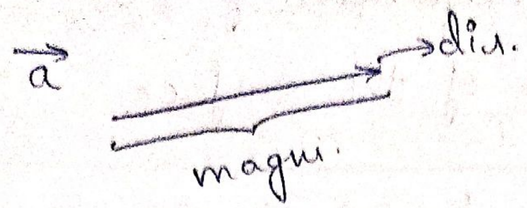
→ VECTOR CALCULUS ←

* **Vector** – quantity with magnitude & direction.



* **scalar** – quantity with only magnitude.

* **Operators** – ① vector addition
  ② vector subtraction
  ③ scalar multiplication
  ④ vector multiplication ↗ $\vec{A}\cdot\vec{B}$ ↘ $\vec{A}\times\vec{B}$

⇒ $\vec{r} = a\hat{i} + b\hat{j} + c\hat{k}$ } vector rep. in 3D space.

NOTE – $\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$

$\vec{b} = b_1\vec{i} + b_2\vec{j} + b_3\vec{k}$

⇒ $\vec{a}\cdot\vec{b} = a_1b_1 + a_2b_2 + a_3b_3$

⇒ $\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

Gradient –

Let $\phi(x,y,z)$ be any scalar point function in a certain region. Then the vector point function is given by –

$$\nabla\phi = \frac{\partial\phi}{\partial x}\vec{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\vec{k}$$

$\nabla\phi \Rightarrow$ ① unit normal

② directional derivative (DD)

③ max. magnitude of DD.

④ angle of the surface.

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* find $\nabla\phi$ if $\phi = \log(x^2+y^2+z^2)$

$$\Rightarrow \nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$$

$$= \frac{2x\vec{i}}{x^2+y^2+z^2} + \frac{2y\vec{j}}{x^2+y^2+z^2} + \frac{2z\vec{k}}{x^2+y^2+k^2}$$

$$\vec{\gamma} = \frac{2}{x^2+y^2+k^2}(\vec{i}x+\vec{j}y+\vec{k}z)$$

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* find $\nabla\phi$, if $\phi = xyz$ at the point $(1,2,3)$

$$\Rightarrow \nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$$

$$= \vec{i}(yz) + \vec{j}(xz) + \vec{k}(xy)$$

$$= \vec{i}(2\cdot3) + \vec{j}(1\cdot3) + \vec{k}(1\cdot2)$$

$$= 6\vec{i} + 3\vec{j} + 2\vec{k} \,//$$

$\Rightarrow$ Unit Normal Vector -

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} \qquad \vec{r} = x\vec{i}+y\vec{j}+z\vec{k}$$

$$|\vec{r}| = \sqrt{x^2+y^2+z^2}$$

* find $\hat{n}$ to the surface $x^2+xy+z^2=4$ at the point $(1,-1,2)$

$\Rightarrow \phi = x^2+xy+z^2-4$

$\nabla\phi = \vec{i}(2x+y)+\vec{j}(x)+\vec{k}(2z)$

$= \vec{i}+\vec{j}+4\vec{k}$

$|\nabla\phi| = |\vec{i}+\vec{j}+4\vec{k}| = \sqrt{1+1+16} = \sqrt{18}$

$$\hat{n} = \frac{\vec{i}+\vec{j}+4\vec{k}}{\sqrt{18}}$$

Directional Derivative (DD) -

dot prod (not cross prod)

$$DD = \nabla\phi \times \frac{\vec{a}}{|\vec{a}|}$$

$\vec{a}$: direction vector

Max. mag. of DD - $|DD|$

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* find DD of $\phi = 3x^2+2y-3z$ at $(1,1,1)$ in the direction $2\vec{i}+2\vec{j}-\vec{k}$.

$\nabla\phi = \vec{i}\,6x+\vec{j}(2)+\vec{k}(-3)$

$\nabla\phi = 6\vec{i}+2\vec{j}-3\vec{k}$; $\vec{a} = 2\vec{i}+2\vec{j}-\vec{k}$

$|\vec{a}| = \sqrt{4+4+1} = \sqrt{9} = 3$ //

$$\Rightarrow (6\vec{i}+2\vec{j}-3\vec{k}) \cdot \frac{(2\vec{i}+2\vec{j}-\vec{k})}{3} = \frac{1}{3}(12+4+3)$$

$$= \frac{19}{3} //$$

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* in what direction from $(3,1,-2)$ is the DD of $\phi = x^2 * y^2 * z^4$ a max. magni. = ?

$\phi = x^2 y^2 z^4$

$\nabla\phi = 2xy^2z^4\vec{i} + 2yx^2z^4\vec{j} + 4z^3x^2y^2\vec{k}$

$\nabla\phi \Rightarrow \nabla\phi_{(3,1,-2)} = 96\vec{i} + 288\vec{j} - 288\vec{k}$

$|\nabla\phi| = \sqrt{96^2 + 288^2 + 288^2} = 96\sqrt{19}$ //

Angle between the surfaces -

$$\cos\theta = \frac{\nabla\phi_1 \cdot \nabla\phi_2}{|\nabla\phi_1|\,|\nabla\phi_2|}$$

* find the angle b/w the surfaces $x\log z = y^2 - 1$ and $x^2 y = 2 - z$, at $(1,1,1)$

$\phi_1 = x\log z - y^2 + 1$

$\phi_2 = x^2 y - 2 + z$

$\nabla\phi_1 = \vec{i}(\log z) + \vec{j}(-2y) + \vec{k}(x/z)$

$\nabla\phi_2 = \vec{i}(2xy) + \vec{j}(x^2) + \vec{k}(1)$

$\nabla\phi_{1(1,1,1)} = -2\vec{j} + \vec{k}$ $\quad |\nabla\phi_1| = \sqrt{5}$

$\nabla\phi_{2(1,1,1)} = 2\vec{i} + \vec{j} + \vec{k}$ $\quad |\nabla\phi_2| = \sqrt{6}$

angle $\Rightarrow \frac{-1}{\sqrt{30}}$ // $\quad \cos\theta = \frac{-1}{\sqrt{30}}$

$\theta = \cos^{-1}\left[\frac{-1}{\sqrt{30}}\right]$ //

→ Divergence & Curl -

There are two kinds of differentiation of vector field $F(x,y,z)$.

① Divergence - div F

② Curl - curl F

Divergence - The divergence of a vector $F = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$ can be defined as div F

$$= \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$$\Rightarrow \boxed{\text{div } F = \nabla \cdot F}$$

$$= \left[\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right] \cdot \vec{F}$$

$$\Rightarrow \boxed{\text{div } F = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}}$$

Curl - The curl of a vector denoted by curl $\vec{F}$ (or) $\nabla \times \vec{F}$ and defined as -

$$\boxed{\nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}}$$

Solenoidal - A vector is said to be Solenoidal if $\nabla \cdot \vec{F} = 0$.

Irrotational - A vector is said to be irrotational if $\nabla \times \vec{F} = 0$.

NOTE - (i) $\nabla \cdot \vec{F} > 0 \Rightarrow$ out flow

(ii) $\nabla \cdot \vec{F} < 0 \Rightarrow$ inflow

(iii) $\nabla \cdot \vec{F} = 0 \Rightarrow$ no flow.

→ if there is no gain or no loss means $\nabla \cdot \vec{F} = 0$ and such a vector field is solenoidal.

NOTE -

→ F is a vector and curl F is a vector.

Then,

* curl F at a point in fluid is a measure of the rotation of the fluid.

* if there is no rotation of fluid anywhere then curl $\times \vec{F} = 0$. Then such a vector field is said to be irrotational.

⇒ Vector Integration -

→ line integration

→ surface integration

→ volume integration

→ Line Integral -

$$\Rightarrow \oint_C f(P)\,ds = \int_a^b f(s)\,ds = \int_C f(x,y,z)\,ds$$

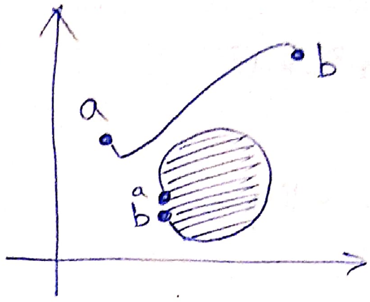
where p has coordinates $x(s), y(s), z(s)$.

The line integral is also known as curve integrals and is denoted as -

$$\int_C \vec{F} \cdot d\vec{r}$$

Thus in a line integral (or) curve integral, the integrand f is integrated along a curve. The curve C is known as path of integration. It's end points a and b are called initial and terminal points.

$\Rightarrow \oint \vec{F} \cdot d\vec{r}$



Green's Theorem -

→ green's theorem gives a relation between a double integral over a region R in the xy-plane and the line integral over a closed curve C enclosing the region R.

Statement of Green's Theorem -

→ if $M(x,y)$ and $N(x,y)$ are continuous function with continuous partial derivatives in region R in the xy-plane and on it's boundary C which is simple closed curve.

Then,

$$\Rightarrow \boxed{\oint (M dx + N dy) = \iint_R \left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right] dx\, dy}$$

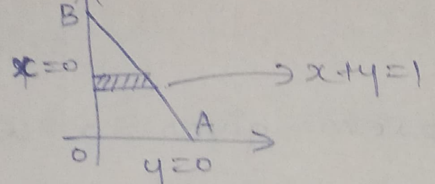
* verify green's theorem in a plane for $\int_C (3x^2-8y^2)\,dx + (4y-6xy)\,dy$ where c is the boundary of the region bounded by the lines $x=0, y=0, x+y=1$.

Sol

$$\Rightarrow \int_C (3x^2-8y^2)\,dx + (4y-6xy)\,dy$$

$M = 3x^2-8y^2$ $\qquad$ $N = 4y-6xy$

$\frac{\partial M}{\partial y} = -16y$ $\qquad$ $\frac{\partial N}{\partial x} = -6y$



always use anti-clockwise direction

$$\Rightarrow \int_0^1 \int_0^{1-y} (-6y+16y)\,dx\,dy$$

$$\Rightarrow \int_0^1 \int_0^{1-y} 10y\,dx\,dy \Rightarrow \int_0^1 10y(1-y)\,dy$$

$$\Rightarrow 10\int_0^1 (y-y^2)\,dy \Rightarrow 10\left[\frac{y^2}{2}-\frac{y^3}{3}\right]_0^1 \Rightarrow 10\times\frac{1}{6} = \underline{5/3}$$

$$\boxed{RHS = 5/3}$$

LHS,

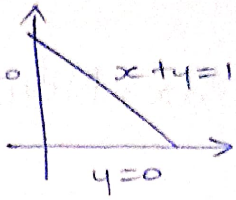
$$\Rightarrow \overrightarrow{OA} + \overrightarrow{AB} + \overrightarrow{BO}$$

compute $\oint M\,dx + N\,dy$ over the simple closed curve a bounding the surface OABO consisting the edges (linear) $\overrightarrow{OA}, \overrightarrow{AB}, \overrightarrow{BO}$.

$$\Rightarrow \oint_C M\,dx + N\,dy = \int_{OABO} M\,dx + N\,dy$$

$$\Rightarrow \underbrace{\int_{OA} M\,dx + N\,dy}_{y=0} + \underbrace{\int_{AB} M\,dx + N\,dy}_{x+y=1} + \underbrace{\int_{BO} M\,dx + N\,dy}_{x=0}$$

On OA - $y=0,\ dy=0$



$$= 3x^2\,dx$$

$$\Rightarrow \int M\,dx + N\,dy = \int_{OA} 3x^2\,dx \Rightarrow \int_0^1 3x^2\,dx$$

$$\Rightarrow \left.\frac{3x^3}{3}\right|_0^1 \Rightarrow 1 //$$

On AB -

$$x+y=1 \Rightarrow y = 1-x \Rightarrow dy = -dx$$

$$\Rightarrow \int M\,dx + N\,dy = \int_{AB} (3x^2-8y^2)\,dx + (4y-6xy)\,dy$$

$$= \int_{AB} 3x^2 - 8(1-x)^2\,dx + 4(1-x) - 6x(1-x)(-dx)$$

$$= \int_{AB} [3x^2 - 8(1-2x+x^2)]\,dx + [(4-4x) + 6x(1-x)]\,dx$$

$$= \int_{AB} (3x^2 - 8 + 16x - 8x^2)\,dx + (4 - 4x + 6x - 6x^2)\,dx$$

$$= \int_{AB} (-11x^2 + 26x - 12)\,dx \Rightarrow \int_1^0 (-11x^2 + 26x - 12)\,dx$$

$$= \left\{ -\frac{11x^3}{3} + \frac{26x^2}{2} - 12x \right\}_1^0$$

$$\Rightarrow -\left[\frac{-11}{3}+13-12\right] \Rightarrow \underline{8/3}$$

<u>On BO</u>-

$x=0, dx=0$

$$\oint_{BO} Mdx+Ndy \Rightarrow (4y-6xy)\,dy \Rightarrow \int_1^0 4y\,dy$$

$$\Rightarrow \left[\frac{4y^2}{2}\right]_{01}^{0} \Rightarrow -2$$

Add $\rightarrow 1+\frac{8}{3}-2=\frac{5}{3} \Rightarrow \boxed{LHS = RHS}$

$$\Rightarrow \left|\int_{OABO} Mdx+Ndy\right| = \underline{5/3}$$

⇒ Gauss Divergence Theorem -

The divergence theorem enables us to convert the surface integral of vector function on a closed surface into volume integral.

⇒ Statement -

Let V be the volume bounded by the closed surface S. If a vector function $\vec{F}$ is continuous & has a continuous potential partial derivatives inside and on S; the surface integral of $\vec{F}$ over S is equal to the volume integral of divergence of $\vec{F}$ taken throughout V.

$$\Rightarrow \iint_S \vec{F}\cdot d\vec{s} = \iiint_V \text{div}.\vec{F}\, dV$$

→ if $\hat{n}$ is the outward normal to the surface, $d\vec{s} = \hat{n}\cdot d\vec{s}$, then gauss divergence theorem.

$$\Rightarrow \iint_S \vec{F}.\hat{n}\, ds = \iiint_V \text{div}\, \vec{F}.dV.$$

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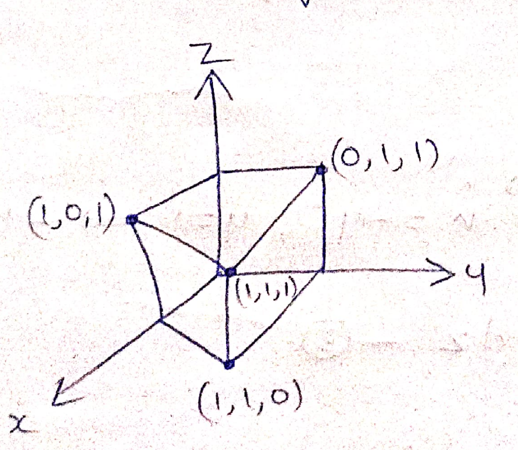
* verify gauss divergence theorem for $\vec{F} = 4xz\hat{i} - y^2\hat{j} + yz\hat{k}$ over the cube bounded by $x=0, x=1, y=0, y=1, z=0$ and $z=1$.

Sol.

gn, $\vec{F} = 4xz\vec{i} - y^2\vec{j} + yz\vec{k}$

gauss divergence theorem -

$$\iint_S \vec{F}.\hat{n}\, ds = \iiint_V \text{div}\, \vec{F}\, dV$$



RHS,

$$\int_0^1\int_0^1\int_0^1 \text{div}\, \vec{F}\, dz\, dy\, dx$$

$\Rightarrow \text{div}\, \vec{F} = \nabla.\vec{F}$

$$\Rightarrow \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right) \cdot (4xz\vec{i} - y^2\vec{j} + yz\vec{k})$$

$$\Rightarrow 4z - 2y + y \Rightarrow 4z - y$$

$$\Rightarrow \int_0^1\int_0^1\int_0^1 (4z - y)\, dz\, dy\, dx$$

$$\Rightarrow \int_0^1\int_0^1 (2z^2 - yz)_0^1\, dy\, dx$$

$$\Rightarrow \int_0^1 \left[2y - \frac{y^2}{2}\right]_0^1 dx \Rightarrow \int_0^1 2 - \tfrac{1}{2}\, dx \Rightarrow \underline{\underline{3/2}}$$

Now,

$$\vec{F} = 4xz\vec{i} - y^2\vec{j} + yz\vec{k}$$

$S_1 \Rightarrow \hat{n} = \vec{i} \Rightarrow \vec{F}\cdot\hat{n} = 4xz,\ x=1 \Rightarrow \vec{F}\cdot\hat{n} = 4z$

$$\Rightarrow \int_0^1\int_0^1 4z\, dy\, dz \quad -(1)$$

$S_2 \Rightarrow \hat{n} = -\vec{i} \Rightarrow \vec{F}\cdot\hat{n} = -4xz,\ x=0 \Rightarrow \vec{F}\cdot\hat{n} = 0$

$$\Rightarrow 0 \quad -(2)$$

$S_3 \Rightarrow \vec{j} = \hat{n} \Rightarrow \vec{F}\cdot\hat{n} = -y^2,\ y=1 \Rightarrow -1$

$$\Rightarrow \int_0^1\int_0^1 (-1)\, dx\, dz \quad -(3)$$

$S_4 \Rightarrow -\vec{j} = \hat{n} \Rightarrow \vec{F}\cdot\hat{n} = y^2,\ y=0 \Rightarrow 0$

$$\Rightarrow 0 \quad -(4)$$

$S_5 \Rightarrow \hat{n} = \vec{k} \Rightarrow \vec{F}\cdot\hat{n} = yz,\ z=1 \Rightarrow y$

$$\Rightarrow \int_0^1\int_0^1 y\, dx\, dy \quad -(5)$$

$S_6 \Rightarrow \hat{n} = -\vec{k} \Rightarrow \vec{F}\cdot\hat{n} = -yz,\ z=0 \Rightarrow 0$

$\Rightarrow 0 \longrightarrow (6)$

$\rightarrow \iint_S \vec{F}\cdot\hat{n}\,ds = (1)+(2)+(3)+(4)+(5)+(6)$

$\Rightarrow \int_0^1\int_0^1 4z\,dy\,dz + 0 + \int_0^1\int_0^1 (-1)\,dx\,dz + 0 + \int_0^1\int_0^1 y\,dx\,dy + 0$

$\Rightarrow 4\int_0^1 z(y)_0^1\,dz - \int_0^1 (x)_0^1\,dz + \int_0^1 y(x)_0^1\,dy$

$\Rightarrow 4\int_0^1 z\,dz - \int_0^1 dz + \int_0^1 y\,dy \Rightarrow \left[4\times\frac{z^2}{2}\right]_0^1 - (z)_0^1 + \left(\frac{y^2}{2}\right)_0^1$

$\Rightarrow \frac{4}{2} - 1 + \frac{1}{2} = 2 - 1 + \frac{1}{2} = 1 + \frac{1}{2} = \frac{3}{2}$

→ Stoke's Theorem

* it gives a relation btw line integral & surface integral, if the region is closed.

Theorem -

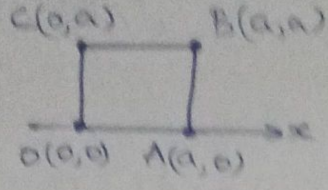
If S is an open surface bounded by a simple closed curve C & if $\vec{F}$ is having continuous partial derivatives in S & on C, then

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S \text{curl}\,\vec{F}\cdot\hat{n}\,dS$$

where '$\hat{n}$' is the unit normal (outward) vector & C is traversed in anti-clockwise direction.

* Verify stokes theorem for $\vec{F} = x^2\hat{i} + xy\hat{j}$ in the region Y bounded by $x=0, x=a, y=0, y=a$.

Sol-



To prove - $\oint_C \vec{F}\cdot d\vec{r} = \iint_S \text{curl}\,\vec{F}\cdot\hat{n}\,ds$

To calculate - $\iint_{OABCO} \text{curl}\,\vec{F}\cdot\hat{n}\,ds$

$$\text{curl}\,\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2 & xy & 0 \end{vmatrix} = \vec{k}y$$

$$\hat{n} = -\vec{k} = \vec{k}$$

$$\Rightarrow \int_0^a\int_0^a y\,dx\,dy \Rightarrow a^3/2$$

On OA - $y=0, dy=0$

$$\int_{OA} \vec{F}\cdot d\vec{r} = \int x^2\vec{i}\cdot(dx\vec{i} + dy\vec{j} + dz\vec{k})$$
$$= \int_0^a x^2\,dx = a^3/3$$

On AB - $x=a, dx=0$

$$\int_{AB} \vec{F}\cdot d\vec{r} = \int_0^a ay\,dy \Rightarrow a^3/2$$

On BC - $y=a$ $dy=0$

$$\int_{BC} \vec{F}\cdot d\vec{r} = \frac{-a^3}{3}$$

On CO - $x=0$ $dx=0$

$\Rightarrow$

Add -

$$\frac{a^3}{3} + \frac{a^3}{2} - \frac{a^3}{3}$$
$$= \frac{a^3}{2}$$
$$= \text{RHS}$$