Continuity of multivariate rational functions

Ali Sinan Sertöz

Abstract

The limiting behavior of a multivariate rational function at its only singularity is read off from the exponents that appear in the expression of the function. We give two proofs of the result, one uses a direct approach and the other uses Lagrange multipliers method.

The behavior of a multivariable rational function at its singularities is erratic. The simplest case where we have a chance of understanding its behavior is when the denominator vanishes only at the origin. In two-variable case this rational function defines a surface which either intersects the z-axis at one point or wraps around it at the origin. To decide which-happens-when is a tricky process. For this reason not many examples float in the literature. For example how do we calculate

$$\lim_{(x,y,z)\to(0,0,0)} \frac{x^3y^2z}{x^4+y^{12}+z^{14}}, \quad \text{or} \quad \lim_{(x,y,z)\to(0,0,0)} \frac{x^3y^2z^2}{x^4+y^{12}+z^{14}}?$$

For a multivariable rational function whose denominator vanishes only at the origin, the continuity of this function at the origin must certainly be encoded in the exponents of the variables. The task is therefore to decipher this code, which is given by the following theorem:

Theorem: *Let $a_1,\dots,a_N$ be non-negative integers, $m_1,\dots,m_N$ be positive integers and $c_1,\dots,c_N$ be positive real numbers, where $N>1$. Then*

$$\lim_{(x_1,\dots,x_N)\to(0,\dots,0)} \frac{x_1^{a_1}\cdots x_N^{a_N}}{c_1x_1^{2m_1}+\cdots+c_Nx_N^{2m_N}} \ \textit{exists if and only if} \ \sum_{i=1}^{N}\frac{a_i}{2m_i}>1.$$

Moreover, when the limit exists, then it is zero.