\int_C \frac{e^z\,dz}{(z+2)(z+1)^2}$$

Soln Let $(z+2)(z+1)^2 = 0$

$z = -2, -1$

Given C is $|z| = 3$

Consider $z = -1$

$|z| = |-1| = 1 < 3$

$\therefore$ $z = -1$ lies inside C

Consider $z = -2$

$|z| = |-2| = 2 < 3$

$\therefore$ $z = -2$ lies inside C

Consider

$$\frac{1}{(z+2)(z+1)^2} = \frac{A}{z+2} + \frac{B}{z+1} + \frac{C}{(z+1)^2}$$

1

$$\frac{1}{(z+2)(z+1)^2} = \frac{A(z+1)^2 + B(z+1)(z+2) + C(z+2)}{(z+1)(z+2)^2}$$

$$1 = A(z+1)^2 + B(z+1)(z+2) + C(z+2) \quad -①$$

put $z=-2$, $\boxed{A=1}$ | put $z=-1$, $1=-C$, $\boxed{C=-1}$ | put $z=0$, $1 = A+2B+2C$, $1 = 1+2B-2$, $2=2B$, $\boxed{B=1}$

$$\frac{1}{(z+2)(z+1)^2} = \frac{1}{z+2} + \frac{1}{z+1} - \frac{1}{(z+1)^2}$$

$$\int_C \frac{e^z}{(z+2)(z+1)^2} = \int_C \frac{e^z dz}{z+2} + \int_C \frac{e^z dz}{z+1} + \int \frac{e^z dz}{(z+1)^2}$$

$$= 2\pi i\,[e^z]_{z=-2} + 2\pi i\,(e^z)_{z=-1} + (-2\pi i)\left(\frac{d}{dz}(e^z)\right)_{z=-1}$$

$$= 2\pi i\,(e^{-2}) + 2\pi i\,(e^{-1}) - 2\pi i\,(e^{-1}) = 2\pi i\, e^{-2}$$

TAYLOR'S SERIES:

$$f(z) = \sum_{n=0} \frac{(z-a)^n}{n!} f^n(a)$$

NOTE: If $a=0$, we get,

$$f(z) = \sum_{n=0} \frac{z^n}{n!} f^n(0) \text{ is called maclurin's series.}$$

for some elementary functions,

Maclurin's series

1) $e^z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \dots \quad |z| < \infty$

2) $\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!}, \dots \quad |z| < \infty$

3) $\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} + \dots \quad |z| < \infty$

4) $\sinh z = z + \frac{z^3}{3!} + \frac{z^5}{5!} + \dots \quad |z| < \infty$

5) $\cosh z = 1 + \frac{z^2}{2!} + \frac{z^4}{4!}$

6) $(1-z)^{-1} = 1 + z + z^2 + z^3 + \dots$

7) $(1+z)^{-1} = 1 - z + z^2 - z^3 + \dots$

8) $(1-z)^{-2} = 1 + 2z + 3z^2 + 4z^3 + \dots$

9) $(1+z)^{-2} = 1 - 2z + 3z^2 - 4z^3 + \dots$

1. Expand $f(z) = \cos z$ as a Taylor's series about $z = \pi/4$.

Soln

$f(z) = \cos z$	$f(\pi/4) = \cos \pi/4 = 1/\sqrt{2}$
$f'(z) = -\sin z$	$f'(\pi/4) = \sin \pi/4 = -1/\sqrt{2}$
$f''(z) = -\cos z$	$f''(\pi/4) = -\cos \pi/4 = -1/\sqrt{2}$
$f'''(z) = \sin z$	$f'''(\pi/4) = \sin \pi/4 = 1/\sqrt{2}$.

$$f(z) = f(a) + \frac{z-a}{1!} f'(a) + \frac{(z-a)^2}{2!} f''(a) + \frac{(z-a)^3}{3!} f'''(a) + \dots$$

At $a = \pi/4$

$$f(z) = f(\pi/4) + (z - \pi/4) f'(\pi/4) + \frac{(z-\pi/4)^2}{2!} f''(\pi/4) + \frac{(z-\pi/4)^3}{3!} f''' + \dots$$

$$f(z) = 1/\sqrt{2} + (z - \pi/4)(-1/\sqrt{2}) + \frac{(z-\pi/4)^2}{2!}(-1/\sqrt{2}) + \frac{(z-\pi/4)^3}{3!}(1/\sqrt{2})$$

2. Expand $f(z) = \log(1+z)$ as a Taylor's series about $z = 0$

Soln

$f(z) = \log(1+z)$	$f(0) = \log(1+0) = 0$
$f'(z) = \frac{1}{1+z}$	$f'(0) = 1$
$f''(z) = \frac{-1}{(1+z)^2}$	$f''(0) = -1$
$f'''(z) = \frac{2}{(1+z)^3}$	$f'''(0) = 2$

$$f(z) = f(a) + \frac{z-a}{1!} f'(a) + \frac{(z-a)^2}{2!} f''(a) + \frac{(z-a)^3}{3!} f'''(a) + \cdots$$

$$= f(0) + \frac{(z-0)}{1!} f'(0) + \frac{(z-0)^2}{2!} f''(0) + \frac{(z-0)^3}{3!} f'''(0) + \cdots$$

$$= 0 + \frac{z}{1!}(1) + \frac{z^2}{2!}(-1) + \frac{z^3}{3}(2) + \cdots$$

$$f(z) = \frac{z}{1!} - \frac{z^2}{2!} + \frac{z^3}{3} + \cdots$$

LAURENT SERIES:-

Let C_1 & C_2 be two concentric circles $|z-a| = R_1$ and $|z-a| = R_2$, where $R_2 < R_1$. Let $f(z)$ be analytic inside and on the annular region R between C_1 and C_2. Then for any $z \in R$.

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + b_n (z-b)^{-n} \quad -①$$

where

$$a_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-a)^{1+n}} dz \ \&$$

$$b_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-a)^{1-n}} dz$$

This series is called Laurent's series of $f(z)$ about the point $z=a$.

Region of Convergence (ROC):

The region in which the expansion of a function $f(z)$ valid is called the region of convergence.

Note: i) In the Laurent series of $f(z)$ about $z=a$ the terms containing the positive term powers is called the regular part.

ii) In the Laurent series of $f(z)$ about $z=a$, the terms containing the negative powers is called the Principal part.

1. Expand the function $f(z) = \dfrac{z^2-1}{(z+2)(z+3)}$ as a Laurent series in i) $|z|>3$. ii) $2<|z|<3$;

Soln

$$f(z) = \frac{z^2-1}{(z+2)(z+3)}$$

$$\frac{z^2-1}{(z+2)(z+3)} = A + \frac{B}{z+2} + \frac{C}{z+3}$$

$$\frac{z^2-1}{(z+2)\cancel{(z+3)}} = \frac{A(z+2)(z+3) + B(z+3) + C(z+2)}{(z+2)\cancel{(z+3)}}$$

$$z^2-1 = A(z+2)(z+3) + B(z+3) + C(z+2) \quad \text{—} (1)$$

Put $z=-2$

$\boxed{B=3}$

Put $z=-3$

$8 = -C$

$\boxed{C=-8}$

Put $z=0$

$-1 = 6A + 3B + 2C$

$-1 = 6A + 9 - 16$

$6A = 6$ $\boxed{A=1}$

$$\frac{z^2-1}{(z+2)(z+3)} = 1 + \frac{3}{z+2} - \frac{8}{z+3}$$

i) $|z|>3 \Rightarrow 3<|z| \Rightarrow \frac{3}{|z|} < 1$

$$\frac{z^2-1}{(z+2)(z+3)} = 1 + \frac{3}{z(1+2/z)} - \frac{8}{z(1+3/z)}$$

$$= 1 + \frac{3}{z}(1+2/z)^{-1} - \frac{8}{z}(1+3/z)^{-1}$$

$= 1 + 3/z \sum_{n=0}^{\infty} (-1)^n (2/z)^n - \frac{8}{z} \sum_{n=0}^{\infty} (-1)^n (3/z)^n$

$= 1 + 3/z \sum_{n=0}^{\infty} (-1)^n \frac{2^n}{z^n} - \frac{8}{z} \sum_{n=0}^{\infty} (-1)^n \frac{3^n}{z^n}$

$f(z) = 1 + \sum_{n=0}^{\infty} \left\{ \frac{3(-1)^n 2^n - 8(-1)^n 3^n}{z^{n+1}} \right\}$

$2 < |z| < |z| < 3$

ii, $2 < |z| < 3$ $\quad \frac{2}{|z|} < 1 \quad \frac{|z|}{3} < 1$

$f(z) = 1 + \frac{3}{z+2} - \frac{8}{z+3}$

$= 1 + \frac{3}{z(1+2/z)} - \frac{8}{3(1+z/3)}$

$= 1 + \frac{3}{z}(1+2/z)^{-1} - \frac{8}{3}(1+z/3)^{-1}$

$f(z) = 1 + 3/z \sum_{n=0}^{\infty} (-1)^n \frac{2^n}{z^n} - \frac{8}{3} \sum_{n=0}^{\infty} (-1)^n \frac{z^n}{3^n}.$

2. Obtain the Laurent's expansion for $f(z) = \frac{1}{(z+2)(z+1)^2}$ in $|z| > 2$ ii, $1 < |z| < 2$.

Soln Given $f(z) = \frac{1}{(z+2)(z+1)^2}$

$\frac{1}{(z+2)(z+1)^2} = \frac{A}{z+2} + \frac{B}{z+1} + \frac{C}{(z+1)^2}$

$\frac{1}{(z+2)(z+1)^2} = \frac{A(z+1)^2 + B(z+1)(z+2) + C(z+2)}{(z+2)(z+1)^2}$

$1 = A(z+1)^2 + B(z+1)(z+2) + C(z+2)$

put $z = -2$ $\quad \boxed{A = 1}$

put $z = -1$ $\quad \boxed{1 = C}$

put $z = 0$

$1 = A + 2B + 2C$

$1 - 3 = 2B$

$\boxed{B = -1}$

$$\frac{1}{(z+2)(z+1)^2} = \frac{1}{z+2} - \frac{1}{z+1} + \frac{1}{(z+1)^2}$$

ii) $|z|>2$

$$f(z) = \frac{1}{(z+1)^2} + \frac{1}{z+2} - \frac{1}{z+1}$$

$$= \frac{1}{z^2(1+1/z)^2} + \frac{1}{z(1+2/z)} - \frac{1}{z(1+1/z)}$$

$$= \frac{1}{z^2}(1+1/z)^{-2} + \frac{1}{z}\left(1+\frac{2}{z}\right)^{-1} - \frac{1}{z}(1+1/z)^{-1}$$

$$= 1/z^2 \sum_{n=0}^{\infty} (-1)^n (n+1)(1/z)^n - 1/z \sum_{n=0}^{\infty} (-1)^n (1/z)^n$$

$$+ 1/z \sum_{n=0}^{\infty} (-1)^n (2/z)^n$$

$$= \sum_{n=0}^{\infty} (-1)^n (n+1)(1/z)^{n+2} - \sum_{n=0}^{\infty} (-1)^n (1/z)^{n+1}$$

$$\sum_{n=0}^{\infty} (-1)^n \frac{1}{z} \cdot \frac{2^n}{z^n}.$$

$$f(z) = \sum_{n=0}^{\infty} (-1)^n (n+1) \frac{1}{z^{n+2}} + \sum_{n=0}^{\infty} \frac{(2^n-1)(-1)^n}{z^{n+1}}$$

3. Obtain the Laurent's expansion for $f(z) = \frac{9z^2-4z+1}{(z-1)(2z-1)(z+2)}$ in $1 < |z-1| < 3$.

Soln

$$f(z) = \frac{9z^2-4z+1}{(z-1)(2z-1)(z+2)}$$

$$\frac{9z^2-4z+1}{(z-1)(2z-1)(z+2)} = \frac{A}{z-1} + \frac{B}{2z-1} + \frac{C}{z+2}$$

$$\frac{9z^2-4z+1}{(z-1)(2z-1)(z+2)} = \frac{A(2z-1)(z+2)+B(z-1)(z+2)+C(z-1)(2z-1)}{(z-1)(2z-1)(z+2)}$$

$$9z^2-4z+1 = A(2z-1)(z+2)+B(z-1)(z+2)+C(z-1)(2z-1) \quad -① $$

Put $z=1$ in eqn ①	put $z=1/2$ in eqn ①
$9-4+1=3A$	$\frac{9}{4}-2+1 = -\frac{5}{4}B$
$\Rightarrow 6=3A$	$\frac{9}{4}-1 = -5/4\,B$
$\boxed{A=2}$	$5/4 = -5/4\,B$
	$\boxed{B=-1}$

put $z=-2$ in eqn ①

$36+8+1 = C(-3)(-5)$

$45 = 15C$

$\boxed{C=3}$

$$f(z) = \frac{2}{z-1} - \frac{1}{2z-1} + \frac{3}{z+2}$$

we have $1<|z-1|<3$

put $z-1=u$; $z=u+1$

$1<|u|<3$

$1<\lvert u\rvert$	$\lvert u\rvert<3$
$\frac{1}{\lvert u\rvert}<1$	$\lvert u/3\rvert<1$

$$f(z) = \frac{2}{u} - \frac{1}{2(u+1)-1} + \frac{3}{u+1+2}$$

$$= \frac{2}{u} - \frac{1}{2u+1} + \frac{3}{u+3}$$

$$= \frac{2}{u} - \frac{1}{2u(1+1/2u)} + \frac{3}{3(1+u/3)}$$

$$= \frac{2}{u} - \frac{1}{2u}\left(1+\frac{1}{2u}\right)^{-1} + \left(1+\frac{u}{3}\right)^{-1}$$

$$= \frac{2}{u} - \frac{1}{2u}\sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{2u}\right)^n + \sum_{n=0}^{\infty}(-1)^n\left(\frac{u}{3}\right)^n$$

$$= \frac{2}{z-1} - \frac{1}{2(z-1)}\sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{2(z-1)}\right)^n +$$

$$\sum_{n=0}^{\infty}\left(\frac{z-1}{3}\right)^n(-1)^n$$

$$f(z) = \frac{2}{z-1} - \sum_{n=0}^{\infty}(-1)^n\left[\frac{1}{2(z-1)}\right]^{n+1} + \sum_{n=0}^{\infty}\left(\frac{z-1}{3}\right)^n$$

4. If $0 < |z-1| < 2$, then $f(z) = \dfrac{z}{(z-1)(z-3)}$ is a series of positive and negative powers of $(z-1)$

Soln

Let $z-1 = u$, $z = u+1$

$0 < |z-1| < 2$ becomes $0 < |u| < 2$

$$f(z) = \frac{z}{(z-1)(z-3)}$$

$$\frac{z}{(z-1)(z-3)} = \frac{A}{z-1} + \frac{B}{z-3}$$

$$\frac{z}{(z-1)\cancel{(z-3)}} = \frac{A(z-3)+B(z-1)}{(z-1)\cancel{(z-3)}}$$

$z = A(z-3) + B(z-1)$ — (1)

put $z=1$ in eqn (1)

$1 = -2A$

$\boxed{A = -1/2}$

put $z=3$ in eqn (1)

$3 = 2B$

$\boxed{B = 3/2}$

$$f(z) = \frac{-1/2}{z-1} + \frac{3/2}{z-3}$$

$$f(z) = \frac{-1}{2(z-1)} + \frac{3}{2(z-3)}$$

$$f(z) = \frac{-1}{2u} + \frac{3}{2(u+1-3)}$$

$$= \frac{-1}{2u} + \frac{3}{2(u-2)}$$

$$= \frac{-1}{2u} + \frac{3}{2(-2)(1-u/2)}$$

$$= \frac{-1}{2u} - \frac{3}{4}(1-u/2)^{-1}$$

$$= \frac{-1}{2u} - \frac{3}{4}\sum_{n=0}^{\infty}(u/2)^n$$

$$f(z) = \frac{-1}{2(z-1)} - \frac{3}{4}\sum_{n=0}^{\infty}\left(\frac{z-1}{2}\right)^n$$

5. Expand $f(z) = \dfrac{e^z}{(z-1)^2}$ as a Laurent series about $z=1$.

Soln Given $f(z) = \dfrac{e^z}{(z-1)^2}$

put $\quad = u \Rightarrow z = u+1$

$$f(z) = \frac{e^{u+1}}{u^2} = \frac{e^u \cdot e}{u^2}$$

$$= \frac{e}{u^2}\left[1 + \frac{u}{1!} + \frac{u^2}{2!} + \cdots\right]$$

$$= e\left[1/u^2 + 1/1!\,u + 1/2! + u/3! + u^2/4! + \cdots\right]$$

$$f(z) = e\left[\frac{1}{(z-1)^2} + \frac{1}{1!(z-1)} + 1/2! + \frac{z-1}{3!} + \cdots\right]$$

poles and Residues

Singular point :-

A point at which a function ceases to be analytic is called a singular point (or) singularity of $f(z)$

Types of singular points:

Isolated singular point:

A singular point $z=a$ of a function $f(z)$ is called an isolated singular point if there exists a circle with center 'a' which contains no other singular point of $f(z)$

when $z=a$ is an isolated singular point of $f(z)$, we can expand $f(z)$ in a Laurent's series about $z=a$,

Thus

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} b_n (z-a)^{-n}$$

pole:

A point $z=a$ is called pole of order m for $f(z)$ if $\lim_{z \to a} (z-a)^m f(z) \neq 0$.

Simple pole:-

A pole of order one is called a simple pole.

Removable singular point:

A point $z=a$ is said to be a removable singular point of $f(z)$ if the Laurent series of $f(z)$

about $z=a$ does not contain the principal parts (all b_n's are zero).

Essential singular point:

A singular point $z=a$ is said to be an essential singular point of $f(z)$ if the Laurent series of $f(z)$ about $z=a$ posses of infinite number of terms in the principal part (terms containing negative powers).

working Rule for finding the types of singularity.

S.NO	Type of singularity	Nature of Laurent series of $f(z)$ about $z=a$
1.	Removable singularity	No negative of $z-a$
2.	pole	A finite no of negative power of $z-a$
3.	Essential singularity	An infinite no of $-ve$ power of $z-a$

Find the singularities of the following functions i) $f(z)=e^z$ ii) $f(z)=\cos z$ iii) $f(z)=e^{z+1}$ iv) $f(z)=\sin(z-2)$

Soln

i) $f(z)=e^z$

WKT $f(z)=e^z=1+\frac{z}{1!}+\frac{z^2}{2!}+\frac{z^3}{3!}+\dots$

Since $f(z)$ has no $1/z$ terms,

$\therefore z=0$ is removable singularity of $f(z)$

ii) $f(z)=\cos z$

WKT $f(z)=\cos z=1-\frac{z^2}{2!}+\frac{z^4}{4!}+\dots$

since $f(z)$ has $\frac{1}{z}$ terms, $z=0$ is removable singularity of $f(z)$

iii) $f(z) = e^{z+1}$

$$f(z) = e^{z+1} = 1 + \frac{(z+1)}{1!} + \frac{(z+2)}{2!} + \dots$$

Since $f(z) = e^{z+1}$ has no $\frac{1}{z+1}$ terms, $z=-1$ is removable singularity of $f(z)$

iv) $f(z) = \sin(z-2)$

$$f(z) = \sin(z-2) = \frac{z-2}{1!} - \frac{(z-2)^3}{3!} + \dots$$

since $f(z)$ has no $\frac{1}{z-2}$ terms, $z=2$ is removable singularity of $f(z)$.

2. Find the singularities for the following i) $e^{1/z}$

ii) $\sin\left(\frac{1}{z+1}\right)$

soln Let $f(z) = e^{1/z}$

$$f(z) = e^{1/z} = 1 + \frac{(1/z)}{1!} + \frac{(1/z)^2}{2!} + \dots$$

$$f(z) = 1 + \frac{1}{z!} + \frac{1}{z^2 \cdot 2!} + \dots$$

since $f(z)$ contains infinite no of $1/z$ terms $z=0$ is an essential singularity of $f(z)$

ii) Let $f(z) = \sin\left(\frac{1}{z+1}\right)$

$$f(z) = \sin\left(\frac{1}{z+1}\right) = \frac{\frac{1}{z+1}}{1!} + \frac{\left(1/z+1\right)^3}{2!} + \dots$$

$$= \frac{1}{(z+1)!} + \frac{1}{(z+1)^3 2!} + \dots$$

since $f(z)$ contains infinite no of $\frac{1}{z+1}$ terms, $z=-1$ is an essential singularity of $f(z)$

3. Find the nature of the singularities for the following

i, $\frac{\sin z}{z}$ ii, $\frac{1-\cos z}{z^2}$ iii, $\frac{1}{z(e^z-1)}$

soln

i, $f(z) = \frac{\sin z}{z}$

$$f(z) = \frac{\frac{z}{1!} - \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots}{z}$$

$$= \frac{1}{z}\left\{\frac{z}{1!} - \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots\right\}$$

$$= 1 - \frac{z^3}{3!} + \frac{z^4}{5!} \cdots$$

since $f(z)$ has no $\frac{1}{z-0}$ terms, $z=0$ is removable singularity of $f(z)$

ii, $\frac{1-\cos z}{z^2}$

$$f(z) = \frac{1-\cos z}{z^2}$$

$$= \frac{1-\left(1-\frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} \cdots\right)}{z^2}$$

$$= \frac{1}{z^2}\left\{\frac{z^2}{2!} - \frac{z^4}{4!} + \frac{z^6}{6!} + \cdots\right\}$$

$$f(z) = 1/2! - \frac{z^2}{4!} + \frac{z^4}{6!} + \cdots$$

since $f(z)$ has $\frac{1}{z-0}$ terms, $z=0$ is a removable singularity of $f(z)$.

iii) $f(z) = \dfrac{1}{z(e^z-1)}$

$z(e^z-1) = 0$

$z=0$; $e^z-1=0$

$z=0$; $e^z=1$

$e^z = e^{\pm 2n\pi i}$ $n=0,1,2\ldots$

$z=0$; $z = \pm 2\pi n i$ $n=0,1,2\ldots\ldots$

$z=0, z=0,$ $z = \pm 2\cancel{n}\pi i, \pm 4\pi i, \ldots$

$z=0$ is a pole of order 2.

$z = \pm 2\pi i, \pm 4\pi i \ldots$ are poles of order 1.

Residue:-

The residue of $f(z)$ at $z=a$ is the coefficient of $\dfrac{1}{z-a}$ in the Laurent series of $f(z)$ about $z=a$

$$[\text{Res } f(z)]_{z=a} = \frac{1}{2\pi i}\int_c f(z)\,dz,$$

where c is a circle with center a.

Calculation of Residues:-

1) If $z=a$ is a pole of order one (simple pole) for $f(z)$, then

$$[\text{Res } f(z)]_{z=a} = \lim_{z\to a} (z-a)\, f(z)$$

2) If $z=a$ is a pole of order 2 for $f(z)$, then

$$[\text{Res } f(z)]_{z=a} = \lim_{z\to a} \frac{d}{dz}\left[(z-a)^2 f(z)\right]$$

3) If $z=a$ is a pole of order m for $f(z)$ then

$$[\text{Res } f(z)]_{z=a} = \frac{1}{(m-1)!} \lim_{z\to a} \frac{d^{m-1}}{dz^{m-1}} [(z-a)^m \cdot f(z)]$$

1. Find the residue of $f(z)$, $f(z) = \dfrac{z^2}{(z-1)^2 (z-2)}$ at each poles.

Soln Given $f(z) = \dfrac{z^2}{(z-1)^2 (z+2)}$

To find poles:-

$(z-1)^2 (z+2) = 0$

$z=-2$ pole of order one

$z=1$ is a pole of order 2.

$$[\text{Res } f(z)]_{z=-2} = \lim_{z\to -2} (z+2) f(z)$$

$$= \lim_{z\to -2} (z+2) \frac{z^2}{(z-1)^2 (z+2)}$$

$$= \lim_{z\to -2} \frac{z^2}{(z-1)^2} = \frac{(-2)^2}{(-2-1)^2} = 4/9$$

$$\therefore [\text{Res } f(z)]_{z=-2} = 4/9$$

$$[\text{Res } f(z)]_{z=1} = \lim_{z\to 1} \frac{d}{dz} [(z-1)^2 \cdot f(z)]$$

$$= \lim_{z\to 1} \frac{d}{dz} \left[\cancel{(z-1)^2} \frac{z^2}{\cancel{(z-1)^2}(z+2)} \right]$$

$$= \lim_{z\to 1} \frac{d}{dz} \left[\frac{z^2}{z+2} \right]$$

$$= \lim_{z\to 1} \left[\frac{(z+2)\, 2z - z^2(1)}{(z+2)^2} \right]$$

$$= \lim_{z \to 1} \left[\frac{2z(z+2) - z^2}{(z+2)^2} \right]$$

$$= \frac{2(3) - 1^2}{3^2} = \frac{6-1}{9} = 5/9$$

$$\left[\text{Res } f(z) \right]_{z=1} = 5/9$$

2. Find the residue of $f(z) = \frac{ze^z}{(z-a)^3}$ at $z = a$

Soln

$$f(z) = \frac{ze^z}{(z-a)^3}$$

To find the pole:

$$(z-a)^3 = 0$$

$z = a$, pole of order 3.

$$\left[\text{Res } f(z) \right]_{z=a} = \frac{1}{(3-1)!} \lim_{z \to a} \frac{d^2}{dz^2} \left((z-a)^3 \cdot f(z) \right)$$

$$= \frac{1}{2!} \lim_{z \to a} \frac{d^2}{dz^2} \left[\cancel{(z-a)^3} \cdot \frac{ze^z}{\cancel{(z-a)^3}} \right]$$

$$= \frac{1}{2} \lim_{z \to a} \frac{d^2}{dz^2} (ze^z)$$

$$= \frac{1}{2} \lim_{z \to a} \frac{d}{dz} \left[\frac{d}{dz} (ze^z) \right]$$

$$= \frac{1}{2} \lim_{z \to a} \frac{d}{dz} \left[ze^z + e^z \right]$$

$$= \frac{1}{2} \lim_{z \to a} \left[ze^z + e^z + e^z \right]$$

$$= \tfrac{1}{2} \left[ae^a + e^a + e^a \right]$$

$$= \frac{e^a (a+2)}{2}$$

Residues using Laurent series :-

$[\text{Res } f(z)]_{z=a}$ = coefficient of $1/z-a$ in the Laurent series of $f(z)$ about $z=a$.

1. Find the residue of $f(z) = \dfrac{1-2z}{(z-1)(z+2)}$ at its isolated sigularities using Laurent series expansion.

Soln Given $f(z) = \dfrac{1-2z}{(z-1)(z+2)}$

$z=1$, $z=-2$ are isolated singularities of $f(z)$

$$\frac{1-2z}{(z-1)(z+2)} = \frac{A}{z-1} + \frac{B}{z+2}$$

$$\frac{1-2z}{\cancel{(z-1)(z+2)}} = \frac{A(z+2)+B(z-1)}{\cancel{(z-1)(z+2)}}$$

$$1-2z = A(z+2) + B(z-1) \quad — ①$$

put $z=1$ in eqn ①

$-1 = 3A$

$\boxed{A = -1/3}$

put $z=-2$ in eqn ①

$5 = -3B$

$\boxed{B = -5/3}$

$$f(z) = \frac{-1}{3(z-1)} - \frac{5}{3(z+2)}$$

Case(i) : At $z=2$

put $z+2 = u$, $z = u-2$

$$f(z) = \frac{-5}{3u} - \frac{1}{3(u-2-1)}$$

$$= \frac{-5}{3u} - \frac{1}{3(u-3)}$$

$$= \frac{-5}{3u} - \frac{1}{9[1-u/3]}$$

$$= \frac{-5}{3u} + \frac{1}{9}\left(1-u/3\right)^{-1}$$

$$= \frac{-5}{3u} + \frac{1}{9}\sum_{n=0}^{\infty}\left(u/3\right)^{n}$$

$$= \frac{-5}{3(z+2)} + \frac{1}{9}\sum_{n=0}^{\infty}\left(\frac{z+2}{3}\right)^{n}$$

$$\left[\text{Res } f(z)\right]_{z=-2} = \text{coeff of } \frac{1}{z+2} = -5/3$$

Case ii, At $z=1$

put $z-1=u$; $z=u+1$

$$f(z) = \frac{-5}{3(u+1+2)} - \frac{1}{3u}$$

$$= \frac{-5}{3(u+3)} - \frac{1}{3u}$$

$$= \frac{-1}{3u} - \frac{5}{3\cdot 3(1+u/3)} \Rightarrow \frac{-1}{3u} - \frac{5}{9(1+u/3)}$$

$$= \frac{-1}{3u} \cdot \frac{-5}{9}\left(1+u/3\right)^{-1}$$

$$= \frac{-1}{3u} - \frac{5}{9}\sum_{n=0}^{\infty}(-1)^{n}\left(u/3\right)^{n}$$

$$f(z) = \frac{-1}{3(z-1)} - \frac{5}{9}\sum_{n=0}^{\infty}(-1)^{n}\left(\frac{z-1}{3}\right)^{n}$$

$$\left[\text{Res } f(z)\right]_{z=1} = \text{coeff of } \frac{1}{z-1} = -1/3$$

2. Find the residues of $f(z) = \dfrac{z^2}{(z-1)(z+2)^2}$ at its isolated singularities using Laurent series expansion.

Soln

$$f(z) = \frac{z^2}{(z-1)(z+2)^2}$$

Let $(z-1)(z+2)^2 = 0$

$z-1=0$; $(z+2)^2 = 0$

$z=1$; $z+2=0 \Rightarrow z=-2$

$$f(z) = \frac{A}{z-1} + \frac{B}{z+2} + \frac{C}{(z+2)^2}$$

$$f(z) = \frac{A(z+2)^2 + B(z-1)(z+2) + C(z+1)}{(z-1)(z+2)^2}$$

$$\frac{z^2}{(z-1)(z+2)^2} = \frac{A(z+2)^2 + B(z-1)(z+2) + C(z-1)}{(z-1)(z+2)^2}$$

$z^2 = A(z+2)^2 + B(z-1)(z+2) + C(z-1)$ — ①

put $z=1$ in eqn ①

$1 = 9A$

$\boxed{A = 1/9}$

put $z=-2$ in eqn ①

$4 = -3C$

$\boxed{C = -4/3}$

put $z=0$ in eqn ①

$0 = 4A - 2B - C$

$0 = 4/9 - 2B + 4/3$

$0 = 16/9 - 2B \Rightarrow 2B = 16/9 \Rightarrow \boxed{B = 8/9}$

$$f(z) = \frac{1}{9(\;-1)} + \frac{8}{9(z+2)} - \frac{4}{3(z+2)^2}$$

case ii, To find $\left[\operatorname{Res} f(z)\right]_{z=1}$

put $z-1=u$; $z=u+1$

$$f(z)=\frac{1}{9u}+\frac{8}{9(u+3)}-\frac{4}{3(u+3)^2}$$

$$=\frac{1}{9u}+\frac{8}{9\cdot 3\cdot(1+u/3)}-\frac{4}{3\cdot 9(1+u/3)^2}$$

$$=\frac{1}{9u}+\frac{8}{27(1+u/3)}-\frac{4}{27(1+u/3)^2}$$

$$=1/9u+\frac{8}{27}(1+u/3)^{-1}-\frac{4}{27}\left[1+u/3\right]^{-2}$$

$$=1/9u+\frac{8}{27}\sum_{n=0}^{\infty}(-1)^n(u/3)^n-$$

$$\frac{4}{27}\sum_{n=0}^{\infty}(-1)^n(n+1)(u/3)^n$$

$$f(z)=\frac{1}{9(z-1)}+\frac{8}{27}\sum_{n=0}^{\infty}(-1)^n\frac{(z-1)^n}{3^n}-\frac{4}{27}(-1)^n(n+1)\frac{(z-1)^n}{3^n}$$

$$\left[\operatorname{Res} f(z)\right]_{z=1}=\text{coeff } \frac{1}{z-1}=1/9$$

$$\boxed{\left[\operatorname{Res} f(z)\right]_{z=1}=1/9}$$

case iii, To find $\left[\operatorname{Res} f(z)\right]_{z=-2}$

$$f(z)=\frac{1}{9(u-2-1)}+\frac{8}{9(u)}-\frac{4}{3u^2}$$

$$=\frac{1}{9(u-3)}+\frac{8}{9u}-\frac{4}{3u^2}$$

$$=\frac{1}{9(-3)(-u/3)}+\frac{8}{9u}-\frac{4}{3u^2}$$

$$= \frac{1}{-27(1-u/3)} + \frac{8}{9u} - \frac{4}{3u^2}$$

$$= \frac{-1}{27}(1-u/3)^{-1} + \frac{8}{9u} - \frac{4}{3u^2}$$

$$= \frac{-1}{27}\sum_{n=0}^{\infty}(u/3)^n + \frac{8}{9u} - \frac{4}{3u^2}$$

$$f(z) = \frac{-1}{27}\sum_{n=0}^{\infty}\frac{(z+2)^n}{3^n} + \frac{8}{9(z+2)} - \frac{4}{3(z+2)^2}$$

$$[\text{Res } f(z)]_{z=-2} = \text{coeff of } \frac{1}{z+2} = 8/9$$

$$[\text{Res } f(z)]_{z=-2} = 8/9$$

3. Find all possible Laurent expansion of the function $f(z) = \frac{-3z+4}{z(-z+1)(2-z)}$ about $z=0$. Indicate the region of convergence in each case. Find also the residue of $f(z)$ at $z=0$. using the approriate Laurent series.

Soln Given $f(z) = \frac{4-3z}{z(1-z)(2-z)}$

$$\frac{4-3z}{z(1-z)(2-z)} = \frac{A}{z} + \frac{B}{\ldots} + \frac{C}{2-z}$$

$$\frac{4-3z}{z(1-z)(2-z)} = \frac{A(1-z)(2-z) + Bz(2-z) + C(1-z)z}{z(1-z)(-z+2)}$$

$$4-3z = A(1-z)(2-z) + Bz(2-z) + Cz(1-z) \quad \text{①}$$

put $z=0$ in eqn ①

$4 = 2A$

$\boxed{A=2}$

put $z=1$ in eqn ①

$1 = B$

$\boxed{B=1}$

put $z=2$ in eqn ①

$-2 = -2C \Rightarrow \boxed{C=1}$

$$f(z) = \frac{2}{z} + \frac{1}{1-z} + \frac{1}{2-z}$$

case i, ROC $|z|<1$

$$f(z) = \frac{2}{z} + (1-z)^{-1} + \frac{1}{2(1-z/2)}$$

$$= \frac{2}{z} + (1-z)^{-1} + \frac{1}{2}(1-z/2)^{-1}$$

$$= \frac{2}{z} + \sum_{n=0}^{\infty} z^n + 1/2 \sum_{n=0}^{\infty} (z/2)^n$$

$$f(z) = \frac{2}{z} + \sum_{n=0}^{\infty} z^n + 1/2 \sum_{n=0}^{\infty} \frac{z^n}{2^n}$$

$[\mathrm{Res}\, f(z)]_{z=0}$ = coeff of $1/z = 2$.

$$\boxed{[\mathrm{Res}\, f(z)]_{z=0} = 2}$$

case ii, ROC $|z|>2$

$$f(z) = \frac{2}{z} + \frac{1}{z(1-1/z)} + \frac{1}{-z(1-2/z)}$$

$$= \frac{2}{z} + \frac{1}{-z}(1-1/z)^{-1} + (-1/z)\left(1-\frac{2}{z}\right)^{-1}$$

$$= \frac{2}{z} - \frac{1}{z}\sum_{n=0}^{\infty}(1/z)^n - 1/z \sum_{n=0}^{\infty}\left(\frac{2}{z}\right)^n$$

$$f(z) = \frac{2}{z} - \sum_{n=0}^{\infty}\frac{1}{z^{n+1}} - \sum_{n=0}^{\infty}\frac{2^n}{z^{n+1}}$$

4. Find the residue of $f(z) = e^{1/z}$ at its isolated singularities using Laurent series expansion.

Soln Given $f(z) = e^{1/z}$

$$f(z) = 1 + \frac{1/z}{1!} + \frac{(1/z)^2}{2!} + \dots$$

$f(z)$ contains infinite numbers of $1/z$ terms, $z=0$ is an essential singularity of $f(z)$

$$\left[\text{Res } f(z)\right]_{z=0} = \text{coeff of } 1/z = 1/1!$$

$$\left[\text{Res } f(z)\right]_{z=0} = 1$$

5. Find the residue of $f(z) = z\sin(1/z)$ at its isolated singularities using Laurent series expansion.

Soln Given $f(z) = z\sin(1/z)$

$$f(z) = z\left[\frac{1/z}{1!} - \frac{(1/z)^3}{3!} + \frac{(1/z)^5}{5!} - \dots\right]$$

$$= z\left[\frac{1}{z1!} - \frac{1}{z^3 \cdot 3!} + \frac{1}{z^5 \cdot 5!} - \dots\right]$$

$$= 1/1! - \frac{1}{z^2 \cdot 3!} + \frac{1}{z^4 \cdot 5!} \dots$$

Since $f(z)$ contains infinite number of $\frac{1}{z}$ terms, $z=0$ is an essential singularities of $f(z)$

$$\therefore \left[\text{Res } f(z)\right]_{z=0} = \text{coeff } \frac{1}{z} = 0$$

$$\therefore \left[\text{Res } f(z)\right]_{z=0} = 0$$

5. Find the residue of $\frac{1}{z-\sin z}$ at its poles

Soln Given $f(z) = \frac{1}{z-\sin z}$.

$$f(z) = \frac{\quad}{z-\left(\frac{z}{1!}-\frac{z^3}{3!}+\frac{z^5}{5!}-\cdots\right)}$$

$$= \frac{1}{z-z+\frac{z^3}{3!}-\frac{z^5}{5!}+\cdots}$$

$$= \frac{1}{\frac{z^3}{6}-\frac{z^5}{120}+\cdots}$$

$$= \frac{1}{\frac{z^3}{6}\left[1-\frac{z^2}{20}+\cdots\right]}$$

$$= \frac{6}{z^3\left[1-\frac{z^2}{20}+\cdots\right]}$$

To find poles put $z^3\left[1-\frac{z^2}{20}+\cdots\right]=0$; $z=0$ is a pole of order 3.

$$f(z) = \frac{6}{z^3}\left[1-\frac{z^2}{20}+\cdots\right]^{-1}$$

$$= \frac{6}{z^3}\left[1-\frac{z^2}{20}+\cdots\right]^{-1}$$

$$= \frac{6}{z^3}\left[1+\frac{z^2}{20}-\cdots\right]$$

$$= \frac{6}{z^3}+\frac{6}{20z}-\cdots$$

$$\left[\operatorname{Res} f(z)\right]_{z=0} = \text{coeff of } 1/z = 6/20$$

$$\therefore \left[\operatorname{Res} f(z)\right]_{z=0} = 3/10 \,//.$$

Cauchy's Residue theorem:-

Statement:

If $f(z)$ is analytic inside and on a simple closed curve C, except at a finite numbers of singular points $z_1, z_2 \dots z_n$ lying inside C, then

$$\int_C f(z)\,dz = 2\pi i\left[R(z_1) + R(z_2) + \dots + R(z_n)\right]$$

where the integral over C is taken in the anticlockwise sense.

Problems:-

1. Evaluate $\int_C \frac{2z-1}{z(z+1)(z-3)}\,dz$, where c is the circle $|z|=2$

Soln Given $f(z) = \frac{2z-1}{z(z+1)(z-3)}$ $|C|$ is given as $|z|=2$

To find Poles:-

$z(z+1)(z-3)=0$

$\Rightarrow z=0, z=-1, z=3$ are all poles of order 1

To find Residues $\left[\operatorname{Res} f(z)\right]_{z=a} = \lim_{z\to a}(z-a)f(z)$

At $z=0$

$z=0$

$\Rightarrow |z|=0 \quad = 0<2$

$\therefore z=0$ lies inside c

$$R_1 = \left[\operatorname{Res} f(z)\right]_{z=0} = \lim_{z\to 0}\left[(z-0)f(z)\right]$$

$$= \lim_{z\to 0}\left[\cancel{z}-0 \; \frac{(2z-1)}{\cancel{z}(z+1)(z-3)}\right]$$

$$= \lim_{z\to 0}\left[\frac{2z-1}{(z+1)(z-3)}\right] = \frac{-1}{1\cdot(-3)} = 1/3$$

At $z=-1$

$z=-1$

$|z| = |-1| = 1 < 2$; $z=-1$ lies inside c.

$R_2 = [\text{Res } f(z)]_{z=-1}$

$= \lim_{z \to -1} [(z+1) f(z)]$

$= \lim_{z \to -1} \left[\cancel{(z+1)} \cdot \frac{2z-1}{z\cancel{(z+1)}(z-3)} \right]$

$= \lim_{z \to -1} \left[\frac{2z-1}{z(z-3)} \right] = \frac{-3}{-1(-1-3)} = -3/4$

Case(iii) $z=3$

$z=3$

$|z| = |3| = 3 > 2$

$\Rightarrow$ $z=3$ lies outside of c

$\therefore$ $\boxed{R_3 = 0}$

$R_3 = [\text{Res } f(z)]_{z=3} = 0$

$\because \int_C \frac{\frac{2z-1}{z(z+1)}}{z-3} dz = 0$

By Cauchy's Residue theorem,

$\int_C \frac{2z-1}{z(z+1)(z-3)} = 2\pi i [R_1 + R_2 + R_3]$

$= 2\pi i [1/3 - 3/4]$

$= 2\pi i [-5/12] = \frac{-5\pi i}{6}$

2. Evaluate $\int_C \frac{3z^2+z-1}{(z^2-1)(z-3)} dz$, using residue theorem, where C is $|z|=2$.

Sol: Given $f(z) = \int_C \frac{3z^2+z-1}{(z^2-1)(z-3)} dz$ and $|z|=2$

To find poles:

$(z^2-1)(z-3)=0$

$z^2-1=0$ | $z-3=0$

$z=\pm 1$ | $z=3$

$z=-1, 1, 3$ are poles of order one.

To find Residues:

case i, $z=-1$:

$|z|=2$

$|z| = |-1| = 1 < 2$, $z=-1$ lies inside C

$R_1 = [\text{Res } f(z)]_{z=-1}$

$= \lim_{z\to -1} \left[(z+1) \frac{3z^2+z-1}{(z+1)(z-1)(z-3)}\right]$

$\lim_{z\to -1} \left[\frac{3z^2+z-1}{(z-1)(z-3)}\right]$

$= \frac{3(-1)^2-1(1)-1}{(-2)(-4)} = \frac{3-2}{8} = 1/8$

Case ii, $z=1$ $|z| = |1| = 1 < 2$ $z=1$ lies inside C

$R_2 = [\text{Res } f(z)]_{z=1}$

$$\Rightarrow \lim_{z\to 1} \left[(z-1)\, f(z) \right]$$

$$= \lim_{z\to 1} \left[\cancel{(z-1)} \frac{3z^2+z-1}{(z+1)\cancel{(z-1)}(z-3)} \right]$$

$$= \lim_{z\to 1} \left[\frac{3z^2+z-1}{(z+1)(z-3)} \right]$$

$$= \frac{3(1^2)+1-1}{2 \cdot -2} = 3/-4 = -3/4$$

$$\boxed{R_2 = -3/4}$$

Case iii, $z = 3$

$|z| = |3| = 3 > 2$

$\therefore z = 3$ lies outside C

$\therefore R_3 = \left[\text{Res } f(z) \right]_{z=3} = 0$

$$\boxed{R_3 = 0}$$

By Cauchy's Residue theorem

$$\int_C \frac{3z^2+z-1}{(z^2-1)(z-3)}\, dz = 2\pi i \left[R_1 + R_2 \right]$$

$$= 2\pi i \left[1/8 - 3/4 \right] = 2\pi i \left[1 - 6/8 \right]$$

$$= 2\pi i \left[-5/8 \right]$$

$$= -\frac{5\pi i}{4} \,//.$$

$$\therefore \int_C \frac{3z^2+z-1}{(z^2-1)(z-3)} = -5\pi i/4$$

8. Evaluate $\int_{|z|=3} \frac{\cos \pi z^2 + \sin \pi z^2}{(z+1)(z+2)} dz$ using cauchy's residue theorem, where C is the unit circle $|z|=3$.

Soln To find Poles:-

$(z+1)(z+2)=0$

$\Rightarrow z=-1,\ z=-2$ are poles of order one

To find Residues:

case i, $z=-1$

$|z|=3$

$|z| = |-1| = 1 < 3$ $z=-1$ lies inside c

$R_1 = [\text{Res } f(z)]_{z=-1}$

$= \lim_{z \to -1} \left[(z+1) \frac{\cos \pi z^2 + \sin \pi z^2}{(z+1)(z+2)} \right]$

$= \lim_{z \to -1} \left[\frac{\cos \pi z^2 + \sin \pi z^2}{z+2} \right]$

$= \frac{\cos \pi (-1)^2 + \sin \pi (-1)^2}{-1+2} = \frac{-1+0}{1} = -1$

$\boxed{R_1 = -1}$

case ii, $z=-2$ $|z| = |-2| = 2 < 3$ $z=-2$ lies inside c.

$R_2 = [\text{Res } f(z)]_{z=-2}$

$= \lim_{z \to -2} \left[\cancel{(z+2)} \frac{\cos \pi z^2 + \sin \pi z^2}{(z+1)\cancel{(z+2)}} \right]$

$= \lim_{z \to -2} \left[\frac{\cos \pi z^2 + \sin \pi z^2}{z+1} \right]$

$= \frac{\cos\pi(-2)^2 + \sin\pi(-2)^2}{-2+1} = \frac{\cos 4\pi - \sin 4\pi}{-1} = \frac{1-0}{-1} = -1$

$\boxed{R_2 = -1}$

By Cauchy's Residue theorem,

$$\int_C \frac{\cos\pi z^2 + \sin\pi z^2}{(z+1)(z+2)} dz = 2\pi i \left[R_1 + R_2\right]$$

$= 2\pi i \left[-1-1\right] = -4\pi i$

$$\boxed{\int_C \frac{\cos\pi z^2 + \sin\pi z^2}{(z+1)(z+2)} = -4\pi i}$$

4. using Cauchy's residue theorem, evaluate $\int_C \frac{e^z}{(z^2+\pi^2)^2} dz$ where C is $|z|=4$.

Soln

To find poles:

$(z^2+\pi^2)^2 = 0$

$z^2+\pi^2 = 0 \Rightarrow z^2 = -\pi^2$

$z = \pm i\pi$ poles of order 2.

To find residues:-

Case (i): $z = +i\pi$

$|z| = 4$

$|z| = |i\pi| = \pi = 3.1424$

$z = +i\pi$ lies inside C

$R_1 = \left[\text{Res } f(z)\right]_{z=+i\pi}$

$= \lim_{z\to i\pi} \frac{d}{dz}\left[(z-\pi i)^2 f(z)\right]$

$$= \lim_{z \to i\pi} \frac{d}{dz}\left[\cancel{(z-\pi i)^2} \cdot \frac{e^z}{\cancel{(z-i\pi)^2} \cdot (z+i\pi)^2}\right]$$

$$= \lim_{z \to i\pi} \frac{d}{dz}\left[\frac{e^z}{(z+i\pi)^2}\right]$$

$$= \lim_{z \to i\pi}\left[\frac{(z+i\pi)^2 e^z - e^z\, 2(z+i\pi)\cdot 1}{(z+i\pi)^4}\right]$$

$$= \frac{(2i\pi)^2 e^{i\pi} - e^{i\pi}\, 2(2i\pi)}{(2i\pi)^4} \Rightarrow \frac{2\cancel{\pi i}\,[2\pi i\, e^{i\pi} - 2e^{i\pi}]}{(2\pi i)^4}$$

$$= \frac{2e^{i\pi}[\pi i - 1]}{2^3 \pi^3 i^3} = \frac{2\,(\cos\pi + i\sin\pi]\,(\pi i - 1)}{2^3 \pi^3 i^3}$$

$$= \frac{-2(\pi i - 1) \times i}{2^3 \pi^3 - i \times -i} = \frac{-i\,(\pi i - 1)}{4\pi^3 \cdot 1}$$

$$\boxed{\therefore R_1 = \frac{-i(\pi i - 1)}{4\pi^3 \cdot 1}}$$

To find R_2: $z = -i\pi$

$|z| = |i\pi| = |i\pi| = \pi = 3.1424$

$z = -i\pi$ lies inside C.

$R_2 = [\operatorname{Res} f(z)]_{z=-i\pi}$

$$= \lim_{z \to i\pi} \frac{d}{dz}\left[\cancel{(z+i\pi)^2} \cdot \frac{e^z}{(z-i\pi)^2 \cancel{(z+i\pi)^2}}\right]$$

$$= \lim_{z \to -i\pi} \frac{d}{dz}\left[\frac{e^z}{(z-i\pi)^2}\right]$$

$$= \lim_{z \to i\pi} \left[\frac{(z-i\pi)^2 e^z - e^z [2(z-i\pi)] \cdot 1}{(z-i\pi)^4} \right]$$

$$\Rightarrow \lim_{z \to -i\pi} \left[\frac{(z-i\pi)(z-i\pi) e^z - 2e^z}{(z-i\pi)^4} \right]$$

$$= \lim_{z \to -i\pi} \left[\frac{e^z [z - i\pi - 2]}{(z-i\pi)^3} \right] = \frac{e^{-i\pi} [-2i\pi - 2]}{(-2i\pi)^3}$$

$$= \frac{[\cos(-\pi) + i\sin(-\pi)] [-2(i\pi+1)]}{-8i^3 \pi^3}$$

$$= \frac{-1 \cdot 2(i\pi+1)}{8i^3\pi^3} = \frac{(1+i\pi)}{4\pi^3}(-i)$$

$$\boxed{R_2 = \frac{-i(1+i\pi)}{4\pi^3}}$$

By Cauchy's Residue theorem,

$$\int_c \frac{e^z}{(z^2+\pi^2)^2} dz = 2\pi i [R_1 + R_2]$$

$$= 2\pi i \left[\frac{-i(\pi i - 1)}{4\pi^3} - \frac{i(\pi i + 1)}{4\pi^3} \right]$$

$$= 2\pi i \left[\frac{-i(\pi i - 1 + \pi i + 1)}{4\pi^3} \right]$$

$$= \frac{\pi [2\pi i]}{2\pi^3} = \frac{i}{\pi}$$

$$\boxed{\int_c \frac{e^z}{(z^2+\pi^2)} dz = \frac{i}{\pi}}$$ //

5. Evaluate $\int_C \frac{dz}{(z^2+4)^2}$ where C is the circle $|z-i|=2$.

Soln Given $f(z) = \frac{1}{(z^2+4)^2}$

$f(z) = \frac{1}{[(z+2i)(z-2i)]^2}$

$f(z) = \frac{1}{(z+2i)^2(z-2i)^2}$

To find poles:

$(z+2i)(z-2i)^2 = 0$

$(z+2i)^2 = 0$ | $(z-2i)^2 = 0$

$z = -2i$ | $z = 2i$

$z = 2i, -2i$ are poles of order 2.

To find residues:

case (i) $z = 2i$

$|z-i| = 2$

$|2i-i| = |i| = 1 < 2$

$\therefore$ $z = 2i$ lies inside C.

$R_1 = \left[\text{Res } f(z)\right]_{z=2i}$

$= \lim_{z \to 2i} \frac{d}{dz}\left[(z-2i)^2 \frac{1}{(z-2i)^2(z+2i)^2}\right]$

$= \lim_{z \to 2i} \frac{d}{dz}\left[\frac{1}{(z+2i)^2}\right]$

$$= \lim_{z\to 2i}\left[\frac{(z+2i)^2(0) - 1\cdot 2\cdot(z+2i)\cdot 1}{(z+2i)^2}\right]$$

$$= \lim_{z\to 2i}\left[\frac{-2(z+2i)}{(z+2i)^4}\right] \Rightarrow \lim_{z\to 2i}\left[\frac{-2}{(z+2i)^3}\right] = \frac{-2}{4^3 i^3 2} = \frac{1}{32i}$$

$$\boxed{R_1 = \frac{1}{32i}}$$

Case ii, $z = -2i$

$|z-i| = 2$

$|-2i-i| = 2 = |-3i| = 3 > 2$

$z = -2i$ lies outside C

$\therefore R_2 = [\text{Res } f(z)]_{z=2i} = 0$

$$\therefore \int_C \frac{dz}{(z^2+4)^2} = 2\pi i\left[\frac{1}{32i}\right] = \pi/16$$

6. Evaluate $\int_C \frac{dz}{z+2}$, where C is the circle $|z|=1$. Hence find the value of $\int_0^{2\pi} \frac{1+2\cos\theta}{5+4\cos\theta}\,d\theta$ and

$$\int_0^{2\pi} \frac{\sin\theta}{5+4\cos\theta}\,d\theta$$

Soln Given $f(z) = \frac{1}{z+2}$

To find Poles:

$z+2 = 0 \Rightarrow z = -2$

At $z = -2$

$|z| = |2| = 2 > 1$

$\therefore z = -2$ lies outside C

$$\int_C \frac{dz}{z+2} = 0 \quad -①$$

put $z = e^{i\theta} \Rightarrow dz = ie^{i\theta}\, d\theta$

C is a circle, $\theta = 0$ to $\theta = 2\pi$

$$\therefore \int_C \frac{dz}{z+2} = \int_0^{2\pi} \frac{i e^{i\theta} d\theta}{e^{i\theta}+2} = 0$$

$$\int_0^{2\pi} \frac{i(\cos\theta + i\sin\theta)\, d\theta}{\cos\theta + i\sin\theta + 2} = 0$$

$$\Rightarrow \int_0^{2\pi} \frac{(i\cos\theta - \sin\theta)\, d\theta}{2+\cos\theta + i\sin\theta} = 0$$

$$\Rightarrow \int_0^{2\pi} \frac{(i\cos\theta - \sin\theta)\, d\theta}{(2+\cos\theta + i\sin\theta)} \cdot \frac{2+\cos\theta - i\sin\theta}{2+\cos\theta - i\sin\theta} = 0$$

$$\Rightarrow \int_0^{2\pi} \frac{(2i\cos\theta - 2\sin\theta + i\cos^2\theta - \cos\theta\sin\theta + \sin^2\theta\, i + \cos\theta\sin\theta)}{(2+\cos\theta)^2 + \sin^2\theta} = 0$$

$$\Rightarrow \int_0^{2\pi} \frac{-2\sin\theta + i(\cos^2\theta + \sin^2\theta) + 2i\cos\theta}{4 + \cos^2\theta + 4\cos\theta + \sin^2\theta}\, d\theta = 0$$

$$\Rightarrow \int_0^{2\pi} \frac{-2\sin\theta + i(2\cos\theta + 1)}{5+4\cos\theta}\, d\theta = 0 \Rightarrow$$

$$\Rightarrow -2\int_0^{2\pi} \frac{\sin\theta}{5+4\cos\theta}\, d\theta + i\int_0^{2\pi} \frac{(2\cos\theta + 1)}{5+4\cos\theta}\, d\theta = 0$$

Equating real and imaginary part,

$$\int_0^{2\pi} \frac{\sin\theta}{5+4\cos\theta}\, d\theta = 0 \quad \& \quad \int_0^{2\pi} \frac{1+2\cos\theta}{5+4\cos\theta}\, d\theta = 0.$$

CONTOUR INTEGRATION

TYPE-I

Integrals of type $\int_0^{2\pi} F(\cos\theta, \sin\theta)\, d\theta$, where F is a rational function of $\cos\theta$ and $\sin\theta$.

Substitution:-

$z = e^{i\theta}$; $dz = ie^{i\theta} d\theta$

$d\theta = \frac{dz}{iz}$; $\cos\theta = \frac{z^2+1}{2z}$

$\sin\theta = \frac{z^2-1}{2iz}$; c is $|z| = 1$

Problems:

1. Evaluate $\int_0^{2\pi} \frac{d\theta}{13+5\sin\theta}$

Soln

Step:1) To find $f(z)$

Put $z = e^{i\theta}$, $d\theta = \frac{dz}{iz}$ | $\cos\theta = \frac{z^2+1}{2z}$; $\sin\theta = \frac{z^2-1}{2iz}$;

C is $|z| = 1$

$$\int_0^{2\pi} \frac{d\theta}{13+5\sin\theta} = \int_c \frac{dz/iz}{13+5\left(\frac{z^2-1}{2iz}\right)}$$

$$= \int_c \frac{dz/iz}{\frac{13(2iz)+5(z^2-1)}{2iz}} \Rightarrow \int_c \frac{dz}{5z^2-5+26iz} \cdot 2$$

$$= 2\int_c \frac{dz}{5z^2+26iz-5}$$

Let $f(z) = \int_c \frac{dz}{5z^2+26iz-5}$

Step: 2 To evaluate $\int_C f(z)dz$

To find poles:

Let $5z^2 + 26iz - 5 = 0$

$a = 5$; $b = 26i$; $c = -5$

$$z = \frac{-26i \pm \sqrt{(26i)^2 - 4 \cdot 5(-5)}}{2 \cdot 5} \Rightarrow \frac{-26i \pm \sqrt{-676 + 100}}{10}$$

$$\Rightarrow \frac{-26i \pm \sqrt{-576}}{10} = \frac{-26i \pm 24i}{10} \Rightarrow \frac{-26i + 24i}{10}, \frac{-26i - 24i}{10}$$

$$= \frac{-2i}{10}, \frac{-50i}{10}$$

$z = -i/5, -5i$ are poles of order one.

To check the region:

$|z| = 1$

$|z| = \left|\frac{-i}{5}\right| = 1/5 < 1$ $\therefore$ $z = -i/5$ lies inside c

Take $z = -5i$ $\Rightarrow$ $|z| = |-5i| = 5 > 1$

$z = -5i$ lies outside c.

To find the residue

$$R_1 = [\operatorname{Res} f(z)]_{z=-i/5}$$

$$= \lim_{z \to -i/5} \left[(z + i/5) \cdot \frac{1}{(z + i/5)(z + 5i)} \right]$$

$$= \lim_{z \to -i/5} \left[\frac{1}{z + 5i} \right] = \frac{1}{-i/5 + 5i} = \frac{1}{\frac{-i + 25i}{5}} = 5/24i$$

$$\int_C f(z)dz = 2\pi i [R_1] = 2\pi i \left[\frac{5}{24i}\right] = \frac{5\pi}{12}$$

$$\int_0^{2\pi} \frac{d\theta}{13 + 5\sin\theta} = 2\int_C f(z)dz = 2 \cdot \frac{5\pi}{12} \Rightarrow 5\pi/6$$

2. using the method of contour integration. show that $\int_0^{2\pi} \frac{d\theta}{5+4\sin\theta} = \frac{2\pi}{3}$

Proof:

Step: 1 To find $f(z)$

use the institution

$z = e^{i\theta}$; $d\theta = \frac{dz}{iz}$; $\cos\theta = \frac{z^2+1}{2z}$

$\sin\theta = \frac{z^2-1}{2iz}$; c is $|z| = 1$

$$\int_0^{2\pi} \frac{d\theta}{5+4\sin\theta} = \int_c \frac{dz/iz}{5+4\left(\frac{z^2-1}{2zi}\right)} = \int_c \frac{dz/iz}{10z+4z^2-4/2zi}$$

$$= \int_c \frac{2dz}{4z^2+10zi-4} = \int_c \frac{\cancel{2}dz}{\cancel{2}(2z^2+5zi-2)}$$

$$\int_0^{2\pi} \frac{d\theta}{5+4\sin\theta} = \int_c \frac{dz}{2z^2+5zi-2}$$

$$\therefore f(z) = \int_c \frac{dz}{2z^2+5zi-2}$$

Step: 2 To find $\int_c f(z)dz$

To find poles:

put $2z^2+5i-2=0$; $a=2$; $b=5z$; $c=-2$

$$z = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{-5i \pm \sqrt{(5i)^2-4\cdot 2(-2)}}{2\cdot 2}$$

$$= \frac{-5i \pm \sqrt{25i+16}}{4} = \frac{-5i \pm \sqrt{-25+16}}{4} = \frac{-5i \pm \sqrt{-}}{4}$$

$$= \frac{-5 \pm 3i}{4} \Rightarrow \frac{-5i+3i}{4}, \frac{-5i-3i}{4} \Rightarrow \frac{-2i}{4}, \frac{-8i}{4}$$

$z = -i/2, -2i$ are poles of order one.

C is $|z|=1$

$z=-i/2$

$|z| = |-i/2| = 1/2 < 1$; $z=-i/2$ lies inside C

$z=-2i$

$|z| = |-2i| = 2 > 1$; $z=-2i$ lies outside C

To find the residue:

$$R_1 = [\text{Res } f(z)]_{z=-i/2}$$

$$= \lim_{z\to -i/2} \left[\cancel{(z+i/2)} \frac{1}{\cancel{(z+i/2)}(4z+5i)} \right] = \lim_{z\to -i/2} \left[\frac{1}{4z+5i} \right]$$

$$\Rightarrow \frac{1}{\frac{-i+4i}{2}} = \frac{1}{3i}$$

$$\int_C f(z)\,dz = 2\pi i \cdot \frac{1}{3i} = 2\pi/3. \;//$$

3. Evaluate $\int_0^{2\pi} \frac{d\theta}{2+\cos\theta}$

Soln Step:1

To find $f(z)$

Put $z=e^{i\theta}$ $d\theta = \frac{dz}{iz}$; $\cos\theta = \frac{z^2+1}{2z}$; C is $|z|=1$

$$\int_0^{2\pi} \frac{d\theta}{2+\cos\theta} = \int_C \frac{dz/iz}{2+\frac{z^2+1}{2z}} = \int_C \frac{dz/iz}{4z+z^2+1/2z}$$

$$= \frac{2}{i}\int_C \frac{dz}{z^2+4z+1} \Rightarrow \int_C f(z)\,dz = \int_C \frac{dz}{z^2+4z+1}$$

Step:2 To Evaluate $\int_C f(z)\,dz$

To find Pole:

Put $(z^2+4z+1)=0$

$a=1$; $b=4$; $c=1$

$$z = \frac{-4 \pm \sqrt{16 - 4\cdot 1\cdot 1}}{2\cdot 1}$$

$$= \frac{-4 \pm \sqrt{12}}{2} \quad = \frac{-4 \pm 2\sqrt{3}}{2} \Rightarrow -2 \pm \sqrt{3}$$

$z = -2+\sqrt{3}$; $z = -2-\sqrt{3}$

C is $|z| = 1$;

$|z| = |-2+\sqrt{3}| = |-2+1.732| = |-0.2684| = 0.2684$

$\therefore$ $z = -2+\sqrt{3}$ lies inside C.

$|z| = |2+\sqrt{3}| = |2+1.732| = 3.1732 > 1$

$z = 2+\sqrt{3}$ lies outside C.

To find residue:

$$R = \left[\operatorname{Res} f(z)\right]_{z=-2+\sqrt{3}}$$

To find residue $\lim_{z \to -2+\sqrt{3}} \left[\frac{1}{D(z^2+4z+1)}\right]$

$$\lim_{z\to -2+\sqrt{3}} \left[\frac{1}{2z+4}\right] = \frac{1}{2(-2+\sqrt{3})+4} = \frac{1}{-4+2\sqrt{3}+4} = \frac{1}{2\sqrt{3}}$$

$$\int_C \frac{dz}{z^2+4z+1} = 2\pi i(R)$$

$$\int_C \frac{dz}{z^2+4z+1} = 2\pi i\left(1/2\sqrt{3}\right) = \pi i/\sqrt{3}$$

$$\int_0^{2\pi} \frac{d\theta}{2+\cos\theta} = \frac{2}{i}\int_C \frac{dz}{z^2+4z+1} = \frac{2}{i}\left(\frac{\pi i}{\sqrt{3}}\right) = 2\pi/\sqrt{3}$$

$$\int_0^{2\pi} \frac{d\theta}{2+\cos\theta} = \frac{2\pi}{\sqrt{3}}$$

11. Show that $\int_0^{2\pi} \frac{(1+2\cos\theta)}{(5+4\cos\theta)} d\theta = 0$

proof:

To find $f(z)$

put $z = e^{i\theta}$; $d\theta = \frac{dz}{iz}$; $\cos\theta = \frac{z^2+1}{2z}$; $\sin\theta = \frac{z^2-1}{2iz}$

c is $|z|=1$

$\cos\theta = RP(e^{i\theta}) = RP(z)$

$\int_0^{2\pi} \frac{1+2\cos\theta}{5+4\cos\theta} d\theta = \int_c \frac{(1+2RP(z))\, dz/iz}{5+4\left(\frac{z^2+1}{2z}\right)}$

$= RP \int_c \frac{(1+2z)\, dz/iz}{(10z+4z^2+4)/2z} = RP \frac{1}{i} \int_c \frac{(1+2z)}{2z^2+5z+2} dz$

$= RP \frac{1}{i} \int_c \frac{(1+2z)\, dz}{2z^2+5z+2}$

let $f(z) = \frac{1+2z}{2z^2+5z+2}$

$a=2$; $b=5$; $c=2$

$z = \frac{-5 \pm \sqrt{25-4\cdot 2\cdot 2}}{2\cdot 2} = \frac{-5\pm\sqrt{9}}{4} = \frac{-5\pm 3}{4}$

$\Rightarrow \frac{-5-3}{4}, \frac{-5+3}{4} \Rightarrow -8/4, -2/4 \Rightarrow -2, -1/2$ are the poles of order 1

c is $|z|=1$

Take $z=-1/2$

$|z| = |-1/2| = 1/2 < 1$

$z=-1/2$ lies inside c

Take $z=2$

$|z| = |-2| = 2 > 1$

$z=-2$ lies outside c

$R = [\text{Res } f(z)]_{z=-1/2}$

$= \lim_{z\to -1/2} \frac{1+2z}{D(2z^2+5z+2)} \Rightarrow \lim_{z\to-1/2}\left[\frac{1+2z}{4z+5}\right]$

$$= \frac{1+2(-1/2)}{4(-1/2)+5} = \frac{0}{-2+5} = 0$$

$R.P \frac{1}{i}(0) = 0$

$$\therefore \boxed{\int_0^{2\pi} \frac{1+2\cos\theta}{5+4\cos\theta} d\theta = 0}$$

5. Evaluate $\int_0^{2\pi} \frac{\cos 2\theta}{1-2a\cos\theta+a^2}$ using contour integration

where $a^2<1$

solⁿ Step: 1 To find $f(z)$

put $z=e^{i\theta}$; $d\theta = \frac{dz}{iz}$; $\cos\theta = \frac{z^2+1}{2}$; c is $|z|=1$

$\cos 2\theta + i\sin 2\theta = e^{i2\theta} \Rightarrow (e^{i\theta})^2$

$R.P$ of $(e^{i\theta})^2 = R.P$ of $z^2 = \cos 2\theta$

$$\boxed{\therefore \cos 2\theta = R.P \text{ of } z^2}$$

$$\int_c \frac{R.P\, z^2}{1-2a\left(\frac{z^2+1}{2z}\right)+a^2} \cdot \frac{dz}{iz}$$

$$\Rightarrow RP \int_c \frac{z^2 \cdot dz}{z - az^2 - ia + a^2 z} \Rightarrow RP \frac{1}{i} \int_c \frac{z^2 dz}{-az^2+z(1+a^2)-a}$$

$$= RP \left[\frac{-1}{i} \int_c \frac{z^2 dz}{az^2 - z(1+a^2)+a} \right]$$

Let $f(z) = \frac{z^2}{az^2 - z(1+a^2)+a}$

Step: 2 To Evaluate $\int_c f(z)\,dz$.

To find poles: put $az^2 - z(1+a^2)+a = 0$

$a=a$, $b=-(1+a^2)$, $c=a$

$$z = \frac{(1+a^2) \pm \sqrt{(1+a)^2 - 4 \cdot a^2}}{2 \cdot a} \Rightarrow \frac{(1+a^2) \pm \sqrt{1+a^2+2a-4a^2}}{2a}$$

$$= \frac{(1+a^2) \pm \sqrt{(1-a^2)^2}}{2a} = \frac{1+a^2 \pm (1-a^2)}{2a}$$

$$= \frac{1+a^2+1-a^2}{2a}, \quad \frac{1+a^2-1+a^2}{2a} \Rightarrow 2/2a, \ \frac{2a^2}{2a} = 1/a, a$$

$z = 1/a, a$ are poles of order one

C is $|z| = 1$

Take $|z| = |1/a| = 1/|a| > 1$

$z = 1/a$ lies outside C

Take $|z| = |a|$

$|z| = |a| < 1$

$z = a$ lies inside C

$$R = \left[\text{Res } f(z)\right]_{z=a}$$

$$= \lim_{z \to a} \frac{z^2}{D\left[az^2 - (1+a^2)z + a\right]} = \lim_{z \to a} \left[\frac{z^2}{2za - (1+a^2)}\right]$$

$$= \frac{a^2}{2a^2-1-a^2} = \frac{a^2}{a^2-1}$$

$$\oint_C f(z)\,dz = 2\pi i \left(\frac{a^2}{a^2-1}\right)$$

$$\int_0^{2\pi} \frac{\cos 2\theta}{1-2a\cos\theta + a^2}\,d\theta = R.P \ \frac{1}{i}\left[2\pi i\left(\frac{a^2}{a^2-1}\right)\right]$$

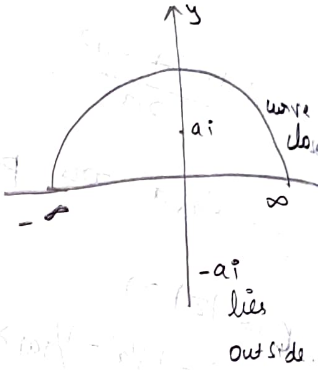
$$= RP\left[\frac{-2\pi a^2}{a^2-1}\right] = RP\left[\frac{2\pi a^2}{1-a^2}\right]$$

CAUCHY'S LEMMA

If $f(z)$ is a continuous function such that $|z f(z)| \to 0$ uniformly $|z| \to \infty$ on S then $\int f(z)\,dz \to 0$ $R \to \infty$ where S is semicircle, $|z| = R$ above the real axis.

TYPE-II

Integrals of the type $\int_{-\infty}^{\infty} \frac{f(x)}{g(x)} dx$, where $f(x)$ and $g(x)$ are polynomials in x such that $\frac{f(x)}{g(x)} \to 0$ as $x \to \infty$ and $g(x)$ has no zeros on the real axis.



PROCEDURE:-

i, write $f(x)$

ii, write $f(z)$

iii, Find the poles of $f(z)$

iv, Takes the poles in upper half of z-plane (+ve poles)

v, Find the residue for positive poles

vi, $\int_C f(z) dz = 2\pi i$ (sum of the residues)

vii, $\int_{-R}^{R} f(x) dx + \int_{\Gamma} f(z) dz = \int_C f(z) dz$

viii, By cauchy's lemma as $R \to \infty$

$\int_{\Gamma} f(z) dz \to 0$

$\therefore \int_{-\infty}^{\infty} f(x) dx = \int_C f(z) dz$

1. Evaluate $\int_{-\infty}^{\infty} \frac{dx}{x^2+a^2}$

Soln let $f(x) = \frac{1}{x^2+a^2}$

$f(z) = \frac{1}{z^2+a^2} = \frac{1}{(z+ia)(z-ia)}$

To find poles:

$(z+ia)(z-ia)=0$

$z = ia, -ia$ poles of order one

$z = ia$ lies in the upperhalf of the z-plane

To find the residue:

$R = \left[\text{Res } f(z)\right]_{z=ia}$

$= \lim_{z \to ia} \left[(z-ia) \cdot \frac{1}{(z-ia)(z+ia)}\right]$

$= \lim_{z \to ia} \left[\frac{1}{z+ia}\right]$

$\boxed{R = 1/2ia}$ $\quad \int_C f(z)\,dz = 2\pi i \left(1/2ia\right) = \pi/a$

$\int_{-R}^{R} f(x)\,dx + \int_{\Gamma} f(z)\,dz = \int_C f(z)\,dz$

$R \to \infty \int_{\Gamma} f(z)\,dz \to 0 = \pi/a$

$$\boxed{\therefore \int_{-\infty}^{\infty} \frac{dx}{x^2+a^2} = \pi/a}$$

2. Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)}$

Soln

$f(x) = \dfrac{1}{(x^2+a^2)(x^2+b^2)}$

$f(z) = \dfrac{1}{(z^2+a^2)(z^2+b^2)}$

$$f(z) = \frac{1}{(z+ia)(z-ia)(z+ib)(z-ib)}$$

To find Poles:

$(z+ia)(z-ia)(z+ib)(z-ib) = 0$

$z = ia, -ia, ib, -ib$ are poles of order one.

$z = ia, ib$ lies upper half of the z-plane.

Case i) $z = ia$

$$R_1 = \left[\text{Res } f(z)\right]_{z=ia}$$

$$= \lim_{z \to ia} \left[(z-ia) \frac{1}{(z+ia)(z-ia)(z+ib)(z-ib)} \right]$$

$$= \lim_{z \to ia} \left[\frac{1}{(z+ia)(z+ib)(z-ib)} \right]$$

$$= \frac{1}{2ia(z^2+b^2)} = \frac{1}{2ia(-a^2+b^2)}$$

$$\boxed{R_1 = \frac{-1}{2ia(a^2-b^2)}}$$

$$R_2 = \left[\text{Res } f(z)\right]_{z=ib}$$

$$R_2 = \lim_{z \to ib} \left[\cancel{(z-ib)} \frac{1}{(z^2+a^2)\cancel{(z-ib)}(z+ib)} \right]$$

$$= \lim_{z \to ib} \left[\frac{1}{(z^2+a^2)(z+ib)} \right] = \frac{1}{2ib(b^2-a^2)}$$

$$\boxed{R_2 = \frac{1}{(a^2-b^2)2ib}}$$

$$\int_C f(z)dz = 2\pi i(R_1+R_2)$$

$$= 2\pi i\left[\frac{-1}{2ia(a^2-b^2)} + \frac{1}{2ib(a^2-b^2)}\right]$$

$$= \frac{2\pi i}{2i}\left[\frac{-1}{a(a^2-b^2)} + \frac{1}{b(a^2-b^2)}\right]$$

$$= \frac{\pi}{a^2-b^2}\left[-1/a + 1/b\right] \Rightarrow \frac{\pi}{a^2-b^2}\left[\frac{-b+a}{ab}\right]$$

$$= \frac{\pi(a-b)}{(a-b)(a+b)ab} = \frac{\pi}{ab(a+b)}$$

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{ab(a+b)} \,//$$

JORDAN'S LEMMA:

If $f(z)$ is continuous function such that $|f(x)| \to 0$ uniformally as $|z| \to \infty$ on S, then

$$\int_S e^{imx} f(x)dx \to 0 \text{ as } R \to \infty$$ where Γ is the semi circle $|z| = R$ above the real axis and $m > 0$.

Type - III

Integrals of the form $\int_{-\infty}^{\infty} \frac{P(x)\sin mx}{Q(x)} dx$.

(or) $\int_{-\infty}^{\infty} \frac{P(x)\cos mx}{Q(x)} dx$

where $P(x)$ and $Q(x)$ polynomial in x.

Problems:-

1. Evaluate $\int_0^{\infty} \frac{x \sin mx}{x^2+a^2}$; $a>0$ $m>0$

Soln Consider $\int_{-\infty}^{\infty} \frac{x e^{imx}}{x^2+a^2} dx$

$f(x) = \frac{x e^{imx}}{x^2+a^2}$; $f(z) = \frac{z e^{imz}}{z^2+a^2}$

$f(z) = \frac{z e^{imz}}{(z+ia)(z-ia)}$

To find the poles:-

$z+ia=0$ | $z=+ia$
$z=-ia$ |

$z = ia, -ia$ are poles of order one.

$z = ia$ lies in upper half of z-plane.

To find the residue:-

$R = [\text{Res } f(z)]_{z=ia}$

$= \lim_{z \to ia} \left[(z-ia) f(z) \right]$

$= \lim_{z \to ia} \left[\cancel{(z-ia)} \frac{z e^{imz}}{(z+ia)\cancel{(z-ia)}} \right]$

$= \lim_{z \to ia} \left[\frac{z e^{imz}}{z+ia} \right]$

$= \frac{ia\, e^{imai}}{2ia} = \frac{e^{-ma}}{2}.$

$$\oint_C f(z)dz = 2\pi i R \quad = 2\pi i \frac{e^{-ma}}{2}$$

$$\oint_C f(z)dz = \pi i e^{-ma}$$

$$\int_{-R}^{R} f(x)dx + \int_{\Gamma} f(z)dz = \int_C f(z)dz$$

By Jordan's lemma $R \to \infty$

$$\int_{\Gamma} f(z)dz \to 0$$

$$\int_{-\infty}^{\infty} \frac{x e^{imx}}{x^2+a^2} dx = \pi i e^{-ma}.$$

$$= \int_{-\infty}^{\infty} \frac{x \sin mx}{x^2+a^2} = \pi e^{-ma} \,//.$$

2. Evaluate $\int_{-\infty}^{\infty} \frac{\cos 2x \, dx}{(x^2+1)(x^2+9)}$ using contour integration.

Soln Consider $\int_{-\infty}^{\infty} \frac{e^{i2x}}{(x^2+1)(x^2+9)} dx$

$$f(x) = \frac{e^{i2x}}{(x^2+1)(x^2+9)}$$

$$f(z) = \frac{e^{i2z}}{(z^2+1)(z^2+9)} \Rightarrow \frac{e^{i2z}}{(z-i)(z+i)(z+3i)(z-3i)}$$

To find poles:

$(z+i)(z-i)(z-3i)(z+3i) = 0$

$z = i, -i, 3i, -3i$ are poles of order one.

$z = i, 3i$ lies in upper half of z-plane

$$R_1 = \left[\text{Res } f(z)\right]_{z=i}$$

$$= \lim_{z \to i} \left[(z-i) f(z) \right]$$

$$= \lim_{z \to i} \left[(z-i) \frac{e^{i2z}}{(z+i)(z-i)(z^2+9)} \right]$$

$$= \lim_{z \to i} \left[\frac{e^{iz}}{(z+i)(z^2+9)} \right]$$

$$= \frac{e^{-2}}{2i \cdot 8} = \frac{e^{-2}}{16i}$$

$$\boxed{R_1 = \frac{e^{-2}}{16i}}$$

$$R_2 = \left[\operatorname{Res} f(z) \right]_{z=3i}$$

$$= \lim_{z \to 3i} \left[(z-3i) f(z) \right]$$

$$= \lim_{z \to 3i} \left[(z-3i) \frac{e^{imz}}{(z^2+1)(z-3i)(z+3i)} \right]$$

$$= \lim_{z \to 3i} \left[\frac{e^{i2z}}{(z^2+1)(z+3i)} \right]$$

$$= \frac{e^{-3 \cdot 2}}{6i(-8)} = \frac{e^{-6}}{-48i}$$

$$\int_c f(z)\, dz = 2\pi i \left[R_1 + R_2 \right]$$

$$= 2\pi i \left[\frac{e^{-2}}{16i} - \frac{e^{-6}}{48i} \right]$$

$$= \frac{2\pi i \left(3e^{-2} - e^{-6} \right)}{48i} = \frac{\pi \left(3e^{-2} - e^{-6} \right)}{24}$$

By Jordan's Lemma,

$$\int_{-\infty}^{\infty} f(x)\,dx = \frac{\pi(3e^{-2} - e^{-6})}{24}$$

$$\int_{-\infty}^{\infty} \frac{(\cos 2x + i\sin 2x)}{(x^2+1)(x^2+9)}\,dx = \frac{\pi[3e^{-2} - e^{-6}]}{24}$$

$$\int_{0}^{\infty} \frac{\cos 2x \, dx}{(x^2+1)(x^2+9)} = \frac{\pi(3e^{-2} - e^{-6})}{12} \;//.$$