Unit-2

1) Verify the GDT, $\vec{F} = x^3\vec{i} + y^3\vec{j} + z^3\vec{k}$ taken over a cube bounded by $x=a,\ x=0;\ y=0,\ y=a;\ z=0,\ z=a$. (12 M)

Sol: Given: $\vec{F} = x^3\vec{i} + y^3\vec{j} + z^3\vec{k}$

we know that: $\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V \nabla\cdot\vec{F}\,dv$

Considering RHS: $\iiint_V \nabla\cdot\vec{F}\,dv$

Here, $\vec{F} = x^3\vec{i} + y^3\vec{j} + z^3\vec{k}$ and we know

$$\nabla = \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}$$

$$\therefore \nabla\cdot\vec{F} = 3x^2+3y^2+3z^2 = 3(x^2+y^2+z^2)$$

Applying the given limits:

$$= \int_0^a\int_0^a\int_0^a 3(x^2+y^2+z^2)\,dz\,dy\,dx = 3\int_0^a\int_0^a\int_0^a (x^2+y^2+z^2)\,dz\,dy\,dx$$

$$= 3\int_0^a\int_0^a \left[x^2z + y^2z + \frac{z^3}{3}\right]_0^a dy\,dx$$

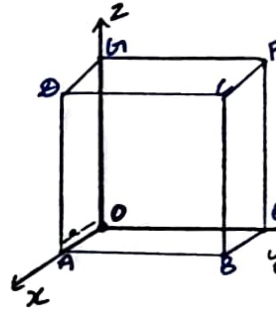
$$= 3\int_0^a\int_0^a ax^2 + ay^2 + \frac{a^3}{3}\,dy\,dx = 3\int_0^a \left[ax^2y + \frac{ay^3}{3} + \frac{ay^3}{3}\right]_0^a dx$$

$$= 3\int_0^a \left[a^2x^2 + a\cdot\frac{a^3}{3} + a\cdot\frac{a^3}{3}\right] dx$$

$$= 3\left[a^2\frac{x^3}{3} + \frac{a^4}{3}x + \frac{a^4}{3}x\right]_0^a = 3\left[\frac{a^5}{3} + \frac{a^5}{3} + \frac{a^5}{3}\right]$$

$$\therefore \iiint_V \nabla\cdot\vec{F}\,dv = 3a^5 \quad -(1)$$

Considering LHS: $\iint_S \vec{F}\cdot\hat{n}\,ds$



Region	surface	$\hat{n}$	$\vec{F}\cdot\hat{n}$	Equation	ds	$\vec{F}\cdot\hat{n}$ as equation
S_1	ABCD	$\vec{i}$	x^3	$x=a$	$dydz$	$\int_0^a\int_0^a a^3\,dydz$
S_2'	OEFG	$-\vec{i}$	$-x^3$	$x=0$	$dydz$	0
S_3	BCFE	$\vec{j}$	y^3	$y=a$	$dxdz$	$\int_0^a\int_0^a a^3\,dxdz$
S_4'	OADG	$-\vec{j}$	$-y^3$	$y=0$	$dxdz$	0
S_5	DCFG	$\vec{k}$	z^3	$z=a$	$dxdy$	$\int_0^a\int_0^a a^3\,dxdy$
S_6'	OABE	$-\vec{k}$	$-z^3$	$z=0$	$dxdy$	0

Integrating all the surfaces:

Note: S_2, S_4, S_6 $\vec{F}\cdot\hat{n}$ is 0.

to $z=a$

$$= \int_0^a\left[\int_0^a a^3\,dy\right]dz = a^3\int_0^a [y]_0^a\,dz = a^3\int_0^a a\,dz$$

$$= a^4\,[z]_0^a = a^5.$$

on S_3: $$\int_0^a\int_0^a a^3\,dx\,dz = a^3\int_0^a\int_0^a dx\,dz$$

$$= a^3\int_0^a [x]_0^a\,dz = a^3\int_0^a a\,dz = a^4[z]_0^a = a^5.$$

on S_5: $$\int_0^a\int_0^a a^3\,dx\,dy = a^3\int_0^a\int_0^a dx\,dy$$

$$= a^3\int_0^a [x]_0^a\,dy = a^4\int_0^a dy = a^4[y]_0^a = a^5.$$

$$\therefore S = S_1+S_2+S_3+S_4+S_5+S_6$$

$$= a^5+0+a^5+0+a^5+0 = 3a^5 \quad \text{—(2)}$$

$\therefore$ (1) = (2)

Hence, it is verified that,

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V \nabla\cdot\vec{F}\,dv.$$

2) Verify GDT for $\vec{F} = (x^2-yz)\vec{i} + (y^2-zx)\vec{j} + (z^2$ … taken over a rectangular parallelepiped bound… by the plane $x=0, x=a$; $y=0, y=b$; $z=0, z$… (12m)

sol: Given: $\vec{F} = (x^2-yz)\vec{i} + (y^2-zx)\vec{j} + (z^2-xy)\vec{k}$

we know that: $$\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V \nabla\cdot\vec{F}\,dv$$

considering RHS: $\iiint_V \nabla\cdot\vec{F}\,dv$

Here, $\vec{F} = (x^2-yz)\vec{i} + (y^2-zx)\vec{j} + (z^2-xy)\vec{k}$ and we know $\nabla = \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}$

$$\therefore \nabla\cdot\vec{F} = 2x+2y+2z = 2(x+y+z)$$

Applying the given limits:

$$= \int_0^a\int_0^b\int_0^c 2(x+y+z)\,dy\,dy\,dx = 2\int_0^a\int_0^b\int_0^c (x+y+z)\,dz\,dy\,d\ldots$$

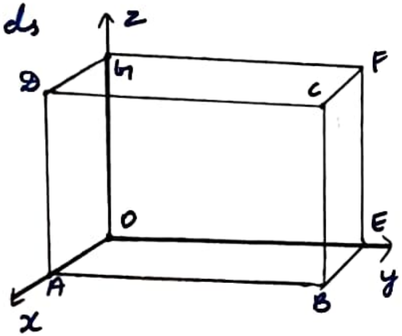
$$= 2\int_0^a\int_0^b\left[xz+yz+\frac{z^2}{2}\right]_0^c dy\,dx = 2\int_0^a\int_0^b\left(xc+yc+\frac{c^2}{2}\right)dy\,dx$$

$$= 2\int_0^a\left[xyc+\frac{y^2c}{2}+\frac{c^2y}{2}\right]_0^b dx = 2\int_0^a xbc+\frac{b^2c}{2}+\frac{c^2b}{2}\,d\ldots$$

$$= 2\left[\frac{x^2bc}{2}+\frac{b^2cx}{2}+\frac{c^2bx}{2}\right]_0^a = \cancel{2}\left[\frac{a^2bc}{\cancel{2}}+\frac{ab^2c}{\cancel{2}}+\frac{ab\ldots}{\cancel{2}}\right.$$

$$= abc(a+b+c) \quad \therefore \iiint \nabla\cdot\vec{F}\,dv = abc(a+b+c) \quad \text{—(1)}$$

sidering LHS: $\iint_S \vec{F}\cdot\hat{n}\,ds$



Region	surface	$\hat{n}$	$\vec{F}\cdot\hat{n}$	Equation	ds	$\vec{F}\cdot\hat{n}$ as eq.
S_1	ABCD	$\vec{i}$	x^2-yz	$x=a$	$dy\,dz$	$\int_0^c\int_0^b a^2-yz\,dy\,dz$
S_2	OEFG	$-\vec{i}$	$-(x^2-yz)$	$x=0$	$dy\,dz$	$\int_0^c\int_0^b yz\,dy\,dz$
S_3	BCFE	$\vec{j}$	y^2-zx	$y=b$	$dx\,dz$	$\int_0^c\int_0^a b^2-zx\,dx\,dz$
S_4	ADG	$-\vec{j}$	$-(y^2-zx)$	$y=0$	$dx\,dz$	$\int_0^c\int_0^a zx\,dx\,dz$
S_5	DCFG	$\vec{k}$	z^2-xy	$z=c$	$dx\,dy$	$\int_0^b\int_0^a c^2-xy\,dx\,dy$
S_6	ABE	$-\vec{k}$	$-(z^2-xy)$	$z=0$	$dx\,dy$	$\int_0^b\int_0^a xy\,dx\,dy$

Integrating all its sides:

on S_1: $\int_0^c\left[\int_0^b a^2-yz\,dy\right]dz = \int_0^c\left[a^2y-\frac{y^2}{2}z\right]_0^b dz$

$= \int_0^c a^2b-\frac{b^2}{2}z\,dz = \left[a^2bz-\frac{b^2}{2}\cdot\frac{z^2}{2}\right]_0^c$

$\therefore S_1 = a^2bc-\frac{b^2c^2}{4}$

on S_2: $\int_0^c\int_0^b yz\,dy\,dz = \int_0^c\left[\frac{y^2}{2}z\right]_0^b dz = \int_0^c \frac{b^2z}{2}\,dz$

$= \left[\frac{b^2}{2}\frac{z^2}{2}\right]_0^c = \frac{b^2c^2}{4} = S_2$

on S_3: $\int_0^c\left[\int_0^a b^2-zx\,dx\right]dz = \int_0^c\left[b^2x-z\frac{x^2}{2}\right]_0^a dz$

$= \int_0^c b^2a-z\frac{a^2}{2}\,dz = \left[b^2za-\frac{z^2}{2}\cdot\frac{a^2}{2}\right]_0^c$

$= ab^2c-\frac{a^2c^2}{4} = S_3$

on S_4: $\int_0^c\left[\int_0^a zx\,dx\right]dz = \int_0^c\left[z\frac{x^2}{2}\right]_0^a dz$

$\int_0^c \frac{a^2z}{2}\,dz = \left[\frac{a^2}{2}\cdot\frac{z^2}{2}\right]_0^c = \frac{a^2c^2}{4} = S_4$

on S_5: $\int_0^b\left[\int_0^a c^2-xy\,dx\right]dy = \int_0^b\left[xc^2-\frac{x^2y}{2}\right]_0^a dy$

$= \int_0^b\left(ac^2-\frac{a^2}{2}y\right)dy = \left[ayc^2-\frac{a^2}{2}\frac{y^2}{2}\right]_0^b = abc^2-\frac{a^2b^2}{4} = S_5$

on S_6: $\int_0^b\left[\int_0^a xy\,dx\right]dy = \int_0^b\left[\frac{x^2}{2}y\right]_0^a dy$

$= \frac{a^2b^2}{4} = S_6$

$\therefore S = S_1+S_2+S_3+S_4+S_5+S_6$

$$S = a^2bc - \cancel{\frac{b^2c^2}{4}} + \cancel{\frac{b^2c^2}{4}} + ab^2c - \cancel{\frac{a^2c^2}{4}} + \cancel{\frac{a^2c^2}{4}} + abc^2 - \cancel{\frac{a^2b^2}{4}} + \cancel{\frac{a^2b^2}{4}} = a^2bc + ab^2c + abc^2$$

$$= abc(a+b+c) \quad — ②$$

$\therefore$ ① = ②

Hence, GDT is verified that: $\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V \nabla\cdot\vec{F}\,dv$

3) Find the work done for $\vec{F} = (x^2-y^2+x)\vec{i} - (2xy+y)\vec{j}$ at point $(0,0)$ to $(1,1)$ along $y^2 = x$.

Sol: Given: $\vec{F} = (x^2-y^2+x)\vec{i} - (2xy+y)\vec{j}$ and

we know that: $\vec{dr} = dx\,\vec{i} + dy\,\vec{j} + dz\,\vec{k}$

work done $= \int \vec{F}\cdot\vec{dr}$ Here,

$$\vec{F}\cdot\vec{dr} = (x^2-y^2+x)\,dx - (2xy+y)\,dy$$

Given $y^2 = x \longrightarrow$ (substitute)

$$\vec{F}\cdot\vec{dr} = (x^2 - \cancel{x} + \cancel{x})\,dx - 2[(y^2)(y) + y]\,dy$$

$$= x^2dx - 2y^3 + y\,dy$$

$$\Rightarrow \int_0^1 x^2dx - \int_0^1 2y^3 + y\,dy = \left[\frac{x^3}{3}\right]_0^1 - \left[\frac{2y^4}{4} + \frac{y^2}{2}\right]_0^1$$

(limits are given)

$$= \frac{1}{3} - \left(\frac{2}{4} + \frac{1}{2}\right) - [0]$$

$$= \frac{1}{3} - \left(\frac{2+2}{4}\right) = \frac{1}{3} - 1 = \frac{-2}{3}$$

$\therefore$ The work done is $-2/3$.

4) Verify the stoke's theorem for the vector $\vec{F} = xy\vec{i} - 2yz\vec{j} - xz\vec{k}$ where 's' is an open surface of a rectangular parallelepiped formed by the plane $x=0, y=0; x=1, y=2, z=3$ above the xoy plane.

Sol: Given: $\vec{F} = xy\,\vec{i} - 2yz\,\vec{j} - xz\,\vec{k}$ and we know by stoke's law: $\iint_S \nabla\times\vec{F}\cdot\hat{n}\,ds = \int_C \vec{F}\cdot\vec{dr}$

considering LHS: $\iint \nabla\times\vec{F}\cdot\hat{n}\,ds$:

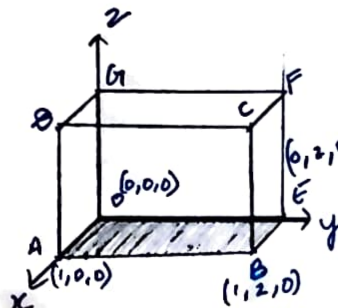
$$\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & -2yz & -xz \end{vmatrix}$$

$$= \vec{i}(0+2y) - \vec{j}(-z-0) + \vec{k}(0-x)$$

$$= 2y\,\vec{i} + z\,\vec{j} - x\,\vec{k}$$

$$\therefore \nabla\times\vec{F}\cdot\hat{n} = 2y + z - x$$

$$\because \hat{n} = (\vec{i} + \vec{j} + \vec{k})$$



n	surface	$\hat{n}$	$(\nabla\times\vec{F})\hat{n}$	Eq.	ds	$(\nabla\times\vec{F})\hat{n}$ as Eq.
S_1	ABCD	$\vec{i}$	$2y$	$x=1$	$dydz$	$\int_0^2\int_0^3 2y\,dz\,dy$
S_2	OEFG	$-\vec{i}$	$-2y$	$x=0$	$dydz$	$\int_0^2\int_0^3 -2y\,dz\,dy$
S_3	BEFC	$\vec{j}$	z	$y=2$	$dxdz$	$\int_0^1\int_0^3 z\,dz\,dx$
S_4	OGDA	$-\vec{j}$	$-z$	$y=0$	$dxdz$	$\int_0^1\int_0^3 -z\,dz\,dx$
S_5	DGFC	$\vec{k}$	$-x$	$z=3$	$dxdy$	$\int_0^1\int_0^2 -x\,dy\,dx$
S_6	OABE	$-\vec{k}$	$-(-x)$	$z=0$	$dxdy$	———

Integrating: on $S_1 = \int_0^2\int_0^3 2y\,dz\,dy = 2\int_0^2 [yz]_0^3\,dy$

$= 2\int_0^2 [3y]\,dy = 6\left[\frac{y^2}{2}\right]_0^2 = 6\cdot\frac{4}{2} = 12$.

on $S_2 = -\int_0^2\int_0^3 2y\,dz\,dy = -12$.

on S_3: $\int_0^1\int_0^3 z\,dz\,dx = \int_0^1\left[\frac{z^2}{2}\right]_0^3 dx = \left[\frac{9}{2}x\right]_0^1 = \frac{9}{2}$.

on S_4: $-\int_0^1\int_0^3 z\,dz\,dx = -\frac{9}{2}$.

on S_5: $-\int_0^1\int_0^2 x\,dy\,dx = \int_0^1 [-xy]_0^2\,dx = \int_0^1 -2x\,dx$

$= -2\left[\frac{x^2}{2}\right]_0^1 = -2\left(\frac{1}{2}\right) = -1$.

$\iint \nabla\times\vec{F}\cdot\hat{n} = S_1+S_2+S_3+S_4+S_5 = 12-12+\frac{9}{2}-\frac{9}{2}-1$

$\therefore \iint \nabla\times\vec{F}\cdot\hat{n} = -1$. — (1)

considering RHS: $\int \vec{F}\cdot\vec{ds}$

Along OA: $(0,0,0)$ to $(1,0,0)$

Here, $y=0 \therefore dy=0$ and $z=0 \therefore dz=0$

x varies from 0 to 1. $= \int_0^1 xy\,dx = 0$ $(\because y=0)$

Along AB: $(1,0,0)$ to $(1,2,0)$

Here, $x=1 \therefore dx=0$ and $z=0 \therefore dz=0$

y varies from 0 to 2. $= \int_0^2 -2yz\,dy = 0$ $(\because z=0)$

Along BE: $(1,2,0)$ to $(0,2,0)$

Here, $y=2 \therefore dy=0$ and $z=0 \therefore dz=0$

x varies from 1 to 0 $= \int_1^0 xy\,dx = -2$.

Along EO: $(0,2,0)$ to $(0,0,0)$

Here, $x=0 \therefore dx=0$ and $z=0 \therefore dz=0$

y varies from 2 to 0 $= \int_2^0 -2yz\,dy = 0$

$\Rightarrow 0+0+0-1 = -1$. $\therefore$ LHS = RHS

$\therefore$ It is verified.

5) Show that $\vec{F} = (6xy+z^3)\vec{i} + (3x^2-z)\vec{j} + (3xz^2-y)\vec{k}$ is irrotational and hence find its scalar potential ϕ.

sol: Given: $\vec{F} = (6xy+z^3)\vec{i} + (3x^2-z)\vec{j} + (3xz^2-y)\vec{k}$

and we know that = Irrotational : $\boxed{\nabla\times\vec{F} = \vec{0}}$

$$\Rightarrow \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 6xy+z^3 & 3x^2-z & 3xz^2-y \end{vmatrix} = \vec{0}$$

$$= \left[\frac{\partial}{\partial y}(3x^2-y) - \frac{\partial}{\partial z}(3x^2-z)\right]\vec{i} - \left[\frac{\partial}{\partial x}(3xz^2-y) - \frac{\partial}{\partial z}(6xy+z^3)\right]\vec{j} + \left[\frac{\partial}{\partial x}(3x^2-z) - \frac{\partial}{\partial y}(6xy+z^3)\right]\vec{k}$$

$$= [(-1)-(-1)]\vec{i} - (3z^2-3z^2)\vec{j} + (6x-6x)\vec{k} = \vec{0}$$

$\therefore$ It is irrotational.

$$\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$$

comparing $\vec{i}$, $\vec{j}$ and $\vec{k}$:

$$\frac{\partial\phi_1}{\partial x} = 6xy+z^3 \; ; \; \frac{\partial\phi_2}{\partial y} = 3x^2-z \; ; \; \frac{\partial\phi_3}{\partial z} = 3xz^2-y$$

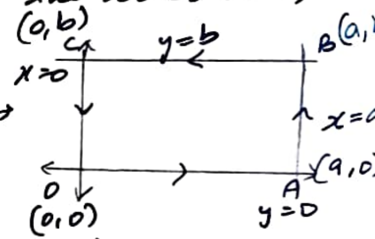
$$\phi_1 = \frac{\overset{3}{\cancel{6}}x^2y}{\cancel{2}} + z^3x = 3x^2y + xz^3$$

$$\phi_2 = 3x^2y - yz \; ; \; \phi_3 = \frac{\cancel{3}xz^3}{\cancel{3}} - yz$$

$\therefore \phi = \phi_1 + \phi_2 + \phi_3$ (Take only common

$$\Rightarrow \phi = 3x^2y + xz^3 - yz$$

6) Verify the Stoke's theorem for $\vec{F} = (x^2+y^2)\vec{i} - 2xy\,\vec{j}$ in a rectangular region $z=0$ plane bounded by the lines $x=0$, $x=a$, $y=0$ and $y=b$.



sol: Given:

$$\vec{F} = (x^2+y^2)\vec{i} - 2xy\,\vec{j}$$

we know that by Stoke's theorem: $\iint_S \nabla\times\vec{F}\cdot\hat{n}\,ds = \int_C \vec{F}\cdot d\vec{r}$

Taking LHS:

$$\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2+y^2 & 2xy & 0 \end{vmatrix}$$

$$= -\frac{\partial}{\partial z}2xy\,\vec{i} - \left[-\frac{\partial}{\partial z}(x^2+y^2)\right]\vec{j} + \left[\frac{\partial}{\partial x}(-2xy) - \frac{\partial}{\partial y}(x^2+y^2)\right]\vec{k} = -2y-2y = -4y\,\vec{k}$$

$$\therefore \nabla\times\vec{F}\cdot\hat{n} = -4y\,\vec{k}\cdot\vec{k} = -4y.$$

$$\int_0^a\int_0^b -4y\,dy\,dx = \int_0^a \left[-4\frac{y^2}{2}\right]_0^b dx = -2ab^2$$

$\vec{F}\cdot\vec{dr} = (x^2+y^2)dx - 2xy\,dy$.

considering RHS:

On OA: $(0,0)$ to $(a,0)$ Here, $y=0 \therefore dy=0$

x varies from 0 to a.

$$= \int_0^a (x^2+y^2)dx = \left[\frac{x^3}{3}+xy^2\right]_0^a = \frac{a^3}{3}+ay^2 \quad (\because y=0)$$

$$\Rightarrow \frac{a^3}{3}$$

On AB: $(a,0)$ to (a,b) Here, $x=a \therefore dx=0$

y: 0 to b

$$= \int_0^b -2xy\,dy = -ab^2$$

On BC: (a,b) to $(0,b)$ Here, $y=b \therefore dy=0$

x varies from a to 0 $\quad \int_a^0 (x^2+y^2)dx = -\frac{a^3}{3} - ab^2$

On CO: $(0,b)$ to $(0,0)$ Here, $x=0 \therefore dx=0$

y varies from b to 0 $\quad \int_b^0 -2xy\,dy = 0$

$$\Rightarrow \int_{OA} + \int_{AB} + \int_{BC} + \int_{CO} = \cancel{\frac{a^3}{3}} - ab^2 - \cancel{\frac{a^3}{3}} - ab^2 = -2ab^2$$

$\therefore$ LHS = RHS and stoke's theorem is verified.

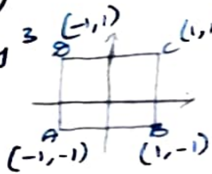
7) verify the Green's theorem $\int_C [x^2(1+y)dx + (x^3+y^3)dy]$ where c is a bounded region by the lines $x=\pm 1$ and $y=\pm 1$.

Sol: Given: $\int_C x^2(1+y)dx + (x^3+y^3)dy$ and

we know that for Green's theorem:

$$\oint_C [P\,dx + Q\,dy] = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

Here, $P = x^2+x^2y$ and $Q = x^3+y^3$



considering RHS:

$$\frac{\partial Q}{\partial x} = 3x^2 \text{ and } \frac{\partial P}{\partial y} = x^2$$

$$\Rightarrow \iint_R (3x^2-x^2)dx\,dy = \int_{-1}^1\int_{-1}^1 2x^2\,dx\,dy$$

$$= 2\int_{-1}^1 \left[\frac{x^3}{3}\right]_{-1}^1 dy = \frac{4}{3}\int_{-1}^1 dy = \frac{4}{3}[y]_{-1}^1$$

$$= \frac{4}{3}[1-(-1)] = \frac{4}{3}(2) = \frac{8}{3}$$

$$\therefore \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy = \frac{8}{3} \quad -(1)$$

considering LHS: $\int_C P\,dx + Q\,dy = \int_{AB} + \int_{BC} + \int_{CD} + \int_{DA}$

Along AB : $(-1,-1)$ to $(1,-1)$ $\quad y=-1 \therefore dy=0$

Here, x varies from -1 to 1.

$$= \int_{-1}^{1} (x^2+x^2y)\,dx + 0 = \int_{-1}^{1} (x^2+x^2y) = x^2 - x^2\,dx$$

$$= \int_{-1}^{1} x^2 + x^2(-1) = \underline{0}.$$

Along BC : $(1,-1)$ to $(1,1)$ $\quad x=1 \therefore dx=0$

Here, y varies from -1 to 1:

$$= \int_{-1}^{1} (1+y^3)\,dy = \left[y + \frac{y^4}{4}\right]_{-1}^{1} = \frac{2}{4} - \left(\frac{-1+(-1)^4}{4}\right)$$

$$= \frac{8}{4} = \underline{2}.$$

Along CD : $(1,1)$ to $(-1,1)$ $\quad y=1 \therefore dy=0$

Here, x varies from 1 to -1.

$$= \int_{1}^{-1} x^2(1+1)\,dx = \int_{1}^{-1} 2x^2\,dx = 2\left[\frac{x^3}{3}\right]_1^{-1} = 2\left(\frac{-2}{3}\right)$$

$$= \underline{\frac{-4}{3}}.$$

Along DA : $(1,1)$ to $(-1,-1)$ $\quad x=-1 \therefore dx=0$

Here, y varies from 1 to -1

$$= \int_{1}^{-1} (-1+y^3)\,dy = \left[-y + \frac{y^4}{4}\right]_1^{-1} = \frac{8}{4} = 2$$

$$\Rightarrow \int_{AB} + \int_{BC} + \int_{CD} + \int_{DA} = 0 + 2 - \frac{4}{3} + 2 = \frac{8}{3} \quad \text{—} (2)$$

$\therefore (1) = (2)$ i.e, LHS = RHS.

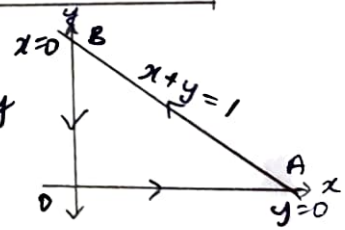
$\therefore$ The Green's theorem is verified.

8) verify the Green's theorem in xy plane for $\int_C (3x-8y^2)\,dx + (4y-6xy)\,dy$ where C is a bounded region by $x=0, y=0; x+y=1$

Sol: Given : $x=0$, $y=0$, $x+y=1$

$$\boxed{\oint M\,dx + N\,dy = \iint \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\,dx\,dy}$$

RHS :



$\frac{\partial N}{\partial x} = -6y$ and $\frac{\partial M}{\partial y} = -16y$

In axis-x : $y=0$

$\therefore x=1$

In y axis : $x=0$

$\therefore y=1-x$

$$\int_0^1\int_0^{1-x} -6y+16y\,dy\,dx = \int_0^1\left[\int_0^{1-x} 10y\,dy\right]dx$$

$4y$

$$= \int_0^1 \left[10\cdot\frac{y^2}{2}\right]_0^{1-x} dx = 5\int_0^1 (1-x^2)dx$$

$$=5\int_0^1 (1-2x+x^2)dx = 5\int_0^1 x+\frac{x^3}{3}-\frac{2x^2}{2}$$

$$= 5\left[1+\frac{1}{3}-\frac{2}{2}\right] = \frac{5}{3} \quad -(1)$$

LHS: $\int_C M dx + N dy = \int_{OA} + \int_{AB} + \int_{BO}$

on OA: $(0,0)$ to $(1,0)$ $[y=0 \therefore dy=0]$

x varies from 0 to 1.

$$\int_0^1 3x\,dx = 3\left[\frac{x^2}{2}\right]_0^1 = \frac{3}{2}$$

on AB: $(1,0)$ to $(0,1)$

$x+y=1$ (Given)

$x=1-y \quad \therefore dx=-dy$

x varies from 1 to 0

Note: Taking 1 to 0 & not 0 to 1. R: cyclic (mentioned in Q)

$$= \int_0^1 3x-8(1-x^2)dx + 4(1-x) -6x(1-x)dx$$

$$= \int_0^1 (3x-8+8x^2)dx + (4-4x-6x+6x^2)dx$$

$$= \int_0^1 (3x-4x-6x-8+4-6+8x^2+6x^2)dx$$

$$= \int_0^1 (-7x-10+14x^2)dx = \int_0^1 -7\cdot\frac{x^2}{2} -10x+\frac{14x^3}{3} dx$$

$$= \left[-7\cdot\frac{1}{2}-10+\frac{14}{3}\right] = \frac{-7}{2}+10+\frac{14}{3}$$

$$= \frac{13}{6}$$

on BO: $(0,1)$ to $(0,0)$

$x=0 \therefore dx=0$ $\quad$ y varies from 1 to 0.

$$\int_1^0 4y\,dy = 4\left[\frac{y^2}{2}\right]_1^0 = 4\left(\frac{-1}{2}\right) = -2$$

$$\therefore \int_{OA} + \int_{AB} + \int_{BO} = \frac{3}{2}+\frac{13}{6}-\frac{2}{1}$$

$$= \frac{9+13-12}{6} = \frac{10}{6} = \frac{5}{3} \quad -(2)$$

$\therefore (1) = (2)$

Hence, Green's theorem is verified.