→ LAPLACE TRANSFORMS

Introduction-

* it was introduced by Pierre - Simon Laplace.
* developed by Olive.
* a tool for solving differential equation.
* changes real to complex variables.

Definition -

→ Laplace transform of a function f(t) defined for all real number, $t \geq 0$ is the function F(s),

$$L\{f(t)\} = F(s) = \int_0^\infty f(t)\, e^{-st}\, dt$$

s → real variable

t → complex variable

* this theorem is applicable on periodic functions.

→ Laplace Transforms of some std. functions -

** $f(t) \longrightarrow L\{f(t)\}$

① $e^{at} \rightarrow 1/s-a,\ s-a>0$

② $e^{-at} \rightarrow 1/s+a,\ s+a>0$

③ $\sin at = a/s^2+a^2,\ s>0$

④ $\cos at = s/s^2+a^2,\ s>0$

⑤ $\sinh at = a/s^2-a^2,\ s>|a|$

⑥ $\cos hat \rightarrow s/s^2-a^2,\ s>|a|$

⑦ $1 \rightarrow 1/s,\ s>0$ ⑨ $t^n (n=int) \rightarrow \dfrac{n!}{s^{n+1}}$

⑧ $t \rightarrow 1/s^2,\ s>0$ ⑩ $t^n (n \neq int) \rightarrow \dfrac{\Gamma(n+1)}{s^{n+1}}$ → gamma

* periodic function with period p $\Bigg\} = \dfrac{1}{1-e^{-sp}} \displaystyle\int_0^p e^{-at} f(t)\,dt$

---

① PT- $L(e^{at}) = \dfrac{1}{s-a}$, $s-a>0$

Proof -

wkt, $L(f(t)) = \displaystyle\int_0^\infty e^{-st} f(t)\,dt$ —①

$f(t) = e^{at}$

$$① \Rightarrow \int_0^\infty e^{-st} e^{at}\,dt \Rightarrow \int_0^\infty e^{-st+at}\,dt \Rightarrow \int_0^\infty e^{-t(s-a)}\,dt$$

$$\Rightarrow \left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_0^\infty \Rightarrow \frac{1}{s-a}\ //$$

Illy for e^{-at} → we get $1/s+a$

---

③ PT $L[\cosh at] = \dfrac{s}{s^2-a^2}$

$$L[\cosh at] = \int_0^\infty e^{-st} \cosh at\,dt$$

$$\cosh at = \frac{e^{at}+e^{-at}}{2}$$

$$\Rightarrow \int_0^\infty e^{-st}\left[\frac{e^{at}+e^{-at}}{2}\right]dt$$

$$\Rightarrow \frac{1}{2}\int_0^\infty (e^{-st}e^{at}+e^{-st}e^{-at})dt$$

$$\Rightarrow \frac{1}{2}\left(\frac{1}{s-a}+\frac{1}{s+a}\right) = \frac{s}{s^2-a^2}$$

→ Properties of Laplace Transforms -

Linear Property -

$$L[a f(t) \pm b g(t)] = aL[f(t)] + bL[g(t)]$$

where a and b are constants.

→ Change of scale Property -

If $L(f(t)) = F(s)$, then $L[f(at)] = \frac{1}{a}F(s/a)$,

$s>0$

→ First Shifting Theorem -

If $L[f(t)] = F(s)$, then

(i) $L[e^{-at}f(t)] = F(s+a)$

(ii) $L[e^{at}f(t)] = F(s-a)$

→ Multiplication by T -

If $L[f(t)] = F(s)$,

then $L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n}(F[s])$

→ Division by 't' -

$$L\left[\frac{f(t)}{t}\right] = \int_s^\infty F(s)\,ds \text{ provided } \lim_{t\to 0} \frac{f(t)}{t} \text{ is finite}$$

→ Transform of Integrals -

$$L\left[\int_0^t f(t)\,dt\right] = \frac{1}{s} L[f(t)] = \frac{L[f(t)]}{s}$$

→ Problems on Linear Property

① find $L[3t^2+2t+1]$

Sol.

$$L[3t^2+2t+1] = 3L[t^2] + 2L[t] + L[1]$$

$$= 3\left[\frac{2!}{s^{2+1}}\right] + 2\left[\frac{1}{s^2}\right] + \frac{1}{s}$$

$$= \frac{6}{s^3} + \frac{2}{s^2} + \frac{1}{s}$$

② find $L((t+2)^3]$

Sol.

$$L[(t+2)^3] = L[(t^3+6t^2+12t+8)]$$

$$= L[t^3] + 6L[t^2] + 12L[t] + 8L[1]$$

$$= \frac{3!}{s^4} + 6\left[\frac{2!}{s^3}\right] + \frac{12}{s^2} + 8\left(\frac{1}{s}\right)$$

③ find $L[\cos^2 2t]$

sol wkt $\cos 2\theta = 2\cos^2\theta - 1$

$$\cos^2\theta = \frac{1+\cos 2\theta}{2}$$

$$\cos^2 2\theta = \frac{1+\cos 4\theta}{2}$$

$$L\left([\cos^2 2t]\right) = L\left[\frac{1+\cos 4t}{2}\right]$$

$$= \frac{1}{2} L[1] + L[\cos 4t]$$

$$L[\cos^2 2t] = \frac{1}{2}\left[\frac{1}{s} + \frac{s}{s^2+4^2}\right]$$

$$= \frac{1}{2}\left\{\frac{1}{s} + \frac{s}{s^2+16}\right\}$$

Problems using first shifting theorem -

$\rightarrow L[e^{-at} f(t)] = L[f(t)]_{s \rightarrow s+a}$ → note the sign

$\rightarrow L[e^{at} f(t)] = L[f(t)]_{s \rightarrow s-a}$

* find $L(te^{-3t})$

$$L[te^{-3t}] = L[t]_{s \rightarrow s+a}$$

$= \left(1/s^2\right)_{s \to s+3}$

$L\left[te^{-3t}\right] = \frac{1}{(s+3)^2}$

(2) find $L\left[e^{4t}\sin 2t\right]$

$L\left[e^{4t}\sin 2t\right] = L\left[\sin 2t\right]_{s \to s-4}$

$= \left[\frac{2}{s^2+2^2}\right]_{s \to s-4}$

$= \frac{2}{(s-4)^2+4} = \frac{2}{s^2-8s+20}$

$L\left[e^{4t}\sin 2t\right] = \frac{2}{s^2+8s+20}$ ||

---

* $L(t^2\sin 2t) = (-1)^2 \frac{d^2}{ds^2} L(\sin 2t)$

$\Rightarrow (-1)^2 \frac{d^2}{ds^2} \times \left[\frac{a^2}{s^2+a^2}\right]$

$\Rightarrow \frac{d^2}{ds^2}\left[\frac{2}{s^2+4}\right] \Rightarrow \frac{d}{ds}\left(\frac{d}{ds}\left[\frac{2}{s^2+4}\right]\right)$

$u = 2 \quad u' = 0$

$v = s^2+a^2 \quad v' = 2s$

* find $L(t\sin 4t)$

$n=1 \Rightarrow (-1)^1 \dfrac{d}{ds} L(\sin 4t)$

$\Rightarrow -1 \dfrac{d}{ds}\left[\dfrac{4}{s^2+16}\right]$

$\Rightarrow -1 \cancel{\dfrac{d}{ds}}\left[\dfrac{-4(2s)}{(s^2+16)^2}\right]$

$\Rightarrow \dfrac{4\times 2s}{(s^2+16)^2} \Rightarrow \dfrac{8s}{(s^2+16)^2}$ //

---

① find $L\left[\dfrac{\cos at}{t}\right]$

→ check $\lim\limits_{t\to 0} \dfrac{f(t)}{t}$ is finite

$\lim\limits_{t\to 0} \dfrac{\cos at}{t} = \dfrac{\cos(0)}{0} = \dfrac{1}{0} = \text{infinite}$

$L\left[\dfrac{\cos at}{t}\right]$ does not exist.

---

② find $L\left[\dfrac{\sin at}{t}\right]$

$\lim\limits_{t\to 0} \dfrac{\sin at}{t} = \dfrac{0}{0} = \text{indefinite}.$

L'Hospital Rule

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{0}{0} \text{ (or) } \frac{\infty}{\infty} \Rightarrow \lim_{x\to a} \frac{f(x)}{g(x)} =$$

$$\lim_{x\to a} \frac{f'(x)}{g'(x)}$$

By applying L'Hospital Rule,

$$\lim_{t\to 0} \frac{\sin at}{t} = \lim_{t\to 0} \frac{a\cos at}{1}$$

$$= \frac{a\cos(0)}{1} = a \text{ (finite)}$$

So $L\left[\frac{\sin at}{t}\right]$ exist.

$$L\left[\frac{\sin at}{t}\right] = \int_s^\infty L[\sin at]\, ds$$

$$= \int_s^\infty \frac{a}{a^2+s^2}\, ds = a\left[\frac{1}{a}\tan^{-1} s/a\right]_s^\infty$$

$$= \tan^{-1}(s/a)_s^\infty$$

$$= \tan^{-1}(\infty) - \tan^{-1}(s/a)$$

$$= \pi/2 - \tan^{-1}(s/a)$$

HW - $L\left[\frac{e^{-at} - e^{-bt}}{t}\right]$

Property -

$$L\left[\int_0^t f(t)\,dt\right] = \frac{1}{s} L[f(t)]$$

① find $L\left[\int_0^t e^{-2t} dt\right] = \frac{1}{s} L[f(t)]$

Sol. $f(t) = e^{-2t}$

$$= \frac{1}{s} L(e^{-2t})$$

$$= \frac{1}{s}\left[\frac{1}{s+2}\right] //$$

② $L\left[\int_0^t t \sin 3t\, dt\right] =$

$f(t) = t \sin 3t$

$$\Rightarrow \frac{1}{s} \times L[t \sin 3t]$$

$$= \frac{1}{s} \times -1 \frac{d}{ds}\left[\frac{3}{s^2+9}\right]$$

$$= \frac{-1}{s} \times \frac{-2s \times 3}{(s^2+9)^2} \Rightarrow \frac{+6\cancel{s}}{\cancel{s}(s^2+9)^2}$$

$$\Rightarrow \frac{+6}{(s^2+9)^2} //$$

Problems using $L\left[\int_0^t f(t)\,dt\right] = \frac{1}{s} L[f(t)]$

---

* find $L\left[\int_0^t e^{-2t}\,dt\right]$

$$L\left[\int_0^t e^{-2t}\,dt\right] = L(e^{-2t}) = \frac{1}{s}\left(1/s+2\right)$$

---

* Problems on integrals using LT-

---

① $\int_0^\infty f(t)e^{-3t}\,dt = L[f(t)]dt$

② $\int_0^\infty f(t)e^{-at}\,dt = L[f(t)]_{s=a}$

③ $\int_0^\infty f(t)\,dt = L[f(t)]_{s=0}$

---

① If $L[f(t)] = \frac{s+2}{s+4}$ the find the value of $\int_0^\infty f(t)dt$

Sol. wkt $\int_0^\infty f(t)dt = L[f(t)]_{s=0}$

$$L[f(t)] = \frac{s+2}{s+4} = \frac{2}{4} = 1/2$$

---

② find $\int_0^\infty t^2 e^{-t} \sin t\,dt$

wkt $\int_0^\infty e^{-at} f(t)\,dt = [L[f(t)]]_{s=a}$

$$\Rightarrow \int_0^\infty e^{-t}\, t^2 \sin t\, dt = L\left[t^2 \sin t\right]_{s=1}$$

$$= (-1)^2 \frac{d^2}{ds^2} L(\sin t)_{s=1}$$

$$= \frac{d^2}{ds^2}\left[\frac{1}{s^2+1}\right]_{s=1} = \frac{d}{ds}\left\{\frac{d}{ds}\left[\frac{1}{s^2+1}\right]\right\}_{s=1}$$

$$= \frac{d}{ds}\left[\frac{-2s}{(s^2+1)^2}\right]_{s=1} \qquad \frac{d}{ds}\left(\frac{u}{v}\right) = \frac{u'v - uv'}{v^2}$$

$u = 2s \quad v = (s^2+1)^2$

$u' = 2 \quad v' = 2(s^2+1)2s$

$v' = 4s(s^2+1)$

$$= -\left[\frac{2(s^2+1)^2 - 2s(4s(s^2+1))}{(s^2+1)^4}\right]$$

$$= \left[\frac{-2(s^2+1)^2 + 8s^2(s^2+1)}{(s^2+1)^4}\right]_{s=1}$$

$$= \frac{-2(4)+8(2)}{(2)^4} = \frac{-8+16}{16} \Rightarrow \frac{1}{2}$$

→ Problems using LT of Piecewise Continuous function -

---

formula -

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt$$

① find the LT of $f(t) = \begin{cases} e^{-t} & 0<t<\pi \\ 0 & t>\pi \end{cases}$

Sol

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt$$

$$= \int_0^\pi e^{-st} f(t)\,dt + \int_\pi^\infty e^{-st} f(t)\,dt$$

$$= \int_0^\pi e^{-st} e^{-t}\,dt + \int_\pi^\infty e^{-st}(0)\,dt$$

$$= \int_0^\pi e^{-t(s+1)}\,dt$$

$$= \left[\frac{e^{-t(s+1)}}{-(s+1)}\right]_0^\pi = \frac{e^{-\pi(s+1)}}{-(s+1)} + \frac{e^0}{s+1}$$

$$= \frac{-e^{-\pi(s+1)}+1}{(s+1)}$$

Problems using LT of unit step function-

$$u(t-a) = \begin{cases} 0 & \text{where } t<a \\ 1 & \text{where } t>a \end{cases}$$

find the LT of unit step function-

$$L[u(t-a)] = \int_0^\infty u(t-a)\, e^{-st}\,dt$$

$$= \int_0^a u(t-a)\, e^{-st}\, dt + \int_a^\infty u(t-a)\, e^{-st}\, dt$$

$$= \int_0^a 0\, e^{-st}\, dt + \int_a^\infty (1)\, e^{-st}\, dt$$

$$= \int_a^\infty e^{-st}\, dt = \left[\frac{e^{-st}}{-s}\right]_a^\infty$$

$$= \frac{e^{-\infty}}{-s} + \frac{e^{-as}}{s} = \frac{e^{-as}}{s} \quad //$$

Problems using LT of Periodic function -

$$L[f(t)] = \frac{1}{1-e^{-sp}} \int_0^p f(t)\, e^{-st}\, dt$$

---

① find the LT of the square wave function defined by -

$$f(t) = \begin{cases} E & 0 < t < a/2 \\ -E & a/2 < t < a \end{cases}$$

period $p = a$

$$L[f(t)] = \frac{1}{1-e^{-sp}} \int_0^p f(t)\, e^{-st}\, dt$$

$$= \frac{1}{1-e^{-sa}} \left[\int_0^{a/2} f(t) e^{-st} dt + \int_{a/2}^{a} f(t) e^{-st} dt \right]$$

$$= \frac{1}{1-e^{-sa}} \left[\int_0^{a/2} E e^{-st} dt + \int_{a/2}^{a} (-E) e^{-st} dt \right]$$

$$= \frac{E}{1-e^{-sa}} \left[\int_0^{a/2} e^{-st} dt - \int_{a/2}^{a} e^{-st} dt \right]$$

$$= \frac{E}{1-e^{-sa}} \left[\left[\frac{e^{-st}}{-s} \right]_0^{a/2} + \left[\frac{e^{-st}}{s} \right]_{a/2}^{a} \right]$$

$$= \frac{E}{s(1-e^{-sa})} \left\{ \left[e^{-st} \right]_{a/2}^{a} - \left[e^{-st} \right]_0^{a/2} \right\}$$

$$= \frac{E}{s(1-e^{-sa})} \left\{ \left(e^{-sa} - e^{-s(a/2)} \right) - \left[e^{-sa/2} - e^{0} \right] \right\}$$

$$= \frac{E}{s(1-e^{-sa})} \left(e^{-sa} - 2e^{-sa/2} + 1 \right) //$$

→ Inverse Laplace Transforms -

* If the LT of a function $f(t)$ is $L[f(t)]$, i.e., $L[f(t)] = F(s)$. Then $f(t)$ is called an inverse Laplace Transform of $F(s)$ and we write symbolically $f(t) = L^{-1}[F(s)]$ where L^{-1} is called the Inverse Laplace Transform (LIT) operator.

Formulas -

1. $L[f(t)] = F(s) \longrightarrow L^{-1}[F(s)] = f(t)$

2. $L(1) = 1/s \longrightarrow L^{-1}(1/s) = 1$

3. $L(t) = 1/s^2 \longrightarrow L^{-1}(1/s^2) = t$

4. $L(t^n) = \frac{n!}{s^{n+1}} \longrightarrow L^{-1}\left[\frac{n!}{s^{n+1}}\right] = t^n$

5. $L(e^{-at}) = \frac{1}{s+a} \longrightarrow L^{-1}\left[\frac{1}{s+a}\right] = e^{-at}$

6. $L(e^{at}) = \frac{1}{s-a} \longrightarrow L^{-1}\left[\frac{1}{s-a}\right] = e^{at}$

7. $L(\sin at) = \frac{a}{s^2+a^2} \longrightarrow L^{-1}\left[\frac{a}{s^2+a^2}\right] = \sin at$

8. $L(\cos at) = \frac{s}{s^2+a^2} \longrightarrow L^{-1}\left[\frac{as}{s^2+a^2}\right] = \cos at$

9. $L[\sinh at] = \frac{a}{s^2-a^2} \longrightarrow L^{-1}\left[\frac{a}{s^2-a^2}\right] = \sinh at$

10. $L[\cosh at] = \frac{s}{s^2-a^2} \longrightarrow L^{-1}\left[\frac{s}{s^2-a^2}\right] = \cosh at.$

Linear Property -

$$L^{-1}[aF(s) + bG(s)] = a L^{-1}[F(s)] + b L^{-1}[G(s)]$$

* $$L^{-1}\left[\frac{s^3 - 3s^2 + 7}{s^4}\right] \Rightarrow L^{-1}\left[\frac{s^3}{s^4} - \frac{3s^2}{s^4} + \frac{7}{s^4}\right]$$

$$\Rightarrow L^{-1}\left[\frac{1}{s} - \frac{3}{s^2} + \frac{7}{s^4}\right]$$

$$\Rightarrow 1 - 3t + 7\left[t^3/3!\right]$$

---

* find $$L^{-1}\left[\frac{3s+5}{s^2+36}\right]$$

$$= L^{-1}\left[\frac{3s}{s^2+36} + \frac{5}{s^2+36}\right]$$

$$= 3\cos 6t + \frac{5}{6}\sin 6t \;//$$

(*) Inverse LT using partial fraction method.

→ find $$L^{-1}\left[\frac{s-2}{s(s+2)(s-1)}\right]$$

$$\frac{s-2}{s(s+2)(s-1)} = \frac{A}{s} + \frac{B}{s+2} + \frac{C}{s-1}$$

$$\frac{s-2}{s(s+2)(s-1)} = \frac{A(s+2)(s-1) + B(s)(s-1) + C(s)(s+2)}{s(s+2)(s-1)}$$

$$s-2 = A(s+2)(s-1) + B(s)(s-1) + C(s)(s+2).$$

put $s=0$

$$-2 = A(2)(-1) + B(0) + C(0)$$

$$-2 = -2A \Rightarrow \boxed{A=1}$$

put $s=1$

$$-1 = A(3)(0) + B(0) + C(1)(3)$$

$$-1 = 3C \Rightarrow \boxed{C = -1/3}$$

put $s=-2$

$$-4 = B(-2)(-3)$$

$$-4 = 6B \Rightarrow \boxed{B = -2/3}$$

$$\frac{s-2}{s(s+2)(s-1)} = \frac{1}{s} - \frac{2}{3(s+2)} - \frac{1}{3(s-1)}$$

$$L^{-1}\left[\frac{s-2}{s(s+2)(s-1)}\right] = L^{-1}\left[\frac{1}{s} - \frac{2}{3(s+2)} - \frac{1}{3(s-1)}\right]$$

$$= 1 - \frac{2}{3}e^{-2t} - \frac{1}{3}e^{t} //$$

→ Convolution of two functions -

The convolution of two functions $f(t)$ and $g(t)$ is denoted by $f(t) * g(t)$ and defined by

$$f(t) * g(t) = \int_0^t f(u)\, g(t-u)\, du$$

→ Convolution theorem on Laplace transform.

If $L[f(t)] = F(s)$ and $L[g(t)] = G(s)$,

then $L[f(t) * g(t)] = F(s)\, G(s)$.

Note - convolution theorem for Inverse LT.

→ $L[f(t) * g(t)] = F(s)\, G(s)$

→ $L^{-1}[F(s)\, G(s)] = f(t) * g(t)$

→ $L^{-1}[F(s)] * L^{-1}[G(s)] = L^{-1}[F(s) G(s)]$.

---

① find $L^{-1}\left[\frac{1}{(s+a)(s+b)}\right]$ using convolution theorem

$L^{-1}[F(s)\, G(s)] = L^{-1}[F(s)] * L^{-1}[G(s)]$

$\Rightarrow L^{-1}\left[\frac{1}{(s+a)(s+b)}\right] = L^{-1}\left[\frac{1}{(s+a)}\right] * L^{-1}\left[\frac{1}{(s+b)}\right]$

$= e^{-at} * e^{-bt}$

wkt, $f(t) * g(t) = \int_0^t f(u)\, g(t-u)\, du$

$f(t) = e^{-at} \qquad g(t) = e^{-bt}$

$f(u) = e^{-au} \qquad \& \; g(t-u) = e^{-b(t-u)}$

$$= \int_0^t e^{-au} \, e^{-b(t-u)} \, du$$

$$= \int_0^t e^{-au} \, e^{-bt} \, e^{bu}$$

$$= e^{-bt} \int_0^t e^{-au} \, e^{bu} \, du$$

$$\Rightarrow e^{-bt} \int_0^t e^{u(b-a)} \, du$$

$$\Rightarrow e^{-bt} \int_0^t e^{u(b-a)} \, du$$

$$\Rightarrow e^{-bt} \left[\frac{e^{u(b-a)}}{(b-a)} \right]_0^t$$

$$\Rightarrow e^{-bt} \, \frac{e^{t(b-a)}}{(b-a)}$$

$$\Rightarrow \frac{e^{-bt} \, e^{bt} * e^{-at}}{(b-a)}$$

$$\Rightarrow \frac{e^{-bt}}{b-a} \left[e^{t(b-a)} - 1 \right] \;//$$

→ Solution of diff. eqn by Laplace Transform

$$L[y'(t)] = sL[y(t)] - y(0)$$

$$L[y''(t)] = s^2 L[y(t)] - s\,y(0) - y'(0)$$

Problem - Solve by using LT $y'' + 5y' + 6y = 2$

given $y(0) = y'(0) = 0$

Sol

Taking LT on both sides,

$$L(y'' + 5y' + 6y) = L(2)$$

$$L(y'') + 5\,L(y') + 6\,L(y) = 2\,L(1)$$

$$s^2 L[y(t) - s y(0) - y'(0)] + 5\,[sL(y(t)) - y(0)]$$

$$+ 6\,L[y(t)] = 2\,L(1)$$

$$\rightarrow s^2 L[y(t)] - 0 - 0 + 5sL[y(t)] - 0$$

$$+ 6\,L[y(t)] = 2L(1)$$

$$= L[y(t)]\,(s^2 + 5s + 6) = 2\,L(1)$$

$$\Rightarrow L[y(t)]\{(s+2)(s+3)\} = 2\,(1/s)$$

$$L[y(t)] = \frac{2}{s(s+2)(s+3)}$$

$$y(t) = L^{-1}\left[\frac{2}{s(s+2)(s+3)}\right]$$

<u>contd..</u>

$$y(t) = L^{-1} \frac{2}{s(s+2)(s+3)}$$

$$\frac{2}{s(s+2)(s+3)} = \frac{A}{s} + \frac{B}{s+2} + \frac{C}{s+3}$$

$$\frac{2}{\cancel{s(s+2)(s+3)}} = \frac{A(s+2)(s+3) + B\,s(s+3) + C\,s(s+2)}{\cancel{s(s+2)(s+3)}}$$

$\cancel{A}$ $s = 0 \Rightarrow A = 1/3$

$s = -2 \Rightarrow B = -1$

$s = -3 \Rightarrow C = 2/3$

$$\frac{2}{s(s+2)(s+3)} = \frac{1}{3(s)} + \frac{-1}{s+2} + \frac{2}{3(s+3)}$$

$$L^{-1}\left(\frac{2}{s(s+2)(s+3)}\right) = 3L^{-1}\left(\frac{1}{s}\right) - L^{-1}\left[\frac{1}{s+2}\right] + \frac{2}{3}L^{-1}\left[\frac{1}{s+3}\right]$$

$$= 1/3 \times 1 - e^{-2t} + \frac{2}{3}e^{-3t}$$

$$= 1/3 - e^{-2t} + \frac{2}{3}e^{-3t}$$ //

→ Initial Value Theorem -

If $L[f(t)] = F(s)$

then $\lim_{t \to 0} f(t) = \lim_{s \to \infty} sF[s]$

→ Final Value Theorem -

If the LT of $f(t)$ & that $f(t)$ exists, then -

$$\lim_{t \to \infty} f(t) = \lim_{s \to 0} sF[s].$$

---

① verify the IVT & FVT for

$$1+e^{-t}(\sin t + \cos t)$$

Sol.

Step 1 - to verify IVT.

$$\lim_{t \to 0} f(t) = \lim_{s \to \infty} sF(s)$$

given $f(t) = 1 + e^{-t}(\sin t + \cos t)$

$$L[f(t)] = F(s)$$

$$L[f(t)] = L\{1 + e^{-t}[(\sin t) + (\cos t)]\}$$

$$= L(1) + L(e^{-t}\sin t) + L(e^{-t}\cos t)$$

$$= 1/s + L(\sin t)_{s \to s+1} + L(\cos t)_{s \to s+1}$$

$$= 1/s + \left.\frac{1}{s^2+1}\right|_{s\to s+1} + \left.\frac{s}{s^2+1}\right|_{s\to s+1}$$

$$F(s) = \frac{1}{s} + \frac{1}{(s+1)^2+1} + \frac{s+1}{(s+1)^2+1} \;//$$

$$\lim_{t\to 0} f(t) = \lim_{t\to 0} \left(1 + e^{-t}(\sin t + \cos t)\right)$$

$$= 1 + e^0 (\sin 0 + \cos 0)$$

$$= 1 + 1\,(0+1) = 2 \;//$$

$$\lim_{s\to\infty} sF(s) = \lim_{s\to\infty} s\left(\frac{1}{s} + \frac{1}{(s+1)^2+1} + \frac{s+1}{(s+1)^2+1}\right)$$

$$= \lim_{s\to\infty}\left[\frac{s}{s} + \frac{s}{s^2+2s+2} + \frac{s(s+1)}{s^2+2s+2}\right]$$

$$= \lim_{s\to\infty}\left[1 + \frac{s}{s^2(1+2/s+2/s^2)} + \frac{s^2(1+1/s)}{s^2(1+2/s+2/s^2)}\right]$$

$$= \lim_{s\to\infty}\left\{1 + \frac{1}{s(1+2/s+2/s^2)} + \frac{(1+1/s)}{1+2/s+2/s^2}\right\}$$

$$= \left\{1 + \frac{1}{\infty(1+2/\infty+2/\infty)} + \frac{1+1/\infty}{1+2/\infty+2/\infty}\right\}$$

$$= 1 + 0 + 1 = 2 \;//$$

Step 2 - to verify FVT.

$$\lim_{t \to \infty} f(t) = \lim_{t \to \infty} 1 + e^{-t}(\sin t + \cos t)$$

$$= 1 + e^{-\infty}(\sin \infty + \cos \infty)$$

$$= 1 + 0(\ ) = 1 + 0 = 1.$$

$$\lim_{s \to 0} sF(s) = \lim_{s \to 0} s\left[\frac{1}{s} + \frac{1}{(s+1)^2+1} + \frac{s+1}{(s+1)^2+1}\right]$$

$$= \lim_{s \to 0}\left[1 + \frac{s}{s^2+2s+2} + \frac{s(s+1)}{(s^2+2s+2)}\right]$$

$$= \left[1 + \frac{0}{0+0+2} + \frac{0(0+1)}{0+0+2}\right]$$

$$= 1 + 0 + 0 = 1$$